Beyond the projection postulate and back: Quantum theories with generalized state-update rules

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Are there consistent and physically reasonable alternatives to the projection postulate? Does it have unique properties compared with acceptable alternatives? We answer these questions by systematically investigating hypothetical state-update rules for quantum systems that Nature could have chosen over the Lüders rule. Among other basic properties, any prospective rule must define unique post-measurement states and not allow for superluminal signaling. Particular attention will be paid to consistently defining post-measurement states when performing local measurements in composite systems. Explicit examples of valid unconventional update rules are presented, each resulting in a distinct, well-defined foil of quantum theory. This framework of state-update rules allows us to identify operational properties that distinguish the projective update rule from all others and to put earlier derivations of the projection postulate into perspective.

I. INTRODUCTION

The main goal of our contribution is to identify operational properties that distinguish the quantum-mechanical projection postulate from hypothetical but physically reasonable state-update rules. Not surprisingly, attempts to justify Lüders rule (for which both Dirac and von Neumann deserve credit [1]) have a long history, and a recent debate revolves around the claim that it can be derived from the other postulates of quantum theory (cf. Sec. II A). Many of these approaches have in common that their focus is on noncomposite systems, either ignoring measurements on composite systems or making implicit assumptions about their consequences.

Our main tool will be the construction of foil theories of quantum theory (cf. Sec. II B), i.e., toy theories that help "to highlight the distinctive characteristics of quantum theory by contrasting with it" (adapted from Ref. [2]). Thus, we investigate the nature of the projection postulate by comparing quantum theory with similar theories that are equipped with different generalized state-update rules (GURs). In other words, we introduce generalized state-update theories (GUTs) for which the post-measurement states associated with a given outcome are not necessarily those prescribed by the Lüders rule. From the outset, we take into account both single and composite systems. Alternative post-measurement rules have been considered occasionally (cf. Sec. II A), but we are not aware of any systematic approach.

To qualify as a legitimate update rule, the assignment of post-measurement states must satisfy a number of well-motivated physical requirements. The framework we present is broad enough to accommodate the standard quantum-mechanical projection postulate as well as many hypothetical rules, including the so-called passive measurements that leave the state of a system untouched while still producing outcome probabilities governed by Born's rule.

In this setting, one can easily compare the properties of different update rules and, importantly, seek to identify those that are unique to the Lüders rule. It also provides a suitable backdrop to assess existing derivations of the Lüders rule summarized in the appendix.

This paper is organized as follows: In Sec. II we set the scene by briefly reviewing earlier discussions related to the quantum-mechanical state-update rule and describe the idea of quantum foil theories. Section III presents a simple axiomatic formulation of quantum theory and a detailed analysis of the state update in composite quantum systems. In Sec. IV, we describe the basic requirements that state-update rules must satisfy and how they give rise to quantum foil theories. Explicit examples of legitimate and invalid update rules are given in Sec. V. In Sec. VI, we explain what distinguishes the quantum-mechanical projection postulate from other valid update rules. In the concluding section, we summarize our results, discuss them, and outline future research.

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II. BACKGROUND AND MOTIVATION

A. The quantum-mechanical state-update rule

Using quantum systems as information carriers has led to applications without classical equivalents. Quantum measurements, in particular, provide a resource that contributes, for example, to secure communication protocols [3], quantum teleportation [4], device-independent random number

generators [5], and speedy algorithms when using measurement-based quantum computation [6].

The advantages of quantum measurements arise from fundamental differences when compared with measurements in classical theories. Classical measurements are deterministic in that they reveal pre-existing values of observables, and the effect on the state of the observed system is negligible, at least in principle. In quantum theory, measurement outcomes are probabilistic, and the state of a system after a measurement coincides with the pre-measurement state only in exceptional cases. Phenomenologically, the quantum-mechanical state update is governed by the projection postulate, which describes a "collapse" of the state to an eigenstate of the operator representing the observed outcome [7–9]. The collapse, also known as state reduction or Lüders rule [9], ensures deterministic repeatability: two measurements of an observable always yield identical outcomes when implemented in quick succession [10].

The standard postulates of quantum theory neither explain whether the state update is a physical process [11,12] nor do they define what constitutes a measurement [13,14]. The foundational questions resulting from this dilemma are often grouped under the term "measurement problem" [15,16]. The present work does *not* attempt to resolve the interpretational challenges posed by this long-standing problem.

Derivations of the Lüders rule have been given [17–21] (see the appendix for a survey), and the difference between von Neumann's and Lüders' formulations of the projection postulate has been discussed repeatedly [9,22,23]. Some authors have argued that the projection postulate follows from the other standard postulates of quantum theory [24,25], and the debate of a recent derivation is ongoing [26–30].

Occasionally, alternative post-measurement rules have been considered, either in the context of standard quantum theory [25,31,32] or of generalized probabilistic theories [33]. A systematic investigation, as proposed here, has apparently not been carried out before.

B. Foil theories

Our approach is inspired by the constructive role of foil theories for the understanding of quantum theory [2]. The strategy in defining foil theories is to embed quantum theory in a larger set of theories by relaxing specific assumptions about the structure of the state space, say, or the form of the time evolution. We briefly describe typical examples of foil theories and point out that a systematic investigation of generalized state-update rules has, apparently, not yet been carried out.

Generalized probabilistic theories (GPTs) [34–36], for example, represent a large and versatile collection of foil theories. They are defined by their own sets of states, observables, and rules for system composition, all of which may differ from those of standard quantum theory. GPTs have become a standard tool to explore and identify distinctive characteristics of quantum theory, which they contain as a special case. Many GPTs share properties such as the no-cloning theorem, uncertainty relations, or teleportation with quantum theory [37,38]. Some of them reproduce the standard quantum-mechanical

correlations [39], while others exhibit superquantum correlations, i.e., Boxworld [40].

Ontological models describe a conceptually different class of foil theories not necessarily based on the familiar Hilbert space structure of quantum theory. They aim to explain their predictions in terms of an observer's incomplete knowledge about the ontic state that provides an objective and complete description of the state of a system. In these models, nonclassical features of quantum theory such as complementarity of observables, superdense coding, interference phenomena, or uncertainty relations may result from epistemic restrictions [41,42].

Other foil theories differ from quantum theory in a single key feature only, such as attempts to modify Born's rule [43,44], or the tensor product for system composition [45]. The consequences of associating observables with non-Hermitian operators have been investigated, using $\mathscr{P}\mathscr{T}$-symmetric [46] or normal [47] operators, for example. Quantum-like theories defined over the real numbers [48,49] or quaternions [50] rather than the field of complex numbers have also been studied.

Nonlinear generalizations of Schrödinger's equation have been suggested [31,51–54], most notably in Weinberg's work [55]. Gisin *et al.* [56–58] pointed out that they imply violations of the no-signaling principle, although some specific nonlinear transformations were later found not to be ruled out by Gisin's argument [31,53,54,59,60].

In the specific context of modifications to the projection postulate, "causal quantum theory" [61] has been proposed, positing that the Lüders collapse is a well-defined physical process that satisfies strict local causality. Additionally, Kent introduced a new type of hypothetical measurement that is capable of revealing complete or partial information about the "local" state of a system, without actually inducing a state update [28,31,32], an idea first presented in [62]. Such a measurement would, effectively, allow for yet other nonlinear state transformations without enabling superluminal signaling.

III. QUANTUM THEORY AND THE PROJECTION POSTULATE

A. Axioms of quantum theory

The foil theories we introduce will differ from quantum theory only in their update rule assigning post-measurement states. The other standard postulates of nonrelativistic, finite-dimensional quantum theory remain unchanged.

(S) The states of a quantum system correspond to density operators $\rho \in \mathcal{S}(\mathcal{H})$, non-negative operators with unit trace acting on a finite-dimensional, separable, and complex Hilbert space $\mathcal{H}$ associated with the system.

(T) The time evolution of states, as well as any other reversible transformation, is described by a unitary operator $U \in \mathcal{U}(\mathcal{H})$.

(C) The state space of a composite system is obtained from tensoring the state spaces describing its constituents, i.e., by $\mathcal{S}_{AB} = \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$. Given a state ρ_{AB} of the composite system, the state of subsystem A is obtained by tracing out the other one, $\rho_A = \text{Tr}_B[\rho_{AB}]$, etc.

(O) Observable quantities are represented by collections $M = \{P_x\}_{x=1}^n$ of mutually orthogonal projectors $P_x \in \mathcal{P}(\mathcal{H})$, $P_x P_y = P_x \delta_{xy}$, that sum to the identity operator on $\mathcal{H}$, i.e., $\sum_x P_x = \mathbb{I}$. Each projector P_x represents a possible outcome of a measurement of the observable M.

(B) The probability that a measurement of M performed on a system in the state ρ yields the outcome represented by P_x is given by the Born rule: $\text{prob}(x) = \text{Tr}[P_x \rho]$.

Finally, the projection postulate describes what happens to the state of a quantum system when an observable is measured.

(L) If a measurement of M yields an outcome represented by the projection P_x, then the initial state $\rho \in \mathcal{S}(\mathcal{H})$ will update in the following way,

$$\rho \overset{x}{\longmapsto} \rho_x = \frac{P_x \rho P_x}{\text{Tr}[P_x \rho P_x]} \in \mathcal{S}(\mathcal{H}). \tag{1}$$

The focus of our work is to investigate the role of postulate (L) by replacing it with alternative update rules. In view of later developments, it will be convenient to suppress the normalization factor in Eq. (1). We introduce the map

$$\rho \overset{x}{\longmapsto} \omega^{\mathsf{L}}(P_x, \rho) = P_x \rho P_x, \tag{2}$$

which sends ρ to a subnormalized state [63] in the space $\bar{\mathcal{S}}(\mathcal{H})$, and call it the Lüders rule, giving postulate (L) its name. The trace of the post-measurement state, $\text{Tr}(\omega^{\mathsf{L}}(P_x, \rho)) = \text{Tr}(P_x \rho)$, yields the outcome probability as prescribed by the Born rule of postulate (B). In addition, the maps (1) and (2) do not require the input state to be normalized. Replacing ρ by $\lambda \rho$, $\lambda \in [0, 1]$, shows that the Lüders rule is 1-homogeneous: $\omega^{\mathsf{L}}(P_x, \lambda \rho) = \lambda \omega^{\mathsf{L}}(P_x, \rho)$.

The alternative update rules $\omega(P_x, \rho)$ considered in Sec. V will be modeled on the Lüders rule in the sense that they are 1-homogeneous, only depend on the measurement outcome P_x and the initial state ρ, and agree with the Born rule for outcome probabilities [64]. Further physically motivated properties that any acceptable update rule should satisfy will be discussed in Sec. IV.

Before defining update rules replacing (L), we must take a closer look at the effect of quantum measurements performed on composite systems. As it stands, the rule (L) does, in fact, *not* specify the post-measurement state of a composite system when measurements are performed on subsystems. The post-measurement state traditionally attributed to a quantum system relies on an additional implicit assumption.

B. Measurements on composite quantum systems

The projection postulate (L) does not explicitly cover local measurements which arise in composite quantum systems, i.e., measurements that are carried out on subsystems. To discuss this situation in detail, we consider a system with two constituents A and B, say, and an associated Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, in line with axiom (C).

Suppose we wish to measure the global observable M_{AB}, represented by an operator acting on the product space $\mathcal{H}_{AB}$. To this effect, a global device $\mathcal{D}_{AB}$ must be used that interacts with the composite system in its entirety. The outcome of the measurement is represented by one of the projection operators P_x^{AB} (associated with M_{AB}), which sums to the identity on the space $\mathcal{H}_{AB}$, as described in axiom (O). The probability to obtain any of these outcomes is found from axiom (B), and the state ρ_{AB} of the overall system will update according to the Lüders rule (L) given in Eq. (1). The projectors P_x^{AB} do not have to respect the product structure of the space $\mathcal{H}_{AB}$: for a two-qubit system, they may project onto the states of the Bell basis, for example. At no point do global measurements refer to the product structure of the space $\mathcal{H}_{AB}$.

However, on a bipartite quantum system, one may also carry out local measurements. They arise naturally when its constituents are in separate locations. Local measurements are implemented by measuring devices $\mathcal{D}_A$ or $\mathcal{D}_B$ that act on subsystems only, i.e., the constituents A or B, respectively. The postulate (L) does not cover this situation: no post-measurement state $\rho_{AB,x} \in \mathcal{S}_{AB} \equiv \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is specified by (L) when using the device $\mathcal{D}_A$ to carry out a measurement of the local observable M_A, resulting in the outcome x. Let us explain why.

According to axiom (O), the local observable M_A corresponds to a collection of mutually orthogonal projectors $\{P_x^A\}_{x=1}^n$, defined on $\mathcal{H}_A$ and summing to the identity, $\sum_x P_x^A = \mathbb{I}_A$. Thus, we associate measurement outcomes of M_A with projectors P_x^A. However, neither Eq. (1) nor Eq. (2) can be used to determine the post-measurement state of the system as a whole: to apply Lüders rule, we need to use (global) projectors P_x^{AB} defined on $\mathcal{H}_{AB}$, but the (local) operators P_x^A act on $\mathcal{H}_A$ only.

For the experimenter carrying out a local measurement on subsystem A, the existence of the other constituent B is irrelevant. If the composite system resides in the state ρ_{AB}, we invoke axiom (C): the initial state of subsystem A is given by the density matrix ρ_A obtained by tracing out subsystem B. Then, we are in a position to apply (L) to find the post-measurement state of subsystem A upon obtaining the local outcome P_x^A, namely,

$$\rho_A \overset{x}{\longmapsto} \rho_{A,x} \propto \omega^{\mathsf{L}}(P_x^A, \rho_A), \tag{3}$$

where the operators $\rho_A = \text{Tr}_B(\rho_{AB})$ and P_x^A are defined on the *same* Hilbert space $\mathcal{H}_A$, as required by the Lüders rule. The relation (3) does, of course, not mean that the system as a whole now resides in a product state. Importantly, when obtaining a specific outcome in a single run, Eq. (3) says nothing about the post-measurement state of the other subsystem, or of the system as a whole [65]. Read in this way, the statement of Lüders rule given by (L) is incomplete because it does not fully prescribe how to update the state of a quantum system when a local measurement is performed [66].

There are, however, consistency requirements when performing measurements on a composite system. If the outcome P_x^A is nondegenerate, then the Lüders rule assigns a pure post-measurement state $\omega^{\mathsf{L}}(P_x^A, \rho_A)$ to subsystem A. Consequently, the post-measurement state of the composite system must be a product state, regardless of the initial state ρ_{AB}. On its own, the projection postulate (L) provides, however, *no* information about the local state at B upon performing a local measurement of M_A on the constituent A. In principle, any density matrix $\rho_B = \text{Tr}_A(\rho_{AB}) \in \mathcal{S}(\mathcal{H}_B)$ is compatible. For example, one could imagine that any local measurement at A throws the state of the composite system into a product state, with

the post-measurement state at B being equal to the maximally mixed state $\rho_B \propto \mathbb{I}_B$.

The correct global post-measurement state after a local measurement on a quantum system is, of course, well-known. Calculating quantum correlations in Bell-type experiments relies on knowing the state of subsystem B if the outcome of a local measurement at A has been found [67]. Formally, upon measuring the local observable M_A and finding P_x^A given $\rho_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$, the state of the composite system updates in this way,

$$\rho_{AB} \overset{x}{\longmapsto} \rho_{AB,x} = \frac{P_x^A \otimes \mathbb{I}_B \, \rho_{AB} \, P_x^A \otimes \mathbb{I}_B}{\mathrm{Tr}\big[P_x^A \otimes \mathbb{I}_B \, \rho_{AB} \, P_x^A \otimes \mathbb{I}_B\big]} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B). \tag{4}$$

This rule generalizes the postulate (L) to cover the update of the global state ρ_{AB} caused by a local measurement,

$$\rho_{AB} \overset{x}{\longmapsto} \omega^{\mathsf{L}}\big(P_x^A, \rho_{AB}\big)$$
$$= P_x^A \otimes \mathbb{I}_B \, \rho_{AB} \, P_x^A \otimes \mathbb{I}_B \in \bar{\mathcal{S}}(\mathcal{H}_A \otimes \mathcal{H}_B). \tag{5}$$

Stated differently, we are making the point that the projection postulate (L) or, equivalently, Lüders rule (2), does *not* say how the state of a composite system updates under local measurements. Additional experimental input is required to arrive at rule (4). For simplicity, we continue to use the symbol ω^{L} to denote the complete update rule, which defines the effects of both local and global measurements.

The update rule (4) does have a simple physical interpretation. Suppose we use a device $\mathcal{D}_{AB}$ to perform a global measurement of the observable $M_A \otimes \mathbb{I}_B$, with outcomes represented by the projector $P_x^A \otimes \mathbb{I}_B$. To find the post-measurement state of the composite system, we can apply the original form of the projection postulate (L), only to find the update rule (4).

The distinction between a local measurement of M_A implemented by the device $\mathcal{D}_A$ and a global one of $M_A \otimes \mathbb{I}_B$, effected by the device $\mathcal{D}_{AB}$, will be important to express the requirements that alternatives to the Lüders rule must satisfy.

IV. GENERALIZING THE STATE-UPDATE RULE

A. Preliminaries

In this section, we define the properties that an acceptable state-update rule must have. The main role of an update rule is, of course, to determine the post-measurement state of the measured system. We need to allow that measurements on a part of a composite system may affect the state of other subsystems not acted upon. This behavior is known to occur in quantum theory and gave rise to the discussion in Sec. III B. In other words, we must ensure that an update rule assigns post-measurement states consistently to composite systems when local measurements are performed. For convenience, we focus on bipartite systems; the extension to multipartite systems is straightforward [68].

A generic update rule will be denoted by the letter ω, with a subscript identifying the possibly composite system to which it applies. Let us consider a composite system with Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, initially residing in the state $\rho_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$. If a local measurement is performed on subsystem $\mathcal{H}_A$ and yields the outcome P_x^A, the operator $\omega_{AB}(P_x^A, \rho_{AB})$ will

represent the subnormalized post-measurement state of the system, in analogy with the Lüders rule of Eq. (5) [69]. The subsystem on which the measurement is performed can be deduced from the Hilbert space associated with the projector P_x^A characterizing the measurement outcome. In the current example, it is the system with label A.

Any update rule must conform with postulate (B) implying a nontrivial constraint. By taking the trace of a subnormalized post-measurement state, the Born rule prescribes the probability to obtain outcome P_x^A,

$$\mathrm{Tr}\big[\omega_{AB}\big(P_x^A, \rho_{AB}\big)\big] = \mathrm{Tr}\big[P_x^A \otimes \mathbb{I}_B \, \rho_{AB}\big] = \mathrm{Tr}\big[P_x^A \, \mathrm{Tr}_B(\rho_{AB})\big]. \tag{6}$$

An analogous condition is, of course, assumed for measurements on subsystem $\mathcal{H}_B$.

When considering sequences of measurements, 1-homogeneity, described following Eq. (2) for the quantum-mechanical update rule, is a convenient mathematical property. It means that an update rule satisfies the constraint

$$\omega_{AB}\big(P_x^A, \lambda \, \rho_{AB}\big) = \lambda \, \omega_{AB}\big(P_x^A, \rho_{AB}\big), \quad \lambda \in [0, 1], \tag{7}$$

for all initial states $\rho_{AB} \in \bar{\mathcal{S}}(\mathcal{H}_A \otimes \mathcal{H}_B)$ and measurement outcomes $P_x^A \in \mathcal{P}(\mathcal{H}_A)$. This property allows us to concatenate update rules without the need to repeatedly normalize post-measurement states. Now suppose that two time-ordered measurements with outcomes P_x and P_y are performed on a composite system in state ρ_{AB}. Then, the final state is simply given by $\omega_{AB}(P_y, \omega_{AB}(P_x, \rho_{AB}))$. Here, the outcomes P_x and P_y may correspond to measurements performed on the same subsystem or on different subsystems.

B. Necessary properties of generalized update rules

We now list physically or operationally motivated requirements for update rules. These assumptions will lead to a concise definition of a generalized update rule, given in Def. 1. Naturally, these requirements are satisfied by the quantum-mechanical Lüders rule.

The requirements will take into account that update rules must hold for single and composite systems. For simplicity, we usually consider only measurements on one subsystem, say $\mathcal{H}_A$, although analogous conditions are assumed to apply to measurements on any subsystem.

The first three assumptions do not explicitly rely on the composite structure. Completeness requires that every conceivable measurement outcome specifies a unique post-measurement state. Context-independence asserts that the observed outcome and the pre-measurement state alone determine the post-measurement state. It introduces a form of noncontextuality [70] that is present in quantum theory—namely, that all operationally equivalent experimental runs (i.e., involving the same initial state and the same measurement outcome) yield identical post-measurement states. Mathematically, completeness and context-independence determine the co-domain and the domain of the update rule ω_{AB} for a composite system, respectively.

A1 and A2. Completeness and context-independence. A complete and context-independent update rule maps any combination of a (local) outcome, $P_x^A \in \mathcal{P}(\mathcal{H}_A)$, and a subnormalized state, $\rho_{AB} \in \bar{\mathcal{S}}(\mathcal{H}_A \otimes \mathcal{H}_B)$, to another subnormalized

state,

$$\omega_{AB} : \mathcal{P}(\mathcal{H}_A) \times \bar{\mathcal{S}}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \bar{\mathcal{S}}(\mathcal{H}_A \otimes \mathcal{H}_B). \quad (8)$$

Third, local covariance ensures that different observers provide consistent descriptions of the same experiment. For a measurement on a noncomposite system, suppose Alice's description involves the state ρ and outcome P_x. In Bob's reference frame, the same measurement will be described by the state $U\rho U^\dagger$ and the outcome $UP_xU^\dagger$, where U represents the frame change. Local covariance ensures that Alice's and Bob's post-measurement states remain related by U, preventing any contradictory predictions concerning future measurements. Effectively, we require that, instead of applying a unitary U on a system after measuring an observable $\{P_x\}_x$, one can equivalently implement it before measuring the suitably transformed observable $\{UP_xU^\dagger\}_x$. The term "local" gains particular significance when considering the more general scenario of a measurement performed on a subsystem of a composite system. To ensure consistency across descriptions of a local measurement, the same joint state must be obtained when applying the unitary transformation $U_A \otimes U_B$ (*i*) after measuring $\mathcal{H}_A$ with the outcome P_x^A, or (*ii*) before measuring $\mathcal{H}_A$ with the rotated outcome $U_A P_x^A U_A^\dagger$. Local unitaries $U_A \otimes U_B$ represent changes to frames that preserve the composite structure of $\mathcal{H}_A \otimes \mathcal{H}_B$.

A3. Local covariance. For all measurement outcomes $P_x^A \in \mathcal{P}(\mathcal{H}_A)$, states $\rho_{AB} \in \bar{\mathcal{S}}(\mathcal{H}_A \otimes \mathcal{H}_B)$ of the joint system, and local transformations $U_A \in \mathcal{U}(\mathcal{H}_A)$ and $U_B \in \mathcal{U}(\mathcal{H}_B)$, we must have

$$\omega_{AB}\big(U_A P_x^A U_A^\dagger, U_A \otimes U_B \rho_{AB} U_A^\dagger \otimes U_B^\dagger\big)$$
$$= (U_A \otimes U_B)\big[\omega_{AB}\big(P_x^A, \rho_{AB}\big)\big]\big(U_A^\dagger \otimes U_B^\dagger\big). \quad (9)$$

The remaining assumptions rely on the composite structure of the system. We require self-consistency of the update rule: the assigned post-measurement states must not depend on whether the probed system is regarded as part of a larger composite system. This property ensures that the assignment of post-measurement states for composite systems is unambiguous. Violating self-consistency would result in an inconsistent treatment of measurements in the framework, wherein an observer's choice of whether to regard a system as a subsystem (and how to partition its environment) would have measurable effects. Mathematically, self-consistency is ensured by the requirement that the state of any system after the measurement could also be obtained by applying ω to a larger system and tracing out the irrelevant subsystems.

A4. Self-consistency. Suppose that the system with label B is, in fact, a composite system, i.e., $\mathcal{H}_B = \mathcal{H}_{B'} \otimes \mathcal{H}_{B''}$, hence the update rule can be relabeled as $\omega_{AB} = \omega_{AB'B''}$. For all $\mathcal{H}_{B''}$, joint states $\rho_{AB'B''} \in \bar{\mathcal{S}}(\mathcal{H}_A \otimes \mathcal{H}_{B'} \otimes \mathcal{H}_{B''})$ and measurement outcomes $P_x \in \mathcal{P}(\mathcal{H}_A)$, we must have

$$\mathrm{Tr}_{B''}\big[\omega_{AB'B''}\big(P_x^A, \rho_{AB'B''}\big)\big] = \omega_{AB'}\big[P_x^A, \mathrm{Tr}_{B''}(\rho_{AB'B''})\big]. \quad (10)$$

Self-consistency entails that the update rule ω_A of a noncomposite system emerges as a special case of the composite-system update rule ω_{AB}: letting $\rho_A = \mathrm{Tr}_B(\rho_{AB})$, we have

$$\omega_A\big(P_x^A, \rho_A\big) = \mathrm{Tr}_B\big[\omega_{AB}\big(P_x^A, \rho_{AB}\big)\big]. \quad (11)$$

Any update rule must respect the (quantum) no-signaling principle, which prohibits spacelike separated parties from communicating through local measurements. Mathematically, we require that unconditional measurements—where no specific outcome is selected—do not alter the reduced state of unmeasured subsystems.

A5. No-signaling. For all joint states $\rho_{AB} \in \bar{\mathcal{S}}(\mathcal{H}_A \otimes \mathcal{H}_B)$ and all local observables $\{P_x^A\}_x$ on $\mathcal{H}_A$, we must have

$$\sum_x \mathrm{Tr}_A\big[\omega_{AB}\big(P_x^A, \rho_{AB}\big)\big] = \mathrm{Tr}_A(\rho_{AB}). \quad (12)$$

Finally, we assume local commutativity: conditioned on fixed outcomes for measurements on different subsystems, the order in which these measurements are performed does not matter. As with no-signaling, local commutativity finds justification in special relativity. Suppose that local measurements on two spacelike separated subsystems yield outcomes P_x^A and P_y^B. Since the time-ordering of the two measurements is relative, we require that the same post-measurement state of the combined system is obtained irrespective of whether P_x^A or P_y^B is measured first. This property also implies that the post-measurement state of the combined system is defined unambiguously if P_x^A and P_y^B are measured simultaneously.

A6. Local commutativity. For all measurement outcomes $P_x^A \in \mathcal{P}(\mathcal{H}_A)$, $P_y^B \in \mathcal{P}(\mathcal{H}_B)$ and joint states $\rho_{AB} \in \bar{\mathcal{S}}(\mathcal{H}_A \otimes \mathcal{H}_B)$, we must have

$$\omega_{AB}\big[P_x^A, \omega_{AB}\big(P_y^B, \rho_{AB}\big)\big] = \omega_{AB}\big[P_y^B, \omega_{AB}\big(P_x^A, \rho_{AB}\big)\big]. \quad (13)$$

Requirements A1–A6 are the core assumptions feeding into the framework of generalized state-update theories. Other features of the Lüders rule of quantum theory will not necessarily be shared by generalized update rules. For example, the resulting foil theories do not have to be deterministically repeatable, and they do not have to allow for quantum measurements to preserve the indistinguishability of preparations (cf. Sec. IV C). As we see in Sec. V, there are generalized state-update theories (GUTs) in which different preparations of the same mixed state do not remain indistinguishable after a measurement. Thus, depending on the update rule, density operators may or may not provide a complete description of an individual system.

We are now in a position to present the definition of generalized state-update rules.

Definition 1. A generalized state-update rule ω is a set of functions $\{\omega_{AB}\}$—where each function ω_{AB} is associated with a pair of Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$—satisfying the following properties: (*i*) consistency with the Born rule [Eq. (6)]; (*ii*) 1-homogeneity [Eq. (7)]; and (*iii*) assumptions A1–A6 [Eqs. (8)–(13)], i.e., $\omega_{AB} : \mathcal{P}(\mathcal{H}_A) \times \bar{\mathcal{S}}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \bar{\mathcal{S}}(\mathcal{H}_A \otimes \mathcal{H}_B)$, etc.

Any update rule chosen to replace the projection postulate of quantum theory will give rise to one specific generalized state-update theory. Multiple update rules may induce the same behavior on noncomposite systems via Eq. (11). This point is illustrated in Sec. V B, where different extensions of the single-system Lüders rule to multipartite systems are presented.

How do the update rules defined here compare with quantum instruments [71], i.e., collections $\{\mathcal{I}_x\}_x$ of quantum operations [72] that describe the effect on states of quantum

measurements? The main difference is that the state-update rules satisfying our definition are *not* required to be linear or completely positive. Consequently, the set of alternative state-update rules is larger than the set of quantum instruments.

C. Updating (im-)proper mixtures

For a pure joint state ρ_{AB}, Eq. (11) applies irrespective of ρ_{AB} being a product state or being entangled. Accordingly, $\omega_A(P_x^A, \rho_A)$ represents the post-measurement state of system $\mathcal{H}_A$ when ρ_A is either a pure state or an improper mixture, i.e., the reduced density operator of an entangled state. In contrast, proper mixtures result from classical uncertainty about the "true" preparation of a system. A proper mixture can be thought of as a *Gemenge* (German for "mixture"), i.e., as a collection $\mathcal{G} = \{(p_1, \rho_1), (p_2, \rho_2), \ldots, (p_n, \rho_n)\}$ where $p_i \geqslant 0$, $\sum_i p_i = 1$, and $\rho_i \in \bar{\mathcal{S}}(\mathcal{H})$ represent pure states or improper mixed states. The *Gemenge* $\mathcal{G}$ describes a system that is prepared in state ρ_i with probability p_i [15]. To any *Gemenge* there corresponds a unique density operator, $\rho_{\mathcal{G}} = \sum_{i=1}^n p_i \rho_i$.

If a system described by $\mathcal{G}$ is measured with outcome P_x^A, the updated description is given by the *Gemenge* $\mathcal{G}_x$ with the same weights p_i but with updated (subnormalized) states, $\mathcal{G}_x = \{p_i, \omega_A(P_x^A, \rho_i)\}$. In other words, if the system resided in the state ρ_i with probability p_i before the measurement with outcome P_x^A, it will reside in the state $\omega_A(P_x^A, \rho_i)$ with probability p_i after the measurement [73]. Thus, the density matrix updates according to the rule

$$\rho_{\mathcal{G}} = \sum_{i=1}^n p_i \rho_i \xrightarrow{P_x^A} \rho_{\mathcal{G}_x} = \sum_{i=1}^n p_i\, \omega_A(P_x^A, \rho_i). \quad (14)$$

A similar expression applies for local measurements on a subsystem when the composite system is described by a proper mixture, i.e., $\mathcal{G} = \{(p_1, \rho_{AB}^1), (p_2, \rho_{AB}^2), \ldots, (p_n, \rho_{AB}^n)\}$.

The update rule is therefore applied differently depending on whether a mixed state is regarded as a proper or an improper mixture. As shown in (14), in the case of a proper mixture, the map acts on each operator ρ_i appearing in the decomposition of $\rho_{\mathcal{G}}$ and included in the associated *Gemenge* $\mathcal{G}$. Typically, the post-measurement state $\rho_{\mathcal{G}_x}$ also represents a proper mixture. Consequently, sequences of measurements on a proper mixture are described by repeatedly applying the update rule to each pure or improper mixed state ρ_i in the initial *Gemenge* $\mathcal{G}$.

The equations (8)–(13) formalizing assumptions A1–A6 were formulated with pure and improper mixed states in mind. Nevertheless, they also ensure a consistent treatment of measurements on proper mixtures. For instance, the completeness A1 of ω guarantees that $\rho_{\mathcal{G}_x}$ in (14) is a valid state of the system. Similar arguments apply to assumptions A3–A6, but not to context-independence A2, which requires minor adaptation. In fact, the post-measurement state $\rho_{\mathcal{G}_x}$ generally does not depend on the initial mixed state $\rho_{\mathcal{G}}$, but rather on the initial *Gemenge* $\mathcal{G}$. When proper mixtures are included, context-independence is generalized as follows: the observed outcome and the pre-measurement *Gemenge* alone determine the post-measurement state [74].

By considering proper and improper mixtures separately, we allow the framework to include foil theories where se-

quential measurements can be used to distinguish between different realizations of the same mixed state. Unlike quantum theory, such theories do not generally preserve the indistinguishability of preparations. In addition, our framework should also include update rules that send pure states to proper mixtures, thus introducing another level of uncertainty into the behavior of a measuring device. In such cases, equations such as (13) for local commutativity A6, which involve sequential measurements, must be adjusted by ensuring that the second update rule is applied to each state in the *Gemenge* resulting from the first measurement.

D. Foil theories from generalized state-update rules

Having defined update rules in Def. 1, we now introduce GUTs.

Definition 2. A generalized state-update theory (GUT) is defined by the quantum postulates for states (S), time evolution (T), system composition (C), observables (O), outcome probabilities (B), and by a generalized state-update rule ω assigning post-measurement states.

V. EXAMPLES OF GENERALIZED STATE-UPDATE RULES

We now introduce several examples of update rules and investigate properties they have in addition to the essential requirements A1–A6 of Sec. IV B. Each of the hypothetical rules defines a foil of quantum theory through Def. 2 in Sec. IV D, characterized by a modified projection postulate.

A. Lüders measurements: Quantum theory

Our framework includes quantum theory, since the properties A1–A6 underlying Def. 1 for generalized update rules have been abstracted from the Lüders rule,

$$\omega_{AB}^{\mathsf{L}}(P_x^A, \rho_{AB}) = P_x^A \otimes \mathbb{I}_B \rho_{AB} P_x^A \otimes \mathbb{I}_B. \quad (15)$$

In Sec. III B, we described an important property of the Lüders rule, that we call composition compatibility. A *local* measurement on subsystem $\mathcal{H}_A$, implemented by a device $\mathcal{D}_A$ and yielding the outcome P_x^A, leads to the same post-measurement state of the composite system as a *global* measurement on $\mathcal{H}_A \otimes \mathcal{H}_B$, implemented by a device $\mathcal{D}_{AB}$ and yielding the outcome represented by $P_x^A \otimes \mathbb{I}_B$.

Definition 3 (Composition compatibility). An update rule ω satisfies composition compatibility if, for every $\mathcal{H}_A$, $\mathcal{H}_B$, joint state $\rho_{AB} \in \bar{\mathcal{S}}(\mathcal{H}_A \otimes \mathcal{H}_B)$, and local outcome $P_x^A \in \mathcal{P}(\mathcal{H}_A)$,

$$\omega_{AB}(P_x^A, \rho_{AB}) = \omega_{AB}(P_x^A \otimes \mathbb{I}_B, \rho_{AB}). \quad (16)$$

We see that not all update rules satisfy this property—in other words, it does not follow from assumptions A1–A6. Note that the two expressions in Eq. (16) capture two different situations: the function on the left-hand side is defined on $\mathcal{P}(\mathcal{H}_A) \times \bar{\mathcal{S}}(\mathcal{H}_A \otimes \mathcal{H}_B)$, while that on the right-hand side on $\mathcal{P}(\mathcal{H}_A \otimes \mathcal{H}_B) \times \bar{\mathcal{S}}(\mathcal{H}_A \otimes \mathcal{H}_B)$. In physical terms, one side of Eq. (16) corresponds to a *local* measurement on $\mathcal{H}_A$, while the other side is associated with a *global* measurement on $\mathcal{H}_A \otimes \mathcal{H}_B$. Equation (16) refers to measurements on $\mathcal{H}_A$ only, but the property is assumed to hold for measurements on any subsystem.

Composition compatibility (16) can therefore be regarded as the additional assumption that extends the "operationally incomplete" postulate (L) for single systems, i.e.,

$$\omega_A^{\mathsf{L}}(P_x^A, \rho_A) = P_x^A \rho_A P_x^A, \quad (17)$$

to the Lüders rule ω_{AB}^{L} for multipartite systems, Eq. (15), which encompasses the effects of both local and global measurements.

Another argument to extend the single-system Lüders rule to composite systems uses the fact that $\omega_{A,x}^{\mathsf{L}}(\cdot) \equiv \omega_A^{\mathsf{L}}(P_x^A, \cdot) : \bar{\mathcal{S}}(\mathcal{H}_A) \to \bar{\mathcal{S}}(\mathcal{H}_A)$, which denotes the Lüders rule for system $\mathcal{H}_A$ conditioned on the measurement outcome P_x^A, is convex-linear over $\bar{\mathcal{S}}(\mathcal{H}_A)$. Therefore, $\omega_{A,x}^{\mathsf{L}}$ has a unique linear extension to the set of bounded operators $\mathcal{L}(\mathcal{H}_A)$ (cf. [71]). The Lüders rule (15) for the composite system $\mathcal{H}_A \otimes \mathcal{H}_B$, conditioned on the local measurement outcome P_x^A, is then obtained by setting

$$\omega_{AB}^{\mathsf{L}}(P_x^A, \rho_{AB}) = (\omega_{A,x}^{\mathsf{L}} \otimes \mathcal{I}_B)(\rho_{AB}) \quad (18)$$

for all $\rho_{AB} \in \bar{\mathcal{S}}(\mathcal{H}_A \otimes \mathcal{H}_B)$, where $(\omega_{A,x}^{\mathsf{L}} \otimes \mathcal{I}_B)(L_A \otimes L_B) = \omega_{A,x}^{\mathsf{L}}(L_A) \otimes L_B$ for all $L_A \in \mathcal{L}(\mathcal{H}_A)$ and $L_B \in \mathcal{L}(\mathcal{H}_B)$. Here, $\mathcal{I}_B$ denotes the identity channel on $\mathcal{H}_B$. Equation (18) is well-defined, as well as linear over $\bar{\mathcal{S}}(\mathcal{H}_A \otimes \mathcal{H}_B)$, because $\omega_{A,x}^{\mathsf{L}}$ can be linearly extended to $\mathcal{L}(\mathcal{H}_A)$. In particular, the completeness A1 of $\omega_{AB}^{\mathsf{L}}(P_x^A, \rho_{AB})$ follows from the complete positivity of $\omega_{A,x}^{\mathsf{L}}$: if $\omega_{A,x}^{\mathsf{L}}$ was not completely positive, then the operator $\omega_{AB}^{\mathsf{L}}(P_x^A, \rho_{AB})$ would not be positive for some ρ_{AB}.

Typically, generalized update rules have to satisfy neither deterministic repeatability nor composition compatibility. They may also lack some other properties, such as preparation indistinguishability, which ensures that different preparations of the same mixed state remain indistinguishable after measurements. This property is equivalent to the statement that "density operators provide complete descriptions of quantum systems." Moreover, ideality (measurement leaves a state unchanged if the outcome is certain) and local tomography (local measurements of M_A and M_B on different subsystems are operationally equivalent to a global measurement of the product observable $M_A \otimes M_B$ if the experimenters are allowed to communicate classically) may or may not be present. The Lüders rule is also endowed with a form of coherence, according to which the outcome probability distribution of an observable M is not affected by a prior coarse-grained measurement of M (see Def. 4 in Sec. VI). Table I summarizes the extent to which these and other properties hold for the hypothetical update rules introduced in the following sections.

B. Locally-Lüders measurements: Correlation-free theory

As noted in Sec. IV, the Lüders rule for single systems does not fix the state-update rule in multipartite systems. Consequently, there will exist other update rules for composite systems that reduce to the single-system Lüders rule. One example is the 1-homogeneous map,

$$\omega_{AB}^{\mathsf{locL}}(P_x^A, \rho_{AB}) = \frac{\omega_A^{\mathsf{L}}\big[P_x^A, \mathrm{Tr}_B(\rho_{AB})\big] \otimes \mathrm{Tr}_A(\rho_{AB})}{\mathrm{Tr}(\rho_{AB})}, \quad (19)$$

TABLE I. Summary of the operational properties of the update rules discussed in Sec. V. Ticks in parentheses indicate that the property holds for measurements on noncomposite systems only.

	ω^{L}	ω^{locL}	ω^{P}	ω^{dep}	ω^{λ}
Deterministic repeatability	✓	✓	×	×	×
Preparation indistinguishability	✓	(✓)	×	✓	×
Composition compatibility	✓	×	✓	×	×
Ideality	✓	(✓)	✓	×	(✓)
Local tomography	✓	×	×	✓	×
Coherence	✓	✓	×	×	×
Nonlocality	✓	×	×	✓	×
Complete positivity	✓	×	×	✓	×
Weak repeatability	✓	✓	×	×	✓

which satisfies assumptions A1–A6 and thus defines a valid update rule. While manifestly different from the standard Lüders rule of Eq. (15), it assigns the same post-measurement states as postulate (L) to noncomposite systems. Using Eq. (11) and letting $\rho_A = \mathrm{Tr}_B(\rho_{AB})$, we obtain

$$\omega_A^{\mathsf{locL}}(P_x^A, \rho_A) = \mathrm{Tr}_B\big[\omega_{AB}^{\mathsf{locL}}(P_x^A, \rho_{AB})\big] = \omega_A^{\mathsf{L}}(P_x^A, \rho_A). \quad (20)$$

In other words, ω^{locL} represents an alternative extension of postulate (L) but, unlike the Lüders rule ω^{L}, it does *not* satisfy composition compatibility and is *not* convex-linear over pre-measurement states of the composite system.

The GUT defined by the update rule (19) is therefore locally indistinguishable from quantum mechanics but creates no correlations between entangled systems. Regardless of the observed outcome, a local measurement on subsystem A leaves the reduced state of subsystem B unchanged. Although measurement breaks the entanglement between two systems—by always mapping ρ_{AB} to a product state—distant parties will observe no correlation between their respective measurement outcomes. In such a correlation-free variant of quantum theory, Bell's inequalities cannot be violated, and entanglement is not available as a resource for distributed tasks.

C. Noncollapsing measurements: Passive quantum theory

We now consider measurements that, instead of updating the quantum state, leave it unchanged [75]. Outcomes still occur probabilistically as dictated by the Born rule, yet, in analogy with classical theory, these measurements do not disturb the pre-measurement state. This situation corresponds to a "passive" update rule ω_A^{P} for noncomposite systems,

$$\omega_A^{\mathsf{P}}(P_x^A, \rho_A) = \mathrm{Tr}(P_x^A \rho_A)\rho_A. \quad (21)$$

Noncollapsing measurements, originally introduced in Ref. [76], cannot be implemented in quantum theory [62]. Indeed, ω_A^{P} is not convex-linear over $\bar{\mathcal{S}}(\mathcal{H}_A)$, which implies that, after a passive measurement, one may be able to distinguish between different preparations of the same mixed state. For example, let $\mathcal{H}_A = \mathbb{C}^2$ and consider the *Gemenge*

$$\mathcal{G} = \{(1/2, |0\rangle\langle 0|), (1/2, |1\rangle\langle 1|)\}$$

$$\text{and} \quad \mathcal{G}' = \{(1/2, |+\rangle\langle +|), (1/2, |-\rangle\langle -|)\}, \quad (22)$$

where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. Both preparations correspond to the maximally mixed state, $\rho_{\mathcal{G}} = \rho_{\mathcal{G}'} = \mathbb{I}_2/2$. According to the *Gemenge*-update rule (14) in Sec. IV C, a passive measurement with outcome $P_x^A = |0\rangle\langle 0|$ will produce two different subnormalized density operators,

$$\rho_{\mathcal{G}_x} = \frac{1}{2}|0\rangle\langle 0| \neq \rho_{\mathcal{G}'_x} = \frac{\mathbb{I}_2}{4}. \qquad (23)$$

Therefore, the noncollapsing update rule ω_A^P does not satisfy preparation indistinguishability, indicating that density operators do not provide complete descriptions of individual systems.

While complete positivity can be defined for both linear and nonlinear transformations [77], it has been described as "physically unfitting" for nonlinear dynamics [78]. In fact, although ω_A^P [or, more precisely, extensions of ω_A^P to $\mathcal{L}(\mathcal{H}_A)$] is not completely positive, the update rule can still be consistently extended to composite systems. Invoking compositional compatibility (Def. 3), we set

$$\omega_{AB}^\mathsf{P}(P_x^A, \rho_{AB}) = \omega_{AB}^\mathsf{P}(P_x^A \otimes \mathbb{I}_B, \rho_{AB}) = \mathrm{Tr}(P_x^A \otimes \mathbb{I}_B \rho_{AB})\rho_{AB}. \qquad (24)$$

The map ω^P satisfies all the conditions in Def. 1 and defines a GUT in which measurements do not modify the state of either the measured system or of the larger composite system to which it belongs. We called this theory passive quantum theory (pQT). An overview of pQT and its key properties has been provided in Ref. [79].

The absence of any measurement-induced state update allows, in principle, for tomographic reconstruction of the state of an individual system. Thus, the cloning of arbitrary states becomes possible and makes computation in pQT more efficient (in some sense) than in standard quantum theory [31,62]. The passive update rule of pQT also satisfies ideality but is not locally tomographic and does not allow for violations of Bell's inequalities (cf. Table I).

D. Depolarizing measurements

Next, consider a theory in which measurements disturb the system to the point that the post-measurement state contains no information about the original state. Repeating measurements on the same system, therefore, extracts no additional information about it.

Updating the pre-measurement state to the maximally mixed state, regardless of the observed outcome, provides one possible realization of such a theory,

$$\omega_A^{\mathsf{dep}}(P_x^A, \rho_A) = \mathrm{Tr}(P_x^A \rho_A)\frac{\mathbb{I}_A}{d_A}, \qquad (25)$$

where $d_A = \dim(\mathcal{H}_A)$. The map in Eq. (25) is both convex-linear over $\bar{\mathcal{S}}(\mathcal{H}_A)$ and completely positive. The set $\{\omega_A^{\mathsf{dep}}(P_x^A)\}_x$ represents, in fact, a quantum instrument [80] compatible with the observable $\{P_x^A\}_x$.

Following Eq. (18), by defining $\omega_{A,x}^{\mathsf{dep}}(\cdot) \equiv \omega_A^{\mathsf{dep}}(P_x^A, \cdot)$, we can extend ω_A^{dep} to composite systems via

$$\omega_{AB}^{\mathsf{dep}}(P_x^A, \rho_{AB}) = (\omega_{A,x}^{\mathsf{dep}} \otimes \mathcal{I}_B)(\rho_{AB}). \qquad (26)$$

In contrast with the Lüders rule, the depolarizing rule ω^{dep} does not satisfy composition compatibility. To see this, con-

sider $\rho_{AB} = |00\rangle\langle 00| \in \mathcal{S}(\mathcal{H}_2 \otimes \mathcal{H}_2)$ and $P_x^A = |0\rangle\langle 0|$. We find that

$$\begin{aligned}\omega_{AB}^{\mathsf{dep}}(|0\rangle\langle 0|, |00\rangle\langle 00|) &= \frac{\mathbb{I}_2}{2} \otimes |0\rangle\langle 0| \neq \frac{\mathbb{I}_2}{2} \otimes \frac{\mathbb{I}_2}{2} \\ &= \omega_{AB}^{\mathsf{dep}}(|0\rangle\langle 0| \otimes \mathbb{I}_B, |00\rangle\langle 00|).\end{aligned}$$
$$(27)$$

This example also demonstrates how composition compatibility could not have been used to extend ω_A^{dep} to composite systems. The right-hand side of Eq. (27) cannot represent the post-measurement state of the composite system following a local measurement on a single qubit, as it would violate the no-signaling condition A5. See Table I for further properties of depolarizing measurements.

E. Probability-amplifying projective measurements

Suppose an observable is measured twice on the same system in quick succession. Being deterministically repeatable, quantum theory predicts that we observe the same outcome for the second measurement with certainty. We can construct update rules that satisfy a weaker notion of repeatability, called weak repeatability: the probability of observing an outcome for the second time is greater than the probability of observing it for the first time. Consider the following one-parameter family of update rules for noncomposite systems:

$$\omega_A^\lambda(P_x^A, \rho_A) = \mathrm{Tr}(P_x^A \rho_A) \frac{G_\lambda^{1/2}(P_x^A)\, \rho_A\, G_\lambda^{1/2}(P_x^A)}{\mathrm{Tr}[G_\lambda(P_x^A)\,\rho_A]}, \qquad (28)$$

where

$$G_\lambda(P_x^A) = (1-\lambda)P_x^A + \lambda\mathbb{I}_A, \quad \lambda \in [0,1]. \qquad (29)$$

If a measurement outcome $P_X^A = \sum_{i\in X}|i\rangle\langle i|$ for $X \subseteq \{0,\ldots,d_A-1\}$ is obtained given the pure state $|\psi\rangle = \sum_{i=0}^{d_A-1} c_i|i\rangle \in \mathcal{H}_A$, the resulting post-measurement state reads

$$\omega_A^\lambda(P_X^A, |\psi\rangle\langle\psi|) = \mathrm{Tr}(P_x^A \rho_A)|\psi_X\rangle\langle\psi_X|, \qquad (30)$$

where

$$|\psi_X\rangle = \frac{1}{N}\left(\sum_{j\in X} c_j|j\rangle + \sqrt{\lambda}\sum_{i\notin X} c_i|i\rangle\right), \qquad (31)$$

with N being the normalization factor. Therefore, the update rule ω_A^λ projects $|\psi\rangle$ to the pure state $|\psi_X\rangle$ and the parameter λ determines how likely a second measurement will yield the same outcome as the first, with a lower value of λ corresponding to a higher probability of repeating the first outcome. The Lüders rule ω_A^L and the passive rule ω_A^P for noncomposite systems are recovered by setting $\lambda = 0$ and $\lambda = 1$, respectively. Thus, for noncomposite systems, the update rule ω_A^λ indeed smoothly interpolates between quantum theory and pQT. Weak repeatability occurs for $\lambda \in [0, 1)$. The map in Eq. (28) is not convex-linear over $\bar{\mathcal{S}}(\mathcal{H}_A)$ for $\lambda \neq 0$, hence the corresponding theories will not satisfy preparation indistinguishability.

Following Eq. (19), we find that a valid extension of Eq. (28) to composite systems is provided by

$$\omega_{AB}^\lambda(P_x^A, \rho_{AB}) = \frac{\omega_A^\lambda[P_x^A, \mathrm{Tr}_B(\rho_{AB})] \otimes \mathrm{Tr}_A(\rho_{AB})}{\mathrm{Tr}(\rho_{AB})}. \qquad (32)$$

TABLE II. Summary of the state-updates described in Sec. V F that do not define an update rule as per Def. 1. For each rule, we record satisfied and violated assumptions using ticks and crosses, respectively. The first two, $\tilde{\omega}^{\mathsf{dep}}$ and $\tilde{\omega}^\lambda$, denote the extensions to composite systems of ω_A^{dep} (25) and ω_A^λ (28), respectively, obtained via composition compatibility (Def. 3).

	$\tilde{\omega}^{\mathsf{dep}}$	$\tilde{\omega}^\lambda$	ω^μ	ω^{vN}	ω^{U}
A1: completeness	✓	✓	✓	✓	✓
A2: context-independence	✓	✓	✓	✗	✓
A3: local covariance	✓	✓	✓	✓	✗
A4: self-consistency	✓	✓	✓	✓	✓
A5: no-signaling	✗	✗	✓	✓	✓
A6: local commutativity	✓	✗	✗	✓	✓

It can be shown that Eq. (32) satisfies all conditions outlined in Def. 1. For each $\lambda \in (0, 1)$, the probability-amplifying projective rule ω^λ defines a weakly repeatable GUT that is manifestly different from quantum theory [81]. While alternative extensions of ω_A^λ may in principle be constructed, none would satisfy composition compatibility (Def. 3). Similarly to the depolarizing rule ω^{dep}, it is easy to show that the extension provided by Eq. (16) violates no-signaling A5, as well as local commutativity A6.

F. Invalid state-update rules

It is instructive to consider examples of state-update rules that violate at least one of the basic requirements A1–A6 of Def. 1 in Sec. IV B. We have, in fact, already acknowledged two of them when presenting depolarizing measurements (Sec. V D) and probability-amplifying projective measurements (Sec. V E). Extending the noncomposite rules ω_A^{dep} (25) and ω_A^λ (28) via compositional compatibility (Def. 3) leads to maps assigning post-measurement states to composite systems in violation of the (quantum) no-signaling principle A5.

Table II summarizes the properties of the update rules we consider in this section.

Consider now a convex mixture of the Lüders ω^{L} and the passive ω^{P} update rules for composite systems,

$$\omega_{AB}^\mu\big(P_x^A, \rho_{AB}\big) = (1 - \mu)\omega_{AB}^{\mathsf{L}}\big(P_x^A, \rho_{AB}\big) + \mu\,\omega_{AB}^{\mathsf{P}}\big(P_x^A, \rho_{AB}\big),$$
$$\mu \in (0, 1), \tag{33}$$

where the right-hand side describes a proper mixture. Equation (33) describes a local measurement that, with some *a priori* probability $(1 - \mu)$, collapses the state of the composite system as prescribed by quantum theory, or, with probability μ, leaves the state unchanged. For $\mu \in (0, 1)$, the probability of observing outcome P_x^A in a possible second measurement is, on average, higher than in the first measurement. In other words, (33) provides an alternative way to satisfy weak repeatability, different from the probability-amplifying projections ω^λ introduced in Sec. V E.

The "real" post-measurement state of the system is either $\omega_{AB}^{\mathsf{L}}(P_x^A, \rho_{AB})$ or $\omega_{AB}^{\mathsf{P}}(P_x^A, \rho_{AB})$; one can imagine that this is determined by the value of some hidden variable $\lambda \in \{0, 1\}$. The probability distribution of this hidden variable is the same across all experimental runs—i.e., prob$(\lambda = 0) = 1 -$

μ, and prob$(\lambda = 1) = \mu$—ensuring that it is independent of the context of a particular experiment. This guarantees that ω^μ satisfies context-independence A2. Moreover, (33) can be shown to satisfy completeness A1, local covariance A3, self-consistency A4 and no-signaling A5, since ω^μ is a convex mixture of mappings that individually satisfy these conditions. Surprisingly, however, we find that (33) violates local commutativity A6, despite both ω^{L} and ω^{P} satisfying it.

The violation can be seen in a simple example. Let $\rho_{AB} = |\Phi^+\rangle\langle\Phi^+| \in \mathcal{S}(\mathcal{H}_2 \otimes \mathcal{H}_2)$, where $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ is a maximally entangled state, and consider $P_x^A = |0\rangle\langle 0|$, and $P_y^B = |1\rangle\langle 1|$. Recall that ω^μ maps pure states to proper mixtures; hence, when representing sequential measurements, the second map must be applied to each pure state in the corresponding *Gemenge*, in line with the discussion in Sec. IV C. If P_x^A is obtained in a local measurement on $\mathcal{H}_A$ before the local measurement of $\mathcal{H}_B$ yields P_y^B, the state of the composite system updates as

$$|\Phi^+\rangle\langle\Phi^+| \overset{(x \prec y)}{\longmapsto} \frac{\mu}{4}[(1 - \mu)|11\rangle\langle 11| + \mu|\Phi^+\rangle\langle\Phi^+|]. \tag{34}$$

If instead the measurement yielding P_y^B precedes the one yielding P_x^A, a different update occurs,

$$|\Phi^+\rangle\langle\Phi^+| \overset{(y \prec x)}{\longmapsto} \frac{\mu}{4}[(1 - \mu)|00\rangle\langle 00| + \mu|\Phi^+\rangle\langle\Phi^+|]. \tag{35}$$

Since the final state depends on the order in which the local measurements are performed, Eq. (33) violates local commutativity A6. As a result, correlations between measurements on different subsystems cannot be unambiguously determined.

The projection postulate introduced by von Neumann [8] is a notable example of state update that does not define a valid update rule. For nondegenerate measurements, von Neumann's and Lüders' postulates coincide. However, according to von Neumann, outcome degeneracy of quantum measurements—represented by projectors of rank greater than one—arises from classical postprocessing of nondegenerate measurements. That is, one does not directly implement a measurement of a degenerate observable M; instead, a measurement of a refinement of M, i.e., a nondegenerate observable M' commuting with M, is carried out, and the outcomes are coarse-grained. Given a degenerate observable, there exist, however, infinitely many refinements. Consequently, the post-measurement state will not only depend solely on the pre-measurement state ρ_A (which we can assume to be pure, without loss of generality) and the observed degenerate outcome P_x^A, but also on the specific refinement chosen. More precisely, letting $\delta = \mathrm{Tr}(P_x^A)$ denote the "cardinality" of the degeneracy of outcome P_x^A, the updated state depends on the subset $\mathcal{B}_x = \{P_{x_i}^A\}_{i=1}^\delta \subseteq M'$ spanning the degenerate subspace $\mathcal{H}_x = P_x^A \mathcal{H}$. The state update for noncomposite systems thus obeys

$$\omega_A^{\mathsf{vN}}\big(P_x^A, \rho_A, \mathcal{B}_x\big) = \sum_{i=1}^\delta \omega_A^{\mathsf{L}}\big(P_{x_i}^A, \rho_A\big), \tag{36}$$

where the right-hand side describes a proper mixture. As a result, von Neumann's postulate—despite being, like Lüders' original postulate (L), concerned only with the state-update of

the measured system—violates context-independence A2 and, therefore, does not give rise to a valid update rule [82].

Finally, we present a simple example of an update rule that violates local covariance A3. Assume that any measurement on a system causes the pre-measurement state to undergo a fixed, observer-independent, and nontrivial [83] unitary transformation $U \in \mathcal{U}(\mathcal{H})$,

$$\omega_{AB}^{\cup}\left(P_x^A, \, \rho_{AB}\right) = \mathrm{Tr}\left[P_x^A \mathrm{Tr}_B(\rho_{AB})\right] (U \otimes \mathbb{I}_B) \, \rho_{AB} \left(U^\dagger \otimes \mathbb{I}_B\right). \tag{37}$$

The rule $\omega_{AB}^{\cup}$ is not locally covariant when using local unitaries U_A that do not commute with U: different observers would provide inconsistent descriptions of the same measurement.

VI. HOW TO RECOVER THE PROJECTION POSTULATE

We have set up a framework of hypothetical update rules that do give rise to consistent foils of quantum theory, and we explored their properties. It is natural to reverse our approach by turning to the question of singling out the Lüders rule of quantum theory among the set of valid updates. The correlation-free version of quantum theory (Sec. V B)—as well as any suitable modification of von Neumann's projection (36)—shows that the Lüders rule cannot be recovered uniquely from assuming deterministic repeatability alone. Similarly, passive quantum theory (Sec. V C) provides another example of an update rule satisfying ideality.

Lüders presents two main objections to von Neumann's projection postulate [9]. First, he argues that the post-measurement state should depend only on the outcome and the initial state. This requirement is equivalent to context-independence A2. Second, he suggests that

> "The measurement of a highly degenerate quantity permits only relatively weak assertions regarding the considered ensemble. For that reason, the resulting change in state should likewise be small." ([84], p. 665)

In the limit of the trivial observable represented by the identity operator $\mathbb{I}$, a measurement reveals no information about the original state and it should have no effect on it. The quoted property describes a form of trade-off between the applied disturbance caused by a measurement and the information acquired from the outcome. The following operational property for noncomposite systems, which we term coherence, implies the desired behavior for measurements of the trivial observable.

Definition 4 (Coherence). An update rule for noncomposite systems ω_A satisfies coherence if the outcome probability distribution of any observable M is not affected by a prior measurement of a coarse-graining of M (i.e., an observable for which M is a refinement). Mathematically, for every $P_x^A, P_y^A \in \mathcal{P}(\mathcal{H}_A)$ such that $P_x^A P_y^A = P_y^A P_x^A = P_x^A$, and every state $\rho_A \in \bar{\mathcal{S}}(\mathcal{H}_A)$, we must have

$$\mathrm{Tr}\left[P_x^A \omega_A\left(P_y^A, \rho_A\right)\right] = \mathrm{Tr}\left(P_x^A \rho_A\right). \tag{38}$$

Consider an infinite ensemble prepared in an arbitrary quantum state, and examine the following two scenarios: (*i*) a measurement of M is performed on each element of the ensemble; (*ii*) a measurement of a coarse-graining of M fol-

lowed by a measurement of M are performed on each element of the ensemble. The property of coherence stipulates that the outcome probabilities observed when measuring M in scenarios (*i*) and (*ii*) must be identical. In fact, Eq. (38) ensures that, for all quantum states and outcomes P_x^A and P_y^A such that $P_x^A P_y^A = P_y^A P_x^A = P_x^A$, the relation

$$\mathrm{prob}\left(o_1 = P_y^A\right) \mathrm{prob}\left(o_2 = P_x^A | o_1 = P_y^A\right) = \mathrm{prob}\left(o_1 = P_x^A\right) \tag{39}$$

is satisfied, where o_1 and o_2 denote the first and second outcomes of two consecutive measurements, respectively. Here, P_x^A denotes the outcome of the M measurement, while P_y^A denotes the outcome of the coarse-graining. If $P_x^A \neq P_y^A$, then $\mathcal{H}_x \subset \mathcal{H}_y$ (where $\mathcal{H}_i = P_i^A \mathcal{H}$), meaning that P_y^A provides less information about the original state than P_x^A. As a result, the effect on the state cannot be arbitrarily strong but must "preserve coherence" in the subspace $\mathcal{H}_y$, in the sense that $\mathrm{prob}(o_2 = P_x^A | o_1 = P_y^A) = \mathrm{const.} \times \mathrm{prob}(P_x^A)$ for all $P_x^A \in \mathcal{P}(\mathcal{H}_y)$. In the limiting case of $P_y^A = \mathbb{I}_A$, one obtains $\omega_A(\mathbb{I}_A, \rho_A) \propto \rho_A$, in agreement with Lüders' remark.

In some form or another, the assumption of coherence has appeared already in the literature, both in the context of quantum theory [17,85,86] and of generalized probabilistic theories [33]. For measurements on noncomposite quantum systems, coherence entails both deterministic repeatability [33] and ideality.

Lemma 1. A coherent update rule for noncomposite systems ω_A satisfies both deterministic repeatability and ideality.

Proof. Consider an arbitrary projector P_x^A and a system residing in some state $\rho_A \in \mathcal{S}(\mathcal{H}_A)$ such that $\mathrm{Tr}(P_x^A \rho) \neq 0$. Letting $P_y^A = P_x^A$ in Eq. (38) leads to $\mathrm{Tr}[P_x^A \omega_A(P_x^A, \rho_A)] = \mathrm{Tr}(P_x^A \rho_A)$, which implies that, if repeated, the measurement will yield the same outcome with probability 1. In other words, $\omega_A(P_x^A, \rho_A)$ has support in the subspace $\mathcal{H}_x = P_x^A \mathcal{H}_A$ only, i.e., $\omega_A(P_x^A, \rho_A) \in \bar{\mathcal{S}}(\mathcal{H}_x)$. Therefore, ω_A satisfies deterministic repeatability.

Following the same argument, we know that $\omega_A(P_y^A, \rho_A)$ has support in $\mathcal{H}_y$ only. Let the initial state ρ_A also have support in $\mathcal{H}_y$ only, i.e., $\mathrm{Tr}(P_y^A \rho_A) = 1$, and consider an informationally complete set of observables on $\mathcal{H}_y$. If $\omega_A(P_y^A, \rho_A) \neq \rho_A$, then for at least one of the outcomes of an observable in the set, say $P_x^A \in \mathcal{P}(\mathcal{H}_y)$ (which must satisfy $P_x^A P_y^A = P_x^A$), we have $\mathrm{Tr}[P_x^A \omega_A(P_y^A, \rho_A)] \neq \mathrm{Tr}(P_x^A \rho_A)$, in contradiction with (38). Therefore, the update rule does *not* disturb the states of measured systems when the outcome is certain, i.e., ω_A satisfies ideality. ∎

Indeed, coherence alone is sufficient to derive the single-system Lüders rule ω_A^{L}, i.e., the original projection postulate (L). This argument was first presented in 1966 [17] and similar versions of it have since been rediscovered several times; see the appendix for a summary. Below, we present a simple proof of this result.

Theorem 1. For noncomposite systems, the Lüders rule ω_A^{L} is the only coherent update rule.

Proof. Substituting ω_A^{L} in (38) leads to

$$\mathrm{Tr}\left(P_x^A P_y^A \rho_A P_y^A\right) = \mathrm{Tr}\left(P_x^A \rho_A\right), \tag{40}$$

which holds for all ρ_A and $P_x^A P_y^A = P_y^A P_x^A = P_x^A$. Hence, the Lüders projection for single systems is coherent.

To show the converse, suppose ω_A satisfies coherence but $\omega_A \neq \omega_A^{\mathsf{L}}$. Then, there must exist some outcome $P_y^A \in \mathcal{P}(\mathcal{H}_A)$ and state $\rho_A \in \mathcal{S}(\mathcal{H}_A)$ such that $\omega_A(P_y^A, \rho_A) \neq \omega_A^{\mathsf{L}}(P_y^A, \rho_A)$. Since both rules are coherent, hence deterministically repeatable by Lemma 1, both updated states must lie within the same subspace defined by P_y^A, i.e., $\omega_A(P_y^A, \rho_A), \omega_A^{\mathsf{L}}(P_y^A, \rho_A) \in \bar{\mathcal{S}}(\mathcal{H}_y)$. Consider a set of observables of $\mathcal{H}_y$ which allow one to reconstruct any quantum state in $\bar{\mathcal{S}}(\mathcal{H}_y)$ tomographically. The states $\omega_A(P_y^A, \rho_A)$ and $\omega_A^{\mathsf{L}}(P_y^A, \rho_A)$ must yield different probabilities for at least one outcome $P_x^A \in \mathcal{P}(\mathcal{H}_y)$ in the set,

$$\mathrm{Tr}\big[P_x^A\,\omega_A\big(P_y^A, \rho_A\big)\big] \neq \mathrm{Tr}\big[P_x^A\,\omega_A^{\mathsf{L}}\big(P_y^A, \rho_A\big)\big]. \qquad (41)$$

Substituting $\omega_A^{\mathsf{L}}(P_y^A, \rho_A) = P_y^A \rho_A P_y^A$ and using the fact that $P_x^A P_y^A = P_y^A P_x^A = P_x^A$, we obtain

$$\mathrm{Tr}\big[P_x^A\,\omega_A\big(P_y^A, \rho_A\big)\big] \neq \mathrm{Tr}\big(P_x^A \rho_A\big), \qquad (42)$$

which contradicts the assumption of coherence. We thus conclude that $\omega_A = \omega_A^{\mathsf{L}}$. ∎

The information-disturbance trade-off principle, as encapsulated by the property of coherence, completely characterizes the update rule for quantum measurements on noncomposite systems. However, there do exist GUTs with different update rules for multipartite systems that reduce to the single-system Lüders rule. The correlation-free theory of Sec. V B defined by the update rule ω^{locL} is an explicit example. Straightforward generalizations of coherence to composite systems do not seem to be sufficiently strong to generalize the proof of Theorem 1. For example, the assumption that a sequence of measurements of P_y^A and P_x^A (see Def. 4) on subsystem $\mathcal{H}_A$ leads to the same post-measurement state for $\mathcal{H}_{AB}$ as a single measurement of P_x^A fails to rule out correlation-free quantum theory.

Additional assumptions are necessary to fully recover the Lüders rule ω_{AB}^{L} for composite systems. One such assumption is composition compatibility (Def. 3)—see Theorem 2 below. This final step in deriving the projection postulate is often overlooked [17,21,85], as the extension of ω_A^{L} to local measurements on composite systems is typically assumed implicitly, despite not being stated in or implied by the standard postulates.

Theorem 2. The Lüders rule ω^{L} is the unique generalized state-update rule that satisfies both coherence and composition compatibility.

Proof. Theorem 1 derives the Lüders projection ω_A^{L} for noncomposite systems from coherence alone. Then, the quantum-mechanical extension to composite systems ω_{AB}^{L} is fixed by Eq. (16) defining composition compatibility, as already discussed in Sec. V A. ∎

This derivation of the Lüders rule does *not* rely on assuming linearity of the update rule. There may, of course, be other physically appealing operational principles ruling out all update rules but the Lüders rule. For instance, it seems promising to explore the implications of nonlocality. Could quantum theory be singled out by imposing specific constraints on the degree of nonlocality exhibited by generalized state-update theories?

VII. CONCLUSIONS

A. Summary

We have introduced a framework that allows us to systematically investigate modifications of the projection postulate. Theories with generalized state-update rules (GURs) differ from quantum theory only in assigning other post-measurement states.

Upon formalizing the concept of state-update rules, we notice that the quantum-mechanical Lüders rule is typically presented in an operationally incomplete way. Spelling out the projection postulate (1) for single systems does not entail a unique update rule when local measurements are performed on composite systems. In standard quantum theory, the necessary extension from single to composite systems is normally assumed implicitly to have a specific form. This observation is a simple example of the subtleties in the definition of update rules. The specific update rule used in composite quantum systems is, of course, well motivated by experimentally verified correlations between measurement outcomes obtained for entangled states.

Any valid state-update rule must satisfy six core requirements, which include the uniqueness of post-measurement states, no-signaling, and consistency for measurements carried out by observers with access to subsystems only. We have shown that quantum theory and passive quantum theory [79] satisfy the requirements. Three new examples of consistent generalized state-update rules have been defined, each one leading to a toy theory structurally different from quantum theory.

Correlation-free quantum theory turns out to be locally indistinguishable from standard quantum theory since the update rule for single systems coincides with the projection postulate. However, the proposed update rule does not extend in the standard way to composite systems. The update rule is "locally Lüders" but local measurements performed on entangled states do not create the correlations known from quantum theory.

Depolarizing measurements are described by an update rule that sends any pre-measurement state to the maximally mixed state, irrespective of the measurement outcome. The update effectively acts as a completely depolarizing channel, flattening the probability distribution for subsequent outcomes of measurements. No information about the pre-measurement state is retained.

Probability-amplifying projective rules lead to post-measurement states that assign a larger probability to the measurement outcome than the pre-measurement state did. The standard projective update rule is the extreme case of this rule: the outcome of the second measurement is necessarily the same as that of the first one, implying deterministic repeatability. The Lüders rule for single systems can be approximated arbitrarily well by probability-amplifying projective state updates.

To avoid signaling in theories with depolarizing or probability-amplifying update rules, careful adjustments are needed when extending them to composite systems. The ways in which state-update rules may violate core requirements have been illustrated by a number of (ultimately invalid) GURs.

Having established a variety of acceptable update rules, it is natural to look for properties that single out the Lüders rule. We have shown that the property of coherence uniquely implies the Lüders rule for noncomposite systems. This result agrees with earlier ones based on similar concepts describing a form of information-disturbance trade-off. However, our systematic approach to GURs highlights the fact that many derivations of the Lüders rule may be considered incomplete by not explicitly addressing composite systems. We find that coherence on its own is insufficient to single out the projection postulate when considering composite systems; equally, composition compatibility alone is not strong enough to do so. However, combining it with coherence implies the projective Lüders rule for composite systems, too (Theorem 2).

B. Discussion and outlook

The framework of generalized state-update rules we present affords us with a bird's-eye view of the projection postulate and physically reasonable modifications thereof, their basic features, and their implications. It is useful to think of update rules as generalizations of quantum instruments that are neither linear nor completely positive, all the while still providing consistent descriptions of measurements on (hypothetical) quantum systems.

Nonlinear update rules may enable observers to distinguish between proper and improper mixtures. In the corresponding foil theories, density operators will *not* provide a complete description of the hypothetical physical systems. Importantly, complete positivity is not strong enough to ensure that such measurement-induced transformations assign post-measurement states to composite systems consistently. In the words of Ref. [78], complete positivity is physically unfitting for this purpose.

Reconstructions of quantum theory that engage—either directly or indirectly—with post-measurement states usually assume measurement-induced state transformations to be linear in the pre-measurement state [25,33,34]. This assumption, often left implicit [26,28], *a priori* excludes measurement behaviors such as the trivial noncollapsing rule of "passive quantum theory" that are, in fact, operationally well-defined and consistent with fundamental principles such as no-signaling (see Sec. IV B). Preliminary results based on the framework of GURs and earlier work (i.e., [24,87]) suggest that the linearity of update rules could be derived from natural operational assumptions, mirroring efforts to establish the linearity of the quantum time evolution [56,88].

It seems promising to investigate the role of nonlocality from the perspective of theories with generalized update rules. Do other update rules exist that are capable of replicating the correlations found in quantum theory? What is more, can they lead to superquantum correlations?

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: DERIVATIONS OF THE LÜDERS RULE–A BRIEF SURVEY

Derivations of the Lüders rule have a long history, with similar arguments being made over the years. We briefly review derivations known to us, highlighting the assumptions made. A key observation is that most of the justifications focus on noncomposite systems, either ignoring measurements on composite systems or making implicit assumptions about them.

In 1966, Bell and Nauenberg showed that the quantum-mechanical state update can be obtained by assuming that the outcome probability distribution of a common refinement for commuting observables M_1 and M_2 equals the joint probability distribution obtained by measuring M_1 and M_2 sequentially, in any order [17]. This assumption is mathematically equivalent to that of coherence (Def. 4) employed in the proof of Theorem 1 in Sec. VI. To the best of our knowledge, the only reference to this early operational derivation of the Lüders rule in noncomposite systems is by Herbut [85]. Over the last 40 years, the argument by Bell and Nauenberg has been rediscovered at least three times, using the concept of coherence.

In 1983, Cassinelli and Zanghì formulated quantum theory in terms of a generalized probability space and showed how to recover the single-system Lüders rule in this context. To do so, they describe the state-update following measurements in terms of a generalized conditional probability defined on the generalized probability space—i.e., a probability measure on $\mathcal{P}(\mathcal{H})$ compatible with Born's rule and satisfying "coherence" [89].

In 2009, Khrennikov showed that the Lüders rule for single systems follows from von Neumann's projection postulate for nondegenerate observables if coherence is assumed [86]. Claiming that "this important observation remained unnoticed for the past 70 years," the author appears to be unaware of Bell and Nauenberg's work.

In 2014, Kleinmann developed a formalism for sequential measurements in a broad class of generalized probabilistic theories (GPTs), introducing a generalization of the Lüders rule based on coherence [33]. A simple derivation of the noncomposite Lüders rule based on coherence is also given, without reference to earlier work. When applied to quantum theory, i.e., in a Hilbert-space setting for states, Kleinmann's approach reproduces convex-linear update rules for single systems. This restriction is due to assuming preparation indistinguishability throughout (see Sec. V A), thereby excluding operationally valid alternatives such as the passive update rule from the outset.

Herbut actually presents two distinct derivations applying to measurements on noncomposite systems in a paper from 1969 [18]. In an information-theoretic approach, he postulates (i) deterministic repeatability (see Sec. II A) and that (ii) the post-measurement state minimizes its distance from the pre-measurement state. The Lüders rule is then shown to map the pre-measurement state to the least distinguish-

able state compatible with outcome repeatability. This result remained largely unnoticed, while anticipating later arguments deriving the state-update from minimization principles, including those by Marchand [90], Dieks [91], and Hadjisavvas [92]. More recently, the single-system Lüders rule was obtained from minimizing nonsymmetric quantum information distances [19,20], thereby establishing a mathematical connection to the Bayes-Laplace rule. Ozawa [24,93] derives the quantum instrument formalism describing state-updates from Bayes' rule by assuming the standard joint probability formula for local quantum measurements. In a sense, the author adopts a stronger interpretation of postulate (B) (cf. Sec. III A) than the one used explicitly by us, and implicitly by other authors [28,31,32,76], where (L) is modified. The stronger interpretation of Born's rule is assumed to govern individual measurement outcomes as well as correlations between local measurements carried out by different experimenters with access to parts of a composite system only. Essentially, it subsumes the property of local tomography (cf. Sec. V A).

Herbut's second derivation in Ref. [18] takes an operational approach. In addition to deterministic repeatability, it is assumed that performing an unconditional measurement of observable M does not affect the expectation values of any observables commuting with M. The argument focuses on measurements on single systems. However, Herbut also briefly considers composite systems, discussing in particular how local Lüders measurements modify joint states without enabling signaling. When doing so, he implicitly adopts the standard extension to composite systems (see Sec. V A). What is more, Herbut takes convex-linearity over states to be a fundamental requirement for state updates, without justifying this assumption.

Herbut revisits the question in a 1974 paper [94]. Here, the Lüders rule for single systems is obtained by combining the assumption of deterministic repeatability with the "preservation of sharp values." An observable M has a sharp value x if $\mathrm{Tr}(P_x^A \rho) = 1$ holds for some $P_x \in M$. This value is preserved if, given an initial state ρ with a sharp M value x, a measurement of any observable commuting with M leaves unchanged the value x, regardless of the observed outcome. When proving this result for improper mixtures, Herbut implicitly adopts "composition compatibility" to describe local measurements.

Martinez [21] offers an early account of known derivations of the Lüders rule, including Herbut's work but not Bell and Nauenberg's contribution. The paper also demonstrates that the converse of Lüders theorem [9,15] implies the quantum mechanical collapse for noncomposite systems: if, for any pair of commuting observables M_1 and M_2 and any initial state, we assume that the sequence of measurements $M_1 \prec M_2 \prec M_1$

always produces identical outcomes for M_1 regardless of the outcome of the M_2 measurement, then the post-measurement state must necessarily be given by the Lüders rule. Martinez further examines a quantum-logical derivation employing the "Sasaki hook" presented in Ref. [95]. The extension of the update rule to local measurements on subsystems is not considered.

The proof that establishes the Lüders instrument as the unique ideal quantum instrument for sharp discrete measurements dates back to Davies [96], as recognized in Refs. [15,97]. The notion of a quantum instrument incorporates two key assumptions: convex-linearity of the single-system state-update rule, and the standard linear extension to composite systems enabled by the complete positivity of ω_A [see Eq. (18) in Sec. V A]. This argument for isolating the Lüders collapse for single systems has been rediscovered occasionally (see Ref. [98], for example).

In 2019, Masanes *et al.* advanced the argument that both the Born rule and the post-measurement state rule can be derived from the other postulates of quantum theory when supplemented by suitable operational constraints [25]. In doing so, they consider alternative rules for computing outcome probabilities, explicitly discussing both composite systems and local measurements. Having established the uniqueness of the Born rule within their framework, they assert that quantum instruments constitute the only consistent description of state updates. This conclusion, along with their broader framework, has attracted some criticism [26–30]. The argument for quantum instruments depends on an assumption regarding sequential measurements that, as already shown in [87], is mathematically equivalent to requiring convex-linearity of state updates.

Conceptually, Masanes *et al.*'s approach mirrors Kleinmann's framework [33] in that it is limited *a priori* to update rules satisfying preparation indistinguishability. This restriction explains why they adopt the completely positive extension (18) for local measurements on composite systems: any other extension would necessarily violate their assumption of preparation indistinguishability.

In retrospect, the Lüders rule is seen to have been derived on the basis of operational, information-theoretic, quantum-logical, and epistemically motivated assumptions. Most of the derivations are limited to single-system update rules, and their extension to local measurements on subsystems is either ignored or tacitly assumed. The extension of state-update rules is, however, not trivial, since the single-system rule alone does *not* determine it. Additional structural assumptions, such as preparation indistinguishability or composition compatibility (cf. Theorem 2), are required to exclude alternative update rules in the multipartite case.

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[1] In view of the historic developments, it would be appropriate to speak of the "Dirac–von Neumann-Lüders rule" [23], but we will continue using "Lüders rule" for simplicity. However, the projection postulate as formulated by Dirac [7] and Lüders

[9] differs from von Neumann's version [8] in key aspects; see Sec. V F.

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[63] A subnormalized state is a non-negative operator on a Hilbert space with trace less than or equal to 1, i.e. $\rho \in \bar{\mathcal{S}}(\mathcal{H}) = \{\lambda\rho : \lambda \in [0,1], \rho \in \mathcal{S}(\mathcal{H})\}$.

[64] Postulate (B) is understood to assign outcome probabilities to each individual measurement. Consequently, Born's rule on its own does not account for correlations between outcomes of multiple measurements. Instead, the update rule is also responsible for the joint probabilities for the outcomes of measurements carried out sequentially, or by distinct parties in composite systems. This subtle point proves significant when examining modifications of the Lüders rule (L); see the appendix for further discussion.

[65] It is only possible to conclude that the pre-measurement state ρ_{AB} assigns a non-zero probability to obtaining the outcome P_x^A.

[66] The update rule also seems to be incomplete if one were to use a different formulation of (L), stating that, after a measurement, the system resides in an eigenstate of the measured observable.

[67] Recall that, mathematically, the state that correctly reproduces the experimental findings is obtained in the following way: Write the initial state corresponding to ρ_{AB} as a superposition of product terms, using the eigenstates of the operator M_A for the first factor. Then, identify the term with the label x, the value of the observed measurement outcome in a given run. The second factor in this expression characterises the correct post-measurement state for subsystem B.

[68] Anticipating the discussion of Sec. IV C on the role of proper and improper mixed states, we assume for now that the density operators we consider represent either pure states or improper mixed states, i.e., those arising from entanglement with other systems.

[69] A different formula is generally required to describe the update of proper mixed states, i.e., those due to the incomplete knowledge about the preparation of a system (cf. Sec. IV C).

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[72] Quantum operations are linear, completely positive and trace non-increasing maps defined on the space $\mathcal{L}(\mathcal{H})$ of bounded operators acting on $\mathcal{H}$. Quantum instruments are usually defined as mappings from an outcome space (X, Σ) to the set of quantum operations. However, since we will only deal with discrete observables, an instrument is completely determined by the finite set of operations $\{\mathcal{I}_x\}_x$.

[73] If the outcome P_x^A is never observed, $\text{Tr}(P_x^A \rho_i) = 0$, then $\omega_A(P_x^A, \rho_i) = O$, where O is the zero operator, and the experimenter learns that the system was *not* prepared in the state ρ_i.

[74] If ρ_A denotes either a pure or an improper mixed state, we can associate the trivial *Gemenge* $\mathcal{G} = \{(1, \rho_A)\}$, describing a system prepared in the state ρ_A with probability 1.

[75] The possibility of noncollapsing measurements was listed by von Neumann [8] as early as 1932 as one of three possible reactions of a physical system to a measurement.

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[80] Specifically, it belongs to the class of trivial instruments.

[81] Setting $\lambda = 0$ recovers the locally-Lüders rule $\omega_{AB}^{\text{locL}}$ of Eq. (19).

[82] One could, however, modify (36) into a valid update rule by, for example, fixing a preferred set $\mathcal{B}_x$ for each experiment yielding the outcome P_x^A, and subsequently extending the map to composite systems via complete positivity.

[83] If $U = \mathbb{I}$, we recover the passive measurements of Sec. V C, which are consistent with A3.

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