Tonal partition algebras: fundamental and geometrical aspects of representation theory

Chwas Ahmed[1,2,*], Paul Martin[2], and Volodymyr Mazorchuk[3]

[1] *Department of Mathematics, University of Sulaimani, Kurdistan Region, Iraq*

[2] *School of Mathematics, University of Leeds, Leeds, UK*

[3] *Department of Mathematics, Uppsala University, Uppsala, Sweden*

[*] *Corresponding author: Chwas Ahmed, chwas.ahmed@univsul.edu.iq*

Abstract

For $l, n \in \mathbb{N}$ we define tonal partition algebra P_n^l over $\mathbb{Z}[\delta]$. We construct modules $\{\Delta_{\underline{\mu}}\}_{\underline{\mu}}$ for P_n^l over $\mathbb{Z}[\delta]$, and hence over any integral domain containing $\mathbb{Z}[\delta]$ (such as $\mathbb{C}[\delta]$), that pass to a complete set of irreducible modules over the field of fractions. We show that P_n^l is semisimple there. That is, we construct for the tonal partition algebras a modular system in the sense of Brauer [6]. Using a 'geometrical' index set for the Δ-modules, we give an order with respect to which the decomposition matrix over $\mathbb{C}$ (with $\delta \in \mathbb{C}^\times$) is upper-unitriangular. We establish several crucial properties of the Δ-modules. These include a tower property, with respect to n, in the sense of Green [22, §6] and Cox *et al* [9]; contravariant forms with respect to a natural involutive antiautomorphism; a highest weight category property; and branching rules.

Keywords— Finite dimensional algebras, diagram algebras, partition algebra, highest weight category, decomposition matrix.

1 Introduction

The partition category is a key paradigm in many areas of representation theory and physics. In particular, for example, it can be regarded as a topological quantum field theory (TQFT) on the one hand; and on the other as controlling the representation theory of all symmetric groups taken together (see for example [46, 13, 40, 28] and references therein). The partition category paradigm is often easier to work with than general TQFTs — in particular its representation theory over the complex

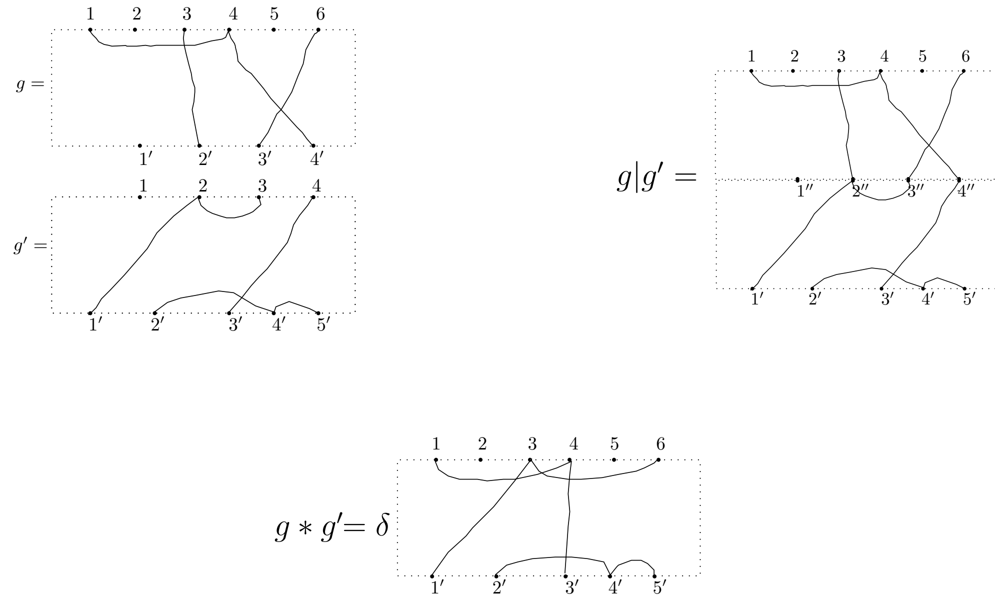


Figure 1: Schematic of composition in category $\mathcal{P}$ using graphs to represent partitions (see Appendix A for details).

field is well understood [38]. From this perspective the ramified partition category [42] can play the role of an extended TQFT (confer for example [48, 51, 7] and references therein). The 'tonal' version discussed in this paper is a corresponding generalisation from the algebraic side. As such its representation theory is interesting to understand, including the non-semisimple cases. The aim here is to investigate the non-semisimple structure of the tonal partition algebras over suitable quotient fields of the natural ground ring, from a geometric perspective (cf. [46]). An obstruction to this has been the greater complexity of the 'modular setup' of the tonal compared to the ordinary case. In this paper we solve this technical problem.

Fix K a commutative ring and $\delta \in K$. Let P_S denote the set of set partitions of a set S. The partition category $\mathfrak{P}$ (as defined in [37]) is a K-linear category: the objects of $\mathfrak{P}$ are finite sets; and the hom-set $Hom_{\mathfrak{P}}(S,T)$ has K-basis $\mathsf{P}_{S \sqcup T}$. (A schematic for the δ-dependent category composition is in Figure 1. Details are recalled in §A.) The full subcategory on objects of form $\underline{n} = \{1, 2, ..., n\}$ is a skeleton in $\mathfrak{P}$, denoted $\mathcal{P}$. In the notation for a category in which $\mathcal{C} = (objects, arrows, composition)$ we summarize the category $\mathcal{P}$ as $\mathcal{P} = (\mathbb{N}_0, K\mathsf{P}_{\underline{n} \sqcup \underline{m}}, *)$.

The category $\mathcal{P}$ has many interesting subcategories, such as the Brauer and Temperley–Lieb categories [5, 53, 37, 40, 35]. Here we study subcategories chosen to serve as testbeds for questions in geometrical representation theory (in the sense arising for example in [10, 41, 16, 15]).

Consider a part $p_i \in p \in \mathsf{P}_{S \sqcup T}$. Define

$$\ker(p_i) = |p_i \cap S| - |p_i \cap T|. \tag{1}$$

For $l \in \mathbb{N}$ part p_i is said to be *l-tone* if $\ker(p_i) \equiv 0 \text{ mod.} l$. A partition is said to be l-tone if every part is l-tone. It is routine to show (see e.g. (3) in §2) that the category composition in $\mathcal{P}$ closes on the span of the subsets $\mathsf{P}^l_{\underline{n} \sqcup \underline{m}}$ of l-tone partitions. Thus for each $l \in \mathbb{N}$ we have a subcategory of $\mathcal{P}$, denoted $\mathcal{P}^l$:

$$\mathcal{P}^l = (\mathbb{N}_0, K\mathsf{P}^l_{\underline{n} \sqcup \underline{m}}, *)$$

There is also a corresponding subcategory of non-crossing partitions, denoted $\mathcal{T}^l$. Together we call these tonal partition categories.

For $\delta = m \in \mathbb{N}$ the partition algebra itself is Schur–Weyl dual to the 'left' symmetric group S_δ [36, 27, 39]. That is, the Potts/tensor action of $\mathcal{P}$ [36, §8.2] is dual to the diagonal action of symmetric group S_δ on $Y^{\otimes n}$, where Y is the $\lambda = (\delta - 1, 1)$ Young module. Neither action is faithful in general. Let Λ denote the set of all integer partitions, $\Lambda^* = \Lambda \setminus \{\emptyset\}$ and

$$\Lambda_i \;=\; \{\lambda \vdash i\}$$

denote the set of integer partitions of i. The natural index sets for simple modules over $\mathbb{C}$ are $\Lambda(P_n) = \sqcup_{i=0,1,..,n}\Lambda_i$ for the partition algebra P_n and $\Lambda(\mathbb{C}S_m) \;=\; \Lambda_m$ for S_m. Let $\mathsf{r} : \Lambda^* \to \Lambda$ denote the map that removes the first part (i.e. removes the first row in the corresponding Young diagram). Fixing m, the duality-induced connection between the index sets is

$$\mathsf{r} : \Lambda(\mathbb{C}S_m) \hookrightarrow \Lambda(P_n) \tag{2}$$

for sufficiently large n (that $\mathbb{C}S_m$ acts faithfully) [20, 39]. This has a useful geometrical realisation — see [46].

In case $K = \mathbb{C}$ the K-algebras $Hom_{\mathcal{P}^l}(S, S)$ are isomorphic to subalgebras of the partition algebra studied by Tanabe [52], Kosuda [31, 32, 33] and Orellana [49]. Tanabe showed that the Schur-Weyl duality between the symmetric group S_m and the partition algebra $P_n(\delta)$ with $\delta = m \in \mathbb{N}$ generalises to a duality between various reflection groups and partition algebras. Kosuda then studied the complex semisimple representation theory of these algebras in the generic case (of $\delta \in \mathbb{C}$) and in certain cases relevant for duality [31, 32]. Orellana also studied the representation theory from the duality perspective [49] (together with an elegant parallel study of the 'coloured' partition algebras).

In the general case of the original S_m/P_n duality (just as for classical Gl_m/S_n duality) the partition algebra does not act faithfully on tensor space. Indeed it clearly acts semisimply when $K = \mathbb{C}$, but it is not generally a semisimple algebra. The way that the tensor space action 'sits inside' the full algebra P_n is (representation theoretically) rather interesting [45, 47], and relates nicely to the geometric-linkage approach to geometric representation theory [26, 46]. Here the aim is to investigate the lift of this geometric approach to the l-tone cases. To this end we construct a corresponding tower of π-modular systems, in the sense of [6, 12, 22, 3, 9, 41].

For our modular system we need first a construction for ordinary irreducible representations over a suitable 'ordinary' ground field. In fact we construct modules directly over an integral ground ring — we do this over $\mathbb{Z}[\delta]$, but the domain of complex polynomials over the indeterminate δ will be adequate for our immediate purposes — and show that they pass by base change to ordinary irreducibles (over the field of fractions). To do this we construct contravariant forms with respect to a natural involutive antiautomorphism; and determine cases where they are non-degenerate. We show that the algebra is semisimple in these cases. We then show that the corresponding decomposition matrix has a unitriangular property with respect to a suitable (partial) order. (To achieve this we must *establish* a suitable partial order.) To verify that our order has the required properties we proceed by showing that a certain quotient algebra A^l_n is semisimple over $\mathbb{C}$. NB (Nota bene), This last step is an addition to the steps needed in the classical P_n and B_n (Brauer algebra) cases. It fulfills our requirements, but it also presents some interesting new features in the representation theory, as we shall elucidate in §7.

Overview: The integral part of the modular tower representation theory of P^l_n follows the same steps as for P_n in [37, 38]. However it is more complex in the detail. The general representation theoretic machinery is collected in §4. In §2 we define the algebra. In §3, §5 we construct a poset of ideals (in P_n this is a chain) with relatively small sections 'controlled' by symmetric groups. In §6 we give a polar decomposition of partitions in an algebra basis that facilitates construction of standard module bases. In §7 we construct our 'standard' modules and cv forms. In §9.1-9.2 we study the algebra that is the top section in a natural tower structure, and hence derive the unitriangularity theorem. In §11 we give restriction rules for our standard modules. Relating these to induction rules, one has potential analogues of the powerful translation functors of Lie theory [26] (to complete this picture and hence connect to Kazhdan-Lusztig Theory, cf. [41], we need a linkage principle - this will be discussed elsewhere).

The main Theorems here are as follows.
Theorem 7.3, which shows that $P^l_{n-l} \cong W^l P^l_n W^l$ for suitable $W^l \in P^l_n$. This tells us that the module category of P^l_{n-l} fully embeds in that of P^l_n, by a 'globalisation' functor. This in turn tells us that we can determine the structure of these module categories iteratively on n. To this end we construct canonical modules in each n that are well-behaved under globalisation — 'standard' modules.
Theorem 9.6, Theorem 9.7 which show that each P^l_n gives a modular system.

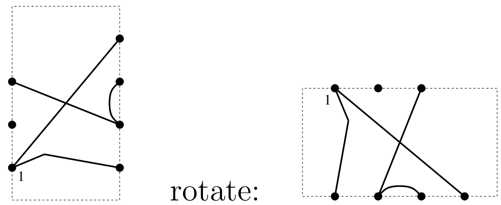


Figure 2: A partition picture (for partition $\{\{1', 4', 1\}, \{2', 3', 3\}, \{2\}\}$) drawn in different orientations.

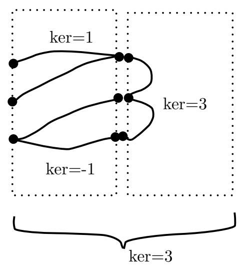


Figure 3: Examples of partitions (here drawn left-to-right) and parts with kernel numbers.

Theorem 9.4: upper-triangularity of the standard module decomposition matrix.
Theorem 10.5: that if $k = \mathbb{C}$ and $\delta \neq 0$ then $P_n^l -$ mod is a highest weight category.
Theorem 11.3: branching rules.

Remark: Here we use the term *tone* as the generalisation to general l of the notion of parity for congruence mod.2. (Kosuda's name of 'modular party algebra' also sounds harmonious, but does not quite fit our purpose.)

2 Tonal partition categories and algebras

Let $\mathcal{P} = (\mathbb{N}_0, \mathcal{P}(n,m), *)$ denote the usual partition category over a given commutative ring K, with parameter $\delta \in K$ [38]. This is a skeleton in $\mathfrak{P}$ obtained by restricting to objects $\underline{n} = \{1, 2, ..., n\}$. Thus $\mathcal{P}(n,m) = K\mathtt{P}_{n,m}$ where $\mathtt{P}_{n,m}$ is the set of set partitions of $\underline{n} \cup \underline{m'}$ (with $\underline{n} = \{1, 2, ..., n\}$, $\underline{m'} = \{1', 2', ..., m'\}$).

We draw pictures of partitions in $\mathtt{P}_{n,m}$ as for example in Fig.2 and 3. Fig.3 also shows the kernel count as in (1). Composition is as in Fig.1.

Write $\otimes$ for the usual monoidal composition in the category $\mathcal{P}$.

Write P_n for the usual partition algebra $P_n = \mathcal{P}(n,n)$, and $\mathtt{P}_n = \mathtt{P}_{n,n}$ for the basis of partitions.

Write $\mathtt{P}^l_{n,m} \subset \mathtt{P}_{n,m}$ for the subset of l-tone partitions.

(**2.1**) THEOREM. (cf. [52, 33]) *Fix $l \in \mathbb{N}$. The restriction of category $\mathcal{P}$ to the span of l-tonal partitions defines a monoidal subcategory, the* tonal partition category $\mathcal{P}^l$. *(Hence defining the tonal partition algebras $P_n^l = \mathcal{P}^l(n,n)$.)*

Proof. Consider the product of composable partitions p, p' in $\mathcal{P}$. Note (e.g. from §A) that in the definition of the product pp' one first forms the concatenation $p|p'$, then discards the 'middle' vertices to form pp'. Thus whenever a part $(pp')_i$ is formed in composition the process is that (in some number of instances) two vertices, one in some $\mathrm{ran}(p_i) = p_i \cap \underline{m}'$ and one in some $\mathrm{cora}(p'_j) = p'_j \cap \underline{m}$, are identified and then discarded from some union of parts. Thus

$$\ker((pp')_i) \;=\; \sum_\pi \ker(\pi) \tag{3}$$

— sum over parts from p, p' involved in $(pp')_i$ (cf. Fig.3). Thus if the incoming parts are all l-tone (ker divisible by l), then the new part is again l-tone.

For the monoidal property note that if a, b are l-tone then so is $a \otimes b$. □

(**2.2**) For $l \in \mathbb{N}$ define

$$b^l \;=\; \{\{1, 2, .., l, 1', 2,', ..l'\}\} \in \mathtt{P}_l$$

Consider Fig.4. Define $u = \mathtt{e}_1^{(2)}$, $a = \mathsf{A}^{12} \in \mathtt{P}_2$. Note that $b^1 = 1_1$, $b^2 = a$, and for $l > 1$ we have $b^l = \mathsf{A}^{12}\mathsf{A}^{23}...\mathsf{A}^{l\text{-}1\,l} \in \mathtt{P}_l$. Fixing n, define $U = \mathtt{e}_1^{(n)}$ as the partition depicted in Fig.4. We have

$$U = u \otimes 1_{n-2}$$

For $p \in \mathtt{P}_{n,m}$ write $p^\star$ for the 'flip' image in $\mathtt{P}_{m,n}$ (given by $i \leftrightarrow i'$ in p) [37].

(**2.3**) A set partition is *even* if all its parts are of even order. For example, all the partitions depicted in Fig.4 are even except for $\varepsilon_i^{(n)}$. Write $\mathtt{E}_n$ for the set of even partitions in P_n. Note that $\mathtt{E}_n = \mathtt{P}_n^2$. Write E_n for P_n^2.

(**2.4**) For given l let w denote the unique partition in $\mathcal{P}^l(l, 0)$; and $w^\star \in \mathcal{P}^l(0, l)$. Let v denote specifically the unique partition in $\mathcal{P}^2(2, 0)$ and v^* the unique partition in $\mathcal{P}^2(0, 2)$. Let σ_1 denote the unique elementary transposition in $\mathcal{P}(2, 2)$. Then $E := \mathcal{P}^2$ is generated as a linear monoidal category by 1_1, v, $v^\star$, σ_1 and a.

(**2.5**) Recall the 'flip' antiisomorphism of P_n, denoted $p \mapsto p^{op}$ (given by $p \mapsto p^\star$ on partitions). This fixes the subset P_n^l, so that P_n^l is isomorphic to its opposite.

Note that the symmetric group S_n is a subgroup of P_n that is also in P_n^l, and that the restriction of the flip antiautomorphism to this subgroup is the usual $g \mapsto g^{-1}$ antiautomorphism.

(**2.6**) Let $w_n \in S_n$ denote the order reversing (Coxeter longest word) element — an involution. Recall the lateral-flip endofunctor on $\mathcal{P}$ given on partitions $p \in \mathtt{P}_{m,n}$ by $p \mapsto \bar{p} = w_m p w_n$. This takes a partition to its mirror image. Note that the endofunctor fixes $\mathcal{P}^l$, and indeed P_n^l.

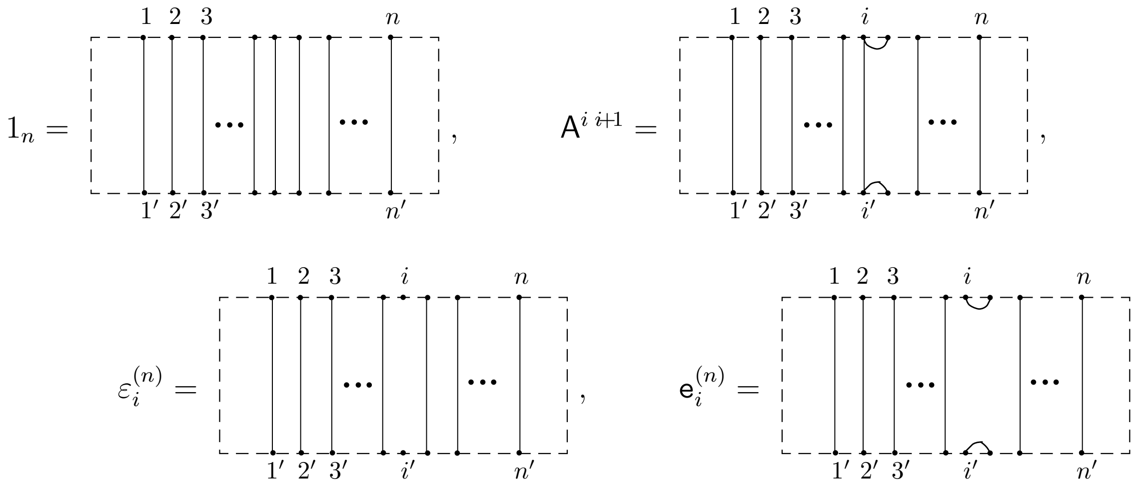


Figure 4: Special elements in the partition algebra P_n.

3 Basic properties of the algebra P_n^l

Here we develop an analogue of the propagating ideals of P_n as in [37, §6.1].

3.1 Set partitions: Co-i parts and propagating numbers

(**3.1**) Recall that, for any n, we write $\# : \mathsf{P}_n \to \mathbb{N}_0$ for the map taking a partition $p \in \mathsf{P}_n$ to the number of propagating parts in p. Note that a product of partitions in $\mathcal{P}$ is a scalar times a partition. The map $\#$ extends to apply to a product of partitions in the obvious way. Recall the 'bottleneck principle':

(**3.2**) Lemma. [37, §6.1] *For $p, p' \in \mathsf{P}_n$ we have $\#(pp') \leq \#(p)$.* □

(**3.3**) Fix l. For $p \in \mathsf{P}_n^l$ a propagating part $s \in p$ is *co-i* if the restriction of s to one 'side' of the underlying set (the set $\underline{n}$ say) is of order congruent to i mod.l. (Note that the definition is independent of the choice of side in case p in P_n^l but not in general in P_n.)

Define $\#^i : \mathsf{P}_n^l \to \mathbb{N}_0$ so that $\#^i(p)$ is the number of co-i propagating parts. Define *propagating vector*

$$\#^-(p) \;=\; (\#^1(p),\ \#^2(p), \#^3(p), ..., \#^l(p))$$

Example: For $l = 3$, $\#^-(1_n) = (n, 0, 0)$.

Consider $p \in \mathsf{P}_n$. Then $p\mathsf{A}^{12}$ is a partition similar to p but with the parts containing vertices $1', 2'$ combined. Thus if vertices $1', 2'$ intersect at most one propagating part in $p \in \mathsf{P}_n^l$ then $\#^-(p\mathsf{A}^{12}) = \#^-(p)$.

3.2 The index set $\gamma^{l,n}$ and corresponding partitions

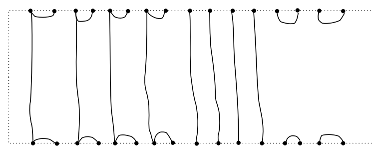
For $l \in \mathbb{N}$ and $\mathbf{m} = (m_1, m_2, ..., m_l) \in \mathbb{Z}^l$ define $r_\mathbf{m} = \sum_{i=1}^{l} i m_i$. For given l and n, define

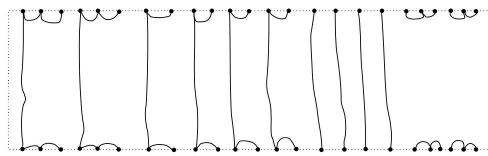
$$\gamma^{l,n} = \{\mathbf{m} \in \mathbb{N}_0^l \ : \ (n - r_\mathbf{m})/l \in \mathbb{N}_0\}$$

For $\mathbf{m} \in \gamma^{l,n}$ define a set partition

$$a^\mathbf{m} \ := \ a_n^\mathbf{m} \ = \ \left(\otimes_{i=l}^{1}(b^i)^{\otimes m_i}\right) \otimes (ww^\star)^{(n-r_\mathbf{m})/l} \tag{4}$$

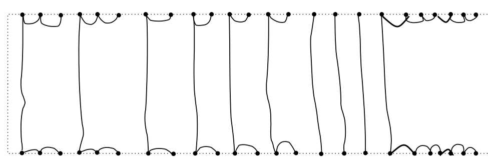
Note that $a^\mathbf{m} \in \mathtt{P}_n$. For example

$a_{16}^{(4,4)} \ =$  .

$a_{24}^{(4,4,2)} \ =$  .

Thus if δ is invertible then $a^\mathbf{m}$ is a (not necessarily normalised) idempotent in P_n.

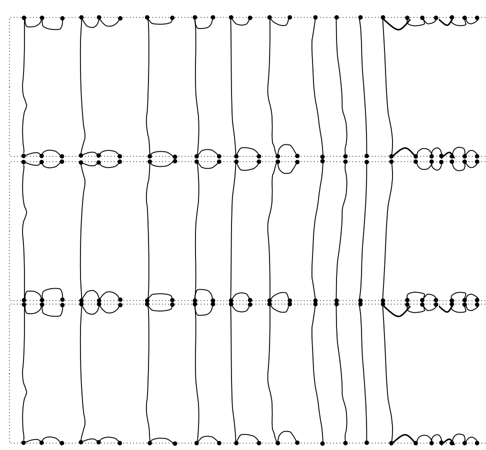
If $\mathbf{m} \neq 0$, so that $a_n^\mathbf{m}$ has at least one propagating part, we also define for each $a_n^\mathbf{m}$ a partition $b_n^\mathbf{m}$, obtained from $a_n^\mathbf{m}$ by combining the last ('rightmost') propagating part with all the non-propagating parts. Thus:

$b_{24}^{(4,4,2)} \ =$  .

Note that $b_n^\mathbf{m}$ is idempotent in P_n (for any δ). Also

$$b^\mathbf{m} a^\mathbf{m} b^\mathbf{m} = b^\mathbf{m} \qquad \text{and} \qquad a^\mathbf{m} b^\mathbf{m} a^\mathbf{m} = a^\mathbf{m} \tag{5}$$

For example

$b_{24}^{(4,4,2)} a_{24}^{(4,4,2)} b_{24}^{(4,4,2)} \ =$  .

3.3 Poset structure on $\gamma^{l,n}$

(**3.4**) Fix l. Define $V \subset \mathbb{Z}^l$ as follows. Define v_{ij} for $1 \leq i \leq j \leq l$ by

$$v_{ij} = (0, 0, ..., 0, \underbrace{1}_{i+j}, 0, ..., 0, \underbrace{-1}_{j}, 0, ..., 0, \underbrace{-1}_{i}, 0, ..., 0) \quad \in \mathbb{Z}^l$$

where the index $i+j$ is understood mod.l. Note that the entry -1 can appear before 1 if $i + j < l$. Then $V = \{v_{ij}\}_{i,j}$. In particular

$$v_{ii} = (0, 0, ..., 0, \underbrace{1}_{2i}, 0, ..., 0, \underbrace{-2}_{i}, 0, ..., 0) \qquad \text{and} \qquad v_{ll} = (0, 0, .., 0, -1)$$

Note that there are a total of $\frac{l(l-1)}{2} + 1$ of these vectors in V.

Define a poset structure on $\gamma^{l,n}$ by $\mathbf{m} \geq \mathbf{m}'$ if $\mathbf{m}' - \mathbf{m}$ lies in the nonnegative integral span of V. For example $(9, 0, 0) > (7, 1, 0)$ since $-(9, 0, 0) + (7, 1, 0) = (-2, 1, 0) = v_{11}$.

Note that $(n, 0, 0, ..., 0)$ is the unique top element in $\gamma^{l,n}$ for any l. The Hasse diagrams in the cases $l = 2, 3$ are indicated in Fig.5. See also Fig.6.

(**3.5**) Note that every element of $\gamma^{l,n}$ is in the positive cone of $(n, 0, 0, ..., 0)$ with respect to the subset $V' = \{v_{11}, v_{12}, ..., v_{1l}\}$. Thus $\gamma^{l,n}$ includes an l-dimensional lattice (in the crystal lattice sense). The subset V' is manifestly a basis for the underlying $\mathbb{R}^l$ containing $\mathbb{Z}^l$. It follows that none of the remaining vectors in V are $\mathbb{R}$-linearly independent of V'. However we claim they are positive-integrally independent. Typically for every three sides in a cube in $\gamma^{l,n}$ then there is an element in $V \setminus V'$ that is the main diagonal in this cube. For example with $l = 3$

$$v_{11} + v_{22} = v_{12} + v_{13}$$

That is to say, $v_{22} = v_{12} + v_{13} - v_{11}$, a non-positive combination of basis elements. This example of v_{22} corresponds to the dashed lines in Fig.5.

(**3.6**) Note from (3.4) that $\gamma^{l,n-l} \hookrightarrow \gamma^{l,n}$. As in Fig.7, consider the subset of $\gamma^{l,n}$

$$h_n^l \;:=\; \{\mathbf{m} \in \gamma^{l,n} \mid \mathbf{m} \not\leq (n-l, 0, 0, .., 0)\} \tag{6}$$

We have $\gamma^{l,n} = h_n^l \sqcup \gamma^{l,n-l}$. Note that no element of $\gamma^{l,n-l}$ lies above any element of h_n^l in the poset $(\gamma^{l,n}, <)$.

We discuss combinatorics of $\gamma^{l,n}$ in [1].

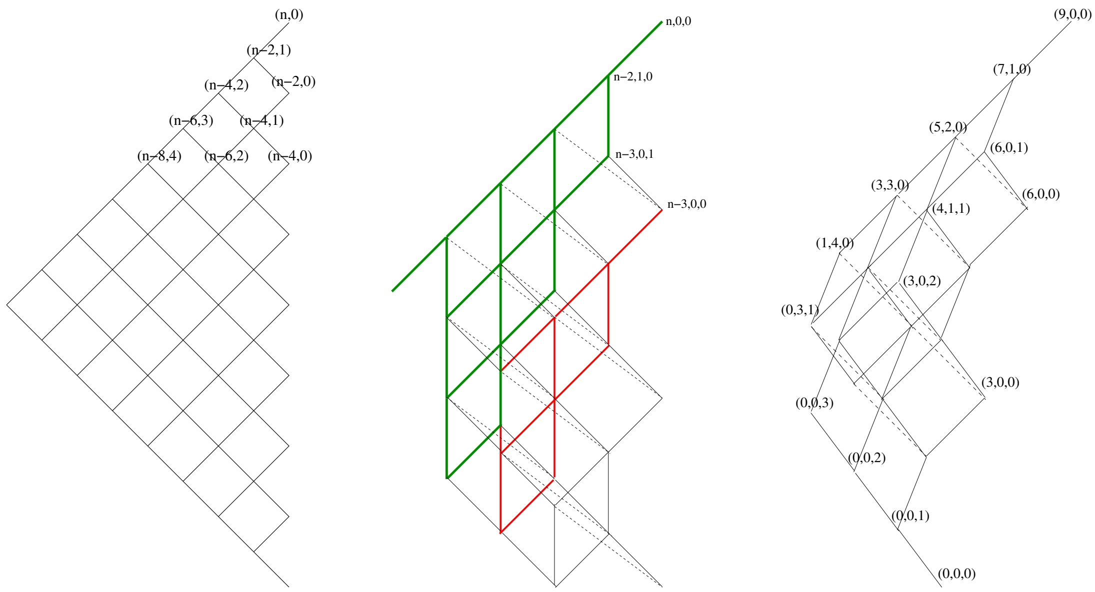


Figure 5: Hasse diagram for inclusion poset for ideals $P_n^l a^{\mathbf{m}} P_n^l$ in cases $l = 2$ (left) and $l = 3$ (right). Vertex $\mathbf{m}$ corresponds to ideal $P_n^l a^{\mathbf{m}} P_n^l$.

3.4 Ideals generated by the $a^{\mathbf{m}}$ elements

(**3.7**) Note that

$$\#^-(a^{\mathbf{m}}) = \mathbf{m}$$

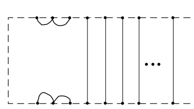
Fix l. Define the subset $\mathsf{P}_n^{\mathbf{m}} \subset \mathsf{P}_n^l$ by

$$\mathsf{P}_n^{\mathbf{m}} = \{p \in \mathsf{P}_n^l | \#^-(p) = \mathbf{m}\} \tag{7}$$

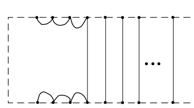
(**3.8**) Fix l and n and define partition W in P_n^l by

$$W \;=\; W^l \;:=\; (ww^\star) \otimes 1_{n-l} \tag{8}$$

NB $W^l \;\in\; S_n a^{(n-l,0,0,\ldots,0)} S_n$. Example:

$W^3 =$ 

We will also use an 'idempotent version' $W_b^l \;=\; \mathsf{A}^{l\,l+1}\, W^l\, \mathsf{A}^{l\,l+1}$:

$W_b^3 =$ 

(**3.9**) Lemma. [52] *For* $l \in \mathbb{N}$, $P_n^l = \langle S_n, \mathsf{A}^{12}, W^l \rangle$. □

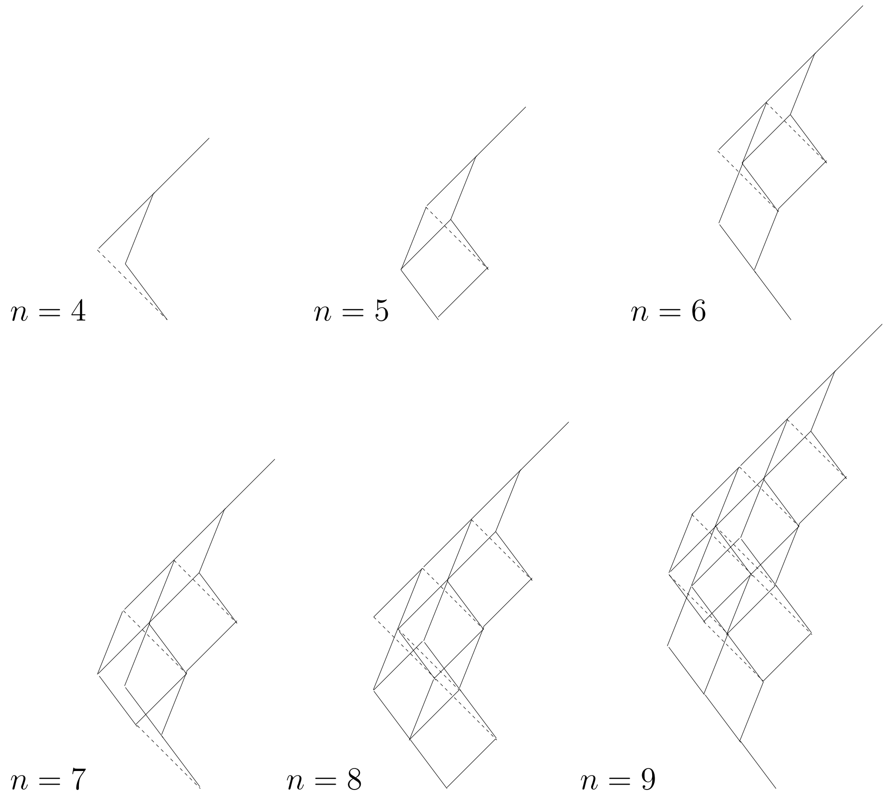


Figure 6: Hasse diagram for inclusion poset for ideals $P_n^l a^{\mathbf{m}} P_n^l$ in case $l = 3$, for $n = 4, 5, ..., 9$. Vertex $\mathbf{m}$ corresponds to ideal $P_n^l a^{\mathbf{m}} P_n^l$.

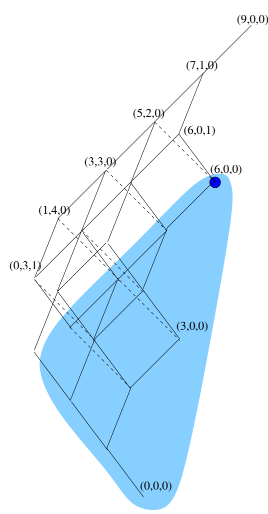
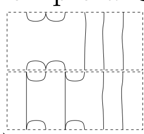


Figure 7: Schematic of the set h_n^l, i.e. $\gamma^{l,n}$ with the poset ideal generated by $(n - l, 0, 0, ..)$ removed. (This case is h_9^3.)

(**3.10**) LEMMA. *(I) The element* $a^{\mathbf{m}} \in P_n^l$ *lies in the ideal* $P_n^l a^{\mathbf{m}'} P_n^l$ *if and only if* $\mathbf{m} \leq \mathbf{m}'$ *in* $\gamma^{l,n}$. *(II) For* $p, p' \in \mathsf{P}_n^l$ *we have* $\#^-(pp') \leq \#^-(p)$.

Proof. (I) ('if' part): If $\mathbf{m} - \mathbf{m}' = v_{ij}$ then, up to a permutation, $a^{\mathbf{m}}$ can be obtained from $a^{\mathbf{m}'}$ by joining a co-i and a co-j propagating part together using some A^{gh}: $a^{\mathbf{m}'} \rightsquigarrow a^{\mathbf{m}'}\mathsf{A}^{gh}$, or cutting a propagating 'line' using W^l.
('only if' part): Consider $p \in \mathsf{P}_n^l$ and the change $\#^-(p) \rightsquigarrow \#^-(ap)$ for $a \in S_n\{1, \mathsf{A}^{12}, W^l\}S_n$. The vector can only be changed by such an a as follows. (1) by combining propagating parts (for example $a \in \{\mathsf{A}^{ij}\}_{ij}$; or $a = W^l$ and p is as in the following case with $l = 3$: $W^l p =$ ); or (2) cutting the propagating line in a $co - l$ part.
But $p \rightsquigarrow ap$ (or pa) for *any* a is a sequence of such 'local' changes by Lemma 3.9.
(II) follows by the same argument. □

(**3.11**) LEMMA. *(I) The set* $\bigsqcup_{\mathbf{m}' \leq \mathbf{m}} \mathsf{P}_n^{\mathbf{m}'}$ *is a basis for the ideal* $P_n^l a^{\mathbf{m}} P_n^l$. *(II) The partitions* p *in the ideal* $P_n^l a^{\mathbf{m}} P_n^l$ *and not in any ideal* $P_n^l a^{\mathbf{m}'} P_n^l$ *with* $\mathbf{m}' < \mathbf{m}$ *are precisely the subset* $\mathsf{P}_n^{\mathbf{m}}$.

Proof. This follows from Lem.3.10. Note that elements of $\mathsf{P}_n^{\mathbf{m}}$ are elements of $S_n a^{\mathbf{m}} S_n$ and elements obtained from these by binding a non-propagating part to a propagating one. □

(**3.12**) LEMMA. *Fix* l.
(1) Every element p *of* P_n^l *containing a part* p_i *with* $\mathrm{cora}(p_i) > l$ *or* $\mathrm{ran}(p_i) > l$ *lies in the ideal* $P_n^l W P_n^l$.
(2) Every element of P_n^l *containing a non-propagating part lies in the ideal* $P_n^l W P_n^l$.
(3) Fix n. *For any* $\mathbf{m}$, *if* $r_{\mathbf{m}} = n$ *then* $a^{\mathbf{m}} \notin P_n^l W P_n^l$.

Proof. (1) Suppose $|\mathrm{cora}(p_i)| > l$ or $|\mathrm{ran}(p_i)| > l$. Then there is a partition $p' \in \mathsf{P}_n^l$ differing from p only in that l elements of p_i are in an isolated non-propagating part. Partition p evidently lies in the ideal generated by p'. Now use (2).

(2) A non-propagating part p_i in $p \in \mathsf{P}_n^l$ necessarily has order at least l. Then there is a partition W' group-conjugate to W such that one of its non-propagating parts exactly meets a subset of p_i in composition, whereupon $pW' = p$ or $W'p = p$. (If the order is exactly l the argument is slightly modified.)

(3) By Lem.3.10, noting that $W \in S_n a^{(n-l,0,0,\ldots,0)} S_n$. □

(**3.13**) For given n, l and $\mathbf{m} \in \gamma^{l,n}$ define the ideal

$$I^{<\mathbf{m}} = \sum_{\mathbf{m}' < \mathbf{m}} P_n^l a^{\mathbf{m}'} P_n^l$$

(**3.14**) LEMMA. *If* $\mathbf{m}' \ngeq \mathbf{m} \in \gamma^{l,n}$ *then* $a^{\mathbf{m}'} P_n^l a^{\mathbf{m}} \subset I^{<\mathbf{m}}$.

Proof. Note that $a^{\mathbf{m}'} P_n^l a^{\mathbf{m}}$ has a basis of partitions. By 3.11 every partition lies in a unique highest ideal of form $P_n^l a^{\mathbf{m}''} P_n^l$. In particular $p \in a^{\mathbf{m}'} P_n^l a^{\mathbf{m}}$ lies in or below $P_n^l a^{\mathbf{m}} P_n^l$, but since $\mathbf{m}' \not\geq \mathbf{m}$ it must be below. □

4 Representation theory generalities

Lemma 3.14 means that P_n^l has what we call the 'core' property. From this property many representation theoretic properties follow quite generally. Here we collect the general arguments.

4.1 Preliminaries: Green's idempotent reciprocity

(**4.1**) THEOREM. [Green localisation theorem, [22, §6.2]] *Let k be a field, let A be a k-algebra, and $e \in A$ idempotent. Let $\Lambda(A), \Lambda(eAe)$ and $\Lambda(A/AeA)$ be index sets for classes of simple modules of the indicated algebras. Then there is a bijection*

$$\Lambda(A) \xrightarrow{\sim} \Lambda(eAe) \sqcup \Lambda(A/AeA)$$

□

Fix an index set $\Lambda(A)$, and a set $\{M_\lambda : \lambda \in \Lambda(A)\}$ that is a complete set of simples up to isomorphism. Let us write $\Lambda_e(A)$ for the subset of $\Lambda(A)$ such that $\lambda \in \Lambda_e(A)$ implies $eM_\lambda \neq 0$ [2, 22]. The map $M \mapsto eM$ is a bijection from $\{M_\lambda : \lambda \in \Lambda_e(A)\}$ to a complete set of simples for eAe. Thus we may take $\Lambda(eAe) = \Lambda_e(A)$. Meanwhile $\{M_\lambda : \lambda \in \Lambda(A) \setminus \Lambda_e(A)\}$ is a complete set for A/AeA. This gives a natural identification

$$\Lambda(A) = \Lambda(eAe) \sqcup \Lambda(A/AeA) \tag{9}$$

(**4.2**) Let k be a field. Given a finite-dimensional k-algebra A, a simple module L and a module M, then $[M : L]$ denotes the composition multiplicity of L in M.

(**4.3**) For A an algebra then $A-$mod denotes its category of left-modules. Given an idempotent e in an algebra A we define, as usual [38, 22], a functor

$$G_e : eAe - \text{mod} \;\; \to \;\; A - \text{mod}$$

by $G_e(M) = Ae \otimes_{eAe} M$. Define functors

$$\begin{aligned} F_e : A - \text{mod} \;\; &\to \;\; eAe - \text{mod} \\ L_e : \text{mod} - A \;\; &\to \;\; \text{mod} - eAe \end{aligned}$$

by $F_e(N) = eN$ and $L_e(N) = Ne$. We have the following standard properties (see e.g. [22, 2, 41]).

(**4.4**) Theorem. *Let algebra A and idempotent $e \in A$ be as above.*
(I) Functor G_e is left-adjoint to F_e; and a right-inverse to F_e.
(II) Functor G_e is right-exact; and F_e is exact.
(III) Functor G_e preserves projectivity and indecomposability.
(IV) Functor G_e preserves simple head.
(V) For L a simple module eL is a simple eAe-module or 0 and if $eL \neq 0$ then

$$[eM : eL] \; = \; [M : L]$$

□

4.2 Algebras with the core property

(**4.5**) Let k be a commutative ring. Let A be a unital k-algebra, with unit 1. Let $\gamma = (\gamma, <)$ be a finite poset and $e_\alpha \in A$ an idempotent for each $\alpha \in \gamma$. Let

$$I^{<\alpha} = \sum_{\beta < \alpha} A e_\beta A$$

(so $I^{<\alpha} = 0$ if α is a lowest element) and (for later use)

$$A^\alpha = A / I^{<\alpha}.$$

The poset γ together with the map $e_- : \gamma \to A$ is called a *core* for A if (AI): $e_\alpha A e_{\alpha'} \subseteq I^{<\alpha}$ for all $\alpha' \not\geq \alpha$; (AII): 1 lies in the image of e_-.

(**4.6**) Theorem. *Suppose δ invertible in K. Then the pair $(\gamma^{l,n}, a^-)$ gives a core for P^l_n.*

Proof. Comparing (4.5) with (3.14) we see that $(\gamma^{l,n}, a^-)$ is a core for P^l_n up to renormalisation of the a^- elements as idempotents. □

(**4.7**) Suppose now that k in (4.5) is a field and A is finite-dimensional with simple index set $\Lambda(A)$. Note the natural inclusion $\Lambda(A^\alpha) \hookrightarrow \Lambda(A)$ giving the (classes of) simple modules M of A such that $e_\beta M = 0$ for $\beta < \alpha$.

By $e_\alpha A^\alpha$ we understand e_α to act on A^α in the natural way, i.e. as $e_\alpha + I^{<\alpha}$. By Green's theorem (4.1) and the construction then we take $\Lambda(e_\alpha A^\alpha e_\alpha) \hookrightarrow \Lambda(A)$ (via $\Lambda(e_\alpha A^\alpha e_\alpha) \hookrightarrow \Lambda(A^\alpha) \hookrightarrow \Lambda(A)$) to index classes of simple modules of A such that $e_\beta M = 0$ if $\beta < \alpha$ and $e_\alpha M \neq 0$.

(**4.8**) Lemma. Let A be an algebra over a field with simple index set $\Lambda(A)$. If algebra A has a core (γ, e_-) then $\Lambda(A) = \cup_{\alpha \in \gamma} \Lambda(e_\alpha A^\alpha e_\alpha)$.

Proof. Consider the classes indexed by $\Lambda(e_\alpha A^\alpha e_\alpha)$ as in (4.7). There may be other simples with $e_\alpha M \neq 0$, but they have $e_\beta M \neq 0$ for some $\beta < \alpha$, and so are 'counted' in some lower $\Lambda(e_\beta A^\beta e_\beta)$ — noting that minimal elements $\omega \in \gamma$ finally exhaust simples with $e_\omega M \neq 0$. Thus the union includes all simples with $e_\alpha M \neq 0$.

But now since $1 \in \gamma$ (by axiom AII) the union includes all simples in $\Lambda(A)$ where $1M \neq 0$, which is all. □

(**4.9**) Remark/caveat: Note that every algebra has a 'core' for every poset with a unique minimal element (call it $\perp$), simply by the constant mapping $e_\alpha = 1$. But all but one $\Lambda(e_\alpha A^\alpha e_\alpha)$ is empty in this case (and our Theorem 4.10 below is trivial). Confer [18, 8, 14, 21] for certain 'tighter' axiomatizations (and indeed cf. for example [23, 4, 17] and references therein for interesting related axiomatizations).

In order for the core formalism to have significant utility, we will need something further like BH reciprocity, as we discuss shortly.

(**4.10**) Theorem. *Let A be an algebra over a field with simple index set $\Lambda(A)$. If algebra A has a core (γ, e_-) then*

$$\Lambda(A) = \sqcup_{\alpha\in\gamma}\Lambda(e_\alpha A^\alpha e_\alpha) \tag{10}$$

Proof. Noting Lem.4.8 it remains to prove disjointness. For $\alpha \in \gamma$ let $\Lambda_\alpha := \Lambda(e_\alpha A^\alpha e_\alpha)$. Recall the natural embedding $\Lambda_\alpha \hookrightarrow \Lambda(A^\alpha) \hookrightarrow \Lambda(A)$. Simple modules M of A coming from the subset Λ_α (if any) obey $e_\alpha M \neq 0$ ($Ae_\alpha M = M$) by (4.1)/(4.7). But now consider $e_{\alpha'}M$. Either $\alpha' \not> \alpha$ and so $e_{\alpha'}M = 0$ by axiom (AI), so $M \not\cong M_\mu$, $\mu \in \Lambda_{\alpha'}$ (for which $Ae_{\alpha'}M_\mu = M_\mu$); or $\alpha' > \alpha$ so $\alpha \not> \alpha'$ and $e_\alpha N = 0$ for all $N = N_\mu$, $\mu \in \Lambda_{\alpha'}$, while $Ae_{\alpha'}N = N$ so $M \not\cong N$. □

4.3 $\daleth^\gamma$-modules over a core

(**4.11**) Let A be an algebra over a field with index set $\Lambda(A)$, and core (γ, e_-) as above. Let $\lceil - \rfloor : \Lambda(A) \to \gamma$ be the map taking $\mu \in \Lambda(A)$ to the part $\alpha = \lceil \mu \rfloor$ to which it belongs in (10). For each $\alpha \in \gamma$ suppose $\{S^\alpha_\mu \ : \ \mu \in \Lambda_\alpha\}$ to be a corresponding set of simple modules of $e_\alpha A^\alpha e_\alpha$.

For each of the algebras $e_\alpha A^\alpha e_\alpha$ we have a corresponding G-functor (as in (4.3)). In particular this lifts the set of simple modules to a set of A^α-modules, and hence A-modules. Thus for each $\mu \in \Lambda(A)$ we have an A-module

$$\daleth^\gamma_\mu = G_{e_\alpha} S^\alpha_\mu \tag{11}$$

for $\alpha = \lceil \mu \rfloor$ denoting the appropriate α. Let $\daleth^\gamma = \{\daleth^\gamma_\mu \mid \mu \in \Lambda(A)\}$.

(**4.12**) Lemma. *Let A be an algebra with index set $\Lambda(A)$, and core (γ, e_-) as above. Let $\{\sqsupset^\gamma_\mu : \mu \in \Lambda(A)\}$ be as in (11). (In particular this supposes fixed the various sets $\{S^\alpha_\mu : \mu \in \Lambda_\alpha\}$.) We have the following:*
(I) The modules $\sqsupset^\gamma_\mu$ have simple heads, denoted L_μ; and their heads are exactly a complete set of simple A-modules.
(II) If $\sqsupset^\gamma_\mu$ has a simple factor L_ν below the head then $e_{\lceil\mu\rfloor} L_\nu = 0$.
(III) Suppose A possesses an involutive antiautomorphism $a \mapsto a^\star$ and hence a contravariant duality $M \mapsto M^o$ [22], so the socle of $(\sqsupset^\gamma_\mu)^o$ is L^o_μ. If $e_{\lceil\mu\rfloor} = e^\star_{\lceil\mu\rfloor}$ then $e_{\lceil\mu\rfloor} L^o_\mu \neq 0$, so either L^o_μ does not appear in $\sqsupset^\gamma_\mu$ and there is no module map $\sqsupset^\gamma_\mu \to (\sqsupset^\gamma_\mu)^o$; or $L^o_\mu = L_\mu$, and there is exactly one module map $\sqsupset^\gamma_\mu \to (\sqsupset^\gamma_\mu)^o$ up to scalars, with image L_μ.

Proof. (I) By Theorem 4.4(IV) and (the proof of) Theorem 4.10.
(II) Follows from Theorem 4.4(V).
(III) If $aM \neq 0$ then $a^\star M^o \neq 0$. □

(**4.13**) Lemma. *Let A be an algebra with core (γ, e_-) as above. Let $d^\gamma_{\mu x} := [\sqsupset^\gamma_\mu : L_x]$. Then $d^\gamma_{\mu\mu} = 1$ and if $\mu \neq x$ and $d^\gamma_{\mu x} \neq 0$ then $\lceil\mu\rfloor < \lceil x\rfloor$.*

That is to say, let $<'$ be any total order on $\Lambda(A)$ in which $\lceil\mu\rfloor < \lceil x\rfloor$ in γ implies $\mu <' x$. Then the matrix of $D^\gamma_A := (d^\gamma_{\mu x})$ with respect to $<'$ is upper-unitriangular.

Proof. Let $\mu, x \in \Lambda(A)$ with $x \neq \mu$. First note that $d^\gamma_{\mu\mu} \neq 0$ by Lemma 4.12(I). Lemma 4.12(II) then implies $d^\gamma_{\mu\mu} = 1$.

It remains to show that if $d^\gamma_{\mu x} \neq 0$ then $\lceil\mu\rfloor < \lceil x\rfloor$. If $d^\gamma_{\mu x} \neq 0$ then $e_{\lceil x\rfloor} \sqsupset^\gamma_\mu \neq 0$. Hence by axiom (AI) of the definition of core we have $\lceil\mu\rfloor \leq \lceil x\rfloor$. Note that again by Lemma 4.12(II) we have $e_{\lceil\mu\rfloor} L_x = 0$. Hence we cannot have $\lceil x\rfloor = \lceil\mu\rfloor$ otherwise we would get $e_{\lceil\mu\rfloor} L_x \neq 0$. Therefore $\lceil\mu\rfloor < \lceil x\rfloor$. □

(**4.14**) In general $\sqsupset^\gamma$ modules are not particularly useful, for computing the Cartan decomposition matrix for example. Below we discuss conditions under which they become useful.

4.4 Modular systems and pivotal sets

(**4.15**) Consider an algebra A over a field with simple modules L_μ, $\mu \in \Lambda(A)$ and indecomposable projective covers P_μ, $\mu \in \Lambda(A)$. Suppose we have another set $\mathcal{D}$ of A-modules $\mathcal{D}_\mu$, $\mu \in \Lambda \supseteq \Lambda(A)$, that span the Grothendieck group. Suppose in particular that there is an expression for the characters of the projective modules P in terms of $\mathcal{D}$, with coefficients denoted $(P : \mathcal{D}_\mu)$. We say A has the *BH reciprocity* property (as in Brauer–Humphreys) if

$$[\mathcal{D}_\mu : L_\nu] = (P_\nu : \mathcal{D}_\mu) \tag{12}$$

In particular we say $\mathcal{D}$ is a *pivotal* set for the BH property.

If $\mathcal{D}$ gives a basis for the Grothendieck group we call this *strong* BH property.

(**4.16**) A Brauer modular system [3] for a k-algebra A is a triple of commutative rings (K, K_0, k) where K is an integral domain, K_0 the field of fractions, and k the quotient by some maximal ideal, with the following properties. Firstly there is an 'integral' version A^K of A over K, and A itself is obtained by base change, $A = k \otimes_K A^K$, and is split. Secondly the base change instead to K_0, the 'ordinary' version, is split semisimple.

(**4.17**) A *BH module* for a Brauer modular system is the image of a simple module of $K_0 \otimes_K A^K$ under an integral lift to A^K followed by the base change to k.

There may be many such lifts in general. But a set of BH modules that is the image of a complete set of ordinary simples is called complete.

(**4.18**) A sufficient condition for the BH property, with pivotal set Δ, is that the set Δ is a complete set of BH modules. (See e.g. [3, Prop.1.9.6].)

4.5 Modular core property and highest weight categories

In this section we define the (strong) modular core property for algebras and show that it implies a highest weight category.

(**4.19**) A *modular core* is an algebra A and a triple of rings (K, K_0, k) as in (4.16) giving a Brauer modular system for A (thus $A = k \otimes_K A^K$) together with core data (γ, e_-) giving a core for A^K.
A *strong* modular core is a modular core that is strong modular, i.e. the simple modules over k and K^0 have the same index set.

(**4.20**) Remark. In the classical modular theory for finite groups one finds the integral representations (that will be reduced mod. the prime defining k) as lattices inside the simples of the rational case. In diagram algebras suitable integral representations can generally be constructed directly (by the good basis properties of such algebras). Nonetheless their base changes to the rational case are the simples of that case. In other words it is the index set for simples in the rational case that labels the 'Brauer/Specht modules'.

This means there are some 'carts and horses' (hypotheses and conclusions) that must be placed carefully in the right order!

(**4.21**) Note that if (K, K_0, k) is such a triple of rings then the core property over K will base change to the field cases.

In particular then consider the algebra $A^0 = K_0 \otimes_K A^K$. This is now split semisimple by hypothesis. We have

$$\Lambda(A^0) = \sqcup_{\alpha\in\gamma} \Lambda(e_\alpha (A^0)^\alpha e_\alpha)$$

by Th.4.10. An idempotent subalgebra of a quotient of a semisimple algebra is semisimple so in this scenario $e_\alpha(A^0)^\alpha e_\alpha$ is semisimple. For each $\alpha \in \gamma$ let us write $\{\hat{S}^\alpha_\mu \mid \mu \in \Lambda^0_\alpha\}$ for a complete set of simples of $e_\alpha(A^0)^\alpha e_\alpha$.

Caveat: Given a modular core there is no reason to suppose that we are able to construct such a set directly.

In this case the corresponding collection of sets of $\beth^\gamma$-modules (here specifically denoted $\hat{\beth}^\gamma$ to distinguish from the other base rings) as in (11) give, by Lemma 4.12, a complete set of simples for A^0.

Working now over K, we have the various algebras $e_\alpha(A^K)^\alpha e_\alpha$. We call a collection of sets of K-algebra modules 'podular' if the appropriate embedding $G_{e_\alpha}-$ followed by base change to K_0 takes them to a complete set of simples as above. Note in particular then that this collection may be indexed in the same way: $\mathsf{S} = \{\{\mathsf{S}^\alpha_\mu \mid \mu \in \Lambda^0_\alpha\} \mid \alpha \in \gamma\}$.
(This is the same as to say that the base change of the podular set gives a collection of sets $\{\{\hat{S}^\alpha_\mu \mid \mu \in \Lambda^0_\alpha\} \mid \alpha \in \gamma\}$.)

(**4.22**) We now have a collection of A^K-modules

$$\check{\beth}^\gamma = \bigcup_{\alpha \in \gamma} \{\check{\beth}^\gamma{}_\mu \mid \mu \in \Lambda^0_\alpha\}$$

obtained by applying the appropriate $G_{e_\alpha}-$ functors to the collection S.
As already noted, this collection passes by base change to K_0 to the $\hat{\beth}^\gamma$-modules.
This collection $\check{\beth}^\gamma$ passes by the other base change to k to a set of A-modules $\bar{\beth}^\gamma$. Once again Lemma 4.12 applies. Note also that these are a complete set of BH modules, thus they are a pivotal set.

(**4.23**) Remark. Note that the implication of semisimplicity flows in one direction only, so the $\bar{\beth}^\gamma$ modules are not simple in general.

(**4.24**) We recall the definition of highest weight category from [29] (the original reference is [8, Definition 3.1] but we use an equivalent definition stated in [29] as it fits our purpose).

Let A be a finite dimensional algebra over a field k. Fix a complete set of pairwise non-isomorphic simple A-modules $\{S_\lambda \mid \lambda \in \Lambda(A)\}$. Let $\trianglelefteq$ be a partial order on the index set $\Lambda(A)$. The category $A-\text{mod}$ is a *highest weight category with respect to* $\trianglelefteq$ if for each $\lambda \in \Lambda(A)$ there exists a left A-module Δ_λ, called *standard module*, such that

1. There exists a surjective morphism $\theta_\lambda : \Delta_\lambda \to S_\lambda$, such that if S_μ is a composition factor of the $ker(\theta_\lambda)$ then $\mu \lhd \lambda$.

2. Let P_λ be the projective cover of S_λ. There exists a surjective morphism $\vartheta_\lambda : P_\lambda \to \Delta_\lambda$ such that the kernel of ϑ_λ is filtered by modules Δ_μ with $\lambda \lhd \mu$.

(**4.25**) LEMMA. *Let A be a k-algebra with a core (γ, e_-). Let $\beth^\gamma = \{\beth^\gamma_\mu \;:\; \mu \in \Lambda(A)\}$ be the set of $\beth$-modules of A for the core (γ, e_-) (in the sense of Equation 11). Suppose the set $\beth^\gamma$ filters projectives and is a pivotal set for the strong BH property. Now let $\prec$ be any order on $\Lambda(A)$ in which $\lceil \mu \rfloor < \lceil x \rfloor$ in γ implies $x \prec \mu$. Then the category A-mod is a highest weight category (HWC) with respect to $\prec$, with $\beth^\gamma$ as the set of standard modules.*

Proof. For Axiom 1 note from Lemma 4.12(I) that there is a map θ_λ; and from Lemma 4.13 that $[\ker(\theta_\lambda) : L_\mu] = 0$ unless $\lceil \lambda \rfloor < \lceil \mu \rfloor$, hence $\mu \prec \lambda$. For Axiom 2, let P_λ be an indecomposable projective module of A, then by the assumptions there is a filtration

$$0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_k = P_\lambda$$

of P_λ by $\beth^\gamma_\mu$-modules. We have $P_\lambda / M_{k-1} \simeq \beth^\gamma_\lambda$, since the $\beth^\gamma$-modules are pairwise non-isomorphic and their heads form a complete set of simple modules of A. By the strong BH *basis* property the multiplicities in the filtrations are uniquely defined, and coincide with the character definition of $(P_\lambda : \beth^\gamma_\mu)$. Thus $(P_\lambda : \beth^\gamma_\mu) = [\beth^\gamma_\mu : S_\lambda]$ by the strong BH property. Hence, if $(M_{k-1} : \beth^\gamma_\mu) \neq 0$ then Lemma 4.13 implies that $\lambda \prec \mu$. □

(**4.26**) THEOREM. *Let A be a k-algebra. Suppose:*
(I) A is a strong modular core algebra with modular system (K, K_0, k) and core (γ, e_-). Let $\bar{\beth}^\gamma$ be the set of $\beth^\gamma$-modules of the k-algebra A for the core (γ, e_-), obtained by base change as in 4.22. Let $\prec$ be any order on $\Lambda(A)$ in which $\lceil \mu \rfloor < \lceil x \rfloor$ in γ implies $x \prec \mu$.
(II) the set $\bar{\beth}^\gamma$ filters projective A-modules.
Then A-mod is a HWC with respect to $\prec$, with $\bar{\beth}^\gamma$ as the set of standard modules.

Proof. By 4.22 the set $\bar{\beth}^\gamma$ is pivotal set for the BH property. Now apply Lemma 4.25. □

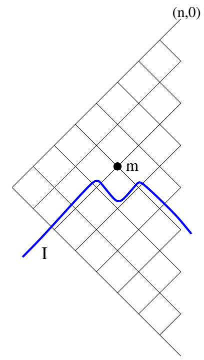
5 Simple index theorem for P^l_n

5.1 The quotient algebras $P^{\mathbf{m}}_n$ of P^l_n

(**5.1**) Define quotient algebra

$$P^{\mathbf{m}}_n \;=\; P^l_n / I^{<\mathbf{m}}$$

Schematically:



(**5.2**) By $a^{\mathbf{m}} \in P_n^{\mathbf{m}}$ we understand the element of which $a^{\mathbf{m}}$ is a representative (and similarly for $b^{\mathbf{m}}$).

(**5.3**) LEMMA. *The ideal $P_n^{\mathbf{m}} a^{\mathbf{m}} P_n^{\mathbf{m}}$ in $P_n^{\mathbf{m}}$ has basis $\mathtt{P}_n^{\mathbf{m}}$.*

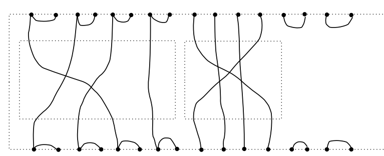
Proof. This follows from Lem.3.11 and the construction. □

(**5.4**) For $\mathbf{m} \in \gamma^{l,n}$ define

$$S_{\mathbf{m}} \;=\; \times_{i=1}^{l} S_{m_i} \tag{13}$$

For $\rho = (\rho_1, \rho_2, ..., \rho_l) \in S_{\mathbf{m}}$, define $w^a_\rho = w_\rho \in P^l_n$ as the image of ρ realised on the propagating lines in $a^{\mathbf{m}}$ as follows. Here is an example for $l = 2$:

$$w_\rho \;=\; \tag{14}$$



— we put ρ_1 on the co-1 propagating lines (the second interior box of lines in the example) in the natural way; ρ_2 on the co-2 lines (the first interior box in the example); and so on.

(**5.5**) For $\mathbf{m} \neq 0$ we define $w^b_\rho \in P^l_n$ analogously to w_ρ, but realised on the propagating lines of $b^{\mathbf{m}}$. We may similarly define w^{ab}_ρ on the propagating lines of $a^{\mathbf{m}} b^{\mathbf{m}}$; and w^{ba}_ρ analogously. Note that

$$w^b_\rho w^b_{\rho'} = w^b_{\rho\rho'}, \qquad\qquad w^a_\rho w^{ba}_{\rho'} = w^a_{\rho\rho'} \tag{15}$$

and so on.

(**5.6**) Consider partitions of form $q = a^{\mathbf{m}} p a^{\mathbf{m}}$, as illustrated in figure 8. (Partitions of form $q = b^{\mathbf{m}} p b^{\mathbf{m}}$ are directly similar.) Note that the number of co-i parts of q cannot be greater than that of $a^{\mathbf{m}}$ for any i, unless this is the result of two or more propagating parts coming together, such that $i' + i'' \equiv i$ mod. l (or $\sum_j i_j \equiv i$).

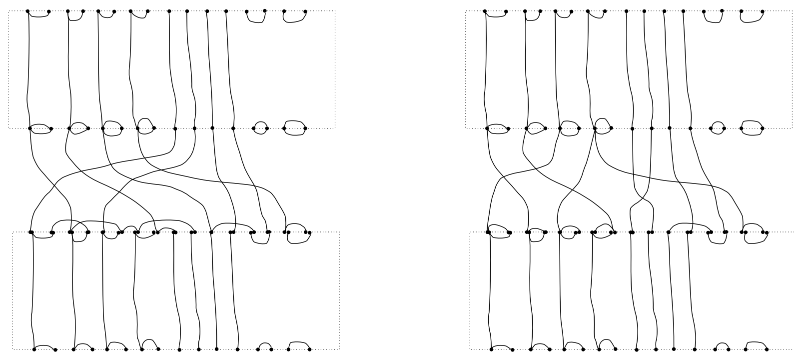


Figure 8: Partitions illustrating $a^{\mathbf{m}} P_n^{\mathbf{m}} a^{\mathbf{m}}$ in case $l = 2$.

Now cf. (3.4) and the definition of $P_n^{\mathbf{m}}$. Thus for example the first element in the figure, $q_1 = a^{\mathbf{m}} p_1 a^{\mathbf{m}}$ say, is in $I^{<\mathbf{m}}$ and hence zero in $P_n^{\mathbf{m}}$ by the quotient. To see this explicitly consider $\#^-(q_1)$, and in particular $\#^-(q_1)_1$. We may follow down the leftmost co-1 part from the top $a^{\mathbf{m}}$ factor. If this meets a co-$i > 1$ part from the lower $a^{\mathbf{m}}$ (as here) then, by the pigeonhole principle, the number of propagating parts decreases.

The second example is essentially a 'permutation' of form w_σ (in this case up to factors of δ). Note then that only permutations within each of the l group factors are possible, if we work in the $P_n^{\mathbf{m}}$ quotient.

(**5.7**) Lemma. *Let K be a commutative ring, and $\delta \in K$. Fix l and n, and $\mathbf{m} \in \gamma^{l,n} \setminus \{0\}$. Then we have the following.*
(bI) The map from $S_{\mathbf{m}}$ to P_n^l given by $\sigma \mapsto w_\sigma^b$ has image in $b^{\mathbf{m}} P_n^l b^{\mathbf{m}}$.
(bII) Idempotent subalgebra $b^{\mathbf{m}} P_n^{\mathbf{m}} b^{\mathbf{m}}$ in $P_n^{\mathbf{m}}$ has basis $\{[w_\sigma^b] = w_\sigma^b + I^{<\mathbf{m}} \mid \sigma \in S_{\mathbf{m}}\}$.
(bII') The map $\sigma \mapsto [w_\sigma^b]$ defines a map from $S_{\mathbf{m}}$ to the basis in $b^{\mathbf{m}} P_n^{\mathbf{m}} b^{\mathbf{m}}$ that is a group isomorphism.
(bIII) This map gives an algebra isomorphism

$$b^{\mathbf{m}} P_n^{\mathbf{m}} b^{\mathbf{m}} \;\cong\; K S_{\mathbf{m}}.$$

Proof. (bI) Note that $w_\sigma^b \in P_n^l$ and $b^{\mathbf{m}} w_\sigma^b b^{\mathbf{m}} = w_\sigma^b$. (bII) Noting (5), this follows essentially from Lemma 3.10 and the pigeonhole principle, as in (5.6) (replacing $a^{\mathbf{m}}$ with $b^{\mathbf{m}}$). (bII') This follows from (bII) and (15). (bIII) Follows from (bII) and (bII'). □

(**5.8**) Note that by the same argument the idempotent subalgebras $b^{\mathbf{m}} P_n^{\mathbf{m}} a^{\mathbf{m}}$, $a^{\mathbf{m}} P_n^{\mathbf{m}} b^{\mathbf{m}}$, and $a^{\mathbf{m}} P_n^{\mathbf{m}} a^{\mathbf{m}}$ are all isomorphic to $K S_{\mathbf{m}}$ (except that the 'aa' case requires δ to be a unit).

(**5.9**) THEOREM. *Suppose K is a field with $\delta \neq 0$. Then we have simple index set* $\Lambda(P_n^l) = \sqcup_{\mathbf{m}\in\gamma^{l,n}} \Lambda(KS_{\mathbf{m}})$.

Proof. Noting Theorem 4.6, now apply Th.(4.10) and substitute using (5.7) and (5.8). □

(**5.10**) Remark. There is a straightforward strengthening of Th.5.9 to the case $\delta = 0$ in most cases. We omit this for brevity.

5.2 Aside on symmetric groups and Specht modules

In light of Lem.5.7 *et seq.*, it is useful to recall some properties of the group algebra $KS_{\mathbf{m}}$ (as defined in (13); for example $KS_{\mathbf{m}} = K(S_{n-2i} \times S_i)$ in case $\mathbf{m} = (n-2i, i)$). We focus for brevity in this exposition on the case $l = 2$. The generalisation is straightforward.

(**5.11**) Define $\Lambda_i = \{\lambda \vdash i\}$, the set of integer partitions; and

$$\Lambda_{\mathbf{m}} = \times_{i=1}^{l} \Lambda_{m_i} \; = \; \Lambda_{m_1} \times \Lambda_{m_2} \times ... \times \Lambda_{m_l}$$

(**5.12**) Here an element e in a k-algebra S (k some commutative ring) is 'preidempotent' if $ee = ce$ for some $c \in k$. If c a unit then e may be renormalised as an idempotent. 'Primitivity' of $e \in S$ means that $ewe = c_w e$ for some $c_w \in k$ for all $w \in S$.

(**5.13**) Recall (cf. [24] and [11, §43], say) that for each $\underline{\lambda} = (\lambda^1, \lambda^2, ..., \lambda^l) \in \Lambda_{\mathbf{m}}$ there exists a primitive preidempotent

$$e_{\underline{\lambda}} = e_\lambda = e'_{\lambda^1} e''_{\lambda^2} ... \tag{16}$$

in $\mathbb{Z}S_{\mathbf{m}}$ (primes indicate belonging to different factors) such that the left ideal

$$\mathtt{Sp}_\lambda = \; KS_{\mathbf{m}} \, e_\lambda$$

is a Specht module for $l = 1$ and hence a generalised Specht module otherwise.

In our case we may choose $e = e_\lambda$ so that $e = e^{op}$, as in (2.5) (this follows from one of the well-known constructions for preidempotents in $\mathbb{Z}S_n$ [25]).

(**5.14**) For definiteness we have in mind a tableau-labelled basis

$$b_\lambda = b_{\lambda^1} \times b_{\lambda^2} \times ...$$

for $\mathtt{Sp}_\lambda$. This is a basis encoded as l-tuples of standard sequences such as $b_{(3,1)} = \{1112, 1121, 1211\}$. The details of the corresponding explicit basis contruction can be found for example in [24], but the full details will not be needed here.

(**5.15**) Returning to (5.12), suppose algebra S has a k-linear map $op : S \to S$, written $s \mapsto s^{op}$, that is an involutive antiautomorphism. Suppose primitive preidempotent $e = e^{op}$. This then allows us to define a k-bilinear form $(-,-)_e$ on Se as follows: for $se, s'e \in Se$ we have

$$(se)^{op}s'e = es^{op}s'e = \; c_{s^{op}s'} \; e. \tag{17}$$

Now set $(se, s'e)_e = c_{s^{op}s'}$. Note that for $a \in S$ we have $(se, as'e)_e = c_{s^{op}as'} = c_{(a^{op}s)^{op}s'} = (a^{op}se, s'e)_e$. That is, the form is *contravariant* with respect to op [22].

(**5.16**) This form is useful in studying the corresponding S-module morphism from Se to its contravariant dual [22]. Note that the choice of e is not unique in our ideal construction $M = Se$, and although Se does not depend on the choice (up to isomorphism) the form does depend on it by an overall factor. Thus the form is not canonical on M. However for symmetric groups there is a good choice of form, due to James, that encodes representation theory within a very useful organisational scheme [24]. Over fields of char.0 the constant c is always a unit and these subtleties can be ignored, as we will see in §9.1.

6 Polar decomposition of partitions in $\mathtt{P}^l_n$

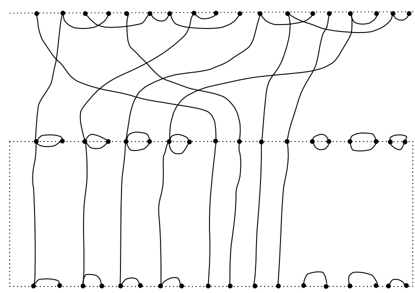
We describe a version for P^l_n of the P_n polar decomposition [37, p79-80].

(**6.1**) Let $\mathbf{m} \in \gamma^{l,n}$. Let $B^{\mathbf{m}}$ denote the natural 'diagram' basis of $P^{\mathbf{m}}_n a^{\mathbf{m}}$ (i.e. the basis of certain partitions $p \in P^l_n a^{\mathbf{m}}$ where p is understood to mean $p + I^{<\mathbf{m}}$). Next we describe $B^{\mathbf{m}}$. Note that the left ideal $P^{\mathbf{m}}_n a^{\mathbf{m}}$ has a natural right action of $S_{\mathbf{m}}$ upon it (see e.g. Lem.5.7 and (5.8)). Indeed it is a free right $KS_{\mathbf{m}}$-module. It follows that we may partition $B^{\mathbf{m}}$ into orbits of the right action of $S_{\mathbf{m}}$.

(**6.2**) The basis $B^{\mathbf{m}}$ consists of elements p representable in the following form.
(1) The restriction of p to the 'top' (unprimed) subset of vertices consists of:
for each $i = 0, l-1, l-2, ..., 1$, m_i co-i parts that belong to propagating parts; and some further number of co-0 non-propagating parts.
The restriction of p to the 'bottom' (primed) subset of vertices consists of:
for each i, m_i order-i parts that belong to propagating parts; and $(n - r_{\mathbf{m}})/l$ further order-l non-propagating parts.
(2) The propagating connection from top to bottom for a propagating part may be drawn as a line from the first (lowest numbered) vertex of the part on the top to the first on the bottom.

(**6.3**) We say that $p \in B^{\mathbf{m}}$ is *relatively non-crossing* if, for each i, the co-i propagating lines in the representation above are pairwise non-crossing.

An example of a relatively non-crossing partition (in case $l = 2$) is given by:



The isolated loops in this picture can be ignored (we assume $\delta \neq 0$ here for simplicity) or replaced with a suitable 'meander'. They are drawn to demonstrate that the element lies in $P_n^{\mathbf{m}} a^{\mathbf{m}}$ (here in case $\mathbf{m} = (m_0, m_1) = (4, 4)$).

(**6.4**) Lemma. *Let $\mathbf{m} \in \gamma^{l,n}$. Each orbit of the right $S_{\mathbf{m}}$ action on basis $B^{\mathbf{m}}$ of $P_n^{\mathbf{m}} a^{\mathbf{m}}$ as in (6.1) contains a unique representative element with the relative non-crossing property. Let $T^{\mathbf{m}}$ denote the relative non-crossing transversal. Then*

$$B^{\mathbf{m}} = \{pw \mid p \in T^{\mathbf{m}};\ w \in S_{\mathbf{m}}\}$$

where w acts in the natural way. In particular $T^{\mathbf{m}}$ is a $kS_{\mathbf{m}}$- basis of the free right $kS_{\mathbf{m}}$-module $P_n^{\mathbf{m}} a^{\mathbf{m}}$. □

(**6.5**) It will be apparent that any partition p in $\mathtt{P}_n^{\mathbf{m}}$ (as defined in (7)) can be written in a generalisation of the usual partition algebra *polar decomposition*. That is, we have the following.

(**6.6**) Lemma. *Each $p \in \mathtt{P}_n^{\mathbf{m}}$ can be written in a factored form as*

$$p = a w_\sigma b$$

where a is relatively non-crossing (as in (6.3)), i.e. $a \in T^{\mathbf{m}}$; $w_\sigma = (w_1, w_2, ...)$ is as in (14); and b is a flipped relatively non-crossing partition ($b^\star \in T^{\mathbf{m}}$). The factorisation in this form is unique. □

(**6.7**) It is convenient to denote the decomposition by

$$p \mapsto |p\rangle\ w(p)\ \langle p| \tag{18}$$

7 Basic integral representation theory of P_n^l

Here we aim to construct a modular system for P_n^l based on analogues of Specht modules.

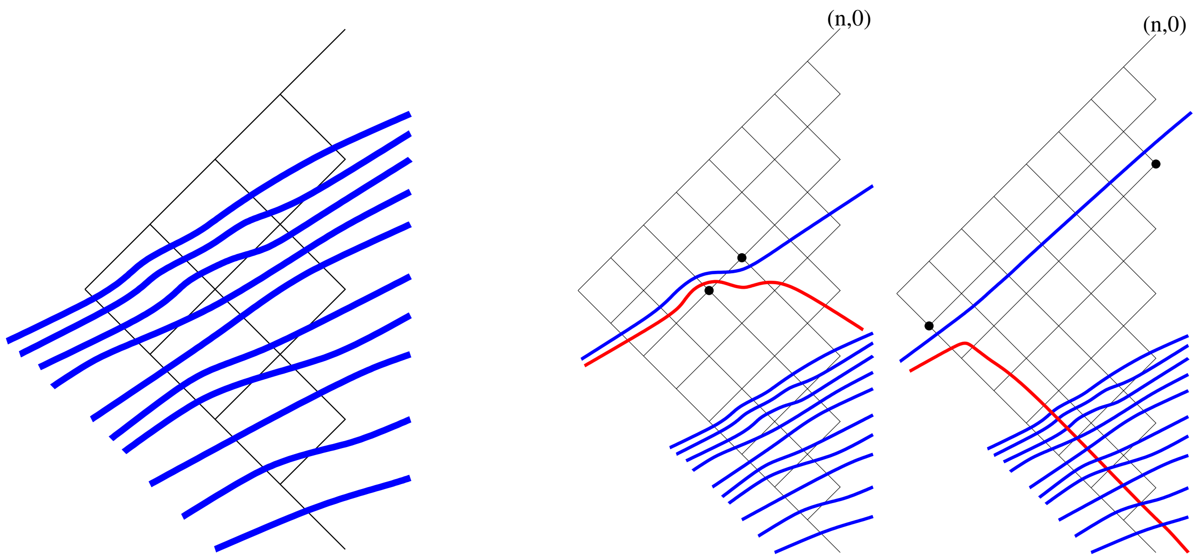


Figure 9: (a) Schematic for $\rhd$-sectioning the $\gamma^{l,n}$ poset ($l = 2$). (b) Schematics for comparing $J_{\mathbf{m}(i)}$ (below blue line) with $I^{<\mathbf{m}(i+1)}$ (below red line).

(**7.1**) Recall that any poset can be refined to a total order. Let $(\gamma^{l,n}, \rhd)$ be a total order refining $(\gamma^{l,n}, >)$ (e.g. as indicated by Fig.9). Let us define

$$J_{\mathbf{m}} = \sum_{\mathbf{m}' \unlhd \mathbf{m}} P^l_n a^{\mathbf{m}'} P^l_n, \qquad\qquad J_{\lhd \mathbf{m}} = \sum_{\mathbf{m}' \lhd \mathbf{m}} P^l_n a^{\mathbf{m}'} P^l_n$$

For convenience define $\mathbf{m}(1), \mathbf{m}(2), ...$ as the elements of $\gamma^{l,n}$ in the total order so

$$J_{\mathbf{m}(1)} \subset J_{\mathbf{m}(2)} \subset J_{\mathbf{m}(3)} \subset ... \subset J_{(n,0,0,...,0)} = P^l_n$$

(**7.2**) Lemma. *Write $P = P^l_n$ for a moment. We have an isomorphism of bimodules*

$$J_{\mathbf{m}(i+1)}/J_{\mathbf{m}(i)} \cong P a^{\mathbf{m}(i+1)} P / I^{<\mathbf{m}(i+1)}. \tag{19}$$

Proof: We have

$$J_{\mathbf{m}(i+1)}/J_{\mathbf{m}(i)} = \frac{P a^{\mathbf{m}(i+1)} P + P a^{\mathbf{m}(i)} P + ...}{P a^{\mathbf{m}(i)} P + P a^{\mathbf{m}(i-1)} P + ...}$$

$$\cong \frac{P a^{\mathbf{m}(i+1)} P}{P a^{\mathbf{m}(i+1)} P \cap (P a^{\mathbf{m}(i)} P + P a^{\mathbf{m}(i-1)} P + ...)} = \frac{P a^{\mathbf{m}(i+1)} P}{P a^{\mathbf{m}(i+1)} P \cap J_{\mathbf{m}(i)}}$$

by the second isomorphism theorem. Thus we may consider the 'numerators' in (19) to be the same, up to isomorphism; and compare the 'denominators' (the submodules that are quotiented by). The argument proceeds in two steps.
(I) For any order $\rhd$ refining $>$ we have that $\mathbf{m}(i+1) > \mathbf{m}$ implies $\mathbf{m}(i+1) \rhd \mathbf{m}$. But for any total order this implies $\mathbf{m}(i) \unrhd \mathbf{m}$. Thus $J_{\mathbf{m}(i)} \supset I^{<\mathbf{m}(i+1)}$. (Cf. Fig.9(b).)
(II) Consider the denominator in the third expression. In particular consider $Pa^{\mathbf{m}(i+1)}P \cap Pa^{\mathbf{m}(j)}P$ for $j = i, i-1, ...$ By Lemma 3.11 every partition p lies in a unique highest ideal of the form $Pa^{\mathbf{m}}P$, and there is a basis of partitions. Since $\mathbf{m}(j) \ngeq \mathbf{m}(i+1)$ we have (cf. (3.14)) that

$$Pa^{\mathbf{m}(i+1)}P \cap Pa^{\mathbf{m}(j)}P \subseteq I^{<\mathbf{m}(i+1)} \qquad (j \leq i)$$

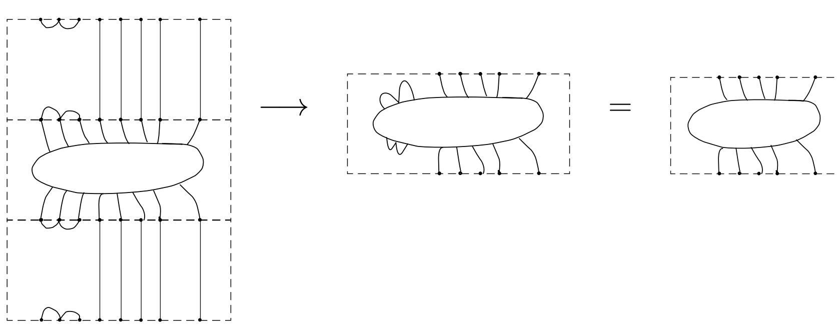
Combining with the inequality in the other direction from (I), noting that $Pa^{\mathbf{m}(i+1)}P \supset I^{<\mathbf{m}(i+1)}$, we see that

$$J_{\mathbf{m}(i+1)}/J_{\mathbf{m}(i)} \cong \frac{Pa^{\mathbf{m}(i+1)}P}{I^{<\mathbf{m}(i+1)}} = P^{\mathbf{m}(i+1)}a^{\mathbf{m}(i+1)}P^{\mathbf{m}(i+1)}$$

□

7.1 Globalisation functors and quotient algebras A_n^l

Note that a partition of a set S determines a partition of a subset S' by restriction. In particular an element p of P_n determines an element $p|_{[1,n-l]}$ of P_{n-l} by restricting to the first $n-l$ pairs of elements 'top and bottom'. Similarly $p|_{[l+1,n]}$ restricts to the last $n-l$ pairs. (Again this determines an element of P_{n-l} in the obvious way.) Note in particular for $n > l$ that the restriction $p|_{[l+1,n]}$ of a partition p in $W_b^l P_n^l W_b^l$ removes l top elements from the same part and l bottom elements from the same part, and hence $p|_{[1,n-l]}$ lies in P_{n-l}^l. A similar property holds on restricting $W^l P_n^l W^l$, or indeed $\overline{W_b^l} P_n^l \overline{W_b^l}$ (using (2.6)), and so on. Given one of these cases, let us write ι_W for this restriction map. Schematically (W^3 case):



(**7.3**) THEOREM. *Fix l. For $n > l$ the maps ι_W give isomorphisms of algebras:*

$$W_b^l P_n^l W_b^l \cong P_{n-l}^l \quad \textit{and} \quad \overline{W_b^l} P_n^l \overline{W_b^l} \cong P_{n-l}^l.$$

Similarly for $n \geq l$ we have the (not necessarily unital) algebra isomorphisms $W^l P_n^l W^l \cong \delta P_{n-l}^l$. and $\overline{W^l} P_n^l \overline{W^l} \cong \delta P_{n-l}^l$.

Proof. Exactly analogous to the P_n case as in [37]. Schematically (W^3 case), consider the figure above. □

(**7.4**) As in (4.3) we define 'short' functors

$$G_{\overline{W}} : \overline{W_b^l} P_n^l \overline{W_b^l} - \text{mod} \quad \rightarrow \quad P_n^l - \text{mod}$$

and $F_{\overline{W}}$. And similarly (for δ invertible) $G_W : W_b^l P_n^l W_b^l - \text{mod} \quad \rightarrow \quad P_n^l - \text{mod}$. By Theorem 7.3 we will consider these as functors between $P_{n-l}^l - \text{mod}$ and $P_n^l - \text{mod}$.

(**7.5**) Fix l. Define the quotient algebra

$$A_n^l \;=\; P_n^l / \overline{W_b^l} P_n^l \;=\; P_n^l / P_n^l W^l P_n^l \;=\; P_n^l / P_n^l W_b^l P_n^l.$$

(**7.6**) Example: In case $l = 1$, $A_n^1 = kS_n$.

(**7.7**) Lemma. *There is a basis for A_n^l consisting of partitions in which every part is propagating with $\text{cora}(p_i) = \text{ran}(p_i)$ and no part has $|\text{cora}(p_i)| > l$ or $|\text{ran}(p_i)| > l$.*

Proof. By Lem.3.12. □

(**7.8**) Let H_n^l denote the subset of idempotents $a^{\mathbf{m}}$ with $\mathbf{m} \in h_n^l$.

Let $\mathsf{P}_n^{l-} \subset \mathsf{P}_n^l$ denote the subset of l-tone partitions of form ShT, where S, T are permutations and $h \in H_n^l$. That is, P_n^{l-} is the set of l-tone partitions having parts with at most l elements per row, and all parts propagating.

Note from Lem.3.10 the following.

(**7.9**) Lemma. *Fix l. Algebra A_n^l has basis the subset P_n^{l-}.* □

(**7.10**) An element of P_n^{l-} is partially characterised by the restricted partition of the upper (resp. lower) row into parts of size $l, l-1, ..., 1$. We say that an ordered pair $(a, b) \in \mathsf{P}_n^l$ are *compatible* if, considering the rows that meet in composition ab, every part from a (resp. b) is a union of parts from b (resp. a).

Note that $ab \equiv 0$ in A_n^l unless compatible.

In case $l > n$ there can be no non-propagating part in a partition in P_n^l. In this case P_n^l coincides with Kosuda's party algebra [32]. We will be interested, though, in general n for each fixed l.

7.2 Long functors from $KS_{\mathbf{m}}$ −mod

(**7.11**) Note from Lem.5.7 and (5.8) that $P_n^{\mathbf{m}} a^{\mathbf{m}}$ is a right $KS_{\mathbf{m}}$ module. (The case $\mathbf{m} = 0$, $\delta = 0$ can be included by identifying KS_0 with the ground ring.)

Now for given $\mathbf{m} \in \gamma^{l,n}$ define the *long functor*

$$G_a : KS_{\mathbf{m}} -\text{mod} \rightarrow P_n^{\mathbf{m}} -\text{mod}$$

by $G_a M = P_n^{\mathbf{m}} a^{\mathbf{m}} \otimes_{KS_{\mathbf{m}}} M$.

Here we want to recall the isomorphism between $G_a M$ and $P_n^{\mathbf{m}} a^{\mathbf{m}} M$ that holds when M is a left ideal, so that standard modules inherit the properties from both constructions. Noting the invariant basis number (IBN) property we write $rank_K(M)$ for the basis number of a free K-module with finite basis.

(**7.12**) Lemma. *Let A be a finite rank K-algebra, M be an A-module with K-basis B, and F be a right free A-module of finite rank with A-basis T. Then*
(I) $T \otimes B = \{t \otimes b \mid t \in T; b \in B\}$ is a basis of $F \otimes_A M$.
If in addition M is a left ideal then
(II) $TB = \{tb \mid t \in T; b \in B\}$ is a K-basis of FM.
(III) The map $\mu : T \otimes B \rightarrow TB$ given by $t \otimes b \mapsto tb$ lifts to a well-defined K-module isomorphism $\mu : F \otimes_A M \rightarrow FM$. If F is an A'-A-bimodule then μ is an isomorphism of left A'-modules.

Proof. (I) The A-basis property says $F = \oplus_{t \in T} tA$ where each tA is a copy of the right regular A-module. Thus $F \otimes_A M = (\oplus_{t \in T} tA) \otimes_A M = \oplus_{t \in T}(tA \otimes_A M)$ by tensor-distributivity (see e.g. [11, §12]) and hence $F \otimes_A M = \oplus_{t \in T} t \otimes M$ since $A \otimes_A {}_A M \cong M$. Thus $rank_K(F \otimes_A M) = |T|.|B|$. Furthermore if G is a K-basis of A then F has K-basis TG, so $\{tg \otimes b \ : \ t \in T, \ g \in G, \ b \in B\}$ spans $F \otimes_A M$. But $ta \otimes b = t \otimes ab$ so, fixing t, the set $\{t \otimes b \ : \ b \in B\}$ spans the same K-module as $\{ta \otimes b \ : \ a \in A, b \in B\}$. Thus $T \otimes B$ spans $F \otimes_A M$. Since $|T \otimes B| = |T|.|B|$ we are done.
(II) We may compare for example with [34, 4.12]. The set FM is a K-module, so $FM = \oplus_t tAM$ is a direct sum of K-modules - specifically $AM = M$ is a K-module and t is a formal symbol. Thus $FM = \oplus_t tM$ is a formal direct sum of copies of M. Since M is itself K-free, we have a formal direct sum of formal direct sums.
(III) Well-definedness follows from the balanced map property of tensor products. The map is surjective by construction, and hence an isomorphism by IBN. Commutativity of the map with the algebra action is also by construction. □

7.3 Standard/Specht modules

(**7.13**) Fix n and l. For $\mathbf{m} \in \gamma^{l,n}$ and $\underline{\mu} = (\mu^1, \mu^2, ..., \mu^l) \in \Lambda_{\mathbf{m}}$ we may define the $\underline{\mu}$-'Specht' module of P_n^l as the module obtained by applying the functor G_a in (7.11) to the $\underline{\mu}$-Specht module $\mathtt{Sp}_{\underline{\mu}}$ of $kS_{\mathbf{m}}$:

$$\Delta_{\underline{\mu}} \;=\; G_a \mathtt{Sp}_{\underline{\mu}} \tag{20}$$

(see §5.2 for details of $\mathtt{Sp}_{\underline{\mu}}$). Note that this is a P_n^l-module since it is a $P_n^{\mathbf{m}}$-module. Define $\Lambda_0(P_n^l) = \cup_{\mathbf{m}\in\gamma^{l,n}} \Lambda_{\mathbf{m}}$ and let $\Delta^{l,n} = \{\Delta_{\underline{\mu}} \mid \underline{\mu} \in \Lambda_0(P_n^l)\}$.

(**7.14**) Lemma. *For $k = \mathbb{C}$ and $\delta \neq 0$ the set $\{L_{\underline{\mu}} = head\ \Delta_{\underline{\mu}}\}_{\underline{\mu}\in\Lambda_0(P_n^l)}$ is a complete set of simple modules.*

Proof. Compare Th.5.9(I) with Lem.4.12(I) and (7.13). □

(**7.15**) Let $\mathbf{m} \in \gamma^{l,n}$ and $\underline{\mu} \in \Lambda_{\mathbf{m}}$, and $e_{\underline{\mu}} \;=\; \prod_i e^i_{\mu^i}$ as in (16). Define

$$\mathcal{S}^n_{\underline{\mu}} \;=\; P_n^{\mathbf{m}} a_n^{\mathbf{m}} w^{ba}_{e_{\underline{\mu}}}$$

where $e_{\underline{\mu}}$ thus acts as in (14 - 15).

(**7.16**) Lemma. *Let $b_{\underline{\mu}}$ be a basis of $\underline{\mu}$-Specht module $\mathtt{Sp}_{\underline{\mu}}$ of $KS_{\mathbf{m}}$ (cf. 5.14, [24]); and $T^{\mathbf{m}}$ be a non-crossing transversal in $P_n^{\mathbf{m}}$ as defined in Lemma 6.4. Then*

$$B^{\underline{\mu}}_{\mathtt{Sp}} \;:=\{\, p\, w^{ba}_{\omega} \mid p \in T^{\mathbf{m}},\ \omega \in b_{\underline{\mu}}\}$$

is a basis of $\mathcal{S}^n_{\underline{\mu}}$.

Proof. By 6.4 the set $T^{\mathbf{m}}$ is a $KS_{\mathbf{m}}$-basis of $P^{\mathbf{m}}a^{\mathbf{m}}$. Now in Lemma 7.12 part II let $A = KS_{\mathbf{m}}$, $M = \mathtt{Sp}_{\underline{\mu}}$ and $F = P_n^{\mathbf{m}} a^{\mathbf{m}}$ to obtain the result. □

(**7.17**) Example. Fig.10 shows a diagrammatic realisation of the basis for $\mathcal{S}^4_{((2),\emptyset)}$. The box labeled + denotes the $\mathbb{Z}$-linear combination corresponding to the preidempotent $e_{(2)} = 1 - \sigma_1 \in \mathbb{Z}S_2$. Note cf. [37] that this diagram calculus is well defined. So far, then, the pictures give combinations of partitions — but then finally these partitions are understood to represent the classes in the module of which they are representative.

(**7.18**) Lemma. (I) *For $\mathbf{m} \in h_n^l$ and $\underline{\mu} \in \Lambda_{\mathbf{m}}$, $F_{\overline{W}} \mathcal{S}^n_{\underline{\mu}} = 0$.* (II) *For $\mathbf{m} \in \gamma^{l,n-l}$ and $\underline{\mu} \in \Lambda_{\mathbf{m}}$, $F_{\overline{W}} \mathcal{S}^n_{\underline{\mu}} = \overline{W}\mathcal{S}^n_{\underline{\mu}} \cong \mathcal{S}^{n-l}_{\underline{\mu}}$. (Here we assume for simplicity that $\delta \neq 0$.)*

Proof. (I) Here $\overline{W} P_n^{\mathbf{m}} \;=\; 0$. (II) Note that for $x \in I^{<\mathbf{m}}_{n-l}$ then $x \otimes ww^\star \in I^{<\mathbf{m}}_n$. Considering the bases one finds that $b \mapsto b \otimes ww^\star$ (with b of form $d + I^{<\mathbf{m}}_{n-l}$) gives rise to an injection $\mathcal{S}^{n-l}_{\underline{\mu}} \to \overline{W}\mathcal{S}^n_{\underline{\mu}}$. The image is fixed by $\overline{W}$ so $F_{\overline{W}}\mathcal{S}^n_{\underline{\mu}}$ contains $\mathcal{S}^{n-l}_{\underline{\mu}}$ as a submodule. We observe complementarily that the map $\overline{W} P_n^l \overline{W} \to P^l_{n-l}$ (here NB the standing assumption) takes $p \in \mathtt{P}_n^{\mathbf{m}}$ to $p' \in \mathtt{P}^{\mathbf{m}}_{n-l}$. Indeed $\overline{W} P_n^l \overline{W} \cap \mathtt{P}_n^{\mathbf{m}} = \mathtt{P}^{\mathbf{m}}_{n-l} \otimes ww^\star$. Furthermore by construction each element of $\overline{W}\mathcal{S}^n_{\underline{\mu}}$ is a subset of $\overline{W} P_n^l \overline{W}$

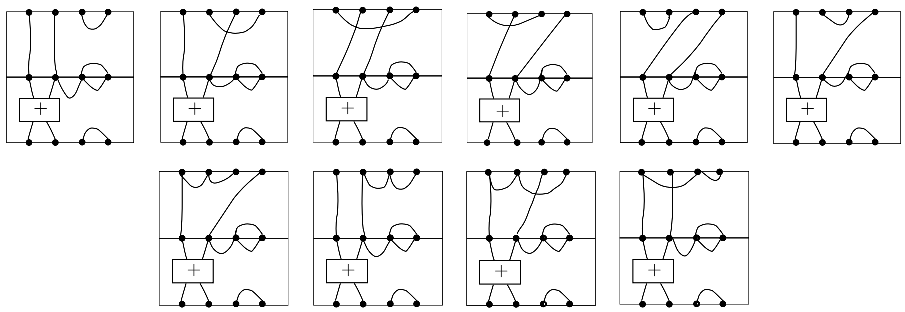


Figure 10: Diagrams for the basis for $\mathcal{S}^4_{((2),\emptyset)}$. See main text.

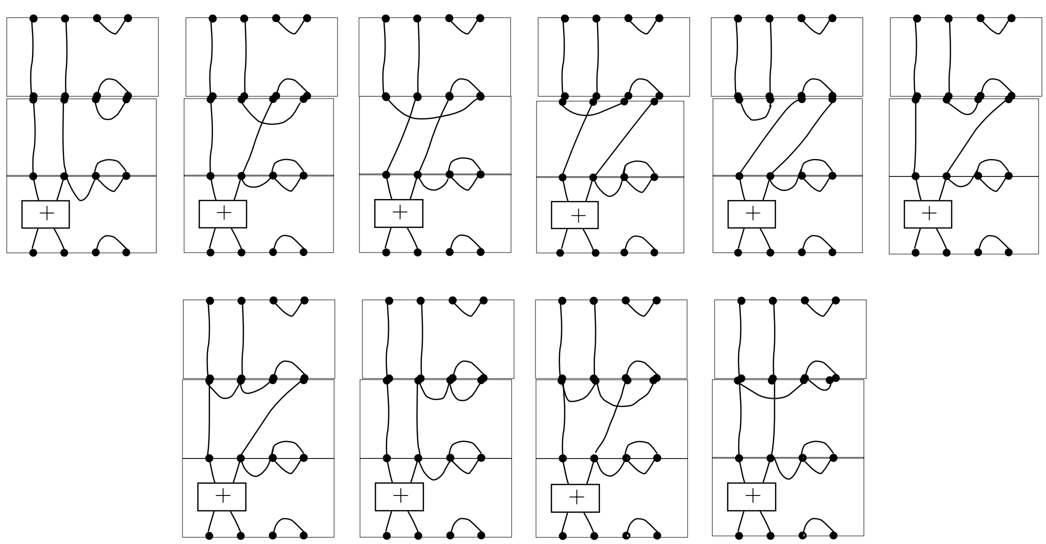


Figure 11: Action of $\overline{W}$ on the basis for $\mathcal{S}^4_{((2),\emptyset)}$.

(to see this consider e.g. Fig.11), so we can apply $x \leftrightarrow x \otimes ww^\star$. On the other hand we know by Lemma 3.10(II) that whenever the propagating index is changed it is reduced. Thus the image of the injection is also spanning.
(To see this less formally, from the example — Fig.11 — one can see the following types of cases for $\overline{W}d$ for $d \in B^{\underline{\mu}}_{\mathtt{Sp}}$: (1) last l vertices already connected in d; (2) propagating lines are at most permuted; (3) the propagating index is reduced.) □

(**7.19**) LEMMA. *We have an isomorphism of P^l_n-modules: $\Delta_{\underline{\mu}} \cong \mathcal{S}^n_{\underline{\mu}}$.*

Proof. By 6.1 the $P^l_n - kS_{\mathbf{m}}$ bimodule $P^{\mathbf{m}}_n a^{\mathbf{m}}$ is free as right $KS_{\mathbf{m}}$-module. Now in Lemma 7.12 part III let F be $P^{\mathbf{m}}_n a^{\mathbf{m}}$ and M be the left ideal $\underline{\mu}$-Specht module $\mathtt{Sp}_{\underline{\mu}}$ to obtain the desired isomorphism. In particular the basis of $\Delta_{\underline{\mu}}$ of form

$$B^{\underline{\mu}}_{\Delta} = \{p \underset{\mathbf{m}}{\otimes} w \mid p \in T^{\mathbf{m}}, w \in b_{\underline{\mu}}\}$$

(Lemma 7.12(I); we write $\underset{\mathbf{m}}{\otimes}$ here to distinguish from the other types of tensor product in this Section) is taken element-wise to $B^{\underline{\mu}}_{\mathtt{Sp}}$. □

(**7.20**) LEMMA. *Let $\delta \neq 0$ and $k = \mathbb{C}$. Let $\mathbf{m} \in \gamma^{l,n}$. As a left P^l_n-module the quotient $P^{\mathbf{m}}_n a^{\mathbf{m}} P^{\mathbf{m}}_n$ is a direct sum of Δ-modules.*

Proof. By Lemma 6.4 we have the following isomorphism of left P^l_n-modules

$$P^{\mathbf{m}}_n a^{\mathbf{m}} P^{\mathbf{m}}_n = \bigoplus_{p \in T^{\mathbf{m}}} P^{\mathbf{m}}_n a^{\mathbf{m}} p^\star \simeq \bigoplus_{T^{\mathbf{m}}} P^{\mathbf{m}}_n a^{\mathbf{m}} \tag{21}$$

where $p^\star$ is as defined in 2.5. Furthermore Lemma 7.19 implies that

$$P^{\mathbf{m}}_n a^{\mathbf{m}} \simeq P^{\mathbf{m}}_n a^{\mathbf{m}} \otimes_{kS_{\mathbf{m}}} kS_{\mathbf{m}} \simeq \bigoplus_{\underline{\mu} \in \Lambda_{\mathbf{m}}} \left(P^{\mathbf{m}}_n a^{\mathbf{m}} e_{\underline{\mu}} \right)^{dim(\mathtt{Sp}_{\underline{\mu}})} \tag{22}$$

as a P^l_n-module. □

(**7.21**) Comparing 3.11, 6.6 and (22) we see that

$$\dim(P^l_n) = \sum_{\underline{\mu} \in \Lambda(P^l_n)} \dim(\Delta_{\underline{\mu}})^2 \tag{23}$$

(cf. a cellular basis of P^l_n in the sense of [21]).

(**7.22**) LEMMA. *Suppose $k = \mathbb{C}$ and $\delta \neq 0$. Recall the simple P^l_n-modules $L_{\underline{\mu}} =$ head $\mathcal{S}^n_{\underline{\mu}}$ from (7.14). The modules $\{\mathcal{S}^n_{\underline{\mu}} : \underline{\mu} \in \Lambda(P^l_n)\}$ have a lower-unitriangular decomposition matrix $([\mathcal{S}^n_{\underline{\mu}} : L_{\underline{\nu}}])_{\underline{\mu},\underline{\nu} \in \Lambda(P^l_n)}$ with respect to any order $\prec$ on $\Lambda(P^l_n)$ in which $\mathbf{m} < \mathbf{m}'$ implies $\underline{\nu} \prec \underline{\mu}$ for $\underline{\mu} \in \Lambda_{\mathbf{m}}$ and $\underline{\nu} \in \Lambda_{\mathbf{m}'}$.*

Proof. By Theorem 5.9 the pair $(\gamma^{l,n}, a^-)$ is a core of P_n^l. When $k = \mathbb{C}$ for each $\underline{\mu} \in \Lambda_{\mathbf{m}}$ the $\underline{\mu}$-Specht module $\mathtt{Sp}_{\underline{\mu}}$ is simple as $\mathbb{C}S_{\mathbf{m}}$-module. By Equation 20 and Lemma 7.19 the set of modules $\{\mathcal{S}^n_{\underline{\mu}} \mid \underline{\mu} \in \Lambda(P_n^l)\}$ are the corresponding long G_a-functor Δ-modules as in Equation 11. Now the result follows from Lemma 4.13. □

8 Globalisation of standard modules

In this section we study the effect of the $G_{\overline{W}}$ functor on standard modules. That is we study the P_n^l-module $G_{\overline{W}}\ \mathcal{S}^n_{\underline{\mu}}$. This is particularly interesting because the non-trivial core property leads to some new departures from the partition algebra argument. The main result is Proposition 8.3 below.

(**8.1**) Recalling $T^{\mathbf{m}}$, let $\mathsf{T}^{\mathbf{m}}_n$ denote the set of representative relative-non-crossing partitions p rather than the classes $p + I^{<\mathbf{m}}$.

(**8.2**) NB our convention is that if R is a ring and S a set then RS generally denotes the free R-module with basis S. However if S is given as a subset of an R-module M then RS means the R-span of S in M.

(**8.3**) Proposition. Let $\mathbf{m} \in \gamma^l_{n-l}$ and $\underline{\mu} \in \Lambda_{\mathbf{m}}$. Applying $G_{\overline{W}}$ from (7.4), consider the subset of $G_{\overline{W}}\ \mathcal{S}^{n-l}_{\underline{\mu}}$ given by $B^{\underline{\mu}}_G = \{t \underset{n-l}{\otimes} a^{\mathbf{m}}_{n-l} w^{ba}_{\omega} \mid t \in \mathsf{T}^{\mathbf{m}}_n, \omega \in b_{\underline{\mu}}\}$ where we use $\underset{n-l}{\otimes}$ for the $G_{\overline{W}}$ tensor product. Then $B^{\underline{\mu}}_G$ is a basis of $G_{\overline{W}}\ \mathcal{S}^{n-l}_{\underline{\mu}}$ and

$$\mathcal{S}^n_{\underline{\mu}} \;\cong\; G_{\overline{W}}\, \mathcal{S}^{n-l}_{\underline{\mu}}\ .$$

Proof. Note from the construction that $P_n^l \overline{W} \mathcal{S}^n_{\underline{\mu}} = \mathcal{S}^n_{\underline{\mu}}$. By Lemma 7.18 $G_{\overline{W}}\, \mathcal{S}^{n-l}_{\underline{\mu}} \cong P_n^l \overline{W} \underset{n-l}{\otimes} \overline{W} \mathcal{S}^n_{\underline{\mu}}$. For the latter form we have a multiplication map $p \otimes s \mapsto ps$ to $P_n^l \overline{W} \mathcal{S}^n_{\underline{\mu}} = \mathcal{S}^n_{\underline{\mu}}$. This gives a surjective P_n^l-module homomorphism. Since $|B^{\underline{\mu}}_G| = |B^{\underline{\mu}}_{\mathtt{Sp}}|$ it is enough to show that $B^{\underline{\mu}}_G$ is spanning. The basis $B^{\underline{\mu}}_{\mathtt{Sp}}$ gives

$$\mathcal{S}^n_{\underline{\mu}} = P^{\mathbf{m}}_n a^{\mathbf{m}}_n w^{ba}_{e_\mu} = K\{t w^{ba}_{\omega} \mid t \in T^{\mathbf{m}}_n;\ \omega \in b_{\underline{\mu}}\} \tag{24}$$

This holds for any n but, as indicated here the non-crossing transversal $T^{\mathbf{m}}_n$ in $P^{\mathbf{m}}_n a^{\mathbf{m}}_n$ of course depends on n. Applying $G_{\overline{W}}$ to the case with n replaced by $n-l$ we have

$$G_{\overline{W}}\, \mathcal{S}^{n-l}_{\underline{\mu}} = P_n^l \overline{W} \underset{n-l}{\otimes} \mathcal{S}^{n-l}_{\underline{\mu}} = K\{P_n^l \overline{W} \underset{n-l}{\otimes} t w^{ba}_{\omega} \mid t \in T^{\mathbf{m}}_{n-l};\ \omega \in b_{\underline{\mu}}\} \tag{25}$$

$$= K\{d \overline{W} \underset{n-l}{\otimes} t w^{ba}_{\omega} \mid d \in \mathtt{P}^l_n;\ t \in T^{\mathbf{m}}_{n-l};\ \omega \in b_{\underline{\mu}}\}$$

Let us 'move' t through the tensor product in (25):

$$= K\{d\overline{W}((t + I^{<\mathbf{m}}_{n-l}) \otimes 1_l) \underset{n-l}{\otimes} a^{\mathbf{m}}_{n-l} w^{ba}_{\omega} \mid d \in \mathsf{P}^l_n;\ t \in \mathsf{T}^{\mathbf{m}}_{n-l};\ \omega \in b_{\underline{\mu}}\}$$

$$= K\{d(t \otimes ww^*) \underset{n-l}{\otimes} a^{\mathbf{m}}_{n-l} w^{ba}_{\omega} \mid d \in \mathsf{P}^l_n;\ t \in \mathsf{T}^{\mathbf{m}}_{n-l}; \omega \in b_{\underline{\mu}}\} \quad (26)$$

Note the recasting of t. We can omit the $+I^{<\mathbf{m}}_{n-l}$ since it does not affect the element. We aim to show that this is spanned by terms of form $t \underset{n-l}{\otimes} a^{\mathbf{m}}_{n-l} w^{ba}_{\omega}$ where $t \in \mathsf{T}^{\mathbf{m}}_n$. Note that we may assume the module is generated by elements of the given form. We proceed as follows. Consider the action of the generators from 4.9 on an element of the claimed spanning set. Let d be such a generator and consider first the 'factor' dt in $dt \underset{n-l}{\otimes} a^{\mathbf{m}}_{n-l} w^{ba}_{\omega}$. Noting Lemma 3.10(II) there are two ways in which dt might pass out of the relative noncrossing transversal: either (A) it has a component with lower propagating index, i.e. a component in $I^{<\mathbf{m}}$; or (B) a relative crossing is introduced.

In Case (A): by definition such a component of dt is spanned by elements of the form $s a^{\mathbf{m}'}_n q$ with $\mathbf{m}' < \mathbf{m}$. Indeed since $\mathbf{m} \in \gamma^l_{n-l}$ we have $t \propto t\overline{W}$ and so dt is spanned by elements of the form $s a^{\mathbf{m}'}_n q(1_{n-l} \otimes ww^\star)$. Since $a^{\mathbf{m}'}_n = a^{\mathbf{m}'}_{n-l} \otimes ww^\star$ $= (a^{\mathbf{m}'}_{n-l} \otimes 1_l)(1_{n-l} \otimes ww^\star)$ this becomes $s(a^{\mathbf{m}'}_{n-l} \otimes 1_l)(1_{n-l} \otimes ww^\star) q (1_{n-l} \otimes ww^\star)$. Noting that

$$(1_{n-l} \otimes ww^\star) q (1_{n-l} \otimes ww^\star) \propto\ q|_{[1,n-l]} \otimes ww^\star,$$

we see that $dt \propto\ \ s((a^{\mathbf{m}'}_{n-l} q|_{[1,n-l]}) \otimes ww^\star)$. But

$$((a^{\mathbf{m}'}_{n-l} q|_{[1,n-l]}) \otimes ww^\star) \underset{n-l}{\otimes} a^{\mathbf{m}}_{n-l} w^{ba}_{\omega} \propto\ \ (a^{\mathbf{m}'}_{n-l} \otimes ww^\star) \underset{n-l}{\otimes} (a^{\mathbf{m}'}_{n-l} q|_{[1,n-l]}) a^{\mathbf{m}}_{n-l} w^{ba}_{\omega}\ \ = 0$$

so $dt \underset{n-l}{\otimes} a^{\mathbf{m}}_{n-l} w^{ba}_{\omega} = 0$.

In Case (B): by the $kS_{\mathbf{m}}$-freeness property (Lemma 6.4) such a crossing may be factored out and passed through the tensor product.

Thus neither case takes us out of the span, and we are done. □

We conclude this section with a remark on our working assumptions.

(**8.4**) Consider bases as in the proof of Lemma 7.19, noting that $G_W G_{a^{\mathbf{m}}_{n-l}} \cong G_{a^{\mathbf{m}}_n}$. That is, $P^l_n W^l \otimes_{P^l_{n-l}} P^{\mathbf{m}}_{n-l} a^{\mathbf{m}}_{n-l} \cong P^{\mathbf{m}}_n a^{\mathbf{m}}_n$, since $a^{\mathbf{m}}_{n-l} \otimes ww^\star = a^{\mathbf{m}}_n$ by (4). At this point the case $\mathbf{m} = 0$, $\delta = 0$ has extra interest.

Note that we cannot apply $P^l_n W^l_b \otimes_{P^l_{n-l}} -$ to $\mathcal{S}^0_{\underline{\mu}}$ since W^l_b requires $n > l$. In this case we could attempt to use W^l instead. This works straightforwardly if δ is a unit. But if $\delta = 0$ then the setup is slightly but interestingly different. Of course our mechanism for making $P^{\mathbf{m}}_l a^{\mathbf{m}}_l$ a right P^l_0 module does not work. And W^l is not normalisable as an idempotent (although the functor given by allowing P^l_{n-l} to act on the right of $P^l_n W^l_b$ by restriction is still well defined). It is an interesting exercise to see what happens if we simply allow $P^l_0 \cong K$ to act as K. In that case we are comparing $G_W G_{a^{\mathbf{m}}_{n-l}}$ with the direct long functor $G_{a^{\mathbf{m}}_n}$. We will leave this for a separate work.

Figure 12: Example partial gram matrix calculation. The table shows part of the case $n = 5$, $(\lambda^1, \lambda^2) = ((2,1), \emptyset)$. We use the tableaux basis for $(2,1)$: $\{112, 121\}$ [24].

9 Properties of standard modules

9.1 Standard module contravariant form

(**9.1**) Lemma. *There is a contravariant form $(-,-)_e$ on each $\mathcal{S}^n_{\underline{\mu}}$ defined by*

$$\Big(xa^{\mathbf{m}}e_{\underline{\mu}}\Big)^{op}\; ya^{\mathbf{m}}e_{\underline{\mu}} \;=\; \Big(e_{\underline{\mu}}\Big)^{op}\, a^{\mathbf{m}}x^{op}y(a^{\mathbf{m}}e_{\underline{\mu}}) \tag{27}$$

$$=\; \Big(e_{\underline{\mu}}\Big)^{op}\,(a^{\mathbf{m}}x^{op}ya^{\mathbf{m}})(e_{\underline{\mu}}) \;=\; (x,y)_e\, a^{\mathbf{m}}e_{\underline{\mu}}$$

Proof. (For (9.1) and (9.2) we have a direct generalisation of the usual partition algebra argument as in, for example, [38].) Note that we are working in $P^{\mathbf{m}}_n$. The well-definedness of the form follows from the construction as in (5.15), using Lemma 5.7 and primitivity at the last step. □

Examples: See Fig.12 and 13.

(**9.2**) Lemma. *For $k = \mathbb{Z}[\delta]$ the determinant of the gram matrix of the form $(,)_e$ on $\mathcal{S}^n_{\underline{\mu}}$ $(\underline{\mu} \in \Lambda(P^l_n))$ is nonzero.*

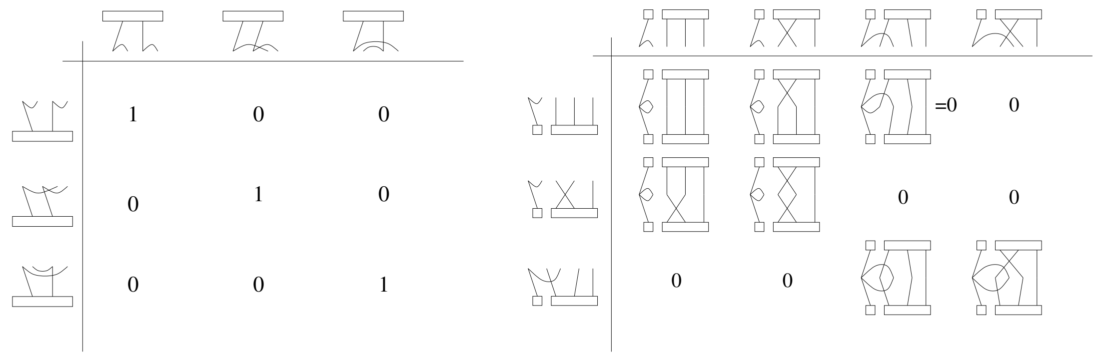


Figure 13: (a) Here we have $\lambda^1 = \emptyset$ and $\lambda^2 = (2)$ or (1^2) in the box (the picture is effectively the same in each case). (b) $\mathcal{S}^5_{((2,1),(1))}$ is 20-dimensional. The top left-hand corner of the gram matrix is shown. Here we have $\lambda^1 = (2,1)$ in the long box, with a choice of idempotent from kS_3 such that e and $\sigma_1 e$ span $kS_3 e$.

Proof. First organise the basis into blocks according to Lemma 7.16 — i.e. each block has a fixed non-crossing partition p, with only the permutation group module basis part w varying. See Fig.12. Note: (QI) If we work for the moment over $\mathbb{Q}[\delta]$ (as it will be clear that we can in investigating the nonzero property) then $kS_{\mathbf{m}}$ is (split) semisimple [24] and we may use a basis for the permutation group part in which the gram matrix of this part is diagonal. We could, for example, use tableau bases [24], as in (5.14), as illustrated in the figure. The numerical *details* of this part of the construction will not be needed here.

For any given choice of ordered basis we arrive at the gram matrix, denoted $G^n_{\underline{\mu}}$. In our $((2,1),\emptyset)$ example we have

$$G^n_{((2,1),\emptyset)} = \left(\begin{array}{c|c|c|c|c|c} \delta G & G & G' & G'' & G & G \;\; ... \\ \hline G & \delta G & G & ... & G & G \;\; ... \\ \hline G' & ... & ... & ... & 0 & 0 \;\; ... \\ \hline ... & ... & ... & & & \end{array} \right)$$

where the entries shown are the block submatrices.

In every row of the gram matrix proper, every entry is a polynomial in δ, indeed an integer multiple of a power of δ. The diagonal entry of the basis element pw (say — cf. Lem.7.16 and its proof), determined by $(w^{op}p^{op})pw = w^{op}(p^{op}p)w$, is a nonzero polynomial whose degree is not exceeded by any other entry. The degree is the same through a given block; and there is at least one row where the block of the diagonal has strictly the highest of all degrees in the row. NB, In case $\underline{\mu} \in \mathbf{m} \in h^l_n$ all entries are constant, but all the off-diagonal blocks are zero by the $I^{<\mathbf{m}}$ quotient. Finally, by (QI) the blocks on the block-diagonal are diagonal. Combining these we see that the gram matrix has full rank for indeterminate δ. Thus the determinant is a nonzero polynomial in δ. □

(**9.3**) PROPOSITION. (I) The standard modules $\mathcal{S}_{\underline{\mu}}$ with $\underline{\mu} \in \sqcup_{\mathbf{m} \in h^l_n} \Lambda_{\mathbf{m}}$ are A^l_n-modules. (II) Over $k = \mathbb{C}$ every such $\mathcal{S}_{\underline{\mu}}$ is simple; and A^l_n is semisimple.

Proof. (I) $W\mathcal{S}_{\underline{\mu}} = 0$ iff $|\underline{\mu}| \in h^l_n$. (II) The contravariant form is non-degenerate here. Indeed, by the compatibility condition (7.10), (x, y) is only non-zero if 'row-parts' match, whereupon the gram matrix consists of blocks corresponding to matches. Within these blocks the entries are the same as for the gram matrices for the (product of) symmetric groups — which gram matrices are of full rank over $\mathbb{C}$ [24]. Finally, by (7.14) we have completeness and pairwise nonisomorphism. □

9.2 Standard module decomposition matrix properties

(**9.4**) THEOREM. *Suppose $k = \mathbb{C}$ and $\delta \neq 0$. Recall the simple P^l_n-modules $L_{\underline{\mu}} =$ head $\mathcal{S}^n_{\underline{\mu}}$ from (7.14). The modules $\{\mathcal{S}^n_{\underline{\mu}} \ : \ \underline{\mu} \in \Lambda(P^l_n)\}$ have an upper-unitriangular decomposition matrix $([\mathcal{S}^n_{\underline{\mu}} : L_{\underline{\nu}}])_{\underline{\mu},\underline{\nu} \in \Lambda(P^l_n)}$ with respect any order $(\Lambda(P^l_n), \preceq)$ in which $\underline{\mu} \prec \underline{\nu}$ if $r_{|\underline{\mu}|} < r_{|\underline{\nu}|}$.*

Proof. This follows from Proposition 8.3, Proposition 9.3 and the construction using Lem.4.12 and Th.5.9. □

(**9.5**) Example: If $r_{|\mu|} = n$ then $\mathcal{S}^n_\mu = L_\mu$ (since there is no $\nu \in \Lambda(P^l_n)$ with $r_{|\nu|} > n$). Meanwhile $(\mathcal{S}_0 : L_0) = 1$ and no other composition factor is precluded for $\mathcal{S}_0$ by this Theorem (and indeed none can be without specifying δ).

(**9.6**) THEOREM. *Consider $k = \mathbb{C}$ and $\delta \in \mathbb{C}$. Each module $\mathcal{S}_{\underline{\mu}}$ is simple for all but finitely many values of δ.*

Proof. By (4.12) it is enough to show that there is a nondegenerate contravariant form on each module (if one of these modules is isomorphic to its contravariant dual then it contains the dual of the head L in the socle; but if these are not the same module then they are not isomorphic, by the unitriangular property Th.9.4; and by completeness there is another such module with head L^o and socle L, contradicting upper-triangularity). Now note Lem.9.2. □

(**9.7**) THEOREM. *Consider $k = \mathbb{C}$ and $\delta \in \mathbb{C}^*$. For all but finitely many values of δ: (I) The set $\{\mathcal{S}_{\underline{\mu}} \mid \underline{\mu} \in \Lambda(P^l_n)\}$ is a complete set of simple modules of P^l_n; (II) P^l_n is semisimple.*

Proof. (I) These modules are (sufficiently often) simple by Th.9.6. By 7.14 it is a complete set. By the embedding property using the functors F, G there are no duplicates (pairwise isomorphisms) in the set.

(Alternatively we may argue using the dimension count (23) and either the completeness or the pairwise nonisomorphism.)
(II) follows from (I) and (23). □

(**9.8**) By Theorem 9.7 and the construction we have, over $\mathbb{C}$, a modular system for each δ (see e.g. [3, §1.9]). Specifically we may take $K = \mathbb{C}[x]$ for the integral ground ring; the field of fractions K_0 as the ordinary case; and $\mathbb{C}$ with x evaluated at δ as the modular case. Thus we have Brauer–Humphreys reciprocity:

$$(P_\lambda : \mathcal{S}_\mu) = [\mathcal{S}_\mu : L_\lambda] \tag{28}$$

where $(P_\lambda : \mathcal{S}_\mu)$ denotes the 'composition multiplicity' of $\mathcal{S}_\mu$ in P_λ (as usual this makes strict sense over the rational field via an idempotent lift, and as a multiplicity in the Grothendieck group in general).

In particular if L_λ is a composition factor of $\mathcal{S}_\mu$ then $\mathcal{S}_\mu$ is a filtration factor of P_λ.

10 On quasi-heredity

Here we prove, in Theorem 10.5, that the P^l_n module categories are highest weight categories (in the sense of Cline, Parshall and Scott [8]) when $\delta \neq 0$ and $k = \mathbb{C}$. Given Theorem 4.26, Theorem 9.4 and so on, it is enough to show that projective modules are filtered by Δ-modules. We do this next. Recall the following.

General Lemmas

(**10.1**) Lemma. [41] (I) Let A be an algebra, M an A-module and S, T sets of A-modules. If M has an S-filtration and every $N \in S$ has a T-filtration then M has a T-filtration.

(**10.2**) Lemma. *Let A be an algebra, f an idempotent, and M a bimodule. Then there are left-module maps $Mf \to M$ given by inclusion; and $M \to Mf$ given by $m \mapsto mf$. Indeed the sequence*

$$0 \to Mf \to M \to M(1-f) \to 0$$

is short-exact and split.

(**10.3**) Lemma. *Let A be a finite-dimenional algebra and suppose $0 \subset J_1 \subset J_2 \subset ... \subset J_k = A$ is a filtration by ideals. Let X be the set of indecomposable summands of all the left-modules J_i/J_{i-1} up to isomorphism. Then every projective left-module of A is filtered by X.*

Proof. It is enough to show for indecomposable projectives, and hence for modules of form Af where f is idempotent. For such a module we have a (possibly degenerate) filtration $0 \subseteq J_1 f \subseteq J_2 f \subseteq ... \subseteq J_k f = Af$ by assumption. Suppose $M \supset N$ are A-bimodules. We claim that there is a left-A-module map $(M/N)f \to Mf/Nf$ given by $\{mf + n : n \in N\} \mapsto \{mf + nf : n \in N\}$ (i.e. act on the set elementwise by f on the right), and that this is a left-A-module isomorphism. To see this note that (i) the image lies in Mf/Nf; (ii) this gives a vector space isomorphism; (iii) it is a left-A-module morphism (the map uses the action on the right which, by the bimodule property commutes with the action on the left).

Thus in particular $(J_i/J_{i-1})f \cong J_i f/J_{i-1} f$, so there is a sectioning of Af with sections isomorphic to modules $(J_i/J_{i-1})f$. By Lemma 10.2 $(J_i/J_{i-1})f$ is a sum of (some) direct summands of an indecomposable direct summand decomposition of J_i/J_{i-1}. And by the Krull–Schmidt Theorem for modules, and our working assumptions, every such decomposition is a sum from X. The Lemma now follows routinely using (10.1), since a direct sum has a filtration by its summands. □

10.1 Quasiheredity/HWC for the tonal algebras

(**10.4**) Lemma. *For $\delta \neq 0$, the indecomposable projective P_n^l-modules are filtered by the set $\Delta^{l,n}$ of Δ-modules (as in (7.13)).*

Proof. Consider the ideal chain from (7.1). The sections are as in Lemma 7.2. By Equations (21) and (22) these are sums of certain modules, and by (7.19) these modules are Δ-modules. Now use Lemma 10.3. □

(**10.5**) Theorem. *If $k = \mathbb{C}$ and $\delta \neq 0$ then $P_n^l - \mathrm{mod}$ is a highest weight category with respect to any order $\prec$ on $\Lambda(P_n^l)$ in which $\mathbf{m} < \mathbf{m}'$ implies $\underline{\nu} \prec \underline{\mu}$ for $\underline{\mu} \in \Lambda_{\mathbf{m}}$ and $\underline{\nu} \in \Lambda_{\mathbf{m}'}$, with the set of standard modules $\{\Delta_{\underline{\mu}} \mid \underline{\mu} \in \Lambda(P_n^l)\}$.*

Proof. We will use Th. 4.26. We thus require to show (I) strong modular core; (II) projective filtration.
(I) Consider K to be the localisation of $\mathbb{C}[\delta]$ at δ, i.e. the ring of Laurent polynomials. Note by Theorem 4.6 that the pair $(\gamma^{l,n}, a^-)$ is a core of P_n^l over K. We take K_0 to be the extension $\mathbb{C}(\delta)$. Then the system is strong modular by Theorem 9.7. Note that our localisation does not prohibit base change to k as required (but does prohibit $\delta = 0$).
(II) By Equation 20 the set of modules $\{\Delta_{\underline{\mu}} \mid \underline{\mu} \in \Lambda_0(P_n^l)\}$ are the corresponding $\sqsupset^{\gamma}$-modules. By Lemma 10.4 the indecomposable projectives of P_n^l are filtered by the Δ-modules. □

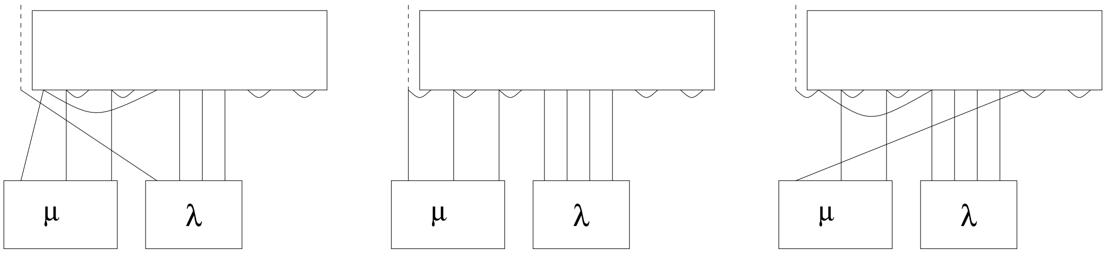


Figure 14: Restricted action on basis elements: cases (1-3). The undecorated box represents the action of the given inclusion $P^2_{n-1} \hookrightarrow P^2_n$ on the element.

11 Standard branching rules for $P^l_n \hookrightarrow P^l_{n+1}$

Consider the algebra inclusion $P^l_n \hookrightarrow P^l_{n+1}$ given by $d \mapsto 1_1 \otimes d$ (or equivalently $d \mapsto d \otimes 1_1$). We give here the 'standard-module branching rule' associated to the corresponding tower of algebras.

In the generic/semisimple case this is the simple branching rule — the edge rule for the Bratteli diagram, and is given by Kosuda for example in [33]. In the semisimple case the restriction of a simple modules is a direct sum of simple modules. In our case we cannot expect this. We will need to make some preparations.

Here we write $M = M_1 + M_2$, or say $M = \mathop{+\!\!\!\!+}_i m_i M_i$, if module M has a filtration by a set $\{M_i\}$, with the indicated multiplicities. This notation does *not* give the filtration series order. (In general there may be a filtration with different multiplicities, but not, say, if the set is a basis for the Grothendieck group.)

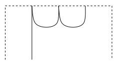
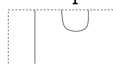
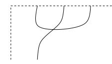
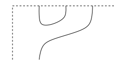
We may write $S_{\mathbf{m}-\epsilon_j}$ for the subgroup of $S_{\mathbf{m}}$ in which factor S_{m_j} is replaced by S_{m_j-1}.

We first deal with the case $l = 2$, i.e. $E_n = P^2_n$, then, more briefly, with the general case.

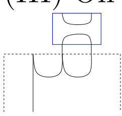
11.1 The case $P^2_n = E_n \hookrightarrow E_{n+1}$

(**11.1**) Set $\underline{\mu} = (\lambda, \mu)$. Consider the set $B^{\underline{\mu}}_{\mathtt{Sp}}$ of basis diagrams of $\mathcal{S}_{(\lambda,\mu)}$. (Because of the e_λ and e_μ in the construction these 'diagrams' are linear combinations of partitions in general, but the well-definedness of the following manipulations will be clear.) We may organise the set $B^{\underline{\mu}}_{\mathtt{Sp}}$ into four subsets $B^{\underline{\mu}}_{\mathtt{Sp}}(i)$, $i = 1, 2, 3, 4$, containing diagrams in which the first vertex is:
(1) a 'propagating singleton' (Fig.14 (1));
(2) part of a co-2 part (Fig.14 (2));

(3) part of a propagating part of higher odd half-order (Fig.14 (3));
(4) part of a non-propagating part (necessarily of even order).

For example, for $\mathcal{S}^3_{((1),0)}$ we have $B_{\mathtt{Sp}}(2) = \emptyset$, $B_{\mathtt{Sp}}(3) = \{$  $\}$; $B_{\mathtt{Sp}}(1) = \{$  $\}$; and $B_{\mathtt{Sp}}(4) = \{$  ,  $\}$.

(**11.2**) Consider the action of $1_1 \otimes E_n$ on $\mathcal{S}^{n+1}_{\underline{\mu}}$.
(I) Since the action of $1_1 \otimes E_n$ is 'trivial' on the first vertex, and so cannot change the propagating singleton property, it closes on the $B(1)$ part of the basis for $\mathcal{S}^{n+1}_{\underline{\mu}}$.
(II) The $B(2)$ part is also closed — the co-2 property at the first vertex can only be changed in principle by combining with a co-1 part, but this gives 0 by the $I^{<\mathbf{m}}$ quotient.
(III) On the other hand $B(3)$ is not closed in general, as illustrated by the example:

 which takes $q \in B(3)$ into $B(1)$.

(IV) Notice however that the subspace spanned by

$$A_{\mathtt{Sp}} \;:=\; \sqcup_{i=1,2,3} B_{\mathtt{Sp}}(i) \tag{29}$$

is a $1_1 \otimes E_n$-submodule.

(**11.3**) Theorem. *Let* $\mathtt{res}_n : E_{n+1}-mod \to E_n-mod$ *denote the natural restriction corresponding to the inclusion* $E_n \hookrightarrow E_{n+1}$ *given by* $d \mapsto 1_1 \otimes d$. *Then* $\mathtt{res}_n \mathcal{S}_{(\lambda,\mu)}$ *has a filtration by standard/Specht modules (as defined in 7.15). The multiplicities are given as follows. Firstly we have a short exact sequence of* E_n*-modules*

$$0 \to kA_{\mathtt{Sp}} \to \mathtt{res}_n \mathcal{S}^{n+1}_{(\lambda,\mu)} \to kB_{\mathtt{Sp}}(4) \to 0 \tag{30}$$

where $A_{\mathtt{Sp}}$ *is the subset of* $\mathcal{S}^{n+1}_{(\lambda,\mu)}$ *defined in (29). Then*

$$kA_{\mathtt{Sp}} \;=\; +_i\, \mathcal{S}_{(\lambda-e_i,\mu)} \;+\; +_{i,j} \mathcal{S}_{(\lambda-e_j,\mu+e_i)} \;+\; +_{i,j} \mathcal{S}_{(\lambda+e_j,\mu-e_i)}$$

and

$$kB_{\mathtt{Sp}}(4) = +_i\, \mathcal{S}_{(\lambda+e_i,\mu)}$$

where sums over $\lambda - e_i$*, say, denote sums over all ways of removing a box from the Young diagram of* λ.

Proof. Firstly (30) follows from (11.2). For the filtration factors, we consider the action of the subalgebra E_n on each of the $B^{\mu}_{\mathtt{Sp}}(i)$ in turn.

Case (1): Here we see from Fig.14(1) that the propagating singleton can essentially be ignored. Each basis element pw (as in (7.16)) is then like a basis element of $+_i \mathcal{S}_{(\lambda-e_i,\mu)}$. In particular the 'inflation' factor associated to the λ label is the restriction of the Specht module of $S_{|\lambda|}$ to $S_{|\lambda|-1}$. One then uses the usual symmetric group restriction rule (valid even integrally [50]):

$$\mathtt{res}^{S_{|\lambda|}}_{S_{|\lambda|-1}} \mathtt{Sp}_\lambda \;=\; +_i \, \mathtt{Sp}_{\lambda - e_i}$$

At the level of bases recall from (5.14) that b_λ is the set of sequence that are perms of 1112233... (say) satisfying the tableau condition. Such a perm still satisfies the condition on removing the last 'letter' i say, leaving a basis element in $b_{\lambda-e_i}$.

We thus have

$$+_i \, \mathcal{S}^n_{(\lambda-e_i,\mu)} \hookrightarrow \mathtt{res}_n \mathcal{S}^{n+1}_{(\lambda,\mu)}$$

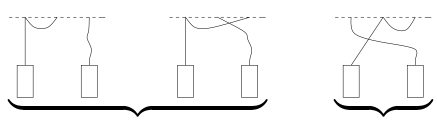
as an injection of P^2_n-modules; and a bijection

$$\bigsqcup_i B_{\mathtt{Sp}}^{(\lambda-e_i,\mu)} \xrightarrow{\sim} B_{\mathtt{Sp}}^{(\lambda,\mu)}(1) \tag{31}$$

The map, on an element pw (as in Lem.6.4), is to add a string starting on the left, then passing over to the box labelled $\lambda - e_i$ and hence to the part of w that is a tableaux sequence for $\lambda - e_i$, then add i to this sequence.

Case (2): Here we can see from the figure that (as far as the restricted action is concerned) the number of propagating even-half-order components effectively goes down by 1 and the number of propagating odd-half-order components goes up by 1. Broadly analogously to the previous case we then have a symmetric group factor with an induction (rather than restriction) on λ, and a restriction on μ.

It will be convenient to have a small complete example to refer to:



$B_{\mathtt{Sp}}^{(1,1)}(2)$ $\quad$ $B_{\mathtt{Sp}}^{(1,1)}(1)$ $\qquad B_{\mathtt{Sp}}(3) = \emptyset, \qquad B_{\mathtt{Sp}}(4) = \emptyset$

We consider this $\mathcal{S}^{n+1}_{(1,1)}$ in case $n = 2$ as a P^l_n module (N.B. $l = 2$). In this case if we quotient by the $B_{\mathtt{Sp}}(1)$ submodule, which is isomorphic to $\mathcal{S}_{(0,1)}$, then the quotient module is already the sum of Specht modules $\mathcal{S}_{((2),0)} + \mathcal{S}_{((1^2),0)}$. (But we can consider how this is realised. We see from (11.2)(II) that we cannot leave the $B_{\mathtt{Sp}}(2)$ part.)

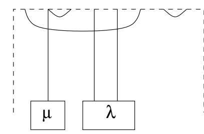
Case (3): Here the number of odd lines goes down and the number of even lines goes up. In the language of the previous cases we have an induction on one factor and a restriction on another.
(However, this time the construction, (11.2)(II), does not preclude a non-split extension. And indeed the extension is non-split in general. To see this consider the example in (11.6) below.)

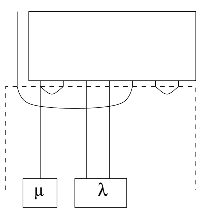
Case (4): Here the number of even propagating lines stays the same and the number of odd lines goes up by 1. Thus we may put the basis elements in correspondence with an induction in the odd position. This gives

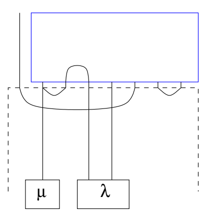
$$\bigsqcup_i B_{\mathtt{Sp}}^{(\lambda+e_i,\mu)} \xrightarrow{\sim} B_{\mathtt{Sp}}^{(\lambda,\mu)}(4) \tag{32}$$

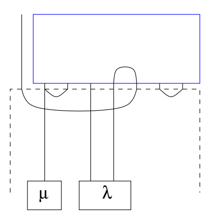
Now compare with the final summand in the identity in the Theorem.

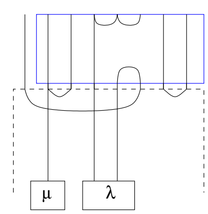
In this case note that the subset does not span a submodule. The quotient in the definition of the module with respect to the γ order has a slightly different effect here (just as in the corresponding classical P_n problem [37]). It will be convenient to articulate the argument using an example.

An example of a basis element of $\mathcal{S}^{n+1}_{\underline{\mu}}$ in case 4 is: $q =$ 

. We consider the action of the $1_1 \otimes P^l_n$ subalgebra as indicated by the following schematic:


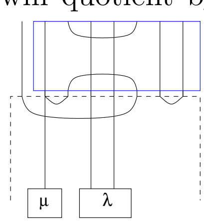
. For example we have, for any completion of the acting partition as indicated here 

$\equiv 0$ by the $I^{<\mathbf{m}}$ quotient. On the other hand the following has completions for which 

$\not\equiv 0$, such as 

.

This is not 0, but lies in $B^{\mu}_{\mathtt{Sp}}(1)$. But if q were a basis element of $\mathcal{S}^n_{(\lambda+e_i,\mu)}$ under the isomorphism (32) then such an element of P^l_n would act as 0. In order to get our filtration here, therefore, we will quotient by a submodule containing $B^{\mu}_{\mathtt{Sp}}(1)$.

Finally consider actions like this: 

. Again this is non-zero, lying in $B^{\mu}_{\mathtt{Sp}}(2)$, but should be 0 in this factor of the restriction. To see that this requirement is satisfied note the position of $B(4)$ in (30). □

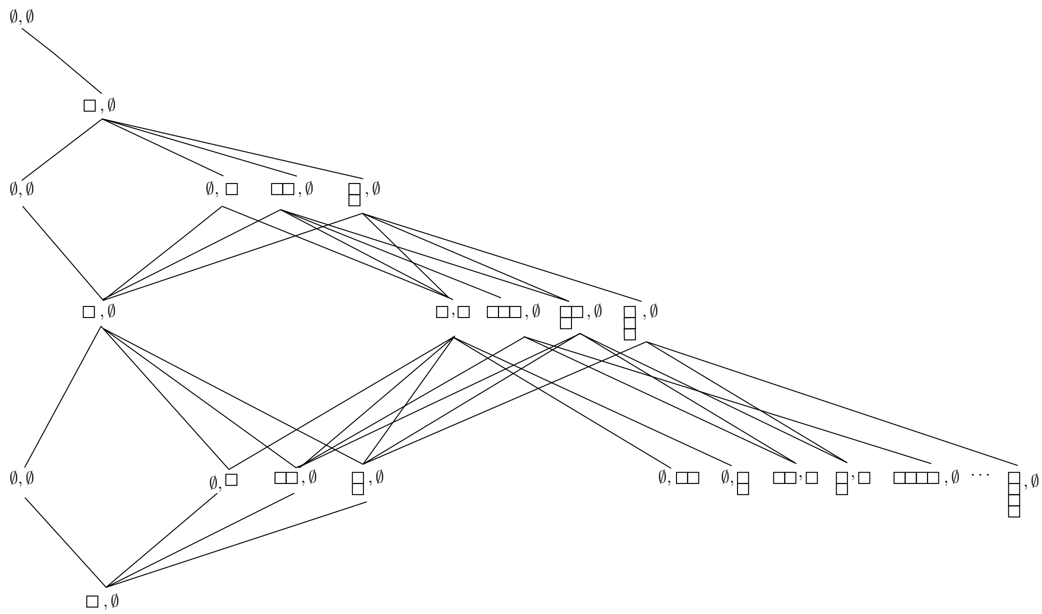


Figure 15: Standard E_n-module restriction diagram (complete up to $n = 3$, partial up to $n = 5$).

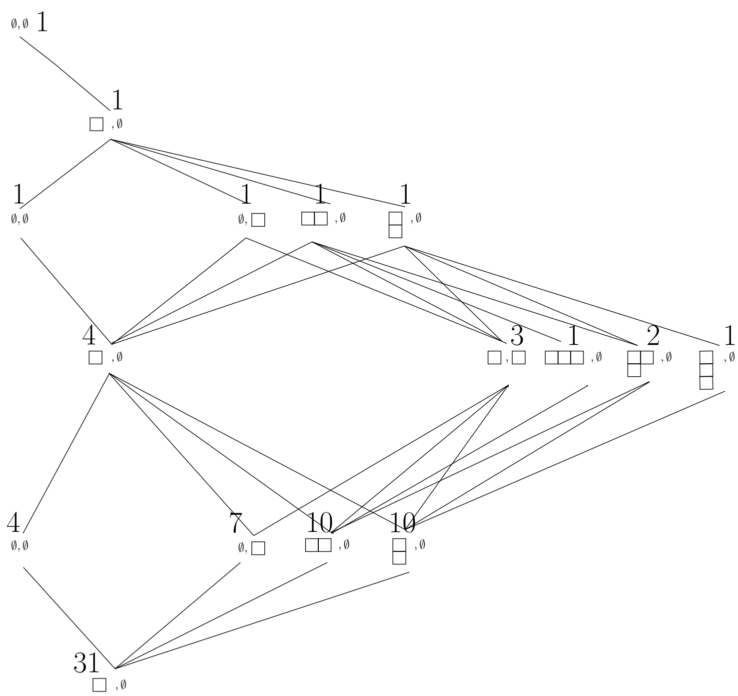
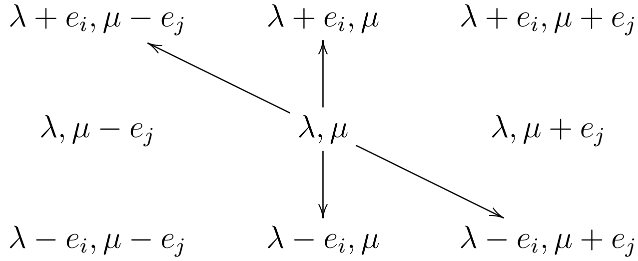


Figure 16: Standard module restriction diagram with dimensions.

(**11.4**) Here is a schematic illustrating restriction in general position:



(**11.5**) It follows from (11.3) that the Bratteli diagram takes the form in Fig.15. It also follows that the 'even Bell numbers' (cf. e.g. [1]) have an intriguing expression as a sum of squares — a generalised Robinson–Schensted correspondence [30, 44]. See Fig.16.

11.2 Examples: applications of modularity

(**11.6**) Example. By Theorem 11.3 we have the following short exact sequences of P_3^2-modules

$$0 \to \mathcal{S}^3_{((1),0)} + \mathcal{S}^3_{((1),(1))} \to \mathtt{res}_3(\mathcal{S}^4_{((2),0)}) \to \mathcal{S}^3_{((3),0)} \oplus \mathcal{S}^3_{((2,1),0)} \to 0 \tag{33}$$

$$0 \to \mathcal{S}^3_{((1),(0))} \to \mathtt{res}_3(\mathcal{S}^4_{((0),(1))}) \to \mathcal{S}^3_{((1),(1))} \to 0. \tag{34}$$

Let us consider the case $k = \mathbb{C}$ and $\delta = 1$. One can see from Fig.16 that P_2^2 is semisimple in this case, so that $\mathcal{S}^2_{((2),0)}$ is simple-projective. It follows that $\mathtt{res}_3(\mathcal{S}^4_{((2),0)})$ is projective. To see this note the following.

(**11.7**) Lemma. *In case $l = 2$ we have the identification of functors* $\mathtt{ind}- \equiv \mathtt{res}\, G_W -$.

Proof. We may essentially use the usual P_n argument as for example in [37] or [38]. The extra requirement is to check that the $l = 2$ constraint is preserved by these manipulations. For this the key point is the alternative characterisation of P^2 as the subset of partitions of even order (as in (2.3)). It follows that $P^2_{n+1}W \cong P^2(n+2, n)$ as k-space. Furthermore the even property is invariant under the disk isomorphism [39, Appendix] $P(n, m) \to P(n-1, m+1)$, so $P^2(n, m) \to P^2(n-1, m+1)$. It follows that

$$_{P_{n+1}}P^2_{n+2}W_{P_n} \cong \ _{P_{n+1}}P^2_{n+1\,P_n}$$

The result now follows by unpacking the various definitions. □

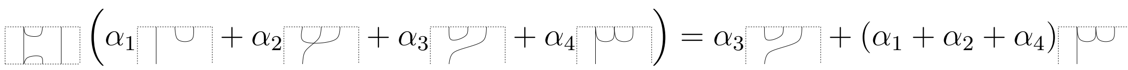
(**11.8**) Since $G_W \mathcal{S}^n_{\underline{\mu}} = \mathcal{S}^{n+2}_{\underline{\mu}}$, by (8.3), and induction preserves projectivity we see that $\mathtt{res}\, \mathcal{S}^{n+2}_{\underline{\mu}} = \mathtt{res}\, G_W \mathcal{S}^n_{\underline{\mu}} = \mathtt{ind}\, \mathcal{S}^n_{\underline{\mu}}$ is projective when $\mathcal{S}^n_{\underline{\mu}}$ is projective.

Now by (11.3) and (7.14) we see that $\texttt{res}\,\mathcal{S}_{\lambda,\mu}$, when projective, contains at least each $P_{\lambda+e_i,\mu}$ (the indecomposable projective cover of $L_{\lambda+e_i,\mu}$). So in our case (33) the restriction contains $P_{((3),0)}$ and $P_{((2,1),0)}$.

(**11.9**) Continuing with (11.6), elementary linear algebra shows that when $\delta = 1$

$$\mathcal{S}_{((1),0)} = L_{((1),0)} + L_{((1),(1))}. \tag{35}$$

The argument is as follows. Firstly, inspection of the gram matrix shows that the socle has dimension 3. Secondly the ideal generated by $a^{((1),(1))}$ acts as 0 on $\mathcal{S}_{\lambda,0}$ for $\lambda \vdash 3$, but the subspace of $\mathcal{S}_{((1),0)}$ on which the ideal generated by $a^{((1),(1))}$ acts as 0 is easily seen to be empty:



so the subspace has $\alpha_3 = 0$ (and $\alpha_1 = \alpha_2 = 0$ by symmetry, and hence $\alpha_4 = 0$).

Meanwhile, we see from (9.3) that all the $\mathcal{S}_{\lambda,\mu}$ except $\mathcal{S}_{((1),0)}$ are simple. Thus in fact $\mathcal{S}_{((3),0)}$ and $\mathcal{S}_{((21),0)}$ are simple-projective and the big sequence in (33) does split. But then by (28) and (35) the restriction includes

$$P_{((1),(1))} = \mathcal{S}_{((1),0)} + \mathcal{S}_{((1),(1))}, \tag{36}$$

so $\mathcal{S}_{((1),0)} + \mathcal{S}_{((1),(1))}$ in (33) does not split.

By (11.8) we have that $\texttt{res}_3(\mathcal{S}^4_{((0),(1))})$ is also projective. Indeed it contains $P_{((1),(1))}$. So by (36) $\texttt{res}_3(\mathcal{S}^4_{((0),(1))}) = P_{((1),(1))}$ and (34) is non-split.

11.3 General l

Let $\underline{\lambda} = (\lambda^1, \lambda^2, \cdots, \lambda^l) \vdash \mathbf{m} \in \gamma^{l,n+1}$ be a multi-partition. For $1 \leq j \leq l$ we define $add_j(\underline{\lambda})$ to be the set of all multi-partitions obtained from $\underline{\lambda}$ by adding an addable box to the Young diagram of λ^j, and $rem_j(\underline{\lambda})$ to be the set of all multi-partitions obtained from $\underline{\lambda}$ by removing a removable box from the Young diagram of λ^j.

(**11.10**) Theorem. *Let* $\texttt{res}_n : P^l_{n+1} - mod \to P^l_n - mod$ *denote the natural restriction corresponding to the inclusion* $P^l_n \hookrightarrow P^l_{n+1}$ *given by* $d \mapsto 1 \otimes d$. *For* $\underline{\lambda} = (\lambda^1, \lambda^2, \cdots, \lambda^l) \vdash \mathbf{m} \in \gamma^{l,n+1}$ *the module* $\texttt{res}_n(\mathcal{S}^{n+1}_{\underline{\lambda}})$ *has a filtration by standard/Specht modules. Specifically there is a short exact sequence of* P^l_n*-modules*

$$0 \to kA_{\texttt{Sp}} \to \texttt{res}_n(\mathcal{S}^{n+1}_{\underline{\lambda}}) \to +_{\underline{\mu} \in add_{l-1}(\underline{\lambda})} \mathcal{S}^n_{\underline{\mu}} \to 0 \tag{37}$$

Where $kA_{\texttt{Sp}} = +_{\substack{\underline{\mu} \in add_i(rem_{i+1}(\underline{\lambda})) \\ 1 \leq i \leq l-1}} \mathcal{S}^n_{\underline{\mu}} \; + \; +_{\underline{\mu} \in add_l(rem_1(\underline{\lambda}))} \mathcal{S}^n_{\underline{\mu}} \; + \; +_{\underline{\mu} \in rem_1(\underline{\lambda})} \mathcal{S}^n_{\underline{\mu}}.$

Proof. We organise the set $B^{\underline{\lambda}}_{\mathtt{sp}}$ of basis diagrams of $\mathcal{S}^{n+1}_{\underline{\lambda}}$ into $l+2$ subsets denoted $B^{\underline{\lambda}}_{\mathrm{sp}}(i)$, $i = 1, 2, \cdots, l, l', 0$, containing diagrams in which the first vertex is:

1. a "propagating singleton". This subset is denoted by $B^{\underline{\lambda}}_{\mathrm{sp}}(1)$.
2. part of a co-i propagating part, for $i = 2, \cdots, l$. This subset is denoted by $B^{\underline{\lambda}}_{\mathrm{sp}}(i)$.
3. part of a co-1 propagating part p with $|\mathrm{cora}(p)| > l$. This subset is denoted by $B^{\underline{\lambda}}_{\mathrm{sp}}(l')$.
4. part of a non-propagating part. This subset is denoted by $B^{\underline{\lambda}}_{\mathrm{sp}}(0)$.

We proceed organisationally as for Theorem 11.3.

Case 1. This is similar to Theorem 11.3. Here the factor associated to λ^1 is the restriction of the Specht module $S_{|\lambda^1|}$ to $S_{|\lambda^1|-1}$, and this gives the submodule $+_{\underline{\mu}\in rem_1(\underline{\lambda})}\mathcal{S}^n_{\underline{\mu}}$ in (37).

Case 2. Here $b \in B^{\underline{\lambda}}_{sp}(i)$ passes on restriction to one fewer co-i line and one more co-$(i-1)$ line (a mild generalisation of the $l = 2$ case). Hence we have restriction of the factor $S_{|\lambda^i|}$ and induction in the factor $S_{|\lambda^{i-1}|}$. This case gives the submodule $+_{\underline{\mu}\in add_{i-1}(rem_i(\underline{\lambda}))}\mathcal{S}^n_{\underline{\mu}}$ of $\mathtt{res}_n(\mathcal{S}^{n+1}_{\underline{\lambda}})$ in (37), for each $i = 2, \cdots, l$.

Case 3. Here we remove from co-1 and add to co-l. Hence we have induction in the factor $S_{|\lambda^l|}$ and restriction in the factor $S_{|\lambda^1|}$. This gives the submodule $+_{\underline{\mu}\in add_l(rem_1(\underline{\lambda}))}\mathcal{S}^n_{\underline{\mu}}$ of $\mathtt{res}_n(\mathcal{S}^{n+1}_{\underline{\lambda}})$ in (37).

Case 4. Here we add to co-$(l-1)$. Hence we have induction in the factor $S_{|\lambda^{l-1}|}$. By the argument we have in the proof of Theorem 11.3 Case 4 the factor $+_{\underline{\mu}\in add_{l-1}(\underline{\lambda})}\mathcal{S}^n_{\underline{\mu}}$ is not necessarily a submodule of $\mathtt{res}_n(\mathcal{S}^{n+1}_{\underline{\lambda}})$. This leads to position of $+_{\underline{\mu}\in add_{l-1}(\underline{\lambda})}\mathcal{S}^n_{\underline{\mu}}$ in (37). □

12 The fusion χ-functor

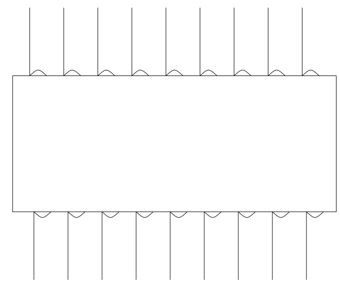
In this paper we have developed tonal analogues of most of the tools used classically to determine the representation theory of the ordinary partition algebra [38]. The remaining key ingredient to develop is a tonal analogue of the alcove geometrics. We address this problem elsewhere. Finally in this paper however, we determine a small but interesting part of the representation theory of the E_n cases, by a striking localisation functor.

Suppose n even and define

$$e_\pi = a^{\otimes n/2} = A^{12}A^{34}...A^{n-1\,n}$$

(12.1) PROPOSITION. $e_\pi E_n e_\pi \cong P_{n/2}$

Proof. Consider the picture:



It will be evident that inserting E_n into the box defines a map to $P_{n/2}$. It is straightforward to check that the map is surjective. □

It follows that $P_{n/2}$-mod fully embeds in E_n-mod, and hence that E_n is non-semisimple whenever $P_{n/2}$ is non-semisimple. The structure of P_n is given in [38]. Thus we have determined another (small but interesting) part of the structure of E_n. In particular we may deduce that non-negative integer δ values are non-semisimple for sufficiently large n. (Confer for example [43], [19].)

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Appendix

A Partition category composition

For S a set let $\mathtt{P}_S$ denote the set of set partitions of S. Let $\underline{n} := \{1, 2, .., n\}$ and $\underline{n'} := \{1', 2', .., n'\}$ and so on; and $P_{n,m} := \mathtt{P}_{\underline{n} \cup \underline{m'}}$.

Given a graph $g = (V, E)$ then $\pi(g) \in \mathtt{P}_V$ denotes the partition according to connected components of g. Given a graph $g = (V, E)$ with $V \supseteq \underline{n} \cup \underline{m'}$ then $\pi_{nm}(g) \in P_{n,m}$ denotes the partition according to connected components of g.

Given graphs $g = (V, E)$, $g' = (V', E')$ then graph

$$g.g' = (V \cup V', E \sqcup E')$$

Given a graph g with $V = \underline{n} \cup \underline{m'}$ and a graph g' with $V' = \underline{m} \cup \underline{l'}$ then

$$g|g' = g_+.g'_- \tag{38}$$

where graph $g_+ = (V_+, E_+)$ is g with vertices i' ($i \in \underline{m}$) replaced by i''; and graph g'_- is g' with vertices $i \in \underline{m}$ replaced by i'' (so $V_+ \cap V'_- = \underline{m''}$).

A connected component of $g|g'$ is *internal* if it intersects neither $\underline{n}$ or $\underline{l'}$. Define $c(g|g')$ as the number of internal components. Define

$$* : P_{n,m} \times P_{m,l} \to kP_{n,l}$$

as follows. For $p \in P_{n,m}$ pick any graph with $\pi(g) = p$; and similarly for $p' \in P_{m,l}$. Then

$$p * p' = \delta^{c(g|g')} \pi_{nl}(g|g')$$

(**A.1**) Theorem. [37] *The composition $*$ is well-defined and, extended k-linearly, makes $kP_{n,n}$ an associative unital algebra; and $(kP_{n,m})_{n,m \in \mathbb{N}_0}$ a k-linear category.* □

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