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Full universal enveloping vertex algebras from factorisation

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Abstract

We give a systematic construction of the symmetries, or observables in the vacuum sector, of a full conformal field theory on an arbitrary real two-dimensional conformal manifold Σ . Specifically, we construct a prefactorisation algebra on Σ which locally encodes the full (non-chiral) version $\mathbb{F}^{\mathfrak{a},\alpha} = \mathbb{V}^{\mathfrak{a},\alpha} \otimes \bar{\mathbb{V}}^{\mathfrak{a},\alpha}$ of a universal enveloping vertex algebra $\mathbb{V}^{\mathfrak{a},\alpha}$, where \mathfrak{a} is a finite-dimensional vector space labelling the set of fields and α is a 2-cocycle controlling the central extension of their Lie brackets. Our construction provides a unified treatment of the three canonical examples of (full) universal enveloping vertex algebras – Kac-Moody, Virasoro and $\beta\gamma$ system – using the notion of unital local Lie algebra. By using the coordinate-invariant nature of prefactorisation algebras we derive an analogue of Huang’s change of variable formula for full vertex algebras. We give a careful treatment of (both Euclidean and Lorentzian) reality conditions in this formalism which allows us, in the Kac-Moody and Virasoro cases, to construct a Hermitian sesquilinear form on these full vertex algebras by using the factorisation product to the global observables on S^2 . We also give an explicit derivation of Borcherds type identities and a construction of the operator formalism for $\mathbb{F}^{\mathfrak{a},\alpha}$.

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1 Introduction

Two-dimensional conformal field theory [BPZ] has been extensively studied by physicists and mathematicians alike over the past four decades. Its original applications in physics ranged from the study of critical phenomena in two-dimensional statistical systems to string theory, see for instance [DMS], but it has also found far-reaching applications in mathematics, from the monstrous moonshine [Bor] to the geometric Langlands correspondence [Fre1, Fre2].

An important distinction to be made is between *chiral* and *full* two-dimensional conformal field theories. Indeed, in both the Euclidean and Lorentzian settings, the infinite dimensional symmetry algebra of a two-dimensional conformal field theory involves a direct sum of two copies of the Virasoro algebra, see e.g. [Sch]. The collection of fields which transform trivially with respect to either of these two copies generates the so-called *chiral* and *anti-chiral* sectors of the conformal field theory, respectively. By contrast, the *full* conformal field theory deals with all of the fields, transforming under both copies of the Virasoro algebra.

The main purpose of this paper is to initiate the construction and study of full conformal field theories within the modern framework of prefactorisation algebras [CG1, CG2] recently introduced by Costello and Gwilliam. We focus on describing the full/non-chiral versions of a broad class of vertex algebras, known as universal enveloping vertex algebras [Pri], which are defined as Verma modules over infinite-dimensional Lie algebras, including centrally extended loop algebras, the Virasoro algebra and the infinite-dimensional Weyl algebra. These full vertex algebras encode the vacuum sectors, or symmetry algebras, of many rational and logarithmic conformal field theories, including the Wess-Zumino-Witten model [WZ, Wit1, Wit2, Nov] and its more recent generalisations associated to non-semisimple Lie algebras [BR, Que], minimal models and the $\beta\gamma$ system; see [DMS] for an extensive review.

The prefactorisation algebra perspective that we follow has several important advantages. It provides a coordinate-invariant description of (full) vertex algebras; this will, in particular, allow us to derive a generalisation of Huang's change of variable formula [Hua] for full vertex algebras which leads to a simple formulation of the operator formalism in full conformal field theories. It also provides an elegant geometric formulation of (full) vertex algebras allowing us to work on an arbitrary two-dimensional conformal manifold; for instance, by paying close attention to reality conditions we will construct a canonical Hermitian sesquilinear form on the full vertex algebra using the factorisation product to the global observables on S^2 .

In the remainder of the introduction we begin in §1.1 by providing a general overview of different mathematical approaches to conformal field theory and their interconnections. In §1.2 we explain the main idea for realising the notion of full vertex algebras in the framework of prefactorisation algebras, based on an observation from [FS]. In §1.3 we describe the class of vertex algebras whose non-chiral versions we will formulate using prefactorisation algebras, and in §1.4 we give an outline the content of the paper to help guide the reader.

1.1 Approaches to conformal field theory

There are various mathematical formulations of both chiral and full two-dimensional conformal field theory, each of which is based on a different axiomatic framework for quantum field theory. These include the Gårding–Wightman axioms [GåWi, SW] or the Osterwalder–Schrader axioms [OS1, OS2] related by analytic continuation, the Haag–Kastler axioms [HaKa] or the closely related Brunetti–Fredenhagen–Verch axioms [BFV] in the locally covariant case, and the Atiyah–Segal–Witten axioms [Seg, Ati, Wi3] for functorial quantum field theory.

1.1.1 (Full) vertex operator algebras

A vertex (operator) algebra is a vector space V equipped with a state-field correspondence, also known as a field map or a vertex operator map, $Y : V \otimes V \rightarrow V((\zeta))$, valued in formal Laurent series in the formal variable ζ with coefficients in V , satisfying certain axioms [Bor, FHL, Hua, Kac2, FB, LL]. The vertex operator map already encodes an ‘algebraic’ version of the Wightman fields [SW] for a chiral two-dimensional conformal field theory, see [Kac2, §1]. If, moreover, V is *unitary* in the sense of [DL], then under certain extra mild assumptions one can use the vertex operator map to associate [CCHW, RTT] to each state $a \in V$ an operator-valued distribution on S^1 , i.e. a Wightman field on S^1 , acting on a dense open subset of the Hilbert space completion \mathcal{H}_V of V with respect to its positive-definite sesquilinear form. In the very recent work [CRTT] a rigorous connection is also established in the broader non-unitary context and without the need for any additional technical assumptions.

Vertex operator algebras only encode *chiral* two-dimensional conformal field theories, but a number of closely related extensions describing full two-dimensional conformal field theories have also been developed: a non-chiral version of vertex algebras was first introduced in [KO] and further studied in [Ros] under the name *OPE-algebras*, the notion of a *full field algebra* was defined in [HuKo, Kon], that of a *full vertex algebra* in [Mor1, Mor2] and that of a *non-chiral vertex operator algebra* in [SS]. All of these non-chiral versions of vertex (operator) algebras are closely related and we refer the reader to the original articles for the various comparisons. A notion of unitarity was also recently introduced in [AMT] to relate the algebraic approach of [Mor1, Mor2] with an analytic one based on the Osterwalder–Schrader axioms.

1.1.2 Conformal nets

A conformal net [FröG, FJ, KL1], by contrast, encodes the local observables of a chiral two-dimensional conformal field theory as a conformal Haag–Kastler net [HaKa] on S^1 , i.e. as a functor from the category of open intervals in S^1 to that of von Neumann subalgebras of $B(\mathcal{H})$ for a fixed Hilbert space \mathcal{H} . There is also a ‘coordinate-free’ formulation of conformal nets [BDH] built instead on the axiomatic framework of Brunetti–Fredenhagen–Verch [BFV], see also [FV, BDRY, Rej, BSW], which rather than working on a fixed spacetime (S^1 in this case) treats all contractible compact 1-dimensional manifolds on an equal footing.

Conformal nets on S^1 similarly only encode *chiral* two-dimensional conformal field theories, whereas full two-dimensional conformal field theories are encoded instead in terms of conformal Haag–Kastler nets on two-dimensional Minkowski space $\mathbb{R}^{1,1}$ or the Einstein cylinder \mathcal{E} . The relationship between a full two-dimensional conformal field theory and its chiral and anti-chiral sectors was studied in the conformal nets setting in [Reh1, KL2, BKL], see also [Reh2] for a review. This was also recently studied from a purely categorical perspective in [BGS] within a general operadic reformulation [BSW] of the Brunetti–Fredenhagen–Verch framework.

1.1.3 Segal's functorial approach

In the chiral setting, unitary vertex operator algebras and conformal nets are intimately related [CKLW, Gui], see also [CGH] in the supersymmetric case. A more geometric perspective on this relationship was also obtained in [Ten1, Ten2] based on ideas from [Hen1], using Segal's functorial definition of a chiral conformal field theory [Seg] to interpolate between the two. Although a similar comparison in the non-chiral case does not yet exist, note that Segal's axioms [Seg] can encode both chiral and full two-dimensional conformal field theories.

1.1.4 Factorisation algebras

In the present work we will, instead, be working in the modern formulation of quantum field theory based on factorisation algebras due to Costello–Gwilliam [CG1, CG2]. It is known to be closely related, at least on a heuristic level [CG1, §1.4], to all of the frameworks mentioned above. The precise relationship between factorisation algebras in the Lorentzian setting and the Brunetti–Fredenhagen–Verch framework was clarified in the recent series of works [GR1, BPS, BMS, GR2]. In the setting of chiral two-dimensional conformal field theories, conformal nets were also shown in [Hen2] to be a particular instance of factorisation algebras on S^1 .

The notion of (pre)factorisation algebra à la Costello–Gwilliam encapsulates, in a simple and general axiomatic framework, the algebra of observables in general quantum field theories [CG1, CG2]. Its development was originally inspired by the theory of factorisation algebras à la Beilinson–Drinfel'd [BD2], see also [FraG] and the book [FB] for a nice review, which provides a sheaf-theoretic coordinate-free formulation of vertex (operator) algebras, and hence of chiral two-dimensional conformal field theory. Indeed, the axioms of a vertex algebra also admit a very natural and elegant reformulation in terms of *holomorphic* prefactorisation algebras à la Costello–Gwilliam on \mathbb{C} , see [CG1, §5] and [Bru1]. There is also a close connection with the theory of geometric vertex (operator) algebra first introduced in [Hua], see [Bru2]. Note that the relation between the two different notions of factorisation algebras of Beilinson–Drinfel'd and Costello–Gwilliam remains to be elucidated, see however [HeKa] for the comparison in the *locally constant* setting corresponding to topological field theories [Lur]. From now on, we shall always consider the notion of (pre)factorisation algebra in the sense of Costello–Gwilliam and we will refer to these simply as (pre)factorisation algebras.

Even though the origins of factorisation algebras are deeply rooted in the theory of vertex algebras, hence in chiral two-dimensional conformal field theory, their scope of applicability is much broader, extending to general quantum field theories in arbitrary space-time dimensions. This is very reminiscent of the axiomatic framework for quantum field theory in curved space-time of Hollands–Wald [HW1, HW2, HW3], see also [HO, HH], in which the *operator product expansion* takes centre stage, exactly as in a vertex (operator) algebra V where it is given by the state-field correspondence $Y : V \otimes V \rightarrow V((\zeta))$. Indeed, factorisation algebras can encode operator product expansions of local observables in a general quantum field theory [CG2, §10].

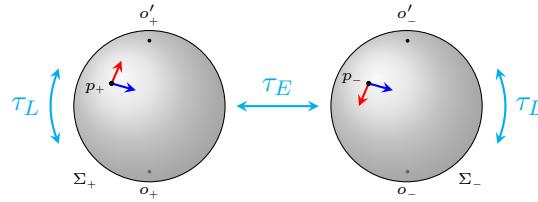
1.2 The orientation double construction

Let Σ be a real two-dimensional conformal manifold. The decomposition of a full conformal field theory on Σ into its chiral and anti-chiral sectors is often motivated from a dynamical point of view by the fact that the space of solutions of the classical field theory factorises into left and right moving solutions in the Lorentzian setting, or holomorphic and anti-holomorphic solutions in the Euclidean setting. We will instead adopt the more geometric perspective, as first advocated in [FS], which relates the chiral and anti-chiral sectors of a full conformal field theory to the two possible choices of orientations one can locally make on Σ . For concreteness we will work with a Euclidean conformal manifold Σ , in which case the choice an orientation

corresponds to a choice of complex structure. However, we will also deal with the Lorentzian case in §4 by working in the Hamiltonian formalism and restricting fields to a Cauchy surface.

The orientation bundle $\text{Or}(\Sigma) \rightarrow \Sigma$ is the principal \mathbb{Z}_2 -bundle whose fibre over any $p \in \Sigma$ consists of two points corresponding to the two possible orientations at p . The *double* of Σ , denoted $\pi : \widehat{\Sigma} \rightarrow \Sigma$, is defined as the quotient of $\text{Or}(\Sigma)$ by the relation which identifies the pair of points above any $p \in \partial\Sigma$. By construction, $\widehat{\Sigma}$ is a 2-dimensional conformal manifold without boundary which is naturally oriented and thus carries a canonical complex structure. Moreover, $\widehat{\Sigma}$ comes naturally equipped with an orientation reversing involution $\tau_E : \widehat{\Sigma} \rightarrow \widehat{\Sigma}$, such that $\pi\tau_E = \pi$, induced by the involution $\text{Or}(\Sigma) \rightarrow \text{Or}(\Sigma)$ exchanging the pair of points in each \mathbb{Z}_2 fibre. In particular, the fixed point set of τ_E corresponds to the boundary $\partial\Sigma \hookrightarrow \widehat{\Sigma}$.

Following [FS], we can describe a full conformal field theory on Σ using a *chiral* conformal field theory on the oriented cover $\widehat{\Sigma}$. For instance, the double of the 2-sphere $\Sigma = S^2$ is given by the disjoint union $\widehat{\Sigma} = \Sigma_+ \sqcup \Sigma_-$ of two copies of Σ equipped with opposite orientations and the chiral and anti-chiral sectors of the full conformal field theory then correspond to Σ_+ and Σ_- , respectively. We will later specialise to this particular case. The subscript ‘ E ’ on the orientation reversing involution $\tau_E : \widehat{\Sigma} \rightarrow \widehat{\Sigma}$ refers to the fact that it then describes reality conditions in Euclidean signature. In this case we shall also consider a different orientation reversing involution $\tau_L : \widehat{\Sigma} \rightarrow \widehat{\Sigma}$ which acts individually on each of the two spheres Σ_\pm , by reflection through the equatorial plane, rather than exchanging them as τ_E does:



We use the subscript ‘ L ’ here since this will correspond to reality conditions in the *Lorentzian* setting (also in the Euclidean setting after Wick rotation from the Lorentzian setting), which send the chiral and anti-chiral fields to themselves rather than exchanging them.

1.3 Universal enveloping vertex algebras

A general class of vertex algebras can be constructed using a vertex-algebraic analogue of the universal enveloping algebra construction for Lie algebras, starting from the simpler notion of a vertex Lie algebra, also known as a Lie conformal algebra, introduced in [Pri]; see also [Kac2, FB]. Specifically, a *vertex Lie algebra* \mathcal{L} is determined by a vector space of fields $\mathbf{a}(z) = \sum_{n \in \mathbb{Z}} \mathbf{a}_{(n)} z^{-n-1}$ labelled by $\mathbf{a} \in \mathfrak{a}$ for some finite-dimensional vector space \mathfrak{a} , which together with all of their derivatives $\partial_z^k \mathbf{a}(z)$ for $k \in \mathbb{Z}_{\geq 1}$ is closed under taking the singular part of their operator product expansions. Then the associated *universal enveloping vertex algebra* [Pri] is the vertex algebra $\mathbb{V}(\mathcal{L})$ that is freely generated by the fields of the vertex Lie algebra \mathcal{L} under taking derivatives and normal ordered products, modulo the relations encoded in \mathcal{L} . More precisely, a general element of the vector space $\mathbb{V}(\mathcal{L})$ is given by a linear combination of states of the form $\mathbf{a}_{(-m_r)}^r \dots \mathbf{a}_{(-m_1)}^1 |0\rangle$ where $\mathbf{a}^i \in \mathfrak{a}$ and $m_i \in \mathbb{Z}_{\geq 1}$ for $i \in \{1, \dots, r\}$, with associated fields

$$Y(\mathbf{a}_{(-m_r)}^r \dots \mathbf{a}_{(-m_1)}^1 |0\rangle, z) = : \frac{1}{(m_r - 1)!} \partial_z^{m_r - 1} \mathbf{a}^r(z) \dots \frac{1}{(m_1 - 1)!} \partial_z^{m_1 - 1} \mathbf{a}^1(z) :. \quad (1.1)$$

The fact that this formula endows $\mathbb{V}(\mathcal{L})$ with the structure of a vertex algebra is an immediate application of the reconstruction theorem, see for instance [Kac2, Theorem 4.5] or [FB, §2.3.11]. The proof of the latter, however, relies on various technical results including Dong’s lemma.

We will see that the formula (1.1) (and in particular its counterpart for full vertex algebras) has a very simple and elegant geometric origin from the prefactorisation algebra perspective on (full) vertex algebras. Therefore, rather than invoke the reconstruction theorem (for which there does not appear to be a full vertex algebra analogue in the literature) to construct (full) vertex algebras from holomorphic prefactorisation algebras, as is done for instance in [CG1, §5] in the chiral setting, we will show directly that the formula (1.1) emerges very naturally from the structure of the prefactorisation algebra itself.

1.3.1 Main examples

We will consider the following three canonical examples of vertex Lie algebras:

Kac-Moody: For any finite-dimensional complex Lie algebra \mathfrak{g} equipped with a symmetric invariant bilinear form $\kappa = \langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{C}$, the Kac-Moody vertex Lie algebra \mathcal{L} is generated by fields $\mathbf{X}(z)$ for each $\mathbf{X} \in \mathfrak{g}$ with operator product expansions given, for any $\mathbf{X}, \mathbf{Y} \in \mathfrak{g}$, by

$$\mathbf{X}(z)\mathbf{Y}(w) \sim \frac{[\mathbf{X}, \mathbf{Y}](w)}{z-w} + \frac{\langle \mathbf{X}, \mathbf{Y} \rangle}{(z-w)^2}.$$

When \mathfrak{g} is reductive the associated universal enveloping vertex algebra $\mathbb{V}(\mathcal{L})$ is known as the affine (Kac-Moody) vertex algebra $\mathbb{V}_\kappa(\mathfrak{g})$, see for instance [LL, Kac2, FB]. This includes the Heisenberg vertex algebra as a special case for the trivial Lie algebra \mathbb{C} with invariant bilinear form given by multiplication. Note, however, that the Lie algebra \mathfrak{g} need not be reductive. For instance, one may take \mathfrak{g} to be a Takiff algebra [Tak], i.e. $\mathfrak{g} = \mathcal{T}^n \mathfrak{f} := \mathfrak{f}[t]/t^{n+1}\mathfrak{f}[t]$ for some reductive Lie algebra \mathfrak{f} and $n \in \mathbb{Z}_{\geq 0}$. This can be equipped with a non-degenerate symmetric invariant bilinear form [Que] and the associated universal enveloping vertex algebra $\mathbb{V}(\mathcal{L})$ gives rise to examples of (logarithmic) chiral conformal field theories first studied in [BR], see also [RR1, RR2], and related to generalised WZW models in [Que].

Virasoro: The Virasoro vertex Lie algebra \mathcal{L} is generated by one field, often called $T(z)$, whose operator product expansion with itself depends on the central charge $c \in \mathbb{R}$ and reads

$$T(z)T(w) \sim \frac{\partial_w T(w)}{z-w} + \frac{2T(w)}{(z-w)^2} + \frac{\frac{1}{2}c}{(z-w)^4}.$$

The associated universal enveloping vertex algebra $\mathbb{V}(\mathcal{L})$ is the Virasoro vertex algebra Vir_c .

$\beta\gamma$ system: The Weyl vertex Lie algebra \mathcal{L} is generated by two free bosonic ghost fields $\beta(z)$ and $\gamma(z)$ with operator product expansion

$$\beta(z)\gamma(w) \sim \frac{1}{z-w}.$$

The associated universal enveloping vertex algebra $\mathbb{V}(\mathcal{L})$ is the $\beta\gamma$ vertex algebra $\mathbb{V}_{\beta\gamma}$, named after the $\beta\gamma$ bosonic ghost system in the physics literature.

1.3.2 Analytic Langlands correspondence

One important motivation for studying the affine vertex algebra $\mathbb{V}_\kappa(\mathfrak{g})$ comes from the fact that at the critical level $\kappa = \kappa_c$ it plays a central role in the study of the geometric Langlands correspondence, as formulated by Beilinson and Drinfel'd in their seminal work [BD1] on the quantisation of Hitchin's integrable system. For an extensive review on the subject we refer the reader to [Fre1, Fre2] and references therein. Much more recently, an analytic version

of the geometric Langlands correspondence was formulated and studied by Etingof, Frenkel and Kazhdan in the series of works [Fre3, EFK1, EFK2, EFK3, EFK4], implementing and extending earlier ideas of Teschner [Tes]. The gauge theoretic interpretation of the geometric Langlands correspondence as electric-magnetic duality in twisted $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions [KW] has also been recently extended in [GW] to the analytic version. It is expected, see for instance [Tes, Fre3], that the role of the vertex algebra $\mathbb{V}_{\kappa_c}(\mathfrak{g})$ in the geometric Langlands correspondence should be replaced by that of the full affine vertex algebra $\mathbb{F}_{\kappa_c}(\mathfrak{g}) = \mathbb{V}_{\kappa_c}(\mathfrak{g}) \otimes \bar{\mathbb{V}}_{\kappa_c}(\mathfrak{g})$ at critical level $\kappa = \kappa_c$ in the analytic version.

1.4 Outline of the paper

In §2 we construct a local Lie algebra $\mathcal{L}_\alpha^{\widehat{\Sigma}}$ over the complex curve $\widehat{\Sigma}$, which is an analogue of the notion of vertex Lie algebra in the present prefactorisation algebra setting. Since we are interested in describing a full vertex algebra on Σ , to forget about the orientation introduced by passing from Σ to the orientation double $\widehat{\Sigma}$, see §1.2, the main object of interest will be the pushforward $\mathcal{L}_\alpha^\Sigma$ of the local Lie algebra along the projection $\pi : \widehat{\Sigma} \rightarrow \Sigma$. We focus on the three examples of vertex Lie algebras described in §1.3.1. Importantly, in all of these the identity operator appears as one term in the operator product expansions. Correspondingly, the local Lie algebra $\mathcal{L}_\alpha^\Sigma$ will be centrally extended and to keep track of its crucial central extension **1** we will work with the notion of *unital* local Lie algebras. The counterpart of the universal enveloping vertex algebra construction is the prefactorisation envelope $\mathcal{U}\mathcal{L}_\alpha^\Sigma$ which we adapt from the case of local Lie algebras in [CG1, Definition 3.6.1] to the unital case. The technical details are relegated to Appendix A. This allows us to deal with (full) vertex algebras over \mathbb{C} rather than over the base ring $\mathbb{C}[s1]$, as in [CG1, Definition 5.5.2]. Our construction thus unifies three key examples of holomorphic prefactorisation algebras in a single framework: the Kac-Moody case [CG1, §5.5], the Virasoro case [Wil] and the $\beta\gamma$ system case [CG1, §5.4].

In §3 we consider the vector space $\mathbb{F}^{\mathfrak{a},\alpha} = \mathbb{V}^{\mathfrak{a},\alpha} \otimes \bar{\mathbb{V}}^{\mathfrak{a},\alpha}$ where $\mathbb{V}^{\mathfrak{a},\alpha}$ denotes the vector space underlying the universal enveloping vertex algebra $\mathbb{V}(\mathcal{L})$ and $\bar{\mathbb{V}}^{\mathfrak{a},\alpha}$ its anti-chiral analogue. We realise this vector space in the limit $\mathcal{F}_p^{\mathfrak{a},\alpha}$ of the prefactorisation algebra $\mathcal{U}\mathcal{L}_\alpha^\Sigma$ over any point $p \in \Sigma^\circ$ in the interior Σ° of Σ . In Proposition 3.9 we show that the state-field correspondence at a point $p \in \Sigma^\circ$ is determined by the factorisation product $\mathcal{F}_q^{\mathfrak{a},\alpha} \rightarrow \mathcal{U}\mathcal{L}_\alpha^\Sigma(Y)$ for some annulus $q \in Y \subset \Sigma^\circ$ around p . In particular, the formula for the state field correspondence of an arbitrary state in $\mathbb{F}^{\mathfrak{a},\alpha}$, i.e. the analogue of (1.1), has a very elegant geometric origin from the prefactorisation algebra perspective. We derive a number of properties satisfied by the modes of the vertex operators of arbitrary states in $\mathbb{F}^{\mathfrak{a},\alpha}$, including ‘Borcherds type’ identities when one of the states involved is (anti-)chiral, i.e. belongs to $\mathbb{V}^{\mathfrak{a},\alpha}$ or $\bar{\mathbb{V}}^{\mathfrak{a},\alpha}$. In §3.4 we introduce the (anti-)conformal states of $\mathbb{F}^{\mathfrak{a},\alpha}$ and use these to derive a non-chiral version of Huang’s “change of variable” formula for the state-field correspondence of $\mathbb{F}^{\mathfrak{a},\alpha}$, see Theorem 3.19 and Corollary 3.20. In §3.5 we specialise to $\Sigma = S^2$ and use the factorisation product to $\mathcal{U}\mathcal{L}_\alpha^\Sigma(S^2)$ to define an invariant bilinear form on $\mathbb{F}^{\mathfrak{a},\alpha}$. We then use this to prove in Theorem 3.26 that $\mathbb{F}^{\mathfrak{a},\alpha}$ satisfies the axioms of a full vertex algebra in the sense of [Mor1, Mor2]. Finally, in §3.6 we discuss reality conditions on $\mathbb{F}^{\mathfrak{a},\alpha}$ and define a Hermitian sesquilinear form on $\mathbb{F}^{\mathfrak{a},\alpha}$.

In §4 we make use of the framework developed in §3 to describe the operator formalism for $\mathbb{F}^{\mathfrak{a},\alpha}$, cf. [DMS, §6]. In particular, we use the change of variable formula applied to the conformal transformation from the plane to the cylinder in order to describe Fourier series decompositions of quantum operators on S^1 associated to generic states in $\mathbb{F}^{\mathfrak{a},\alpha}$. In §4.2 we also derive ‘Borcherds type’ identities for the Fourier modes of states in $\mathbb{F}^{\mathfrak{a},\alpha}$, again in the case when one of the states involved is (anti-)chiral. Finally, in §4.3 we discuss reality conditions, define a natural notion of Hermitian conjugate for quantum operators on S^1 and show that it corresponds to the adjoint with respect to the Hermitian sesquilinear form on $\mathbb{F}^{\mathfrak{a},\alpha}$.

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2 Prefactorisation algebra $\mathcal{UL}_\alpha^\Sigma$

Throughout this section we let Σ be an arbitrary connected 2-dimensional conformal manifold but which could be non-orientable and possibly with boundary.

Let $\mathbf{Vec}_{\mathbb{C}}$ denote the symmetric monoidal category of complex vector spaces with the tensor product over \mathbb{C} serving as the monoidal product. We shall also need the symmetric monoidal category $\mathbf{dgVec}_{\mathbb{C}}$ of DG vector spaces over \mathbb{C} , also equipped with the tensor product over \mathbb{C} as the monoidal product. Let $\mathbf{dgLie}_{\mathbb{C}}$ denote the symmetric monoidal category of DG Lie algebras over \mathbb{C} with the direct sum of DG Lie algebras serving as monoidal product.

2.1 Unital local Lie algebra $\mathcal{L}_\alpha^\Sigma$

Let Ω_c^\bullet denote the cosheaf of compactly supported sections of the de Rham complex on the complex manifold $\widehat{\Sigma}$. Let $\Omega_c^{0,\bullet}$ be the cosheaf of compactly supported sections of the Dolbeault complex on $\widehat{\Sigma}$, equipped with the Dolbeault differential $\bar{\partial}$, namely the anti-holomorphic part of the de Rham differential $d = \partial + \bar{\partial}$. These are cosheaves of commutative DG algebras.

2.1.1 General setup

Let L be a holomorphic vector bundle over $\widehat{\Sigma}$. We will make a number of assumptions about L , motivated in part by the list of examples considered in §2.1.2 below, which will allow us to define a unital local Lie algebra $\mathcal{L}_\alpha^\Sigma$ on Σ that will form the basis for the rest of the paper.

Assumption 1 *The compactly supported sections of the L -valued Dolbeault complex $U \mapsto \Omega_c^{0,\bullet}(U, L)$ define a presheaf of DG Lie algebras on $\widehat{\Sigma}$, whose Lie bracket we denote by*

$$[\cdot, \cdot] : \Omega_c^{0,\bullet}(U, L) \otimes \Omega_c^{0,\bullet}(U, L) \longrightarrow \Omega_c^{0,\bullet}(U, L). \quad (2.1)$$

Assumption 2 *The holomorphic vector bundle L is locally holomorphically trivial over any coordinate chart on $\widehat{\Sigma}$, that is for any open subset $U \subset \widehat{\Sigma}$ equipped with a local holomorphic coordinate $\xi : U \rightarrow \mathbb{C}$ we have $L|_U \cong U \times \mathfrak{a}_\xi$ for some finite-dimensional vector space \mathfrak{a}_ξ with the Dolbeault operator $\bar{\partial}_L$ of L given locally by $\bar{\partial}$.*

Although the vector space \mathfrak{a}_ξ will typically depend on the choice of coordinate ξ , hence the use of the subscript ξ in the notation, there is a canonical isomorphism $\mathfrak{a}_\eta \xrightarrow{\cong} \mathfrak{a}_\xi$ between the vectors spaces associated with different local holomorphic coordinates $\xi, \eta : U \rightarrow \mathbb{C}$ determined by the the change of coordinate from η to ξ . We then have the isomorphism of DG vector spaces

$$\Omega_c^{0,\bullet}(U, L) \cong \mathfrak{a}_\xi \otimes \Omega_c^{0,\bullet}(U). \quad (2.2)$$

It will be useful to also introduce a canonical copy \mathfrak{a} of the finite-dimensional vector spaces \mathfrak{a}_ξ with a corresponding isomorphism $(\cdot)_\xi : \mathfrak{a} \xrightarrow{\cong} \mathfrak{a}_\xi$. However, it is important to note at this stage that the induced map $(\cdot)_{\eta \rightarrow \xi} := (\cdot)_\xi \circ ((\cdot)_\eta)^{-1} : \mathfrak{a}_\eta \xrightarrow{\cong} \mathfrak{a}_\xi$ will generally be different from the canonical change of coordinate isomorphism $\mathfrak{a}_\eta \xrightarrow{\cong} \mathfrak{a}_\xi$ described above.

Assumption 3 *For each $U \in \widehat{\Sigma}$, the DG Lie algebra $\Omega_c^{0,\bullet}(U, L)$ is equipped with a 2-cocycle*

$$\alpha : \Omega_c^{0,\bullet}(U, L) \otimes \Omega_c^{0,\bullet}(U, L) \longrightarrow \mathbb{C}. \quad (2.3)$$

We use this 2-cocycle to introduce the *twisted* presheaf of DG Lie algebras $\mathcal{L}_\alpha^{\widehat{\Sigma}}$ as a central extension of the presheaf of DG Lie algebras $U \mapsto \Omega_c^{0,\bullet}(U, L)$ following [CG1]. Specifically, for any open $U \subset \widehat{\Sigma}$ we define the DG vector space

$$\mathcal{L}_\alpha^{\widehat{\Sigma}}(U) := \Omega_c^{0,\bullet}(U, L) \oplus \mathbb{C}\mathbf{1},$$

where $\mathbf{1}$ is a formal variable of cohomological degree 1 so that $\mathbb{C}\mathbf{1}$ is a 1-dimensional complex DG vector space concentrated in degree 1, isomorphic to the inverse suspension $\mathbb{C}[-1]$ of \mathbb{C} .

We have a morphism of DG vector spaces $\text{ext}_{U,V} : \mathcal{L}_\alpha^{\widehat{\Sigma}}(U) \rightarrow \mathcal{L}_\alpha^{\widehat{\Sigma}}(V)$ defined as the extension by zero on the first summand and as the identity on $\mathbb{C}\mathbf{1}$. We define the \mathbb{C} -linear operation

$$[\cdot, \cdot]_\alpha : \mathcal{L}_\alpha^{\widehat{\Sigma}}(U) \otimes \mathcal{L}_\alpha^{\widehat{\Sigma}}(U) \longrightarrow \mathcal{L}_\alpha^{\widehat{\Sigma}}(U) \quad (2.4a)$$

by declaring $\mathbb{C}\mathbf{1}$ to be central and for every $a, b \in \Omega_c^{0,\bullet}(U, L)$ setting

$$[a, b]_\alpha := [a, b] + \alpha(a, b) \mathbf{1}. \quad (2.4b)$$

It is straightforward to show that the above structure makes $\mathcal{L}_\alpha^{\widehat{\Sigma}}$ into a presheaf of DG Lie algebras on $\widehat{\Sigma}$. In fact, since the central extension will play a key role in what follows, it will be important to keep track of it by introducing the notion of a unital DG Lie algebra¹. A unital DG Lie algebra can be described as a DG Lie algebra L equipped with a map $\eta \in \text{Map}_{\text{dgVec}_{\mathbb{C}}}(\mathbb{C}, L)$ of degree 1 whose image is both central and closed in L . In other words, a DG Lie algebra L is unital if it has a central cocycle in degree 1, namely $\eta(1)$. We will refer to the degree 1 map $\eta : \mathbb{C} \rightarrow L$ as the *unit* of the unital DG Lie algebra. See §A.1 for more details. Finally, a unital local Lie algebra is a presheaf of unital DG Lie algebras with the property that sections over disjoint opens commute. Let $\text{uLocLie}_{\mathbb{C}}(D)$ denote the category of unital local Lie algebras on a manifold D , see §A.2 for more details. It is straightforward to check by direct computation that $\mathcal{L}_\alpha^{\widehat{\Sigma}}$ is a unital local Lie algebra on $\widehat{\Sigma}$.

In order to get rid of the dependence on the orientation which was introduced by going from Σ to its oriented cover $\widehat{\Sigma}$, from now on we will only consider the pushforward

$$\mathcal{L}_\alpha^\Sigma := \pi_* \mathcal{L}_\alpha^{\widehat{\Sigma}} \in \text{uLocLie}_{\mathbb{C}}(\Sigma) \quad (2.5)$$

of the unital local Lie algebra $\mathcal{L}_\alpha^{\widehat{\Sigma}}$ on $\widehat{\Sigma}$ introduced above, along the projection $\pi : \widehat{\Sigma} \rightarrow \Sigma$. In other words, the sections of (2.5) over any open $U \subset \Sigma$ are given by

$$\mathcal{L}_\alpha^\Sigma(U) = \mathcal{L}_\alpha^{\widehat{\Sigma}}(\pi^{-1}(U)). \quad (2.6)$$

Note that on a general 2-dimensional conformal manifold Σ , with possibly non-empty boundary $\partial\Sigma$ and interior region $\Sigma^\circ := \Sigma \setminus \partial\Sigma$, there are two types of connected open subsets $U \subset \Sigma$ one may consider:

- (i) $U \subset \Sigma^\circ$ in which case $\pi^{-1}(U) \subset \widehat{\Sigma}$ consists of two connected components,
- (ii) $U \subset \Sigma$ with $U \cap \partial\Sigma \neq \emptyset$ for which $\pi^{-1}(U) \subset \widehat{\Sigma}$ has a single connected component.

Throughout this paper we will only consider bulk fields in the twisted prefactorisation envelope of $\mathcal{L}_\alpha^\Sigma$, to be introduced in §2.3.2, namely observables living on open subsets $U \subset \Sigma^\circ$ of type (i). We leave the study of boundary fields associated with open subsets $U \subset \Sigma$ of type (ii), and their interplay with bulk fields, for future work.

¹I thank Alex Schenkel for this suggestion which makes some of the later constructions more elegant.

2.1.2 Main examples

The three primary examples of the above general setup which we will consider throughout this paper correspond to the three canonical examples of vertex Lie algebras discussed in §1.3.1. Upon taking their twisted prefactorisation envelopes in §2.3 below, these will lead to the full versions of the Kac-Moody, Virasoro and $\beta\gamma$ vertex algebras, respectively. We now describe the data underlying the unital local Lie algebra $\mathcal{L}_\alpha^\Sigma$ in each of these cases.

Kac-Moody: Let \mathfrak{g} be any finite-dimensional complex Lie algebra and consider the trivial vector bundle

$$L = \widehat{\Sigma} \times \mathfrak{g}. \quad (2.7)$$

This clearly satisfies the assumption of local triviality over any open $U \subset \widehat{\Sigma}$ equipped with a holomorphic coordinate $\xi : U \rightarrow \mathbb{C}$, where $\mathfrak{a}_\xi = \mathfrak{g}$ is actually independent of the coordinate ξ in this case. The Lie bracket (2.1) on $\Omega_c^{0,\bullet}(U, L) \cong \mathfrak{a}_\xi \otimes \Omega_c^{0,\bullet}(U)$ is given by

$$[\mathbf{X} \otimes \omega, \mathbf{Y} \otimes \eta] := [\mathbf{X}, \mathbf{Y}] \otimes \omega \wedge \eta \quad (2.8a)$$

for $\mathbf{X}, \mathbf{Y} \in \mathfrak{g}$ and $\omega, \eta \in \Omega_c^{0,\bullet}(U)$.

We also define $\mathfrak{a} = \mathfrak{g}$ and the isomorphism $(\cdot)_\xi : \mathfrak{a} \xrightarrow{\cong} \mathfrak{a}_\xi$ is the identity.

We fix a symmetric invariant bilinear form $\kappa : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{C}$ on \mathfrak{g} . Using this we can define a 2-cocycle (2.3) on $\Omega_c^{0,\bullet}(U, L)$ by the following formula, see [CG1, §5.5.1],

$$\alpha(\mathbf{X} \otimes \omega, \mathbf{Y} \otimes \eta) := -\frac{\kappa(\mathbf{X}, \mathbf{Y})}{2\pi i} \int_U \partial\omega \wedge \eta \quad (2.8b)$$

for any $\mathbf{X}, \mathbf{Y} \in \mathfrak{g}$ and $\omega, \eta \in \Omega_c^{0,\bullet}(U)$. The integral over U in (2.8b) is trivially zero on degree grounds if $|\omega| + |\eta| \neq 1$. Indeed, if $\omega \in \Omega_c^{0,r}(U)$ and $\eta \in \Omega_c^{0,s}(U)$ then $\partial\omega \wedge \eta \in \Omega_c^{1,r+s}(U)$, which can be integrated over U only if $r + s = 1$.

The unital local Lie algebra $\mathcal{L}_\alpha^{\widehat{\Sigma}}$ in this case corresponds to the affine Kac-Moody example considered in [CG1, §5.5].

Virasoro: Consider the holomorphic tangent bundle

$$L = T^{1,0}\widehat{\Sigma}. \quad (2.9)$$

Over any open $U \subset \widehat{\Sigma}$ equipped with a holomorphic coordinate $\xi : U \rightarrow \mathbb{C}$ we have the local trivialisation $L|_U \cong U \times \mathfrak{a}_\xi$ with $\mathfrak{a}_\xi = \text{span}_{\mathbb{C}}\{\partial_\xi\}$. The induced isomorphism (2.2) then allows us to represent an element of $\Omega_c^{0,\bullet}(U, L)$ in the form $\partial_\xi \otimes u \in \mathfrak{a}_\xi \otimes \Omega_c^{0,\bullet}(U)$ with $u \in \Omega_c^{0,\bullet}(U)$ and the Lie bracket (2.1) on $\Omega_c^{0,\bullet}(U, L)$ explicitly reads

$$[\partial_\xi \otimes u, \partial_\xi \otimes v] = \partial_\xi \otimes (u \wedge \partial_\xi v - v \wedge \partial_\xi u) \quad (2.10a)$$

for any $u, v \in \Omega_c^{0,\bullet}(U)$. One can view $\partial_\xi \otimes u$ as a compactly supported Dolbeault form valued vector field on U , which would be more conventionally denoted $u\partial_\xi$. The Lie bracket (2.10a) is then simply the commutator of vector fields combined with the wedge product on forms.

We let $\mathfrak{a} = \text{span}_{\mathbb{C}}\{\Omega\}$ and the isomorphism $(\cdot)_\xi : \mathfrak{a} \xrightarrow{\cong} \mathfrak{a}_\xi$ is given by $\Omega \mapsto -\partial_\xi$.

Following [Wil], a formula for the 2-cocycle (2.3) on $\Omega_c^{0,\bullet}(U, L)$ is obtained by generalising the formula for the Gelfand-Fuchs 2-cocycle on the Lie algebra of vector fields on S^1 . First, we suppose that $U \subset \widehat{\Sigma}$ is covered by a local coordinate chart $\xi : U \rightarrow \mathbb{C}$. A pair of elements in $\Omega_c^{0,\bullet}(U, L)$ can then be represented as $\partial_\xi \otimes u$ and $\partial_\xi \otimes v$, whose coefficients $u, v \in \Omega_c^{0,\bullet}(U)$ transform as components of $(1, 0)$ vector fields on U . We then define the 2-cocycle [Wil, §2.2.2]

$$\alpha_\xi(\partial_\xi \otimes u, \partial_\xi \otimes v) := \frac{c}{24\pi i} \int_U \partial(\partial_\xi u) \wedge \partial_\xi v, \quad (2.10b)$$

where $c \in \mathbb{R}$ is fixed. As in the previous example, the integral over U in (2.10b) is trivially zero on degree grounds if $|u| + |v| \neq 1$. The subscript ξ on the left hand side of (2.10b) is used to indicate that this 2-cocycle explicitly depends on the chosen coordinate ξ . However, it depends on the coordinate ξ only up to a 2-coboundary. Indeed, given any other local coordinate $\eta : U \rightarrow \mathbb{C}$ on the open $U \subset \widehat{\Sigma}$ and introducing the 1-cochain

$$\beta_{\eta,\xi} : \Omega_c^{0,\bullet}(U, L) \longrightarrow \mathbb{C}, \quad \partial_\xi \otimes u \longmapsto \frac{c}{24\pi i} \int_U (S\eta)(\xi) d\xi \wedge u, \quad (2.11)$$

where $(S\eta)(\xi) = \frac{\partial^3 \eta}{\partial \xi^3} - \frac{3}{2} \left(\frac{\partial^2 \eta}{\partial \xi^2} \right)^2$ denotes the Schwarzian derivative, it is straightforward to show that

$$\alpha_\eta(\partial_\eta \otimes \tilde{u}, \partial_\eta \otimes \tilde{v}) = \alpha_\xi(\partial_\xi \otimes u, \partial_\xi \otimes v) + \beta_{\eta,\xi}([\partial_\xi \otimes u, \partial_\xi \otimes v])$$

for any pair of vector fields $\partial_\eta \otimes \tilde{u} = \partial_\xi \otimes u$ and $\partial_\eta \otimes \tilde{v} = \partial_\xi \otimes v$ in $\Omega_c^{0,\bullet}(U, L)$. In other words, $\alpha_\eta = \alpha_\xi + \delta\beta_{\eta,\xi}$ so that we have an isomorphism of unital local Lie algebras

$$\Phi_{\eta \rightarrow \xi} : \mathcal{L}_{\alpha_\eta}^{\widehat{\Sigma}}(U) \xrightarrow{\cong} \mathcal{L}_{\alpha_\xi}^{\widehat{\Sigma}}(U), \quad a + x\mathbf{1} \longmapsto a + (x - \beta_{\eta,\xi}(a))\mathbf{1}. \quad (2.12)$$

The 2-cocycle (2.10b) thus leads to a well-defined and coordinate independent central extension (2.4). We refer to [Wil, §5.2] for further details.

The resulting unital local Lie algebra $\mathcal{L}_\alpha^{\widehat{\Sigma}}$ coincides with the local Lie algebra underlying the Virasoro factorisation algebra in [Wil].

$\beta\gamma$ system: Consider the direct sum of the trivial 1-dimensional line bundle on $\widehat{\Sigma}$ with the holomorphic cotangent bundle, namely

$$L = (\widehat{\Sigma} \times \mathbb{C}) \oplus T^{*1,0}\widehat{\Sigma}. \quad (2.13)$$

Over any open $U \subset \widehat{\Sigma}$ equipped with a holomorphic coordinate $\xi : U \rightarrow \mathbb{C}$ we have the local trivialisation $L|_U \cong U \times \mathfrak{a}_\xi$ where $\mathfrak{a}_\xi = \text{span}_{\mathbb{C}}\{1, d\xi\}$. The Lie bracket (2.1) on $\Omega_c^{0,\bullet}(U, L)$ is taken to be trivial. The isomorphism (2.2) here implies $\Omega_c^{0,\bullet}(U, L) \cong \Omega_c^{0,\bullet}(U) \oplus \Omega_c^{1,\bullet}(U)$.

We let $\mathfrak{a} = \text{span}_{\mathbb{C}}\{\beta, \gamma\}$ and the isomorphism $(\cdot)_\xi : \mathfrak{a} \xrightarrow{\cong} \mathfrak{a}_\xi$ is given by $\beta \mapsto 1, \gamma \mapsto d\xi$.

A 2-cocycle (2.3) on $\Omega_c^{0,\bullet}(U, L)$ is given by

$$\alpha(\omega, \eta) := \frac{1}{2\pi i} \int_U \omega \wedge \eta \quad (2.14)$$

for any $\omega, \eta \in \Omega_c^{0,\bullet}(U) \oplus \Omega_c^{1,\bullet}(U)$, where once again the integral is understood to vanish unless $\omega \wedge \eta \in \Omega_c^{1,1}(U)$ so that if $\omega \in \Omega_c^{0,r}(U)$, say, then we should have $\eta \in \Omega_c^{1,s}(U)$ with $r + s = 1$.

This corresponds to the $\beta\gamma$ system example discussed in [CG1, §5.4].

2.2 Cohomology of $\mathcal{L}_\alpha^\Sigma$

In this section we compute the cohomology of the DG vector space $\mathcal{L}_\alpha^\Sigma(U)$ for connected open subsets $U \subset \Sigma^\circ$ in the interior Σ° of Σ , equipped with a local holomorphic coordinate, see the last part of Theorem 2.5 below. Although this result is well known, see e.g. [LR, §9] or [CG1, Theorem 5.4.7], we shall give the details of the proof in the present context since the full statement of Theorem 2.5 will be crucial for us in §3.

Given any $U \subset \mathbb{C}$ or $U \subset \widehat{\Sigma}$ we denote by \overline{U} its closure. The following result is standard.

Lemma 2.1. *Let $U \subset \mathbb{C}$ be a bounded open subset with C^1 boundary.*

(a) For any $f \in \Omega^0(\overline{U})$ we have the Cauchy-Pompeiu formula

$$\frac{1}{2\pi i} \int_{\partial U} \frac{f d\mu}{\mu - \lambda} + \frac{1}{2\pi i} \int_U \frac{d\mu \wedge \bar{\partial}f}{\mu - \lambda} = \begin{cases} f(\lambda) & \text{if } \lambda \in U, \\ 0 & \text{if } \lambda \in \mathbb{C} \setminus \overline{U}. \end{cases}$$

(b) For any $\omega \in \Omega_c^{0,1}(U)$, the $\bar{\partial}$ -problem $\bar{\partial}f = \omega$ for $f \in \Omega^{0,0}(\mathbb{C})$ on \mathbb{C} is solved by

$$f(\lambda) = \omega^\downarrow(\lambda) := \frac{1}{2\pi i} \int_U \frac{d\mu \wedge \omega}{\mu - \lambda}. \quad \square$$

We denote the complement of any subset $S \subset \mathbb{C}P^1 := \mathbb{C} \cup \{\infty\}$ of the Riemann sphere by $S^c := \mathbb{C}P^1 \setminus S$. For any $S \subset \mathbb{C}$, let $\mathcal{O}_\infty(S^c)$ denote the algebra of germs of holomorphic functions on S^c vanishing at ∞ . Explicitly, an element of $\mathcal{O}_\infty(S^c)$ is the equivalence class $[f]$ of a holomorphic function $f \in \mathcal{O}_{\mathbb{C}P^1}(\Delta_f)$ defined on an open neighbourhood $\Delta_f \supset S^c \ni \infty$ and such that $f(\infty) = 0$, where two such functions f and g are considered equivalent if $f|_V = g|_V$ for some open neighbourhood $V \subset \Delta_f \cap \Delta_g$ of S^c .

Lemma 2.2. Let $W \subset \mathbb{C}$ be a bounded open subset. For any $\omega \in \Omega_c^{0,1}(W)$ we have:

- (i) $\omega^\downarrow \in \Omega^{0,0}(\mathbb{C})$ defines a germ $[\omega^\downarrow] \in \mathcal{O}_\infty(W^c)$,
- (ii) $[\omega^\downarrow] = 0$ in $\mathcal{O}_\infty(W^c)$ if and only if ω is $\bar{\partial}$ -exact.

Proof. Firstly, since ω has compact support the integral entering the definition of $\omega^\downarrow(\lambda)$ is well defined, in particular the integrand is locally integrable near the singularity at λ . And by Lemma 2.1(b), this function provides a solution to the $\bar{\partial}$ -problem for ω . We then also deduce from the support property of ω that ω^\downarrow is holomorphic on an open neighbourhood $\Delta_{\omega^\downarrow}$ of W^c . Moreover, since for every $\mu \in \text{supp } \omega \subset W$ we have $|\mu - \lambda| \geq \text{dist}(\lambda, \text{supp } \omega)$, it follows that

$$|\omega^\downarrow(\lambda)| \leq \frac{C \sup_{\mu \in U} |\omega(\mu)|}{\text{dist}(\lambda, \text{supp } \omega)}$$

for some $C > 0$, and hence $\omega^\downarrow(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$. The statement (i) now follows.

If $[\omega^\downarrow] = 0$ then ω^\downarrow vanishes on an open neighbourhood of W^c and so it has compact support in the bounded open $W \subset \mathbb{C}$, i.e. $\omega^\downarrow \in \Omega_c^{0,0}(W)$. The ‘only if’ part of (ii) now follows as ω^\downarrow solves the $\bar{\partial}$ -problem for ω .

Conversely, if $\omega = \bar{\partial}\eta$ for some $\eta \in \Omega_c^{0,0}(W)$ then it is immediate from the Cauchy-Pompeiu formula that $\omega^\downarrow = \eta$. The ‘if’ part of (ii) now follows since $\omega^\downarrow \in \Omega_c^{0,0}(W)$. \square

Given any bounded open subset $W \subset \mathbb{C}$, let us fix a basis of the space of germs $\mathcal{O}_\infty(W^c)$ and for each basis element $[f] \in \mathcal{O}_\infty(W^c)$ we pick a representative $f \in \mathcal{O}_{\mathbb{C}P^1}(\Delta_f)$ defined on an open neighbourhood Δ_f of W^c and we choose a smooth bump function $\rho_f \in \Omega_c^{0,0}(W)$ which is equal to 1 on some open neighbourhood of $(\Delta_f)^c \subset W$. Since $f|_{\Delta_f \cap W} \in \Omega^{0,0}(\Delta_f \cap W)$ is defined on $\text{supp } \bar{\partial}\rho_f \subset \Delta_f \cap W$ we can set

$$[f] := -f \bar{\partial}\rho_f \in \Omega_c^{0,1}(W) \quad (2.15)$$

for every basis element $[f] \in \mathcal{O}_\infty(W^c)$ and then extend by linearity to all of $\mathcal{O}_\infty(W^c)$.

Lemma 2.3. Let $W \subset \mathbb{C}$ be a bounded open subset. For any germ $[f] \in \mathcal{O}_\infty(W^c)$, the 1-form $[f] \in \Omega_c^{0,1}(W)$ is independent of the choices made, up to $\bar{\partial}$ -exact terms. In particular, for any representative $f \in \mathcal{O}_{\mathbb{C}P^1}(\Delta_f)$ of $[f]$ and any smooth bump function $\rho \in \Omega_c^{0,0}(W)$ equal to 1 on some open neighbourhood of $(\Delta_f)^c \subset W$ we have that $[f] + f \bar{\partial}\rho$ is $\bar{\partial}$ -exact.

Proof. Let $[f] \in \mathcal{O}_\infty(W^c)$. In terms of the fixed basis of $\mathcal{O}_\infty(W^c)$, we can write it as a finite linear combination $[f] = \sum_i \alpha_i [f_i]$ of basis elements $[f_i] \in \mathcal{O}_\infty(W^c)$. Then $f|_V = \sum_i \alpha_i f_i|_V$ for some neighbourhood $V \subset \Delta_f \cap \bigcap_i \Delta_{f_i}$ of W^c . Let $\rho' \in \Omega_c^{0,0}(W)$ be a smooth bump function equal to 1 on some neighbourhood of $V^c \subset W$. By definition $[f] = -\sum_i \alpha_i f_i \bar{\partial} \rho_{f_i}$ so that

$$[f] + f \bar{\partial} \rho = -\sum_i \alpha_i f_i \bar{\partial}(\rho_{f_i} - \rho') - \left(\sum_i \alpha_i f_i - f \right) \bar{\partial} \rho' + f \bar{\partial}(\rho - \rho').$$

The second term on the right hand side is equal to zero since $\text{supp } \bar{\partial} \rho' \subset V \cap W$ and $\sum_i \alpha_i f_i - f$ vanishes on V . On the other hand, the first term on the right hand side is $\bar{\partial}$ -exact because $\rho_{f_i} - \rho'$ vanishes on $(\Delta_{f_i})^c \subset V^c$ so that f_i is holomorphic on the support of $\rho_{f_i} - \rho'$. Likewise, the last term is also $\bar{\partial}$ -exact, as required. \square

Lemma 2.4. *Let $W \subset \mathbb{C}$ be a bounded open subset. For $[f] \in \mathcal{O}_\infty(W^c)$ we have $[\lceil f \rceil^\downarrow] = [f]$. Moreover, for $\omega \in \Omega_c^{0,1}(W)$ we have $\omega - [\lceil \omega \rceil^\downarrow] = \bar{\partial}((\omega - [\lceil \omega \rceil^\downarrow])^\downarrow)$.*

Proof. For any $f \in \mathcal{O}_{\mathbb{C}P^1}(\Delta_f)$ tending to zero at infinity, we have

$$-(f \bar{\partial} \rho_f)^\downarrow = (f \bar{\partial}(1 - \rho_f))^\downarrow = (\bar{\partial}((1 - \rho_f)f))^\downarrow = (1 - \rho_f)f,$$

where in the second step we used the fact that $\bar{\partial}f = 0$ on $\text{supp}(1 - \rho_f) \subset \Delta_f$. In order to see the last step, consider the Cauchy-Pompeiu formula applied to the function $(1 - \rho_f)f \in \Omega^0(\overline{D_R})$ with $D_R \supset W$ an open disc of radius $R > 0$, namely

$$\frac{1}{2\pi i} \int_{\partial D_R} \frac{f d\mu}{\mu - \lambda} + \frac{1}{2\pi i} \int_W \frac{d\mu \wedge \bar{\partial}((1 - \rho_f)f)}{\mu - \lambda} = (1 - \rho_f(\lambda))f(\lambda) \quad \text{if } \lambda \in D_R,$$

where in the first integral on the left hand side we used the fact that $(1 - \rho_f)f = f$ on ∂D_R and the domain of the second integral was restricted from D_R to W using the fact that the integrand has support in W . The first integral is bounded by $\sup_{\mu \in \partial D_R} |f(\mu)|$ which tends to zero as $R \rightarrow \infty$. And in this limit, the right hand side of the Cauchy-Pompeiu formula tends to $(1 - \rho_f(\lambda))f(\lambda)$ for all $\lambda \in \mathbb{C}$. Finally, we note that $(1 - \rho_f)f = f$ on the open neighbourhood $(\text{supp } \rho_f)^c$ of W^c from which the first claim now follows.

For the ‘moreover’ part, it follows using Lemma 2.2(i) and the above that ω^\downarrow and $[\lceil \omega \rceil^\downarrow]^\downarrow$ define the same germ in $\mathcal{O}_\infty(W^c)$. The desired result now follows by using Lemma 2.2(ii), see in particular its proof. \square

Let $U \subset \Sigma^\circ$ be a connected open subset of the interior $\Sigma^\circ \subset \Sigma$, i.e. not intersecting the boundary $\partial\Sigma$ of Σ , with a bounded holomorphic coordinate on each connected component of $\pi^{-1}(U) \subset \widehat{\Sigma}$. That is, $\pi^{-1}(U) = \sqcup_{i=1}^m W_i$ with $m \in \{1, 2\}$, where each $W_i \subset \widehat{\Sigma}$ is a connected open subset equipped with a holomorphic coordinate $\xi_i : W_i \rightarrow \mathbb{C}$ such that $\xi_i(W_i) \subset \mathbb{C}$ is bounded. We will denote this set of holomorphic coordinates collectively as $\xi : \pi^{-1}(U) \rightarrow \mathbb{C}$. Consider the DG vector space concentrated in degree 0 given by

$$\mathcal{L}_0^\xi(\pi^{-1}(U)) := \bigoplus_{i=1}^m \mathfrak{a} \otimes \mathcal{O}_\infty(\xi_i(W_i)^c) \oplus \mathbb{C}.$$

We define a morphism in the category of pointed vector spaces $\text{udgVec}_{\mathbb{C}}$, see §A.1, as

$$\begin{aligned} i_U : \mathcal{L}_0^\xi(\pi^{-1}(U))[-1] &\longrightarrow \mathcal{L}_\alpha^\Sigma(U) \\ s^{-1}(\mathfrak{a}_i \otimes [f_i])_{i=1}^m &\longmapsto \sum_{i=1}^m (\mathfrak{a}_i)_{\xi_i} \otimes \xi_i^* \lceil f_i \rceil \\ s^{-1}a &\longmapsto a\mathbf{1} \end{aligned} \tag{2.16a}$$

for any $a_i \in \mathfrak{a}$, $a \in \mathbb{C}$ and $[f_i] \in \mathcal{O}_\infty(\xi_i(W_i)^c)$. We also define another morphism in $\text{udgVec}_{\mathbb{C}}$, going in the other direction, as

$$\begin{aligned}
p_U : \mathcal{L}_\alpha^\Sigma(U) &\longrightarrow \mathcal{L}_\mathcal{O}^\xi(\pi^{-1}(U))[-1] & (2.16b) \\
\sum_{i=1}^m (\mathbf{a}_i)_{\xi_i} \otimes \omega_i &\longmapsto s^{-1} \left(\mathbf{a}_i \otimes \left[((\xi_i^{-1})^* \omega_i)^\downarrow \right] \right)_{i=1}^m \\
\sum_{i=1}^m (\mathbf{b}_i)_{\xi_i} \otimes \eta_i &\longmapsto 0 \\
a \mathbf{1} &\longmapsto s^{-1} a
\end{aligned}$$

for any $a_i, b_i \in \mathfrak{a}$, $a \in \mathbb{C}$, $\eta_i \in \Omega_c^{0,0}(W_i)$ and $\omega_i \in \Omega_c^{0,1}(W_i)$. Finally, we define a degree -1 linear map

$$\begin{aligned}
h_U : \mathcal{L}_\alpha^\Sigma(U) &\longrightarrow \mathcal{L}_\alpha^\Sigma(U) & (2.16c) \\
\sum_{i=1}^m (\mathbf{a}_i)_{\xi_i} \otimes \omega_i &\longmapsto \sum_{i=1}^m (\mathbf{a}_i)_{\xi_i} \otimes \xi_i^* \left((\xi_i^{-1})^* \omega_i - \lceil ((\xi_i^{-1})^* \omega_i)^\downarrow \rceil \right)^\downarrow \\
\sum_{i=1}^m (\mathbf{b}_i)_{\xi_i} \otimes \eta_i &\longmapsto 0 \\
a \mathbf{1} &\longmapsto 0
\end{aligned}$$

for any $a_i, b_i \in \mathfrak{a}$, $a \in \mathbb{C}$, $\eta_i \in \Omega_c^{0,0}(W_i)$ and $\omega_i \in \Omega_c^{0,1}(W_i)$.

Theorem 2.5. *For any connected open subset $U \subset \Sigma^\circ$ equipped with holomorphic coordinates $\xi : \pi^{-1}(U) \rightarrow \mathbb{C}$ as above, we have a strong deformation retract*

$$\mathcal{L}_0^\xi(\pi^{-1}(U))[-1] \xrightleftharpoons[\substack{i_U \\ p_U}]{\hspace{1cm}} \mathcal{L}_\alpha^\Sigma(U) \xhookrightarrow{\hspace{1cm}} h_U$$

in $\text{udgVec}_{\mathbb{C}}$. In other words, the collection of linear maps in (2.16) define a deformation retract in the sense that $p_U i_U = \text{id}$ and $i_U p_U - \text{id} = -(\bar{\partial} h_U + h_U \bar{\partial})$, and this deformation retract is strong in the sense that $p_U h_U = h_U i_U = h_U h_U = 0$.

In particular, $H^0(\mathcal{L}_\alpha^\Sigma(U)) = 0$ and $H^1(\mathcal{L}_\alpha^\Sigma(U)) \cong \mathcal{L}_\alpha^\xi(\pi^{-1}(U))$.

Proof. Let $a_i, b_i \in \mathfrak{a}$, $a \in \mathbb{C}$, $\eta_i \in \Omega_c^{0,0}(W_i)$, $\omega_i \in \Omega_c^{0,1}(W_i)$ and $[f_i] \in \mathcal{O}_\infty(\xi(W_i)^c)$. Then

$$p_U i_U \left(s^{-1} \left((\mathsf{a}_i \otimes [f_i])_{i=1}^m, a \right) \right) = s^{-1} \left(\left(\mathsf{a}_i \otimes \left[[f_i]^\downarrow \right] \right)_{i=1}^m, a \right),$$

so using Lemma 2.4 we deduce that $p_U i_U = \text{id}$. Next, we have

$$\begin{aligned}
(i_U p_U - \text{id}) & \left(\sum_{i=1}^m ((\mathbf{a}_i)_{\xi_i} \otimes \omega_i + (\mathbf{b}_i)_{\xi_i} \otimes \eta_i) + a \mathbf{1} \right) \\
& = - \sum_{i=1}^m (\mathbf{a}_i)_{\xi_i} \otimes \left(\omega_i - \xi_i^* \lceil ((\xi_i^{-1})^* \omega_i)^\downarrow \rceil \right) - \sum_{i=1}^m (\mathbf{b}_i)_{\xi_i} \otimes \eta_i \\
& = -\bar{\partial} \left(h_U \left(\sum_{i=1}^m ((\mathbf{a}_i)_{\xi_i} \otimes \omega_i + (\mathbf{b}_i)_{\xi_i} \otimes \eta_i) + a \mathbf{1} \right) \right) - \sum_{i=1}^m h_U ((\mathbf{b}_i)_{\xi_i} \otimes \bar{\partial} \eta_i),
\end{aligned}$$

where the first sum in last step has been rewritten using the ‘moreover’ part of Lemma 2.4 and the definition of h_U . To rewrite the second sum, we note that by Lemma 2.2(ii), see in

particular its proof, we have $((\xi_i^{-1})^* \bar{\partial} \eta_i)^\downarrow = (\xi_i^{-1})^* \eta_i \in \Omega_c^{0,0}(\mathbb{C})$ which defines the trivial germ in $\mathcal{O}_\infty(\xi_i(W_i)^c)$ and hence $\lceil ((\xi_i^{-1})^* \bar{\partial} \eta_i)^\downarrow \rceil = 0$. Therefore by definition of h_U we have

$$\sum_{i=1}^m h_U((\mathbf{b}_i)_{\xi_i} \otimes \bar{\partial} \eta_i) = h_U \left(\sum_{i=1}^m (\mathbf{b}_i)_{\xi_i} \otimes \bar{\partial} \eta_i \right) = \sum_{i=1}^m (\mathbf{b}_i)_{\xi_i} \otimes \eta_i.$$

We have shown $i_U p_U - \text{id} = -(\bar{\partial} h_U + h_U \bar{\partial})$ so that we have a deformation retract.

It remains to show that this deformation retract is strong. But it is immediate on degree grounds that $p_U h_U = 0$ and $h_U h_U = 0$. On the other hand, we have

$$h_U i_U \left(s^{-1} ((\mathbf{a}_i \otimes [f_i])_{i=1}^m, a) \right) = \sum_{i=1}^m (\mathbf{a}_i)_{\xi_i} \otimes \xi_i^* \left(\lceil f_i \rceil - \lceil \lceil f_i \rceil^\downarrow \rceil \right)^\downarrow.$$

Now by Lemma 2.4 we know that $[f_i] = \lceil \lceil f_i \rceil^\downarrow \rceil$ and hence the right hand side vanishes, so we have a strong deformation retract. \square

2.3 Twisted prefactorisation envelope of $\mathcal{L}_\alpha^\Sigma$

We now introduce the prefactorisation algebra $\mathcal{UL}_\alpha^\Sigma$, referred to as the twisted prefactorisation envelope of the unital local Lie algebra $\mathcal{L}_\alpha^\Sigma$, by adapting the general construction in [CG1]. To formulate the notion of prefactorisation algebra most succinctly it is convenient to use the framework of *multicategories*, which are also known as *coloured operads*. These consist of classes of objects and arrows, just like an ordinary category, but where the arrows describe n -ary operations with an arbitrary number $n \in \mathbb{Z}_{\geq 0}$ of inputs but still just a single output. We refer the reader to [Lei, §2] or [CG1, §A.3.3] for an introduction to multicategories.

2.3.1 Prefactorisation algebras

Let $\mathbf{Top}(\Sigma)$ denote the category whose objects consist of Σ itself and all open subsets which are homeomorphic to either \mathbb{C} or $\mathbb{C} \setminus \{0\}$, and whose morphisms are given by inclusions, i.e. for all $U, V \in \mathbf{Top}(\Sigma)$ the set of morphisms $\text{Hom}_{\mathbf{Top}(\Sigma)}(U, V)$ contains a single morphism $U \rightarrow V$ if $U \subset V$ and is empty otherwise. Note that we do not restricting attention to open subsets homeomorphic to \mathbb{C} , as is done for instance in the locally constant setting of [AF, Lur] which describes topological quantum field theories. The reason for also allowing open subsets homeomorphic to $\mathbb{C} \setminus \{0\}$ will become clear later in §3.2 below, see in particular Proposition 3.9 where the state-field correspondence will be constructed as a factorisation product from a disc to an annulus. Likewise, the reason for including Σ itself as an open subset in $\mathbf{Top}(\Sigma)$ will become clear in §3.5 when we introduce the notion of invariant bilinear form. Note that our choice of category $\mathbf{Top}(\Sigma)$ also differs from the one used in [CG1, CG2] whose objects consist of all open subsets of Σ . However, unlike [CG1, CG2], where the focus is on general perturbative quantum field theories, here we restrict to conformal field theories, for which our choice of category $\mathbf{Top}(\Sigma)$ is particularly well-adapted.

We define an associated multicategory $\mathbf{Top}(\Sigma)^\sqcup$ as follows. It has the same set of objects as $\mathbf{Top}(\Sigma)$ and for any finite collection of open subsets U_i with $i \in \{1, \dots, n\}$ and V in $\mathbf{Top}(\Sigma)^\sqcup$, with $n \in \mathbb{Z}_{\geq 0}$, its set of n -operations is

$$P_n^{\mathbf{Top}(\Sigma)^\sqcup}((U_i)_{i=1}^n, V) := \begin{cases} \text{Hom}_{\mathbf{Top}(\Sigma)}(\sqcup_{i=1}^n U_i, V) & \text{if } U_i \cap U_j = \emptyset \text{ for all } i \neq j, \\ \emptyset & \text{otherwise.} \end{cases}$$

In other words, it contains the single element $\sqcup_{i=1}^n U_i \rightarrow V$ if all the U_i are pairwise disjoint and contained within V , and is empty otherwise. By convention, for any open $V \subset \Sigma$ there is a unique 0-operation $\emptyset \rightarrow V$ from the empty collection of disjoint open subsets to V . Since the

disjoint union is symmetric we have $\sqcup_{i=1}^n U_i = \sqcup_{i=1}^n U_{\sigma(i)}$ for any $\sigma \in S_n$ and the corresponding action of the symmetric group on the n -operations

$$\sigma^* : P_n^{\text{Top}(\Sigma)^\sqcup}((U_i)_{i=1}^n, V) \xrightarrow{\cong} P_n^{\text{Top}(\Sigma)^\sqcup}((U_{\sigma(i)})_{i=1}^n, V)$$

is simply the identity.

Given any symmetric monoidal category \mathbf{C} with monoidal product denoted \otimes , we denote the associated multicategory by \mathbf{C}^\otimes . A \mathbf{C}^\otimes -valued *prefactorisation algebra* \mathcal{F} on Σ , is an object in the category $\text{Alg}_{\text{Top}(\Sigma)^\sqcup}(\mathbf{C}^\otimes)$ of $\text{Top}(\Sigma)^\sqcup$ -algebras in \mathbf{C}^\otimes , i.e. it is a multifunctor

$$\mathcal{F} : \text{Top}(\Sigma)^\sqcup \longrightarrow \mathbf{C}^\otimes.$$

More explicitly, this is an assignment of:

- an object $\mathcal{F}(U) \in \mathbf{C}$ to each open subset $U \subset \Sigma$ in $\text{Top}(\Sigma)$, and
- a morphism $m_{(U_i),V}^{\mathcal{F}} : \bigotimes_{i=1}^n \mathcal{F}(U_i) \rightarrow \mathcal{F}(V)$ in \mathbf{C} , called a *factorisation product*, to each inclusion of $n \in \mathbb{Z}_{\geq 0}$ disjoint open sets $\sqcup_{i=1}^n U_i \subset V$ in $\text{Top}(\Sigma)$, such that the diagram

$$\begin{array}{ccc} \bigotimes_{i=1}^n \bigotimes_{j=1}^{m_i} \mathcal{F}(U_{ij}) & \xrightarrow{\bigotimes_{i=1}^n m_{(U_{ij}),V_i}^{\mathcal{F}}} & \bigotimes_{i=1}^n \mathcal{F}(V_i) \\ & \searrow m_{(U_{ij}),W}^{\mathcal{F}} & \downarrow m_{(V_i),W}^{\mathcal{F}} \\ & & \mathcal{F}(W) \end{array} \quad (2.17)$$

in \mathbf{C} commutes, for any inclusion of disjoint open subsets $\sqcup_{j=1}^{m_i} U_{ij} \subset V_i$ for $i = 1, \dots, n$ and $\sqcup_{i=1}^n V_i \subset W$ in $\text{Top}(\Sigma)^\sqcup$. In particular, the unique 0-operation $\emptyset \rightarrow V$ is assigned a morphism $u \rightarrow \mathcal{F}(V)$ from the identity object u of \mathbf{C} , i.e. $\mathcal{F}(V)$ is *pointed*.

When \mathcal{F} is clear from the context we denote the factorisation products simply as $m_{(U_i),V}$. We also use the abbreviation $m_{U,V} := m_{(U),V}$ for any inclusion of open subsets $U \subset V$ in $\text{Top}(\Sigma)$. Note that a prefactorisation algebra \mathcal{F} is, in particular, a precosheaf on Σ restricted to $\text{Top}(\Sigma)$ with extension morphisms $m_{U,V}$ for every inclusion of open subsets $U \subset V$ in $\text{Top}(\Sigma)$.

A *morphism of prefactorisation algebras* $\phi : \mathcal{F} \rightarrow \mathcal{G}$ is a natural transformation

$$\begin{array}{ccc} & \mathcal{F} & \\ \text{Top}(\Sigma)^\sqcup & \downarrow \phi & \mathbf{C}^\otimes \\ & \mathcal{G} & \end{array}$$

of multifunctors $\mathcal{F}, \mathcal{G} : \text{Top}(\Sigma)^\sqcup \rightarrow \mathbf{C}^\otimes$. Let $\text{PFac}(\Sigma, \mathbf{C}^\otimes) := \text{Alg}_{\text{Top}(\Sigma)^\sqcup}(\mathbf{C}^\otimes)$ denote the category of prefactorisation algebras on Σ valued in the multicategory \mathbf{C}^\otimes . Given any morphisms of prefactorisation algebras $\phi : \mathcal{F} \rightarrow \mathcal{G}$ and $\psi : \mathcal{G} \rightarrow \mathcal{H}$ with $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \text{PFac}(\Sigma, \mathbf{C}^\otimes)$ we denote by $\psi \circ \phi : \mathcal{F} \rightarrow \mathcal{H}$ their composition, i.e. the vertical composition of these natural transformations of multifunctors. The horizontal composition of natural transformations will be denoted by concatenation.

2.3.2 Twisted prefactorisation envelope of $\mathcal{L}_\alpha^\Sigma$

Any local Lie algebra on Σ defines, in a canonical way, a prefactorisation algebra on Σ valued in $\text{dgLie}_{\mathbb{C}}^\oplus$. Similarly, any unital local Lie algebra on Σ defines a prefactorisation algebra on Σ valued in $\text{udgLie}_{\mathbb{C}}^\oplus$, see Proposition A.3 for details. In [CG1, Definition 3.6.4], the notion of twisted prefactorisation envelope is defined by applying the Chevalley-Eilenberg functor

CE_\bullet for Lie algebra homology to such a unital DG Lie algebra (or more generally to a DG Lie algebra with a 1-dimensional central extension in any degree $-k \in \mathbb{Z}$). This produces a prefactorisation algebra over the base ring $\mathbb{C}[s\mathbf{1}]$ and ultimately in the holomorphic setting of [CG1, §5.5] to vertex algebra structures over the base ring $\mathbb{C}[s\mathbf{1}]$. In order to obtain vertex algebras over \mathbb{C} , we introduce a variant $\overline{\text{CE}}_\bullet$ of the homological Chevalley-Eilenberg functor in §A.1.2, in which we additionally quotient by the ideal generated by $s\mathbf{1} - 1$, see Proposition A.2. Applying this functor to the unital local Lie algebra $\mathcal{L}_\alpha^\Sigma$ viewed as a prefactorisation algebra on Σ valued in $\text{udgLie}_\mathbb{C}^\oplus$, and taking 0th cohomology to obtain a prefactorisation algebra valued in vector spaces, leads to

$$\mathcal{U}\mathcal{L}_\alpha^\Sigma := H^0(\overline{\text{CE}}_\bullet(\mathcal{L}_\alpha^\Sigma)) \in \text{PFac}(\Sigma, \text{Vec}_\mathbb{C}^\otimes).$$

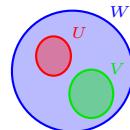
By a slight abuse of terminology, in this paper we will still refer to this construction as taking the *twisted prefactorisation envelope* of the unital local Lie algebra $\mathcal{L}_\alpha^\Sigma$. We refer the reader to §A for the full details of this construction, leading to the definition (A.6).

Since $\mathcal{L}_\alpha^\Sigma(U)$ is concentrated in degrees 0 and 1 for any open subset U in $\text{Top}(\Sigma)$, it follows that for every $i > 0$ we have $\text{CE}_i(\mathcal{L}_\alpha^\Sigma(U)) = 0$ and hence also $\overline{\text{CE}}_i(\mathcal{L}_\alpha^\Sigma(U)) = 0$. In particular, any $\mathcal{A} \in \overline{\text{CE}}_0(\mathcal{L}_\alpha^\Sigma(U))$ is closed and we will denote by $[\mathcal{A}]_U \in \mathcal{U}\mathcal{L}_\alpha^\Sigma(U)$ its 0th cohomology class.

The factorisation products $m_{(U_i),V}$ of $\mathcal{U}\mathcal{L}_\alpha^\Sigma$ can be described explicitly as follows. For any inclusion $U \subset V$ of open subsets in $\text{Top}(\Sigma)$, the factorisation product $m_{U,V}$ is induced by the extension morphism $\mathcal{L}_\alpha^\Sigma(U) \rightarrow \mathcal{L}_\alpha^\Sigma(V)$ of the presheaf $\mathcal{L}_\alpha^\Sigma$. For any inclusion $U \sqcup V \subset W$ in $\text{Top}(\Sigma)^\sqcup$, the factorisation product $m_{(U,V),W}$ is given by the composition

$$\begin{aligned} \mathcal{U}\mathcal{L}_\alpha^\Sigma(U) \otimes \mathcal{U}\mathcal{L}_\alpha^\Sigma(V) &\xrightarrow{\cong} H^0\left(\overline{\text{CE}}_\bullet(\mathcal{L}_\alpha^\Sigma(U)) \otimes \overline{\text{CE}}_\bullet(\mathcal{L}_\alpha^\Sigma(V))\right) \\ &\xrightarrow{\cong} H^0\left(\overline{\text{CE}}_\bullet(\mathcal{L}_\alpha^\Sigma(U) \bar{\oplus} \mathcal{L}_\alpha^\Sigma(V))\right) \longrightarrow \mathcal{U}\mathcal{L}_\alpha^\Sigma(W), \end{aligned} \quad (2.18)$$

where we have first applied the canonical isomorphism given by $[\mathcal{A}]_U \otimes [\mathcal{B}]_V \mapsto [\mathcal{A} \otimes \mathcal{B}]$ for any $\mathcal{A} \in \overline{\text{CE}}_0(\mathcal{L}_\alpha^\Sigma(U))$ and $\mathcal{B} \in \overline{\text{CE}}_0(\mathcal{L}_\alpha^\Sigma(V))$ where we denote by $[\mathcal{A} \otimes \mathcal{B}]$ the 0th cohomology class of $\mathcal{A} \otimes \mathcal{B} \in \overline{\text{CE}}_0(\mathcal{L}_\alpha^\Sigma(U)) \otimes \overline{\text{CE}}_0(\mathcal{L}_\alpha^\Sigma(V))$. The second isomorphism in (2.18) is induced by the Sym-product $\mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{A} \mathcal{B}$. The final morphism in (2.18) is induced by the factorisation product $\mathcal{L}_\alpha^\Sigma(U) \bar{\oplus} \mathcal{L}_\alpha^\Sigma(V) \rightarrow \mathcal{L}_\alpha^\Sigma(W)$ of the unital local Lie algebra $\mathcal{L}_\alpha^\Sigma$ regarded as a prefactorisation algebra valued in $\text{udgLie}_\mathbb{C}^\oplus$, which is given explicitly by extending by zero elements of $\mathcal{L}_\alpha^\Sigma(U)$ and $\mathcal{L}_\alpha^\Sigma(V)$ to $\mathcal{L}_\alpha^\Sigma(W)$ then taking their sum in $\mathcal{L}_\alpha^\Sigma(W)$ and identifying their central extensions. In summary, the above factorisation product is given by

$$\begin{aligned} m_{(U,V),W} : \mathcal{U}\mathcal{L}_\alpha^\Sigma(U) \otimes \mathcal{U}\mathcal{L}_\alpha^\Sigma(V) &\longrightarrow \mathcal{U}\mathcal{L}_\alpha^\Sigma(W), \\ [\mathcal{A}]_U \otimes [\mathcal{B}]_V &\mapsto [\mathcal{A} \mathcal{B}]_W. \end{aligned}$$


The factorisation product $m_{(U_i),V} : \bigotimes_{i=1}^n \mathcal{U}\mathcal{L}_\alpha^\Sigma(U_i) \rightarrow \mathcal{U}\mathcal{L}_\alpha^\Sigma(V)$ for any inclusion $\sqcup_{i=1}^n U_i \subset V$ in $\text{Top}(\Sigma)^\sqcup$ with $n \in \mathbb{Z}_{\geq 3}$ is obtained recursively from the above factorisation product.

2.4 Reality conditions

To describe reality conditions on the prefactorisation algebra $\mathcal{U}\mathcal{L}_\alpha^\Sigma$, which will be needed later in §3.6, we will show in Proposition 2.6 below that $\mathcal{U}\mathcal{L}_\alpha^\Sigma$ is equivariant under a certain action of the group $\mathbb{Z}_2 = \langle t \rangle$ in which t acts by an anti-linear isomorphism of prefactorisation algebras. We begin by constructing a \mathbb{Z}_2 -action on the unital local Lie algebra $\mathcal{L}_\alpha^\Sigma$ after making some further assumptions on the holomorphic vector bundle L from §2.1.1. We will show in §2.4.2 that all these assumptions hold in each of the main examples from §2.1.2.

In this section we take $\tau : \widehat{\Sigma} \rightarrow \widehat{\Sigma}$ to be any orientation reversing involution of $\widehat{\Sigma}$ such that there is an involution $\tau : \Sigma \rightarrow \Sigma$ of Σ with the property that $\pi\tau = \tau\pi$. The main examples are the Euclidean involution $\tau_E : \widehat{\Sigma} \rightarrow \widehat{\Sigma}$, introduced in §1.2, together with the identity involution $\text{id} : \Sigma \rightarrow \Sigma$ or the Lorentzian involution $\tau_L : \widehat{\Sigma} \rightarrow \widehat{\Sigma}$ also introduced in §1.2, in the case $\Sigma = S^2$ so that $\widehat{\Sigma} = \Sigma_+ \sqcup \Sigma_-$, together with the orientation reversing involution $\tau : \Sigma \rightarrow \Sigma$ defined in the same way as τ_L on each copy Σ_\pm .

2.4.1 \mathbb{Z}_2 -equivariance of $\mathcal{UL}_\alpha^\Sigma$

Recall from §2.1.1 that we are assuming the anti-holomorphic vector bundle L on $\widehat{\Sigma}$ to be such that $\Omega_c^{0,\bullet}(U, L)$ is a DG Lie algebra for every open $U \subset \widehat{\Sigma}$ with Lie bracket denoted by (2.1). We will make two further assumptions on L below.

Assumption 4 *The orientation reversing involution $\tau : \widehat{\Sigma} \rightarrow \widehat{\Sigma}$ lifts to an anti-linear involutive holomorphic vector bundle automorphism*

$$\begin{array}{ccc} L & \xrightarrow{\tilde{\tau}} & L \\ \downarrow & & \downarrow \\ \widehat{\Sigma} & \xrightarrow{\tau} & \widehat{\Sigma} \end{array} \quad (2.19)$$

over $\tau : \widehat{\Sigma} \rightarrow \widehat{\Sigma}$ such that the induced isomorphism

$$\hat{\tau} : \Omega_c^{0,\bullet}(U, L) \xrightarrow{\cong} \Omega_c^{0,\bullet}(\tau(U), L), \quad \sigma \otimes \omega \mapsto \tilde{\tau}\sigma \otimes \overline{\tau^*\omega} \quad (2.20)$$

for any open subset $U \subset \widehat{\Sigma}$, with $\sigma \in \Gamma_c(U, L)$ a smooth compactly supported section of L and $\omega \in \Omega_c^{0,\bullet}(U)$, is an anti-linear morphism of DG Lie algebras.

Note that since τ reverses the orientation we have $\tau^*I = -I\tau^*$ where $I : \Omega_c^1 \rightarrow \Omega_c^1$ denotes the complex structure of $\widehat{\Sigma}$, and hence we obtain an anti-linear isomorphism of cosheaves $\Omega_c^{0,\bullet} \xrightarrow{\cong} \tau^*\Omega_c^{0,\bullet}$, $\omega \mapsto \overline{\tau^*\omega}$. The fact that (2.20) commutes with the differential $\bar{\partial}$ follows because $\tilde{\tau}$ is a holomorphic vector bundle automorphism so that $\bar{\partial}_L \tilde{\tau} = \tilde{\tau} \bar{\partial}_L$ and for every $\omega \in \Omega_c^{0,\bullet}(U)$ with $U \subset \widehat{\Sigma}$ we have

$$\overline{\tau^*\bar{\partial}\omega} = \overline{\frac{1}{2}\tau^*\mathbf{d}\omega + \frac{1}{2}\mathbf{i}\tau^*I(\mathbf{d}\omega)} = \frac{1}{2}\mathbf{d}\overline{\tau^*\omega} + \frac{1}{2}\mathbf{i}I(\mathbf{d}\overline{\tau^*\omega}) = \bar{\partial}(\overline{\tau^*\omega}), \quad (2.21)$$

where in the third equality we have used the fact that $\tau^*I = -I\tau^*$.

Recall from §2.1.1 that we are also assuming L to be locally trivial over coordinate charts on $\widehat{\Sigma}$, i.e. for every open subset $U \subset \widehat{\Sigma}$ equipped with a holomorphic coordinate $\xi : U \rightarrow \mathbb{C}$ we have $L|_U \cong U \times \mathfrak{a}_\xi$. Similarly, on the image open $\tau(U) \subset \widehat{\Sigma}$ we have the holomorphic coordinate $\hat{\tau}\xi := \xi \circ \tau : \tau(U) \rightarrow \mathbb{C}$ with the induced local trivialisation $L|_{\tau(U)} \cong \tau(U) \times \mathfrak{a}_{\hat{\tau}\xi}$. Since the anti-linear vector bundle automorphism $\tilde{\tau} : L \rightarrow L$ acts fibrewise, it induces an anti-linear map $\tau : \mathfrak{a}_\xi \rightarrow \mathfrak{a}_{\hat{\tau}\xi}$ and under the isomorphism (2.20) we can represent the isomorphism (2.20) locally by

$$\hat{\tau} : \mathfrak{a}_\xi \otimes \Omega_c^{0,\bullet}(U) \xrightarrow{\cong} \mathfrak{a}_{\hat{\tau}\xi} \otimes \Omega_c^{0,\bullet}(\tau(U)), \quad \mathfrak{a} \otimes \omega \mapsto \tau\mathfrak{a} \otimes \overline{\tau^*\omega}. \quad (2.22)$$

Using the isomorphisms $(\cdot)_\xi : \mathfrak{a} \xrightarrow{\cong} \mathfrak{a}_\xi$ and $(\cdot)_{\hat{\tau}\xi} : \mathfrak{a} \xrightarrow{\cong} \mathfrak{a}_{\hat{\tau}\xi}$ with the canonical copy \mathfrak{a} introduced in §2.1.1, we obtain an induced anti-linear involution $\tau : \mathfrak{a} \rightarrow \mathfrak{a}$.

Recall finally from §2.1.1 that we are also assuming L to be such that the DG Lie algebra $\Omega_c^{0,\bullet}(U, L)$ is equipped with a 2-cocycle (2.3).

Assumption 5 *The 2-cocycle (2.3) satisfies the reality condition*

$$\overline{\alpha(a, b)} = \alpha(\hat{\tau}a, \hat{\tau}b) \quad (2.23)$$

for all $a, b \in \Omega_c^{0,\bullet}(U, L)$.

It then follows that the anti-linear morphism of DG Lie algebras (2.20) extends to an anti-linear isomorphism of unital local Lie algebras

$$\hat{\tau} : \mathcal{L}_\alpha^{\widehat{\Sigma}} \xrightarrow{\cong} \tau^* \mathcal{L}_\alpha^{\widehat{\Sigma}},$$

defined by sending $\mathbf{1} \mapsto \mathbf{1}$, which satisfies $(\tau^* \hat{\tau}) \circ \hat{\tau} = \text{id}$. Note here that the pullback $\tau^* \mathcal{L}_\alpha^{\Sigma}$ of the precosheaf of unital DG Lie algebras $\mathcal{L}_\alpha^{\Sigma}$ also satisfies the local Lie algebra condition (A.5), so that $\tau^* \mathcal{L}_\alpha^{\Sigma} \in \mathbf{uLocLie}_{\mathbb{C}}(\Sigma)$. Since $\pi\tau = \tau\pi$ we have an induced anti-linear isomorphism

$$\hat{\tau} : \mathcal{L}_\alpha^{\Sigma} \xrightarrow{\cong} \tau^* \mathcal{L}_\alpha^{\Sigma} \tag{2.24}$$

also satisfying $(\tau^* \hat{\tau}) \circ \hat{\tau} = \text{id}$. We will say that $\mathcal{L}_\alpha^{\Sigma} \in \mathbf{uLocLie}_{\mathbb{C}}(\Sigma)$ is \mathbb{Z}_2 -equivariant.

Recall from §2.3 that we denote the composition of morphisms in $\mathbf{PFac}(\Sigma, \mathbf{C}^\otimes)$ by \circ .

Proposition 2.6. *We have an anti-linear isomorphism of prefactorisation algebras*

$$\hat{\tau} : \mathcal{U}\mathcal{L}_\alpha^{\Sigma} \xrightarrow{\cong} \tau^* \mathcal{U}\mathcal{L}_\alpha^{\Sigma}$$

satisfying $(\tau^* \hat{\tau}) \circ \hat{\tau} = \text{id}_{\mathcal{U}\mathcal{L}_\alpha^{\Sigma}}$, i.e. the prefactorisation algebra $\mathcal{U}\mathcal{L}_\alpha^{\Sigma}$ is \mathbb{Z}_2 -equivariant.

Proof. Firstly, the anti-linear isomorphism (2.24) of unital local Lie algebras on Σ induces, by Proposition A.3, an anti-linear isomorphism of $\mathbf{PFac}(\Sigma, \mathbf{udgLie}_{\mathbb{C}}^{\widehat{\oplus}})$, i.e. a natural isomorphism of multifunctors $\mathbf{Top}(\Sigma)^\sqcup \rightarrow \mathbf{udgLie}_{\mathbb{C}}^{\widehat{\oplus}}$. Abusing notation slightly, we denote it in the same way, as $\hat{\tau} : \mathcal{L}_\alpha^{\Sigma} \xrightarrow{\cong} \tau^* \mathcal{L}_\alpha^{\Sigma}$, but where now we view $\mathcal{L}_\alpha^{\Sigma}$ as a prefactorisation algebra on Σ valued in $\mathbf{udgLie}_{\mathbb{C}}^{\widehat{\oplus}}$ and $\tau^* \mathcal{L}_\alpha^{\Sigma} = \mathcal{L}_\alpha^{\Sigma} \tau$ as its pre-composition with (the identity natural transformation of) the induced multifunctor $\tau : \mathbf{Top}(\Sigma)^\sqcup \rightarrow \mathbf{Top}(\Sigma)^\sqcup$. The property $(\tau^* \hat{\tau}) \circ \hat{\tau} = \text{id}$ of the morphism (2.24) of unital local Lie algebras on Σ then turns into the property $(\tau^* \hat{\tau}) \circ \hat{\tau} = \text{id}_{\mathcal{L}_\alpha^{\Sigma}}$ for the natural isomorphism $\hat{\tau} : \mathcal{L}_\alpha^{\Sigma} \xrightarrow{\cong} \tau^* \mathcal{L}_\alpha^{\Sigma}$ of multifunctors $\mathbf{Top}(\Sigma)^\sqcup \rightarrow \mathbf{udgLie}_{\mathbb{C}}^{\widehat{\oplus}}$.

The horizontal post-composition of this natural isomorphism with (the identity natural transformation of) the composite functor $H^0 \overline{\mathbf{CE}}_\bullet$ yields a natural isomorphism

$$H^0 \overline{\mathbf{CE}}_\bullet \hat{\tau} : \mathcal{U}\mathcal{L}_\alpha^{\Sigma} \xrightarrow{\cong} \mathcal{U}\mathcal{L}_\alpha^{\Sigma} \circ \tau \tag{2.25}$$

of multifunctors $\mathbf{Top}(\Sigma)^\sqcup \rightarrow \mathbf{Vec}_{\mathbb{C}}^\otimes$. Then by the interchange law for the vertical and horizontal composition of natural transformations we have

$$(H^0 \overline{\mathbf{CE}}_\bullet (\tau^* \hat{\tau})) \circ (H^0 \overline{\mathbf{CE}}_\bullet \hat{\tau}) = H^0 \overline{\mathbf{CE}}_\bullet ((\tau^* \hat{\tau}) \circ \hat{\tau}) = H^0 \overline{\mathbf{CE}}_\bullet \text{id}_{\mathcal{L}_\alpha^{\Sigma}} = \text{id}_{\mathcal{U}\mathcal{L}_\alpha^{\Sigma}}. \tag{2.26}$$

If by a slight abuse of notation we denote the morphism in (2.25) also by $\hat{\tau}$, so that the first morphism on the left hand side of (2.26) is the pre-composition of $\hat{\tau}$ with the identity natural transformation of the multifunctor τ , i.e. $\tau^* \hat{\tau} = \hat{\tau} \tau$, then the relation (2.26) is then equivalent to the desired identity $(\tau^* \hat{\tau}) \circ \hat{\tau} = \text{id}_{\mathcal{U}\mathcal{L}_\alpha^{\Sigma}}$. \square

2.4.2 Examples

We now show how to construct, for each of the three main examples of holomorphic vector bundles L over $\widehat{\Sigma}$ from §2.1.2, the vector bundle automorphism $\tilde{\tau} : L \rightarrow L$ in (2.19) satisfying the further conditions assumed in the general discussion of §2.4.1.

Kac-Moody: We fix an anti-linear involution $\tau : \mathfrak{g} \xrightarrow{\cong} \mathfrak{g}$. This lifts the orientation reversing involution $\tau : \widehat{\Sigma} \rightarrow \widehat{\Sigma}$ to an anti-linear involutive automorphism of the trivial bundle (2.7). The induced isomorphism (2.22) is clearly a morphism of DG Lie algebra since

$$\begin{aligned}\hat{\tau}([\mathbf{X} \otimes \omega, \mathbf{Y} \otimes \eta]) &= \tau[\mathbf{X}, \mathbf{Y}] \otimes \overline{\tau^*(\omega \wedge \eta)} = [\tau\mathbf{X}, \tau\mathbf{Y}] \otimes \overline{\tau^*\omega} \wedge \overline{\tau^*\eta} \\ &= [\tau\mathbf{X} \otimes \overline{\tau^*\omega}, \tau\mathbf{Y} \otimes \overline{\tau^*\eta}] = [\hat{\tau}(\mathbf{X} \otimes \omega), \hat{\tau}(\mathbf{Y} \otimes \eta)].\end{aligned}$$

In the second equality we used the fact that $\tau : \mathfrak{g} \xrightarrow{\cong} \mathfrak{g}$ is an (anti-linear) automorphism of \mathfrak{g} .

In order to show that the 2-cocycle in (2.8b) satisfies (2.23), we assume that the bilinear form $\kappa : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{C}$ is such that

$$\kappa(\tau\mathbf{X}, \tau\mathbf{Y}) = \overline{\kappa(\mathbf{X}, \mathbf{Y})} \quad (2.27)$$

for every $\mathbf{X}, \mathbf{Y} \in \mathfrak{g}$. Note that, $\overline{\tau^*\partial\omega} = \partial(\overline{\tau^*\omega})$ for every $\omega \in \Omega_c^{0,\bullet}(U)$ by the same computation as in (2.21) and so for any $\omega, \eta \in \Omega_c^{0,\bullet}(U)$ we then have

$$-\frac{1}{2\pi i} \overline{\int_{\widehat{\Sigma}} \partial\omega \wedge \eta} = -\frac{1}{2\pi i} \int_{\widehat{\Sigma}} \tau^*(\overline{\tau^*\partial\omega} \wedge \overline{\tau^*\eta}) = \frac{1}{2\pi i} \int_{\widehat{\Sigma}} \partial(\overline{\tau^*\omega}) \wedge \overline{\tau^*\eta} \quad (2.28)$$

where in the second equality we have also used the fact that $\tau : \widehat{\Sigma} \rightarrow \widehat{\Sigma}$ is orientation reversing. Note here that we have taken the integration over all of $\widehat{\Sigma}$ rather than just U by implicitly using the extension morphism $\Omega_c^{0,\bullet}(U) \rightarrow \Omega_c^{0,\bullet}(\widehat{\Sigma})$. It now follows from combining (2.27) and (2.28) that the 2-cocycle in (2.8b) satisfies (2.23).

We observe for later that the bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{C}$, $\langle \mathbf{X}, \mathbf{Y} \rangle := -\kappa(\tau\mathbf{X}, \mathbf{Y})$ is a Hermitian sesquilinear form on \mathfrak{g} , i.e. it is anti-linear in the first argument, linear in the second and $\langle \mathbf{X}, \mathbf{Y} \rangle = \overline{\langle \mathbf{Y}, \mathbf{X} \rangle}$. It is non-degenerate if κ is. In particular, $\langle \mathbf{X}, \mathbf{X} \rangle \in \mathbb{R}$ for all $\mathbf{X} \in \mathfrak{g}$.

Virasoro: Since $\tau : \widehat{\Sigma} \rightarrow \widehat{\Sigma}$ is orientation reversing, its differential defines a vector bundle isomorphism $d\tau : T^{1,0}\widehat{\Sigma} \xrightarrow{\cong} T^{0,1}\widehat{\Sigma}$. Postcomposing this with complex conjugation we obtain the desired anti-linear involutive vector bundle automorphism $\tilde{\tau} := \overline{d\tau(\cdot)} : L \rightarrow L$ in (2.19) of the holomorphic tangent bundle $L = T^{1,0}\widehat{\Sigma}$ from (2.9).

Recalling that $\mathfrak{a}_\xi = \text{span}_{\mathbb{C}}\{\partial_\xi\}$, and likewise $\mathfrak{a}_{\hat{\tau}\xi} = \text{span}_{\mathbb{C}}\{\partial_{\hat{\tau}\xi}\}$, the induced anti-linear map $\tau : \mathfrak{a}_\xi \rightarrow \mathfrak{a}_{\hat{\tau}\xi}$ along the fibres is given explicitly by $\partial_\xi \mapsto \partial_{\hat{\tau}\xi}$. The isomorphism (2.22) then sends $\partial_\xi \otimes u$ for any $u \in \Omega_c^{0,\bullet}(U)$ to $\partial_{\hat{\tau}\xi} \otimes \overline{\tau^*u}$. To see that this is a morphism of DG Lie algebras, with respect to the Lie bracket (2.10a), note that for $u, v \in \Omega_c^{0,\bullet}(U)$ we have

$$\begin{aligned}\hat{\tau}([\partial_\xi \otimes u, \partial_\xi \otimes v]) &= \hat{\tau}(\partial_\xi \otimes (u\partial_\xi v - v\partial_\xi u)) = \partial_{\hat{\tau}\xi} \otimes (\overline{\tau^*u} \partial_{\hat{\tau}\xi} \overline{\tau^*v} - \overline{\tau^*v} \partial_{\hat{\tau}\xi} \overline{\tau^*u}) \\ &= [\partial_{\hat{\tau}\xi} \otimes \overline{\tau^*u}, \partial_{\hat{\tau}\xi} \otimes \overline{\tau^*v}] = [\hat{\tau}(\partial_\xi \otimes u), \hat{\tau}(\partial_\xi \otimes v)].\end{aligned}$$

The anti-linear involution $\tau : \mathfrak{a} \rightarrow \mathfrak{a}$ induced on the canonical copy $\mathfrak{a} = \text{span}_{\mathbb{C}}\{\Omega\}$ is simply given by complex conjugation $x\Omega \mapsto \bar{x}\Omega$.

To see that the 2-cocycle (2.10b) satisfies the condition (2.23), we use the identity (2.28) applied to $\omega = \partial_\xi u$ and $\eta = \partial_\xi v$, and noting that $\overline{\tau^*\partial_\xi u} = \partial_{\hat{\tau}\xi} \overline{\tau^*u}$ and $\overline{\tau^*\partial_\xi v} = \partial_{\hat{\tau}\xi} \overline{\tau^*v}$ we find

$$\overline{\alpha_\xi(\partial_\xi \otimes u, \partial_\xi \otimes v)} = \alpha_{\hat{\tau}\xi}(\partial_{\hat{\tau}\xi} \otimes \overline{\tau^*u}, \partial_{\hat{\tau}\xi} \otimes \overline{\tau^*v}) = \alpha_{\hat{\tau}\xi}(\hat{\tau}(\partial_\xi \otimes u), \hat{\tau}(\partial_\xi \otimes v)).$$

The fact that $c \in \mathbb{R}$ was used in the first equality. We then recall from §2.1.2 that the 2-cocycle (2.10b) depends on the coordinate ξ only up to a 2-coboundary, explicitly $\alpha_{\hat{\tau}\xi} = \alpha_\xi + \delta\beta_{\hat{\tau}\xi, \xi}$. The 2-cocycle property (2.23) is therefore satisfied in this example only up to a 2-coboundary.

$\beta\gamma$ system: The pullback by the orientation reversing map $\tau : \widehat{\Sigma} \rightarrow \widehat{\Sigma}$ defines a vector bundle isomorphism $\tau^* : T^{*1,0}\widehat{\Sigma} \xrightarrow{\cong} T^{*0,1}\widehat{\Sigma}$ so that its postcomposition with complex conjugation yields an anti-linear involutive vector bundle automorphism $\tilde{\tau} := \overline{\tau^*(\cdot)} : T^{*1,0}\widehat{\Sigma} \rightarrow T^{*1,0}\widehat{\Sigma}$ over $\tau : \widehat{\Sigma} \rightarrow \widehat{\Sigma}$ of the holomorphic cotangent bundle $T^{*1,0}\widehat{\Sigma}$. Notice that this vector bundle automorphism naturally covers $\tau^{-1} : \widehat{\Sigma} \rightarrow \widehat{\Sigma}$, however since τ is an involution we have $\tau^{-1} = \tau$ so that it also defines a vector bundle automorphism over $\tau : \widehat{\Sigma} \rightarrow \widehat{\Sigma}$. Combining this with the canonical anti-linear isomorphism of the trivial vector bundle $\widehat{\Sigma} \times \mathbb{C} \rightarrow \widehat{\Sigma} \times \mathbb{C}$ given by τ on the base $\widehat{\Sigma}$ and complex conjugation on the fibre \mathbb{C} , we obtain the desired anti-linear involutive vector bundle automorphism $\tilde{\tau} : L \rightarrow L$ of the holomorphic vector bundle (2.13).

Recalling that in the present case we have $\mathfrak{a}_\xi = \text{span}_{\mathbb{C}}\{1, d\xi\}$ and $\mathfrak{a}_{\hat{\tau}\xi} = \text{span}_{\mathbb{C}}\{1, d(\hat{\tau}\xi)\}$, the induced anti-linear involution $\tau : \mathfrak{a}_\xi \rightarrow \mathfrak{a}_{\hat{\tau}\xi}$ is given on the basis elements by $1 \mapsto 1$ and $d\xi \mapsto d(\hat{\tau}\xi)$. The induced anti-linear involution $\tau : \mathfrak{a} \rightarrow \mathfrak{a}$ on $\mathfrak{a} = \text{span}_{\mathbb{C}}\{\beta, \gamma\}$ is again just given by complex conjugation $x\beta + y\gamma \mapsto \bar{x}\beta + \bar{y}\gamma$.

Since the Lie bracket is trivial the only thing to check is that the 2-cocycle (2.14) satisfies the condition (2.23). But under the canonical isomorphism $\Omega_c^{0,\bullet}(U, L) \cong \Omega_c^{0,\bullet}(U) \oplus \Omega_c^{1,\bullet}(U)$, the isomorphism (2.22) acts simply as $\omega \mapsto \overline{\tau^*\omega}$ on any $\omega \in \Omega_c^{0,\bullet}(U) \oplus \Omega_c^{1,\bullet}(U)$. It then follows, exactly as in (2.28), that

$$\overline{\alpha(\omega, \eta)} = -\frac{1}{2\pi i} \overline{\int_{\widehat{\Sigma}} \omega \wedge \eta} = \frac{1}{2\pi i} \int_{\widehat{\Sigma}} \overline{\tau^*\omega} \wedge \overline{\tau^*\eta} = \alpha(\hat{\tau}\omega, \hat{\tau}\eta)$$

for any $\omega, \eta \in \Omega_c^{0,\bullet}(U) \oplus \Omega_c^{1,\bullet}(U)$, where as before we implicitly use the extension morphism $\Omega_c^{0,\bullet}(U) \rightarrow \Omega_c^{0,\bullet}(\widehat{\Sigma})$, as required.

3 Full vertex algebra $\mathcal{F}_p^{\mathfrak{a}, \alpha}$

As in §2, we will keep working with an arbitrary connected 2-dimensional conformal manifold Σ which could be non-orientable and also with boundary. However, throughout this paper we will not treat boundary conditions and therefore ignore points on the boundary $\partial\Sigma$ by only working locally arounds points in the interior $\Sigma^\circ \subset \Sigma$. Later, towards the end of §3.4, we will specialise to the case of the 2-sphere $\Sigma = S^2$.

3.1 Underlying vector space

As mentioned in §1.3, we are interested in full vertex algebras whose underlying vector spaces are built as induced modules of certain infinite-dimensional Lie algebras. In what follows we will focus on the three main examples of infinite-dimensional Lie algebras given by centrally extended loop algebras which includes affine Kac-Moody algebras, the Virasoro algebra and the $\beta\gamma$ system. These will closely correspond to the three examples of unital local Lie algebra $\mathcal{L}_\alpha^\Sigma$ given in §2.1.2. It will be convenient to describe all these examples uniformly as follows.

Recall the finite-dimensional vector space \mathfrak{a} introduced in §2.1.1, and consider the associated infinite-dimensional vector spaces

$$\widehat{\mathfrak{a}} := \mathfrak{a} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}, \quad \widehat{\mathfrak{a}} := \mathfrak{a} \otimes \mathbb{C}[\bar{t}, \bar{t}^{-1}] \oplus \mathbb{C}\bar{\mathbf{k}} \quad (3.1)$$

where t and \bar{t} are independent formal variables referred to as loop parameters. These vector spaces come equipped with Lie algebra structures $[\cdot, \cdot] : \widehat{\mathfrak{a}} \otimes \widehat{\mathfrak{a}} \rightarrow \widehat{\mathfrak{a}}$ and $[\cdot, \cdot] : \widehat{\mathfrak{a}} \otimes \widehat{\mathfrak{a}} \rightarrow \widehat{\mathfrak{a}}$, respectively, with respect to which \mathbf{k} and $\bar{\mathbf{k}}$ are central. We will describe these explicitly below in each of the three main examples in terms of the loop generators

$$\mathfrak{a}_{(n)} := \mathfrak{a} \otimes t^n \in \widehat{\mathfrak{a}}, \quad \bar{\mathfrak{a}}_{(n)} := \mathfrak{a} \otimes \bar{t}^n \in \widehat{\mathfrak{a}}, \quad (3.2)$$

for $\mathfrak{a} \in \mathfrak{a}$ and $n \in \mathbb{Z}$. In all cases, the Lie algebras (3.1) admit a decomposition into a direct sum $\widehat{\mathfrak{a}} = \widehat{\mathfrak{a}}_+ \oplus \widehat{\mathfrak{a}}_-$ and $\widehat{\mathfrak{a}} = \widehat{\mathfrak{a}}_+ \oplus \widehat{\mathfrak{a}}_-$ of Lie subalgebras

$$\begin{aligned}\widehat{\mathfrak{a}}_+ &:= \mathfrak{a} \otimes \mathbb{C}[t] \oplus \mathbb{C}\mathbf{k}, & \widehat{\mathfrak{a}}_- &:= \mathfrak{a} \otimes t^{-1}\mathbb{C}[t^{-1}] \\ \widehat{\mathfrak{a}}_+ &:= \mathfrak{a} \otimes \mathbb{C}[\bar{t}] \oplus \mathbb{C}\bar{\mathbf{k}}, & \widehat{\mathfrak{a}}_- &:= \mathfrak{a} \otimes \bar{t}^{-1}\mathbb{C}[\bar{t}^{-1}].\end{aligned}$$

Let $\mathbb{C}|0\rangle$ be the 1-dimensional module over the direct sum Lie algebra $\widehat{\mathfrak{a}}_+ \oplus \widehat{\mathfrak{a}}_-$ on which $\mathfrak{a} \otimes \mathbb{C}[t]$ and $\mathfrak{a} \otimes \mathbb{C}[\bar{t}]$ act trivially and the central elements \mathbf{k} and $\bar{\mathbf{k}}$ both act by multiplication by 1. Define the *full affine vertex algebra* $\mathbb{F}^{\mathfrak{a},\alpha}$ as the induced module over the direct sum Lie algebra $\widehat{\mathfrak{a}} \oplus \widehat{\mathfrak{a}}$, namely

$$\mathbb{F}^{\mathfrak{a},\alpha} := \text{Ind}_{\widehat{\mathfrak{a}}_+ \oplus \widehat{\mathfrak{a}}_+}^{\widehat{\mathfrak{a}} \oplus \widehat{\mathfrak{a}}} \mathbb{C}|0\rangle.$$

Note that we have included the 2-cocycle α from (2.3) in the notation. This is a slight abuse of notation since α is not directly used in the definition but it will be closely related to the central extension of the Lie algebra structures on (3.1) in each of the examples described below. We therefore use the subscript α in the notation to emphasise this choice of central extension. In each of the three cases we will have an isomorphism of vector spaces $\mathbb{F}^{\mathfrak{a},\alpha} \cong U(\widehat{\mathfrak{a}}_- \oplus \widehat{\mathfrak{a}}_-)|0\rangle$ so that a general element of $\mathbb{F}^{\mathfrak{a},\alpha}$ is given by a linear combination of expressions of the form

$$\mathfrak{a}_{(-m_r)}^r \dots \mathfrak{a}_{(-m_1)}^1 \bar{\mathfrak{b}}_{(-n_r)}^{\bar{r}} \dots \bar{\mathfrak{b}}_{(-n_1)}^1 |0\rangle \quad (3.3)$$

for any $\mathfrak{a}^i \in \mathfrak{a}$, $m_i \in \mathbb{Z}_{\geq 1}$ with $i \in \{1, \dots, r\}$ and $\mathfrak{b}^j \in \mathfrak{a}$, $n_j \in \mathbb{Z}_{\geq 1}$ with $j \in \{1, \dots, \bar{r}\}$ where $r, \bar{r} \in \mathbb{Z}_{\geq 0}$. The full affine vertex algebra $\mathbb{F}^{\mathfrak{a},\alpha}$ is canonically isomorphic to the tensor product of two copies of the usual affine vertex algebra, namely we have an isomorphism of complex vector spaces $\mathbb{F}^{\mathfrak{a},\alpha} \cong \mathbb{V}^{\mathfrak{a},\alpha} \otimes \bar{\mathbb{V}}^{\mathfrak{a},\alpha}$ where

$$\mathbb{V}^{\mathfrak{a},\alpha} := \text{Ind}_{\widehat{\mathfrak{a}}_+}^{\widehat{\mathfrak{a}}} \mathbb{C}|0\rangle, \quad \bar{\mathbb{V}}^{\mathfrak{a},\alpha} := \text{Ind}_{\widehat{\mathfrak{a}}_+}^{\widehat{\mathfrak{a}}} \mathbb{C}|0\rangle$$

are the affine vertex algebra and its ‘anti-chiral’ copy.

We now describe the Lie algebra structures on (3.1) in all three cases of interest, where the finite-dimensional vector space \mathfrak{a} was defined in each case in §2.1.1.

Kac-Moody: We can endow (3.1) for $\mathfrak{a} = \mathfrak{g}$ with Lie brackets described in terms of the loop generators (3.2) by

$$[\mathbf{X}_{(m)}, \mathbf{Y}_{(n)}] = [\mathbf{X}, \mathbf{Y}]_{(m+n)} + m \kappa(\mathbf{X}, \mathbf{Y}) \delta_{m+n,0} \mathbf{k}, \quad (3.4a)$$

$$[\bar{\mathbf{X}}_{(m)}, \bar{\mathbf{Y}}_{(n)}] = \overline{[\mathbf{X}, \mathbf{Y}]_{(m+n)}} + m \kappa(\mathbf{X}, \mathbf{Y}) \delta_{m+n,0} \bar{\mathbf{k}}. \quad (3.4b)$$

for every $\mathbf{X}, \mathbf{Y} \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$. In this case $\widehat{\mathfrak{a}}$ and $\widehat{\mathfrak{a}}$ are two copies of the centrally extended loop algebra associated with \mathfrak{g} . Since the case when \mathfrak{g} is a simple Lie algebra corresponds to a pair of untwisted affine Kac-Moody algebras, by a slight abuse of terminology we will keep referring to the general case with \mathfrak{g} arbitrary as the ‘Kac-Moody’ case.

Virasoro: We endow (3.1) for $\mathfrak{a} = \text{span}_{\mathbb{C}}\{\Omega\}$ with the Lie brackets given in terms of the loop generators (3.2) of the fixed basis element Ω by

$$[\Omega_{(m)}, \Omega_{(n)}] = (m-n)\Omega_{(m+n-1)} + \frac{m(m-1)(m-2)}{12} c \delta_{m+n,2} \mathbf{k}, \quad (3.5a)$$

$$[\bar{\Omega}_{(m)}, \bar{\Omega}_{(n)}] = (m-n)\bar{\Omega}_{(m+n-1)} + \frac{m(m-1)(m-2)}{12} c \delta_{m+n,2} \bar{\mathbf{k}}. \quad (3.5b)$$

In this case $\widehat{\mathfrak{a}}$ and $\widehat{\mathfrak{a}}$ are two copies of the Virasoro algebra, where the usual generators $L_{(n)}$ and $\bar{L}_{(n)}$ satisfying the more familiar looking Virasoro relations are given by a simple shift in the indices, namely $L_{(n)} = \Omega_{(n+1)}$ and $\bar{L}_{(n)} = \bar{\Omega}_{(n+1)}$.

$\beta\gamma$ system: We endow (3.1) for $\mathfrak{a} = \text{span}_{\mathbb{C}}\{\beta, \gamma\}$ with the Lie brackets given in terms of the loop generators (3.2) of the fixed basis elements β and γ by

$$[\beta_{(m)}, \gamma_{(n)}] = \delta_{m+n, -1} \mathbf{k}, \quad (3.6a)$$

$$[\bar{\beta}_{(m)}, \bar{\gamma}_{(n)}] = \delta_{m+n, -1} \bar{\mathbf{k}}. \quad (3.6b)$$

In this case $\widehat{\mathfrak{a}}$ and $\widehat{\bar{\mathfrak{a}}}$ are two copies of the infinite-dimensional Weyl algebra, also known as the $\beta\gamma$ system, whose generators are often denoted by $a_{(n)}$ and $a_{(n)}^*$ with the more standard Lie algebra relations obtained from (3.6) by a simple shift of indices

$$a_{(n)} = \beta_{(n)}, \quad a_{(n)}^* = \gamma_{(n-1)}, \quad \bar{a}_{(n)} = \bar{\beta}_{(n)}, \quad \bar{a}_{(n)}^* = \bar{\gamma}_{(n-1)}. \quad (3.7)$$

3.1.1 Geometric realisation in $\mathcal{UL}_\alpha^\Sigma(U)$

Let $U \subset \Sigma^\circ$ be a connected open subset of the interior $\Sigma^\circ \subset \Sigma$. Its preimage under the projection $\pi : \widehat{\Sigma} \rightarrow \Sigma$ is a disjoint union of two copies of U which we denote $\pi^{-1}(U) = U_+ \sqcup U_-$. Suppose that U_+ is equipped with a holomorphic coordinate $\xi : U_+ \rightarrow \mathbb{C}$ such that $\xi(U_+) \subset \mathbb{C}$ is bounded. Since U_- comes equipped with the opposite complex structure to U_+ , the same map ξ viewed as a function on U_- defines an anti-holomorphic coordinate $\bar{\xi} : U_- \rightarrow \mathbb{C}$ on $U_- \subset \pi^{-1}(U)$. In other words, the complex conjugate map $\bar{\xi} : U_- \rightarrow \mathbb{C}$ defines a holomorphic coordinate on U_- . We then refer to $\xi : U \rightarrow \mathbb{C}$ as a *local (holomorphic) coordinate* on $U \subset \Sigma^\circ$.

Given any $p \in U \subset \Sigma^\circ$, we let $p_\pm \in U_\pm$ denote its preimages under $\pi : \widehat{\Sigma} \rightarrow \Sigma$. We refer to $\xi(p) := \xi(p_+)$, i.e. the value of $\xi : U_+ \rightarrow \mathbb{C}$ at $p_+ \in U_+$, as its *holomorphic coordinate*. Similarly, we refer to $\bar{\xi}(p) := \bar{\xi}(p_-)$, i.e. the value of $\bar{\xi} : U_- \rightarrow \mathbb{C}$ at $p_- \in U_-$, as its *anti-holomorphic coordinate*. We also define *shifted local (anti-)holomorphic coordinates* as

$$\xi_p := \xi - \xi(p) : U_+ \longrightarrow \mathbb{C}, \quad \bar{\xi}_p := \bar{\xi} - \bar{\xi}(p) : U_- \longrightarrow \mathbb{C} \quad (3.8)$$

so that any $q \in U \subset \Sigma$ has holomorphic coordinate $\xi_p(q) = \xi(q) - \xi(p)$ and anti-holomorphic coordinate $\bar{\xi}_p(q) = \bar{\xi}(q) - \bar{\xi}(p)$.

In view of giving a geometric description of the vector space $\mathbb{F}^{\mathfrak{a}, \alpha}$ using the prefactorisation algebra $\mathcal{UL}_\alpha^\Sigma$ it is useful to first consider the level 1 subspace $(\widehat{\mathfrak{a}}_- \oplus \widehat{\bar{\mathfrak{a}}_-})|0\rangle \subset \mathbb{F}^{\mathfrak{a}, \alpha}$. We have an injective linear map

$$\begin{aligned} (\widehat{\mathfrak{a}}_- \oplus \widehat{\bar{\mathfrak{a}}_-})|0\rangle &\hookrightarrow \mathfrak{a} \otimes \mathcal{O}_\infty(\xi_p(U_+)^c) \oplus \mathfrak{a} \otimes \mathcal{O}_\infty(\bar{\xi}_p(U_-)^c), \\ \mathfrak{a}_{(-m)}|0\rangle + \bar{\mathfrak{b}}_{(-n)}|0\rangle &\longmapsto (\mathfrak{a} \otimes [\lambda^{-m}], \mathfrak{b} \otimes [\bar{\lambda}^{-n}]) \end{aligned} \quad (3.9)$$

for $\mathfrak{a}, \mathfrak{b} \in \mathfrak{a}$ and $m, n \in \mathbb{Z}_{\geq 1}$, where $[\lambda^{-m}] \in \mathcal{O}_\infty(\xi_p(U_+)^c)$ denotes the germ of the holomorphic function $\xi_p(U_+)^c \subset \mathbb{C}P^1 \setminus \{0\} \rightarrow \mathbb{C}$, $\lambda \mapsto \lambda^{-m}$ and similarly $[\bar{\lambda}^{-n}] \in \mathcal{O}_\infty(\bar{\xi}_p(U_-)^c)$ denotes the germ of the holomorphic function $\bar{\xi}_p(U_-)^c \subset \mathbb{C}P^1 \setminus \{0\} \rightarrow \mathbb{C}$, $\bar{\lambda} \mapsto \bar{\lambda}^{-n}$.

Recall the $(0, 1)$ -form $[f] \in \Omega_c^{0,1}(W)$ on a bounded open subset $W \subset \mathbb{C}$ associated with a given germ $[f] \in \mathcal{O}_\infty(W^c)$ defined in §2.2. In the case of the germ $[\lambda^{-m}] \in \mathcal{O}_\infty(\xi_p(U_+)^c)$ we can pick the representative $f : \Delta_f := \mathbb{C}P^1 \setminus \{0\} \rightarrow \mathbb{C}$, $\lambda \mapsto \lambda^{-m}$. By Lemma 2.3, the $(0, 1)$ -form $[\lambda^{-m}] \in \Omega_c^{0,1}(\xi_p(U_+))$ is then given, up to $\bar{\partial}$ -exact terms, by $-f \bar{\partial} \rho$ where $\rho \in \Omega_c^{0,0}(\xi_p(U_+))$ is a smooth bump function equal to 1 on some neighbourhood of $\{0\} = (\Delta_f)^c \subset \xi_p(U_+)$. The pullback of $f|_{\xi_p(U_+) \setminus \{0\}}$ along $\xi_p : U_+ \rightarrow \xi_p(U_+)$ is $\xi_p^* f : U_+ \setminus \{p_+\} \rightarrow \mathbb{C}$, $q \mapsto \xi_p(q)^{-m}$, i.e. the function $\xi_p^{-m} : U_+ \setminus \{p_+\} \rightarrow \mathbb{C}$. Then $\xi_p^* [\lambda^{-m}] \in \Omega_c^{0,1}(U_+)$, which for brevity we denote by $[\xi_p^{-m}]$, is given by Lemma 2.3 up to $\bar{\partial}$ -exact terms by $\xi_p^{-m} \bar{\partial}(\xi_p^* \rho)$ where $\xi_p^* \rho \in \Omega_c^{0,0}(U_+)$ is equal to 1 in a neighbourhood of $p_+ \in U_+$. Similarly, we let $[\bar{\xi}_p^{-n}]$ stand for $\bar{\xi}_p^* [\bar{\lambda}^{-n}] \in \Omega_c^{0,1}(U_-)$ which is given again by Lemma 2.3 up to $\bar{\partial}$ -exact terms by $\bar{\xi}_p^{-n} \bar{\partial}(\bar{\xi}_p^* \rho')$ for some smooth bump function $\rho' \in \Omega_c^{0,0}(\bar{\xi}_p(U_-))$ equal to 1 in some neighbourhood of $0 \in \bar{\xi}_p(U_-)$.

Combining (3.9) with the isomorphism from Theorem 2.5 defined using the local holomorphic coordinate on $\pi^{-1}(U) = U_+ \sqcup U_-$ given by (3.8) we obtain an injective linear map

$$\begin{aligned} (\widehat{\mathfrak{a}}_- \oplus \widehat{\mathfrak{a}}_-) |0\rangle &\hookrightarrow H^1(\mathcal{L}_\alpha^\Sigma(U)) = H^0(\mathcal{L}_\alpha^\Sigma(U)[1]), \\ \mathfrak{a}_{(-m)} |0\rangle + \bar{\mathfrak{b}}_{(-n)} |0\rangle &\longmapsto [\mathfrak{a}_{\xi_p} \otimes [\xi_p^{-m}]_{U_+} + \mathfrak{b}_{\bar{\xi}_p} \otimes [\bar{\xi}_p^{-n}]_{U_-}]_U. \end{aligned} \quad (3.10)$$

Note, in particular, that although $[\xi_p^{-m}]$ and $[\bar{\xi}_p^{-n}]$ were described above only up to $\bar{\partial}$ -exact terms, this ambiguity drops out from taking the cohomology class $[\cdot]_U$ in (3.10).

If we regard $H^0(\mathcal{L}_\alpha^\Sigma(U)[1])$ as a subspace of $\mathcal{U}\mathcal{L}_\alpha^\Sigma(U)$ then (3.10) gives an embedding of the subspace $(\widehat{\mathfrak{a}}_- \oplus \widehat{\mathfrak{a}}_-) |0\rangle \subset \mathbb{F}^{\mathfrak{a},\alpha}$ into $\mathcal{U}\mathcal{L}_\alpha^\Sigma(U)$ for any neighbourhood $U \subset \Sigma^\circ$ of p equipped with the local coordinate ξ . A general element of $\mathbb{F}^{\mathfrak{a},\alpha}$ involves products of elements from $\widehat{\mathfrak{a}}_-$ and $\widehat{\mathfrak{a}}_-$ in some order, and we can realise such an ordering geometrically in $\mathcal{U}\mathcal{L}_\alpha^\Sigma(U)$ through the choice of supports of the smooth bump functions entering the definitions of $[\xi_p^{-m}] \in \Omega_c^{0,1}(U_+)$ and $[\bar{\xi}_p^{-n}] \in \Omega_c^{0,1}(U_-)$. To describe this ordering explicitly we introduce the important notion of *nested* open subsets that will be used repeatedly throughout the rest of the paper.

Definition 3.1. *Given two open subsets $V, W \subset U_\pm$, we say that V is nested in W if V is relatively compact in W , i.e. if $\bar{V} \subset W$ and \bar{V} is compact, and we write this as $V \Subset W$.*

Let $\Omega_c^{0,0}(W)_V^1 \subset \Omega_c^{0,0}(W)$ denote the subset consisting of elements equal to 1 on V . Given any nested neighbourhoods $p_+ \in V \Subset W \subset U_+$ we pick a $\rho_W^V \in \Omega_c^{0,0}(W)_V^1$ and define

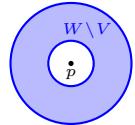
$$[\xi_p^n]_W^V := -\xi_p^n \bar{\partial} \rho_W^V \in \Omega_c^{0,1}(W) \quad (3.11)$$

for all $n \in \mathbb{Z}$. It follows from Lemma 2.3 that $[\xi_p^n]_W^V$ differs by $\bar{\partial}$ -exact terms from $[\xi_p^n]$ defined above (3.10). We similarly define $[\bar{\xi}_p^n]_W^V$ for nested neighbourhoods $p_- \in V \Subset W \subset U_-$.

Lemma 3.2. *Let $U \subset \Sigma^\circ$ be a neighbourhood of $p \in \Sigma^\circ$ equipped with a holomorphic coordinate $\xi : U \rightarrow \mathbb{C}$. For any $n \in \mathbb{Z}$ we have $\frac{1}{2\pi i} \int_{U_+} [\xi_p^n] \wedge d\xi_p = \delta_{n,-1}$ and $\frac{1}{2\pi i} \int_{U_-} [\bar{\xi}_p^n] \wedge d\bar{\xi}_p = \delta_{n,-1}$.*

Proof. Consider the integral over U_+ . Let $p_+ \in V \Subset W \subset U_+$ be nested neighbourhoods and pick a smooth bump function $\rho_W^V \in \Omega_c^{0,0}(W)_V^1$. Since $[\xi_p^n] - [\xi_p^n]_W^V$ is $\bar{\partial}$ -exact by Lemma 2.3, we can replace $[\xi_p^n]$ in the integrand by $[\xi_p^n]_W^V$. The given integral over U_+ thus evaluates to

$$\begin{aligned} \frac{1}{2\pi i} \int_{U_+} [\xi_p^n] \wedge d\xi_p &= \frac{1}{2\pi i} \int_{U_+} \xi_p^n d\xi_p \wedge \bar{\partial} \rho_W^V = \frac{1}{2\pi i} \int_{W \setminus V} \xi_p^n d\xi_p \wedge \bar{\partial} \rho_W^V \\ &= -\frac{1}{2\pi i} \int_{W \setminus V} d(\rho_W^V \xi_p^n d\xi_p) = \frac{1}{2\pi i} \int_{\partial V} \xi_p^n d\xi_p = \delta_{n,-1}, \end{aligned}$$



where the second equality follows from the support property of $\bar{\partial} \rho_W^V$. The third equality uses the fact that $\xi_p^n d\xi_p$ is closed on $W \setminus V$ and in the fourth equality we used Stokes's theorem together with the support properties of ρ_W^V . The last integral is evaluated using the residue theorem, noting that ∂V is oriented counterclockwise.

The evaluation of the integral over U_- is completely analogous. \square

We are now in a position to describe the geometric realisation of the vector space $\mathbb{F}^{\mathfrak{a},\alpha}$ in $\mathcal{U}\mathcal{L}_\alpha^\Sigma(U)$ around a point $p \in U$ in any open subset $U \subset \Sigma^\circ$ of $\text{Top}(\Sigma)$ equipped with a local coordinate $\xi : U \rightarrow \mathbb{C}$ such that $\xi(U) \subset \mathbb{C}$ is bounded. Specifically, we define a linear map

$$(\cdot)_U^{\xi_p} : \mathbb{F}^{\mathfrak{a},\alpha} \longrightarrow \mathcal{U}\mathcal{L}_\alpha^\Sigma(U), \quad A \longmapsto A_U^{\xi_p} \quad (3.12a)$$

as follows. It sends the state of the form (3.3) to

$$\left[\prod_{i=1}^r s\left(\mathfrak{a}_{\xi_p}^i \otimes [\xi_p^{-m_i}]_{U_i}^{U_{i-1}}\right) \prod_{j=1}^{\bar{r}} s\left(\mathfrak{b}_{\bar{\xi}_p}^j \otimes [\bar{\xi}_p^{-n_j}]_{V_j}^{V_{j-1}}\right) \right]_U, \quad (3.12b)$$

where $p_+ \in U_0 \Subset \dots \Subset U_r \subset U_+$ and $p_- \in V_0 \Subset \dots \Subset V_{\bar{r}} \subset U_-$ are choices of nested sequences of open subsets. In particular, the vacuum $|0\rangle$ is sent to the class $[1]_U$.

Proposition 3.3. *The linear map (3.12) is well defined. In particular, we have*

$$\dots (a_{(m)}b_{(n)} - b_{(n)}a_{(m)}) \dots |0\rangle_U^{\xi_p} = \dots [a_{(m)}, b_{(n)}] \dots |0\rangle_U^{\xi_p}, \quad (3.13a)$$

$$\dots (a_{(m)}\bar{b}_{(n)} - \bar{b}_{(n)}a_{(m)}) \dots |0\rangle_U^{\xi_p} = 0, \quad (3.13b)$$

$$\dots (\bar{a}_{(m)}\bar{b}_{(n)} - \bar{b}_{(n)}\bar{a}_{(m)}) \dots |0\rangle_U^{\xi_p} = \dots [\bar{a}_{(m)}, \bar{b}_{(n)}] \dots |0\rangle_U^{\xi_p} \quad (3.13c)$$

for all $a, b \in \mathfrak{a}$ and $m, n \in \mathbb{Z}$, where on the right hand sides of (3.13a) and (3.13c) we use the Lie brackets on (3.1) and in each equation the ellipses on either side of each term represent the same sum of product of elements from $\widehat{\mathfrak{a}} \oplus \widehat{\mathfrak{a}}$. We also have, for any $k \in \mathbb{Z}_{\geq 0}$,

$$\dots a_{(k)}|0\rangle_U^{\xi_p} = 0, \quad \dots \bar{a}_{(k)}|0\rangle_U^{\xi_p} = 0. \quad (3.13d)$$

Proof. Focusing on the chiral part of the state, first note that the cohomology class in (3.12b) is independent of the choice of smooth bump functions $\rho_{U_i}^{U_{i-1}} \in \Omega_c^{0,0}(U_i)_{U_{i-1}}^1$ and of the nested sequences of open subsets $p_+ \in U_0 \Subset \dots \Subset U_r \subset U_+$. Indeed, Lemma 2.3 allows one to modify the smooth bump functions $\rho_{U_i}^{U_{i-1}}$ one by one while maintaining disjoint supports

$$\text{supp}(\bar{\partial}\rho_{U_i}^{U_{i-1}}) \cap \text{supp}(\bar{\partial}\rho_{U_j}^{U_{j-1}}) = \emptyset, \quad (3.14)$$

for all $i \neq j$, until we arrive at a desired set of new smooth bump functions $\rho_{U'_i}^{U'_{i-1}} \in \Omega_c^{0,0}(U'_i)_{U'_{i-1}}^1$ associated with another choice of nested sequence of open subsets $p_+ \in U'_0 \Subset \dots \Subset U'_r \subset U_+$ in $\text{Top}(\Sigma)$. Keeping the supports disjoint as in (3.14) ensures that there is never any contribution from the differential $\mathsf{d}_{[.,.]}$, as defined in §A.1, when working up to d_{CE} -exact terms, so that we can effectively work up to $\bar{\partial}$ -exact terms. The same applies to the anti-chiral part.

Next, we must check that the cohomology class $A_U^{\xi_p} \in \mathcal{UL}_\alpha^\Sigma(U)$ as given in (3.12b) does not depend on the way in which the input state $A \in \mathbb{F}^{\mathfrak{a}, \alpha}$ is written. It is enough to check the identities (3.13). We will focus on (3.13a) and the first identity in (3.13d), the proof of the other identities being completely analogous.

Choosing a nested sequence of open subsets $p_+ \in U_0 \Subset U_1 \Subset U_2 \Subset U_3 \subset U_+$ we can write the left hand side of (3.13a) as

$$\begin{aligned} & \left[\dots s(a_{\xi_p} \otimes \xi_p^m \bar{\partial} \rho_{U_2}^{U_1}) s(b_{\xi_p} \otimes \xi_p^n \bar{\partial} \rho_{U_1}^{U_0}) \dots \right]_U - \left[\dots s(b_{\xi_p} \otimes \xi_p^n \bar{\partial} \rho_{U_3}^{U_2}) s(a_{\xi_p} \otimes \xi_p^m \bar{\partial} \rho_{U_2}^{U_1}) \dots \right]_U \\ &= \left[\dots s(b_{\xi_p} \otimes \bar{\partial}(\xi_p^n(\rho_{U_1}^{U_0} - \rho_{U_3}^{U_2}))) s(a_{\xi_p} \otimes \xi_p^m \bar{\partial} \rho_{U_2}^{U_1}) \dots \right]_U, \end{aligned} \quad (3.15)$$

where in the first term on the left hand side we chose to use the nested sequence of open subsets $p_+ \in U_0 \Subset U_1 \Subset U_2 \subset U_+$ in $\text{Top}(\Sigma)$ and in the second term we used $p_+ \in U_1 \Subset U_2 \Subset U_3 \subset U_+$ instead. In each of the three terms in (3.15), the ellipses on either side represent the same sum of products of degree 0 elements from $\mathcal{L}_\alpha^\Sigma(U)[1]$, whose support is disjoint from $U_3 \setminus U_0$. In the second line of (3.15) we have used the fact that ξ_p^n is holomorphic on the support of $\rho_{U_1}^{U_0} - \rho_{U_3}^{U_2}$ since the latter vanishes in a neighbourhood of p .

By definition of the differential d_{CE} in the Chevalley-Eilenberg complex, see §A.1, we find

$$\begin{aligned} & \mathsf{d}_{\text{CE}} \left(\dots s(b_{\xi_p} \otimes \xi_p^n(\rho_{U_1}^{U_0} - \rho_{U_3}^{U_2})) s(a_{\xi_p} \otimes \xi_p^m \bar{\partial} \rho_{U_2}^{U_1}) \dots \right) \\ &= \dots \left(-s(b_{\xi_p} \otimes \bar{\partial}(\xi_p^n(\rho_{U_1}^{U_0} - \rho_{U_3}^{U_2}))) s(a_{\xi_p} \otimes \xi_p^m \bar{\partial} \rho_{U_2}^{U_1}) \right. \\ & \quad \left. - s \left[b_{\xi_p} \otimes \xi_p^n(\rho_{U_1}^{U_0} - \rho_{U_3}^{U_2}), a_{\xi_p} \otimes \xi_p^m \bar{\partial} \rho_{U_2}^{U_1} \right]_\alpha \right) \dots \end{aligned}$$

where the minus sign in the last line comes from the definition of the differential $\mathbf{d}_{[.,.]_\alpha}$ in §A.1 and using the fact that the first argument in the Lie bracket has degree 0 in $\mathcal{L}_\alpha^\Sigma(U)$. Note also that the action of \mathbf{d}_{CE} on the ellipses terms vanishes since these consist of sums of products of degree 0 elements from $\mathcal{L}_\alpha^\Sigma(U)[1]$, so they are killed by $\bar{\partial}$, and whose support is disjoint from $U_3 \setminus U_0$, so they also do not contribute to the action of $\mathbf{d}_{[.,.]_\alpha}$. It now follows that the cohomology class on the right hand side of (3.15) can be rewritten as

$$\begin{aligned} & \left[\dots s(\mathbf{a}_{\xi_p} \otimes \xi_p^m \bar{\partial} \rho_{U_2}^{U_1}) s(\mathbf{b}_{\xi_p} \otimes \xi_p^n \bar{\partial} \rho_{U_1}^{U_0}) \dots \right]_U - \left[\dots s(\mathbf{b}_{\xi_p} \otimes \xi_p^n \bar{\partial} \rho_{U_3}^{U_2}) s(\mathbf{a}_{\xi_p} \otimes \xi_p^m \bar{\partial} \rho_{U_2}^{U_1}) \dots \right]_U \\ &= - \left[\dots s \left[\mathbf{b}_{\xi_p} \otimes \xi_p^n (\rho_{U_1}^{U_0} - \rho_{U_3}^{U_2}), \mathbf{a}_{\xi_p} \otimes \xi_p^m \bar{\partial} \rho_{U_2}^{U_1} \right]_\alpha \dots \right]_U. \end{aligned} \quad (3.16)$$

In order to complete the proof, it remains to explicitly evaluate the Lie bracket in $\mathcal{L}_\alpha^\Sigma(U)$ on the second line of the right hand side, using the formulae in §2.1.2, and relate it to the Lie bracket in $\hat{\mathbf{a}}$ on the right hand side of (3.13a). We proceed by separately considering the three main examples of §2.1.2, namely the Kac-Moody, Virasoro and $\beta\gamma$ system cases.

Kac-Moody: Recall from §2.1.1 that the map $(\cdot)_{\xi_p} : \mathfrak{a} \xrightarrow{\cong} \mathfrak{a}_{\xi_p}$ in this case is the identity so that with $\mathbf{a} = \mathbf{X} \in \mathfrak{g}$ and $\mathbf{b} = \mathbf{Y} \in \mathfrak{g}$ we find

$$\begin{aligned} & -s \left[\mathbf{Y} \otimes \xi_p^n (\rho_{U_1}^{U_0} - \rho_{U_3}^{U_2}), \mathbf{X} \otimes \xi_p^m \bar{\partial} \rho_{U_2}^{U_1} \right]_\alpha \\ &= -s([\mathbf{X}, \mathbf{Y}] \otimes \xi_p^{m+n} \bar{\partial} \rho_{U_2}^{U_1}) - \frac{\kappa(\mathbf{X}, \mathbf{Y})}{2\pi i} \int_{U_+} \partial(\xi_p^n) \wedge \xi_p^m \bar{\partial} \rho_{U_2}^{U_1} \\ &= s([\mathbf{X}, \mathbf{Y}] \otimes [\xi_p^{m+n}]_{U_2}^{U_1}) + m \kappa(\mathbf{X}, \mathbf{Y}) \delta_{m+n,0}, \end{aligned}$$

where in the first step we used the explicit form of the Lie bracket on $\mathcal{L}_\alpha^\Sigma(U)$ given by (2.8) and the fact that

$$(\rho_{U_1}^{U_0} - \rho_{U_3}^{U_2}) \bar{\partial} \rho_{U_2}^{U_1} = -\bar{\partial} \rho_{U_2}^{U_1}. \quad (3.17)$$

In the last term we also used the fact that $\partial \rho_{U_1}^{U_0}$ and $\partial \rho_{U_3}^{U_2}$ have disjoint supports with $\bar{\partial} \rho_{U_2}^{U_1}$. The integral can be evaluated using Lemma 3.2 to give $2\pi i n \delta_{m+n,0}$. For the first term on the last line we recall the minus sign in the definition (3.11). Using (3.4a) and the definition of the map (3.12), the resulting expression on the right hand side of (3.16) is therefore precisely the right hand side of (3.13a) in the case at hand.

Virasoro: Recall from §2.1.2 that in this case the 2-cocycle (2.10b) explicitly depends on the coordinate used. Since we are working in the local coordinate ξ_p we therefore use the 2-cocycle α_{ξ_p} relative to this coordinate, i.e. we work in the unital local Lie algebra $\mathcal{L}_{\alpha_{\xi_p}}^\Sigma(U)$.

Here $(\cdot)_{\xi_p} : \mathfrak{a} \xrightarrow{\cong} \mathfrak{a}_{\xi_p}$ is given by $\Omega \mapsto -\partial_{\xi_p}$ so that with $\mathbf{a} = \mathbf{b} = \Omega \in \mathfrak{a}$ we find

$$\begin{aligned} & -s \left[\partial_{\xi_p} \otimes \xi_p^n (\rho_{U_1}^{U_0} - \rho_{U_3}^{U_2}), \partial_{\xi_p} \otimes \xi_p^m \bar{\partial} \rho_{U_2}^{U_1} \right]_{\alpha_{\xi_p}} \\ &= s \left(\partial_{\xi_p} \otimes (m-n) \xi_p^{m+n-1} \bar{\partial} \rho_{U_2}^{U_1} \right) - \bar{\partial} \left(s(\partial_{\xi_p} \otimes \xi_p^{m+n} \partial_{\xi_p} \rho_{U_2}^{U_1}) \right) \\ & \quad + \frac{c}{24\pi i} \int_{U_+} \partial(n \xi_p^{n-1}) \wedge m \xi_p^{m-1} \bar{\partial} \rho_{U_2}^{U_1} \\ &= (m-n)s \left(-\partial_{\xi_p} \otimes [\xi_p^{m+n-1}]_{U_2}^{U_1} \right) + \frac{m(m-1)(m-2)}{12} c \delta_{m+n,2} - \bar{\partial} \left(s(\partial_{\xi_p} \otimes \xi_p^{m+n} \partial_{\xi_p} \rho_{U_2}^{U_1}) \right), \end{aligned}$$

where in the first step we used again the identity (3.17). Here, in the term coming from the Lie bracket (2.10a) we note that the term involving $\partial_{\xi_p}(\rho_{U_1}^{U_0} - \rho_{U_3}^{U_2})$ vanishes using the fact that it disjoint support with $\bar{\partial} \rho_{U_2}^{U_1}$. The evaluation of the term coming from the 2-cocycle (2.10b) is

similar to the above Kac-Moody case but for the terms involving $\partial_{\xi_p}(\rho_{U_1}^{U_0} - \rho_{U_3}^{U_2})$ we used the additional fact that this has disjoint support with $\bar{\partial}\rho_{U_2}^{U_1}$ and for the term involving $\bar{\partial}(\partial_{\xi_p}\rho_{U_2}^{U_1})$ we note that this vanishes using Stokes's theorem, cf. the proof of Lemma 3.2. In the second step the integral is then evaluated using Lemma 3.2. The exact term on the last line drops out in the cohomology class (3.16) since $\partial_{\xi_p}\rho_{U_2}^{U_1}$ has disjoint support with the remaining terms in the ellipses. Comparing the above with (3.5a) using the definition of the map (3.12), the resulting expression on the right hand side of (3.16) also agrees with the right hand side of (3.13a) in the present case.

$\beta\gamma$ system: Recall that here the map $(\cdot)_{\xi_p} : \mathfrak{a} \xrightarrow{\cong} \mathfrak{a}_{\xi_p}$ sends $\beta \mapsto 1$ and $\gamma \mapsto \mathbf{d}\xi_p$ so that with $\mathfrak{a} = \beta$ and $\mathbf{b} = \gamma$ in \mathfrak{a} we find

$$-s \left[\mathbf{d}\xi_p \otimes \xi_p^n (\rho_{U_1}^{U_0} - \rho_{U_3}^{U_2}), 1 \otimes \xi_p^m \bar{\partial}\rho_{U_2}^{U_1} \right]_\alpha = \frac{1}{2\pi i} \int_{U_+} \xi_p^{m+n} \mathbf{d}\xi_p \wedge \bar{\partial}\rho_{U_2}^{U_1} = \delta_{m+n, -1}.$$

where in the first step we used the 2-cocycle (2.14) and again the identity (3.17). Comparing this with (3.6a) using the definition of the map (3.12), the resulting expression on the right hand side of (3.16) again agrees with the right hand side of (3.13a), as required.

Finally, in all three cases the first identity in (3.13d) similarly follows using the fact that $[\xi_p^k]_W^V$ is $\bar{\partial}$ -exact when $k \in \mathbb{Z}_{\geq 0}$. \square

Proposition 3.4. *The linear map (3.12) is injective and for every inclusion of open subsets $p \in U \subset V \subset \Sigma^\circ$ in $\mathbf{Top}(\Sigma)$ covered by the coordinate ξ , we have the commutative diagram*

$$\begin{array}{ccc} & \mathbb{F}^{\mathfrak{a}, \alpha} & \\ (\cdot)_{\xi_U}^{\xi_p} \nearrow & & \searrow (\cdot)_{\xi_V}^{\xi_p} \\ \mathcal{UL}_\alpha^\Sigma(U) & \xrightarrow{m_{U,V}} & \mathcal{UL}_\alpha^\Sigma(V) \end{array} \quad (3.18)$$

Proof. To begin with, we note that the Poincaré-Birkhoff-Witt theorem yields an isomorphism $\text{PBW} : \mathbb{F}^{\mathfrak{a}, \alpha} \xrightarrow{\cong} \text{Sym}(\widehat{\mathfrak{a}}_- \oplus \widehat{\mathfrak{a}}_-)|0\rangle$ which is given by the inverse of the total symmetrisation map $\text{Sym}(\widehat{\mathfrak{a}}_- \oplus \widehat{\mathfrak{a}}_-)|0\rangle \xrightarrow{\cong} U(\widehat{\mathfrak{a}}_- \oplus \widehat{\mathfrak{a}}_-)|0\rangle$, or more explicitly by writing a given element in $\mathbb{F}^{\mathfrak{a}, \alpha}$ as a linear combination of totally symmetrised monomials acting on $|0\rangle$ which then maps canonically to $\text{Sym}(\widehat{\mathfrak{a}}_- \oplus \widehat{\mathfrak{a}}_-)|0\rangle$. Applying the functor (A.4) to the strong deformation retract in $\mathbf{dgVec}_{\mathbb{C}}$ from Theorem 2.5, where here $\xi = (\xi, \bar{\xi})$, we obtain a strong deformation retract

$$\text{Sym} \left(\mathfrak{a} \otimes \mathcal{O}_\infty(\xi(U_+)^c) \oplus \mathfrak{a} \otimes \mathcal{O}_\infty(\bar{\xi}(U_-)^c) \right) \xrightarrow[P_U]{I_U} \overline{\text{Sym}}_\bullet(\mathcal{L}_\kappa^\Sigma(U)) \xleftarrow[H_U]{} \quad (3.19)$$

in $\mathbf{dgVec}_{\mathbb{C}}$, where the left hand side is a DG vector space concentrated in degree 0. Here P_U and I_U are morphisms of $\mathbf{dgVec}_{\mathbb{C}}$ whose components in Sym-degree $n \in \mathbb{Z}_{\geq 0}$ are given explicitly by $P_U^n := p_U[1]^{\otimes n}$ and $I_U^n := i_U[1]^{\otimes n}$, respectively. An explicit expression for the homotopy H_U can be found, for instance, in [BSV, (4.11b)]. In order to perturb the differential on the right hand side of (3.19) by $\mathbf{d}_{[.,.]}$ we apply the homological perturbation lemma [Cra, LV], noting that this perturbation is small since it lowers by 1 the Sym-degree which is bounded below by 0. We obtain the perturbed strong deformation retract

$$\text{Sym} \left(\mathfrak{a} \otimes \mathcal{O}_\infty(\xi(U_+)^c) \oplus \mathfrak{a} \otimes \mathcal{O}_\infty(\bar{\xi}(U_-)^c) \right) \xleftarrow[\widehat{P}_U]{I_U} \overline{\text{CE}}_\bullet(\mathcal{L}_\kappa^\Sigma(U)) \xleftarrow[\widehat{H}_U]{} \quad (3.19)$$

where $\widehat{P}_U := \sum_{j \geq 0} P_U(\mathbf{d}_{[.,.]})^j H_U$ and the expression for the homotopy \widehat{H}_U will not be needed. Notice that neither I_U nor the differential on the left hand side are perturbed, for the same

reason as in the proof of [BSV, Proposition 4.4]. Since \widehat{P}_U is a quasi-isomorphism, applying the 0th-cohomology functor we obtain an isomorphism

$$H^0 \widehat{P}_U : \mathcal{UL}_\kappa^\Sigma(U) \xrightarrow{\cong} \text{Sym} \left(\mathfrak{a} \otimes \mathcal{O}_\infty(\xi(U_+)^c) \oplus \mathfrak{a} \otimes \mathcal{O}_\infty(\bar{\xi}(U_-)^c) \right). \quad (3.20)$$

We will show that we have a commutative diagram

$$\begin{array}{ccc} \mathbb{F}^{\mathfrak{a}, \alpha} & \xrightarrow{(\cdot)_U^{\xi_p}} & \mathcal{UL}_\kappa^\Sigma(U) \\ \text{PBW} \downarrow \cong & & \cong \downarrow H^0 \widehat{P}_U \\ \text{Sym}(\widehat{\mathfrak{a}}_- \oplus \widehat{\bar{\mathfrak{a}}}_-) |0\rangle & \xhookrightarrow{\iota} & \text{Sym} \left(\mathfrak{a} \otimes \mathcal{O}_\infty(\xi(U_+)^c) \oplus \mathfrak{a} \otimes \mathcal{O}_\infty(\bar{\xi}(U_-)^c) \right) \end{array} \quad (3.21)$$

in $\mathsf{Vec}_{\mathbb{C}}$, where the bottom morphism is obtained by applying the Sym-functor to the injection (3.9). Since this bottom morphism is injective, see Remark 3.5 below, it will then follow that the linear map (3.12) is also injective. By the Poincaré-Birkhoff-Witt theorem any state in $\mathbb{F}^{\mathfrak{a}, \alpha}$ can be written as a linear combination of totally symmetrised monomials acting on $|0\rangle$, so it suffices to consider a single such monomial, say a totally symmetrised version of (3.3), i.e.

$$\frac{1}{r! \bar{r}!} \sum_{\sigma \in S_r} \sum_{\bar{\sigma} \in S_{\bar{r}}} \mathfrak{a}_{(-m_{\sigma(r)})}^{\sigma(r)} \cdots \mathfrak{a}_{(-m_{\sigma(1)})}^{\sigma(1)} \bar{\mathfrak{b}}_{(-n_{\bar{\sigma}(\bar{r})})}^{\bar{\sigma}(\bar{r})} \cdots \bar{\mathfrak{b}}_{(-n_{\bar{\sigma}(1)})}^{\bar{\sigma}(1)} |0\rangle \in \mathbb{F}^{\mathfrak{a}, \alpha}. \quad (3.22)$$

We can make a further important simplification, similar to [BSV, §4.2]. Since we are focusing on totally symmetrised monomials, it is sufficient to consider ones built out of a single pair of elements in $\widehat{\mathfrak{a}}_-$ and $\widehat{\bar{\mathfrak{a}}}_-$. Namely, we may focus on states in $\mathbb{F}^{\mathfrak{a}, \alpha}$ of the form

$$A := \mathfrak{A}^r \mathfrak{B}^{\bar{r}} |0\rangle, \quad \mathfrak{A} = \sum_{i=1}^r t_i \mathfrak{a}_{(-m_i)}^i \in \widehat{\mathfrak{a}}_-, \quad \mathfrak{B} = \sum_{j=1}^{\bar{r}} \bar{t}_j \bar{\mathfrak{b}}_{(-n_j)}^j \in \widehat{\bar{\mathfrak{a}}}_-, \quad (3.23)$$

where $t_i, \bar{t}_j \in \mathbb{C}$, $\mathfrak{a}^i, \bar{\mathfrak{b}}^j \in \mathfrak{a}$ and $m_i, n_j \in \mathbb{Z}_{\geq 1}$ for each $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, \bar{r}\}$ with $r, \bar{r} \in \mathbb{Z}_{\geq 0}$. Indeed, the more general state (3.22) is recovered by polarization, i.e. it reads

$$\frac{1}{r! \bar{r}!} \frac{\partial^r}{\partial t_1 \cdots \partial t_r} \Big|_{t_1, \dots, t_r=0} \frac{\partial^{\bar{r}}}{\partial \bar{t}_1 \cdots \partial \bar{t}_{\bar{r}}} \Big|_{\bar{t}_1, \dots, \bar{t}_{\bar{r}}=0} \left(\sum_{i=1}^r t_i \mathfrak{a}_{(-m_i)}^i \right)^r \left(\sum_{j=1}^{\bar{r}} \bar{t}_j \bar{\mathfrak{b}}_{(-n_j)}^j \right)^{\bar{r}} |0\rangle.$$

Applying the isomorphism on the left hand side of (3.21) to the special state (3.23) then gives $\text{PBW}(A) = \mathfrak{A}^r \mathfrak{B}^{\bar{r}} |0\rangle \in \text{Sym}(\widehat{\mathfrak{a}}_- \oplus \widehat{\bar{\mathfrak{a}}}_-) |0\rangle$. Subsequently applying the injection at the bottom of the diagram (3.21) gives an element in the symmetric algebra on the bottom right corner, given by realising \mathfrak{A} and \mathfrak{B} in terms of germs of holomorphic functions on $\xi(U_+)^c$ and $\bar{\xi}(U_-)^c$, respectively, i.e. in terms of the notation after (3.9) we have

$$\iota(\text{PBW}(A)) = \left(\sum_{i=1}^r t_i \mathfrak{a}^i \otimes \lambda^{-m_i} \right)^r \left(\sum_{j=1}^{\bar{r}} \bar{t}_j \bar{\mathfrak{b}}^j \otimes \mu^{-n_j} \right)^{\bar{r}}. \quad (3.24)$$

Going instead along the top right of the diagram (3.21), applying the first map (3.12) we find

$$A_U^{\xi_p} = \left[s \left(\sum_{i=1}^r t_i \mathfrak{a}_{\xi_p}^i \otimes [\xi_p^{-m_i}]_{U_i}^{U_{i-1}} \right)^r s \left(\sum_{j=1}^{\bar{r}} \bar{t}_j \bar{\mathfrak{b}}_{\xi_p}^j \otimes [\bar{\xi}_p^{-n_j}]_{V_j}^{V_{j-1}} \right)^{\bar{r}} \right]_U,$$

Finally, one checks that upon applying the isomorphism $H^0 \widehat{P}_U$ to the above, we obtain exactly the same result as in (3.24). For this one needs to consider the explicit form of H_U given in

[BSV, (4.11b)]. One can make a further simplification by choosing the smooth bump functions $\rho_f \in \Omega_c^{0,0}(U_+)$ in (2.15), entering the definition of the strong deformation retract in Theorem 2.5, for every $f \in \mathcal{O}_\infty(\mathbb{C}P^1 \setminus \{0\})$ to coincide and to be equal to 1 on U_r , and likewise for the smooth bump functions $\rho_g \in \Omega_c^{0,0}(U_-)$. We then observe that only the $j = 0$ term in the sum defining \hat{P}_U contributes, as a consequence of the very special form of the state A in (3.23).

The commutativity of the diagram (3.18) is immediate from the explicit description of the morphisms $(\cdot)_U^{\xi_p}, (\cdot)_V^{\xi_p}$ and of the factorisation products $m_{U,V}$ described in §2.3. \square

Remark 3.5. In general, the symmetric algebra functor $\text{Sym} : \text{Mod}_R \rightarrow \mathbf{g}^{\geq 0}\text{CAlg}_R$ from the category Mod_R of modules over a commutative ring R to the category $\mathbf{g}^{\geq 0}\text{CAlg}_R$ of positively graded commutative algebras over R is right exact and hence preserves surjections. However, it is not left exact and so it does *not*, in general, preserve injections. In the present setting we are working over $R = \mathbb{C}$ and in this case the Sym functor *does* preserve injections. Indeed, every injective linear map $i : V \hookrightarrow W$ between complex vector spaces has a left inverse, i.e. a linear map $r : W \rightarrow V$ such that $ri = \text{id}_V$ and so applying the functor Sym we deduce that $\text{Sym}(i) : \text{Sym } V \rightarrow \text{Sym } W$ is also injective. \triangleleft

3.1.2 Local copies of $\mathbb{F}^{\alpha,\alpha}$

Let $\text{Top}(\Sigma)_p$ be the category of neighbourhoods $U \subset \Sigma^\circ$ of $p \in \Sigma^\circ$, with inclusions $U \subseteq V$ as morphisms $U \rightarrow V$. It is a subcategory of the multicategory $\text{Top}(\Sigma)^\sqcup$ from §2.3, so that we may restrict the multifunctor $\mathcal{UL}_\alpha^\Sigma : \text{Top}(\Sigma)^\sqcup \rightarrow \text{Vec}_\mathbb{C}^\otimes$ to this subcategory to form a diagram $(\mathcal{UL}_\alpha^\Sigma)_p : \text{Top}(\Sigma)_p \rightarrow \text{Vec}_\mathbb{C}$ of shape $\text{Top}(\Sigma)_p$. Since the category $\text{Vec}_\mathbb{C}$ is complete and $\text{Top}(\Sigma)_p$ is small, we can form the limit $\lim (\mathcal{UL}_\alpha^\Sigma)_p$ in $\text{Vec}_\mathbb{C}$, which for any open subset U in $\text{Top}(\Sigma)_p$ comes with a canonical linear map $m_{p,U} : \lim (\mathcal{UL}_\alpha^\Sigma)_p \rightarrow \mathcal{UL}_\alpha^\Sigma(U)$. We also have such a map for any open subset $U \subset \Sigma$ in $\text{Top}(\Sigma)$ containing p by post-composing the latter with a factorisation product.

By the universal property of limits there is a unique injective linear map

$$(\cdot)_p^\xi : \mathbb{F}^{\alpha,\alpha} \hookrightarrow \lim (\mathcal{UL}_\alpha^\Sigma)_p, \quad A \mapsto A_p^\xi \quad (3.25)$$

for $p \in \Sigma^\circ$ such that the following diagram

$$\begin{array}{ccccc} & & \mathbb{F}^{\alpha,\alpha} & & \\ & \swarrow (\cdot)_U^{\xi_p} & \downarrow \exists! (\cdot)_p^\xi & \searrow (\cdot)_V^{\xi_p} & \\ \lim (\mathcal{UL}_\alpha^\Sigma)_p & & & & \\ \downarrow m_{p,U} \quad \quad \quad \quad \downarrow m_{p,V} & & & & \downarrow \\ \mathcal{UL}_\alpha^\Sigma(U) & \xrightarrow{m_{U,V}} & \mathcal{UL}_\alpha^\Sigma(V) & & \end{array}$$

is commutative, for every inclusion of open neighbourhoods $p \in U \subset V \subset \Sigma^\circ$.

Proposition 3.6. *The image of the linear map (3.25) does not depend on the local coordinate ξ used on a neighbourhood of $p \in \Sigma^\circ$. We denote this image as $\mathcal{F}_p^{\alpha,\alpha} \in \text{Vec}_\mathbb{C}$.*

Proof. Let ξ and η be two local coordinates on a neighbourhood U of p . It suffices to show that for any $A \in \mathbb{F}^{\alpha,\alpha}$, the vector $A_U^{\xi_p}$ also lies in the image of the linear map (3.25) defined relative to the local coordinate η . And by linearity it is enough to consider states of the form (3.3). Its image $A_U^{\xi_p}$ is given by (3.12b). Writing $\xi = \varrho^{\eta \rightarrow \xi} \circ \eta$ for some holomorphic function $\varrho^{\eta \rightarrow \xi} : \eta(U) \rightarrow \xi(U)$, so that also $\bar{\xi} = \bar{\varrho}^{\eta \rightarrow \xi} \circ \bar{\eta}$, we have expansions (see (3.88) later for the exact expressions)

$$\xi_p^{-m_i} = \sum_{k \geq 0} \alpha_{k,m_i} \eta_p^{-m_i+k}, \quad \bar{\xi}_p^{-n_j} = \sum_{\ell \geq 0} \bar{\alpha}_{\ell,n_j} \bar{\eta}_p^{-n_j+\ell}$$

for all $i \in \{1, \dots, r\}$, $j \in \{1, \dots, \bar{r}\}$. Here the coefficients $\alpha_{k,n}$ are complex numbers for every $k \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}_{\geq 1}$ whose exact expressions will not be needed for the argument below, but for instance we have $\alpha_{0,n} = (\varrho^{\eta \rightarrow \xi})'(\eta(p))^{-n}$. Substituting these expansions into (3.12b) we obtain the formal expression

$$A_U^{\xi_p} = \left(\sum_{k_r \geq 0} \alpha_{k_r, m_r} \mathbf{a}_{(-m_r + k_r)}^r \right) \dots \left(\sum_{k_1 \geq 0} \alpha_{k_1, m_1} \mathbf{a}_{(-m_1 + k_1)}^1 \right) \times \left(\sum_{\ell_{\bar{r}} \geq 0} \overline{\alpha_{\ell_{\bar{r}}, n_{\bar{r}}}} \bar{\mathbf{b}}_{(-n_{\bar{r}} + \ell_{\bar{r}})}^{\bar{r}} \right) \dots \left(\sum_{\ell_1 \geq 0} \overline{\alpha_{\ell_1, n_1}} \bar{\mathbf{b}}_{(-n_1 + \ell_1)}^1 \right) |0\rangle_U^{\eta_p}, \quad (3.26)$$

It remains to observe that the right hand side is of the form $B_U^{\eta_p}$ for some well defined state $B \in \mathbb{F}^{\mathbf{a}, \alpha}$ since only finitely many terms contribute from each of the infinite sums. Indeed, this follows from repeatedly applying the identities (3.13) from Proposition 3.3. \square

For any point $p \in \Sigma^\circ$, we refer to $\mathcal{F}_p^{\mathbf{a}, \alpha}$ as the ‘*local copy of $\mathbb{F}^{\mathbf{a}, \alpha}$ attached to p* ’, and for any $A \in \mathbb{F}^{\mathbf{a}, \alpha}$ we will refer to $A_p^\xi \in \mathcal{F}_p^{\mathbf{a}, \alpha}$ as the ‘*state A prepared at p in the local coordinate ξ* ’. Explicitly, the state (3.3) in $\mathbb{F}^{\mathbf{a}, \alpha}$ prepared at p in the local coordinate ξ will be denoted

$$\mathbf{a}_{(-m_r)}^r \dots \mathbf{a}_{(-m_1)}^1 \bar{\mathbf{b}}_{(-n_{\bar{r}})}^{\bar{r}} \dots \bar{\mathbf{b}}_{(-n_1)}^1 |0\rangle_p^\xi.$$

For any neighbourhood $U \subset \Sigma^\circ$ of p with local coordinate $\xi : U \rightarrow \mathbb{C}$, we refer to $A_U^{\xi_p} \in \mathcal{UL}_\alpha^\Sigma(U)$ as the ‘*state A prepared at p in the local chart (U, ξ)* ’.

Let ξ and η be two local holomorphic coordinates in the neighbourhood of a point $p \in \Sigma^\circ$. Given a state $A \in \mathbb{F}^{\mathbf{a}, \alpha}$, it can be prepared at p in either the local coordinate ξ to give the element $A_p^\xi \in \mathcal{F}_p^{\mathbf{a}, \alpha}$, or in the local coordinate η to give the element $A_p^\eta \in \mathcal{F}_p^{\mathbf{a}, \alpha}$. These will, in general, describe different elements of $\mathcal{F}_p^{\mathbf{a}, \alpha}$ so we obtain a linear isomorphism

$$(\cdot)_p^{\eta \rightarrow \xi} := (\cdot)_p^\xi \circ ((\cdot)_p^\eta)^{-1} : \mathcal{F}_p^{\mathbf{a}, \alpha} \longrightarrow \mathcal{F}_p^{\mathbf{a}, \alpha}, \quad A_p^\eta \longmapsto A_p^\xi \quad (3.27)$$

which takes as input any element of $\mathcal{F}_p^{\mathbf{a}, \alpha} = \text{im}(\cdot)_p^\eta$, identifies the state $A \in \mathbb{F}^{\mathbf{a}, \alpha}$ corresponding to it in the local coordinate η near p , which is unique by the injectivity of (3.25), and then prepares the same state A at the same point p but in the different local coordinate ξ . We note here that the isomorphism (3.27) plays a similar role for the local copy $\mathcal{F}_p^{\mathbf{a}, \alpha}$ of $\mathbb{F}^{\mathbf{a}, \alpha}$ as the map $(\cdot)_{\eta \rightarrow \xi} : \mathbf{a}_\eta \xrightarrow{\cong} \mathbf{a}_\xi$ defined in §2.1.1 does for the holomorphic vector bundle L .

3.1.3 Translation operators

In the vertex algebra setting it is customary to introduce the *infinitesimal translation operator* D acting as an endomorphism of the vertex algebra. In the present non-chiral setting we have two such endomorphisms

$$D, \bar{D} : \mathbb{F}^{\mathbf{a}, \alpha} \longrightarrow \mathbb{F}^{\mathbf{a}, \alpha} \quad (3.28a)$$

defined as follows. For any $A \in \mathbb{F}^{\mathbf{a}, \alpha}$ of the form (3.3) we set

$$DA := \sum_{i=1}^r m_i \mathbf{a}_{(-m_r)}^r \dots \mathbf{a}_{(-m_{i-1})}^i \dots \mathbf{a}_{(-m_1)}^1 \bar{\mathbf{b}}_{(-n_{\bar{r}})}^{\bar{r}} \dots \bar{\mathbf{b}}_{(-n_1)}^1 |0\rangle, \quad (3.28b)$$

$$\bar{D}A := \sum_{j=1}^{\bar{r}} n_j \mathbf{a}_{(-m_r)}^r \dots \mathbf{a}_{(-m_1)}^1 \bar{\mathbf{b}}_{(-n_{\bar{r}})}^{\bar{r}} \dots \bar{\mathbf{b}}_{(-n_{j-1})}^j \dots \bar{\mathbf{b}}_{(-n_1)}^1 |0\rangle. \quad (3.28c)$$

In particular, $D|0\rangle = \bar{D}|0\rangle = 0$. We call a state $A \in \mathbb{F}^{\mathbf{a}, \alpha}$ *chiral* if $\bar{D}A = 0$ and *anti-chiral* if $DA = 0$. We let $\mathbb{V}^{\mathbf{a}, \alpha}$ denote the subspace of chiral states since it is isomorphic to the affine vertex algebra and, correspondingly, we let $\bar{\mathbb{V}}^{\mathbf{a}, \alpha}$ denote the subspace of anti-chiral states.

Proposition 3.7. *For any $A \in \mathbb{F}^{\mathfrak{a},\alpha}$ and any local coordinate ξ in the neighbourhood of a point $p \in \Sigma^\circ$, we have the relations*

$$(DA)_p^\xi = \partial_{\xi(p)} A_p^\xi, \quad (\bar{D}A)_p^\xi = \partial_{\bar{\xi}(p)} A_p^\xi$$

in $\mathcal{F}_p^{\mathfrak{a},\alpha}$. In particular, if A is (anti-)chiral then A_p^ξ depends (anti-)holomorphically on p .

Proof. It is enough to prove the relations

$$(DA)_U^{\xi_p} = \partial_{\xi(p)} A_U^{\xi_p}, \quad (\bar{D}A)_U^{\xi_p} = \partial_{\bar{\xi}(p)} A_U^{\xi_p}$$

in $\mathcal{UL}_\alpha^\Sigma(U)$ for an open neighbourhood $U \subset \Sigma^\circ$ of p . This is immediate from the definition of the operators D, \bar{D} in (3.28) and of the map $(\cdot)_U^{\xi_p}$ in (3.12), noting that for any $m \in \mathbb{Z}$ we have $\partial_{\xi(p)} \xi_p^{-m} = m \xi_p^{-m-1}$ and $\partial_{\bar{\xi}(p)} \bar{\xi}_p^{-m} = m \bar{\xi}_p^{-m-1}$. \square

3.2 Vertex modes and vertex operators

For any $R \in \mathbb{R}_{>0}$ we use the notation $D_R := \{\zeta \in \mathbb{C} \mid |\zeta| < R\} \subset \mathbb{C}$ for the open disc of radius R around the origin in \mathbb{C} . Similarly, for any $R^-, R^+ \in \mathbb{R}_{>0}$ with $R^- < R^+$ we denote by $A_{R^-, R^+} := \{\zeta \in \mathbb{C} \mid R^- < |\zeta| < R^+\} \subset \mathbb{C}$ the open annulus with radii R^- and R^+ around the origin. Note that $A_{R^-, R^+} = D_{R^+} \setminus \overline{D_{R^-}}$.

Given a local coordinate $\xi : U \rightarrow \mathbb{C}$ on a neighbourhood $U \subset \Sigma^\circ$ of $p \in \Sigma^\circ$, we say that $V \subset U$ is a *disc shaped open subset* around p in the local coordinate $\xi : U \rightarrow \mathbb{C}$ if $V = \xi_p^{-1}(D_R)$ for some $R > 0$. Likewise, we say that $Y \subset U$ is an *annulus shaped open subset* around p in the local coordinate $\xi : U \rightarrow \mathbb{C}$ if $Y = \xi_p^{-1}(A_{R^-, R^+})$ for some $0 < R^- < R^+$. We can then define disc shaped open subsets $V^\pm := \xi_p^{-1}(D_{R^\pm})$ so that $Y = V^+ \setminus \overline{V^-}$.

By definition of the category $\mathbf{Top}(\Sigma)$, see the start of §2.3, disc shaped open subsets and annulus shaped open subsets around a given point p are both objects in $\mathbf{Top}(\Sigma)$ homeomorphic to \mathbb{C} and $\mathbb{C} \setminus \{0\}$, respectively.

3.2.1 Vertex modes

To every $\mathfrak{a} \in \mathfrak{a}$, $n, \bar{n} \in \mathbb{Z}$ and any local coordinate ξ in the neighbourhood of $p \in \Sigma^\circ$, we associate a pair of endomorphisms of the vector space $\mathcal{F}_p^{\mathfrak{a},\alpha}$, called the *chiral vertex n^{th} -mode* and *anti-chiral vertex \bar{n}^{th} -mode* of \mathfrak{a} at p , defined as follows. Given an annulus shaped open subset $Y \subset \Sigma^\circ$ around p in the local coordinate ξ , we let $V^\pm \subset \Sigma^\circ$ denote the disc shaped open subsets around p such that $Y = V^+ \setminus \overline{V^-}$. The preimages of V^\pm under the projection $\pi : \widehat{\Sigma} \rightarrow \Sigma$ is a disjoint union of disc shaped open subsets around p_+ and p_- , respectively, which we denote as $\pi^{-1}(V^\pm) = V_+^\pm \sqcup V_-^\pm$, as in §3.1. We define the elements

$$\mathfrak{a}_{(n)}^{\xi_p} := \left[s \left(\mathfrak{a}_{\xi_p} \otimes [\xi_p^n]_{V_+^\pm}^{V_-^\pm} \right) \right]_Y, \quad \bar{\mathfrak{a}}_{(\bar{n})}^{\xi_p} := \left[s \left(\bar{\mathfrak{a}}_{\bar{\xi}_p} \otimes [\bar{\xi}_p^{\bar{n}}]_{V_-^\pm}^{V_+^\pm} \right) \right]_Y \quad (3.29)$$

in $\mathcal{UL}_\alpha^\Sigma(Y)$. We do not include the subset Y in the notation for the elements (3.29) of $\mathcal{UL}_\alpha^\Sigma(Y)$ since the endomorphisms of $\mathcal{F}_p^{\mathfrak{a},\alpha}$ they define will be independent of Y . Explicitly, given any state $B \in \mathbb{F}^{\mathfrak{a},\alpha}$ and any neighbourhood $U \subset \Sigma^\circ$ of p we may form the factorisation products

$$m_{(Y,p),U}(\mathfrak{a}_{(n)}^{\xi_p} \otimes B_p^\xi) = (\mathfrak{a}_{(n)} B)_U^{\xi_p}, \quad m_{(Y,p),U}(\bar{\mathfrak{a}}_{(\bar{n})}^{\xi_p} \otimes B_p^\xi) = (\bar{\mathfrak{a}}_{(\bar{n})} B)_U^{\xi_p}$$

in $\mathcal{UL}_\alpha^\Sigma(U)$ for every $\mathfrak{a} \in \mathfrak{a}$, where $Y \subset U$ is any annulus shaped open subset around p and the map $m_{(Y,p),U} : \mathcal{UL}_\alpha^\Sigma(Y) \otimes \lim(\mathcal{UL}_\alpha^\Sigma)_p \rightarrow \mathcal{UL}_\alpha^\Sigma(U)$ is defined as $m_{(Y,p),U} := m_{(Y,V),U} \circ (\text{id} \otimes m_{p,V})$

for any open neighbourhood $V \ni p$ disjoint from Y . The above equalities both follow from the definitions (3.12) and (3.29), see also Proposition 3.3. In particular, we obtain linear maps

$$\mathbf{a}_{(n)}^{\xi_p} : \mathcal{F}_p^{\mathbf{a}, \alpha} \longrightarrow \mathcal{F}_p^{\mathbf{a}, \alpha}, \quad B_p^\xi \longmapsto \mathbf{a}_{(n)}^{\xi_p} B_p^\xi = (\mathbf{a}_{(n)} B)_p^\xi, \quad (3.30a)$$

$$\bar{a}_{(\bar{n})}^{\xi_p} : \mathcal{F}_p^{\mathbf{a}, \alpha} \longrightarrow \mathcal{F}_p^{\mathbf{a}, \alpha}, \quad B_p^\xi \longmapsto \bar{a}_{(\bar{n})}^{\xi_p} B_p^\xi = (\bar{a}_{(\bar{n})} B)_p^\xi \quad (3.30b)$$

for any $a \in \mathfrak{a}$, $n, \bar{n} \in \mathbb{Z}$ and any local coordinate ξ in a neighbourhood of p .

The infinite sequences of chiral vertex n^{th} -modes in (3.29) can be described collectively, for $n \geq 0$ and $n < 0$ respectively, as the coefficients in the expansion of single elements of $\mathcal{UL}_\alpha^\Sigma(Y)$ as follows. The same will be true of the anti-chiral vertex \bar{n}^{th} -modes, see below, but focusing on the chiral vertex n^{th} -modes $a_{(n)}^{\xi_p}$ of $a \in \mathfrak{a}$ we consider

$$\left[s \left(\mathbf{a}_{\xi_q} \otimes \lceil \xi_q^{-1} \rceil \begin{smallmatrix} V_- \\ V_+ \end{smallmatrix} \right) \right]_Y \in \mathcal{UL}_\alpha^\Sigma(Y) \quad (3.31)$$

for some other point $q \in U$ in the local coordinate patch covered by $\xi : U \rightarrow \mathbb{C}$, where the notation used is the same as in (3.29). Since ξ_q^{-1} is singular at q , the latter should not lie in the closure of the annulus shaped subset $Y = V^+ \setminus \overline{V^-}$, so there are two cases to consider:

(v_{<0}) If $q \in V^-$ then $|\xi_p(t)| > |\xi_p(q)|$ for any $t \in Y$ so that the function $\xi_q^{-1} = (\xi_p - \xi_p(q))^{-1}$ can be expanded in small $\xi_p(q)$ and hence (3.31) has the following expansion

$$\left[s \left(a_{\xi_q} \otimes \left[\xi_q^{-1} \right]_{V_+^+}^{V_+^-} \right) \right]_Y = \sum_{n < 0} a_{(n)}^{\xi_p} \xi_p(q)^{-n-1} =: a[\xi_p(q)]_+.$$

(v_{≥0}) If $q \notin \overline{V^+}$ then instead $|\xi_p(t)| < |\xi_p(q)|$ for any $t \in Y$ so that $\xi_q^{-1} = (\xi_p - \xi_p(q))^{-1}$ can be expanded in small ξ_p , and hence (3.31) now has the following expansion

$$-\left[s \left(\mathbf{a}_{\xi_q} \otimes \lceil \xi_q^{-1} \rceil_{V_+^+}^{V_-^-} \right) \right]_Y = \sum_{n>0} \mathbf{a}_{(n)}^{\xi_p} \xi_p(q)^{-n-1} =: \mathbf{a}[\xi_p(q)]_-.$$

Remark 3.8. When \mathbf{a}_{ξ_q} explicitly depends on the coordinate ξ_q , as in the Virasoro or $\beta\gamma$ system cases, we should also re-express this in terms of the local coordinate ξ_p near p in both of the above expansions in $(v_{<0})$ and $(v_{\geq 0})$. However, since the local coordinates ξ_q and ξ_p are related simply by a constant shift $\xi_q = \xi_p - \xi_p(q)$, in all three examples we consider it is the case that $\mathbf{a}_{\xi_q} = \mathbf{a}_{\xi_p}$ for every $\mathbf{a} \in \mathfrak{a}$. Indeed, in the Virasoro case we have $\Omega_{\xi_q} = -\partial_{\xi_q} = -\partial_{\xi_p} = \Omega_{\xi_p}$, and likewise in the $\beta\gamma$ system case we have $\gamma_{\xi_q} = \mathbf{d}\xi_q = \mathbf{d}\xi_p = \gamma_{\xi_p}$. \triangleleft

It is convenient to gather the negative modes $a_{(n)}^{\xi_p}$ for $n < 0$ and positive modes $a_{(n)}^{\xi_p}$ for $n \geq 0$ together into a chiral vertex operator, the single doubly infinite series

$$\mathbf{a}[\xi_p(q)] := \mathbf{a}[\xi_p(q)]_+ + \mathbf{a}[\xi_p(q)]_-, \quad (3.32)$$

with coefficients in $\text{End } \mathcal{F}_p^{\mathbf{a}, \alpha}$. Strictly speaking, the nested sequence of open neighbourhoods $p_+ \in V_+^- \Subset V_+^+ \subset U_+$ used in defining the modes $\mathbf{a}_{(n)}^{\xi_p}$ should be different here depending on whether $n < 0$ or $n \geq 0$ so that the point $q \in U$, which is fixed in (3.32), satisfies $q \in V^-$ in the first term and $q \notin \overline{V^+}$ in the second. See the proof of Proposition 3.9 for details.

We can similarly assemble all the anti-chiral elements $\bar{a}_{(\bar{n})}^{\xi_p} \in \mathcal{UL}_\alpha^\Sigma(Y)$ in (3.29) into series $\bar{a}[\bar{\xi}_p(q)]_\pm$ for $\bar{n} < 0$ and $\bar{n} \geq 0$, respectively, and formally set $\bar{a}[\bar{\xi}_p(q)] := \bar{a}[\bar{\xi}_p(q)]_+ + \bar{a}[\bar{\xi}_p(q)]_-$.

3.2.2 Homogeneous vertex modes

Since the vertex modes (3.29) living in $\mathcal{UL}_\alpha^\Sigma(Y)$ are supported on an annulus shaped open subset $Y \subset \Sigma^\circ$ around p in the local coordinate ξ , i.e. the shifted local coordinate ξ_p provides a bijection $\xi_p : Y \xrightarrow{\cong} A_{R^-, R^+}$ to an open annulus with radii $0 < R^- < R^+$, we have an action of the circle $S^1 = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ on Y induced by the obvious S^1 -action on $A_{R^-, R^+} \subset \mathbb{C}$ given by multiplication by $\lambda \in S^1$. In other words, in the local coordinate $\xi_p : Y \rightarrow \mathbb{C}$ this action is given by $\xi_p \mapsto \lambda \xi_p$. We will say that $\mathbf{a} \in \mathfrak{a}$ has *conformal dimension* $\Delta_{\mathbf{a}}$ if the behaviour of the associated chiral vertex n^{th} -modes in (3.29) under this S^1 -action on Y takes the form

$$\mathbf{a}_{(n)}^{\lambda \xi_p} = \lambda^{n-\Delta_{\mathbf{a}}+1} \mathbf{a}_{(n)}^{\xi_p}$$

for every $n \in \mathbb{Z}$. Note that since $|\lambda| = 1$ we have $\bar{\xi}_p \mapsto \lambda^{-1} \bar{\xi}_p$ so that the anti-chiral vertex n^{th} -modes associated with $\mathbf{a} \in \mathfrak{a}$ transform for any $n \in \mathbb{Z}$ as

$$\bar{\mathbf{a}}_{(n)}^{\lambda \xi_p} = \lambda^{-n+\Delta_{\mathbf{a}}-1} \bar{\mathbf{a}}_{(n)}^{\xi_p}.$$

For any $\mathbf{a} \in \mathfrak{a}$ of definite conformal dimension $\Delta_{\mathbf{a}}$, it is convenient to define its associated *homogeneous chiral vertex n^{th} -mode* and *homogeneous anti-chiral vertex n^{th} -mode* at p in the local coordinate ξ , respectively, as

$$\mathbf{a}_{[n]}^{\xi_p} := \mathbf{a}_{(n+\Delta_{\mathbf{a}}-1)}^{\xi_p}, \quad \bar{\mathbf{a}}_{[n]}^{\xi_p} := \bar{\mathbf{a}}_{(n+\Delta_{\mathbf{a}}-1)}^{\xi_p}. \quad (3.33)$$

The reason that these (anti-)chiral vertex n^{th} -modes are called ‘homogeneous’ comes from the fact that their behaviour under the S^1 -action is just a rescaling by $\lambda^{\pm n}$, which is independent of $\mathbf{a} \in \mathfrak{a}$. In terms of homogeneous chiral vertex n^{th} -modes, the chiral vertex operator (3.32) and its anti-chiral counterpart take the form

$$\mathbf{a}[\xi_p(q)] = \sum_{n \in \mathbb{Z}} \mathbf{a}_{[n]}^{\xi_p} \xi_p(q)^{-n-\Delta_{\mathbf{a}}}, \quad \bar{\mathbf{a}}[\bar{\xi}_p(q)] = \sum_{n \in \mathbb{Z}} \bar{\mathbf{a}}_{[n]}^{\xi_p} \bar{\xi}_p(q)^{-n-\Delta_{\mathbf{a}}}. \quad (3.34)$$

We also define the homogeneous loop generators of the Lie algebra $\widehat{\mathfrak{a}} \oplus \widehat{\bar{\mathfrak{a}}}$ as, cf. (3.2),

$$\mathbf{a}_{[n]} := \mathbf{a} \otimes t^{n+\Delta_{\mathbf{a}}-1} \in \widehat{\mathfrak{a}}, \quad \bar{\mathbf{a}}_{[n]} := \mathbf{a} \otimes \bar{t}^{n+\Delta_{\mathbf{a}}-1} \in \widehat{\bar{\mathfrak{a}}}.$$

Given any monomial state $A \in \mathbb{F}^{\mathfrak{a}, \alpha}$ as in (3.3) such that \mathbf{a}^i and \mathbf{b}^j have definite conformal dimensions $\Delta_{\mathbf{a}^i}$ and $\Delta_{\mathbf{b}^j}$ for each $i = 1, \dots, r$ and $j = 1, \dots, \bar{r}$, we will say that A has *chiral conformal dimension* $\Delta_A := \sum_{i=1}^r (\Delta_{\mathbf{a}^i} + m_i - 1)$ and that it has *anti-chiral conformal dimension* $\bar{\Delta}_A := \sum_{j=1}^{\bar{r}} (\Delta_{\mathbf{b}^j} + n_j - 1)$. In all cases we shall consider, every possible rewriting of the state A is given by a linear combination of monomial states with the same chiral conformal dimension Δ_A and anti-chiral conformal dimension $\bar{\Delta}_A$ so we have a well defined \mathbb{Z}^2 -grading on $\mathbb{F}^{\mathfrak{a}, \alpha}$. Rewriting the relations (3.33) as

$$\mathbf{a}_{(-n)}^{\xi_p} = \mathbf{a}_{[-n-\Delta_{\mathbf{a}}+1]}^{\xi_p}, \quad \bar{\mathbf{a}}_{(-n)}^{\xi_p} = \bar{\mathbf{a}}_{[-n-\Delta_{\mathbf{a}}+1]}^{\xi_p}$$

we see that the \mathbb{Z}^2 -grading on $\mathbb{F}^{\mathfrak{a}, \alpha}$ is measured by the negative of the sums of the chiral and anti-chiral mode numbers of a homogeneous state A , respectively, when written in terms of homogeneous chiral and anti-chiral vertex modes.

We say that $A_p^\xi \in \mathcal{F}_p^{\mathfrak{a}, \alpha}$ is *homogeneous in the coordinate ξ* and call $\text{wt}(A_p^\xi) := \Delta_A$ the *chiral weight* of A_p^ξ in the coordinate ξ and $\overline{\text{wt}}(A_p^\xi) := \bar{\Delta}_A$ its *anti-chiral weight* in the coordinate ξ . Every choice of local coordinate ξ in a neighbourhood of the point $p \in \Sigma^\circ$ induces a \mathbb{Z}^2 -grading on the vector space $\mathcal{F}_p^{\mathfrak{a}, \alpha}$. These \mathbb{Z}^2 -gradings of $\mathcal{F}_p^{\mathfrak{a}, \alpha}$ depend on the coordinate used because if $A_p^\xi \in \mathcal{F}_p^{\mathfrak{a}, \alpha}$ is homogeneous in the coordinate ξ then it will generally not be homogeneous in another local coordinate η near p . More precisely, if $A_p^\xi = B_p^\eta$ for some $B \in \mathbb{F}^{\mathfrak{a}, \alpha}$ as in the proof of Proposition 3.6, then B will generally not be of definite bi-grade.

The conformal dimensions of basis elements in \mathfrak{a} and their corresponding homogeneous vertex operators in the three main examples from §2.1.2 are as follows.

Kac-Moody: Recall that $\mathfrak{a} = \mathfrak{g}$. Since $X_{\xi_p} = X$ is independent of the local coordinate for any $X \in \mathfrak{g}$, it follows that $\Delta_X = 1$. In particular, homogeneous vertex n^{th} -modes coincide with the vertex n^{th} -modes, namely $X_{[n]} = X_{(n)}$ for all $n \in \mathbb{Z}$ and the Lie algebra relations of $\widehat{\mathfrak{a}} \oplus \widehat{\mathfrak{a}}$ in (3.4) take the exact same form

$$[X_{[m]}, Y_{[n]}] = [X, Y]_{[m+n]} + m \kappa(X, Y) \delta_{m+n,0} k, \quad (3.35a)$$

$$[\bar{X}_{[m]}, \bar{Y}_{[n]}] = \overline{[X, Y]}_{[m+n]} + m \kappa(X, Y) \delta_{m+n,0} \bar{k}. \quad (3.35b)$$

for every $X, Y \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$. It is evident from these relations that the \mathbb{Z}^2 -grading on $\mathbb{F}^{\mathfrak{a}, \alpha}$ is well defined since each term has the same total homogeneous mode number.

Virasoro: We have $\mathfrak{a} = \text{span}_{\mathbb{C}}\{\Omega\}$ and because $\Omega_{\xi_p} = -\partial_{\xi_p}$ we deduce that $\Delta_{\Omega} = 2$. The homogeneous chiral and anti-chiral vertex n^{th} -modes of Ω are then given $\Omega_{[n]} = \Omega_{(n+1)}$ and $\bar{\Omega}_{[n]} = \bar{\Omega}_{(n+1)}$. The defining relations (3.5) of $\widehat{\mathfrak{a}} \oplus \widehat{\mathfrak{a}}$ then take the form

$$[\Omega_{[m]}, \Omega_{[n]}] = (m-n)\Omega_{[m+n]} + \frac{m^3 - m}{12}c \delta_{m+n,0} k, \quad (3.36a)$$

$$[\bar{\Omega}_{[m]}, \bar{\Omega}_{[n]}] = (m-n)\bar{\Omega}_{[m+n]} + \frac{m^3 - m}{12}c \delta_{m+n,0} \bar{k}, \quad (3.36b)$$

which are the usual Virasoro algebra relations for $L_{(n)} = \Omega_{[n]}$ and $\bar{L}_{(n)} = \bar{\Omega}_{[n]}$. In particular, it is again clear from these relations that the \mathbb{Z}^2 -grading on $\mathbb{F}^{\mathfrak{a}, \alpha}$ is well defined.

$\beta\gamma$ system: Here $\mathfrak{a} = \text{span}_{\mathbb{C}}\{\beta, \gamma\}$ with $\beta_{\xi_p} = 1$ and $\gamma_{\xi_p} = d\xi_p$ so that $\Delta_{\beta} = 1$ and $\Delta_{\gamma} = 0$. The homogeneous (anti-)chiral vertex n^{th} -modes are then $\beta_{[n]} = \beta_{(n)}$, $\bar{\beta}_{[n]} = \bar{\beta}_{(n)}$, $\gamma_{[n]} = \gamma_{(n-1)}$ and $\bar{\gamma}_{[n]} = \bar{\gamma}_{(n-1)}$, cf. (3.7). The defining relations (3.6) of $\widehat{\mathfrak{a}} \oplus \widehat{\mathfrak{a}}$ then read

$$[\beta_{[m]}, \gamma_{[n]}] = \delta_{m+n,0} k, \quad (3.37a)$$

$$[\bar{\beta}_{[m]}, \bar{\gamma}_{[n]}] = \delta_{m+n,0} \bar{k}, \quad (3.37b)$$

which are the usual infinite-dimensional Weyl algebra relations. Again, it is immediate from these relations that the \mathbb{Z}^2 -grading on $\mathbb{F}^{\mathfrak{a}, \alpha}$ is well defined.

3.2.3 Vertex operators

The vertex operator associated with a state $A \in \mathbb{F}^{\mathfrak{a}, \alpha}$ can be described geometrically in terms of the prefactorisation algebra $\mathcal{UL}_{\alpha}^{\Sigma}$ by preparing this state at a point $q \in U \subset \Sigma^{\circ}$ in some local coordinate $\xi : U \rightarrow \mathbb{C}$, to obtain the element $A_q^{\xi} \in \mathcal{F}_q^{\mathfrak{a}, \alpha}$, and then applying the factorisation product $m_{q,Y} : \mathcal{F}_q^{\mathfrak{a}, \alpha} \rightarrow \mathcal{UL}_{\alpha}^{\Sigma}(Y)$ to an annulus shaped open subset $Y \subset U$ with $q \in Y$ and encircling another point $p \in U$. The resulting element of $\mathcal{UL}_{\alpha}^{\Sigma}(Y)$ then naturally acts on $\mathcal{F}_p^{\mathfrak{a}, \alpha}$ via the factorisation products $m_{(Y,p),U}$, cf. the action of vertex modes on $\mathcal{F}_p^{\mathfrak{a}, \alpha}$ described in (3.30), and encodes the vertex operator of the state $A \in \mathbb{F}^{\mathfrak{a}, \alpha}$, see the next proposition.

For any $\lambda \in \mathbb{C}$ we use the notation $\underline{\lambda}$ as a shorthand for the pair $(\lambda, \bar{\lambda})$. Similarly, we use the notation $\underline{\zeta}$ for a pair of formal variables $(\zeta, \bar{\zeta})$. Recall that the normal ordered product of $r \in \mathbb{Z}_{\geq 2}$ operators $\mathcal{O}_i(\zeta)$ for $i \in \{1, \dots, r\}$ with a decomposition $\mathcal{O}_i(\zeta) = \mathcal{O}_i(\zeta)_+ + \mathcal{O}_i(\zeta)_-$ into creation and annihilation operators $\mathcal{O}_i(\zeta)_{\pm}$ is defined recursively by

$$:\mathcal{O}_r(\zeta) \dots \mathcal{O}_1(\zeta): := \mathcal{O}_r(\zeta)_+ :\mathcal{O}_{r-1}(\zeta) \dots \mathcal{O}_1(\zeta): + :\mathcal{O}_{r-1}(\zeta) \dots \mathcal{O}_1(\zeta): \mathcal{O}_r(\zeta)_-.$$

Proposition 3.9. Let $q \in Y \subset \Sigma^\circ$ be an annulus shaped open subset around a point $p \in \Sigma^\circ$ in a local coordinate ξ . For any $A_q^\xi \in \mathcal{F}_q^{\alpha, \alpha}$, the element $m_{q, Y}(A_q^\xi) \in \mathcal{U}\mathcal{L}_\alpha^\Sigma(Y)$ expands as

$$m_{q,Y}(A_q^\xi) = \mathcal{Y}\big(A_q^\xi, \underline{\xi_p(q)}\big),$$

where the right hand side is the vertex operator defined as the usual normal ordered product

$$\begin{aligned} \mathcal{Y}(A_q^\xi, \underline{\xi_p(q)}) &:= : \frac{1}{(m_r-1)!} \partial_{\xi_p(q)}^{m_r-1} \mathbf{a}^r [\xi_p(q)] \dots \frac{1}{(m_1-1)!} \partial_{\xi_p(q)}^{m_1-1} \mathbf{a}^1 [\xi_p(q)]: \\ &\quad \times : \frac{1}{(n_{\bar{r}}-1)!} \partial_{\bar{\xi}_p(q)}^{n_{\bar{r}}-1} \bar{\mathbf{b}}^{\bar{r}} [\bar{\xi}_p(q)] \dots \frac{1}{(n_1-1)!} \partial_{\bar{\xi}_p(q)}^{n_1-1} \bar{\mathbf{b}}^1 [\bar{\xi}_p(q)]: \end{aligned}$$

for any monomial state $A \in \mathbb{F}^{\mathbf{a}, \alpha}$ as in (3.3), and extended by linearity to all of $\mathbb{F}^{\mathbf{a}, \alpha}$.

Proof. By linearity it is sufficient to consider a state $A \in \mathbb{F}^{\alpha, \alpha}$ as in (3.3). In fact, we will focus on proving the statement for a chiral state $A = a_{(-m_r)}^r \dots a_{(-m_1)}^1 |0\rangle$ since the treatment of the anti-chiral part is completely analogous.

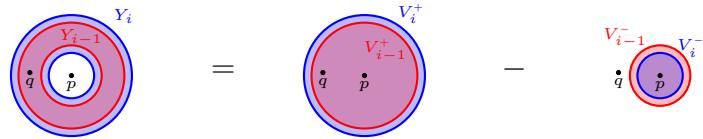
Let $q_+ \in Y_0 \Subset \dots \Subset Y_r \subset Y_+$ be a nested sequence of annuli shaped open subsets around the point p_+ . We have

$$m_{q,Y}(A_q^\xi) = \left[\prod_{i=1}^r s\left(\mathbf{a}_{\xi_q}^i \otimes \lceil \xi_q^{-m_i} \rceil_{Y_i}^{Y_{i-1}}\right) \right]_Y.$$

Each open subset Y_j for $j \in \{0, \dots, r\}$ can be written as a difference $Y_j = V_j^+ \setminus \overline{V_j^-}$ for disc shaped open subsets V_j^\pm around the point p_+ . We can thus write each $\rho_{Y_i}^{Y_{i-1}} \in \Omega_c^{0,0}(Y_i)_{Y_{i-1}}^1$ for $i \in \{1, \dots, r\}$ as a difference of two smooth bump functions

$$\rho_{Y_i}^{Y_{i-1}} = \rho_{V_i^+}^{V_{i-1}^+} - \rho_{V_{i-1}^-}^{V_i^-}, \quad \text{with} \quad \rho_{V_i^+}^{V_{i-1}^+} \in \Omega_c^{0,0}(V_i^+)_1^1_{V_{i-1}^+}, \quad \rho_{V_{i-1}^-}^{V_i^-} \in \Omega_c^{0,0}(V_{i-1}^-)_1^1_{V_i^-}$$

where $p_+ \in V_i^- \Subset V_{i-1}^- \Subset V_{i-1}^+ \Subset V_i^+$ is a nested sequence of disc shaped open subsets around p_+ in the local coordinate ξ such that $q_+ \in V_j^+$ and $q_+ \notin V_j^-$ for all $j \in \{0, \dots, r\}$, which we can depict schematically as



We may then write

$$m_{q,Y}(A_q^\xi) = \left[\prod_{i=1}^r s \left(\mathfrak{a}_{\xi_q}^i \otimes \left(\lceil \xi_q^{-m_i} \rceil_{V_i^+}^{V_{i-1}^+} - \lceil \xi_q^{-m_i} \rceil_{V_{i-1}^-}^{V_i^-} \right) \right) \right]_Y. \quad (3.38)$$

Expanding this out we get a sum of 2^r terms, each of which is the cohomology class of a r -fold Sym-product of terms of the form

$$s\left(\mathbf{a}_{\xi_q}^i \otimes \lceil \xi_q^{-m_i} \rceil_{V_{i-1}^+}^{V_i^-}\right) \quad \text{or} \quad -s\left(\mathbf{a}_{\xi_q}^i \otimes \lceil \xi_q^{-m_i} \rceil_{V_{i-1}^+}^{V_i^-}\right), \quad (3.39)$$

for each $i \in \{1, \dots, r\}$. Due to the support properties of the different smooth bump functions present, in each case we can expand $\xi_q^{-m_i} = \frac{1}{(m_i-1)!} \partial_{\xi(q)}^{m_i-1} (\xi_p - \xi_p(q))^{-1}$ in the region where

$|\xi_p(q)| < |\xi_p|$ or $|\xi_p(q)| > |\xi_p|$, respectively. Using also the fact that $\mathbf{a}_{\xi_q}^i = \mathbf{a}_{\xi_p}^i$ by Remark 3.8, we may write (3.39) as

$$\begin{aligned} & \frac{1}{(m_i - 1)!} \partial_{\xi(q)}^{m_i-1} \left(\sum_{n < 0} \xi_p(q)^{-n-1} s \left(\mathbf{a}_{\xi_p}^i \otimes \lceil \xi_p^n \rceil_{V_i^+}^{V_{i-1}^+} \right) \right) \\ \text{or} \quad & \frac{1}{(m_i - 1)!} \partial_{\xi(q)}^{m_i-1} \left(\sum_{n \geq 0} \xi_p(q)^{-n-1} s \left(\mathbf{a}_{\xi_p}^i \otimes \lceil \xi_p^n \rceil_{V_{i-1}^-}^{V_i^-} \right) \right). \end{aligned} \quad (3.40)$$

Define the annulus shaped open subsets $Y_i^+ := V_i^+ \setminus \overline{V_{i-1}^+}$ and $Y_i^- := V_{i-1}^- \setminus \overline{V_i^-}$ for $i \in \{1, \dots, r\}$. Then taking the cohomology classes $[\cdot]_{Y_i^+}$ or $[\cdot]_{Y_i^-}$ of the expressions in (3.40) gives

$$\frac{1}{(m_i - 1)!} \partial_{\xi(q)}^{m_i-1} \mathbf{a}[\xi_p(q)]_+ \in \mathcal{UL}_\alpha^\Sigma(Y_i^+) \quad \text{or} \quad \frac{1}{(m_i - 1)!} \partial_{\xi(q)}^{m_i-1} \mathbf{a}[\xi_p(q)]_- \in \mathcal{UL}_\alpha^\Sigma(Y_i^-). \quad (3.41)$$

In summary, we can rewrite (3.38) explicitly as a sum of 2^r terms

$$m_{q,Y}(A_q^\xi) = \sum_{(\varepsilon_1, \dots, \varepsilon_r) \in \{+, -\}^r} m_{(Y_1^{\varepsilon_1}, \dots, Y_r^{\varepsilon_r}), Y} \left(\bigotimes_{i=1}^r \frac{1}{(m_i - 1)!} \partial_{\xi(q)}^{m_i-1} \mathbf{a}[\xi_p(q)]_{\varepsilon_i} \right) \quad (3.42)$$

where the support of the i^{th} term in each of the above factorisation products is $Y_i^{\varepsilon_i}$, namely it is determined by the sign $\varepsilon_i \in \{+, -\}$ according to (3.41). The desired result now follows from observing that the relative ordering of the $2r$ annuli Y_i^\pm for $i \in \{1, \dots, r\}$ appearing in each of the 2^r terms of the sum (3.42) coincides with the relative ordering of the $2r$ operators appearing in each of the 2^r terms of the desired normal ordered product. \square

The coefficients in the expansion of the vertex operator from Proposition 3.9, namely

$$\mathcal{Y}(A_q^\xi, \xi_p(q)) = \sum_{n, \bar{n} \in \mathbb{Z}} A_{(n, \bar{n})}^{\xi_p} \xi_p(q)^{-n-1} \bar{\xi}_p(q)^{-\bar{n}-1}, \quad (3.43)$$

define the *vertex modes* $A_{(n, \bar{n})}^{\xi_p} \in \mathcal{UL}_\alpha^\Sigma(Y)$ of the state $A \in \mathbb{F}^{\mathbf{a}, \alpha}$ at $p \in \Sigma^\circ$ in the local coordinate ξ for any $n, \bar{n} \in \mathbb{Z}$. Just as the elements (3.29) of $\mathcal{UL}_\alpha^\Sigma(Y)$ did, each vertex mode of any given state $A \in \mathbb{F}^{\mathbf{a}, \alpha}$ also gives rise to an endomorphism

$$A_{(n, \bar{n})}^{\xi_p} : \mathcal{F}_p^{\mathbf{a}, \alpha} \longrightarrow \mathcal{F}_p^{\mathbf{a}, \alpha}, \quad B_p^\xi \mapsto A_{(n, \bar{n})}^{\xi_p} B_p^\xi$$

of $\mathcal{F}_p^{\mathbf{a}, \alpha}$ for each $n, \bar{n} \in \mathbb{Z}$, defined by forming the factorisation product $m_{(Y, p), U}(A_{(n, \bar{n})}^{\xi_p} \otimes B_p^\xi)$ with the input state $B \in \mathbb{F}^{\mathbf{a}, \alpha}$ prepared at p in the local coordinate ξ . The fact that this defines an element of $\mathcal{F}_p^{\mathbf{a}, \alpha}$ follows from the next lemma.

Lemma 3.10. *In any local coordinate ξ in the neighbourhood of a point $p \in \Sigma^\circ$ and for any $A, B \in \mathbb{F}^{\mathbf{a}, \alpha}$ we have*

$$A_{(n, \bar{n})}^{\xi_p} B_p^\xi = (A_{(n, \bar{n})} B)_p^\xi$$

for some unique $A_{(n, \bar{n})} B \in \mathbb{F}^{\mathbf{a}, \alpha}$. Moreover, we have $A_{(n, \bar{n})} B = 0$ if either n or \bar{n} is sufficiently large. In particular, we have $A_{(n, \bar{n})} |0\rangle = 0$ if either $n \geq 0$ or $\bar{n} \geq 0$.

Proof. We use the same notation as in the proof of Proposition 3.9. And just as in the latter, it is sufficient to consider only the chiral part of the state $B \in \mathbb{F}^{\mathbf{a}, \alpha}$, i.e. we can suppose that $B = \mathbf{Y}_{(-n_s)}^s \dots \mathbf{Y}_{(-n_1)}^1 |0\rangle$, since the anti-chiral part can be treated completely analogously.

Now for every $n \in \mathbb{Z}$, the vertex mode $A_{(n, -1)}^{\xi_p} \in \mathcal{UL}_\alpha^\Sigma(Y)$ is the coefficient of $\xi_p(q)^{-n-1}$ in the operator (3.42) from the proof of Proposition 3.9, consisting of a sum of 2^r terms. Aside

from the two extremal terms with $\varepsilon_i = \pm$ for all $i \in \{1, \dots, r\}$, the remaining $2^r - 2$ terms will all contribute an infinite sum to the coefficient of $\xi_p(q)^{-n-1}$. However, it can be seen by repeatedly applying the identities (3.13) from Proposition 3.3 that only finitely many terms in these infinite sums will contribute non-trivially to the factorisation product

$$m_{(Y,p),U} \left(A_{(n,-1)}^{\xi_p} \otimes B_p^\xi \right) \in \mathcal{UL}_\alpha^\Sigma(U)$$

into a disc shaped open subset $U \supset Y$. It follows that this factorisation product can be written as $(A_{(n,-1)}B)_U^{\xi_p} \in \mathcal{UL}_\alpha^\Sigma(U)$ for some state $A_{(n,-1)}B \in \mathbb{F}^{\mathbf{a},\alpha}$ which is unique by the injectivity of (3.12). If $n \in \mathbb{Z}_{\geq 0}$ is too large then $A_{(n,-1)}B = 0$ since none of the 2^r terms in the sum (3.42) will contribute to the above factorisation product. Finally, for the ‘in particular’ part, terms from the sum (3.42) which contribute to the coefficient of $\xi_p(q)^{-n-1}$ for $n \in \mathbb{Z}_{\geq 0}$ must have $\varepsilon_i = -$ for some $i \in \{1, \dots, r\}$ and all such terms annihilate the vacuum by (3.13d). \square

In light of the mode expansion (3.43) and the definition of the states $A_{(n,\bar{n})}B \in \mathbb{F}^{\mathbf{a},\alpha}$ given in Lemma 3.10, it will be convenient to define the formal vertex operator map which gathers all these states into a formal sum defined as

$$Y(A, \underline{\zeta})B = \sum_{n, \bar{n} \in \mathbb{Z}} A_{(n, \bar{n})}B \zeta^{-n-1} \bar{\zeta}^{-\bar{n}-1}, \quad (3.44)$$

where ζ and $\bar{\zeta}$ are formal variables. This is related to the vertex operator in (3.43) by

$$\mathcal{Y}(A_q^\xi, \underline{\xi_p(q)})B_p^\xi = \left(Y(A, \underline{\xi_p(q)})B \right)_p^\xi. \quad (3.45)$$

In other words, by preparing all the states $A_{(n, \bar{n})}B \in \mathbb{F}^{\mathbf{a},\alpha}$ for $n, \bar{n} \in \mathbb{Z}$, appearing in (3.44), at the point $p \in U$ in the local coordinate chart (U, ξ) and replacing the formal variables $\zeta, \bar{\zeta}$ in (3.44) by the complex numbers $\xi_p(q), \bar{\xi}_p(q)$ for some other point $q \in U$, we get back the expansion of the factorisation product $m_{(p,q),U}(A_q^\xi \otimes B_p^\xi)$ in q near p .

3.2.4 Homogeneous vertex operators

Recall from §3.2.2 the notion of homogeneous (anti-)chiral vertex modes for an element $\mathbf{a} \in \mathbf{a}$ of definite conformal dimension $\Delta_{\mathbf{a}}$. Given a monomial state $A \in \mathbb{F}^{\mathbf{a},\alpha}$, it is convenient to similarly introduce a notion of homogeneous vertex modes of A by expanding the vertex operator from Proposition 3.9 as

$$\mathcal{Y}(A_q^\xi, \underline{\xi_p(q)}) = \sum_{n, \bar{n} \in \mathbb{Z}} A_{[n, \bar{n}]}^{\xi_p} \xi_p(q)^{-n-\Delta_A} \bar{\xi}_p(q)^{-\bar{n}-\bar{\Delta}_A}, \quad (3.46)$$

which is to be compared with (3.43). The coefficients $A_{[n, \bar{n}]}^{\xi_p} \in \mathcal{UL}_\alpha^\Sigma(Y)$ for $n, \bar{n} \in \mathbb{Z}$ are called the *homogeneous vertex modes* of the monomial state $A \in \mathbb{F}^{\mathbf{a},\alpha}$ at $p \in \Sigma$ in the local coordinate ξ . They are related to the vertex modes by a simple shift, cf. (3.33),

$$A_{[n, \bar{n}]}^{\xi_p} = A_{(n+\Delta_A-1, \bar{n}+\bar{\Delta}_A-1)}^{\xi_p}. \quad (3.47)$$

We also introduce the notation $A_{[n, \bar{n}]} = A_{(n+\Delta_A-1, \bar{n}+\bar{\Delta}_A-1)} \in \text{End } \mathbb{F}^{\mathbf{a},\alpha}$, see Lemma 3.10.

Lemma 3.11. *Let $A \in \mathbb{F}^{\mathbf{a},\alpha}$ be any monomial state and $n, \bar{n} \in \mathbb{Z}$. Its vertex modes at $p \in \Sigma$ in any local coordinate ξ are homogeneous elements of $\text{End } \mathbb{F}_p^{\mathbf{a},\alpha}$ in the local coordinate ξ , of chiral and anti-chiral weights*

$$\text{wt}(A_{(n, \bar{n})}^{\xi_p}) = \Delta_A - n - 1, \quad \overline{\text{wt}}(A_{(n, \bar{n})}^{\xi_p}) = \bar{\Delta}_A - \bar{n} - 1.$$

Likewise, the homogeneous vertex modes of A at p in the local coordinate ξ are homogeneous elements of $\text{End } \mathcal{F}_p^{\mathbf{a}, \alpha}$ in the local coordinate ξ , of chiral and anti-chiral weights

$$\text{wt}(A_{[n, \bar{n}]}^{\xi_p}) = -n, \quad \overline{\text{wt}}(A_{[n, \bar{n}]}^{\xi_p}) = -\bar{n}.$$

Proof. By definition of the \mathbb{Z}^2 -grading on $\mathcal{F}_p^{\mathbf{a}, \alpha}$ relative to the local coordinate ξ , it is clear that the homogeneous chiral and anti-chiral vertex modes (3.30) are homogeneous with weights

$$\text{wt}(a_{[n]}^{\xi_p}) = -n, \quad \overline{\text{wt}}(a_{[n]}^{\xi_p}) = 0, \quad \text{wt}(\bar{a}_{[\bar{n}]}^{\xi_p}) = 0, \quad \overline{\text{wt}}(\bar{a}_{[\bar{n}]}^{\xi_p}) = -\bar{n}$$

for every $n, \bar{n} \in \mathbb{Z}$. The last statement now follows using the explicit expression for the vertex operator of the monomial state $A \in \mathbb{F}^{\mathbf{a}, \alpha}$ from Proposition 3.9 and the definition (3.46) of the homogeneous vertex modes, recalling that $\Delta_A = \sum_{i=1}^r (\Delta_{\mathbf{a}_i} + m_i - 1)$ and noting using (3.34) that the chiral factor $\partial_{\xi_p(q)}^{m_i-1} a^i[\xi_p(q)]$ for $i \in \{1, \dots, r\}$ consists of operators of the form

$$a_{[k]}^i \xi_p(q)^{-k - \Delta_{\mathbf{a}_i} - m_i + 1}$$

with $k \in \mathbb{Z}$ and likewise for the anti-chiral factors. The first result then follows by (3.47). \square

Given any monomial state $A \in \mathbb{F}^{\mathbf{a}, \alpha}$, we will refer to

$$\mathcal{X}(A_q^\xi, \underline{\xi_p(q)}) := \xi_p(q)^{\Delta_A} \bar{\xi}_p(q)^{\bar{\Delta}_A} \mathcal{Y}(A_q^\xi, \underline{\xi_p(q)}) = \sum_{n, \bar{n} \in \mathbb{Z}} A_{[n, \bar{n}]}^{\xi_p} \xi_p(q)^{-n} \bar{\xi}_p(q)^{-\bar{n}} \quad (3.48)$$

as the *homogeneous vertex operator of $A_q^\xi \in \mathcal{F}_q^{\mathbf{a}, \alpha}$ at q in the coordinate ξ* , see e.g. [LL, Remark 3.1.25] in the chiral case. We extend this notion by linearity to all states in $\mathcal{F}_q^{\mathbf{a}, \alpha}$.

3.2.5 Translation operators

Recall the translation operators $D, \bar{D} : \mathbb{F}^{\mathbf{a}, \alpha} \rightarrow \mathbb{F}^{\mathbf{a}, \alpha}$ introduced in (3.28). Their action on $\mathcal{F}_p^{\mathbf{a}, \alpha}$ was identified in Proposition 3.7 with that of the derivative operators $\partial_{\xi(p)}, \bar{\partial}_{\xi(p)} : \mathcal{F}_p^{\mathbf{a}, \alpha} \rightarrow \mathcal{F}_p^{\mathbf{a}, \alpha}$, respectively. In terms of the vertex modes (3.43) of the vertex operator from Proposition 3.9 their actions is given by the usual formula.

Lemma 3.12. *For any $A \in \mathbb{F}^{\mathbf{a}, \alpha}$ and $n, \bar{n} \in \mathbb{Z}$ we have*

$$(DA)_{(n, \bar{n})}^{\xi_p} = -n A_{(n-1, \bar{n})}^{\xi_p}, \quad (\bar{D}A)_{(n, \bar{n})}^{\xi_p} = -\bar{n} A_{(n, \bar{n}-1)}^{\xi_p}.$$

in $\text{End } \mathcal{F}_p^{\mathbf{a}, \alpha}$, for any local coordinate ξ in the neighbourhood of a point $p \in \Sigma^\circ$.

Proof. The first result follows from comparing the mode expansion of both sides of the relation

$$\mathcal{Y}((DA)_q^\xi, \underline{\xi_p(q)}) = m_{q, Y}((DA)_q^\xi) = m_{q, Y}(\partial_{\xi(q)} A_q^\xi) = \partial_{\xi(q)} \mathcal{Y}(A_q^\xi, \underline{\xi_p(q)}),$$

where the first and last equalities make use of Proposition 3.9 and the second equality is by Proposition 3.7. The proof of the second result is completely analogous. \square

Lemma 3.13. *We have $Y(|0\rangle, \underline{\zeta}) = \text{id}_{\mathbb{F}^{\mathbf{a}, \alpha}}$, and for any $A \in \mathbb{F}^{\mathbf{a}, \alpha}$ we have the Taylor expansion $Y(A, \underline{\zeta})|0\rangle = e^{\zeta D} e^{\bar{\zeta} \bar{D}} A$. In particular, $DA = A_{(-2, -1)}|0\rangle$ and $\bar{D}A = A_{(-1, -2)}|0\rangle$.*

Proof. Since $m_{q, Y}(|0\rangle_q^\xi) = [1]_Y$ acts as the identity in $\text{End } \mathcal{F}^{\mathbf{a}, \alpha}$ the first result is immediate from the definition of \mathcal{Y} in Proposition 3.9. In any chart $\xi : U \rightarrow \mathbb{C}$ with $p, q \in U$ we have

$$\begin{aligned} \mathcal{Y}(A_q^\xi, \underline{\xi_p(q)})|0\rangle_U^{\xi_p} &= m_{(Y, p), U}(m_{q, Y}(A_q^\xi) \otimes |0\rangle_p^{\xi_p}) = m_{q, U}(A_q^\xi) = A_U^{\xi_q} \\ &= e^{\xi_p(q) \partial_{\xi(p)}} e^{\bar{\xi}_p(q) \bar{\partial}_{\xi(p)}} A_U^{\xi_p} = (e^{\xi_p(q) D} e^{\bar{\xi}_p(q) \bar{D}} A)_U^{\xi_p} \end{aligned}$$

where Y is an annulus shaped open subset containing q and encircling the point p . In the second last equality we have Taylor expanded the expression $A_U^{\xi_q}$ in q near p and the final step is by Proposition 3.7, see in particular its proof. The result now follows from (3.45) and the injectivity of the linear map (3.12) by Proposition 3.4. \square

Proposition 3.14. *For any $A, B \in \mathbb{F}^{\mathfrak{a}, \alpha}$ we have $Y(B, \underline{\zeta})A = e^{\zeta D} e^{\bar{\zeta} \bar{D}} Y(A, -\underline{\zeta})B$.*

Proof. By associativity (2.17) of the factorisation product we have

$$\mathcal{Y}(B_q^\xi, \underline{\xi_p}(q)) A_p^\xi = \mathcal{Y}(A_p^\xi, \underline{\xi_q}(p)) B_q^\xi$$

since both sides are given by $m_{(p,q),U}(A_p^\xi \otimes B_q^\xi)$ in any local chart $\xi : U \rightarrow \mathbb{C}$ containing p and q . In other words, in terms of the map Y defined in (3.44) we have

$$\left(Y(B, \underline{\xi_p}(q)) A \right)_p^\xi = \left(Y(A, \underline{\xi_q}(p)) B \right)_q^\xi = \left(e^{\xi_p(q)D} e^{\bar{\xi}_p(q)\bar{D}} Y(A, \underline{\xi_q}(p)) B \right)_p^\xi$$

where the last step is by Lemma 3.13, see in particular its proof. \square

3.2.6 (Anti-)chiral states

The chiral vertex operator (3.32) and its anti-chiral analogue for any $\mathfrak{a} \in \mathfrak{a}$ correspond to the states $\mathfrak{a}_{(-1)}|0\rangle_q^\xi$ and $\bar{\mathfrak{a}}_{(-1)}|0\rangle_q^\xi$, respectively, namely

$$\mathfrak{a}[\xi_p(q)] = \mathcal{Y}(\mathfrak{a}_{(-1)}|0\rangle_q^\xi, \xi_p(q)), \quad \bar{\mathfrak{a}}[\bar{\xi}_p(q)] = \mathcal{Y}(\bar{\mathfrak{a}}_{(-1)}|0\rangle_q^\xi, \bar{\xi}_p(q)) \quad (3.49)$$

and comparing the mode expansion of both sides it follows that for any $n, \bar{n} \in \mathbb{Z}$ we have

$$(\mathfrak{a}_{(-1)}|0\rangle)_{(n, \bar{n})}^{\xi_p} = \mathfrak{a}_{(n)}^{\xi_p} \delta_{\bar{n}, -1}, \quad (\bar{\mathfrak{a}}_{(-1)}|0\rangle)_{(n, \bar{n})}^{\xi_p} = \bar{\mathfrak{a}}_{(\bar{n})}^{\xi_p} \delta_{n, -1}. \quad (3.50)$$

Notice $\Delta_{\mathfrak{a}_{(-1)}|0\rangle} = \bar{\Delta}_{\bar{\mathfrak{a}}_{(-1)}|0\rangle} = \Delta_{\mathfrak{a}}$ and $\Delta_{\bar{\mathfrak{a}}_{(-1)}|0\rangle} = \bar{\Delta}_{\mathfrak{a}_{(-1)}|0\rangle} = 0$ for any $\mathfrak{a} \in \mathfrak{a}$ so that also

$$(\mathfrak{a}_{(-1)}|0\rangle)_{[n, \bar{n}]}^{\xi_p} = \mathfrak{a}_{[n]}^{\xi_p} \delta_{\bar{n}, 0}, \quad (\bar{\mathfrak{a}}_{(-1)}|0\rangle)_{[n, \bar{n}]}^{\xi_p} = \bar{\mathfrak{a}}_{[\bar{n}]}^{\xi_p} \delta_{n, 0}.$$

More generally, if $A \in \mathbb{F}^{\mathfrak{a}, \alpha}$ is chiral then by Proposition 3.7 the expansion of $m_{q,Y}(A_q^\xi)$ from Proposition 3.9 is holomorphic, i.e. of the form $\mathcal{Y}(A_q^\xi, \xi_p(q)) = \sum_{n \in \mathbb{Z}} A_{(n)}^{\xi_p} \xi_p(q)^{-n-1}$, in which case its modes can be extracted as contour integrals

$$A_{(n)}^{\xi_p} := A_{(n, -1)}^{\xi_p} = \frac{1}{2\pi i} \int_{c_p} m_{q,Y}(A_q^\xi) \xi_p(q)^n d\xi(q) \quad (3.51a)$$

for every $n \in \mathbb{Z}$, using any counterclockwise oriented contour c_p in Y encircling the point p . If instead $A \in \mathbb{F}^{\mathfrak{a}, \alpha}$ is anti-chiral then we similarly have $\mathcal{Y}(A_q^\xi, \bar{\xi}_p(q)) = \sum_{n \in \mathbb{Z}} A_{(n)}^{\xi_p} \bar{\xi}_p(q)^{-n-1}$ whose modes we can extract as contour integrals

$$A_{(n)}^{\xi_p} := A_{(-1, n)}^{\xi_p} = -\frac{1}{2\pi i} \int_{c_p} m_{q,Y}(A_q^\xi) \bar{\xi}_p(q)^n d\bar{\xi}(q). \quad (3.51b)$$

Although we have introduced the same abbreviated notation in both (3.51a) and (3.51b), this should not lead to confusion since the full correct notation $A_{(n, -1)}^{\xi_p}$ or $A_{(-1, n)}^{\xi_p}$ can be restored by noting that A is chiral or anti-chiral, respectively.

Recall that the vacuum $|0\rangle$ is both chiral and anti-chiral. It follows from Lemma 3.13 that its modes are $|0\rangle_{(n, \bar{n})}^{\xi_p} = [1]_Y \delta_{n, -1} \delta_{\bar{n}, -1}$ for all $n, \bar{n} \in \mathbb{Z}$ which is consistent with (3.51).

3.3 Borcherds type identities

In an ordinary vertex algebra the modes of arbitrary states satisfy algebraic relations known as the Borcherds identities [Bor, FHL, LL]. In the present setting, the vertex algebra modes of both chiral and anti-chiral states (3.51) also satisfy the same standard Borcherds identities. In fact, slightly more general ‘Borcherds type’ identities also hold, given below in Proposition 3.15, between the vertex modes of any (anti-)chiral state and those of a generic state. However, there does not appear to be algebraic ‘Borcherds type’ identities relating the vertex modes of two non-chiral states (see [Mor2, Remark 1.9]). This is the reason why the Borcherds identity is replaced by a different axiom in the various existing frameworks for full vertex operator algebras [HuKo, Mor2, SS]. See §3.5.3 for details.

3.3.1 Moving points

We can keep track of the dependence of the linear map (3.12) on the location of the point $p \in U$ where the state A is inserted as follows. Given any open subset $U \subset \Sigma^\circ$ in $\text{Top}(\Sigma)$, equipped with a local coordinate $\xi : U \rightarrow \mathbb{C}$, we define the map

$$\Phi_U^{1,\xi} : U \times \mathbb{F}^{\mathfrak{a},\alpha} \longrightarrow \mathcal{UL}_\alpha^\Sigma(U), \quad (p, A) \longmapsto m_{p,U}(A_p^\xi). \quad (3.52)$$

The superscript ξ refers to the coordinate used to prepare the state A at the point p . We can also generalise the linear map (3.12) to describe the insertion of $n \in \mathbb{Z}_{\geq 1}$ states from $\mathbb{F}^{\mathfrak{a},\alpha}$ at a subset of n points in Σ° as follows. Given any open subset $U \subset \Sigma^\circ$ in $\text{Top}(\Sigma)$ and a collection of points $p_i \in U$ with $i \in \{1, \dots, n\}$ for $n \in \mathbb{Z}_{\geq 1}$, we define morphisms

$$m_{(p_i),U} : \bigotimes_{i=1}^n \lim (\mathcal{UL}_\alpha^\Sigma)_{p_i} \longrightarrow \mathcal{UL}_\alpha^\Sigma(U) \quad (3.53)$$

as the composition $m_{(p_i),U} := m_{(U_i),U} \circ (\bigotimes_{i=1}^n m_{p_i,U_i})$ for any inclusion of n disjoint subsets $\sqcup_{i=1}^n U_i \subset U$. Combining the linear map (3.53) with the local realisations (3.25) at each p_i in the local coordinate ξ , we then obtain a linear map

$$\bigotimes_{i=1}^n \mathbb{F}^{\mathfrak{a},\alpha} \longrightarrow \mathcal{UL}_\alpha^\Sigma(U), \quad (A_i)_{i=1}^n \longmapsto m_{(p_i),U} \left(\bigotimes_{i=1}^n (A_i)_{p_i}^\xi \right). \quad (3.54)$$

In fact, one could more generally consider different local coordinates ξ_i , $i \in \{1, \dots, n\}$ around each of the points p_i to obtain a linear map

$$\bigotimes_{i=1}^n \mathbb{F}^{\mathfrak{a},\alpha} \longrightarrow \mathcal{UL}_\alpha^\Sigma(U), \quad (A_i)_{i=1}^n \longmapsto m_{(p_i),U} \left(\bigotimes_{i=1}^n (A_i)_{p_i}^{\xi_i} \right). \quad (3.55)$$

It is useful, as in (3.52), to keep track of the location of the points p_i at which the individual states A_i are prepared. For any open subset $U \subset \Sigma^\circ$ in $\text{Top}(\Sigma)$ and any $n \in \mathbb{Z}_{\geq 1}$ we let

$$\text{Conf}_n(U) := \{(p_i)_{i=1}^n \in U^{\times n} \mid p_i \neq p_j \text{ for all } i \neq j \in \{1, \dots, n\}\}$$

denote the configuration space of n distinct points in U . In particular, for $n = 1$ this is just $\text{Conf}_1(U) = U$. We can now define the analogue of (3.52) for n points as

$$\Phi_U^{n,\xi} : \text{Conf}_n(U) \times \bigotimes_{i=1}^n \mathbb{F}^{\mathfrak{a},\alpha} \longrightarrow \mathcal{UL}_\alpha^\Sigma(U), \quad ((p_i)_{i=1}^n, (A_i)_{i=1}^n) \longmapsto m_{(p_i),U} \left(\bigotimes_{i=1}^n (A_i)_{p_i}^\xi \right), \quad (3.56)$$

where again the superscript ξ refers to the coordinate used to prepare each state A_i at p_i . Again, more generally, we could use different local coordinates ξ_i , $i \in \{1, \dots, n\}$ around each of the points p_i as in (3.55), in which case we may define a map

$$\Phi_U^{n,\xi} : \text{Conf}_n(U) \times \bigotimes_{i=1}^n \mathbb{F}^{\alpha,\alpha} \longrightarrow \mathcal{UL}_\alpha^\Sigma(U), \quad ((p_i)_{i=1}^n, (A_i)_{i=1}^n) \longmapsto m_{(p_i),U} \left(\bigotimes_{i=1}^n (A_i)_{p_i}^{\xi_i} \right), \quad (3.57)$$

which depends on the collection $\xi = (\xi_i)_{i=1}^n$ of local coordinates around each point p_i .

3.3.2 ‘Borcherds type’ identities and consequences

Recall the state $A_{(n,\bar{n})}B \in \mathbb{F}^{\alpha,\alpha}$ given by Lemma 3.10 for any $A, B \in \mathbb{F}^{\alpha,\alpha}$ and $n, \bar{n} \in \mathbb{Z}$.

Proposition 3.15. *Let $A, B, C \in \mathbb{F}^{\alpha,\alpha}$ and $k, \bar{k}, m, \bar{m}, n \in \mathbb{Z}$. We have the identities in $\mathbb{F}^{\alpha,\alpha}$:*

(i) *If A is chiral then*

$$\begin{aligned} & \sum_{j \geq 0} \binom{m}{j} (A_{(n+j)}B)_{(m+k-j,\bar{k})} C \\ &= \sum_{j \geq 0} (-1)^j \binom{n}{j} A_{(m+n-j)} B_{(k+j,\bar{k})} C - \sum_{j \geq 0} (-1)^{n+j} \binom{n}{j} B_{(n+k-j,\bar{k})} A_{(m+j)} C. \end{aligned}$$

(ii) *If A is anti-chiral then*

$$\begin{aligned} & \sum_{j \geq 0} \binom{\bar{m}}{j} (A_{(n+j)}B)_{(k,\bar{m}+\bar{k}-j)} C \\ &= \sum_{j \geq 0} (-1)^j \binom{n}{j} A_{(\bar{m}+n-j)} B_{(k,\bar{k}+j)} C - \sum_{j \geq 0} (-1)^{n+j} \binom{n}{j} B_{(k,n+\bar{k}-j)} A_{(\bar{m}+j)} C. \end{aligned}$$

Proof. We prove only the chiral case (i). The proof of the anti-chiral case (ii) is completely analogous. So let $A, B \in \mathbb{F}^{\alpha,\alpha}$ and suppose that $\bar{D}A = 0$.

Let $U \subset \Sigma^\circ$ be an open subset equipped with a local coordinate $\xi : U \rightarrow \mathbb{C}$. Recall the map $\Phi_U^{3,\xi}$ in (3.56) and define $F_{A,B,C} : \text{Conf}_3(U) \rightarrow \mathcal{UL}_\alpha^\Sigma(U)$ by

$$\begin{aligned} F_{A,B,C}(q, p, t) &:= \Phi_U^{3,\xi}((q, p, t), (A, B, C)) \xi_p(q)^n \xi_t(q)^m \xi_t(p)^k \bar{\xi}_t(p)^{\bar{k}} \\ &= m_{(q,p,t),U} (A_q^\xi \otimes B_p^\xi \otimes C_t^\xi) \xi_p(q)^n \xi_t(q)^m \xi_t(p)^k \bar{\xi}_t(p)^{\bar{k}} \end{aligned} \quad (3.58)$$

for pairwise distinct points $q, p, t \in U$. Since we are assuming that $\bar{D}A = 0$, the above depends holomorphically on q by Proposition 3.7. By Cauchy’s theorem we then have the identity

$$\frac{1}{2\pi i} \int_{c_p} F_{A,B,C}(q, p, t) d\xi(q) = \frac{1}{2\pi i} \int_{c_t} F_{A,B,C}(q, p, t) d\xi(q) - \frac{1}{2\pi i} \int_{c'_t} F_{A,B,C}(q, p, t) d\xi(q) \quad (3.59)$$

using the standard deformation of contour argument

where c_t and c'_t are counterclockwise oriented contours around t , as depicted, and c_p is a small counterclockwise oriented contour around the point p .

On the left hand side of (3.59), since q is closer to p than p is to t , we can expand the factor $\xi_t(q)^m = (\xi_t(p) + \xi_p(q))^m$ of (3.58) in small $\xi_p(q)$. Using also Proposition 3.9 gives

$$F_{A,B,C}(q, p, t) = \sum_{j \geq 0} \binom{m}{j} \mathcal{Y}(A_q^\xi, \xi_p(q)) B_p^\xi, \xi_t(p) C_t^\xi \xi_p(q)^{n+j} \xi_t(p)^{m+k-j} \bar{\xi}_t(p)^{\bar{k}}.$$

In the first integral on the right hand side of (3.59), since p is closer to t than q is, we can expand the factor $\xi_p(q)^n = (\xi_t(q) - \xi_t(p))^n$ of (3.58) in small $\xi_t(p)$. Using also Proposition 3.9 we have the following expansion

$$F_{A,B,C}(q, p, t) = \sum_{j \geq 0} (-1)^j \binom{n}{j} \mathcal{Y}(A_q^\xi, \xi_t(q)) \mathcal{Y}(B_p^\xi, \xi_t(p)) C_t^\xi \xi_t(p)^{k+j} \xi_t(q)^{m+n-j} \bar{\xi}_t(p)^{\bar{k}}.$$

Likewise, in the second integral on the right hand side of (3.59) we can expand the same factor $\xi_p(q)^n = (-1)^n (\xi_t(p) - \xi_t(q))^n$ of (3.58) in small $\xi_t(q)$. Using also Proposition 3.9 yields

$$F_{A,B,C}(q, p, t) = \sum_{j \geq 0} (-1)^{n+j} \binom{n}{j} \mathcal{Y}(B_p^\xi, \xi_t(p)) \mathcal{Y}(A_q^\xi, \xi_t(q)) C_t^\xi \xi_t(q)^{m+j} \xi_t(p)^{n+k-j} \bar{\xi}_t(p)^{\bar{k}}.$$

Upon integrating the above expansions in $\xi(q)$ along the contours c_p , c_t and c'_t , respectively, and using the definition of the vertex modes in (3.43) we obtain the three terms of the desired identity as the coefficient of $\xi_t(p)^0 \bar{\xi}_t(p)^0$. \square

Remark 3.16. It is clear from the proof of Proposition 3.15 that ‘Borcherds type’ identities can only be derived in the case when one of the two states involved is either chiral or anti-chiral. If this is not the case then we cannot use Cauchy’s residue theorem as in (3.59) to relate the three terms in the Borcherds identity (see [Mor2, Remark 1.9]). \triangleleft

The following consequences of the ‘Borcherds type’ identities are equivalent to those in [Mor2, Lemma 3.11], in the case of a chiral state A . We have also explicitly stated the version for an anti-chiral state A for completeness.

Corollary 3.17. *Let $A, B, C \in \mathbb{F}^{\mathfrak{a}, \alpha}$ and $k, \bar{k}, m, \bar{m} \in \mathbb{Z}$. We have the identities in $\mathbb{F}^{\mathfrak{a}, \alpha}$:*

(i) *If A is chiral then*

$$\begin{aligned} [A_{(m)}, B_{(k, \bar{k})}] C &= \sum_{j \geq 0} \binom{m}{j} (A_{(j)} B)_{(m+k-j, \bar{k})} C, \\ (A_{(-1)} B)_{(k, \bar{k})} C &= \sum_{j \geq 0} A_{(-j-1)} B_{(k+j, \bar{k})} C + \sum_{j < 0} B_{(k+j, \bar{k})} A_{(-j-1)} C. \end{aligned}$$

(ii) *If A is anti-chiral then*

$$\begin{aligned} [A_{(\bar{m})}, B_{(k, \bar{k})}] C &= \sum_{j \geq 0} \binom{\bar{m}}{j} (A_{(j)} B)_{(k, \bar{m}+\bar{k}-j)} C, \\ (A_{(-1)} B)_{(k, \bar{k})} C &= \sum_{j \geq 0} A_{(-j-1)} B_{(k, \bar{k}+j)} C + \sum_{j < 0} B_{(k, \bar{k}+j)} A_{(-j-1)} C. \end{aligned}$$

Proof. The commutator formulae both follow from taking $n = 0$ in the Borcherds type identities of Proposition 3.15. The second relation in (i) (resp. (ii)) follows from Proposition 3.15(i) (resp. (ii)) in the case $n = -1$ and $m = 0$ (resp. $\bar{m} = 0$). \square

It is instructive to see that we recover the Lie algebra relations of the generators of $\widehat{\mathfrak{a}}$ and $\widehat{\mathfrak{a}}$ from §3.1 as a special case of the commutation relations in Corollary 3.17. We consider the three main examples from §2.1.2 separately.

Kac-Moody: It follows from the action of the vertex modes of any $\mathbf{X} \in \mathfrak{g}$ in (3.30) and using Proposition 3.3 that for any $\mathbf{Y} \in \mathfrak{g}$, $n \in \mathbb{Z}_{\geq 0}$ and $p \in \Sigma^\circ$ we have

$$\mathbf{X}_{(n)}^{\xi_p} \mathbf{Y}_{(-1)} |0\rangle_p^\xi = [\mathbf{X}, \mathbf{Y}]_{(-1)} |0\rangle_p^\xi \delta_{n,0} + \kappa(\mathbf{X}, \mathbf{Y}) |0\rangle_p^\xi \delta_{n,1}, \quad (3.60a)$$

$$\bar{\mathbf{X}}_{(n)}^{\xi_p} \bar{\mathbf{Y}}_{(-1)} |0\rangle_p^\xi = [\bar{\mathbf{X}}, \bar{\mathbf{Y}}]_{(-1)} |0\rangle_p^\xi \delta_{n,0} + \kappa(\mathbf{X}, \mathbf{Y}) |0\rangle_p^\xi \delta_{n,1} \quad (3.60b)$$

in $\mathcal{F}_p^{\mathfrak{a},\alpha}$ for any local coordinate ξ in the neighbourhood of p . Then as a simple application of Corollary 3.17, recalling also (3.50), we recover (3.4) with \mathbf{k} and $\bar{\mathbf{k}}$ both set to 1.

Virasoro: Again, by definition of the vertex modes in (3.30) and using Proposition 3.3, for any $n \in \mathbb{Z}_{\geq 0}$, $p \in \Sigma^\circ$ and working in any local coordinate ξ near p we have

$$\Omega_{(n)}^{\xi_p} \Omega_{(-1)} |0\rangle_p^\xi = \Omega_{(-2)} |0\rangle_p^\xi \delta_{n,0} + 2\Omega_{(-1)} |0\rangle_p^\xi \delta_{n,1} + \frac{c}{2} |0\rangle_p^\xi \delta_{n,3}, \quad (3.61a)$$

$$\bar{\Omega}_{(n)}^{\xi_p} \bar{\Omega}_{(-1)} |0\rangle_p^\xi = \bar{\Omega}_{(-2)} |0\rangle_p^\xi \delta_{n,0} + 2\bar{\Omega}_{(-1)} |0\rangle_p^\xi \delta_{n,1} + \frac{c}{2} |0\rangle_p^\xi \delta_{n,3}. \quad (3.61b)$$

As a consequence of Corollary 3.17 we then recover the Virasoro algebra in the form (3.5) with \mathbf{k} and $\bar{\mathbf{k}}$ both set to 1. To deal with the first term on the right hand sides of (3.61) we use the fact that $\Omega_{(-2)} |0\rangle = D(\Omega_{(-1)} |0\rangle)$ and $\bar{\Omega}_{(-2)} |0\rangle = \bar{D}(\bar{\Omega}_{(-1)} |0\rangle)$ by definition of the translation operators in (3.28) and then apply Lemma 3.12 and the identities (3.50).

$\beta\gamma$ system: For any $n \in \mathbb{Z}_{\geq 0}$ and any local coordinate ξ near $p \in \Sigma^\circ$ we find

$$\beta_{(n)}^{\xi_p} \gamma_{(-1)} |0\rangle_p^\xi = |0\rangle_p^\xi \delta_{n,0}, \quad (3.62a)$$

$$\bar{\beta}_{(n)}^{\xi_p} \bar{\gamma}_{(-1)} |0\rangle_p^\xi = |0\rangle_p^\xi \delta_{n,0}. \quad (3.62b)$$

We then immediately recover the infinite-dimensional Weyl algebra relations (3.6), again with \mathbf{k} and $\bar{\mathbf{k}}$ both set to 1, by applying Corollary 3.17.

3.4 Conformal and anti-conformal states

In this subsection we introduce the conformal and anti-conformal states $\Omega, \bar{\Omega} \in \mathbb{F}^{\mathfrak{a},\alpha}$, which are respectively chiral and anti-chiral, and whose vertex n^{th} -modes generate infinitesimal local coordinate transformations. An important use of the non-negative shifted vertex n^{th} -modes of the conformal and anti-conformal states, namely $\Omega_{(n+1)}$ and $\bar{\Omega}_{(n+1)}$ for $n \geq 0$, is in establishing a generalisation of Huang's change of variable formula to the present case of full vertex operator algebras. This is the content of Corollary 3.20 below. As an application we also show how to define an invariant bilinear form on $\mathbb{F}^{\mathfrak{a},\alpha}$, in the sense of Proposition 3.23.

3.4.1 (Anti-)conformal states

We call a chiral state $\Omega \in \mathbb{V}^{\mathfrak{a},\alpha} \subset \mathbb{F}^{\mathfrak{a},\alpha}$ *conformal* if

$$\Omega_{(n)} \Omega = D\Omega \delta_{n,0} + 2\Omega \delta_{n,1} + \frac{c}{2} |0\rangle \delta_{n,3} \quad (3.63a)$$

for every $n \geq 0$ and, moreover, for any $A \in \mathbb{F}^{\mathfrak{a},\alpha}$ of chiral conformal dimension Δ_A we have

$$\Omega_{(0)} A = D A, \quad \Omega_{(1)} A = \Delta_A A. \quad (3.63b)$$

In particular, it follows that $\Delta_\Omega = 2$.

Similarly, an *anti-conformal state* is an anti-chiral state $\bar{\Omega} \in \bar{\mathbb{V}}^{\mathfrak{a},\alpha} \subset \mathbb{F}^{\mathfrak{a},\alpha}$ such that

$$\bar{\Omega}_{(n)} \bar{\Omega} = \bar{D}\bar{\Omega} \delta_{n,0} + 2\bar{\Omega} \delta_{n,1} + \frac{c}{2} |0\rangle \delta_{n,3} \quad (3.64a)$$

for every $n \geq 0$, and for any $A \in \mathbb{F}^{\mathfrak{a},\alpha}$ of anti-chiral conformal dimension $\bar{\Delta}_A$ we have

$$\bar{\Omega}_{(0)}A = \bar{D}A, \quad \bar{\Omega}_{(1)}A = \bar{\Delta}_AA. \quad (3.64b)$$

In particular, $\bar{\Delta}_{\bar{\Omega}} = 2$. Because Ω is chiral and $\bar{\Omega}$ is anti-chiral we also have that $\Omega_{(n)}\bar{\Omega} = 0$ for all $n \geq 0$. The parameter $c \in \mathbb{R}$ entering (3.64a) and (3.64b) is the *central charge*.

Since the (anti-)chiral conformal dimensions of the states $\Omega, \bar{\Omega} \in \mathbb{F}^{\mathfrak{a},\alpha}$ are different from 1, it is convenient to work with their homogeneous vertex n^{th} -modes introduced in §3.2.4 which we will denote by

$$L_{(n)} := \Omega_{[n]} = \Omega_{(n+1)}, \quad \bar{L}_{(n)} := \bar{\Omega}_{[n]} = \bar{\Omega}_{(n+1)} \quad (3.65)$$

for every $n \in \mathbb{Z}$. The notations $L_{(n)}, \bar{L}_{(n)} \in \text{End } \mathbb{F}^{\mathfrak{a},\alpha}$ are slightly misleading since L and \bar{L} do not represent states in $\mathbb{F}^{\mathfrak{a},\alpha}$: the states in question are $\Omega, \bar{\Omega} \in \mathbb{F}^{\mathfrak{a},\alpha}$ and (3.65) represent their homogeneous vertex n^{th} -modes. On the other hand, the more standard notation L_n, \bar{L}_n for these endomorphisms will be reserved for the Fourier modes introduced in §4.1 below so we will keep using the notation (3.65), hoping this will not cause any confusion. It will also be useful to introduce the notation

$$L_{(n)}^{\xi_p} := \Omega_{[n]}^{\xi_p} = \Omega_{(n+1)}^{\xi_p}, \quad \bar{L}_{(n)}^{\xi_p} := \bar{\Omega}_{[n]}^{\xi_p} = \bar{\Omega}_{(n+1)}^{\xi_p} \quad (3.66)$$

for any local coordinate $\xi : U \rightarrow \mathbb{C}$ in a neighbourhood of a point $p \in U$.

It follows from the first equations in (3.63b) and (3.64b) that $L_{(-1)} = D$ and $\bar{L}_{(-1)} = \bar{D}$ act as the translation operators (3.28), and from the second equations in (3.63b) and (3.64b) that $L_{(0)}$ and $\bar{L}_{(0)}$ measure the chiral and anti-chiral conformal dimensions of a state. It follows also from Lemma 3.10 that for any given state $A \in \mathbb{F}^{\mathfrak{a},\alpha}$ there exists a $N_A \in \mathbb{Z}_{\geq 0}$ such that $L_{(k)}A = 0$ and $\bar{L}_{(k)}A = 0$ for all $k \geq N_A$. Moreover, it follows from Lemma 3.11 that $L_{(k)}$ and $\bar{L}_{(k)}$ lower the conformal dimension when $k \geq 1$ and hence both they act locally nilpotently on $\mathbb{F}^{\mathfrak{a},\alpha}$ in the sense that for every $A \in \mathbb{F}^{\mathfrak{a},\alpha}$ there exists $r_A, \bar{r}_A \in \mathbb{Z}_{\geq 1}$ such that

$$(L_{(k)})^{r_A}A = 0, \quad (\bar{L}_{(k)})^{\bar{r}_A}A = 0. \quad (3.67)$$

As an application of Corollary 3.17 using (3.63a), (3.64a) and Lemma 3.12, we deduce that $L_{(n)}, \bar{L}_{(n)} \in \text{End } \mathbb{F}^{\mathfrak{a},\alpha}$ generate a direct sum of two copies of the Virasoro algebra, i.e.

$$\begin{aligned} [L_{(m)}, L_{(n)}] &= (m-n)L_{(m+n)} + \frac{m^3 - m}{12}c\delta_{m+n,0}, \\ [\bar{L}_{(m)}, \bar{L}_{(n)}] &= (m-n)\bar{L}_{(m+n)} + \frac{m^3 - m}{12}c\delta_{m+n,0} \end{aligned}$$

and $[L_{(m)}, \bar{L}_{(n)}] = 0$ for any $m, n \in \mathbb{Z}$. More importantly for what follows, when $m, n \in \mathbb{Z}_{\geq -1}$ we have a direct sum of two copies of the simpler Witt algebra

$$[L_{(m)}, L_{(n)}] = (m-n)L_{(m+n)}, \quad [\bar{L}_{(m)}, \bar{L}_{(n)}] = (m-n)\bar{L}_{(m+n)}. \quad (3.68)$$

The (anti-)conformal states in each of the three main examples from §2.1.2 are as follows.

Kac-Moody: For simplicity we assume here that the Lie algebra \mathfrak{g} is simple. The extension to the reductive case is straightforward, decomposing the Lie algebra $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i \oplus \mathfrak{z}$ as a direct sum of its simple components \mathfrak{g}_i and its centre \mathfrak{z} . The construction of (anti-)conformal states in the more general case when $\mathfrak{g} = \mathcal{T}^n\mathfrak{f} := \mathfrak{f}[t]/t^{n+1}\mathfrak{f}[t]$ is a Takiff algebra [Tak] for any reductive Lie algebra \mathfrak{f} and any $n \in \mathbb{Z}_{\geq 0}$ can be treated similarly along the lines of [Que].

Let $\{I_a\}_{a=1}^{\dim \mathfrak{g}}$ and $\{I^a\}_{a=1}^{\dim \mathfrak{g}}$ be dual bases of \mathfrak{g} with respect to a non-degenerate symmetric invariant bilinear form $\kappa_0 : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{C}$. For any $\mathsf{X} \in \mathfrak{g}$ we then have $\kappa_0(\mathsf{X}, I^b)I_b = \mathsf{X}$, where we use summation convention for Lie algebra indices $b = 1, \dots, \dim \mathfrak{g}$. Since every non-degenerate

symmetric invariant bilinear form on the simple Lie algebra \mathfrak{g} is proportional to the Killing form $\kappa_{\mathfrak{g}} : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{C}$, we denote by $\frac{\kappa}{\kappa'}$ the relative coefficient of proportionality between any two non-degenerate symmetric invariant bilinear forms $\kappa, \kappa' : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{C}$. The *critical* level is defined as $\kappa_c := -\frac{1}{2}\kappa_{\mathfrak{g}} : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{C}$. We assume that the level $\kappa : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{C}$ entering the definition of $\mathbb{F}^{\mathfrak{a},\alpha}$ is non-critical, i.e. $\kappa \neq \kappa_c$. We then define the *conformal* and *anti-conformal states* of $\mathbb{F}^{\mathfrak{a},\alpha}$, respectively, as

$$\Omega := \frac{\kappa_0}{2(\kappa - \kappa_c)} I_{a(-1)} I_{(-1)}^a |0\rangle, \quad \bar{\Omega} := \frac{\kappa_0}{2(\kappa - \kappa_c)} \bar{I}_{a(-1)} \bar{I}_{(-1)}^a |0\rangle. \quad (3.69)$$

It is a standard fact that these satisfy the relations (3.63a) and (3.63b) with central charge $c := \kappa \dim \mathfrak{g} / (\kappa - \kappa_c)$.

It is straightforward to check, for instance using the commutation relations from Corollary 3.17 in the case where $A = X_{(-1)}|0\rangle$ and $B = \Omega$ along with the defining relations (3.4) in \mathfrak{a} , that the homogeneous vertex n^{th} -modes (3.65) satisfy

$$[L_{(k)}, X_{(-n)}] = n X_{(k-n)}, \quad [\bar{L}_{(k)}, \bar{X}_{(-n)}] = n \bar{X}_{(k-n)}. \quad (3.70)$$

for every $k, n \in \mathbb{Z}$. It then immediately follows that (3.63b) and (3.64b) hold for any state $A \in \mathbb{F}^{\mathfrak{a},\alpha}$ of definite (anti-)chiral conformal dimension.

Although the formulae (3.69) for the (anti-)conformal states of $\mathbb{F}^{\mathfrak{a},\alpha}$ take a different form in the case when \mathfrak{g} is reductive, the resulting relations (3.70) remain true for arbitrary reductive \mathfrak{g} . We also expect (3.70) to hold more generally in the case of a Takiff algebra $\mathfrak{g} = \mathcal{T}^n \mathfrak{f}$ for any reductive Lie algebra \mathfrak{f} and any $n \in \mathbb{Z}_{\geq 0}$.

Virasoro: We define the *conformal* and *anti-conformal states* of $\mathbb{F}^{\mathfrak{a},\alpha}$ as

$$\Omega := \Omega_{(-1)}|0\rangle, \quad \bar{\Omega} := \bar{\Omega}_{(-1)}|0\rangle. \quad (3.71)$$

We are abusing notation slightly by using Ω to denote both the basis element of $\mathfrak{a} = \text{span}_{\mathbb{C}}\{\Omega\}$ and the above conformal state in $\mathbb{F}^{\mathfrak{a},\alpha}$. Since these live in different spaces, it should be clear from the context which we mean. This abuse of notation is also justified by the identity (3.50) which implies that for any $n \in \mathbb{Z}$ we have $\Omega_{(n)} = (\Omega_{(-1)}|0\rangle)_{(n)}$ with $\Omega \in \mathfrak{a}$, so that $\Omega_{(n)}$ denotes the same element of $\text{End } \mathbb{F}^{\mathfrak{a},\alpha}$ whether $\Omega \in \mathfrak{a}$ or $\Omega \in \mathbb{F}^{\mathfrak{a},\alpha}$.

It follows from the defining relations (3.5) in \mathfrak{a} that (3.71) satisfy the relations (3.63a) and (3.63b) for any central charge c . The relations (3.5), with the central elements k and \bar{k} set to 1 as in the definition of $\mathbb{F}^{\mathfrak{a},\alpha}$ from §3.1, can also be rewritten as

$$[L_{(k)}, \Omega_{(-n)}] = (n+k+1)\Omega_{(k-n)} + \frac{k^3 - k}{12}c \delta_{k-n,1}, \quad (3.72a)$$

$$[\bar{L}_{(k)}, \bar{\Omega}_{(-n)}] = (n+k+1)\bar{\Omega}_{(k-n)} + \frac{k^3 - k}{12}c \delta_{k-n,1}. \quad (3.72b)$$

for every $k, n \in \mathbb{Z}$. Since the terms proportional to c on the right hand sides vanish when $k = -1$ or $k = 0$, it follows that (3.63b) and (3.64b) hold for any state $A \in \mathbb{F}^{\mathfrak{a},\alpha}$ of definite (anti-)chiral conformal dimension.

$\beta\gamma$ system: Here we define the *conformal* and *anti-conformal states* of $\mathbb{F}^{\mathfrak{a},\alpha}$ as

$$\Omega = \beta_{(-1)}\gamma_{(-2)}|0\rangle, \quad \bar{\Omega} = \bar{\beta}_{(-1)}\bar{\gamma}_{(-2)}|0\rangle. \quad (3.73)$$

It follows again from Corollary 3.17 and the defining relations (3.6) in \mathfrak{a} , that the homogeneous vertex n^{th} -modes (3.65) satisfy

$$[L_{(k)}, \beta_{(-n)}] = n\beta_{(k-n)}, \quad [L_{(k)}, \gamma_{(-n)}] = (n-k-1)\gamma_{(k-n)}, \quad (3.74a)$$

$$[\bar{L}_{(k)}, \bar{\beta}_{(-n)}] = n\bar{\beta}_{(k-n)}, \quad [\bar{L}_{(k)}, \bar{\gamma}_{(-n)}] = (n-k-1)\bar{\gamma}_{(k-n)} \quad (3.74b)$$

for every $k, n \in \mathbb{Z}$, and hence that (3.63b) and (3.64b) hold for any state $A \in \mathbb{F}^{\mathbf{a}, \alpha}$ of definite (anti-)chiral conformal dimension.

3.4.2 Change of variable formula

Given two local coordinates $\xi, \eta : U \rightarrow \mathbb{C}$ on an open subset $U \subset \Sigma^\circ$, we denote the change of coordinate from η to ξ by $\varrho^{\eta \rightarrow \xi} : \xi \circ \eta^{-1} : \eta(U) \rightarrow \xi(U)$, or simply ϱ for short when the coordinates η and ξ are clear from the context. Furthermore, given any point $p \in U$, we denote the corresponding change of shifted local coordinate from η_p to ξ_p by $\varrho_p^{\eta \rightarrow \xi} : \eta_p(U) \rightarrow \xi_p(U)$, or simply by ϱ_p for short. Explicitly, we have

$$\xi_p = \varrho_p(\eta_p) = \varrho(\eta_p + \eta(p)) - \xi(p) = \varrho'(\eta(p)) \left(\eta_p + \sum_{n \geq 2} \frac{1}{n!} \frac{\varrho^{(n)}(\eta(p))}{\varrho'(\eta(p))} \eta_p^n \right) \quad (3.75a)$$

where we note that $\varrho'(\eta(p)) \neq 0$ because the coordinate transformation ϱ_p is invertible. Since the set of local (holomorphic) coordinate transformations fixing the origin is generated by holomorphic vector fields vanishing at 0, i.e. $\ell_k^{\eta_p} := -\eta_p^{k+1} \partial_{\eta_p}$ for $k \geq 0$, following [Hua, §2.1], see also [FB, §6.3.1], it is natural to rewrite the above power series in the form

$$\xi_p = \exp \left(- \sum_{k \geq 1} b_k \ell_k^{\eta_p} \right) b_0^{-\ell_0^{\eta_p}} \eta_p. \quad (3.75b)$$

By equating this power series term by term with the one in (3.75a) we find that the first few coefficients are given explicitly by

$$\begin{aligned} b_0 &= \varrho'(\eta(p)), & b_1 &= \frac{1}{2}(P\varrho)(\eta(p)), & b_2 &= \frac{1}{3!}(S\varrho)(\eta(p)), \\ b_3 &= \frac{1}{4!} \left((S\varrho)'(\eta(p)) - (S\varrho)(\eta(p))(P\varrho)(\eta(p)) \right), & \dots & & & \end{aligned} \quad (3.75c)$$

where $(Pf)(t) := f''(t)/f'(t)$ is the pre-Schwarzian derivative of f at t , i.e. the logarithmic derivative of f' , and $(Sf)(t) := (Pf)'(t) - \frac{1}{2}(Pf)(t)^2$ is the Schwarzian derivative of f at t .

Recall from §2.1.1 that in the shifted local (anti-)holomorphic coordinates $\xi_p : U_+ \rightarrow \mathbb{C}$ and $\bar{\xi}_p : U_- \rightarrow \mathbb{C}$ around a point $p \in U$, see (3.8), we have the isomorphism of DG vector spaces $\Omega_c^{0,\bullet}(\pi^{-1}(U), L) \cong \mathbf{a}_{\xi_p} \otimes \Omega_c^{0,\bullet}(U_+) \oplus \mathbf{a}_{\bar{\xi}_p} \otimes \Omega_c^{0,\bullet}(U_-)$. Letting L_X denote the Lie derivative of a vector field X on U , we obtain linear maps

$$L_{\ell_k^{\xi_p}}, L_{\bar{\ell}_k^{\xi_p}} : \mathcal{L}_\alpha^\Sigma(U) \longrightarrow \mathcal{L}_\alpha^\Sigma(U) \quad (3.76)$$

for $k \geq -1$. These extend by the Leibniz rule to endomorphisms of $\mathcal{U}\mathcal{L}_\alpha^\Sigma(U)$ and in turn by taking limits over neighbourhoods $U \ni p$ to linear maps

$$L_{\ell_k^{\xi_p}}, L_{\bar{\ell}_k^{\xi_p}} : \mathcal{F}_p^{\mathbf{a}, \alpha} \longrightarrow \mathcal{F}_p^{\mathbf{a}, \alpha}. \quad (3.77)$$

The following proposition is a generalisation of Proposition 3.7, which concerned only the translation operators $L_{(-1)} = D$ and $\bar{L}_{(-1)} = \bar{D}$, to all operators $L_{(k)}$ and $\bar{L}_{(k)}$ for $k \geq -1$.

Proposition 3.18. *For any $A \in \mathbb{F}^{\mathbf{a}, \alpha}$ and local coordinate ξ in the neighbourhood of $p \in \Sigma^\circ$, we have the relations*

$$(L_{(k)} A)_p^\xi = L_{\ell_k^{\xi_p}} A_p^\xi, \quad (\bar{L}_{(k)} A)_p^\xi = L_{\bar{\ell}_k^{\xi_p}} A_p^\xi \quad (3.78)$$

in $\mathcal{F}_p^{\mathbf{a}, \alpha}$ for every $k \in \mathbb{Z}_{\geq -1}$.

Proof. By linearity it is sufficient to consider a state $A \in \mathbb{F}^{\mathfrak{a}, \alpha}$ of the form (3.3). And just as in the proof of Proposition 3.9 we may focus on a chiral state $A = \mathfrak{a}_{(-m_r)}^r \dots \mathfrak{a}_{(-m_1)}^1 |0\rangle$ since the treatment of the anti-chiral part is completely analogous. Since both sides of (3.78) are defined using the Leibniz rule it then suffices to show that

$$[L_{(k)}^{\xi_p}, \mathfrak{a}_{(-m_i)}^i] = L_{\ell_k^{\xi_p}} \mathfrak{a}_{(-m_i)}^i = L_{\ell_k^{\xi_p}} \left[s \left(\mathfrak{a}_{\xi_p}^i \otimes \left(-\xi_p^{-m_i} \bar{\partial} \rho_{V_+^+}^{V_-} \right) \right) \right]_Y, \quad (3.79)$$

where the second step is by definition of the vertex modes in (3.29). We now consider separately the three main examples from §2.1.2.

Kac-Moody: Here $\mathfrak{a}^i = \mathbb{X}^i \in \mathfrak{g}$ for all $i = 1, \dots, r$. And by (3.70) we have

$$[L_{(k)}, \mathfrak{a}_{(-m_i)}^i] = [L_{(k)}, \mathbb{X}_{(-m_i)}^i] = m_i \mathbb{X}_{(k-m_i)}^i \quad (3.80)$$

for the left hand side of (3.79). As for the right hand side, by definition of the Lie derivative we have $L_{\ell_k^{\xi_p}} \xi_p^{-m_i} = \ell_k^{\xi_p} (\xi_p^{-m_i}) = m_i \xi_p^{k-m_i}$ so this reads

$$\begin{aligned} L_{\ell_k^{\xi_p}} \mathbb{X}_{(-m_i)}^i &= m_i \left[\mathbb{X}^i \otimes \left(-\xi_p^{k-m_i} \bar{\partial} \rho_{V_+^+}^{V_-} \right) \right]_Y - \left[s \left(\mathbb{X}^i \otimes \left(\xi_p^{-m_i} \bar{\partial} (\ell_k^{\xi_p} \rho_{V_+^+}^{V_-}) \right) \right) \right]_Y \\ &= m_i \mathbb{X}_{(k-m_i)}^i - \left[\bar{\partial} \left(s \left(\mathbb{X}^i \otimes \left(\xi_p^{-m_i} \ell_k^{\xi_p} \rho_{V_+^+}^{V_-} \right) \right) \right) \right]_Y. \end{aligned}$$

The first term on the right is exactly the vertex mode at p in the coordinate ξ associated with the loop generator in $\hat{\mathfrak{a}}$ on the right hand side of (3.80). The second term on the right hand side is the cohomology of a $\bar{\partial}$ -exact element of $\mathfrak{a}_{\xi_p} \otimes \Omega_c^{0,1}(Y_+)$ which therefore vanishes.

Virasoro: Now $\mathfrak{a}^i = \Omega$ for all $i = 1, \dots, r$ and by (3.72), the left hand side of (3.79) reads

$$[L_{(k)}, \mathfrak{a}_{(-m_i)}^i] = [L_{(k)}, \Omega_{(-m_i)}] = (m_i + k + 1) \Omega_{(k-m_i)} + \frac{k^3 - k}{12} c \delta_{k-m_i, 1}. \quad (3.81)$$

For the right hand side of (3.79) we recall that $\Omega_{\xi_p} = -\partial_{\xi_p}$ on which the Lie derivative of $\ell_k^{\xi_p}$ acts as $-L_{\ell_k^{\xi_p}} \partial_{\xi_p} = -(k+1) \xi_p^k \partial_{\xi_p}$. Using also $L_{\ell_k^{\xi_p}} \xi_p^{-m_i} = m_i \xi_p^{k-m_i}$ as above we find

$$\begin{aligned} L_{\ell_k^{\xi_p}} \Omega_{(-m_i)}^{\xi_p} &= (m_i + k + 1) \left[s \left(\Omega_{\xi_p} \otimes \left(-\xi_p^{k-m_i} \bar{\partial} \rho_{V_+^+}^{V_-} \right) \right) \right]_Y - \left[s \left(\Omega_{\xi_p} \otimes \left(\xi_p^{-m_i} \bar{\partial} (\ell_k^{\xi_p} \rho_{V_+^+}^{V_-}) \right) \right) \right]_Y \\ &\quad + \frac{k^3 - k}{24\pi i} c \int_U \xi_p^{k-m_i-2} d\xi_p \wedge \bar{\partial} \rho_{V_+^+}^{V_-}. \end{aligned} \quad (3.82)$$

To explain the origin of the last term, recall from §2.1.2 that in the Virasoro case the 2-cocycle (2.10b) depends explicitly on the coordinate up to a 2-coboundary (2.11). And under a change of coordinate $\eta \mapsto \xi$ this induces an isomorphism of unital local Lie algebras (2.12) given by a shift along the central element. Here we are considering the Lie derivative action of the vector field $\ell_k^{\xi_p}$ which corresponds to the infinitesimal change of coordinate $\xi_p \mapsto \xi'_p := \xi_p - \epsilon \xi_p^{k+1}$. But for an infinitesimal coordinate transformation $\eta \mapsto \xi = \eta + \epsilon v(\eta)$, the Schwarzian derivative to leading order in ϵ is $(S\xi)(\eta) = \epsilon v'''(\eta)$ which here takes the form $-(k^3 - k) \xi_p^{k-2}$. This explains the last term on the right hand side of (3.82). Using Lemma 3.2 to evaluate the integral we obtain exactly the last term on the right hand side of (3.81). As in the Kac-Moody case, the first term on the right hand side of (3.82) corresponds to that of (3.81) and the second term on the right hand side of (3.81) vanishes.

$\beta\gamma$ system: Here we have either $\mathbf{a}^i = \beta$ or $\mathbf{a}^i = \gamma$ for each $i = 1, \dots, r$.

In the first case, since $\mathbf{a}_{\xi_p}^i = \beta_{\xi_p} = 1$ is constant, the computation works exactly as in the Kac-Moody case and we find that (3.79) matches the first equation in (3.74).

In the second case, the left hand side of (3.79) reads

$$[L_{(k)}, \mathbf{a}_{(-m_i)}^i] = [L_{(k)}, \gamma_{(-m_i)}] = (m_i - k - 1)\gamma_{(k-m_i)} \quad (3.83)$$

by virtue of the second equation in (3.74). Recalling from §2.1.2 that we have $\mathbf{a}_{\xi_p}^i = \mathbf{d}\xi_p$ and using the fact that $L_{\ell_k^{\xi_p}} \mathbf{d}\xi_p = -(k+1)\xi_p^k \mathbf{d}\xi_p$ we deduce

$$L_{\ell_k^{\xi_p}} \gamma_{(-m_i)}^{\xi_p} = (m_i - k - 1) \left[s \left(\gamma_{\xi_p} \otimes \left(-\xi_p^{k-m_i} \bar{\partial} \rho_{V_+^+}^{V_+^-} \right) \right) \right]_Y. \quad (3.84)$$

This agrees exactly with vertex mode at p in the coordinate ξ of the loop generator in $\hat{\mathbf{a}}$ on the right hand side of (3.83). \square

We define a linear map $R_p^{\eta \rightarrow \xi} : \mathbb{F}^{\mathbf{a}, \alpha} \rightarrow \mathbb{F}^{\mathbf{a}, \alpha}$ by, cf. [FB, (6.3.3)],

$$R_p^{\eta \rightarrow \xi} := \exp \left(- \sum_{k \geq 1} (b_k L_{(k)} + \bar{b}_k \bar{L}_{(k)}) \right) b_0^{-L_{(0)}} \bar{b}_0^{-\bar{L}_{(0)}}. \quad (3.85)$$

This is well defined since the last two factors act diagonally as $b_0^{-\Delta_A} \bar{b}_0^{-\bar{\Delta}_A}$ on any monomial state A as in (3.3), of chiral and anti-chiral conformal dimensions Δ_A and $\bar{\Delta}_A$ respectively, and the exponential is well defined since $L_{(k)}, \bar{L}_{(k)} : \mathbb{F}^{\mathbf{a}, \alpha} \rightarrow \mathbb{F}^{\mathbf{a}, \alpha}$ are locally nilpotent for $k \geq 1$.

It follows immediately from the definitions (3.65), (3.66) and from Lemma 3.10 that for any state $A \in \mathbb{F}^{\mathbf{a}, \alpha}$ and any $k \in \mathbb{Z}$ we have

$$L_{(k)}^{\xi_p} A_p^\xi = (L_{(k)} A)_p^\xi, \quad \bar{L}_{(k)}^{\xi_p} A_p^\xi = (\bar{L}_{(k)} A)_p^\xi. \quad (3.86)$$

It is then also convenient to introduce the linear map $\mathcal{R}_p^{\eta \rightarrow \xi} : \mathcal{F}_p^{\mathbf{a}, \alpha} \rightarrow \mathcal{F}_p^{\mathbf{a}, \alpha}$ defined by

$$\mathcal{R}_p^{\eta \rightarrow \xi} := \exp \left(- \sum_{k \geq 1} (b_k L_{(k)}^{\eta_p} + \bar{b}_k \bar{L}_{(k)}^{\eta_p}) \right) b_0^{-L_{(0)}^{\eta_p}} \bar{b}_0^{-\bar{L}_{(0)}^{\eta_p}}, \quad (3.87)$$

with the property $\mathcal{R}_p^{\eta \rightarrow \xi} A_p^\eta = (R_p^{\eta \rightarrow \xi} A)_p^\eta$ as a result of the identity (3.86). In particular, the operator $\mathcal{R}_p^{\eta \rightarrow \xi} : \mathcal{F}_p^{\mathbf{a}, \alpha} \rightarrow \mathcal{F}_p^{\mathbf{a}, \alpha}$ naturally takes as input a state $A \in \mathbb{F}^{\mathbf{a}, \alpha}$ prepared at p in the local coordinate η and returns another state $R_p^{\eta \rightarrow \xi} A \in \mathbb{F}^{\mathbf{a}, \alpha}$ also prepared at p in the same local coordinate η . This is to be contrasted with the linear map $(\cdot)_p^{\eta \rightarrow \xi} : \mathcal{F}_p^{\mathbf{a}, \alpha} \rightarrow \mathcal{F}_p^{\mathbf{a}, \alpha}$ defined in (3.27) which given a state $A \in \mathbb{F}^{\mathbf{a}, \alpha}$ prepared at p in the local coordinate η returns the same state A also prepared at p but now in the new coordinate ξ . These two maps, in fact, coincide.

Theorem 3.19. *Let $p \in U \subset \Sigma^\circ$ be a point in a connected open subset equipped with two local coordinates $\xi, \eta : U \rightarrow \mathbb{C}$. For any state $A \in \mathbb{F}^{\mathbf{a}, \alpha}$ we have $A_p^\xi = \mathcal{R}_p^{\eta \rightarrow \xi} A_p^\eta$.*

Proof. Consider the operator

$$\hat{\varrho}_p^{\eta \rightarrow \xi} := \exp \left(- \sum_{k \geq 1} (b_k L_{\ell_k^{\eta_p}} + \bar{b}_k \bar{L}_{\bar{\ell}_k^{\eta_p}}) \right) b_0^{-L_{\ell_0^{\eta_p}}} \bar{b}_0^{-\bar{L}_{\bar{\ell}_0^{\eta_p}}} : \mathcal{F}_p^{\mathbf{a}, \alpha} \longrightarrow \mathcal{F}_p^{\mathbf{a}, \alpha}. \quad (3.88)$$

Since the Lie derivatives along the vector fields $\ell_k^{\eta_p}$ and $\bar{\ell}_k^{\eta_p}$ generate local (anti-)holomorphic coordinate transformations fixing the point p , it follows by definition of the coefficients b_k and \bar{b}_k in (3.75) that (3.88) implements the coordinate transformation $\eta_p \mapsto \xi_p$ in $\mathcal{F}_p^{\mathbf{a}, \alpha}$. Therefore

$$A_p^\xi = \hat{\varrho}_p^{\eta \rightarrow \xi} A_p^\eta = (R^{\eta \rightarrow \xi} A)_p^\eta = \mathcal{R}^{\eta \rightarrow \xi} A_p^\eta$$

where the second equality is by definition (3.85) and Proposition 3.18, while the last equality is by definition (3.87). \square

The following gives an analogue of Huang's "change of variable" formula [Hua, p.176-177], see also [FB, Lemma 6.5.6], in the present full vertex algebra setting.

Corollary 3.20. *With the same setting as in Theorem 3.19, for any $A, B \in \mathbb{F}^{\alpha, \alpha}$ we have*

$$\mathcal{R}_p^{\eta \rightarrow \xi} \left(\left(\mathcal{Y}(A_q^\xi, \underline{\xi}_p(q)) B_p^\xi \right)_p^{\xi \rightarrow \eta} \right) = \mathcal{Y}(\mathcal{R}_q^{\eta \rightarrow \xi} A_q^\eta, \underline{\eta}_p(q)) \mathcal{R}_p^{\eta \rightarrow \xi} B_p^\eta.$$

Remark 3.21. The slightly awkward use of the linear map $(\cdot)_p^{\xi \rightarrow \eta} : \mathcal{F}_p^{\alpha, \alpha} \rightarrow \mathcal{F}_p^{\alpha, \alpha}$ on the left hand side ensures that this is an equality between states prepared at p in the same local coordinate η . Indeed, the left hand side above could also be written simply as $\mathcal{Y}(A_q^\xi, \underline{\xi}_p(q)) B_p^\xi$ but then the equality in Corollary 3.20 would be comparing a set of states prepared at p in the local coordinate ξ with a set of states prepared at p in the local coordinate η . In other words, in terms of the formal vertex operator $Y(A, \underline{\zeta})B$ introduced in (3.44) for any $A, B \in \mathbb{F}^{\alpha, \alpha}$, the identity in Corollary 3.20 takes the more recognisable form

$$R_p^{\eta \rightarrow \xi} \left(Y(A, \underline{\xi}_p(q)) B \right) = Y(R_q^{\eta \rightarrow \xi} A, \underline{\eta}_p(q)) R_p^{\eta \rightarrow \xi} B. \quad \square$$

Proof of Corollary 3.20. By Theorem 3.19 and the definition of the linear map $(\cdot)_p^{\xi \rightarrow \eta}$, for any state $C \in \mathbb{F}^{\alpha, \alpha}$ we have $C_p^\xi = \mathcal{R}_p^{\eta \rightarrow \xi} ((C_p^\xi)_p^{\xi \rightarrow \eta})$. In particular, it follows that

$$\mathcal{Y}(A_q^\xi, \underline{\xi}_p(q)) B_p^\xi = \mathcal{R}_p^{\eta \rightarrow \xi} \left(\left(\mathcal{Y}(A_q^\xi, \underline{\xi}_p(q)) B_p^\xi \right)_p^{\xi \rightarrow \eta} \right).$$

On the other hand, we have

$$\begin{aligned} \mathcal{Y}(A_q^\xi, \underline{\xi}_p(q)) B_p^\xi &= m_{(Y, p), U} (m_{q, Y}(A_q^\xi) \otimes B_p^\xi) = m_{(Y, p), U} \left(m_{q, Y} \left((R_q^{\eta \rightarrow \xi} A)_q^\eta \right) \otimes (R_p^{\eta \rightarrow \xi} B)_p^\eta \right) \\ &= \mathcal{Y} \left((R_q^{\eta \rightarrow \xi} A)_q^\eta, \underline{\eta}_p(q) \right) (R_p^{\eta \rightarrow \xi} B)_p^\eta = \mathcal{Y}(\mathcal{R}_q^{\eta \rightarrow \xi} A_q^\eta, \underline{\eta}_p(q)) \mathcal{R}_p^{\eta \rightarrow \xi} B_p^\eta, \end{aligned}$$

where in the first and third equalities we used Proposition 3.9 and the second equality is by Theorem 3.19 again. \square

3.5 Invariant bilinear form

From now on we will specialise to the case of the 2-sphere

$$\Sigma = S^2. \quad (3.89)$$

Since Σ is orientable, in this case the double is a disjoint union $\widehat{\Sigma} = \Sigma_+ \sqcup \Sigma_-$ of two copies of Σ equipped with opposite orientations. As a simple application of Theorem 3.19, in §3.5.2 below we will construct a canonical invariant bilinear form on the vector space $\mathbb{F}^{\alpha, \alpha}$. In particular, this will be used in §3.5.3 to show that $\mathbb{F}^{\alpha, \alpha}$ satisfies the axioms of a full vertex algebra [Mor2]. To define this invariant bilinear form, we first need to a notion of vacuum state.

3.5.1 Vacuum state

Recall from §2.3 that $\Sigma \in \mathbf{Top}(\Sigma)$. However, $\Sigma = S^2$ cannot be covered by a single coordinate patch, so Theorem 2.5 does not apply to the open subset $U = \Sigma$ and in particular $\mathcal{UL}_\alpha^\Sigma(\Sigma)$ is not given by the isomorphism in (3.20). Instead we have the following result which depends on the choice of holomorphic vector bundle L among the three examples in §2.1.2.

Lemma 3.22. *In the Kac-Moody and Virasoro cases we have an isomorphism*

$$\langle \cdot \rangle : \mathcal{UL}_\alpha^\Sigma(\Sigma) \xrightarrow{\cong} \mathbb{C} \quad (3.90)$$

such that $\langle [1]_\Sigma \rangle = 1$. In the $\beta\gamma$ system case we have instead $\mathcal{UL}_\alpha^\Sigma(\Sigma) \cong 0$.

Proof. We consider the three cases separately.

Kac-Moody: We have $H^0(\mathcal{L}_\alpha^\Sigma(\Sigma)) = \mathfrak{g} \oplus \mathfrak{g}$ and $H^1(\mathcal{L}_\alpha^\Sigma(\Sigma)) = \mathbb{C}$ since

$$H^0(\mathfrak{g} \otimes \Omega^{0,\bullet}(\widehat{\Sigma})) = \mathfrak{g} \oplus \mathfrak{g}, \quad H^1(\mathfrak{g} \otimes \Omega^{0,\bullet}(\widehat{\Sigma})) = 0.$$

Consider the unital DG Lie algebra $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathbb{C}\mathbf{1}$ with $\mathbf{1}$ of degree 1 and equipped with the trivial Lie bracket. We then have a quasi-isomorphism of unital DG Lie algebras

$$\phi : \mathfrak{g} \oplus \mathfrak{g} \oplus \mathbb{C}\mathbf{1} \xrightarrow{\simeq} \mathcal{L}_\alpha^\Sigma(\Sigma)$$

sending $(X, Y) \in \mathfrak{g} \oplus \mathfrak{g}$ to the constant function in $\mathfrak{g} \otimes \Omega^{0,0}(\widehat{\Sigma}) = \mathfrak{g} \otimes \Omega^{0,0}(\Sigma_+) \oplus \mathfrak{g} \otimes \Omega^{0,0}(\Sigma_-)$ equal to X on Σ_+ and Y on Σ_- , and $\mathbf{1}$ to $\mathbf{1} \in \mathcal{L}_\alpha^\Sigma(\Sigma)$. Since the functor $\overline{\text{CE}}_\bullet$ introduced in §A.1 preserved quasi-isomorphisms by Proposition A.2 we obtain a quasi-isomorphism

$$\overline{\text{CE}}_\bullet(\phi) : \text{Sym}((\mathfrak{g} \oplus \mathfrak{g})[1]) \xrightarrow{\simeq} \overline{\text{CE}}_\bullet(\mathcal{L}_\alpha^\Sigma(\Sigma))$$

where the differential d_{CE} on the domain is trivial. Taking the 0th-cohomology then yields an isomorphism $\mathbb{C} \xrightarrow{\simeq} \mathcal{U}\mathcal{L}_\alpha^\Sigma(\Sigma)$ which sends $1 \in \mathbb{C}$ to $[1]_\Sigma \in \mathcal{U}\mathcal{L}_\alpha^\Sigma(\Sigma)$ and whose inverse then gives the required isomorphism (3.90).

Virasoro: We have $H^0(\mathcal{L}_\alpha^\Sigma(\Sigma)) = \mathbb{C}^6$ and $H^1(\mathcal{L}_\alpha^\Sigma(\Sigma)) = \mathbb{C}$ since

$$H^0(\Omega^{0,\bullet}(\widehat{\Sigma}, T^{1,0}\widehat{\Sigma})) = \mathbb{C}^6, \quad H^1(\Omega^{0,\bullet}(\widehat{\Sigma}, T^{1,0}\widehat{\Sigma})) = 0$$

where the 0th cohomology is given by global (anti-)holomorphic vector fields on $\mathbb{C}P^1$. Consider therefore the unital DG Lie algebra $\mathbb{C}^6 \oplus \mathbb{C}\mathbf{1}$, with $\mathbf{1}$ of degree 1, again equipped with the trivial Lie bracket. We then have a quasi-isomorphism of unital DG Lie algebras

$$\phi : \mathbb{C}^6 \oplus \mathbb{C}\mathbf{1} \xrightarrow{\simeq} \mathcal{L}_\alpha^\Sigma(\Sigma)$$

which sends $(a, b, c, a', b', c') \in \mathbb{C}^6$ to the pair of global (anti-)holomorphic vector fields given by $\partial_\xi \otimes (a\xi^2 + b\xi + c) \in \mathfrak{a}_\xi \otimes \Omega^{0,0}(\Sigma_+)$ and $\partial_{\bar{\xi}} \otimes (a'\bar{\xi}^2 + b'\bar{\xi} + c') \in \mathfrak{a}_{\bar{\xi}} \otimes \Omega^{0,0}(\Sigma_-)$ in any choice of holomorphic coordinate $\xi : \mathbb{C} = \mathbb{C}P^1 \setminus \{\infty\} \rightarrow \mathbb{C}$, and $\mathbf{1}$ to $\mathbf{1} \in \mathcal{L}_\alpha^\Sigma(\Sigma)$. Applying the functor $\overline{\text{CE}}_\bullet$ we obtain a quasi-isomorphism

$$\overline{\text{CE}}_\bullet(\phi) : \text{Sym}(\mathbb{C}^6[1]) \xrightarrow{\simeq} \overline{\text{CE}}_\bullet(\mathcal{L}_\alpha^\Sigma(\Sigma))$$

where the differential d_{CE} on the domain is again trivial so that taking the 0th cohomology we again obtain the inverse of the required isomorphism (3.90).

$\beta\gamma$ system: We have $H^0(\mathcal{L}_\alpha^\Sigma(\Sigma)) = \mathbb{C}^2$ and $H^1(\mathcal{L}_\alpha^\Sigma(\Sigma)) = \mathbb{C}^3$ coming from the fact that

$$H^0(\Omega^{0,\bullet}(\widehat{\Sigma}, L)) = \mathbb{C}^2, \quad H^1(\Omega^{0,\bullet}(\widehat{\Sigma}, L)) = \mathbb{C}^2.$$

We then form the unital DG Lie algebra $\mathfrak{a} \oplus \bar{\mathfrak{a}} \oplus \mathbb{C}\mathbf{1}$ with $\mathfrak{a} = \text{span}_{\mathbb{C}}\{\beta, \gamma\}$ and where we set $\bar{\mathfrak{a}} := \text{span}_{\mathbb{C}}\{\bar{\beta}, \bar{\gamma}\}$. Here β and $\bar{\beta}$ of degree 0 and $\gamma, \bar{\gamma}$ and $\mathbf{1}$ all of degree 1. The Lie bracket is given by $[\beta, \gamma] = \mathbf{1} = [\bar{\beta}, \bar{\gamma}]$ and $[\beta, \bar{\gamma}] = 0 = [\bar{\beta}, \gamma]$. We have a quasi-isomorphism of unital DG Lie algebras

$$\phi : \mathfrak{a} \oplus \bar{\mathfrak{a}} \oplus \mathbb{C}\mathbf{1} \xrightarrow{\simeq} \mathcal{L}_\alpha^\Sigma(\Sigma)$$

which sends β and $\bar{\beta}$ to the pair of constant functions $1 \in \Omega^{0,0}(\Sigma_+)$ and $1 \in \Omega^{0,0}(\Sigma_-)$, and sends γ and $\bar{\gamma}$ to the pair of $(1,1)$ -forms $d\xi \wedge \xi^{-1}\bar{\partial}\chi \in \Omega^{1,1}(\Sigma_+)$ and $d\bar{\xi} \wedge \bar{\xi}^{-1}\bar{\partial}\chi' \in \Omega^{1,1}(\Sigma_-)$ where $\chi \in \Omega^{0,0}(\Sigma_+)$ is constant equal to 1 in a neighbourhood of $\xi = 0$ and constant equal to 0 in a neighbourhood of $\xi = \infty$, and likewise for $\chi' \in \Omega^{0,0}(\Sigma_-)$ in the holomorphic coordinate $\bar{\xi}$ on Σ_- . Applying the functor $\overline{\text{CE}}_\bullet$ we obtain a quasi-isomorphism

$$\overline{\text{CE}}_\bullet(\phi) : \text{Sym}((\mathfrak{a} \oplus \bar{\mathfrak{a}})[1]) \xrightarrow{\simeq} \overline{\text{CE}}_\bullet(\mathcal{L}_\alpha^\Sigma(\Sigma))$$

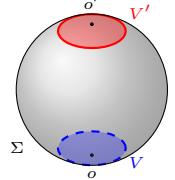
where the differential on the domain is $d_{CE} = d_{[.,.]}$. A degree 0 element in the domain is a linear combination of elements of the form $(s\gamma)^n(s\bar{\gamma})^{\bar{n}}$ with $n, \bar{n} \in \mathbb{Z}_{\geq 0}$. But the latter is always d_{CE} -exact since we can write it, for instance, as $-d_{CE}(\frac{1}{n+1}(s\beta)(s\gamma)^{n+1}(s\bar{\gamma})^{\bar{n}})$ using the fact that $-d_{CE}((s\beta)(s\gamma)) = s[\beta, \gamma] = s\mathbf{1} = 1$ where the last step is from working in the quotient in the definition of the functor \overline{CE}_\bullet in (A.3). The result for the $\beta\gamma$ system now follows from taking the 0th cohomology of the above quasi-isomorphism. \square

A linear map $\mathcal{UL}_\alpha^\Sigma(\Sigma) \rightarrow \mathbb{C}$, as in (3.90) but not necessarily an isomorphism, defines the notion of a *state* for the prefactorisation algebra $\mathcal{UL}_\alpha^\Sigma$ in the sense of [CG1, Definition 4.9.1]. We will use this below to define an invariant bilinear form on $\mathbb{F}^{\alpha, \alpha}$.

Note that while Lemma 3.22 provides us with a state in both the Kac-Moody and Virasoro cases, it fails to provide a suitable notion of state for the $\beta\gamma$ system. It is clear from the proof of Lemma 3.22 that the culprit for the vanishing of $\mathcal{UL}_\alpha^\Sigma(\Sigma)$ are the global holomorphic sections of $\Omega^{0,0}(\Sigma_\pm)$ and $\Omega^{1,1}(\Sigma_\pm)$ which pair non-trivially to 1 under the differential d_{CE} . Removing these zero modes can be achieved by adding a mass term which will have the effect of modifying the global observables of the factorisation algebra $\mathcal{UL}_\alpha^\Sigma$ over $\Sigma = S^2$ while preserving the observables over small open sets [Gwi, §6.1.2], see also [CG1, Lemma 2.5.1]. We will not pursue this idea further here and from now on, when discussing the invariant bilinear form on $\mathbb{F}^{\alpha, \alpha}$, we will therefore only focus on the Kac-Moody and Virasoro cases.

3.5.2 Invariant bilinear form on $\mathbb{F}^{\alpha, \alpha}$

Let us fix antipodal points o, o' on Σ and a coordinate $u : \Sigma \setminus \{o'\} \rightarrow \mathbb{C}$ with $u(o) = 0$. We also let $u^{-1} : \Sigma \setminus \{o\} \rightarrow \mathbb{C}$ be its inverse defined by $u^{-1}(p) := u(p)^{-1}$ so that, in particular, $u^{-1}(o') = 0$. Consider the linear map (3.55) in the case $n = 2$ and $U = \Sigma$, associated with the points $o, o' \in \Sigma$ and with the local coordinates u and u^{-1} near these points, respectively. We obtain a linear map

$$\begin{aligned} (\cdot, \cdot) : \mathbb{F}^{\alpha, \alpha} \otimes \mathbb{F}^{\alpha, \alpha} &\longrightarrow \mathbb{C}, \\ (B, C) &\coloneqq \left\langle m_{(o', o), \Sigma}(B_{o'}^{u^{-1}} \otimes C_o^u) \right\rangle. \end{aligned} \quad (3.91)$$


Specifically, the state $B \in \mathbb{F}^{\alpha, \alpha}$ is prepared at $o' \in V'$ in the local chart (V', u^{-1}) while the state $C \in \mathbb{F}^{\alpha, \alpha}$ is prepared at $o \in V$ in the local chart (V, u) with $V \cap V' = \emptyset$. We then apply the factorisation product $m_{(V', V), \Sigma} : \mathcal{UL}_\alpha^\Sigma(V') \otimes \mathcal{UL}_\alpha^\Sigma(V) \rightarrow \mathcal{UL}_\alpha^\Sigma(\Sigma)$ to the element $B_{V'}^{u^{-1}} \otimes C_V^u$ followed by the isomorphism $\langle \cdot \rangle$ from Lemma 3.22 to obtain a complex number.

Proposition 3.23. *In the Kac-Moody and Virasoro cases, the bilinear form (3.91) is invariant in the sense that*

$$(B, Y(A, \underline{\zeta})C) = \left(Y(e^{\zeta L_{(1)} + \bar{\zeta} \bar{L}_{(1)}} (-1)^{L_{(0)} + \bar{L}_{(0)}} \zeta^{-2L_{(0)}} \bar{\zeta}^{-2\bar{L}_{(0)}} A, \underline{\zeta}^{-1}) B, C \right)$$

for any states $A, B, C \in \mathbb{F}^{\alpha, \alpha}$.

Remark 3.24. This identity coincides exactly with [Mor1, §3.1] or [AMT, eq. (4)], except for the fact that the operator $(-1)^{L_{(0)} + \bar{L}_{(0)}}$ here is replaced by $(-1)^{L_{(0)} - \bar{L}_{(0)}}$ there. However, since for us $\bar{L}_{(0)}$ acts as an integer it follows that these two operators coincide in our case. \square

Remark 3.25. In the $\beta\gamma$ system case the identity is vacuous as both sides vanish identically. \square

Proof of Proposition 3.23. The change of coordinate $u^{-1} \rightarrow u$ on $U = \Sigma \setminus \{o, o'\}$ is given simply by $\varrho(t) = t^{-1}$. Letting $p \in U$ and denoting its coordinate by $w := u(p) \in \mathbb{C}^\times$ we find

that $b_0 = -w^2$ and $b_1 = -w$ while $b_n = 0$ for all $n \geq 2$. In particular, using Theorem 3.19 and definition (3.85), for any state $A \in \mathbb{F}^{\mathfrak{a},\alpha}$ we have

$$A_p^u = \exp \left(w L_{(1)}^{u_p^{-1}} + \bar{w} \bar{L}_{(1)}^{u_p^{-1}} \right) (-w^2)^{-L_{(0)}^{u_p^{-1}}} (-\bar{w}^2)^{-\bar{L}_{(0)}^{u_p^{-1}}} A_p^{u^{-1}} = \tilde{A}_p^{u^{-1}}, \quad (3.92)$$

where in the last step we have used (3.86) and introduced the state

$$\tilde{A} := \exp (w L_{(1)} + \bar{w} \bar{L}_{(1)}) (-1)^{L_{(0)} + \bar{L}_{(0)}} w^{-2L_{(0)}} \bar{w}^{-2\bar{L}_{(0)}} A \in \mathbb{F}^{\mathfrak{a},\alpha}.$$

Consider now the linear map (3.55) for $n = 3$ distinct points $o, o', p \in \Sigma$ with $U = \Sigma$. At the points o and o' we still use the local coordinates u and u^{-1} , respectively, and at the point p we will also use u . The linear map (3.55) for $U = \Sigma$ combined with the isomorphism from Lemma 3.22 then gives

$$\mathbb{F}^{\mathfrak{a},\alpha} \otimes \mathbb{F}^{\mathfrak{a},\alpha} \otimes \mathbb{F}^{\mathfrak{a},\alpha} \longrightarrow \mathbb{C}, \quad (B, A, C) \longmapsto \left\langle m_{(o',p,o),\Sigma} (B_{o'}^{u^{-1}} \otimes A_p^u \otimes C_o^u) \right\rangle.$$

Using the associativity (2.17) of the factorisation product, we can compute the factorisation product on the right hand side by first computing the factorisation product

$$m_{(p,o),V} (A_p^u \otimes C_o^u) = \mathcal{Y}(A_p^u, \underline{u}_o(p)) C_o^u = (Y(A, \underline{w}) C)_V^u$$

for some open $V \subset \Sigma \setminus \{o'\}$ containing p and o , where we have used the fact that $u(o) = 0$ so that $u_o(p) = w$, followed by the factorisation product $m_{(o',V),\Sigma} (B_{o'}^{u^{-1}} \otimes m_{(p,o),V} (A_p^u \otimes C_o^u))$. By definition (3.91) of the bilinear form on $\mathbb{F}^{\mathfrak{a},\alpha}$, the result of this computation can be written simply as $(B, Y(A, \underline{w}) C)$. On the other hand, we can compute the same factorisation product by first using (3.92) to rewrite the state A prepared at p in the coordinate u in terms of states prepared in the coordinate u^{-1} . This allows us to first compute the factorisation product

$$m_{(o',p),V'} (B_{o'}^{u^{-1}} \otimes A_p^u) = \mathcal{Y}(\tilde{A}_p^{u^{-1}}, \underline{u}_{o'}^{-1}(p)) B_{o'}^{u^{-1}} = (Y(\tilde{A}, \underline{w}^{-1}) B)_{V'}^{u^{-1}}$$

for some choice of open subset $V' \subset \Sigma \setminus \{o\}$ containing the points p and o' , where we have used the fact that $u^{-1}(o') = 0$ and $u^{-1}(p) = w^{-1}$. By subsequently performing the remaining factorisation product with C_o^u , namely $m_{(V',o),\Sigma} (m_{(o',p),V'} (B_{o'}^{u^{-1}} \otimes A_p^u) \otimes C_o^u)$, and comparing the result with the other computation described above we deduce the claim. \square

3.5.3 Full vertex algebra axioms

As recalled in §1.1.1, there are a number of mathematical formulations of the notion of full, or non-chiral, vertex operator algebras capable of describing both chiral and anti-chiral states in a full conformal field theory. Since the emphasis in each formulation is slightly different they also go by distinct names: *OPE-algebras* in [KO, Ros], *full field algebras* in [HuKo, Kon], *full vertex algebras* in [Mor1, Mor2] and *non-chiral vertex operator algebras* in [SS]. We now show that our geometric realisation (3.12) of the vector space $\mathbb{F}^{\mathfrak{a},\alpha}$ using the prefactorisation algebra $\mathcal{UL}_\alpha^\Sigma$ naturally endows it with the structure of a full vertex algebra in the sense of [Mor1, Mor2], justifying the use of the name ‘full vertex algebra’ for the vector space $\mathcal{F}_p^{\mathfrak{a},\alpha}$.

Recall from [Mor1, Mor2], see also [Mor3], that a full vertex algebra is an \mathbb{R}^2 -graded vector space $F = \bigoplus_{h,\bar{h} \in \mathbb{R}^2} F_{h,\bar{h}}$ over \mathbb{C} equipped with a linear map

$$Y(-, \underline{z}) : F \longrightarrow \text{End}(F)[[z^{\pm 1}, \bar{z}^{\pm 1}, |z|^\mathbb{R}]], \quad A \longmapsto Y(A, \underline{z}) = \sum_{\substack{r,s \in \mathbb{R} \\ r-s \in \mathbb{Z}}} A_{(r,s)} z^{-r-1} \bar{z}^{-s-1},$$

where the space $\text{End}(F)[[z^{\pm 1}, \bar{z}^{\pm 1}, |z|^\mathbb{R}]]$ consists of formal sums as above with $A_{(r,s)} \in \text{End}(F)$, and a non-zero element $|0\rangle \in F_{0,0}$ satisfying the following axioms:

FV1) For any $A, B \in F$,

- (a) There exists $N \in \mathbb{R}$ such that $A_{(r,s)}B = 0$ for any $r \geq N$ or $s \geq N$, and
- (b) For any $H \in \mathbb{R}$, the set $\{(r, s) \in \mathbb{R}^2 \mid A_{(r,s)}B \neq 0 \text{ and } r + s \geq H\}$ is finite,

FV2) For any $h, \bar{h} \in \mathbb{R}$, $F_{h,\bar{h}} \neq 0$ implies $h - \bar{h} \in \mathbb{Z}$,

FV3) For any $A \in F$, $Y(A, \underline{z})|0\rangle \in F[[z, \bar{z}]]$ and $\lim_{z \rightarrow 0} Y(A, \underline{z})|0\rangle = A_{(-1,-1)}|0\rangle = A$,

FV4) $Y(|0\rangle, \underline{z}) = \text{id}_F \in \text{End}(F)$,

FV5) For any $A, B, C \in F$ and $u \in F^\vee := \bigoplus_{h, \bar{h} \in \mathbb{R}} F_{h,\bar{h}}^*$ with $F_{h,\bar{h}}^*$ the dual of $F_{h,\bar{h}}$, the formal power series

$$u(Y(A, \underline{z}_1)Y(B, \underline{z}_2)C), \quad u(Y(B, \underline{z}_2)Y(A, \underline{z}_1)C), \quad u(Y(Y(A, \underline{z}_1 - \underline{z}_2)B, \underline{z}_2)C)$$

are the expansions of the same single-valued real analytic function on

$$Y_2 := \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 \neq 0, z_2 \neq 0, z_1 \neq z_2\}$$

in the respective regions $\{(z_1, z_2) \in Y_2 \mid |z_1| > |z_2|\}$, $\{(z_1, z_2) \in Y_2 \mid |z_2| > |z_1|\}$ and $\{(z_1, z_2) \in Y_2 \mid |z_2| > |z_1 - z_2|\}$ of Y_2 ,

FV6) For any $h, h', \bar{h}, \bar{h}', r, s \in \mathbb{R}$, $(F_{h,\bar{h}})_{(r,s)}F_{h',\bar{h}'} \subset F_{h+h'-r-1, \bar{h}+\bar{h}'-s-1}$.

Recall the formal vertex operator $Y(-, \underline{\zeta})$ defined in (3.44) using the geometric realisation (3.12) of the vector space $\mathbb{F}^{\alpha, \alpha}$ in terms of the prefactorisation algebra $\mathcal{UL}_\alpha^\Sigma$.

Theorem 3.26. *In the Kac-Moody and Virasoro cases, the tuple $(\mathbb{F}^{\alpha, \alpha}, Y(-, \underline{\zeta}), |0\rangle)$ is a full vertex algebra.*

Remark 3.27. In the $\beta\gamma$ system case our proof of axiom (FV5) does not apply since we make explicit use of the isomorphism (3.90) from Lemma 3.22. \square

Proof. Recall from §3.2.2 that in all cases we consider there is a natural \mathbb{Z}^2 -grading on the vector space $\mathbb{F}^{\alpha, \alpha}$ such that $A \in (\mathbb{F}^{\alpha, \alpha})_{h, \bar{h}}$ if $A \in \mathbb{F}^{\alpha, \alpha}$ is a homogeneous state with chiral and anti-chiral conformal dimensions $\Delta_A = h$ and $\bar{\Delta}_A = \bar{h}$. Axiom (FV2) is then immediate.

Part (a) of axiom (FV1) follows from the ‘Moreover’ part of Lemma 3.10 and part (b) then also follows since in our case we have $A_{(n, \bar{n})}B = 0$ unless $(n, \bar{n}) \in \mathbb{Z}^2$.

Axioms (FV3) and (FV4) follow from the second and first statements in Lemma 3.13.

To see axiom (FV5), let $p, q \in \Sigma \setminus \{o', o\}$ be distinct points. Let $u_1 := u(q)$ and $u_2 := u(p)$. Let $W, A, B, C \in \mathbb{F}^{\alpha, \alpha}$ and consider the expression

$$\mu(u_1, u_2) := \left\langle \Phi_U^{4, (u^{-1}, u, u, u)}((o', q, p, o), (W, A, B, C)) \right\rangle \in \mathbb{C}$$

defined by using the map (3.57) for $n = 4$ and the tuple of coordinates $\xi = (u^{-1}, u, u, u)$, and also the isomorphism (3.90) from Lemma 3.22. By Proposition 3.9 and the definition of the bilinear form in (3.91), this has the three expansions

$$(W, Y(A, \underline{u}_1)Y(B, \underline{u}_2)C), \quad (W, Y(B, \underline{u}_2)Y(A, \underline{u}_1)C), \quad (W, Y(Y(A, \underline{u}_1 - \underline{u}_2)B, \underline{u}_2)C)$$

in the respective regions of $\text{Conf}_3(U)$ determined by $|u_1| > |u_2|$, $|u_2| > |u_1|$ and $|u_2| > |u_1 - u_2|$.

Finally, Lemma 3.11 implies that for any homogeneous states $A, B \in \mathbb{F}^{\alpha, \alpha}$ and any $n, \bar{n} \in \mathbb{Z}$, the state $A_{(n, \bar{n})}B \in \mathbb{F}^{\alpha, \alpha}$ is homogeneous with chiral conformal dimension $\Delta_A + \Delta_B - n - 1$ and anti-chiral conformal dimension $\bar{\Delta}_A + \bar{\Delta}_B - \bar{n} - 1$, from which axiom (FV6) follows. \square

3.6 Reality conditions

In this subsection we construct an anti-linear involution on $\mathbb{F}^{\mathfrak{a},\alpha}$ and describe its action on the vertex operator products. We then combine it with the invariant bilinear form (3.91) to define a Hermitian sesquilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{F}^{\mathfrak{a},\alpha}$.

We let $\tau : \widehat{\Sigma} \rightarrow \widehat{\Sigma}$ be an orientation reversing involution of $\widehat{\Sigma}$ with respect to which the coordinate u used at the end of §3.4 has the property that

$$\overline{u \circ \tau} = u^{-1}. \quad (3.93)$$

A concrete example will be described at the start of §4 below, where τ will correspond to the Lorentzian involution $\tau_L : \widehat{\Sigma} \rightarrow \widehat{\Sigma}$ from §1.

3.6.1 Anti-linear involution on $\mathbb{F}^{\mathfrak{a},\alpha}$

Recall from §2.4 that we have an anti-linear involution $\tau : \mathfrak{a} \xrightarrow{\cong} \mathfrak{a}$. We extend it to an anti-linear involution of the untwisted Kac-Moody algebras $\widehat{\mathfrak{a}}$ and $\widehat{\mathfrak{a}}$ defined in §3.1 by letting it act by complex conjugation on the coefficients of the Laurent polynomials in the second tensor factor, i.e. $x t^n \mapsto \bar{x} t^n$ and $x \bar{t}^n \mapsto \bar{x} \bar{t}^n$ for any $x \in \mathbb{C}$ and $n \in \mathbb{Z}_{>0}$, where \bar{x} denotes the complex conjugate of x , and fixing the central extensions $\mathbf{k} \mapsto \mathbf{k}$ and $\bar{\mathbf{k}} \mapsto \bar{\mathbf{k}}$. In turn, this induces an anti-linear involution $\hat{\tau} : \mathbb{F}^{\mathfrak{a},\alpha} \xrightarrow{\cong} \mathbb{F}^{\mathfrak{a},\alpha}$ on the full affine vertex algebra defined by the Leibniz rule, namely it is given on a monomial state $A \in \mathbb{F}^{\mathfrak{a},\alpha}$ as in (3.3) by

$$\hat{\tau}A := (\tau \mathbf{a}^r)_{(-m_r)} \dots (\tau \mathbf{a}^1)_{(-m_1)} \overline{(\tau \mathbf{b}^{\bar{r}})_{(-n_{\bar{r}})}} \dots \overline{(\tau \mathbf{b}^1)_{(-n_1)}} |0\rangle \quad (3.94)$$

and extended by anti-linearity to all of $\mathbb{F}^{\mathfrak{a},\alpha}$.

Recall also the anti-linear isomorphism of prefactorisation algebras $\hat{\tau} : \mathcal{U}\mathcal{L}_\alpha^\Sigma \xrightarrow{\cong} \tau^* \mathcal{U}\mathcal{L}_\alpha^\Sigma$ from Proposition 2.6. It induces an anti-linear isomorphism $\hat{\tau} : (\mathcal{U}\mathcal{L}_\alpha^\Sigma)_p \xrightarrow{\cong} (\tau^* \mathcal{U}\mathcal{L}_\alpha^\Sigma)_p$, i.e. a natural isomorphism of diagrams $\mathbf{Top}(\Sigma)_p \rightarrow \mathbf{Vec}_\mathbb{C}$ of shape the category $\mathbf{Top}(\Sigma)_p$ defined in §3.1, whose components are all anti-linear maps. By the universal property of limits we obtain a unique anti-linear isomorphism

$$\hat{\tau} : \lim(\mathcal{U}\mathcal{L}_\alpha^\Sigma)_p \xrightarrow{\cong} \lim(\tau^* \mathcal{U}\mathcal{L}_\alpha^\Sigma)_p \quad (3.95)$$

such that for every neighbourhood $U \in \mathbf{Top}(\Sigma)_p$ of $p \in \Sigma$ we have the commutative diagram

$$\begin{array}{ccc} \lim(\mathcal{U}\mathcal{L}_\alpha^\Sigma)_p & \xrightarrow{m_{p,U}} & \mathcal{U}\mathcal{L}_\alpha^\Sigma(U) \\ \hat{\tau} \downarrow & & \downarrow \hat{\tau} \\ \lim(\tau^* \mathcal{U}\mathcal{L}_\alpha^\Sigma)_p & \xrightarrow{m_{\tau(p),\tau(U)}} & \mathcal{U}\mathcal{L}_\alpha^\Sigma(\tau(U)) \end{array} \quad (3.96)$$

We denote by $m_{\tau(p),\tau(U)}$ the canonical linear map $\lim(\tau^* \mathcal{U}\mathcal{L}_\alpha^\Sigma)_p \rightarrow \mathcal{U}\mathcal{L}_\alpha^\Sigma(\tau(U))$ in the bottom of this diagram since, for any inclusion of open subsets $U \subset V \subset \Sigma$, the factorisation product $\mathcal{U}\mathcal{L}_\alpha^\Sigma(\tau(U)) \rightarrow \mathcal{U}\mathcal{L}_\alpha^\Sigma(\tau(V))$ of the prefactorisation algebra $\tau^* \mathcal{U}\mathcal{L}_\alpha^\Sigma$ is given by $m_{\tau(U),\tau(V)}$ in terms of the factorisation products of the prefactorisation algebra $\mathcal{U}\mathcal{L}_\alpha^\Sigma$, in other words

$$m_{U,V}^{\tau^* \mathcal{U}\mathcal{L}_\alpha^\Sigma} = m_{\tau(U),\tau(V)}^{\mathcal{U}\mathcal{L}_\alpha^\Sigma}.$$

Since $\tau : \Sigma_\pm \rightarrow \Sigma_\pm$ is an anti-holomorphic map, given any local holomorphic coordinate $\xi : U \rightarrow \mathbb{C}$ on an open subset $U \subset \Sigma$, the coordinate $\hat{\tau}\xi = \overline{\xi \circ \tau} : \tau(U) \rightarrow \mathbb{C}$ is also holomorphic.

Lemma 3.28. *For any $A \in \mathbb{F}^{\mathfrak{a},\alpha}$ and a local coordinate ξ in a neighbourhood of $p \in \Sigma$ we have $\hat{\tau}(A_p^\xi) = (\hat{\tau}A)_{\tau(p)}^{\hat{\tau}\xi}$. In particular, (3.95) induces an anti-linear isomorphism $\hat{\tau} : \mathcal{F}_p^{\mathfrak{a},\alpha} \xrightarrow{\cong} \mathcal{F}_{\tau(p)}^{\mathfrak{a},\alpha}$.*

Proof. By linearity we can assume that $A \in \mathbb{F}^{\mathfrak{a},\alpha}$ is a monomial state and we will only consider the chiral case $A = \mathfrak{a}_{(-m_r)}^r \cdots \mathfrak{a}_{(-m_1)}^1 |0\rangle$ for some $\mathfrak{a}^i \in \mathfrak{a}$ and $m_i \in \mathbb{Z}_{\geq 1}$ with $i \in \{1, \dots, r\}$ and $r \in \mathbb{Z}_{\geq 0}$, since the treatment of the anti-chiral part is completely analogous. For any neighbourhood U of the point p equipped with the local coordinate $\xi : U \rightarrow \mathbb{C}$ we then have

$$\begin{aligned} m_{\tau(p),\tau(U)}(\hat{\tau}(A_p^\xi)) &= \hat{\tau}(A_U^\xi) = \left[\prod_{i=1}^r s\left(\tau(\mathfrak{a}_{\xi_p}^i) \otimes \hat{\tau}\left(\lceil \xi_p^{-m_i} \rceil_{U_i}^{U_{i-1}}\right)\right) \right]_{\tau(U)} \\ &= \left[\prod_{i=1}^r s\left((\tau \mathfrak{a})_{(\hat{\tau}\xi)_{\tau(p)}}^i \otimes \left[(\hat{\tau}\xi)_{\tau(p)}^{-m_i} \right]_{\tau(U_i)}^{\tau(U_{i-1})}\right) \right]_{\tau(U)} = (\hat{\tau}A)_{\tau(U)}^{\hat{\tau}\xi} = m_{\tau(p),\tau(U)}((\hat{\tau}A)_{\tau(p)}^{\hat{\tau}\xi}) \end{aligned}$$

where in the first step we have used the commutativity of the diagram (3.96). The second step is by definition of the anti-linear isomorphism $\hat{\tau} : \mathcal{UL}_\alpha^\Sigma \xrightarrow{\cong} \tau^* \mathcal{UL}_\alpha^\Sigma$ in Proposition 2.6. In the third step we have used the fact that $\hat{\tau} : \Omega_c^{0,\bullet} \xrightarrow{\cong} \tau^* \Omega_c^{0,\bullet}$ is an anti-linear isomorphism of cosheaves of commutative DG algebras, which in particular commutes with $\bar{\partial}$, and

$$\hat{\tau}(\xi_p^{-m_i}) = (\hat{\tau}\xi - (\hat{\tau}\xi)(\tau(p)))^{-m_i} = ((\hat{\tau}\xi)_{\tau(p)})^{-m_i}.$$

We have also used the fact that $\hat{\tau}(\rho_{U_i}^{U_{i-1}}) \in \Omega_c^{0,0}(\tau(U_i))_{\tau(U_{i-1})}^1$. In the second last step we have used the definition of $\hat{\tau} : \mathbb{F}^{\mathfrak{a},\alpha} \xrightarrow{\cong} \mathbb{F}^{\mathfrak{a},\alpha}$ in (3.94). \square

Proposition 3.29. *For any $A, B \in \mathbb{F}^{\mathfrak{a},\alpha}$ and $n, \bar{n} \in \mathbb{Z}$, we have*

$$\hat{\tau}(A_{(n,\bar{n})}B) = (\hat{\tau}A)_{(n,\bar{n})}\hat{\tau}B.$$

In other words, $\hat{\tau} : \mathbb{F}^{\mathfrak{a},\alpha} \xrightarrow{\cong} \mathbb{F}^{\mathfrak{a},\alpha}$ is an anti-linear automorphism of the full vertex algebra $\mathbb{F}^{\mathfrak{a},\alpha}$.

Proof. Let $p, q \in U$ be two points in an open subset $U \subset \Sigma$ equipped with a local coordinate $\xi : U \rightarrow \mathbb{C}$. Using Proposition 3.9 and the definition (3.43) of the vertex modes, we have

$$\begin{aligned} \hat{\tau}(m_{(q,p),U}(A_q^\xi \otimes B_p^\xi)) &= \hat{\tau}\left(\mathcal{Y}(A_q^\xi, \underline{\xi_p(q)})B_p^\xi\right) = \hat{\tau}\left(\sum_{n,\bar{n} \in \mathbb{Z}} A_{(n,\bar{n})}^{\xi_p} B_p^\xi \xi_p(q)^{-n-1} \bar{\xi}_p(q)^{-\bar{n}-1}\right) \\ &= \sum_{n,\bar{n} \in \mathbb{Z}} \hat{\tau}(A_{(n,\bar{n})}^{\xi_p} B_p^\xi) (\hat{\tau}\xi)_{\tau(p)}(\tau(q))^{-n-1} (\hat{\tau}\xi)_{\tau(p)}(\tau(q))^{-\bar{n}-1} \\ &= \sum_{n,\bar{n} \in \mathbb{Z}} \hat{\tau}\left((A_{(n,\bar{n})}B)_p^\xi\right) (\hat{\tau}\xi)_{\tau(p)}(\tau(q))^{-n-1} (\hat{\tau}\xi)_{\tau(p)}(\tau(q))^{-\bar{n}-1} \\ &= \sum_{n,\bar{n} \in \mathbb{Z}} \left(\hat{\tau}(A_{(n,\bar{n})}B)\right)_{\tau(p)}^{\hat{\tau}\xi} (\hat{\tau}\xi)_{\tau(p)}(\tau(q))^{-n-1} (\hat{\tau}\xi)_{\tau(p)}(\tau(q))^{-\bar{n}-1} \end{aligned}$$

where in the third step we used $\overline{\xi_p(q)} = \overline{\xi(q)} - \overline{\xi(p)} = (\hat{\tau}\xi)(\tau(q)) - (\hat{\tau}\xi)(\tau(p)) = (\hat{\tau}\xi)_{\tau(p)}(\tau(q))$, in the fourth step we used Lemma 3.10 and in the last step we used Lemma 3.28. On the other hand, we can also compute this as follows

$$\begin{aligned} \hat{\tau}(m_{(q,p),U}(A_q^\xi \otimes B_p^\xi)) &= m_{(\tau(q),\tau(p)),\tau(U)}\left((\hat{\tau}A)_{\tau(q)}^{\hat{\tau}\xi} \otimes (\hat{\tau}B)_{\tau(p)}^{\hat{\tau}\xi}\right) \\ &= \mathcal{Y}\left((\hat{\tau}A)_{\tau(q)}^{\hat{\tau}\xi}, (\hat{\tau}\xi)_{\tau(p)}(\tau(q))\right)(\hat{\tau}B)_{\tau(p)}^{\hat{\tau}\xi} \\ &= \sum_{n,\bar{n} \in \mathbb{Z}} (\hat{\tau}A)_{(n,\bar{n})}^{(\hat{\tau}\xi)_{\tau(p)}} (\hat{\tau}B)_{\tau(p)}^{\hat{\tau}\xi} (\hat{\tau}\xi)_{\tau(p)}(\tau(q))^{-n-1} (\hat{\tau}\xi)_{\tau(p)}(\tau(q))^{-\bar{n}-1} \\ &= \sum_{n,\bar{n} \in \mathbb{Z}} \left((\hat{\tau}A)_{(n,\bar{n})}\hat{\tau}B\right)_{\tau(p)}^{\hat{\tau}\xi} (\hat{\tau}\xi)_{\tau(p)}(\tau(q))^{-n-1} (\hat{\tau}\xi)_{\tau(p)}(\tau(q))^{-\bar{n}-1} \end{aligned}$$

where in the first step we used Proposition 2.6 and Lemma 3.28, in the second step Proposition 3.9, in the third step the definition (3.43) and in the final step we used Lemma 3.10. \square

3.6.2 Hermitian sesquilinear form on $\mathbb{F}^{\mathfrak{a},\alpha}$

Using the bilinear map (3.91) and the anti-linear involution $\hat{\tau} : \mathbb{F}^{\mathfrak{a},\alpha} \xrightarrow{\cong} \mathbb{F}^{\mathfrak{a},\alpha}$ we introduce the sesquilinear form

$$\langle \cdot, \cdot \rangle := (\hat{\tau}(\cdot), \cdot) : \mathbb{F}^{\mathfrak{a},\alpha} \otimes \mathbb{F}^{\mathfrak{a},\alpha} \longrightarrow \mathbb{C}, \quad \langle B, C \rangle := \left\langle m_{(o',o),\Sigma}((\hat{\tau}B)^{u^{-1}}_{o'} \otimes C^u_o) \right\rangle \quad (3.97)$$

which is anti-linear in the first argument and linear in the second. The vertex algebra $\mathbb{F}^{\mathfrak{a},\alpha}$ equipped with the anti-linear involution $\hat{\tau} : \mathbb{F}^{\mathfrak{a},\alpha} \xrightarrow{\cong} \mathbb{F}^{\mathfrak{a},\alpha}$ is said to be *unitary* if the sesquilinear form (3.97) is positive definite [Mor1, AMT].

Lemma 3.30. *The isomorphism $\langle \cdot \rangle : \mathcal{UL}_\alpha^\Sigma(\Sigma) \xrightarrow{\cong} \mathbb{C}$ from Lemma 3.22, in the Kac-Moody and Virasoro cases, is such that for any $\mathcal{A} \in \mathcal{UL}_\alpha^\Sigma(\Sigma)$ we have $\overline{\langle \mathcal{A} \rangle} = \langle \hat{\tau}\mathcal{A} \rangle$.*

Proof. Recall that the isomorphism from Lemma 3.22 is the inverse of $\mathbb{C} \xrightarrow{\cong} \mathcal{UL}_\alpha^\Sigma(\Sigma)$, $a \mapsto [a]_\Sigma$ which is clearly \mathbb{Z}_2 -equivariant, with the action of $\mathbf{t} \in \mathbb{Z}_2$ given by complex conjugation on \mathbb{C} and by the anti-linear isomorphism $\hat{\tau} : \mathcal{UL}_\alpha^\Sigma(\Sigma) \xrightarrow{\cong} \mathcal{UL}_\alpha^\Sigma(\Sigma)$ from Proposition 2.6. \square

Proposition 3.31. *The sesquilinear form (3.97) is Hermitian, in the Kac-Moody and Virasoro cases.*

Proof. Let $B, C \in \mathbb{F}^{\mathfrak{a},\alpha}$. Using Lemma 3.28 with $p = o$ and $\xi = u$, recalling that $\tau(o) = o'$ and $\hat{\tau}u = u^{-1}$ by (3.93), we can rewrite (3.97) as $\langle B, C \rangle = \langle m_{(o',o),\Sigma}(\hat{\tau}(B^u_o) \otimes C^u_o) \rangle$. Then

$$\overline{\langle B, C \rangle} = \left\langle \hat{\tau} \left(m_{(o',o),\Sigma}(\hat{\tau}(B^u_o) \otimes C^u_o) \right) \right\rangle = \left\langle m_{(o,o'),\Sigma}(B^u_o \otimes \hat{\tau}(C^u_o)) \right\rangle = \langle C, B \rangle$$

where the first step is by definition (3.97) and Lemma 3.30, the second step is by Proposition 2.6 and the last step is again by definition (3.97). \square

In the remainder of this section we focus only on the Kac-Moody case since the statement of Proposition (3.32) relates to a similar formula, in the chiral setting, for a Hermitian form on a Kac-Moody algebra module, see for instance [Kac1, §9.4] or [Kac1, (11.5.1)].

Recall from §3.1 that any state $A \in \mathbb{F}^{\mathfrak{a},\alpha}$ can be written as $A = g|0\rangle$ for a unique element $g \in U(\widehat{\mathfrak{a}}_- \oplus \widehat{\mathfrak{a}}_-)$. The negative of the anti-linear involution $\tau : \mathfrak{g} \xrightarrow{\cong} \mathfrak{g}$ defines an anti-linear anti-involution $-\tau : \mathfrak{g} \xrightarrow{\cong} \mathfrak{g}$. We extend this to an anti-linear anti-involution $(\cdot)^\dagger : \widehat{\mathfrak{a}} \oplus \widehat{\mathfrak{a}} \xrightarrow{\cong} \widehat{\mathfrak{a}} \oplus \widehat{\mathfrak{a}}$ of $\widehat{\mathfrak{a}} \oplus \widehat{\mathfrak{a}}$ defined by (cf. [Kac1, §§2.7, 7.6] and [DL, §4.2] in the chiral case, where ω_0 denotes the compact anti-linear involution)

$$(\mathsf{X}_{(-m)} + \bar{\mathsf{Y}}_{(-n)} + a \mathsf{k} + b \bar{\mathsf{k}})^\dagger := (-\tau \mathsf{X})_{(m)} + (\overline{-\tau \mathsf{Y}})_{(n)} + \bar{a} \mathsf{k} + \bar{b} \bar{\mathsf{k}} \quad (3.98)$$

for any $\mathsf{X}, \mathsf{Y} \in \mathfrak{g}$, $m, n \in \mathbb{Z}$ and $a, b \in \mathbb{C}$. We then extend this to an anti-linear anti-involution of $U(\widehat{\mathfrak{a}}_- \oplus \widehat{\mathfrak{a}}_-)$ by letting $(gg')^\dagger = g'^\dagger g^\dagger$ for any $g, g' \in U(\widehat{\mathfrak{a}}_- \oplus \widehat{\mathfrak{a}}_-)$, cf. [Kac1, §11.5].

Proposition 3.32. *For any $g, g' \in U(\widehat{\mathfrak{a}} \oplus \widehat{\mathfrak{a}})$ we have $\langle g|0\rangle, g'|0\rangle \rangle = \langle m_{o,\Sigma}(g^\dagger g'|0\rangle^u_o) \rangle$.*

Proof. By anti-linearity in $g \in U(\widehat{\mathfrak{a}} \oplus \widehat{\mathfrak{a}})$, it is enough to consider monomial g . Let $A = g|0\rangle \in \mathbb{F}^{\mathfrak{a},\alpha}$ be the monomial state in (3.3) so that $g = \mathsf{X}_{(-m_r)}^r \dots \mathsf{X}_{(-m_1)}^1 \bar{\mathsf{Y}}_{(-n_r)}^r \dots \bar{\mathsf{Y}}_{(-n_1)}^1 \in U(\widehat{\mathfrak{a}} \oplus \widehat{\mathfrak{a}})$, and let $C = g'|0\rangle \in \mathbb{F}^{\mathfrak{a},\alpha}$ be arbitrary.

Now by definition of (3.97) we have $\langle g|0\rangle, g'|0\rangle \rangle = (\hat{\tau}A, C)$. To compute the right hand side, note that for any open subset $o' \in V' \subset \Sigma$ we have

$$(\hat{\tau}A)^{u^{-1}}_{V'} = \left[\prod_{i=1}^r s\left(\tau \mathsf{X}^i \otimes [u^{m_i}]_{U_i}^{U_{i-1}}\right) \prod_{j=1}^{\bar{r}} s\left(\tau \bar{\mathsf{Y}}^j \otimes [\bar{u}^{n_j}]_{V_j}^{V_{j-1}}\right) \right]_{V'}, \quad (3.99)$$

where recall that $o'_+ \in U_0 \Subset \dots \Subset U_r \subset V'_+ \subset \Sigma_+$ and $o'_- \in V_0 \Subset \dots \Subset V_{\bar{r}} \subset V'_- \subset \Sigma_-$ is a choice of nested sequence of open subsets. In simplifying the above expression we have used the fact that $u^{-1}(o') = 0$ so that the shifted coordinate $u_{o'}^{-1}$ is the same as the unshifted one, namely $u_{o'}^{-1} = u^{-1}$. Note that each factor in the above cohomology now has a positive power of u since $(u^{-1})^{-m_i} = u^{m_i}$ and similarly for the anti-chiral part.

For $i \in \{0, \dots, r\}$ we define the open subset $U'_i := (\overline{U}_i)^c$ as the complement in Σ of the closure of U_i and similarly for each $j \in \{0, \dots, \bar{r}\}$ we define $V'_j := (\overline{V}_j)^c$. These are open subsets of some open neighbourhoods $W_{\pm} \subset \Sigma_{\pm} \setminus \{o'_{\pm}\}$ of $o_{\pm} \in \Sigma_{\pm}$. In fact, they form nested sequences of open subsets $o_+ \in U'_r \Subset \dots \Subset U'_0 \subset W_+ \subset \Sigma_+$ and $o_- \in V'_{\bar{r}} \Subset \dots \Subset V'_0 \subset W_- \subset \Sigma_-$. We can then introduce smooth bump functions

$$\rho_{U'_{i-1}}^{U'_i} := 1 - \rho_{U_i}^{U_{i-1}} \in \Omega_c^{0,0}(U'_{i-1})_{U'_i}^1, \quad \rho_{V'_{j-1}}^{V'_j} := 1 - \rho_{V_j}^{V_{j-1}} \in \Omega_c^{0,0}(V'_{j-1})_{V'_j}^1$$

for each $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, \bar{r}\}$. We may now rewrite (3.99) as

$$(\hat{\tau}A)^{u^{-1}}_{V'} = \left[\prod_{i=1}^r s\left(-\tau \mathsf{X}^i \otimes [u^{m_i}]_{U'_{i-1}}^{U'_i} \right) \prod_{j=1}^{\bar{r}} s\left(-\tau \mathsf{Y}^j \otimes [\bar{u}^{n_j}]_{V'_{j-1}}^{V'_j} \right) \right]_{V'} \quad (3.100)$$

where we note, crucially, that an extra minus sign appeared in each term through the replacement of smooth bump functions in (3.11) since

$$\bar{\partial} \rho_{U_i}^{U_{i-1}} = -\bar{\partial} \rho_{U'_{i-1}}^{U'_i}, \quad \bar{\partial} \rho_{V_j}^{V_{j-1}} = -\bar{\partial} \rho_{V'_{j-1}}^{V'_j}.$$

Recalling the definition of the vertex modes in (3.29), the cohomology class (3.100) represents the product of vertex modes $(-\tau \mathsf{X}^i)_{(m_i)}^u$ and $(-\tau \mathsf{Y}^j)_{(n_j)}^u$ but in the reverse order compared to (3.99) due to the reversal of the nested sequences of open subsets U'_i and V'_j compared to U_i and V_j . By associativity (2.17) of the factorisation product and definition (3.98) we now have

$$\begin{aligned} m_{(o', o), \Sigma}((\hat{\tau}A)^{u^{-1}}_{o'} \otimes C_o^u) &= m_{(V', V), \Sigma}((\hat{\tau}A)^{u^{-1}}_{V'} \otimes C_V^u) \\ &= m_{(Y, V), \Sigma}((g^\dagger)_Y^u \otimes C_V^u) = (g^\dagger C)_\Sigma^u \end{aligned}$$

where in the second step $Y \subset W \setminus \overline{V}$ is an annulus shaped open subset which encircles the open subset $V \ni o$. The result follows by applying the isomorphism $\langle \cdot \rangle$ from Lemma 3.22. \square

4 Operator formalism for $\mathbb{F}^{\mathbf{a}, \alpha}$

In this section we use the prefactorisation algebra $\mathcal{U}\mathcal{L}_\alpha^\Sigma \in \text{PFac}(\Sigma, \text{Vec}_{\mathbb{C}}^\otimes)$ from §2.3 to describe the operator formalism [DMS, §6] for $\mathbb{F}^{\mathbf{a}, \alpha}$. That is, we describe quantum operators associated with states in $\mathbb{F}^{\mathbf{a}, \alpha}$ on the cylinder $\Sigma' := \mathbb{R} \times S^1 \cong \mathbb{C}/2\pi\mathbb{Z}$, where \mathbb{R} corresponds to the time direction and $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ to the compactified space direction. It will be convenient to extend Σ' to a 2-sphere $\Sigma = S^2$ by adding the points at ‘infinity’ on both ends of the cylinder. Recall that in (3.89) we had specialised to the real 2-dimensional manifold $\Sigma = S^2$ and we will thus keep doing so throughout this section. We make the identification $\Sigma' = \Sigma \setminus \{o, o'\}$, where o and o' are the points added to the ‘bottom’ and ‘top’ of the infinite cylinder, respectively.

4.1 Fourier modes and quantum operators

4.1.1 From the cylinder to the plane

On a simply connected open subset $U \subset \Sigma'$ we define a local holomorphic coordinate $\vartheta : U \rightarrow \mathbb{C}$ by choosing a representative of the coset $\mathbb{C}/2\pi\mathbb{Z}$ continuously over U . For later convenience,

we choose the orientation on Σ'_+ such that $\vartheta \rightarrow i\infty$ as we approach the bottom of the cylinder, i.e. the point $o \in \Sigma$. This coordinate satisfies

$$\bar{\vartheta} = \vartheta \circ \tau, \quad (4.1)$$

where $\tau : \Sigma' \rightarrow \Sigma'$ denotes the orientation reversing involution given by complex conjugation on $\Sigma' = \mathbb{C}/2\pi\mathbb{Z}$, extended to an orientation reversing involution $\tau : \Sigma \rightarrow \Sigma$ as $\tau(o) = o'$. We also define $\tau : \widehat{\Sigma} \rightarrow \widehat{\Sigma}$ by letting it act as $\tau : \Sigma \rightarrow \Sigma$ on each copy Σ_\pm of Σ . This orientation reversing involution corresponds to the Lorentzian involution $\tau_L : \widehat{\Sigma} \rightarrow \widehat{\Sigma}$ from §1.

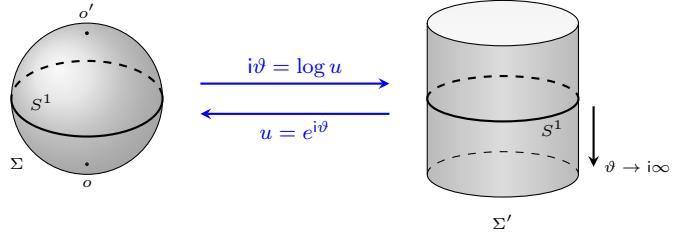
It is crucial to note that while the coordinate $\vartheta : U \rightarrow \mathbb{C}$ on any simply connected open subset $U \subset \Sigma$ is defined only up to an additive multiple of 2π , the shifted local coordinate $\vartheta_p := \vartheta - \vartheta(p) : U \rightarrow \mathbb{C}$, introduced in (3.8), is unambiguously defined. This will mean that we can use the geometric realisation linear maps (3.12) or (3.25) to prepare states in $\mathbb{F}^{\alpha,\alpha}$ at any $p \in \Sigma'$ using the local coordinate ϑ .

However, on any open subset $W \subset \Sigma'$ wrapping around the cylinder, a continuous representative $\vartheta : W \rightarrow \mathbb{C}$ of the coset $\mathbb{C}/2\pi\mathbb{Z}$ over W is necessarily multi-valued, and hence does not provide a well defined coordinate on W . Moreover, ϑ is not defined at $o, o' \in \Sigma$. We thus introduce two other holomorphic coordinates

$$u : \Sigma \setminus \{o'\} \rightarrow \mathbb{C}, \quad u^{-1} : \Sigma \setminus \{o\} \rightarrow \mathbb{C} \quad (4.2)$$

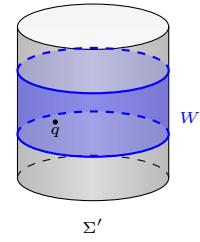
related to the above local coordinate $\vartheta : U \rightarrow \mathbb{C}$ on a simply connected open subset $U \subset \Sigma'$ by the holomorphic changes of coordinate $u = e^{i\vartheta}$ and $u^{-1} = e^{-i\vartheta}$. Note, in particular, that the multi-valuedness of the local coordinate ϑ drops out from the local coordinates (4.2). These coordinates satisfy all the properties of the charts with the same name from §3.4 and §3.6. For instance, $u(o) = 0$ and $u^{-1}(o') = 0$, and the property (3.93) is a consequence of (4.1).

We summarise the above setup in the following picture



Proposition 4.1. *Let $A \in \mathbb{F}^{\alpha,\alpha}$, $q \in \Sigma'$ and let $W \ni q$ be an annulus shaped open subset around $o \in \Sigma$ in the coordinate u . The expansion of $m_{q,W}(A_q^{i\vartheta})$ in the coordinate $u(q)$, namely*

$$A(\vartheta(q)) := \mathcal{Y}(A_q^{i\vartheta}, u(q)) = \sum_{n, \bar{n} \in \mathbb{Z}} A_{n, \bar{n}} e^{-in\vartheta(q)} e^{i\bar{n}\bar{\vartheta}(q)},$$



is a homogeneous vertex operator at $o \in \Sigma$ in the coordinate u . That is, its Fourier coefficients $A_{n, \bar{n}}$ are the homogeneous vertex modes $\tilde{A}_{[n, \bar{n}]}^u$ of some state $\tilde{A} \in \mathbb{F}^{\alpha,\alpha}$ at $o \in \Sigma$. We will refer to this as the quantum operator on Σ' associated with the state $A \in \mathbb{F}^{\alpha,\alpha}$.

Proof. Let $A \in \mathbb{F}^{\alpha,\alpha}$ be any state of definite (anti-)chiral conformal dimensions Δ_A and $\bar{\Delta}_A$. Using Theorem 3.19 to change from the local coordinate u to $i\vartheta = \log u$ we have

$$A_q^{i\vartheta} = \mathcal{R}_q^{u \rightarrow i\vartheta} A_q^u = (R^{u \rightarrow i\vartheta} A)_q^u = u(p)^{\Delta_A} \bar{u}(p)^{\bar{\Delta}_A} \left(\exp \left(- \sum_{k \geq 1} (b_k L_{(k)} + \bar{b}_k \bar{L}_{(k)}) \right) A \right)_q^u \quad (4.3)$$

where in the last step we used the explicit expression for the operator $R^{u \rightarrow i\vartheta}$ from (3.85) and the facts that $b_0 = u(p)^{-1}$, $L_{(0)}A = \Delta_A A$ and $\bar{L}_{(0)}A = \bar{\Delta}_A A$.

Now notice that $L_{(k)}$ and $\bar{L}_{(k)}$ lower the (anti-)chiral conformal dimension of a state by k since $[L_{(0)}, L_{(k)}] = -kL_{(k)}$ by (3.68) and similarly for $\bar{L}_{(k)}$. Also observe that b_k is proportional to $u(p)^{-k}$ for all $k \geq 1$ since

$$\begin{aligned} u(p)^{-1} \exp \left(\sum_{k \geq 1} b_k u_p^{k+1} \partial_{u_p} \right) u_p &= i\vartheta_p = \log(u_p + u(p)) - \log(u(p)) \\ &= u(p)^{-1} \left(u_p + \sum_{k \geq 1} \frac{(-1)^k u_p^{k+1}}{(k+1) u(p)^k} \right), \end{aligned}$$

where the first equality is by definition of the b_k for $k \geq 1$ in (3.75b) and using the fact that $b_0 = u(p)^{-1}$. We then immediately deduce from (4.3) that we can write $A_q^{i\vartheta}$ as a finite sum

$$A_q^{i\vartheta} = \sum_i u(p)^{\Delta_{B_i}} \bar{u}(p)^{\bar{\Delta}_{B_i}} (B_i)_q^u$$

for some states $B_i \in \mathbb{F}^{\mathfrak{a}, \alpha}$ of definite (anti-)chiral conformal dimensions Δ_{B_i} and $\bar{\Delta}_{B_i}$. Then

$$\mathcal{Y}(A_q^{i\vartheta}, \underline{u}(q)) = \sum_i u(p)^{\Delta_{B_i}} \bar{u}(p)^{\bar{\Delta}_{B_i}} \mathcal{Y}((B_i)_q^u, \underline{u}(q)) = \mathcal{X}(\tilde{A}_q^u, \underline{u}(q))$$

with $\tilde{A} := \sum_i B_i$, where in the last equality we used the definition of the homogeneous vertex operator in (3.48). The result now follows. \square

4.1.2 (Anti-)chiral states

Recall from §3.2.6 that the chiral vertex operators (3.32) and its anti-chiral analogue for any $\mathfrak{a} \in \mathfrak{a}$ correspond to the states $\mathfrak{a}_{(-1)}|0\rangle_q^\xi$ and $\bar{\mathfrak{a}}_{(-1)}|0\rangle_q^\xi$, respectively, see (3.49). Similarly, we introduce the quantum operators associated to any $\mathfrak{a} \in \mathfrak{a}$ as

$$\begin{aligned} \mathfrak{a}(\vartheta(q)) &= \mathcal{Y}(\mathfrak{a}_{(-1)}|0\rangle_q^{i\vartheta}, u(q)) = \sum_{n \in \mathbb{Z}} \mathfrak{a}_n e^{-in\vartheta(q)}, \\ \bar{\mathfrak{a}}(\bar{\vartheta}(q)) &= \mathcal{Y}(\bar{\mathfrak{a}}_{(-1)}|0\rangle_q^{i\vartheta}, \bar{u}(q)) = \sum_{n \in \mathbb{Z}} \bar{\mathfrak{a}}_n e^{in\bar{\vartheta}(q)}. \end{aligned}$$

In other words, comparing with the general mode expansion from (4.1), the Fourier modes \mathfrak{a}_n and $\bar{\mathfrak{a}}_n$ are defined as, cf. (3.50),

$$\mathfrak{a}_n := (\mathfrak{a}_{(-1)}|0\rangle)_{n,0}, \quad \bar{\mathfrak{a}}_n := (\bar{\mathfrak{a}}_{(-1)}|0\rangle)_{0,n}.$$

We can describe these Fourier modes more explicitly in each of the three examples from §2.1.2.

Kac-Moody: We note using (3.70) that for any $\mathsf{X} \in \mathfrak{g}$ we have

$$L_{(n)} \mathsf{X}_{(-1)}|0\rangle = \mathsf{X}_{(-1)}|0\rangle \delta_{n,0}, \quad \bar{L}_{(n)} \bar{\mathsf{X}}_{(-1)}|0\rangle = \bar{\mathsf{X}}_{(-1)}|0\rangle \delta_{n,0} \quad (4.4)$$

for all $n \geq 0$. Using this we may apply the formula (4.3) for the change coordinates $u \mapsto i\vartheta$ to the states $\mathsf{X}_{(-1)}|0\rangle$ and $\bar{\mathsf{X}}_{(-1)}|0\rangle$ to obtain

$$\mathsf{X}_{(-1)}|0\rangle_p^{i\vartheta} = u(p) \mathsf{X}_{(-1)}|0\rangle_p^u, \quad \bar{\mathsf{X}}_{(-1)}|0\rangle_p^{i\vartheta} = \bar{u}(p) \bar{\mathsf{X}}_{(-1)}|0\rangle_p^u.$$

It then follows from (3.46) that the quantum operators from Proposition 4.1, associated to the (anti-)chiral states $X_{(-1)}|0\rangle$ and $\bar{X}_{(-1)}|0\rangle$, are respectively given by

$$X(\vartheta(p)) = \sum_{n \in \mathbb{Z}} X_n e^{-in\vartheta(p)}, \quad \bar{X}(\bar{\vartheta}(p)) = \sum_{n \in \mathbb{Z}} \bar{X}_n e^{in\bar{\vartheta}(p)}, \quad (4.5)$$

where the Fourier modes are related to the homogeneous vertex modes in the coordinate u as

$$X_n = X_{[n]}^u, \quad \bar{X}_n = \bar{X}_{[n]}^u$$

for every $n \in \mathbb{Z}$.

Virasoro: Using the formula (4.3) for changing coordinates $u \mapsto i\vartheta$, we find

$$\Omega_p^{i\vartheta} = u(p)^2 \left(\exp \left(- \sum_{k \geq 1} (b_k L_{(k)} + \bar{b}_k \bar{L}_{(k)}) \right) \Omega \right)_p^u = u(p)^2 \Omega_p^u - \frac{c}{24} |0\rangle_p^u, \quad (4.6)$$

and similarly for the anti-chiral state $\bar{\Omega}$. To see the last step note that by the defining property (3.63a) of a conformal state Ω we have $L_{(k)}\Omega = \frac{c}{2}|0\rangle\delta_{k,2}$ for all $k \geq 1$ and also $b_2 = \frac{1}{12}u(p)^{-2}$. Expanding both sides of (4.6) in $u(p)$ we obtain

$$\sum_{n \in \mathbb{Z}} L_n e^{-in\vartheta(p)} = \sum_{n \in \mathbb{Z}} \Omega_{[n]}^u e^{-in\vartheta(p)} - \frac{c}{24}$$

where on the left hand side we used the definition of the Fourier coefficients in Proposition 4.1, which are conventionally denoted as $L_n := \Omega_n$ and $\bar{L}_n := \bar{\Omega}_n$, and on the right hand side, for the first term we used the definition of the homogeneous vertex n^{th} -modes, see (3.46), in the coordinate u on the plane and for the second term the fact that $\mathcal{Y}(|0\rangle_p^u, u(p))$ is the identity operator. The above is the well known relation between the stress energy tensor on the cylinder and on the plane, see e.g. [DMS, (5.138)]. In particular, we have

$$L_n = \Omega_{[n]}^u - \frac{c}{24} \delta_{n,0}, \quad \bar{L}_n = \bar{\Omega}_{[n]}^u - \frac{c}{24} \delta_{n,0}$$

for every $n \in \mathbb{Z}$.

$\beta\gamma$ system: Using (3.74) we find

$$\begin{aligned} L_{(n)} \beta_{(-1)}|0\rangle &= \beta_{(-1)}|0\rangle \delta_{n,0}, & \bar{L}_{(n)} \bar{\beta}_{(-1)}|0\rangle &= \bar{\beta}_{(-1)}|0\rangle \delta_{n,0}, \\ L_{(n)} \gamma_{(-1)}|0\rangle &= 0, & \bar{L}_{(n)} \bar{\gamma}_{(-1)}|0\rangle &= 0 \end{aligned}$$

and therefore apply the formula (4.3) for the change coordinates $u \mapsto i\vartheta$ to the states $\beta_{(-1)}|0\rangle$, $\bar{\beta}_{(-1)}|0\rangle$, $\gamma_{(-1)}|0\rangle$ and $\bar{\gamma}_{(-1)}|0\rangle$ we find

$$\begin{aligned} \beta_{(-1)}|0\rangle_p^{i\vartheta} &= u(p) \beta_{(-1)}|0\rangle_p^u, & \bar{\beta}_{(-1)}|0\rangle_p^{i\vartheta} &= \bar{u}(p) \bar{\beta}_{(-1)}|0\rangle_p^u \\ \gamma_{(-1)}|0\rangle_p^{i\vartheta} &= \gamma_{(-1)}|0\rangle_p^u, & \bar{\gamma}_{(-1)}|0\rangle_p^{i\vartheta} &= \bar{\gamma}_{(-1)}|0\rangle_p^u. \end{aligned}$$

It then follows from (3.46) that the quantum operators from Proposition 4.1 associated to these (anti-)chiral states are given by

$$\beta(\vartheta(p)) = \sum_{n \in \mathbb{Z}} \beta_n e^{-in\vartheta(p)}, \quad \bar{\beta}(\bar{\vartheta}(p)) = \sum_{n \in \mathbb{Z}} \bar{\beta}_n e^{in\bar{\vartheta}(p)}, \quad (4.7a)$$

$$\gamma(\vartheta(p)) = \sum_{n \in \mathbb{Z}} \gamma_n e^{-in\vartheta(p)}, \quad \bar{\gamma}(\bar{\vartheta}(p)) = \sum_{n \in \mathbb{Z}} \bar{\gamma}_n e^{in\bar{\vartheta}(p)} \quad (4.7b)$$

where the Fourier modes are related to the homogeneous vertex modes in the coordinate u as

$$\beta_n = \beta_{[n]}^u, \quad \bar{\beta}_{\bar{n}} = \bar{\beta}_{[\bar{n}]}^u, \quad \gamma_n = \gamma_{[n]}^u, \quad \bar{\gamma}_{\bar{n}} = \bar{\gamma}_{[\bar{n}]}^u$$

for every $n \in \mathbb{Z}$.

More generally, if $A \in \mathbb{F}^{\alpha,\alpha}$ is a chiral monomial state then its associated quantum operator from Proposition 4.1 is holomorphic in this case so that

$$A(\vartheta(q)) := \mathcal{Y}(A_q^{i\vartheta}, u(q)) = \sum_{n \in \mathbb{Z}} A_n e^{-in\vartheta(q)},$$

i.e. A has Fourier modes $A_{n,\bar{n}} = A_n \delta_{\bar{n},0}$. Moreover, since $m_{q,W}(A_q^{i\vartheta})$ depends holomorphically on $\vartheta(q)$ by Proposition 3.7, we can extract the Fourier modes using a contour integral

$$A_n = \frac{1}{2\pi} \int_{\gamma} m_{q,W}(A_q^{i\vartheta}) e^{in\vartheta(q)} d\vartheta(q) \quad (4.8)$$

along any contour γ wrapping once around the cylinder Σ' . Likewise, if $A \in \mathbb{F}^{\alpha,\alpha}$ is anti-chiral then the quantum operator from Proposition 4.1 is anti-holomorphic and we have

$$A(\bar{\vartheta}(q)) := \mathcal{Y}(A_q^{i\bar{\vartheta}}, \bar{u}(q)) = \sum_{n \in \mathbb{Z}} A_n e^{in\bar{\vartheta}(q)},$$

i.e. A has Fourier modes $A_{n,\bar{n}} = A_{\bar{n}} \delta_{n,0}$. Since in this case $m_{q,W}(A_q^{i\bar{\vartheta}})$ depends holomorphically on $\bar{\vartheta}(q)$ by Proposition 3.7, we can extract these Fourier modes using a similar contour integral

$$A_n = \frac{1}{2\pi} \int_{\gamma} m_{q,W}(A_q^{i\bar{\vartheta}}) e^{-in\bar{\vartheta}(q)} d\bar{\vartheta}(q).$$

As in the case of vertex modes discussed in §3.2, we use the same notation for Fourier modes of chiral and anti-chiral states. This should not lead to confusion for the same reason as before, namely the type of Fourier mode in question is determined by the chirality of the state.

The vacuum state $|0\rangle \in \mathbb{F}^{\alpha,\alpha}$ is both chiral and anti-chiral, with Fourier modes given by $|0\rangle_{n,\bar{n}} = \delta_{n,0} \delta_{\bar{n},0}$ for $n, \bar{n} \in \mathbb{Z}$. Here we are omitting the factor of $[1]_W \in \mathcal{UL}_\alpha^\Sigma(W)$ for brevity since it corresponds to the identity operator.

4.1.3 Translation operators

Recall the endomorphisms $D, \bar{D} : \mathbb{F}^{\alpha,\alpha} \rightarrow \mathbb{F}^{\alpha,\alpha}$ in (3.28) and Proposition 3.7. The following is the analogue of Lemma 3.12 for Fourier modes.

Lemma 4.2. *For any $A \in \mathbb{F}^{\alpha,\alpha}$ and $n, \bar{n} \in \mathbb{Z}$ we have*

$$(DA)_{n,\bar{n}} = -n A_{n,\bar{n}}, \quad (\bar{D}A)_{n,\bar{n}} = -\bar{n} A_{n,\bar{n}}.$$

Proof. This is very similar to the proof of Lemma 3.12. We have

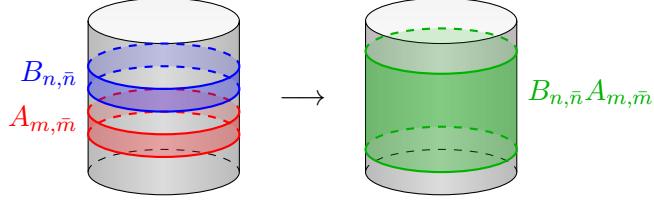
$$\begin{aligned} \mathcal{Y}((DA)_q^{i\vartheta}, \underline{u}(q)) &= m_{q,W}((DA)_q^{i\vartheta}) = m_{q,W}(-i\partial_{\vartheta(q)} A_q^{i\vartheta}) = -i\partial_{\vartheta(q)} \mathcal{Y}(A_q^{i\vartheta}, \underline{u}(q)), \\ \mathcal{Y}((\bar{D}A)_q^{i\bar{\vartheta}}, \underline{u}(q)) &= m_{q,W}((\bar{D}A)_q^{i\bar{\vartheta}}) = m_{q,W}(i\partial_{\bar{\vartheta}(q)} A_q^{i\bar{\vartheta}}) = i\partial_{\bar{\vartheta}(q)} \mathcal{Y}(A_q^{i\bar{\vartheta}}, \underline{u}(q)) \end{aligned}$$

where the first and last steps in both cases use Proposition 3.9 and the second steps are by Proposition 3.7. The result follows from the definition of Fourier modes in Proposition 4.1. \square

4.2 Borcherds type identities

In Proposition 3.15 we showed that the vertex modes of states in $\mathbb{F}^{\alpha,\alpha}$ satisfy the Borcherds type identities when one of the states involved is (anti-)chiral. In Theorem 4.3 below we derive similar identities for Fourier modes.

Recall from §3.1 that the relative ordering of the (anti-)chiral vertex modes in a state in $\mathbb{F}^{\alpha,\alpha}$ prepared at a point $p \in \Sigma^\circ$ in a local coordinate ξ is encoded in terms of the prefactorisation algebra $\mathcal{UL}_\alpha^\Sigma$ by associating each factor with disjoint concentric annuli shaped open subsets around p . In the case of Fourier modes, the annuli shaped open subsets around $o \in \Sigma$ in the local coordinate u correspond to open strips around the cylinder Σ' , as depicted below. In particular, the centre of these annuli is $o \in \Sigma$ so the product of Fourier modes is from left to right if their corresponding open strips are ordered from top to bottom along Σ' . In pictures,



The Nørlund polynomials $\mathfrak{B}_n^{(a)}(x)$, for any $n \in \mathbb{Z}_{\geq 0}$ and $a \in \mathbb{Z}$, are defined by

$$\frac{e^{xt}}{(e^t - 1)^a} = \sum_{n \geq 0} \frac{\mathfrak{B}_n^{(a)}(x)}{n!} t^{n-a}. \quad (4.9)$$

They generalise the Bernoulli polynomials $\mathfrak{B}_n(x)$ for $n \in \mathbb{Z}_{\geq 0}$ which correspond to the special case $a = 1$, i.e. $\mathfrak{B}_n(x) = \mathfrak{B}_n^{(1)}(x)$. The values $\mathfrak{B}_n(1)$ are the Bernoulli numbers. An alternative definition of these numbers that we shall also use is $\mathfrak{B}_n(0)$, which differ from the first definition only in its first term: $\mathfrak{B}_1(0) = -\frac{1}{2}$ and $\mathfrak{B}_1(1) = \frac{1}{2}$ while $\mathfrak{B}_n(0) = \mathfrak{B}_n(1)$ for all $n \geq 2$.

In Theorem 4.3 and Proposition 4.4 below we use the convention that $0^0 = 1$.

Theorem 4.3. *Let $A, B \in \mathbb{F}^{\alpha,\alpha}$ and $k, m, n, \bar{n} \in \mathbb{Z}$. We have the following identities:*

(i) *If A is chiral then*

$$\begin{aligned} \sum_{j \geq 0} (-1)^j \binom{k}{j} A_{m-j} B_{n+j, \bar{n}} - \sum_{j \geq 0} (-1)^{k+j} \binom{k}{j} B_{n+k-j, \bar{n}} A_{m-k+j} \\ = \sum_{j, r \geq 0} \frac{\mathfrak{B}_j^{(-k)}(-k)m^r}{j!r!} (A_{(j+k+r)} B)_{m+n, \bar{n}}. \end{aligned}$$

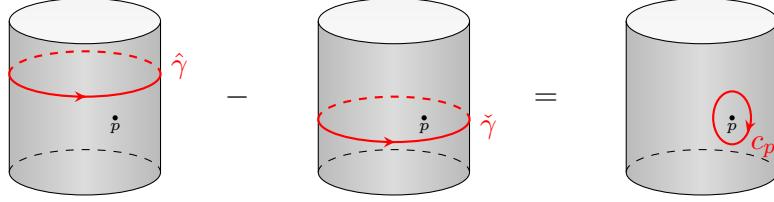
(ii) *If A is anti-chiral then*

$$\begin{aligned} \sum_{j \geq 0} (-1)^j \binom{k}{j} A_{m-j} B_{n, \bar{n}+j} - \sum_{j \geq 0} (-1)^{k+j} \binom{k}{j} B_{n, \bar{n}+k-j} A_{m-k+j} \\ = \sum_{j, r \geq 0} \frac{\mathfrak{B}_j^{(-k)}(-k)m^r}{j!r!} (A_{(j+k+r)} B)_{n, m+\bar{n}}. \end{aligned}$$

Proof. Suppose first that $\bar{D}A = 0$. Let $W \subset \Sigma'$ be an open subset in $\text{Top}(\Sigma)$ wrapping around Σ' . Recall the map in (3.56) and define $F_{A,B} : \text{Conf}_2(W) \rightarrow \mathcal{UL}_\alpha^\Sigma(W)$ by

$$\begin{aligned} F_{A,B}(q, p) &:= \Phi_W^{2, \text{i}\vartheta}((q, p), (A, B)) u_p(q)^k u(p)^n u(q)^{m-k} \bar{u}(p)^{\bar{n}} \\ &= m_{(q,p), W}(A_q^{\text{i}\vartheta} \otimes B_p^{\text{i}\vartheta}) u_p(q)^k u(p)^n u(q)^{m-k} \bar{u}(p)^{\bar{n}}. \end{aligned}$$

This depends holomorphically on $q \in W$ by Proposition 3.7. By the standard deformation of contour argument



we then have the identity

$$\frac{1}{2\pi} \left(\int_{\hat{\gamma}} - \int_{\check{\gamma}} \right) F_{A,B}(q, p) d\vartheta(q) = \frac{1}{2\pi i} \int_{c_p} F_{A,B}(q, p) d(i\vartheta(q)), \quad (4.10)$$

where $\hat{\gamma}$ and $\check{\gamma}$ are contours wrapping the cylinder which pass, respectively, above and below the point p and c_p is a small *clockwise* oriented contour around p .

On the right hand side of (4.10), since $q \in c_p$ is close to p we have the expansion

$$F_{A,B}(q, p) = \sum_{j,r \geq 0} \frac{\mathfrak{B}_j^{(-k)}(-k)m^r}{j!r!} \mathcal{Y} \left(\mathcal{Y}(A_q^{i\vartheta}, i\vartheta_p(q)) B_p^{i\vartheta}, \underline{u(p)} \right) (i\vartheta_p(q))^{j+r+k} u(p)^{m+n} \bar{u}(p)^{\bar{n}},$$

where we used the definition (4.9) of the Nørlund polynomials. We can perform the integral on the right hand side of (4.10) using (3.51a). For this, note that the orientation of the integration variable $i\vartheta(q)$, see the choice of orientation on Σ'_+ we made at the start of §4.1, matches the clockwise orientation of the contour c_p . By definition of Fourier modes in Proposition 4.1, the right hand side of the desired identity then corresponds to the coefficient of $u(p)^0 \bar{u}(p)^0$.

In the first integral on the left hand side of (4.10), by the relative positioning of $q \in \hat{\gamma}$ and the point p as depicted in the above picture, we have $|u(p)| < |u(q)|$ and hence the expansion

$$F_{A,B}(q, p) = \sum_{j \geq 0} (-1)^j \binom{k}{j} \mathcal{Y}(A_q^{i\vartheta}, u(q)) \mathcal{Y}(B_p^{i\vartheta}, \underline{u(p)}) u(q)^{m-j} u(p)^{n+j} \bar{u}(p)^{\bar{n}}.$$

Performing the integral over $\hat{\gamma}$ using the property (4.8) and using the definition of the Fourier modes in Proposition 4.1, the first term on the left hand side of the desired identity is given again by the coefficient of $u(p)^0 \bar{u}(p)^0$.

Likewise, in the second integral on the left hand side of (4.10) we have $q \in \check{\gamma}$ which lies below p so that $|u(p)| > |u(q)|$ and hence we have the expansion

$$F_{A,B}(q, p) = \sum_{j \geq 0} (-1)^{k+j} \binom{k}{j} \mathcal{Y}(B_p^{i\vartheta}, \underline{u(p)}) \mathcal{Y}(A_q^{i\vartheta}, u(q)) u(q)^{m-k+j} u(p)^{n+k-j} \bar{u}(p)^{\bar{n}}.$$

Upon integrating over $\check{\gamma}$ and extracting the coefficient of $u(p)^0 \bar{u}(p)^0$ this then gives the second term on the left hand side of the desired identity, using again (4.8) and Proposition 4.1.

The proof in the anti-chiral case $DA = 0$ is very similar. \square

As with vertex modes, cf. Corollary 3.17, the ‘Borcherds type’ identities for Fourier modes lead to commutator and normal ordering formulae for Fourier modes of composite states.

Proposition 4.4. *Let $A, B \in \mathbb{F}^{a,\alpha}$ and $m, n, \bar{n} \in \mathbb{Z}$. We have the following identities:*

(i) *If A is chiral then*

$$[A_m, B_{n,\bar{n}}] = \sum_{r \geq 0} \frac{m^r}{r!} (A_{(r)} B)_{m+n,\bar{n}},$$

$$(A_{(-1)} B)_{n,\bar{n}} = \sum_{j \leq 0} A_j B_{n-j,\bar{n}} + \sum_{j > 0} B_{n-j,\bar{n}} A_j + \sum_{r \geq 0} \frac{\zeta(-r)}{r!} (A_{(r)} B)_{n,\bar{n}}.$$

(ii) If A is anti-chiral then

$$[A_m, B_{n,\bar{n}}] = \sum_{r \geq 0} \frac{m^r}{r!} (A_{(r)} B)_{n,m+\bar{n}},$$

$$(A_{(-1)} B)_{n,\bar{n}} = \sum_{j \leq 0} A_j B_{n,\bar{n}-j} + \sum_{j > 0} B_{n,\bar{n}-j} A_j + \sum_{r \geq 0} \frac{\zeta(-r)}{r!} (A_{(r)} B)_{n,\bar{n}}.$$

Proof. Taking $k = 0$ in the identities from Theorem 4.3 (i) and (ii) we obtain the first desired relations in (i) and (ii), respectively. Taking instead $m = 0$ and $k = -1$ in the identities from Theorem 4.3(i) and (ii) gives the second desired relations in (i) and (ii) upon noting the fact that $\zeta(-r) = -\frac{\mathfrak{B}_{r+1}(1)}{r+1}$ for every $r \in \mathbb{Z}_{\geq 0}$. \square

Remark 4.5. The combination of the commutator and normal ordering formulae for Fourier modes in $\mathbb{F}^{\alpha,\alpha}$ from Proposition 4.4 have the following interesting heuristic interpretation.

Replacing n in the first relation of Proposition 4.4(i) by $n - m$, we can rewrite it as

$$A_m B_{n-m,\bar{n}} = B_{n-m,\bar{n}} A_m + \sum_{r \geq 0} \frac{m^r}{r!} (A_{(r)} B)_{n,\bar{n}}.$$

Taking the formal sum of the latter over $m \in \mathbb{Z}_{>0}$ and formally “replacing” the infinite sum $\sum_{m>0} m^r$ by the ζ -value $\zeta(-r)$, we see that the right hand side coincides exactly with the last two terms on the right hand side of the second relation in Proposition 4.4(i). Therefore, by virtue of the first relation in Proposition 4.4(i), the right hand side of the second relation in Proposition 4.4(i) can be interpreted as a ζ -function regularisation of the formal infinite sum $\sum_{m \in \mathbb{Z}} A_m B_{n-m,\bar{n}}$. In other words, we have the formal equality

$$(A_{(-1)} B)_{n,\bar{n}} \text{ “=} \sum_{m \in \mathbb{Z}} A_m B_{n-m,\bar{n}}$$

where the non-sensical infinite sum on the right hand side, which corresponds to the Fourier $(n, \bar{n})^{\text{th}}$ -mode of the naive product $A(\vartheta(q))B(\underline{\vartheta}(q))$, is given meaning by bringing all its terms into normal ordered form and using ζ -function regularisation on the resulting divergent sum. The same reasoning applies to the relations in Proposition 4.4(ii). \triangleleft

We have the following immediate applications of Proposition 4.4.

Corollary 4.6. *The Fourier modes of the (anti-)conformal states satisfy*

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{m^3}{12}c \delta_{m+n,0}, \quad [\bar{L}_m, \bar{L}_n] = (m - n)\bar{L}_{m+n} + \frac{m^3}{12}c \delta_{m+n,0}$$

and $[L_m, \bar{L}_n] = 0$ for $m, n \in \mathbb{Z}$. In the Kac-Moody case we have the non-trivial commutators

$$[L_m, X_n] = -n X_{m+n}, \quad [X_m, Y_n] = [X, Y]_{m+n} + m \kappa(X, Y) \delta_{m+n,0},$$

$$[\bar{L}_m, \bar{X}_n] = -n \bar{X}_{m+n}, \quad [\bar{X}_m, \bar{Y}_n] = [\bar{X}, \bar{Y}]_{m+n} + m \kappa(X, Y) \delta_{m+n,0}$$

for any $X, Y \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$. In the $\beta\gamma$ system case we have

$$[L_m, \beta_n] = -n \beta_{m+n}, \quad [L_m, \gamma_n] = -(m + n) \gamma_{m+n}, \quad [\beta_m, \gamma_n] = \delta_{m+n,0},$$

$$[\bar{L}_m, \bar{\beta}_n] = -n \bar{\beta}_{m+n}, \quad [\bar{L}_m, \bar{\gamma}_n] = -(m + n) \bar{\gamma}_{m+n}, \quad [\bar{\beta}_m, \bar{\gamma}_n] = \delta_{m+n,0}$$

Proof. These all follow immediately from Proposition 4.4. For the first set of equations we use in particular the defining property (3.63a) of the (anti-)conformal states and Lemma 4.2. For the second set of equations we use the identities (3.60) and the fact that

$$\Omega_{(n)} \mathsf{X}_{(-1)} |0\rangle = D(\mathsf{X}_{(-1)} |0\rangle) \delta_{n,0} + \mathsf{X}_{(-1)} |0\rangle \delta_{n,1}$$

for all $n \geq 0$, and similarly for the anti-chiral version, cf. (4.4). For the third set of equations we use the identities (3.62) and the fact that

$$\begin{aligned} \Omega_{(n)} \beta_{(-1)} |0\rangle &= D(\beta_{(-1)} |0\rangle) \delta_{n,0} + \beta_{(-1)} |0\rangle \delta_{n,1}, \\ \Omega_{(n)} \gamma_{(-1)} |0\rangle &= D(\gamma_{(-1)} |0\rangle) \delta_{n,0} \end{aligned}$$

for all $n \geq 0$, and similarly for the anti-chiral versions. \square

4.3 Reality conditions

In this section we discuss reality conditions on quantum operators defined in Proposition 4.1. Specifically, we introduce a natural notion of adjoint operator, defined by the action of the anti-linear isomorphism $\hat{\tau}$ of $\mathcal{UL}_\alpha^\Sigma$ from Proposition 2.6 and show in Proposition 4.9 that it matches the usual adjoint with respect to the Hermitian sesquilinear form introduced in §3.6.

Lemma 4.7. *For any $A \in \mathbb{F}^{\mathbf{a},\alpha}$ and $p \in \Sigma'$ we have $\hat{\tau}(A^{i\vartheta}) = (-1)^{\Delta_A + \bar{\Delta}_A} (\hat{\tau}A)^{i\vartheta}_{\tau(p)}$.*

Proof. Using Lemma 3.28 and the fact that $\hat{\tau}(i\vartheta) = -i\vartheta$ which follows from (4.1), we deduce $\hat{\tau}(A_p^{i\vartheta}) = (\hat{\tau}A)^{-i\vartheta}_{\tau(p)}$. By the explicit expression (3.12b) for $A_{\tau(p)}^{-i\vartheta}$, using the fact that

$$\begin{aligned} (\tau\mathbf{a})_{-i\vartheta_{\tau(p)}}^i \otimes \lceil (-i\vartheta_{\tau(p)})^{-m_i} \rceil_{U_i}^{U_{i-1}} &= (-1)^{m_i - \Delta_{\mathbf{a}} + 1} (\tau\mathbf{a})_{i\vartheta_{\tau(p)}}^i \otimes \lceil (i\vartheta_{\tau(p)})^{-m_i} \rceil_{U_i}^{U_{i-1}}, \\ (\tau\mathbf{b})_{i\bar{\vartheta}_{\tau(p)}}^j \otimes \lceil (i\bar{\vartheta}_{\tau(p)})^{-n_j} \rceil_{V_j}^{V_{j-1}} &= (-1)^{n_j - \Delta_{\mathbf{b}} + 1} (\tau\mathbf{b})_{-i\bar{\vartheta}_{\tau(p)}}^j \otimes \lceil (-i\bar{\vartheta}_{\tau(p)})^{-n_j} \rceil_{V_j}^{V_{j-1}} \end{aligned}$$

for $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, \bar{r}\}$, the result now follows. \square

Given any $\mathcal{O} \in \mathcal{UL}_\alpha^\Sigma(W)$ on an annulus shaped open subset $W \subset \Sigma'$ around $o \in \Sigma$ in the coordinate u , we say that it acts on $\mathcal{F}_o^{\mathbf{a},\alpha}$ if it induces an endomorphism

$$\mathcal{O} : \mathcal{F}_o^{\mathbf{a},\alpha} \longrightarrow \mathcal{F}_o^{\mathbf{a},\alpha}, \quad B_o^u \longmapsto (\mathcal{O} B)_o^u$$

where the state $\mathcal{O} B \in \mathbb{F}^{\mathbf{a},\alpha}$ is defined by the factorisation product $m_{(W,o),U}(\mathcal{O} \otimes B_o^u) = (\mathcal{O} B)_U^u$ into a larger open subset $U \subset \Sigma$ of $o \in \Sigma$. We then obtain an endomorphism $\mathcal{O} : \mathbb{F}^{\mathbf{a},\alpha} \rightarrow \mathbb{F}^{\mathbf{a},\alpha}$. For instance, the vertex modes $A_{(n,\bar{n})}^u$ for $n, \bar{n} \in \mathbb{Z}$ of any state $A \in \mathbb{F}^{\mathbf{a},\alpha}$ act on $\mathcal{F}_o^{\mathbf{a},\alpha}$ by Lemma 3.10. We define the adjoint of $\mathcal{O} \in \mathcal{UL}_\alpha^\Sigma(W)$ as

$$\mathcal{O}^\dagger := \hat{\tau}(\mathcal{O}) \in \mathcal{UL}_\alpha^\Sigma(\tau(W)), \quad (4.11)$$

where we can also view $\tau(W) \subset \Sigma'$ as an annulus shaped open subset around $o \in \Sigma$ in the coordinate u . In particular, \mathcal{O}^\dagger then also acts on $\mathcal{F}_o^{\mathbf{a},\alpha}$.

Recall the quantum operator $A(\underline{\vartheta}(p)) \in \mathcal{UL}_\alpha^\Sigma(W)$ associated to a state $A \in \mathbb{F}^{\mathbf{a},\alpha}$, as defined in Proposition 4.1. Its adjoint is given by the following.

Proposition 4.8. *For a monomial state $A \in \mathbb{F}^{\mathbf{a},\alpha}$ we have*

$$A(\underline{\vartheta}(p))^\dagger = (-1)^{\Delta_A + \bar{\Delta}_A} (\hat{\tau}A)(\underline{\vartheta}(p)) \in \mathcal{UL}_\alpha^\Sigma(\tau(W)), \quad (4.12)$$

or equivalently, for any $n, \bar{n} \in \mathbb{Z}$ we have $A_{n,\bar{n}}^\dagger = (-1)^{\Delta_A + \bar{\Delta}_A} (\hat{\tau}A)_{-n,-\bar{n}}$.

Proof. By definition (4.11) of the adjoint we have

$$\begin{aligned}
A(\underline{\vartheta}(p))^\dagger &= \hat{\tau}\left(A(\underline{\vartheta}(p))\right) = \hat{\tau}\left(\mathcal{Y}(A_p^{i\vartheta}, \underline{u}(p))\right) = \hat{\tau}(m_{p,W}(A_p^{i\vartheta})) \\
&= m_{\tau(p), \tau(W)}(\hat{\tau}A_p^{i\vartheta}) = (-1)^{\Delta_A + \bar{\Delta}_A} m_{\tau(p), \tau(W)}((\hat{\tau}A)_{\tau(p)}^{i\vartheta}) \\
&= (-1)^{\Delta_A + \bar{\Delta}_A} \mathcal{Y}((\hat{\tau}A)_{\tau(p)}^{i\vartheta}, \underline{u}(\tau(p))) = (-1)^{\Delta_A + \bar{\Delta}_A} (\hat{\tau}A)(\underline{\vartheta}(\tau(q)))
\end{aligned}$$

where the fourth step we uses Proposition 2.6 and the fifth step is by Lemma 4.7. The result (4.12) now follows by using the property (4.1). Using the definition of the Fourier modes in Proposition 4.1 to rewrite both sides of the last expression we find

$$\sum_{n, \bar{n} \in \mathbb{Z}} A_{n, \bar{n}}^\dagger e^{in\bar{\vartheta}(q)} e^{-i\bar{n}\vartheta(q)} = (-1)^{\Delta_A + \bar{\Delta}_A} \sum_{n, \bar{n} \in \mathbb{Z}} (\hat{\tau}A)_{n, \bar{n}} e^{-in\vartheta(\tau(q))} e^{i\bar{n}\bar{\vartheta}(\tau(q))},$$

where on the left hand side we applied the anti-linear map $(\cdot)^\dagger$ defined in (4.11) to each term of the Fourier series from Proposition 4.1. The final result about the Fourier modes now follows from comparing both sides of the above using the property (4.1). \square

The above notation $(\cdot)^\dagger$ for the adjoint of Fourier modes is consistent with the anti-linear anti-involution $(\cdot)^\dagger$ on the modes of $\hat{\mathfrak{a}} \oplus \hat{\bar{\mathfrak{a}}}$ introduced in the Kac-Moody case in (3.98).

Indeed, recall from after (4.5) that the Fourier modes X_n and \bar{X}_n coincide with the vertex modes (3.29) at $o \in \Sigma$ in the u coordinate for all $n \in \mathbb{Z}$, which in turn have the same action as modes in $\hat{\mathfrak{a}}$ and $\hat{\bar{\mathfrak{a}}}$, respectively, when acting on states prepared at $o \in \Sigma$ in the u coordinate by (3.30). Also, the commutation relations of the Fourier modes in Corollary 4.6 coincide with the Lie algebra relations (3.4) in $\hat{\mathfrak{a}} \oplus \hat{\bar{\mathfrak{a}}}$, with the central elements \mathbf{k} and $\bar{\mathbf{k}}$ both replaced by 1. Now the definition (3.98) implies that $(X_n)^\dagger = -(\tau X)_{-n}$ and $(\bar{Y}_{\bar{n}})^\dagger = -(\bar{\tau}Y)_{-\bar{n}}$, which are equivalent to the statement of Proposition 4.8 for the states $X_{(-1)}|0\rangle$ and $\bar{Y}_{(-1)}|0\rangle$. Furthermore, the property from Proposition 3.32 of the Hermitian sesquilinear form on $\mathbb{F}^{a, \alpha} \cong U(\hat{\mathfrak{a}} \oplus \hat{\bar{\mathfrak{a}}})|0\rangle$ defined in (3.97) follows from the next two propositions.

Proposition 4.9. *Suppose $\mathcal{O} \in \mathcal{UL}_\alpha^\Sigma(W)$, for an annulus shaped open subset $W \subset \Sigma'$ around $o \in \Sigma$ in the coordinate u , acts on $\mathcal{F}_o^{a, \alpha}$. Then for any $B, C \in \mathbb{F}^{a, \alpha}$ we have*

$$\langle B, \mathcal{O} C \rangle = \langle \mathcal{O}^\dagger B, C \rangle.$$

In particular, for all states $A, B, C \in \mathbb{F}^{a, \alpha}$ we have $\langle B, A(\underline{\zeta})C \rangle = \langle A(\underline{\zeta})^\dagger B, C \rangle$ or, equivalently, we have $\langle B, A_{n, \bar{n}}C \rangle = \langle A_{n, \bar{n}}^\dagger B, C \rangle$ for any $n, \bar{n} \in \mathbb{Z}$.

Proof. This is very similar to the proof of Proposition 3.23. For any $B, C \in \mathbb{F}^{a, \alpha}$, we compute

$$\left\langle m_{(o', W, o), \Sigma}((\hat{\tau}B)_{o'}^{u-1} \otimes \mathcal{O} \otimes C_o^u) \right\rangle \in \mathbb{C}$$

in two different ways using the associativity (2.17) of the factorisation product.

On the one hand, by definition of the action of \mathcal{O} on $\mathcal{F}_o^{a, \alpha}$ we have

$$\left\langle m_{(o', U), \Sigma}((\hat{\tau}B)_{o'}^{u-1} \otimes m_{(W, o), U}(\mathcal{O} \otimes C_o^u)) \right\rangle = \left\langle m_{(o', U), \Sigma}((\hat{\tau}B)_{o'}^{u-1} \otimes (\mathcal{O} C)_U^u) \right\rangle$$

for some open subset $o \in U \subset \Sigma \setminus \{o'\}$. By definition (3.97) of the Hermitian sesquilinear form on $\mathbb{F}^{a, \alpha}$, the above is simply $\langle B, \mathcal{O} C \rangle$.

On the other hand, since $\mathcal{O} = \hat{\tau}(\mathcal{O}^\dagger)$ we can compute the same factorisation product as

$$\left\langle m_{(o', W, o), \Sigma}(\hat{\tau}(B_o^u) \otimes \hat{\tau}(\mathcal{O}^\dagger) \otimes C_o^u) \right\rangle = \left\langle m_{(U', o), \Sigma}((\hat{\tau}(\mathcal{O}^\dagger B))_{U'}^{u-1} \otimes C_o^u) \right\rangle$$

where in the second expression we introduced an open $o' \in U' \subset \Sigma \setminus \{o\}$. By definition (3.97) of the Hermitian sesquilinear form on $\mathbb{F}^{a, \alpha}$, the above is just $\langle \mathcal{O}^\dagger B, C \rangle$, hence the result. \square

Proposition 4.10. *For any $A, B \in \mathbb{F}^{\alpha, \alpha}$ and $m, \bar{m}, n, \bar{n} \in \mathbb{Z}$ we have*

$$(A_{m, \bar{m}} B_{n, \bar{n}})^\dagger = B_{n, \bar{n}}^\dagger A_{m, \bar{m}}^\dagger.$$

In other words, taking the adjoint (4.11) is an anti-linear anti-involution on Fourier modes.

Proof. Recall that the product of Fourier modes, described explicitly at the start of §4.2, is induced by the factorisation product of $\mathcal{UL}_\alpha^\Sigma$. By Proposition 2.6, for any inclusion $U \sqcup V \subset W$ of disjoint annuli shaped open subsets around $o \in \Sigma$ in the u coordinate into another larger such annuli shaped open subset $W \subset \Sigma'$, we have the commutative diagram

$$\begin{array}{ccc} \mathcal{UL}_\alpha^\Sigma(U) \otimes \mathcal{UL}_\alpha^\Sigma(V) & \xrightarrow{m_{(U, V), W}} & \mathcal{UL}_\alpha^\Sigma(W) \\ \hat{\tau} \otimes \hat{\tau} \downarrow & & \downarrow \hat{\tau} \\ \mathcal{UL}_\alpha^\Sigma(\tau(U)) \otimes \mathcal{UL}_\alpha^\Sigma(\tau(V)) & \xrightarrow{m_{(\tau(U), \tau(V)), \tau(W)}} & \mathcal{UL}_\alpha^\Sigma(\tau(W)) \end{array}$$

The result now follows since τ reverses the ordering along the cylinder Σ' of annuli shaped open subsets around $o \in \Sigma$ in the u coordinate. \square

A Twisted prefactorisation envelopes

A.1 Unital DG Lie algebras

Following [BS, Example 2.9], let \mathbf{uLie} be the DG operad generated by an element $b \in \mathbf{uLie}(2)$ in degree 0 and an element $u \in \mathbf{uLie}(0)$ in degree 1, which we represent graphically as trees

$$b \longleftrightarrow \begin{array}{c} \text{---} \\ | \\ \bullet \end{array}, \quad u \longleftrightarrow \begin{array}{c} \text{---} \\ | \\ \circ \end{array},$$

such that $db = 0$, $du = 0$ and subject to three relations which we can represent most easily in graphical form as

$$\begin{array}{c} \text{---} \\ | \\ \bullet \\ 1 \quad 2 \end{array} + \begin{array}{c} \text{---} \\ | \\ \bullet \\ 2 \quad 1 \end{array} = 0, \quad \begin{array}{c} \text{---} \\ | \\ \bullet \\ 1 \quad 2 \quad 3 \end{array} + \begin{array}{c} \text{---} \\ | \\ \bullet \\ 2 \quad 3 \quad 1 \end{array} + \begin{array}{c} \text{---} \\ | \\ \bullet \\ 3 \quad 1 \quad 2 \end{array} = 0, \quad \begin{array}{c} \text{---} \\ | \\ \bullet \\ \quad \quad \circ \end{array} = 0, \quad (\text{A.1})$$

where the numbers below the trees indicate input permutations. We shall refer to an algebra in $\mathbf{dgVec}_{\mathbb{C}}$ over the DG operad \mathbf{uLie} as a *unital DG Lie algebra*. We let $\mathbf{udgLie}_{\mathbb{C}} := \mathbf{Alg}_{\mathbf{uLie}}(\mathbf{dgVec}_{\mathbb{C}})$ denote the category of unital DG Lie algebras. Explicitly, a unital DG Lie algebra L is described by a multifunctor $\mathbf{uLie} \rightarrow \mathbf{dgVec}_{\mathbb{C}}^\otimes$. More explicitly, such a multifunctor singles out an object $L \in \mathbf{dgVec}_{\mathbb{C}}$ as the image of the single object of \mathbf{uLie} and closed linear maps $[\cdot, \cdot] : L \otimes L \rightarrow L$ and $\eta : \mathbb{C} \rightarrow L$, of degrees 0 and 1 respectively, as the images of $b \in \mathbf{uLie}(2)$ and $u \in \mathbf{uLie}(0)$. In particular, the first two relations in (A.1) make L into a DG Lie algebra and the last relation says that the image of the *unit* $\eta : \mathbb{C} \rightarrow L$, which is closed in L , is also central in L .

A.1.1 Monoidal structure

For any unital DG Lie algebras $L, L' \in \mathbf{udgLie}_{\mathbb{C}}$, with respective units $\eta : \mathbb{C} \rightarrow L$ and $\eta' : \mathbb{C} \rightarrow L'$, we define the unital DG Lie algebra

$$L \bar{\oplus} L' := (L \oplus L') / \text{im}(\eta - \eta')$$

where the quotient is by the image of the linear map $\eta - \eta' : \mathbb{C} \rightarrow L \oplus L'$, which is a DG Lie ideal of $L \oplus L'$. The unit in $L \bar{\oplus} L'$ is the map induced by $\eta : \mathbb{C} \rightarrow L \oplus L'$, or equivalently by $\eta' : \mathbb{C} \rightarrow L \oplus L'$. The identity object for the monoidal product $\bar{\oplus}$ on $\mathbf{udgLie}_{\mathbb{C}}$ is the trivial unital DG Lie algebra $\mathbb{C}[-1]$.

We will use the following universal property of the direct sum of unital DG Lie algebras.

Lemma A.1. Let $f_i : L_i \rightarrow L$ for $i \in I$ be any collection of morphisms of unital DG Lie algebras, with indexing set I , such that $[\text{im } f_i, \text{im } f_j] = 0$ for every $i \neq j \in I$. There exists a unique morphism of unital DG Lie algebras $\overline{\bigoplus}_{j \in I} L_j \rightarrow L$ such that the diagram

$$\begin{array}{ccc} & & L \\ & \nearrow f_i & \uparrow \exists! \\ L_i & \xrightarrow{\iota_i} & \overline{\bigoplus}_{j \in I} L_j \end{array}$$

is commutative for each $i \in I$, where $\iota_i : L_i \rightarrow \overline{\bigoplus}_{j \in I} L_j$ is the canonical embedding.

Moreover, suppose $f'_i : L'_i \rightarrow L'$ for $i \in I$ is another collection of morphisms of unital DG Lie algebras such that $[\text{im } f'_i, \text{im } f'_j] = 0$ for every $i \neq j \in I$ and suppose we are given morphisms of unital DG Lie algebras $\phi : L \rightarrow L'$ and $\phi_i : L_i \rightarrow L'_i$ for each $i \in I$ such that $\phi \circ f_i = f'_i \circ \phi_i$. Then there exists a unique morphism of unital DG Lie algebras $\overline{\bigoplus}_{j \in I} \phi_j : \overline{\bigoplus}_{j \in I} L_j \rightarrow \overline{\bigoplus}_{j \in I} L'_j$ making the following diagram

$$\begin{array}{ccccc} & & L & & \\ & \nearrow f_i & \uparrow & \searrow \phi & \\ L_i & \xrightarrow{\iota_i} & \overline{\bigoplus}_{j=1}^n L_j & & L' \\ & \searrow \phi_i & \nearrow f'_i & \nearrow \exists! & \\ & & L'_i & \xrightarrow{\iota'_i} & \overline{\bigoplus}_{j=1}^n L'_j \end{array}$$

commute for each $i \in I$, where the vertical morphisms are defined as above.

Proof. We first define $h : \overline{\bigoplus}_{j \in I} L_j \rightarrow L$ by $(x_i)_{i \in I} \mapsto \sum_{i \in I} f_i(x_i)$. This is well defined since the sum over $i \in I$ is finite by virtue of x_i being zero for all but finitely many $i \in I$. And since f_i are morphisms of unital DG Lie algebras we have $f_i \circ \eta_i = \eta$ where η_i denotes the unit in each L_i and η the unity in L . It follows that h factors through a map $\bar{h} : \overline{\bigoplus}_{j \in I} L_j \rightarrow L$, defined by $[(x_i)_{i \in I}] \mapsto \sum_{i \in I} f_i(x_i)$ where $[(x_i)_{i \in I}] \in \overline{\bigoplus}_{j \in I} L_j$ is the class of $(x_i)_{i \in I} \in \overline{\bigoplus}_{j \in I} L_j$. In other words, $\bar{h}([(x_i)_{i \in I}]) = h([(x_i)_{i \in I}])$. But we also have

$$\begin{aligned} [h((x_i)_{i \in I}), h((y_i)_{i \in I})] &= \sum_{i, j \in I} [f_i(x_i), f_j(y_j)] = \sum_{i \in I} [f_i(x_i), f_i(y_i)] \\ &= \sum_{i \in I} f_i([x_i, y_i]) = h([(x_i, y_i)]_{i \in I}) = h([(x_i)_{i \in I}, (y_i)_{i \in I}]) \end{aligned}$$

where the second step is by the assumption that $[\text{im } f_i, \text{im } f_j] = 0$ for $i \neq j \in I$. The third step follows since each f_i is a morphism of unital DG Lie algebras and the last step uses the DG Lie algebra structure on $\overline{\bigoplus}_{i \in I} L_i$. So h is a morphism of DG Lie algebras and hence \bar{h} is a morphism of unital DG Lie algebras. By construction, the latter is unique such that $h \circ \iota_i = f_i$.

Let us consider now the second claim. Since $\text{im}(\iota'_i \circ \phi_i) \subset \text{im } \iota'_i$ for each $i \in I$ it follows that $[\text{im}(\iota'_i \circ \phi_i), \text{im}(\iota'_j \circ \phi_j)] = 0$ for each $i \neq j \in I$. By the first part of the lemma applied to the morphisms of unital DG Lie algebras $\iota'_i \circ \phi_i : L_i \rightarrow \overline{\bigoplus}_{j=1}^n L'_j$ we thus have a unique morphism of unital DG Lie algebras $\overline{\bigoplus}_{j \in I} \phi_j : \overline{\bigoplus}_{j \in I} L_j \rightarrow \overline{\bigoplus}_{j \in I} L'_j$ which makes the bottom square of the second diagram in the statement commute. It remains to show that it also makes the square on the right of the diagram commute, i.e. that $\bar{h}' \circ (\overline{\bigoplus}_{j \in I} \phi_j) = \phi \circ \bar{h}$ where \bar{h} and \bar{h}' are the two vertical morphisms defined as above.

Now $\phi \circ f_i = f'_i \circ \phi_i$ and $[\text{im } f'_i, \text{im } f'_j] = 0$ for every $i \neq j \in I$ from which it follows that $[\text{im}(\phi \circ f_i), \text{im}(\phi \circ f_j)] = 0$ for every $i \neq j \in I$. Therefore, applying the first part of the lemma

to the collection of morphisms of unital DG Lie algebras $\phi \circ f_i : L_i \rightarrow L'$ we obtain a unique morphism of unital DG Lie algebras $g : \overline{\bigoplus}_{j=1}^n L_j \rightarrow L'$ such that $g \circ \iota_i = \phi \circ f_i$ for each $i \in I$. Yet we have

$$\bar{h}' \circ \left(\overline{\bigoplus}_{j \in I} \phi_j \right) \circ \iota_i = \bar{h}' \circ \iota'_i \circ \phi_i = f'_i \circ \phi_i = \phi \circ f_i = \phi \circ \bar{h} \circ \iota_i$$

from which we deduce that $\bar{h}' \circ (\overline{\bigoplus}_{j \in I} \phi_j) = g = \phi \circ \bar{h}$ by uniqueness of g , as required. \square

A.1.2 Chevalley-Eilenberg functor for $\text{udgLie}_{\mathbb{C}}$

The homological Chevalley-Eilenberg functor is the symmetric monoidal functor

$$\begin{aligned} \text{CE}_{\bullet} : (\text{dgLie}_{\mathbb{C}}, \oplus) &\longrightarrow (\text{dgVec}_{\mathbb{C}}, \otimes) \\ L &\longmapsto \text{CE}_{\bullet}(L) := \left((\text{Sym}(L[1]))^{\bullet}, \mathbf{d}_{\text{CE}} \right), \end{aligned} \quad (\text{A.2})$$

where $(\text{Sym } V)^{\bullet} := \bigoplus_{n \geq 0} \text{Sym}^n V$ is the free commutative graded algebra on a graded vector space V and $\text{Sym}^n V$ is its component of word length n . Here $L[1] := \mathbb{C}[1] \otimes L \in \text{dgVec}_{\mathbb{C}}$ denotes the suspension of the DG vector space L where $\mathbb{C}[1]$ is the DG vector space with \mathbb{C} placed in degree -1 . For any element $v \in V$ of a DG vector space V we let $sv := 1 \otimes v \in V[1]$ denote its suspension, and we use the notation $s^{-1}v := 1 \otimes v \in V[-1] := \mathbb{C}[-1] \otimes V$ to denote its inverse suspension. The product in $(\text{Sym } V)^{\bullet}$ is denoted by concatenation. The differential on $\text{CE}_{\bullet}(L)$ is $\mathbf{d}_{\text{CE}} := \mathbf{d}_{L[1]} + \mathbf{d}_{[\cdot, \cdot]}$ where $\mathbf{d}_{L[1]} : L[1] \rightarrow L[1]$ is induced by the differential $\mathbf{d}_L : L \rightarrow L$ of the DG Lie algebra L and $\mathbf{d}_{[\cdot, \cdot]} : (\text{Sym}(L[1]))^{\bullet} \rightarrow (\text{Sym}(L[1]))^{\bullet}$ is the unique coderivation of the graded coalgebra $(\text{Sym}(L[1]))^{\bullet}$ extending the degree 1 map $\text{Sym}^2(L[1]) \rightarrow L[1]$, where $\text{Sym}^2(L[1])$ is the graded symmetric tensor square, induced by the Lie bracket $[\cdot, \cdot] : L \otimes L \rightarrow L$, see for instance [FHT, Lemma 22.2]. Explicitly, $\text{Sym}^2(L[1]) \rightarrow L[1]$ is given by $sx sy \mapsto (-1)^{|x|+1} s[x, y]$ for any homogeneous $x, y \in L$. This is well defined since $sx sy = (-1)^{|x||y|+|x|+|y|+1} sy sx$ is sent to $(-1)^{|x||y|+|x|+|y|+1} (-1)^{|y|+1} s[y, x] = (-1)^{|x|+1} s[x, y]$ using the graded skew-symmetry of the Lie bracket, namely $[y, x] = -(-1)^{|x||y|} [x, y]$. Note that $\mathbf{d}_{L[1]} \mathbf{d}_{[\cdot, \cdot]} = -\mathbf{d}_{[\cdot, \cdot]} \mathbf{d}_{L[1]}$ and $\mathbf{d}_{[\cdot, \cdot]}^2 = 0$.

We shall need a variant of the functor (A.2) for unital DG Lie algebras defined in the next proposition. Let $I_L^i := \{\mathcal{A} s(\eta(1)) - \mathcal{A} \mid \mathcal{A} \in \text{CE}_i(L)\}$ for every $i \in \mathbb{Z}$. This is a subspace of $\text{CE}_i(L)$ since $\eta(1)$ is of degree 1. Moreover, since $\eta(1)$ is a cocycle, i.e. $\mathbf{d}_L \eta(1) = 0$, and is central in L , it follows that I_L^{\bullet} is a DG vector subspace of $\text{CE}_{\bullet}(L)$.

Proposition A.2. *We have a symmetric monoidal functor*

$$\begin{aligned} \overline{\text{CE}}_{\bullet} : (\text{udgLie}_{\mathbb{C}}, \bar{\oplus}) &\longrightarrow (\text{dgVec}_{\mathbb{C}}, \otimes) \\ L &\longmapsto \overline{\text{CE}}_{\bullet}(L) := \text{CE}_{\bullet}(L) / I_L^{\bullet} \end{aligned} \quad (\text{A.3})$$

which preserves quasi-isomorphisms.

Proof. For any morphism of unital DG Lie algebras $f : L \rightarrow L'$, the morphism of DG vector spaces $\text{CE}_{\bullet}(f) : \text{CE}_{\bullet}(L) \rightarrow \text{CE}_{\bullet}(L')$ is given in degree $i \in \mathbb{Z}$ by $\text{CE}_i(f) = (\text{Sym } f[1])^i$, where the morphism of DG vector spaces $f[1] : L[1] \rightarrow L'[1]$ is the suspension of $f : L \rightarrow L'$. Since $f(\eta(1)) = \eta'(1)$ it follows that $\text{CE}_{\bullet}(f)(I_L^{\bullet}) \subset I_{L'}^{\bullet}$, and therefore $\text{CE}_{\bullet}(f)$ induces a morphism of DG vector spaces $\overline{\text{CE}}_{\bullet}(f) : \overline{\text{CE}}_{\bullet}(L) \rightarrow \overline{\text{CE}}_{\bullet}(L')$.

Since the functor (A.2) preserves quasi-isomorphisms, if $f : L \xrightarrow{\sim} L'$ is a quasi-isomorphism then so is $\text{CE}_{\bullet}(f) : \text{CE}_{\bullet}(L) \xrightarrow{\sim} \text{CE}_{\bullet}(L')$. Now the morphism $\overline{\text{CE}}_{\bullet}(f) : \overline{\text{CE}}_{\bullet}(L) \rightarrow \overline{\text{CE}}_{\bullet}(L')$ is a retract of the latter since we have a commutative diagram

$$\begin{array}{ccccc} \overline{\text{CE}}_{\bullet}(L) & \xrightarrow{i_L} & \text{CE}_{\bullet}(L) & \xrightarrow{q_L} & \overline{\text{CE}}_{\bullet}(L) \\ \overline{\text{CE}}_{\bullet}(f) \downarrow & & \downarrow \text{CE}_{\bullet}(f) & & \downarrow \overline{\text{CE}}_{\bullet}(f) \\ \overline{\text{CE}}_{\bullet}(L') & \xrightarrow{i_{L'}} & \text{CE}_{\bullet}(L') & \xrightarrow{q_{L'}} & \overline{\text{CE}}_{\bullet}(L') \end{array}$$

where $q_L : \text{CE}_\bullet(L) \rightarrow \overline{\text{CE}}_\bullet(L)$ is the canonical map and $i_L : \overline{\text{CE}}_\bullet(L) \rightarrow \text{CE}_\bullet(L)$ is given by taking the representative with no factors of $s(\eta(1))$ so that we clearly have $q_L \circ i_L = \text{id}_{\overline{\text{CE}}_\bullet(L)}$ and similarly $q_{L'} \circ i_{L'} = \text{id}_{\overline{\text{CE}}_\bullet(L')}$. Hence $\overline{\text{CE}}_\bullet(f) : \overline{\text{CE}}_\bullet(L) \xrightarrow{\sim} \overline{\text{CE}}_\bullet(L')$ is a quasi-isomorphism.

It remains to show that the functor $\overline{\text{CE}}_\bullet$ is symmetric monoidal. In particular, we must show that given any unital DG Lie algebras $L, L' \in \text{udgLie}_\mathbb{C}$ we have a canonical isomorphism of DG vector spaces

$$\overline{\text{CE}}_\bullet(L \bar{\oplus} L') \cong \overline{\text{CE}}_\bullet(L) \otimes \overline{\text{CE}}_\bullet(L').$$

To see this, let $J^i := \{\mathcal{A} s(\eta(1)) - \mathcal{A} s(\eta'(1)) \mid \mathcal{A} \in \text{CE}_i(L \oplus L')\}$ for every $i \in \mathbb{Z}$. Since $\eta(1)$ and $\eta'(1)$, thought of as elements in $L \oplus L'$, are both central cocycles of degree 1, it follows that J^\bullet is a DG vector subspace of $\text{CE}_\bullet(L \oplus L')$. Moreover, we have a canonical isomorphism of DG vector spaces $\text{CE}_\bullet(L \bar{\oplus} L') \cong \text{CE}_\bullet(L \oplus L')/J^\bullet$. Also, introducing the DG vector subspace K^\bullet of $\text{CE}_\bullet(L \oplus L')$ with components $K^i := \{\mathcal{A} s(\eta(1)) - \mathcal{A} + \mathcal{A}' s(\eta'(1)) - \mathcal{A}' \mid \mathcal{A}, \mathcal{A}' \in \text{CE}_i(L \oplus L')\}$ for every $i \in \mathbb{Z}$, of which J^\bullet is an obvious DG vector subspace, we have a canonical isomorphism $I_{L \bar{\oplus} L'}^\bullet \cong K^\bullet/J^\bullet$. We therefore have

$$\begin{aligned} \overline{\text{CE}}_\bullet(L \bar{\oplus} L') &= \text{CE}_\bullet(L \bar{\oplus} L')/I_{L \bar{\oplus} L'}^\bullet \cong (\text{CE}_\bullet(L \oplus L')/J^\bullet)/(K^\bullet/J^\bullet) \cong \text{CE}_\bullet(L \oplus L')/K^\bullet \\ &\cong (\text{CE}_\bullet(L) \otimes \text{CE}_\bullet(L'))/(I_L^\bullet \otimes \text{CE}_\bullet(L') + \text{CE}_\bullet(L) \otimes I_{L'}^\bullet) = \overline{\text{CE}}_\bullet(L) \otimes \overline{\text{CE}}_\bullet(L'), \end{aligned}$$

where in the first isomorphism we made use of the two isomorphisms stated above. The next isomorphism is by the third isomorphism theorem and the last isomorphism uses the fact that the homological Chevalley-Eilenberg functor CE_\bullet is symmetric monoidal.

Also, the result of applying the functor $\overline{\text{CE}}_\bullet$ to the identity object $\mathbb{C}[-1]$ of the symmetric monoidal product $\bar{\oplus}$ on $\text{udgLie}_\mathbb{C}$ is isomorphic to $\mathbb{C}[0]$, i.e. the identity object of the symmetric monoidal product \otimes on $\text{dgVec}_\mathbb{C}$. Hence $\overline{\text{CE}}_\bullet$ is a symmetric monoidal functor. \square

It will be useful to also introduce, mainly for the purpose of proving part of Proposition 3.4, the category of *pointed* DG vector spaces, denoted $\text{udgVec}_\mathbb{C}$, whose objects are vector spaces V equipped with a degree 1 map $\eta : \mathbb{C} \rightarrow V$ and whose morphisms preserve this structure. We also introduce the symmetric monoidal functor

$$\begin{aligned} \overline{\text{Sym}}_\bullet : (\text{udgVec}_\mathbb{C}, \bar{\oplus}) &\longrightarrow (\text{dgVec}_\mathbb{C}, \otimes) \\ V &\longmapsto \overline{\text{Sym}}_\bullet(V) := (\text{Sym}(V[1]))^\bullet/J_V^\bullet \end{aligned} \tag{A.4}$$

where the differential on $(\text{Sym}(V[1]))^\bullet$ is the one induced from $\text{d}_{V[1]} : V[1] \rightarrow V[1]$ and we set $J_V^i := \{\mathcal{A} s(\eta(1)) - \mathcal{A} \mid \mathcal{A} \in (\text{Sym}(V[1]))^i\}$ for each $i \in \mathbb{Z}$, which is a subspace of $(\text{Sym}(V[1]))^i$ since $\eta(1)$ has degree 1. Moreover, since $\eta(1)$ is a cocycle, namely $\text{d}_V \eta(1) = 0$, these form a DG vector subspace J_V^\bullet of $(\text{Sym}(V[1]))^\bullet$. The proof that (A.4) is symmetric monoidal is completely analogous to that of Proposition A.2.

A.2 Unital local Lie algebras

Let \mathcal{L} be a presheaf of unital DG Lie algebras on a manifold D , with extension morphisms denoted by $\text{ext}_{U,V} : \mathcal{L}(U) \rightarrow \mathcal{L}(V)$ for any inclusion of subsets $U \subset V$ in D . We say that \mathcal{L} is a *unital local Lie algebra* on D if for any finite collection $\{U_i\}_{i=1}^n$ of disjoint open subsets $U_i \subset V$ of an open subset $V \subset D$ we have

$$[\text{im}(\text{ext}_{U_i,V}), \text{im}(\text{ext}_{U_j,V})] = 0, \tag{A.5}$$

for every $i \neq j$. We denote by $\text{uLocLie}_\mathbb{C}(D)$ the category of unital local Lie algebras on D , where a morphism of unital local Lie algebras is defined as a morphism of the underlying presheaves of unital DG Lie algebras.

Proposition A.3. *We have a canonical functor*

$$\mathbf{uLocLie}_{\mathbb{C}}(D) \longrightarrow \mathbf{PFac}(D, \mathbf{udgLie}_{\mathbb{C}}^{\oplus}).$$

More explicitly, any $\mathcal{L} \in \mathbf{uLocLie}_{\mathbb{C}}(D)$ defines an object in $\mathbf{PFac}(D, \mathbf{udgLie}_{\mathbb{C}}^{\oplus})$ which by a slight abuse of notation we also denote by \mathcal{L} . Moreover, every morphism $\mathcal{L} \rightarrow \mathcal{L}'$ of $\mathbf{uLocLie}_{\mathbb{C}}(D)$ induces a morphism of $\mathbf{PFac}(D, \mathbf{udgLie}_{\mathbb{C}}^{\oplus})$ which we also denote $\mathcal{L} \rightarrow \mathcal{L}'$.

Proof. Let $\mathcal{L} \in \mathbf{uLocLie}_{\mathbb{C}}(D)$. Given any open subsets $\sqcup_{i=1}^n U_i \subset V$, by Lemma A.1 we obtain a unique morphism of DG Lie algebras $m_{(U_i), V}^{\mathcal{L}} : \overline{\bigoplus}_{j=1}^n \mathcal{L}(U_j) \rightarrow \mathcal{L}(V)$ such that the diagram

$$\begin{array}{ccc} & \mathcal{L}(V) & \\ \text{ext}_{U_i, V}^{\mathcal{L}} \nearrow & \uparrow \exists! m_{(U_i), V}^{\mathcal{L}} & \\ \mathcal{L}(U_i) \longrightarrow & \overline{\bigoplus}_{j=1}^n \mathcal{L}(U_j) & \end{array}$$

commutes. The composition property of these morphisms, required in order for \mathcal{L} to define a prefactorisation algebra, then follows from their uniqueness property. Explicitly, we have the following commutative diagram

$$\begin{array}{ccccc} & & \mathcal{L}(W) & \leftarrow \cdots & \\ & \text{ext}_{V_i, W}^{\mathcal{L}} \nearrow & \uparrow \exists! m_{(V_i), W}^{\mathcal{L}} & \nearrow \exists! m_{(U_{ij}), W}^{\mathcal{L}} & \\ \mathcal{L}(V_i) & \longrightarrow & \overline{\bigoplus}_{i=1}^n \mathcal{L}(V_i) & & \\ \text{ext}_{U_{ij}, V_i}^{\mathcal{L}} \nearrow & \uparrow \exists! m_{(U_{ij}), V_i}^{\mathcal{L}} & \uparrow \overline{\bigoplus}_{i=1}^n m_{(U_{ij}), V_i}^{\mathcal{L}} & \nearrow \cdots & \\ \mathcal{L}(U_{ij}) \longrightarrow & \overline{\bigoplus}_{j=1}^{m_i} \mathcal{L}(U_{ij}) \longrightarrow & \overline{\bigoplus}_{i=1}^n \overline{\bigoplus}_{j=1}^{m_i} \mathcal{L}(U_{ij}) & & \end{array}$$

where the two small commutative triangles are given by the universal property in Lemma A.1. The square on the bottom right of the diagram is commutative by definition of the morphism of unital DG Lie algebras $\overline{\bigoplus}_{i=1}^n m_{(U_{ij}), V_i}^{\mathcal{L}}$, in the second part of Lemma A.1. The extra morphism from the bottom right to the top right is given by the universal property from Lemma A.1 applied to the big outside triangle. Its uniqueness implies the required composition property. It follows that \mathcal{L} defines an element of $\mathbf{PFac}(D, \mathbf{udgLie}_{\mathbb{C}}^{\oplus})$, as required.

Let $\phi : \mathcal{L} \rightarrow \mathcal{L}'$ be a morphism of $\mathbf{uLocLie}_{\mathbb{C}}(D)$. By using the second part of Lemma A.1 we have a morphism $\overline{\bigoplus}_{j=1}^n \phi_{U_j} : \overline{\bigoplus}_{j=1}^n \mathcal{L}(U_j) \rightarrow \overline{\bigoplus}_{j=1}^n \mathcal{L}'(U_j)$ making the following diagram

$$\begin{array}{ccc} & \mathcal{L}(V) & \\ \nearrow & \uparrow m_{(U_i), V}^{\mathcal{L}} & \searrow \phi_V \\ \mathcal{L}(U_i) \longrightarrow & \overline{\bigoplus}_{j=1}^n \mathcal{L}(U_j) & \longrightarrow \mathcal{L}'(V) \\ \phi_{U_i} \searrow & \nearrow \exists! & \uparrow m_{(U_i), V}^{\mathcal{L}'} \\ \mathcal{L}'(U_i) \longrightarrow & \overline{\bigoplus}_{j=1}^n \mathcal{L}'(U_j) & \end{array}$$

commutative. In particular, the commutativity of the right hand square is equivalent to the statement that $\phi : \mathcal{L} \rightarrow \mathcal{L}'$ is a morphism of $\mathbf{PFac}(D, \mathbf{udgLie}_{\mathbb{C}}^{\oplus})$, as required. \square

Viewing any $\mathcal{L} \in \mathbf{uLocLie}_{\mathbb{C}}(D)$ as a multifunctor $\mathbf{Top}(D)^{\sqcup} \rightarrow \mathbf{udgLie}_{\mathbb{C}}^{\oplus}$, by Proposition A.3, we can consider its post-composition with the symmetric monoidal functor (A.3) from Proposition A.2 to obtain a prefactorisation algebra $\overline{\mathbf{CE}}_{\bullet} \mathcal{L} \in \mathbf{PFac}(D, \mathbf{dgVec}_{\mathbb{C}}^{\otimes})$. Post-composing the

latter with the lax monoidal 0th cohomology functor $H^0 : \mathbf{dgVec}_{\mathbb{C}} \rightarrow \mathbf{Vec}_{\mathbb{C}}$ then yields the *twisted prefactorisation envelope* of the unital local Lie algebra \mathcal{L} , denoted

$$\mathcal{U}\mathcal{L} := H^0 \overline{\mathbf{CE}}_{\bullet} \mathcal{L} \in \mathbf{PFac}(D, \mathbf{Vec}_{\mathbb{C}}^{\otimes}). \quad (\text{A.6})$$

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