The 3d Mixed BF Lagrangian 1-Form: A Variational Formulation of Hitchin's Integrable System

Vincent Caudrelier[1] , **Derek Harland**[1], **Anup Anand Singh**[1,3] , **Benoît Vicedo**[2]

[1] School of Mathematics, University of Leeds, Leeds LS2 9JT, UK.
[2] Department of Mathematics, University of York, York YO10 5GH, UK.
     E-mail: v.caudrelier@leeds.ac.uk
[3] Department of Mathematical Sciences, Loughborough University, Loughborough LE11 3TU, UK

Abstract: We introduce the concept of gauged Lagrangian 1-forms, extending the notion of Lagrangian 1-forms to the setting of gauge theories. This general formalism is applied to a natural geometric Lagrangian 1-form on the cotangent bundle of the space of holomorphic structures on a smooth principal G-bundle $\mathcal{P}$ over a compact Riemann surface C of arbitrary genus g, with or without marked points, in order to gauge the symmetry group of smooth bundle automorphisms of $\mathcal{P}$. The resulting construction yields a multiform version of the 3d mixed BF action with so-called type A and B defects, providing a variational formulation of Hitchin's completely integrable system over C. By passing to holomorphic local trivialisations and going partially on-shell, we obtain a *unifying action* for a hierarchy of Lax equations describing the Hitchin system in terms of meromorphic Lax matrices. The cases of genus 0 and 1 with marked points are treated in greater detail, producing explicit Lagrangian 1-forms for the rational Gaudin hierarchy and the elliptic Gaudin hierarchy, respectively, with the elliptic spin Calogero–Moser hierarchy arising as a special subcase.

Contents

1. Introduction and Summary of Main Results

The Hamiltonian formalism has historically played a more prominent role in shaping the modern theory of integrable systems than the Lagrangian one. Indeed, it naturally provides a description of *integrable hierarchies* as commuting Hamiltonian flows and its algebraic nature has been pivotal in the problem of quantisation. However, over the past 15 years, there has been a resurgence of interest in both applying and further developing Lagrangian methods in the context of integrable systems, driven by two major developments.

The first of these is the development of the theory of *Lagrangian multiforms*, first introduced by Lobb and Nijhoff in the setting of integrable lattice models [51]. Since then, significant progress was made in extending the multiform framework to various realms of integrability: discrete and continuous finite-dimensional systems [12,15,57, 69,72], integrable field theories in 1 + 1 dimensions [16,17,19,58,64,65,70] and 2 + 1 dimensions [54,66], and semi-discrete systems [67]. The concept was even extended to non-commuting flows in [13,14].

Unlike traditional action functionals in d spacetime dimensions, which capture the dynamics of a single d-dimensional integrable field theory (the $d = 1$ case corresponding to finite-dimensional systems), a multiform action functional is given by the integral of a Lagrangian d-form over a d-dimensional *sub*manifold Γ in a so-called *multi-time* space of dimension strictly greater than d. The resulting action functional $S[q, \Gamma]$ depends not only on the field configurations q but also on the submanifold Γ. Applying a generalised variational principle to this action produces the so-called *multi-time Euler-Lagrange equations*, encoding the dynamics of several d-dimensional integrable field theories from the same hierarchy, together with the so-called *closure relation* which provides a variational analogue of the Poisson involutivity of the Hamiltonians of this hierarchy.

The second development concerns the deep relationship between the notion of Lax integrability and certain so-called *mixed holomorphic-topological (HT) gauge theories*. The general philosophy is that the Lax formalism of an integrable field theory on a d-dimensional manifold M (again the $d = 1$ case corresponds to finite-dimensional systems), depending on a spectral parameter living on a Riemann surface C, arises from the equations of motion of a suitable gauge theory on $M \times C$ which is holomorphic along C and (possibly a mixture of holomorphic and) topological along M.

The first concrete example, discovered by Costello and further developed more recently with Witten and Yamazaki in [22–25], relates integrable lattice models to a 4-dimensional semi-holomorphic variant of Chern–Simons theory in the presence of line defects. This idea has been extensively studied and generalised over the past 5 years. Most notably, integrable field theories in 1 + 1 dimensions were shown to arise from surface defects in the same 4-dimensional Chern–Simons theory in the seminal paper [26] and many subsequent works, see for instance [4,7,18,27,29–32,40,42,46,47,50,60]. Integrable field theories in higher dimensions arising as reductions of the 4-dimensional anti-self-dual Yang–Mills equations were similarly derived from various defects in 6-dimensional holomorphic Chern–Simons theory on twistor space [8,20,21]. But the

example most relevant for the purpose of the present paper is the relationship between the finite (rational) Gaudin model and the 3-dimensional mixed holomorphic-topological BF theory with line defects [71].

It is important to stress that, while both developments are rooted in the Lagrangian formalism, their scopes are very different and each has their own advantages and drawbacks. Indeed, the main purpose of the Lagrangian multiform framework is to encode *hierarchies* of integrable systems, but the only known systematic constructions [12,15,19] rely on the algebraic machinery of the classical *r*-matrix method. In contrast, the framework of mixed HT gauge theories provides an elegant geometric origin for the spectral parameter and a powerful method for constructing many integrable systems, but it is currently limited to producing individual integrable systems rather than hierarchies. It is therefore very tempting to try to combine the two frameworks.

The primary purpose of the present paper is to take the first step in this direction,[1] in the setting of finite-dimensional integrable systems. Specifically, we will focus on Hitchin's integrable system [41] and its generalisations with marked points introduced in subsequent works (see, for instance, [9,28,52,55]) which provide master systems from which prototypical examples of completely integrable finite-dimensional Hamiltonian systems can be derived. Focusing in the introduction on the case without marked points for conciseness, the phase space of the Hitchin system is described by the cotangent bundle $T^*\mathrm{Bun}_G(C)$ of the moduli space $\mathrm{Bun}_G(C)$ of holomorphic principal G-bundles of a fixed topological type on a Riemann surface C of arbitrary genus $g \geq 2$. The Hitchin map describes a complete set of Poisson-commuting Hamiltonians on this phase space, thereby encoding a finite-dimensional integrable hierarchy.

Our main goal is to construct a Lagrangian 1-form which encodes the entire integrable hierarchy of Hitchin's system variationally. In doing so, we will find that we are naturally led to a multiform version of the 3d mixed BF theory introduced in [71], see also [34,39,73]. We thus generalise [71] in three key respects: (1) we consider Hitchin systems with marked points on Riemann surfaces C of *arbitrary* genus rather than just $\mathbb{C}P^1$, (2) we encode the entire hierarchy of such Hitchin systems rather than one individual flow in this hierarchy, and (3) we *derive* the 3d mixed BF action as a variational description of Hitchin systems rather than taking that action as a given. The present paper thus naturally sits at the intersection of the following three topics: Hitchin's integrable systems, mixed HT gauge theories and Lagrangian multiforms.
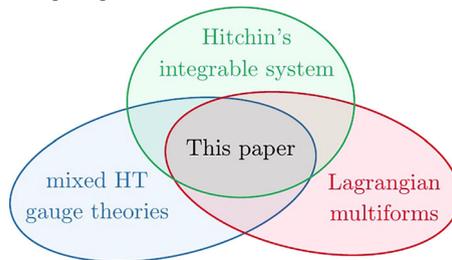
As such, it provides a first instance of the merging of the framework of Lagrangian multiforms with mixed HT gauge theories, using Hitchin's integrable system as a driving example.

---

[1] In connection with combining the framework of Lagrangian multiforms with gauge-theoretic ideas, we note the work [53]. However, it is based on a different approach from mixed HT gauge theories. Lagrangian multiforms for the Darboux–Kadomtsev–Petviashvili system were derived from a hierarchy of Chern–Simons actions in an infinite-dimensional space of Miwa variables.

In the remainder of the introduction we give a detailed outline of the structure of the paper and summarise the main results. For ease of presentation in this introduction, we will focus here on the case of Hitchin's system on a Riemann surface C of arbitrary genus $g \geq 2$ *without* marked points. The case with marked points is dealt with in detail in the main body of the paper.

In the absence of marked points, the phase space of Hitchin's system is given by the cotangent bundle $T^*\mathrm{Bun}_G(C)$. A point in the base $\mathrm{Bun}_G(C)$ is a stable holomorphic principal G-bundle $\mathcal{P}_{\mathrm{hol}}$, of a fixed topological type, which can always be described using a single holomorphic transition function $\gamma : U_0 \cap U_1 \to G$ relative to an open cover $\{U_0, U_1\}$ of C with U_0 an open neighbourhood of a fixed point $\mathsf{p} \in C$ and $U_1 := C\backslash\{\mathsf{p}\}$. A point in the fibre $T^*_{\mathcal{P}_{\mathrm{hol}}}\mathrm{Bun}_G(C)$ above $\mathcal{P}_{\mathrm{hol}}$ is called a holomorphic Higgs field L. Identifying $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ using a fixed nondegenerate invariant bilinear pairing on $\mathfrak{g}$, it is given explicitly by a pair of $\mathfrak{g}$-valued $(1, 0)$-forms L^0 and L^1 on U_0 and U_1, respectively, which are related via the adjoint action $L^0 = \gamma L^1 \gamma^{-1}$ on the overlap $U_0 \cap U_1$. The pair $(\mathcal{P}_{\mathrm{hol}}, L)$ is an example of a stable Higgs bundle. The Hitchin map, which is also known as the Hitchin fibration, provides a complete set of Poisson-commuting Hamiltonians H_i for $i = 1, \ldots, n$, where n denotes half the dimension of the phase space. These Hamiltonians induce commuting time flows $\partial_{t^i}, i = 1, \ldots, n$ on $T^*\mathrm{Bun}_G(C)$ whose actions on the pair (L^0, γ) are given by

$$\partial_{t^i} L^0 = \left[M_i^0, L^0\right], \qquad M_i^0 = \gamma M_i^1 \gamma^{-1} + \partial_{t^i}\gamma\gamma^{-1} \tag{1.1}$$

for $i = 1, \ldots, n$, where M_i^1 are $\mathfrak{g}$-valued meromorphic functions on U_1 each with a simple pole at a fixed marked point $\mathsf{q}_i \in U_1$ whose residue there is determined by the Hitchin Hamiltonian H_i, and M_i^0 are $\mathfrak{g}$-valued holomorphic functions on U_0.

One of the main goals of the present paper is to give a variational description of the hierarchy of commuting flows (1.1). To do so, we will exploit the fact that the Hitchin phase space $T^*\mathrm{Bun}_G(C)$ can be obtained as a symplectic reduction of the infinite-dimensional cotangent bundle $T^*\mathcal{M}$ of the space $\mathcal{M}$ of stable holomorphic structures on a fixed smooth principal G-bundle $\mathcal{P}$ by the action of the group $\mathcal{G} = \mathrm{Aut}\,\mathcal{P}$ of smooth bundle automorphisms of $\mathcal{P}$, as we now briefly recall.

A holomorphic structure on $\mathcal{P}$ can be specified in terms of a partial[2] $(0, 1)$-connection A'' on $\mathcal{P}$ given in local coordinate patches by $\mathfrak{g}$-valued $(0, 1)$-forms on C. A smooth bundle automorphism $g \in \mathcal{G}$ acts on such a holomorphic structure A'' by gauge transformations $A'' \mapsto gA''g^{-1} - \bar\partial gg^{-1}$ and two holomorphic structures related in this way define the same holomorphic principal G-bundle $\mathcal{P}_{\mathrm{hol}}$. In particular, the action of $\mathcal{G}$ on the space $\mathcal{M}$ of stable holomorphic structures is free and we have an isomorphism $\mathcal{M}/\mathcal{G} \cong \mathrm{Bun}_G(C)$. Moreover, the action of $\mathcal{G}$ on $\mathcal{M}$ lifts to a Hamiltonian action on $T^*\mathcal{M}$ with moment map given by $\mu(B, A'') = \bar\partial^{A''}B$, where B denotes the smooth Higgs field parametrising the fibre of $T^*_{A''}\mathcal{M}$ over a fixed holomorphic structure $A'' \in \mathcal{M}$. We prefer to use the symbol B over the more standard Φ as the Higgs field will shortly become identified with the corresponding field with the same name in 3d mixed BF theory. The starting point for our construction is then the fact that the Hitchin phase space is given by the symplectic quotient

$$T^*\mathrm{Bun}_G(C) \cong \mu^{-1}(0)/\mathcal{G}. \tag{1.2}$$

Specifically, in Sect. 3.2 we introduce a natural lift of the Hitchin map to the cotangent bundle $T^*\mathcal{M}$ which induces n commuting flows on this infinite-dimensional symplectic

---

[2] The notion of partial connection is explained below after (3.43).

manifold and that can be described variationally using a natural geometric multiform action $S_\Gamma[B, A'', t]$ on $T^*\mathcal{M}$. Upon performing the symplectic reduction to $\mu^{-1}(0)/\mathcal{G}$ at the level of the action, which is the content of Sect. 3, we obtain a multiform version of the action for the gauge theory known as 3d mixed holomorphic-topological BF theory. After subsequently passing to holomorphic local trivialisations and going partially on-shell to obtain an equivalent description of the dynamics on $T^*\mathrm{Bun}_G(C)$ via the isomorphism (1.2), which is the content of Sect. 4, this will lead to the desired action for the dynamics (1.1) of the Hitchin system on $T^*\mathrm{Bun}_G(C)$.

In order to describe this in more detail, it is useful to first recall how the analog of symplectic reduction can be implemented in the Lagrangian formalism. Consider a symplectic manifold T^*M equipped with a Hamiltonian action of a Lie group G with moment map $\mu : T^*M \to \mathfrak{g}^*$. If the Hamiltonian is G-invariant then so is the first-order action $S[p, q] = \int_0^1 \left(p_\mu \dot{q}^\mu - H(p, q)\right) dt$ and symplectic reduction to $\mu^{-1}(0)/G$ is implemented by adding the term $\int_0^1 \langle \mu(p, q), \mathcal{A} \rangle$ to the action where $\mathcal{A} = \mathcal{A}_t dt$ is a $\mathfrak{g}$-valued Lagrange multiplier. Indeed, the restriction to the level set $\mu^{-1}(0)$ is implemented dynamically via the equation of motion for $\mathcal{A}$ while the quotient by G is implemented as a gauge symmetry since the global G-symmetry of the original action is promoted to a gauge symmetry of the new action if we let $\mathcal{A}$ transform as gauge field $\mathcal{A} \mapsto g\mathcal{A}g^{-1} - \partial_t g g^{-1} dt$ for any $g \in C^\infty\big((0, 1), G\big)$. In Sect. 2 we generalise this to the setting of Lagrangian multiforms.

The symplectic reduction in (1.2) is implemented at the level of the action in Sect. 3.3. Specifically, starting from the geometric multiform action $S_\Gamma[B, A'', t]$ on $T^*\mathcal{M}$, where in the present multiform setting the fields B, A'' and also the time variables t^i depend on an auxiliary space of multi-times $\mathbb{R}^n$, we introduce a $\mathfrak{g}$-valued gauge field $\mathcal{A} = \mathcal{A}_j du^j$ (with an implicit sum over $j = 1, \ldots, n$) which transforms as a connection along $\mathbb{R}^n$, namely $\mathcal{A} \mapsto g\mathcal{A}g^{-1} - d_{\mathbb{R}^n} g g^{-1}$. To implement the symplectic reduction by the group $\mathcal{G}$ we then add the new term $\int_C \langle \mu, \mathcal{A} \rangle = -\int_C \langle B, \bar{\partial}^{A''}\mathcal{A} \rangle$ to the Lagrangian 1-form $\int_C \langle B, d_{\mathbb{R}^n} A'' \rangle - H_i(B, A'') d_{\mathbb{R}^n} t^i$ underlying the original action $S_\Gamma[B, A'', t]$. Remarkably, the resulting Lagrangian 1-form is exactly a multiform version of the Lagrangian for 3d mixed BF theory; see Theorem 3.4. In particular, the partial $(0, 1)$-connection A'' and $\mathbb{R}^n$-connection $\mathcal{A}$ combine into a partial connection $A = A'' + \mathcal{A}$ which together with the Higgs field B form the field content of 3d mixed BF theory. Moreover, the lift of the Hitchin Hamiltonians H_i to the cotangent bundle $T^*\mathcal{M}$ now play the role of type B defects in the language of [71].

By construction, the Lagrangian 1-form for 3d mixed BF theory encodes the dynamics of the Hitchin system but in terms of the degrees of freedom of the cotangent bundle $T^*\mathcal{M}$, i.e. in terms of a smooth partial $(0, 1)$-connection A'' specifying the holomorphic structure of P parametrising the base, a smooth Higgs field B parametrising the fibre, and a smooth $\mathbb{R}^n$-connection $\mathcal{A}$ serving as Lagrange multiplier for the constraint $\bar{\partial}^{A''} B = 0$. To obtain the sought-after Lagrangian 1-form describing the dynamics of the Hitchin system on the actual Hitchin phase space $T^*\mathrm{Bun}_G(C)$, which is parametrised by pairs $(\mathcal{P}_{\mathrm{hol}}, L)$ with $\mathcal{P}_{\mathrm{hol}}$ a stable holomorphic principal G-bundle on C and L a holomorphic Higgs fields, we need to perform one final step detailed in Sect. 4. Specifically, the holomorphic principal G-bundle $\mathcal{P}_{\mathrm{hol}}$ is obtained by moving to a local trivialisation of the smooth principal G-bundle $\mathcal{P}$ in which $A'' = 0$. Moreover, to turn the smooth Higgs field B into the holomorphic Higgs field L we simply enforce the constraint $\bar{\partial}^{A''} B = 0$.
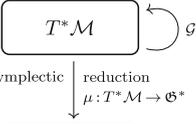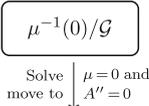
Hitchin's system on	Phase-space parametrised by	Variational description
1) $T^*\mathcal{M}$ $\circlearrowleft \mathcal{G}$	$\mathcal{P}$ smooth principal G-bundle A'' holomorphic structure B smooth Higgs field	canonical action $S_\Gamma[B, A'', t]$ (§3.2)
Symplectic reduction $\mu : T^*\mathcal{M} \to \mathfrak{G}^*$		
2) $\mu^{-1}(0)/\mathcal{G}$	$\mathcal{P}$ smooth principal G-bundle A'' holomorphic structure $\Big\}A$ $\mathcal{A}$ Lagrange multiplier for $\mu = 0$ B smooth Higgs field	$3d$ mixed BF action $S_{3d,\Gamma}[B, A, t]$ (§3.3)
Solve $\mu = 0$ and move to $A'' = 0$		
3) $T^*\mathrm{Bun}_G(C)$	$\mathcal{P}_{\mathrm{hol}}$ holomorphic principal G-bundle L holomorphic Higgs field/Lax matrix	$1d$ unifying action $S_{\mathrm{H},\Gamma}[L^0, \gamma, t]$ (§4.2)

Fig. 1. The three levels of Hitchin's integrable system and their corresponding actions

The resulting Lagrangian 1-form, derived in Theorem 4.1, reads

$$\mathscr{L}_{\mathrm{H}} = -\frac{1}{2\pi i} \int_{c_{\mathrm{p}}} \langle L^0, \mathrm{d}_{\mathbb{R}^n} \gamma \gamma^{-1} \rangle - H_i \mathrm{d}_{\mathbb{R}^n} t^i, \qquad (1.3)$$

where c_{p} is a small counter-clockwise oriented circle around the marked point p. More precisely, in Theorem 4.1 we derive the more general Lagrangian 1-form for a Hitchin system with marked points $\mathrm{p}_\alpha \in C$, for $\alpha = 1, \ldots, N$ with $N \in \mathbb{Z}_{\geq 1}$, at which we attach degrees of freedom φ_α living in coadjoint orbits of fixed elements $\Lambda_\alpha \in \mathfrak{g}^*$. See Sect. 3.4 for details. In Theorem 4.3 we show that the variation of the corresponding action reproduces the equations of motion (1.1) of the Hitchin system, as expected. A visual summary of the whole procedure, detailing the passage from the geometrical action on $T^*\mathcal{M}$ to that of the Hitchin system on $T^*\mathrm{Bun}_G(C)$, is sketched in Fig. 1.

Finally, in Sect. 5 we explicitly compute our unifying Lagrangian 1-form $\mathscr{L}_{\mathrm{H}}$, or its multiform action $S_{\mathrm{H},\Gamma}[L^0, \gamma, (\varphi_\alpha), t]$ from Sect. 4.2, in special cases. Specifically, we exploit the invariance of the action under changes of holomorphic local trivialisations to fix particularly nice representatives for the transition function γ. In the $g = 0$ case, our Lagrangian 1-form recovers that of the rational Gaudin hierarchy first obtained in [12] by a completely different method. In the $g = 1$ case, we obtain a novel Lagrangian 1-form for the elliptic Gaudin hierarchy—and the elliptic spin Calogero–Moser hierarchy as a special case—thus filling a gap in the landscape of Lagrangian multiforms.

We also include a technical Appendix A in which we compare, in the $G = SL_m(\mathbb{C})$ case, the differential notion of the stability condition we use in the main text to the more familiar algebraic notion of the stability condition used in the literature.

2. Gauged Univariational Principle and Symmetry Reduction

We recall in Sect. 2.1 the phase-space Lagrangian 1-form introduced in [13]. In particular, in Sect. 2.1.1 we formulate the univariational principle of [13]. In Sect. 2.1.2 we explain how to incorporate a symmetry, represented by the free action of a connected Lie group G on M, and recast the known relation between Noether charges and the moment map $\mu : T^*M \to \mathfrak{g}^*$ in the context of Lagrangian 1-forms. Then, in Sect. 2.2 we introduce the gauged univariational principle and explain how it describes, purely in the variational language of Lagrangian 1-forms, the symplectic reduction procedure to $\mu^{-1}(0)/G$ traditionally presented in Hamiltonian terms.

2.1. Lagrangian 1-forms, Univariational Principle and Symmetry.

2.1.1. Univariational Principle for Lagrangian 1-forms

Let M be an m-dimensional manifold with coordinates q^μ for $\mu = 1, \ldots, m$. The cotangent bundle T^*M is parametrised by coordinates (q^μ, p_μ) where p_μ for $\mu = 1, \ldots, m$ are dual coordinates along the fibres. The tautological 1-form α and symplectic form ω of T^*M are given by $\alpha = p_\mu dq^\mu$ and $\omega = dp_\mu \wedge dq^\mu$. Here and in what follows, we always use the summation convention according to which repeated upstairs and downstairs indices of any kind are summed over. Given any function f on T^*M, the associated Hamiltonian vector field $\mathcal{X}_f$ is defined by the property $\mathcal{X}_f \lrcorner \omega = df$ and is given explicitly by

$$\mathcal{X}_f = \frac{\partial f}{\partial q^\mu}\frac{\partial}{\partial p_\mu} - \frac{\partial f}{\partial p_\mu}\frac{\partial}{\partial q^\mu}. \tag{2.1}$$

For any pair of functions f and g, their Poisson bracket is then given by $\{f, g\} = -\mathcal{X}_f g = \mathcal{X}_g f$.

Let us introduce the phase-space Lagrangian 1-form on $T^*M \times \mathbb{R}^n$ given by

$$\mathcal{L} := \alpha - H_i dt^i = p_\mu dq^\mu - H_i dt^i, \tag{2.2}$$

where $H_1, \ldots, H_n$ are $n \in \mathbb{Z}_{\geq 1}$ real functions on T^*M and t^i are Cartesian coordinates on $\mathbb{R}^n$. (We do not necessarily assume here any relation between the integers n and $m = \dim M$; the case $n = 1$ corresponds to an ordinary Hamiltonian system while the case $n = m$ will correspond to that of a Liouville integrable system.) Note that in (2.2), the phase space variables p_μ, q^μ and the time variables t^i are treated on an equal footing. Accordingly, the action associated to a parametrised curve $\gamma : (0, 1) \to T^*M \times \mathbb{R}^n$ is now

$$S_0[\gamma] = \int_0^1 \gamma^* \mathcal{L} = \int_0^1 \left(p_\mu \frac{dq^\mu}{ds} - H_i(p, q)\frac{dt^i}{ds} \right) ds. \tag{2.3}$$

If we were to apply the usual principle of least action, or variational principle, to (2.3) then we would seek a curve $\gamma : (0, 1) \to T^*M \times \mathbb{R}^n$ which is a critical point of this action. That is to say, under an arbitrary variation $\delta\gamma(s) = (\delta p_\mu(s), \delta q^\mu(s), \delta t^i(s))$ satisfying the boundary conditions $0 = \lim_{s\to 0,1} \delta q(s) = \lim_{s\to 0,1} \delta t(s)$ we want

$$\begin{aligned}
0 = \delta S_0[\gamma] &= \int_0^1 \left(\delta p_\mu \frac{dq^\mu}{ds} + p_\mu \frac{d\delta q^\mu}{ds} - \left(\frac{\partial H_i}{\partial p_\mu}\delta p_\mu + \frac{\partial H_i}{\partial q^\mu}\delta q^\mu \right)\frac{dt^i}{ds} - H_i \frac{d\delta t^i}{ds} \right) ds \\
&= \int_0^1 \left(\delta p_\mu \left(\frac{dq^\mu}{ds} - \frac{\partial H_i}{\partial p_\mu}\frac{dt^i}{ds} \right) - \delta q^\mu \left(\frac{dp_\mu}{ds} + \frac{\partial H_i}{\partial q^\mu}\frac{dt^i}{ds} \right) + \frac{dH_i}{ds}\delta t^i \right) ds ,
\end{aligned} \tag{2.4}$$

where in the second line we integrated by parts and used the boundary conditions. This leads to the Euler–Lagrange equations

$$\gamma' \lrcorner d\mathcal{L} = 0 \iff \frac{dq^\mu}{ds} = \frac{\partial H_i}{\partial p_\mu}\frac{dt^i}{ds}, \quad \frac{dp_\mu}{ds} = -\frac{\partial H_i}{\partial q^\mu}\frac{dt^i}{ds}, \quad \frac{dH_i}{ds} = 0 \tag{2.5}$$

where γ' denotes the tangent vector to the curve γ. It is clear from the first two Euler–Lagrange equations that locally along the curve we must always have $\frac{dt^i}{ds} \neq 0$ for some

$i \in \{1, \ldots, n\}$ since otherwise we would have $\gamma' = 0$. We could then look for solutions of (2.5) such that $\frac{dt^i}{ds} \neq 0$ for a fixed i which up to a change of variable amounts to working with the parameter $s = t^i$ along the curve. However, this would inevitably single out H_i as our 'preferred' Hamiltonian.

Rather than looking for individual curves $\gamma : (0, 1) \to T^*M \times \mathbb{R}^n$ which are critical points of the action $S_0[\gamma]$ in (2.3), the univariational principle takes on board the fundamental idea of the multiform framework and replaces the curve γ by an *immersion*

$$\Sigma : \mathbb{R}^n \longrightarrow T^*M \times \mathbb{R}^n, \quad (u^j) \longmapsto \big(p_\mu(u), q^\mu(u), t^i(u)\big) \tag{2.6}$$

i.e. a map Σ such that its tangent map

$$d_u \Sigma : T_u \mathbb{R}^n \to T_{\Sigma(u)}\big(T^*M \times \mathbb{R}^n\big) \tag{2.7}$$

is injective for every $u \in \mathbb{R}^n$. In particular, Σ is then a (local) diffeomorphism onto its image in $T^*M \times \mathbb{R}^n$. In view of applying the univariational principle, Σ is required to satisfy boundary conditions that fix the functions $\lim_{\|u\| \to \infty} q^\mu(u)$ and $\lim_{\|u\| \to \infty} t^i(u)$ of S^{n-1} at spatial infinity.

Introducing also a curve

$$\Gamma : (0, 1) \longrightarrow \mathbb{R}^n, \quad s \longmapsto \big(t^i(s)\big) \tag{2.8}$$

such that $\lim_{s \to 0, 1} \|\Gamma(s)\| = \infty$, we can evaluate the action (2.3) on the composite map

$$\gamma := \Sigma \circ \Gamma : (0, 1) \xrightarrow{\Gamma} \mathbb{R}^n \xrightarrow{\Sigma} T^*M \times \mathbb{R}^n \,,$$

leading now to a family of multiform actions labelled by Γ, namely

$$S_\Gamma[\Sigma] := S_0[\Sigma \circ \Gamma] = \int_0^1 \left(p_\mu \frac{\partial q^\mu}{\partial u^j} - H_i(p, q) \frac{\partial t^i}{\partial u^j} \right) \frac{du^j}{ds} ds. \tag{2.9}$$

Crucially, we view this family of actions as depending only on the map Σ in (2.6) and we treat the curve Γ in (2.8) as a parameter labelling the family. The distinction between the roles of Σ and Γ, reflected in the notation $S_\Gamma[\Sigma]$, underpins the *univariational principle* which can now be formulated as seeking an immersion Σ which is simultaneously a critical point of the family of actions $S_\Gamma[\Sigma]$ for all curves Γ in $\mathbb{R}^n$. In other words, it consists in the single-step procedure:

Find Σ such that $\delta_\Sigma S_\Gamma[\Sigma] = 0$ for all curves Γ in $\mathbb{R}^n$.

As shown in [13], the univariational principle implies that the map Σ in (2.6) can be written as $\Sigma = (f_\Sigma, \mathrm{id}_{\mathbb{R}^n})$ for some map $f_\Sigma : \mathbb{R}^n \to T^*M, (t^i) \mapsto \big(p_\mu(t), q^\mu(t)\big)$ and the univariational principle is equivalent to the system of equations

$$\frac{\partial q^\mu}{\partial t^i} = \frac{\partial H_i}{\partial p_\mu}, \quad \frac{\partial p_\mu}{\partial t^i} = -\frac{\partial H_i}{\partial q^\mu}, \quad i = 1, \ldots, n, \quad \mu = 1, \ldots, m, \tag{2.10a}$$

$$\frac{\partial H_i}{\partial p_\mu} \frac{\partial H_j}{\partial q^\mu} - \frac{\partial H_i}{\partial q^\mu} \frac{\partial H_j}{\partial p_\mu} = \{H_i, H_j\} = 0, \quad i, j = 1, \ldots, n. \tag{2.10b}$$

We refer to the proof of Theorem 2.6 below for details in the more general gauge invariant setting. We see from (2.10b) that the univariational principle already encodes the closure relation through the variation of Σ, as a result of having included the times t^i among the dynamical variables in the map Σ. In particular, the univariational principle admits

solutions if and only if the Hamiltonian functions $H_1, \ldots, H_n$ mutually Poisson commute. We also see from (2.10a) that the time coordinate t^i parametrises the Hamiltonian flow of H_i for each $i = 1, \ldots, n$. The univariational principle applied to the Lagrangian 1-form (2.2) therefore encodes multitime Hamiltonian mechanics, which includes the case of Liouville integrable systems when $n = m$.

Remark 2.1. Readers familiar with Lagrangian multiforms may wonder how the univariational principle adopted here connects to the bivariational principle that has been employed in the literature until the recent work [13]. In the bivariational formulation, one views Σ as $(f_\Sigma, \mathrm{id}_{\mathbb{R}^n})$ for some map $f_\Sigma : \mathbb{R}^n \to T^*M, (t^i) \mapsto (p_\mu(t), q^\mu(t))$ *from the outset* so that $S_0[\Sigma \circ \Gamma]$ in (2.9) becomes the (perhaps more familiar) action

$$S[f_\Sigma, \Gamma] := S_0[f_\Sigma \circ \Gamma] = \int_0^1 \left(p_\mu \frac{\partial q^\mu}{\partial t^j} - H_j(p, q) \right) \frac{\mathrm{d}t^j}{\mathrm{d}s} \mathrm{d}s. \qquad (2.11)$$

The price to pay in this formulation is that one has to vary with respect to both f_Σ *and* Γ to obtain the full set of equations (2.10).

We also note that the univariational principle is a natural generalisation to the multitime case of the well-known *extended phase space* approach to classical mechanics. That approach corresponds to having a single time t in our case, *i.e.* $n = 1$, so that $\Gamma : (0, 1) \to \mathbb{R}, s \mapsto t$ is a (re)parametrisation of time. This idea applied in the Lagrangian formalism can be found for instance in [68] but is likely much older.

2.1.2. Group Actions and Symmetry
Suppose now that M admits a free right action of a connected Lie group G, $\rho : G \times M \to M$. Let X_a be a basis of the Lie algebra $\mathfrak{g}$ of G and let $X_a^\mu(q) \frac{\partial}{\partial q^\mu}$ denote the corresponding fundamental vector fields generating the action of $\mathfrak{g}$ on M. The right action of G lifts to T^*M, which we also denote by $\rho : G \times T^*M \to T^*M$, and the corresponding action of $\mathfrak{g}$ is generated by vector fields

$$X_a^\sharp = X_a^\mu(q) \frac{\partial}{\partial q^\mu} - p_\nu \frac{\partial X_a^\nu}{\partial q^\mu} \frac{\partial}{\partial p_\mu} \qquad (2.12)$$

satisfying $[X_a^\sharp, X_b^\sharp] = f_{ab}{}^c X_c^\sharp$, where $f_{ab}{}^c$ denote the structure constants of $\mathfrak{g}$. We further lift this to a right action $\rho : G \times T^*M \times \mathbb{R}^n \to T^*M \times \mathbb{R}^n$ of G on $T^*M \times \mathbb{R}^n$ by letting G act trivially on $\mathbb{R}^n$ so that the infinitesimal action of $\mathfrak{g}$ is still generated by the same vector fields (2.12).

In what follows we will consider only the corresponding left action of the group G on $T^*M \times \mathbb{R}^n$, given by $G \times T^*M \times \mathbb{R}^n \to T^*M \times \mathbb{R}^n, (g, x) \mapsto g \cdot x := \rho_{g^{-1}}(x)$ and which is infinitesimally generated by the vector fields $\delta_{X_a} := -X_a^\sharp$ for $a = 1, \ldots, \dim \mathfrak{g}$.

Proposition 2.2. *The action* (2.9) *is invariant under the infinitesimal action of G on $T^*M \times \mathbb{R}^n$ generated by* (2.12) *if and only if each H_i is invariant under this infinitesimal group action, i.e.*

$$\mathcal{L}_{X_a^\sharp} H_i = 0, \quad i = 1, \ldots, n, \quad a = 1, \ldots, \dim \mathfrak{g}. \qquad (2.13)$$

The Noether charges associated with this global G symmetry are then given by $\mu_a(p, q) = -p_\nu X_a^\nu(q)$.

Proof. The variation of any given map $\Sigma : \mathbb{R}^n \to T^*M \times \mathbb{R}^n$, $(u^j) \mapsto \big(p_\mu(u), q^\mu(u),$ $t^i(u)\big)$ under the infinitesimal left action of G on $T^*M \times \mathbb{R}^n$ generated by the vector fields (2.12) is given by $\delta_X \Sigma(u)$ $= -X^\sharp\big(\Sigma(u)\big)$, or more explicitly in components

$$\delta_X q^\mu(u) = -\lambda^a X_a^\mu(q(u)) \tag{2.14a}$$

$$\delta_X p_\mu(u) = \lambda^a p_\nu(u) \frac{\partial X_a^\nu}{\partial q^\mu}(q(u)) \tag{2.14b}$$

$$\delta_X t^i(u) = 0 \tag{2.14c}$$

for arbitrary $X = \lambda^a X_a \in \mathfrak{g}$. Noting that

$$\delta_X \left(p_\mu \frac{\partial q^\mu}{\partial u^j} \right) = \lambda^a p_\nu \frac{\partial X_a^\nu}{\partial q^\mu} \frac{\partial q^\mu}{\partial u^j} - \lambda^a p_\nu \frac{\partial X_a^\nu}{\partial u^j} = 0, \tag{2.15}$$

where the last step is by the chain rule for the function $X_a^\nu(q(u))$, the corresponding variation of the action $S_\Gamma[\Sigma]$ in (2.9) is then

$$\delta_X S_\Gamma[\Sigma] = \lambda^a \int_0^1 \left(X_a^\mu \frac{\partial H_i}{\partial q^\mu} - p_\nu \frac{\partial X_a^\nu}{\partial q^\mu} \frac{\partial H_i}{\partial p_\mu} \right) \frac{dt^i}{ds} ds = \lambda^a \int_0^1 \big(\mathcal{L}_{X_a^\sharp} H_i \big) \frac{dt^i}{ds} ds , \tag{2.16}$$

where the second step is by definition (2.12) of $X_a^\sharp$. Now the resulting expression in (2.16) should be zero for any λ^a, any map Σ and any curve Γ from which (2.13) follows.

The expression $\mu_a = -p_\nu X_a^\nu$ for the Noether charge associated with this global symmetry can be obtained by using the standard trick of promoting the constant parameters $\lambda^a \in \mathbb{C}$ to functions $\lambda^a : \mathbb{R}^n \to \mathbb{C}$. This leads to an additional term $\frac{\partial \lambda^a}{\partial u^j} \mu_a$ on the right-hand side of (2.15) so that the variation of the action now reads $\delta_X S_\Gamma[\Sigma] = \int_0^1 \frac{d\lambda^a}{ds} \mu_a ds$, as required. $\square$

In the Hamiltonian formalism, the Noether charges μ_a from Proposition 2.2 are encoded in a moment map $\mu : T^*M \to \mathfrak{g}^*$ such that

$$\langle \mu(p,q), X_a \rangle = \mu_a(p,q) = -p_\nu X_a^\nu(q), \tag{2.17}$$

where $\langle \, , \, \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{C}$ denotes the canonical pairing. In general, a moment map is required to satisfy the equations

$$\big\langle d\mu, X_a \big\rangle = X_a^\sharp \lrcorner \omega \quad \text{and} \quad \mathcal{L}_{X_a^\sharp} \mu + \mathrm{ad}_{X_a}^* \mu = 0. \tag{2.18}$$

In our case, the first of these follows by a direct calculation from (2.12), (2.17) and $\omega = dp_\mu \wedge dq^\mu$. The second is proved as follows. First we note that $\langle \mu, X_b \rangle = -X_b^\sharp \lrcorner \alpha$ and that $\mathcal{L}_{X_a^\sharp} \alpha = 0$. So

$$\big\langle \mathcal{L}_{X_a^\sharp} \mu, X_b \big\rangle = -\mathcal{L}_{X_a^\sharp} \big(X_b^\sharp \lrcorner \alpha \big) = -\big[X_a^\sharp, X_b^\sharp \big] \lrcorner \alpha = \big\langle \mu, [X_a, X_b] \big\rangle = -\big\langle \mathrm{ad}_{X_a}^* \mu, X_b \big\rangle, \tag{2.19}$$

where in the second last step we used the fact that $[X_a^\sharp, X_b^\sharp] = [X_a, X_b]^\sharp$. Hence the second equation of (2.18) holds.

2.2. Gauged Lagrangian 1-form and Gauged Univariational Principle. When a physical system is invariant under the action of a Lie group G, such as in the context of Sect. 2.1.2, this symmetry can be used to reduce the number of degrees of freedom. Indeed, since the components of the moment map $\mu : T^*M \to \mathfrak{g}^*$ are preserved under all the Hamiltonian flows $\mathcal{X}_{H_i}$, i.e. $\mathcal{X}_{H_i}\mu_a = 0$, one may consistently restrict to the zero-level set $\mu^{-1}(0) \subset T^*M$ of this moment map by imposing the constraint $\mu(p, q) = 0$. Furthermore, by the G-equivariance property of the moment map, the zero-level set $\mu^{-1}(0)$ is invariant under the action of G so that we may further reduce the number of degrees of freedom by working on the quotient space $\mu^{-1}(0)/G$.

In order to implement the above symplectic reduction procedure in the variational setting, and thereby construct a Lagrangian 1-form and action for the reduced system, we can impose the constraint $\mu(p, q) = 0$ in $\mathfrak{g}^*$ by introducing a $\mathfrak{g}$-valued Lagrange multiplier. Specifically, since we want to gauge the action of G on the map $\Sigma : \mathbb{R}^n \to T^*M \times \mathbb{R}^n$ described infinitesimally in (2.14), we introduce a $\mathfrak{g}$-valued Lagrange multiplier 1-form on $\mathbb{R}^n$ which we denote by

$$\mathcal{A} = \mathcal{A}_j \mathrm{d}u^j = \mathcal{A}_j^a X_a \mathrm{d}u^j \in \Omega^1(\mathbb{R}^n, \mathfrak{g}). \tag{2.20}$$

Using the canonical pairing $\langle \ , \ \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{C}$ we can combine (2.20) with the moment map $\mu : T^*M \to \mathfrak{g}^*$, which we view as a map $\mu : T^*M \times \mathbb{R}^n \to \mathfrak{g}^*$ that is constant along $\mathbb{R}^n$, to obtain a 1-form $\langle \Sigma^*\mu, \mathcal{A} \rangle \in \Omega^1(\mathbb{R}^n)$. The gauging procedure then simply consists in adding this term to the pullback $\Sigma^*\mathscr{L} \in \Omega^1(\mathbb{R}^n)$ of the Lagrangian 1-form $\mathscr{L}$. We obtain a family of actions for a map $\Sigma : \mathbb{R}^n \to T^*M \times \mathbb{R}^n$ and a 1-form $\mathcal{A} \in \Omega^1(\mathbb{R}^n, \mathfrak{g})$, parametrised by curves $\Gamma : (0, 1) \to \mathbb{R}^n$,

$$S_\Gamma[\Sigma, \mathcal{A}] = \int_0^1 \Gamma^*\Big(\Sigma^*\mathscr{L} + \langle \Sigma^*\mu, \mathcal{A} \rangle\Big). \tag{2.21}$$

The constraint $\mu(p, q) = 0$ will now be enforced dynamically through the equations of motion for $\mathcal{A}$, see Sect. 2.2.2 below. Moreover, the effect of adding this new term to the Lagrangian is to promote the G-symmetry we started with, from Sect. 2.1.2, to a gauge symmetry as we now show.

2.2.1. Gauging a Symmetry in a Lagrangian 1-form We consider local gauge transformations parametrised by smooth maps $g : \mathbb{R}^n \to G$, which act pointwise on the map $\Sigma : \mathbb{R}^n \to T^*M \times \mathbb{R}^n$ and on the Lagrange multiplier $\mathcal{A} \in \Omega^1(\mathbb{R}^n, \mathfrak{g})$ as a gauge transformation, i.e.

$$\Sigma(u) \longmapsto g(u) \cdot \Sigma(u), \tag{2.22a}$$

$$\mathcal{A} \longmapsto g \, \mathcal{A} \, g^{-1} - \mathrm{d}_{\mathbb{R}^n} g \, g^{-1}, \tag{2.22b}$$

where $\mathrm{d}_{\mathbb{R}^n}$ denotes the de Rham differential on $\mathbb{R}^n$. Due to its transformation property (2.22b), we will henceforth often refer to the Lagrange multiplier 1-form $\mathcal{A} \in \Omega^1(\mathbb{R}^n, \mathfrak{g})$ as a gauge field.

Proposition 2.3. *The action $S_\Gamma[\Sigma, \mathcal{A}]$ is invariant under an infinitesimal version of the G-valued local gauge transformation in (2.22) if and only if the functions H_i are G-invariant, i.e.*

$$\mathcal{L}_{X_a^\sharp} H_i = 0, \quad i = 1, \ldots, n, \quad a = 1, \ldots, \dim \mathfrak{g}. \tag{2.23}$$

Proof. Let $\Sigma : \mathbb{R}^n \to T^*M \times \mathbb{R}^n$, $(u^j) \mapsto \big(p_\mu(u), q^\mu(u), t^i(u)\big)$ be a given map as in the proof of Proposition 2.2. The action (2.21) for this Σ and any curve $\Gamma : (0, 1) \to \mathbb{R}^n$ then explicitly reads

$$S_\Gamma[\Sigma, \mathcal{A}] = \int_0^1 \left(p_\mu \frac{\partial q^\mu}{\partial u^j} - H_i(p, q) \frac{\partial t^i}{\partial u^j} + \mu_a(p, q) \mathcal{A}_j^a \right) \frac{\mathrm{d} u^j}{\mathrm{d} s} \mathrm{d} s, \qquad (2.24)$$

where p_μ, q^μ, t^i and A_j^a all depend on s through their dependence on the functions $u^j(s)$. Consider an infinitesimal gauge transformation parametrised by a $\mathfrak{g}$-valued function $X : \mathbb{R}^n \to \mathfrak{g}$, which we write in components as $X = \lambda^a(u) X_a$. The variations of the components of Σ and $\mathcal{A}$ under the infinitesimal left action of G on $T^*M \times \mathbb{R}^n$ then read (cf. the variations (2.14) where λ^a was constant)

$$\delta_X q^\mu(u) = -\lambda^a(u) X_a^\mu\big(q(u)\big) \qquad (2.25a)$$

$$\delta_X p_\mu(u) = \lambda^a(u) p_\nu(u) \frac{\partial X_a^\nu}{\partial q^\mu}\big(q(u)\big) \qquad (2.25b)$$

$$\delta_X t^i(u) = 0 \qquad (2.25c)$$

$$\delta_X \mathcal{A}_j^a(u) = -\frac{\partial \lambda^a}{\partial u^j} - f_{bc}{}^a \mathcal{A}_j^b(u) \lambda^c(u) . \qquad (2.25d)$$

By the exact same computation as in the proof of Proposition 2.2, where in the end of that proof λ^a was already treated as a function $\lambda^a : \mathbb{R}^n \to \mathbb{C}$, the corresponding variation of $S_\Gamma[\Sigma, \mathcal{A}]$ reads

$$\delta_X S_\Gamma[\Sigma, \mathcal{A}] = \int_0^1 \left[\lambda^a \big(\mathcal{L}_{X_a^\sharp} H_i\big) \frac{\partial t^i}{\partial u^j} - \lambda^a \mathcal{A}_j^b \big(\mathcal{L}_{X_a^\sharp} \mu_b - f_{ab}{}^c \mu_c \big) \right] \frac{\mathrm{d} u^j}{\mathrm{d} s} \mathrm{d} s. \quad (2.26)$$

We note, in particular, that the term $-\mu_a \frac{\partial \lambda^a}{\partial u^j}$ appearing in the variation $\mu_a \delta_X A_j^a$ cancels with the term $-p_\nu X_a^\nu \frac{\partial \lambda^a}{\partial u^j}$ from the variation (2.15) when λ^a is not constant. The right-hand side of (2.26) needs to vanish for all functions λ^a, $\mathcal{A}_i^a$ and all maps Σ and curves Γ, but the last bracketed term vanishes by the second relation in (2.18) since

$$\langle \mathcal{L}_{X_a^\sharp} \mu, X_b \rangle + \langle \mathrm{ad}^*_{X_a} \mu, X_b \rangle = \mathcal{L}_{X_a^\sharp} \mu_b - \langle \mu, [X_a, X_b] \rangle = \mathcal{L}_{X_a^\sharp} \mu_b - f_{ab}{}^c \mu_c. \quad (2.27)$$

Therefore gauge invariance is equivalent to the condition (2.23). $\qquad \square$

By combining Propositions 2.2 and 2.3 we see that the gauged action $S_\Gamma[\Sigma, \mathcal{A}]$ in (2.21) is invariant under the gauge group $C^\infty(\mathbb{R}^n, G)$ if and only if the original action $S_\Gamma[\Sigma]$ in (2.9) is invariant under the Lie group G.

2.2.2. Gauged Univariational Principle Recall from Sect. 2.1.1 that the univariational principle seeks an immersion $\Sigma : \mathbb{R}^n \to T^*M \times \mathbb{R}^n$ such that for all curves $\Gamma : (0, 1) \to \mathbb{R}^n$ we have $\delta_\Sigma S_\Gamma[\Sigma] = 0$. We introduce the gauged version in the following definition. To do so we need to introduce the gauge-covariant derivative of Σ. This is given by the formula

$$D_u^\mathcal{A} \Sigma = \mathrm{d}_u \Sigma - \mathcal{A}_u^\sharp : T_u \mathbb{R}^n \to T_{\Sigma(u)}(T^*M \times \mathbb{R}^n), \qquad (2.28)$$

in which $\mathrm{d}_u \Sigma$ is the usual differential of Σ and $\mathcal{A}_u^\sharp$ is defined by

$$\mathcal{A}_u^\sharp : T_u \mathbb{R}^n \to T_{\Sigma(u)}(T^*M \times \mathbb{R}^n), \qquad \frac{\partial}{\partial u^i} \mapsto \mathcal{A}_i^a(u) X_a^\sharp(\Sigma(u)). \qquad (2.29)$$

where we recall that $X_a^\sharp$ is given in (2.12). Put differently, $\mathcal{A}_u^\sharp$ is the composition of $\mathcal{A} : T_u \mathbb{R}^n \to \mathfrak{g}$, the map $\mathfrak{g} \to \Gamma(T(T^*M \times \mathbb{R}^n))$ induced by the action of G on $T^*M \times \mathbb{R}^n$, and the map $\Gamma(T(T^*M \times \mathbb{R}^n)) \to T_{\Sigma(u)}(T^*M \times \mathbb{R}^n)$ given by evaluation at $\Sigma(u)$. The derivative (2.28) is gauge-covariant in the sense that

$$D_u^{g\mathcal{A}g^{-1}-\mathrm{d}gg^{-1}}(g \cdot \Sigma) = \mathrm{d}_{\Sigma(u)}g(u) \circ D_u^\mathcal{A}\Sigma, \tag{2.30}$$

where $\mathrm{d}_{\Sigma(u)}g(u) : T_{\Sigma(u)}(T^*M \times \mathbb{R}^n) \to T_{g(u)\cdot\Sigma(u)}(T^*M \times \mathbb{R}^n)$ is the differential of $g(u) : T^*M \times \mathbb{R}^n \to T^*M \times \mathbb{R}^n$.

Definition 2.4. The *gauged univariational principle* seeks a map

$$\Sigma : \mathbb{R}^n \longrightarrow T^*M \times \mathbb{R}^n$$

and a gauge field $\mathcal{A} \in \Omega^1(\mathbb{R}^n, \mathfrak{g})$ such that the linear map $D_u^\mathcal{A}\Sigma = \mathrm{d}_u\Sigma - \mathcal{A}_u^\sharp$ is injective for every $u \in \mathbb{R}^n$, and such that the pair $(\Sigma, \mathcal{A})$ is simultaneously a critical point of the family of actions $S_\Gamma[\Sigma, \mathcal{A}]$ for all curves $\Gamma : (0, 1) \to \mathbb{R}^n$, namely such that

$$\delta_\Sigma S_\Gamma[\Sigma, \mathcal{A}] = 0 \quad \text{and} \quad \delta_\mathcal{A} S_\Gamma[\Sigma, \mathcal{A}] = 0.$$

Remark 2.5. The condition that $\mathrm{d}_u\Sigma - \mathcal{A}_u^\sharp$ is injective is the natural gauge-covariant analogue of the requirement that the map Σ in (2.6) is an immersion.

Theorem 2.6. *The gauged univariational principle applied to the gauge invariant action $S_\Gamma[\Sigma, \mathcal{A}]$ in (2.21) gives rise to the following set of equations*

$$\mu(p, q) = 0, \tag{2.31a}$$

$$\frac{\partial q^\mu}{\partial t^i} - \frac{\partial H_i}{\partial p_\mu} = \widetilde{\mathcal{A}}_i^a X_a^\mu, \tag{2.31b}$$

$$\frac{\partial p_\mu}{\partial t^i} + \frac{\partial H_i}{\partial q^\mu} = -\widetilde{\mathcal{A}}_i^a p_\nu \frac{\partial X_a^\nu}{\partial q^\mu}, \tag{2.31c}$$

$$\{H_i, H_j\} = 0, \tag{2.31d}$$

where the composition $\pi_{\mathbb{R}^n} \circ \Sigma : \mathbb{R}^n \to \mathbb{R}^n$, $(u^j) \mapsto (t^i(u))$ with the projection $\pi_{\mathbb{R}^n} : T^*M \times \mathbb{R}^n \to \mathbb{R}^n$, $(p_\mu, q^\mu, t^i) \mapsto (t^i)$ to the second factor is a (local) diffeomorphism and is used to define $\widetilde{\mathcal{A}}_i^a = \frac{\partial u^j}{\partial t^i} \mathcal{A}_j^a$.

Moreover, the $\mathfrak{g}$-valued connection $\mathrm{d}_{\mathbb{R}^n} + \mathcal{A}$ is flat, i.e. it satisfies the zero-curvature equation $F = \mathrm{d}_{\mathbb{R}^n}\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] = 0$ or in components

$$F_{ij}^a := \frac{\partial \widetilde{\mathcal{A}}_j^a}{\partial t^i} - \frac{\partial \widetilde{\mathcal{A}}_i^a}{\partial t^j} + f_{bc}{}^a \widetilde{\mathcal{A}}_i^b \widetilde{\mathcal{A}}_j^c = 0, \quad a = 1, \ldots, \dim \mathfrak{g}, \quad i, j = 1, \ldots, n. \tag{2.32}$$

Proof. Let $\Sigma : \mathbb{R}^n \to T^*M \times \mathbb{R}^n$, $(u^j) \mapsto (p_\mu(u), q^\mu(u), t^i(u))$ and $\Gamma : (0, 1) \to \mathbb{R}^n$ be arbitrary maps. The action (2.21) for these can be written as in (2.24).

We first perform a variation of the action (2.24) with respect to the functions $\mathcal{A}_j^a : \mathbb{R}^n \to \mathbb{R}$

$$\delta_\mathcal{A} S_\Gamma[\Sigma, \mathcal{A}] = \int_0^1 \mu_a(p, q) \delta \mathcal{A}_j^a(u(s)) \frac{\mathrm{d}u^j}{\mathrm{d}s} \mathrm{d}s. \tag{2.33}$$

In order for this to vanish for arbitrary variations $\delta \mathcal{A}_j^a$ we must have

$$\mu_a(p, q) = 0, \tag{2.34}$$

which is the expected constraint equation $\mu = 0$.

Next, consider the variation of the action (2.24) with respect to the map $\Sigma : \mathbb{R}^n \to T^*M \times \mathbb{R}^n$. Under an arbitrary variation $\delta \Sigma(u) = \left(\delta p_\mu(u), \delta q^\mu(u), \delta t^i(u)\right)$ satisfying the boundary conditions $0 = \lim_{\|u\| \to \infty} \delta q^\mu(u) = \lim_{\|u\| \to \infty} \delta t^i(u)$, the action varies as

$$\delta_\Sigma S_\Gamma[\Sigma, \mathcal{A}] = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\left[p_\mu \delta q^\mu - H_i \delta t^i\right] \mathrm{d}s + \int_0^1 \left[\left(\frac{\partial q^\mu}{\partial u^j} - \frac{\partial t^i}{\partial u^j}\frac{\partial H_i}{\partial p_\mu} + \frac{\partial \mu_a}{\partial p_\mu}\mathcal{A}_j^a\right)\delta p_\mu \right.$$
$$\left. - \left(\frac{\partial p_\mu}{\partial u^j} + \frac{\partial t^i}{\partial u^j}\frac{\partial H_i}{\partial q^\mu} - \frac{\partial \mu_a}{\partial q^\mu}\mathcal{A}_j^a\right)\delta q^\mu + \left(\frac{\partial H_i}{\partial p_\mu}\frac{\partial p_\mu}{\partial u^j} + \frac{\partial H_i}{\partial q^\mu}\frac{\partial q^\mu}{\partial u^j}\right)\delta t^i\right]\frac{\mathrm{d}u^j}{\mathrm{d}s}\mathrm{d}s. \tag{2.35}$$

The total derivative term on the right-hand side of this equation vanishes due to the boundary conditions on $\delta \Sigma$ and the boundary condition $\lim_{s \to 0,1} \|\Gamma(s)\| = \infty$ on Γ. We now want the above variation to vanish for all $\delta \Sigma$ and all curves Γ, and hence for arbitrary $\frac{\mathrm{d}u^j}{\mathrm{d}s}$, which is equivalent to the system of equations

$$\frac{\partial q^\mu}{\partial u^j} = \frac{\partial t^i}{\partial u^j}\frac{\partial H_i}{\partial p_\mu} - \frac{\partial \mu_a}{\partial p_\mu}\mathcal{A}_j^a, \tag{2.36a}$$

$$\frac{\partial p_\mu}{\partial u^j} = -\frac{\partial t^i}{\partial u^j}\frac{\partial H_i}{\partial q^\mu} + \frac{\partial \mu_a}{\partial q^\mu}\mathcal{A}_j^a, \tag{2.36b}$$

$$\frac{\partial H_i}{\partial p_\mu}\frac{\partial p_\mu}{\partial u^j} + \frac{\partial H_i}{\partial q^\mu}\frac{\partial q^\mu}{\partial u^j} = 0. \tag{2.36c}$$

Next, we claim that these equations of motion imply that the Jacobian matrix $\left(\frac{\partial t^j}{\partial u^j}\right)$ must be invertible, thus ensuring that the composition $\pi_{\mathbb{R}^n} \circ \Sigma : \mathbb{R}^n \to \mathbb{R}^n$ is a (local) diffeomorphism. To show this we adapt the similar argument [13, §2.2] to the present case with gauge field. The differential of Σ at $u \in \mathbb{R}^n$ reads

$$\mathrm{d}_u \Sigma : T_u \mathbb{R}^n \to T_{\Sigma(u)}(T^*M \times \mathbb{R}^n), \quad \frac{\partial}{\partial u^i} \mapsto \frac{\partial p_\mu}{\partial u^i}\frac{\partial}{\partial p_\mu} + \frac{\partial q^\mu}{\partial u^i}\frac{\partial}{\partial q^\mu} + \frac{\partial t^j}{\partial u^i}\frac{\partial}{\partial t^j}, \tag{2.37}$$

and therefore proving the claim is equivalent to showing that the tangent map of the composition $\pi_{\mathbb{R}^n} \circ \Sigma : \mathbb{R}^n \to \mathbb{R}^n, u^j \mapsto t^i(u)$ at every $u \in \mathbb{R}^n$, namely

$$\mathrm{d}_u(\pi_{\mathbb{R}^n} \circ \Sigma) = \mathrm{d}_{\Sigma(u)}\pi_{\mathbb{R}^n} \circ \mathrm{d}_u \Sigma : T_u \mathbb{R}^n \to T_{\pi_{\mathbb{R}^n}(\Sigma(u))}\mathbb{R}^n, \quad \frac{\partial}{\partial u^i} \mapsto \frac{\partial t^j}{\partial u^i}\frac{\partial}{\partial t^j}, \tag{2.38}$$

is injective when the univariational equations (2.36a)–(2.36c) hold. And to do this we will use the fact that the latter can equivalently be written as

$$\mathrm{d}_u \Sigma\left(\frac{\partial}{\partial u^i}\right) \lrcorner \mathrm{d}\mathcal{L} - \mathcal{A}_u^\sharp\left(\frac{\partial}{\partial u^i}\right) \lrcorner \omega = 0 \quad \forall i, \tag{2.39}$$

in which

$$\mathrm{d}\mathscr{L} = \omega - \mathrm{d}H_i \wedge \mathrm{d}t^i = \mathrm{d}p_\mu \wedge \mathrm{d}q^\mu - \frac{\partial H_i}{\partial p_\mu}\mathrm{d}p_\mu \wedge \mathrm{d}t^i - \frac{\partial H_i}{\partial q^\mu}\mathrm{d}q^\mu \wedge \mathrm{d}t^i. \quad (2.40)$$

So let $V \in \ker\left(\mathrm{d}_{\Sigma(u)}\pi_{\mathbb{R}^n} \circ \mathrm{d}_u\Sigma\right)$. On the one hand, (2.39) and (2.40) imply

$$0 = (\mathrm{d}_u\Sigma - \mathcal{A}_u^\sharp)(V)\lrcorner\omega - [\mathrm{d}_u\Sigma(V)\lrcorner\mathrm{d}H_i]\mathrm{d}t^i + [\mathrm{d}_u\Sigma(V)\lrcorner\mathrm{d}t^i]\mathrm{d}H_i. \quad (2.41)$$

On the other hand, $\mathrm{d}_u\Sigma(V)\lrcorner\mathrm{d}t^i = (\mathrm{d}_{\Sigma(u)}\pi_{\mathbb{R}^n} \circ \mathrm{d}_u\Sigma)(V)\lrcorner\mathrm{d}u^i = 0$ since $V \in \ker\left(\mathrm{d}_{\Sigma(u)}\pi_{\mathbb{R}^n} \circ \mathrm{d}_u\Sigma\right)$. Thus we are left with

$$(\mathrm{d}_u\Sigma - \mathcal{A}_u^\sharp)(V)\lrcorner\omega - [\mathrm{d}_u\Sigma(V)\lrcorner\mathrm{d}H_i]\mathrm{d}t^i = 0 \quad (2.42)$$

and each term must individually vanish since the first belongs to $T^*(T^*M)$ and the second belongs to $T^*\mathbb{R}^n$. In particular, since ω is nondegenerate, we must have $(\mathrm{d}_u\Sigma - \mathcal{A}_u^\sharp)(V) = 0$, from which it follows that $V = 0$ since $\mathrm{d}_u\Sigma - \mathcal{A}_u^\sharp$ is injective. Note that since $\mathcal{A}_u^\sharp\left(\frac{\partial}{\partial u^i}\right)\lrcorner\mathrm{d}H_k = 0$ (invariant Hamiltonians), (2.39) can be compactly written using the gauge-covariant derivative (2.28)

$$D_u^{\mathcal{A}}\Sigma\left(\frac{\partial}{\partial u^i}\right)\lrcorner\mathrm{d}\mathscr{L} = 0 \quad \forall i. \quad (2.43)$$

As a consequence, performing a suitable coordinate transformation on $\mathbb{R}^n$, we may bring Σ to the canonical form $(f_\Sigma, \mathrm{id}_{\mathbb{R}^n})$ for some function $f_\Sigma : \mathbb{R}^n \to T^*M$, $t^i \mapsto (p_\mu(t), q^\mu(t))$. We may then rewrite (2.36) as

$$\frac{\partial q^\mu}{\partial t^i} = \frac{\partial H_i}{\partial p_\mu} - \frac{\partial \mu_a}{\partial p_\mu}\widetilde{\mathcal{A}}_i^a, \quad (2.44\mathrm{a})$$

$$\frac{\partial p_\mu}{\partial t^i} = -\frac{\partial H_i}{\partial q^\mu} + \frac{\partial \mu_a}{\partial q^\mu}\widetilde{\mathcal{A}}_i^a, \quad (2.44\mathrm{b})$$

$$\frac{\partial H_i}{\partial p_\mu}\frac{\partial p_\mu}{\partial t^j} + \frac{\partial H_i}{\partial q^\mu}\frac{\partial q^\mu}{\partial t^j} = 0, \quad (2.44\mathrm{c})$$

where $\widetilde{\mathcal{A}}_i^a := \frac{\partial u^j}{\partial t^i}\mathcal{A}_j^a$ is defined using the inverse $\left(\frac{\partial u^j}{\partial t^i}\right)$ of the matrix $\left(\frac{\partial t^i}{\partial u^j}\right)$. Inserting the expression (2.17) for the moment map into (2.44a) and (2.44b) gives (2.31b) and (2.31c). To obtain (2.31d) we substitute (2.44a) and (2.44b) into the left-hand side of (2.44c), yielding

$$\{H_i, H_j\} = \{H_i, \mu_a\}\widetilde{\mathcal{A}}_j^a.$$

The right-hand side vanishes using Proposition 2.2 since $\{H_i, \mu_a\} = \mathcal{L}_{X_a^\sharp}H_i = 0$, where the first equality follows using the definitions (2.12), (2.1) and (2.17). It follows that $\{H_i, H_j\} = 0$, which completes the derivation of the Euler–Lagrange equations (2.31).

Finally, relation (2.32) is a consequence of the vanishing of the Lie bracket of the coordinate vector fields on $\mathbb{R}^n$ pushed forward by the map Σ and restricted to the solution space. More explicitly, under the map Σ a coordinate vector field $\frac{\partial}{\partial u^j}$ is sent to the vector

field $Z_j = \frac{\partial q^\mu}{\partial u^j}\frac{\partial}{\partial q^\mu} + \frac{\partial p_\mu}{\partial u^j}\frac{\partial}{\partial p_\mu} + \frac{\partial t^k}{\partial u^j}\frac{\partial}{\partial t^k}$ on $T^*M \times \mathbb{R}^n$ (see (2.37)) whose restriction to the solution space reads, recalling (2.1) and (2.12),

$$Z_j = \left(\frac{\partial t^i}{\partial u^j}\frac{\partial H_i}{\partial p_\mu} - \frac{\partial \mu_a}{\partial p_\mu}\mathcal{A}_j^a\right)\frac{\partial}{\partial q^\mu} + \left(-\frac{\partial t^i}{\partial u^j}\frac{\partial H_i}{\partial q^\mu} + \frac{\partial \mu_a}{\partial q^\mu}\mathcal{A}_j^a\right)\frac{\partial}{\partial p_\mu} + \frac{\partial t^i}{\partial u^j}\frac{\partial}{\partial t^i}$$

$$= \frac{\partial t^i}{\partial u^j}\left(-\mathcal{X}_{H_i} + \widetilde{\mathcal{A}}_i^a X_a^\sharp + \frac{\partial}{\partial t^i}\right). \tag{2.45}$$

Now $\left[\frac{\partial}{\partial u^j}, \frac{\partial}{\partial u^k}\right] = 0$ implies $[Z_j, Z_k] = 0$ which in turn yields

$$0 = \left[\mathcal{X}_{H_i} - \widetilde{\mathcal{A}}_i^a X_a^\sharp - \frac{\partial}{\partial t^i}, \mathcal{X}_{H_j} - \widetilde{\mathcal{A}}_j^a X_a^\sharp - \frac{\partial}{\partial t^j}\right]$$

$$= [\mathcal{X}_{H_i}, \mathcal{X}_{H_j}] - \widetilde{\mathcal{A}}_i^a[X_a^\sharp, \mathcal{X}_{H_j}] + \widetilde{\mathcal{A}}_j^a[X_a^\sharp, \mathcal{X}_{H_i}] + \left(\frac{\partial}{\partial t^i}\widetilde{\mathcal{A}}_j^a - \frac{\partial}{\partial t^j}\widetilde{\mathcal{A}}_i^a + f_{bc}{}^a \widetilde{\mathcal{A}}_i^b \widetilde{\mathcal{A}}_j^c\right) X_a^\sharp$$

$$= [\mathcal{X}_{H_i}, \mathcal{X}_{H_j}] - \widetilde{\mathcal{A}}_i^a[X_a^\sharp, \mathcal{X}_{H_j}] + \widetilde{\mathcal{A}}_j^a[X_a^\sharp, \mathcal{X}_{H_i}] + F_{ij}^a X_a^\sharp \tag{2.46}$$

where F_{ij}^a are the components of the curvature of $\mathcal{A}$ defined in the statement of the theorem. The second and third terms in (2.46) vanish because $[X_a^\sharp, \mathcal{X}_{H_i}] = \mathcal{X}_{X_a^\sharp H_i}$ and $X_a^\sharp H_i = \mathcal{L}_{X_a^\sharp} H_i = 0$, where the last step uses the condition (2.23) from Proposition 2.3 since we are assuming that $S_\Gamma[\Sigma, \mathcal{A}]$ is gauge invariant. The first term in (2.46) vanishes because $[\mathcal{X}_{H_i}, \mathcal{X}_{H_j}] = \mathcal{X}_{\{H_i, H_j\}} = 0$ using the fact that $\{H_i, H_j\} = 0$, as we have already established. Therefore $F_{ij}^a X_a^\sharp = 0$. Since the action of G is free, the tangent vectors $X_a^\sharp$ are linearly independent at each point (p_μ, q^μ) of T^*M, and so we deduce the relation (2.32), as required. $\qquad\square$

The equation (2.31a) together with (2.31b)-(2.31c) for any $i = 1, \ldots, n$ represent Hamiltonian flow equations for a constrained Hamiltonian system. Indeed, the effect of the first equation (2.31a) is to impose the set of constraints $\mu^a(p, q) = 0$ on the phase space T^*M, restricting the dynamics to the submanifold $\mu^{-1}(0) \subset T^*M$. On the other hand, the equations (2.31b)–(2.31c) written in the form (2.44a)–(2.44b) describe, for each $i = 1, \ldots, n$, the flow equation for the time-dependent Hamiltonian

$$\widetilde{H}_i(p, q, t) := H_i(p, q) - \mu_a(p, q)\widetilde{\mathcal{A}}_i^a(t). \tag{2.47}$$

This flow is a linear combination of the Hamiltonian flow $\mathcal{X}_{H_i}$ of the unconstrained system and an arbitrary time-dependent linear combination of the flows $\mathcal{X}_{\mu_a}$ for $a = 1, \ldots, \dim \mathfrak{g}$ along the orbits of the G-action, which implements the quotienting by G to $\mu^{-1}(0)/G$. The last equation (2.31d) encodes the Poisson commutativity of the flows for different $i = 1, \ldots, n$ and was obtained in our variational setting (it relates to the so-called closure relation of Lagrangian multiform theory, see e.g. [12] for a detailed exposition).

3. 3d Mixed BF Theory and Hitchin's System

We will now generalise the construction of Sect. 2 to an infinite-dimensional context, where the role of the underlying manifold M in Sect. 2 will be played here by the space

$\mathcal{M}$ of holomorphic structures on a principle G-bundle $\mathcal{P} \to C$ over a compact Riemann surface C, for some connected Lie group G, and the role of the symmetry group G in Sect. 2 will be played here by the group $\mathcal{G} = \text{Aut } \mathcal{P}$ of (fibre-preserving) automorphisms of $\mathcal{P}$.

The cotangent bundle $T^*\mathcal{M}$ equipped with its free action of $\mathcal{G}$ is the well-known setup for the famous Hitchin system [41] and by selecting the Hamiltonians of this system as our invariant functions H_i on $T^*\mathcal{M}$, we will arrive in Sect. 3.2 at a variational description of the Hitchin system. The first main result of this section is that the Lagrangian 1-form for the Hitchin system on the symplectic quotient $\mu^{-1}(0)/\mathcal{G}$ is given by the 3d mixed BF Lagrangian 1-form with type B line defects, in the terminology of [71], associated with each of the Hitchin Hamiltonians. We then go on in Sect. 3.4 to extend the construction to the case of Hitchin systems on a compact Riemann surface C with marked points. Our second main result is to show that this leads to a 3d mixed BF Lagrangian 1-form with type B *and* type A defects, in the terminology of [71].

Since the construction of examples discussed later in Sects. 4 and 5 will crucially rely on some intricacies of the definition of $T^*\mathcal{M}$ and $\mathcal{G}$, we start by recalling the relevant details in Sects. 3.1.1 and 3.1.2, thereby also setting our conventions.

3.1. Geometric Setup for the Hitchin System. Let G be a complex connected Lie group with Lie algebra $\mathfrak{g}$. Let C be a compact Riemann surface and fix a holomorphic atlas $\{(U_I, z_I)\}_{I \in \mathcal{I}}$ of C with $z_I : U_I \to \mathbb{C}$ local holomorphic coordinates on each open subset $U_I \subset C$ and $\mathcal{I}$ some indexing set.

3.1.1. Holomorphic Structures on a Principal G-Bundle We fix a smooth principal G-bundle $\pi : \mathcal{P} \to C$ which is specified relative to the open cover $\{U_I\}$ of C by local trivialisations $\psi_I : \pi^{-1}(U_I) \xrightarrow{\cong} U_I \times G$, $p \mapsto (\pi(p), f_I(p))$. The principal bundle is equipped with a free right action $G \times \mathcal{P} \to \mathcal{P}$, $p \mapsto p \cdot g$ and the local trivialisations should be G-equivariant, i.e. $f_I(p \cdot g) = f_I(p)g$ for any $g \in G$. The transition between local trivialisations ψ_J and ψ_I on overlapping charts $U_I \cap U_J \neq \emptyset$ is given by

$$\psi_I \circ \psi_J^{-1} : (U_I \cap U_J) \times G \longrightarrow (U_I \cap U_J) \times G, \quad (x, g) \longmapsto \big(x, g_{IJ}(x)g\big)$$

with smooth transition functions $g_{IJ} : U_I \cap U_J \to G$, given by $g_{IJ}(x) = f_I(p)f_J(p)^{-1}$ for any $p \in \mathcal{P}$ with $\pi(p) = x \in U_I \cap U_J$, satisfying the Čech cocycle condition $g_{IJ}g_{JK} = g_{IK}$ on triple overlaps $U_I \cap U_J \cap U_K \neq \emptyset$.

A *change of local trivialisation* of $\mathcal{P}$ is specified by a family of smooth maps $h_I : U_I \to G$, i.e. a Čech 0-cochain $h = (h_I)_{I \in \mathcal{I}} \in \check{C}^0(C, G)$. Indeed, given local trivialisations $\psi_I : \pi^{-1}(U_I) \xrightarrow{\cong} U_I \times G$, $p \mapsto (\pi(p), f_I(p))$ we can define new local trivialisations by

$$\tilde{\psi}_I : \pi^{-1}(U_I) \xrightarrow{\cong} U_I \times G, \quad p \longmapsto \big(\pi(p), h_I(\pi(p))f_I(p)\big). \tag{3.1}$$

The transition functions of $\mathcal{P}$ relative to these new local trivialisations are the smooth maps

$$\tilde{g}_{IJ} = h_I g_{IJ} h_J^{-1} : U_I \cap U_J \to G. \tag{3.2}$$

Said differently, the change of local trivialisations from $\{\psi_I\}_{I \in \mathcal{I}}$ to $\{\tilde{\psi}_I\}_{I \in \mathcal{I}}$ on the fixed bundle $\mathcal{P}$ can be seen as producing a new principal G-bundle $\tilde{\mathcal{P}} \to C$ that is smoothly isomorphic to $\mathcal{P}$.

An automorphism of $\mathcal{P}$, or more precisely a fibre-preserving automorphism of $\mathcal{P}$ which we will sometimes refer to as a *gauge transformation*, is a Čech 0-cochain $g = (g_I)_{I \in \mathcal{I}} \in \check{C}^0(C, G)$ which preserves the transition functions of P in the sense that

$$g_I = g_{IJ} g_J g_{IJ}^{-1} \tag{3.3}$$

on any overlap $U_I \cap U_J \neq \emptyset$. We can describe the action of g on $\mathcal{P}$ relative to a fixed choice of local trivialisations $\{\psi_I\}_{I \in \mathcal{I}}$ as sending $\psi_I(p) = (\pi(p), f_I(p))$ to $\psi_I(g \cdot p) := \big(\pi(p), g_I(\pi(p)) f_I(p)\big)$. The compatibility condition (3.3) ensures that this is well-defined on $\mathcal{P}$, in the sense that we can either perform the gauge transformation directly in the local trivialisation ψ_I or we can first move to the local trivialisation ψ_J, perform the gauge transformation there and then move back to the local trivialisation ψ_I. Both give the same result. We let

$$\mathcal{G} := \mathrm{Aut}\, \mathcal{P} \subset \check{C}^0(C, G) \tag{3.4}$$

denote the infinite-dimensional group of automorphisms of the principal G-bundle $\mathcal{P}$. Note that, by the condition (3.3), we can equally describe automorphisms of $\mathcal{P}$ as sections of the fibre bundle $\mathcal{P} \times_{\mathrm{Ad}} G$ associated with the adjoint representation of G on itself.

A *holomorphic structure* on $\mathcal{P}$ is a choice of local trivialisations $\{\psi_I\}_{I \in \mathcal{I}}$ with respect to which the transition functions $g_{IJ} : U_I \cap U_J \to G$ are holomorphic. It can equally be described [2] as a family of $\mathfrak{g}$-valued $(0, 1)$-forms $A''_{\bar{z}_I}(z_I, \bar{z}_I) \mathrm{d}\bar{z}_I \in \Omega^{0,1}(U_I, \mathfrak{g})$ relative to a choice of local trivialisations $\{\psi_I\}_{I \in \mathcal{I}}$, denoted collectively as A'', such that

$$A''_{\bar{z}_I} \mathrm{d}\bar{z}_I = g_{IJ} A''_{\bar{z}_J} g_{IJ}^{-1} \mathrm{d}\bar{z}_J - \bar{\partial} g_{IJ} g_{IJ}^{-1}. \tag{3.5}$$

on $U_I \cap U_J \neq \emptyset$. We let $\mathcal{M}$ denote the infinite-dimensional space of holomorphic structures on $\mathcal{P}$.

Under a change of local trivialisations $h \in \check{C}^0(C, G)$, the holomorphic structure A'' is described in the new local trivialisations (3.1) by the family of $\mathfrak{g}$-valued $(0, 1)$-forms

$$\tilde{A}''_{\bar{z}_I} \mathrm{d}\bar{z}_I = h_I A''_{\bar{z}_I} h_I^{-1} \mathrm{d}\bar{z}_I - \bar{\partial} h_I h_I^{-1} \in \Omega^{0,1}(U_I, \mathfrak{g}). \tag{3.6}$$

In particular, by solving the equations $A''_{\bar{z}_I} = h_I^{-1} \partial h_I / \partial \bar{z}_I$, which is always possible locally [2, Section 5], we obtain smooth maps $h_I : U_I \to G$ which define a new local trivialisation where

$$\tilde{A}''_{\bar{z}_I}(z_I, \bar{z}_I) = 0. \tag{3.7}$$

This represents the same holomorphic structure A'' of $\mathcal{P}$ but now in an adapted local trivialisation of $\mathcal{P}$ where its components vanish. In particular, it now follows from (3.5) that in this new local trivialisation the transition functions $\tilde{g}_{IJ} : U_I \cap U_J \to G$ of the bundle are holomorphic.

Under a gauge transformation by $g \in \mathcal{G}$, a holomorphic structure $A'' \in \mathcal{M}$ is transformed to a new holomorphic structure on $\mathcal{P}$ given by the family of $\mathfrak{g}$-valued $(0, 1)$-forms

$$^{g_I} A''_{\bar{z}_I} \mathrm{d}\bar{z}_I := g_I A''_{\bar{z}_I} g_I^{-1} \mathrm{d}\bar{z}_I - \bar{\partial} g_I g_I^{-1} \in \Omega^{0,1}(U_I, \mathfrak{g}). \tag{3.8}$$

Let $^g A'' := g A'' g^{-1} - \bar{\partial} g g^{-1} \in \mathcal{M}$ denote this transformed holomorphic structure. We have a left action of $\mathcal{G}$ on $\mathcal{M}$ given by

$$\mathcal{G} \times \mathcal{M} \longrightarrow \mathcal{M}, \qquad (g, A'') \longmapsto g \cdot A'' := {}^g A'' = g A'' g^{-1} - \bar{\partial} g g^{-1}. \tag{3.9}$$

Any $A'' \in \mathcal{M}$ determines a Dolbeault operator $\bar{\partial}^{A''}$ on any vector bundle $V_{\mathcal{P}} := \mathcal{P} \times_\rho V$ associated with $\mathcal{P}$ in some representation $\rho : G \to \mathrm{Aut}\, V$, which acts on local sections over U_I as $\bar{\partial} + \rho(A''_{\bar{z}_I})\mathrm{d}\bar{z}_I$. In terms of Dolbeault operators, the left action (3.9) reads $(g, \bar{\partial}^{A''}) \mapsto g\bar{\partial}^{A''}g^{-1}$.

The Lie algebra $\mathfrak{G} := \mathrm{Lie}(\mathcal{G})$ of $\mathcal{G}$ consists of sections X of the vector bundle $\mathfrak{g}_{\mathcal{P}} = \mathcal{P} \times_{\mathrm{ad}} \mathfrak{g}$ associated with $\mathcal{P}$ in the adjoint representation. Explicitly, this is given by a family of $\mathfrak{g}$-valued functions $X^I \in C^\infty(U_I, \mathfrak{g})$ in each local trivialisation such that on each overlap $U_I \cap U_J \neq \emptyset$ we have the relation $X^I = g_{IJ}X^J g_{IJ}^{-1}$. The left action (3.9) of the group $\mathcal{G}$ induces an infinitesimal left action of a Lie algebra element $X \in \mathfrak{G}$ on $A'' \in \mathcal{M}$ given by

$$\delta_X A'' = -\bar{\partial}^{A''}X = -\bar{\partial}X - [A'', X], \qquad (3.10)$$

or in local trivialisations by $\delta_{X^I}A''^I = -\bar{\partial}X^I - [A''^I, X^I]$.

3.1.2. Cotangent Bundle $T^*\mathcal{M}$ and Action of $\mathcal{G}$

The tangent space $T_{A''}\mathcal{M}$ at any point $A'' \in \mathcal{M}$ is given by the space of sections of the bundle $\bigwedge^{0,1} C \otimes \mathfrak{g}_{\mathcal{P}}$. Likewise, the cotangent space $T^*_{A''}\mathcal{M}$ is the space of sections of $\bigwedge^{1,0} C \otimes \mathfrak{g}^*_{\mathcal{P}}$, where $\mathfrak{g}^*_{\mathcal{P}} := \mathcal{P} \times_{\mathrm{ad}^*} \mathfrak{g}^*$ is the vector bundle associated with $\mathcal{P}$ in the coadjoint representation. Concretely, any $X \in T_{A''}\mathcal{M}$ is described by a family of $\mathfrak{g}$-valued $(0, 1)$-forms $X^I = X_{\bar{z}_I}(z_I, \bar{z}_I)\mathrm{d}\bar{z}_I \in \Omega^{0,1}(U_I, \mathfrak{g})$ and any $Y \in T^*_{A''}\mathcal{M}$ by a family of $\mathfrak{g}^*$-valued $(1, 0)$-form $Y^I = Y_{z_I}(z_I, \bar{z}_I)\mathrm{d}z_I \in \Omega^{1,0}(U_I, \mathfrak{g}^*)$ such that on any overlap $U_I \cap U_J \neq \emptyset$ we have the relations

$$X^I = g_{IJ}X^J g_{IJ}^{-1}, \qquad Y^I = \mathrm{Ad}^*_{g_{IJ}} Y^J, \qquad (3.11)$$

respectively. Using the canonical pairing $\langle\,,\,\rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{C}$ we obtain a family of local $(1, 1)$-forms $\langle Y^I, X^I \rangle \in \Omega^{1,1}(U_I)$, where we suppress a wedge product between the 1-forms Y^I and X^I. It follows from (3.11) that these local $(1, 1)$-forms agree on overlaps, i.e. $\langle Y^I, X^I \rangle = \langle Y^J, X^J \rangle$, and hence define a global $(1, 1)$-form on C which we denote by $\langle Y, X \rangle \in \Omega^{1,1}(C)$. In particular, we can integrate the latter over the compact Riemann surface C to obtain a pairing

$$T^*_{A''}\mathcal{M} \times T_{A''}\mathcal{M} \longrightarrow \mathbb{C}, \qquad (Y, X) \longmapsto \frac{1}{2\pi i} \int_C \langle Y, X \rangle. \qquad (3.12)$$

A point in the cotangent bundle $T^*\mathcal{M}$ is given by a pair (B, A''), with $A'' \in \mathcal{M}$ a holomorphic structure on $\mathcal{P}$ parametrising the base and B a section of $\bigwedge^{1,0} C \otimes \mathfrak{g}^*_{\mathcal{P}}$ parametrising the fibre. To describe vector fields on $T^*\mathcal{M}$ we note that we have the canonical isomorphism

$$T_{(B,A'')}(T^*\mathcal{M}) \cong T^*_{A''}\mathcal{M} \oplus T_{A''}\mathcal{M}. \qquad (3.13)$$

The differential at $(B, A'') \in T^*\mathcal{M}$ of the projection $\pi_{\mathcal{M}} : T^*\mathcal{M} \to \mathcal{M}, (B, A'') \mapsto A''$ is a linear map $\delta_{(B,A'')}\pi_{\mathcal{M}}$ from the tangent space of $T^*\mathcal{M}$ at (B, A'') to the tangent space of $\mathcal{M}$ at A''. Under the isomorphism (3.13), it is given simply by the projection onto the second summand. By a standard abuse of notation, we will identify the map $\pi_{\mathcal{M}}$ with its value A'' at a generic point (B, A'') and denote this differential by

$$\delta A'' : T_{(B,A'')}(T^*\mathcal{M}) \longrightarrow T_{A''}\mathcal{M}, \qquad (Y, X) \longmapsto X. \qquad (3.14)$$

The tautological 1-form on $T^*\mathcal{M}$ is then defined using the pairing (3.12) as

$$\alpha_{(B,A'')} := \frac{1}{2\pi i} \int_C \langle B, \delta A'' \rangle. \tag{3.15}$$

More explicitly, we can describe this as a map $\alpha_{(B,A'')} : T_{(B,A'')}(T^*\mathcal{M}) \to \mathbb{C}$ given by $\alpha_{(B,A'')}(Y, X) := \frac{1}{2\pi i} \int_C \langle B, X \rangle$. The corresponding symplectic form $\omega := \delta\alpha$ is given by

$$\omega_{(B,A'')}\big((Y_1, X_1), (Y_2, X_2)\big) = \frac{1}{2\pi i} \int_C \langle Y_1, X_2 \rangle - \frac{1}{2\pi i} \int_C \langle Y_2, X_1 \rangle. \tag{3.16}$$

Recall the left action (3.9) of the group of gauge transformations $\mathcal{G}$ on the space of holomorphic structures $\mathcal{M}$. We can lift this to an action of $\mathcal{G}$ on $T^*\mathcal{M}$ as follows. In a local trivialisation, an element $g \in \mathcal{G}$ is represented by smooth maps $g_I : U_I \to G$ and a section $B \in T^*_{A''}\mathcal{M}$ of the bundle $\bigwedge^{1,0} C \otimes \mathfrak{g}^*_{\mathcal{P}}$ is described by a family of $\mathfrak{g}^*$-valued $(1,0)$-forms $B^I \in \Omega^{1,0}(U_I, \mathfrak{g}^*)$. Since

$$\mathrm{Ad}^*_{g_I} B^I = \mathrm{Ad}^*_{g_I g_{IJ}} B^J = \mathrm{Ad}^*_{g_{IJ} g_J} B^J = \mathrm{Ad}^*_{g_{IJ}} \big(\mathrm{Ad}^*_{g_J} B^J \big), \tag{3.17}$$

we obtain a well-defined left action of $g = (g_I)_{I \in \mathcal{I}} \in \mathcal{G}$ on the fibres

$$T^*_{A''}\mathcal{M} \longrightarrow T^*_{g \cdot A''}\mathcal{M}, \qquad B \longmapsto g \cdot B := \mathrm{Ad}^*_g B \tag{3.18}$$

given explicitly in the local trivialisation over U_I by $B^I \mapsto \mathrm{Ad}^*_{g_I} B^I$. Combining this with the left action of $\mathcal{G}$ on the base $\mathcal{M}$, we obtain the desired left action of $\mathcal{G}$ on $T^*\mathcal{M}$ given by

$$\mathcal{G} \times T^*\mathcal{M} \longrightarrow T^*\mathcal{M}, \qquad \big(g, (B, A'')\big) \longmapsto g \cdot (B, A'') := \big(\mathrm{Ad}^*_g B, {}^g A'' \big) \tag{3.19}$$

This induces an infinitesimal left action of a Lie algebra element $X \in \mathfrak{G}$ on $(B, A'') \in T^*\mathcal{M}$ given by

$$\big(\delta_X B, \delta_X A''\big) := \big(\mathrm{ad}^*_X B, -\bar\partial^{A''} X\big), \tag{3.20}$$

where $\delta_X B = \mathrm{ad}^*_X B$ is given in local trivialisations by $\delta_{X^I} B^I = \mathrm{ad}^*_{X^I} B^I \in \Omega^{1,0}(U_I, \mathfrak{g}^*)$.

As in Sect. 2, we will need a notion of freeness for the action of the group $\mathcal{G}$ on $T^*\mathcal{M}$. We will say that $(B, A'') \in T^*\mathcal{M}$ is *stable* if

$$\bar\partial X + [A'', X] = 0, \quad \mathrm{ad}^*_X B = 0 \implies X = 0. \tag{3.21}$$

for all $X \in \mathfrak{G}$. This means that the stabiliser of (B, A'') is not a continuous subgroup of $\mathcal{G}$, so it is an infinitesimal version of freeness.

In the literature on Higgs bundles, the notion of a stable principal Higgs bundle has been introduced in [11]. We expect that any Higgs bundle that is stable in the sense of [11] satisfies (3.21). This was proved for semisimple irreducible reductive algebraic groups G in the case $B = 0$ in [59], Proposition 3.2. In the appendix, we prove this statement for $B \neq 0$ and $G = SL_m(\mathbb{C})$ using the Kobayashi–Hitchin correspondence of Simpson. It would be interesting to prove it for more general G.

3.2. Lagrangian 1-form on $T^*\mathcal{M}$. In the present infinite-dimensional setting, the analogue of the Lagrangian 1-form (2.2) is

$$\mathscr{L} := \alpha_{(B, A'')} - H_i(B, A'')\mathrm{d}t^i = \frac{1}{2\pi i}\int_C \langle B, \delta A''\rangle - H_i(B, A'')\mathrm{d}t^i. \quad (3.22)$$

To write down the corresponding action, i.e. the analogue of (2.9), let $\Sigma : \mathbb{R}^n \to T^*\mathcal{M} \times \mathbb{R}^n$ be an immersion. For any $u \in \mathbb{R}^n$ we write $\Sigma(u) = \big(B(u), A''(u), t(u)\big)$. It is helpful to take a moment to describe each component of $\Sigma(u)$ in more detail. Firstly, $B(u)$ describes an $\mathbb{R}^n$-dependent section of $\bigwedge^{1,0} C \otimes \mathfrak{g}^*_{\mathcal{P}}$ given in each local trivialisation of $\mathcal{P}$ over U_I by $\mathfrak{g}^*$-valued $(1, 0)$-forms $B_{z_I}(z_I, \bar{z}_I, u)\mathrm{d}z_I \in \Omega^{1,0}(U_I \times \mathbb{R}^n, \mathfrak{g}^*)$. These are related in chart overlaps $U_I \cap U_J \neq \emptyset$ by the second relation in (3.11), explicitly

$$B_{z_I}(z_I, \bar{z}_I, u)\mathrm{d}z_I = \mathrm{Ad}^*_{g_{IJ}} B_{z_J}(z_J, \bar{z}_J, u)\mathrm{d}z_J. \quad (3.23)$$

Secondly, $A''(u)$ describes an $\mathbb{R}^n$-dependent element of $\mathcal{M}$ given in each local trivialisation of $\mathcal{P}$ over U_I by $\mathfrak{g}$-valued $(0, 1)$-forms $A''_{\bar{z}_I}(z_I, \bar{z}_I, u)\mathrm{d}\bar{z}_I \in \Omega^{0,1}(U_I, \mathfrak{g})$. These are related in chart overlaps $U_I \cap U_J \neq \emptyset$ by (3.5), explicitly

$$A''_{\bar{z}_I}(z_I, \bar{z}_I, u)\mathrm{d}\bar{z}_I = g_{IJ}A''_{\bar{z}_J}(z_J, \bar{z}_J, u)g_{IJ}^{-1}\mathrm{d}\bar{z}_J - \bar{\partial}g_{IJ}g_{IJ}^{-1}. \quad (3.24)$$

Finally, $t(u)$ describes an $\mathbb{R}^n$-dependent point in $\mathbb{R}^n$ with components $t^i(u)$ for $i = 1, \ldots, n$.

Note that since the transition functions $g_{IJ} : U_I \cap U_J \to G$ of $\mathcal{P}$ obviously do not depend on the parameter $u = (u^j) \in \mathbb{R}^n$, differentiating the relation (3.24) with respect to u^j we obtain

$$\partial_{u^j}A''_{\bar{z}_I}(z_I, \bar{z}_I, u)\mathrm{d}\bar{z}_I = g_{IJ}\partial_{u^j}A''_{\bar{z}_J}(z_J, \bar{z}_J, u)g_{IJ}^{-1}\mathrm{d}\bar{z}_J. \quad (3.25)$$

Thus, for every $j = 1, \ldots, n$, the family of $\mathfrak{g}$-valued $(0, 1)$-forms $\partial_{u^j}A''_{\bar{z}_I}\mathrm{d}\bar{z}_I \in \Omega^{0,1}(U_I, \mathfrak{g})$ defines an $\mathbb{R}^n$-dependent section of $\bigwedge^{0,1} C \otimes \mathfrak{g}_{\mathcal{P}}$ which we denote by $\partial_{u^j}A''(u)$. Similarly, we denote by $\partial_{u^j}B(u)$ the family of $\mathfrak{g}^*$-valued $(1, 0)$-forms $\partial_{u^j}B_{z_I}\mathrm{d}z_I \in \Omega^{1,0}(U_I, \mathfrak{g}^*)$. Now let $\Sigma_1 : \mathbb{R}^n \to T^*\mathcal{M}, u \mapsto \big(B(u), A''(u)\big)$ be the component of the map Σ in $T^*\mathcal{M}$. Its differential at $u \in \mathbb{R}^n$ is

$$\mathrm{d}_u\Sigma_1 : T_u\mathbb{R}^n \longrightarrow T_{\Sigma_1(u)}(T^*\mathcal{M}) \cong T^*_{A''(u)}\mathcal{M} \oplus T_{A''(u)}\mathcal{M}, \qquad \frac{\partial}{\partial u^j} \longmapsto \big(\partial_{u^j}B(u), \partial_{u^j}A''(u)\big).$$

The pullback of the differential $\delta A''$ defined in (3.14) by the map Σ_1 is then given by the composition $\big(\Sigma^*(\delta A'')\big)(u) = \delta_{\Sigma_1(u)}A'' \circ \mathrm{d}_u\Sigma_1$ and hence $\Sigma^*(\delta A'') = \partial_{u^j}A''(u) \wedge \mathrm{d}u^j = -\mathrm{d}_{\mathbb{R}^n}A''(u)$. We now find that the pullback $\Sigma^*\mathscr{L}$ of the Lagrangian (3.22) by the map Σ is given by

$$\Sigma^*\mathscr{L} = \frac{1}{2\pi i}\int_C \langle B(u), \mathrm{d}_{\mathbb{R}^n}A''(u)\rangle - H_i\big(B(u), A''(u)\big)\mathrm{d}_{\mathbb{R}^n}t^i. \quad (3.26)$$

Given an arbitrary curve $\Gamma : (0, 1) \to \mathbb{R}^n, s \mapsto \big(u^j(s)\big)$ can now finally write down the analogue of the action (2.9) in the present case, which reads

$$S_\Gamma[\Sigma] = \int_0^1 (\Sigma \circ \Gamma)^*\mathscr{L}$$

$$= \int_0^1 \left(-\frac{1}{2\pi i} \int_C \langle B(u), \partial_{u^j} A''(u) \rangle - H_i \big(B(u), A''(u) \big) \frac{\partial t^i}{\partial u^j} \right) \frac{\mathrm{d} u^j}{\mathrm{d} s} \mathrm{d} s.$$

(3.27)

Remark 3.1. At this stage, a few important remarks on notations and terminology are in order to avoid confusion. Note that the kinetic part $\frac{1}{2\pi i} \int_C \langle B, \delta A'' \rangle$ of (3.22) is the direct analogue of $p_\mu \mathrm{d} q^\mu$ in (2.2). In particular, the integration over C in the present setting is the analogue of the summation over $\mu \in \{1, \ldots, m\}$ in the finite-dimensional setting of Sect. 2. This means that, although the first term in the action (3.27) involves the integral of a 3-form over $C \times (0, 1)$, from the point of view of Lagrangian multiform theory we should really regard the whole action $S_\Gamma[\Sigma]$ as the integral of a 1-form on $(0, 1)$, namely the pullback of (3.22) along $\Sigma \circ \Gamma : (0, 1) \to T^*\mathcal{M} \times \mathbb{R}^n$.

The terminology "gauge group" for the group $\mathcal{G}$ of fibre-preserving automorphisms is standard in the geometric formulation of the Hitchin system and that is why we used it here. However, it is crucial to note that, at this stage of the construction, the group $\mathcal{G}$ is the analogue of what we called the *global* symmetry group G in Sect. 2, which we will later gauge by considering transformations parametrised by maps $g : \mathbb{R}^n \to \mathcal{G}$ by analogy with the finite-dimensional setting of Sect. 2.2.1; see Sect. 3.3 for details. A crude way to say this is that we will "gauge the gauge group $\mathcal{G}$".

With this in mind, we shall now prove the analogue of Proposition 2.2 in the present infinite-dimensional context. In fact, the setting of Sect. 2.1.2 was very generic and the infinitesimal action of a Lie algebra element $X \in \mathfrak{g}$ on M was specified only implicitly through the vector fields $X^\sharp$. By contrast, in the present context, we have an explicit description of the action of the Lie algebra $\mathfrak{G}$ on $T^*\mathcal{M}$ in (3.20) and even of the action of the group $\mathcal{G}$ on $T^*\mathcal{M}$ in (3.19). We can therefore prove a stronger statement than Proposition 2.2 in the present case. We first need to lift the action of $\mathcal{G}$ to $T^*\mathcal{M} \times \mathbb{R}^n$ by letting it act trivially on $\mathbb{R}^n$, i.e. we set $g \cdot t = t$ for any $g \in \mathcal{G}$ and $t \in \mathbb{R}^n$.

Proposition 3.2. *The action* (3.27) *is invariant under the action of* $\mathcal{G}$ *on* $T^*\mathcal{M} \times \mathbb{R}^n$ *given in* (3.19) *if and only if each* H_i *for* $i = 1, \ldots, n$ *is invariant under the group action, i.e.*

$$H_i \big(\mathrm{Ad}_g^* B, {}^g A'' \big) = H_i(B, A'')$$

(3.28)

for any $(B, A'') \in T^*\mathcal{M}$ *and* $g \in \mathcal{G}$. *Moreover, the Noether charge associated with an infinitesimal bundle morphism* $X \in \mathfrak{G}$ *is given by*

$$\mu_{(B, A'')}(X) = \frac{1}{2\pi i} \int_C \langle B, \bar{\partial}^{A''} X \rangle.$$

(3.29)

Proof. We closely follow the proof of Proposition 2.2. Let $\Sigma : \mathbb{R}^n \to T^*\mathcal{M} \times \mathbb{R}^n$ be given by $\Sigma(u) = \big(B(u), A''(u), t(u) \big)$. By (3.19), its pointwise image under the left action of any $g \in \mathcal{G}$ is $g \cdot \Sigma : \mathbb{R}^n \to T^*\mathcal{M} \times \mathbb{R}^n$ given by

$$(g \cdot \Sigma)(u) = \big(\mathrm{Ad}_g^* B(u), {}^g A''(u), t(u) \big).$$

(3.30)

Note, in particular, that this implies $\partial_{u^j}(g \cdot \Sigma)(u) = \big(\mathrm{Ad}_g^* \partial_{u^j} B(u), g \partial_{u^j} A''(u) g^{-1}, \partial_{u^j} t(u) \big)$. The action for the transformed map $g \cdot \Sigma$ therefore reads

$$S_\Gamma[g \cdot \Sigma] = \int_0^1 \left(-\frac{1}{2\pi i} \int_C \langle \mathrm{Ad}_g^* B(u), g \partial_{u^j} A''(u) g^{-1} \rangle - H_i \big(\mathrm{Ad}_g^* B(u), {}^g A''(u) \big) \frac{\partial t^i}{\partial u^j} \right) \frac{\mathrm{d} u^j}{\mathrm{d} s} \mathrm{d} s$$

$$= \int_0^1 \left(-\frac{1}{2\pi i} \int_C \langle B(u), \partial_{u^j} A''(u) \rangle - H_i \left(\mathrm{Ad}_g^* B(u), {}^g A''(u) \right) \frac{\partial t^i}{\partial u^j} \right) \frac{\mathrm{d}u^j}{\mathrm{d}s} \mathrm{d}s$$

$$= S_\Gamma[\Sigma] + \int_0^1 \left(H_i \left(B(u), A''(u) \right) - H_i \left(\mathrm{Ad}_g^* B(u), {}^g A''(u) \right) \right) \frac{\mathrm{d}t^i}{\mathrm{d}s} \mathrm{d}s$$

The result now follows since for $g \in \mathcal{G}$ to be a symmetry means that $S_\Gamma[g \cdot \Sigma] = S_\Gamma[\Sigma]$ for any map Σ and any curve Γ and hence the integral on the right-hand side must vanish for any curve Γ but this, in turn, is equivalent to the condition (3.28).

To work out the Noether charge associated with the infinitesimal symmetry generated by a Lie algebra element $X \in \mathfrak{G}$, we introduce an arbitrary smooth function $\lambda : \mathbb{R}^n \to \mathbb{C}$ and consider now the pointwise variations of the map $\Sigma(u) = \left(B(u), A''(u), t(u) \right)$ given by

$$\delta_{\lambda(u)X} B(u) = \lambda(u) \mathrm{ad}_X^* B(u),$$

$$\delta_{\lambda(u)X} A''(u) = -\lambda(u) \bar\partial^{A''(u)} X = -\lambda(u) \left(\bar\partial X + [A''(u), X] \right).$$

The variation of the action (3.27) then only has a contribution from the kinetic term, which reads

$$\delta_{\lambda(u)X} S_\Gamma[\Sigma] = \delta_{\lambda(u)X} \int_0^1 \frac{-1}{2\pi i} \int_C \langle B(u), \partial_{u^j} A''(u) \rangle \frac{\mathrm{d}u^j}{\mathrm{d}s} \mathrm{d}s$$

$$= \int_0^1 \partial_{u^j} \lambda(u) \left(\frac{1}{2\pi i} \int_C \langle B(u), \bar\partial^{A''(u)} X \rangle \right) \frac{\mathrm{d}u^j}{\mathrm{d}s} \mathrm{d}s,$$

cf. the end of the proof of Propostion 2.2. From this we read off the desired expression (3.29). □

Remark 3.3. We make a trivial but important comment concerning the expression (3.29), to help avoid potential confusion later. Note that this expression is well-defined since $\bar\partial^{A''} X$ is a section of $\bigwedge^{0,1} C \otimes \mathfrak{g}_\mathcal{P}$. Indeed, using the relations (3.5) and $X^I = g_{IJ} X^J g_{IJ}^{-1}$, respectively, between the local expressions of $A'' \in \mathcal{M}$ and $X \in \mathfrak{G}$ in overlapping charts $U^I \cap U^J \neq \emptyset$, we find that

$$\bar\partial X^I + \left[A''_{\bar z_I}, X^I \right] \mathrm{d}\bar z_I = g_{IJ} \left(\bar\partial X_J + \left[A''_{\bar z_J}, X_J \right] \mathrm{d}\bar z^J \right) g_{IJ}^{-1}.$$

We thus have $\langle B^I, \bar\partial^{A''} X^I \rangle = \langle B^J, \bar\partial^{A''} X^J \rangle$ on overlaps $U_I \cap U_J \neq \emptyset$ so that these define a global $(1,1)$-form $\langle B, \bar\partial^{A''} X \rangle \in \Omega^{1,1}(C)$ which can be integrated over the compact Riemann surface C. Likewise, we have a well-defined global $(1,1)$-form $\langle \bar\partial^{A''} B, X \rangle \in \Omega^{1,1}(C)$ given by the expression $\langle \bar\partial^{A''} B^I, X^I \rangle$ in each local chart U_I. Moreover, these are related by

$$\langle B, \bar\partial^{A''} X \rangle - \langle \bar\partial^{A''} B, X \rangle = -\mathrm{d}_C \langle B, X \rangle \tag{3.31}$$

where $\langle B, X \rangle \in \Omega^{1,0}(C)$ is a well-defined $(1,0)$-form on C. The relative sign on the left-hand side comes from the fact that B is a 1-form and the operator $\bar\partial^{A''}$ has cohomological degree 1. Integrating both sides over C and using Stokes's theorem on the right-hand side, noting that C has no boundary, we deduce that the Noether charge (3.29) can equivalently be rewritten as

$$\mu_{(B, A'')}(X) = \frac{1}{2\pi i} \int_C \langle \bar\partial^{A''} B, X \rangle. \tag{3.32}$$

In other words, the value of the corresponding moment map $\mu : T^*\mathcal{M} \to \mathfrak{G}^*$ at $(B, A'') \in T^*\mathcal{M}$ is the element of $\mathfrak{G}^*$ given by the linear map $X \mapsto \frac{1}{2\pi i} \int_C (\bar{\partial}^{A''} B, X)$ which takes in any vector $X \in \mathfrak{G}$, i.e. a section of $\mathfrak{g}_{\mathcal{P}}$, and integrates it against the section $\bar{\partial}^{A''} B$ of $\bigwedge^{1,1} C \otimes \mathfrak{g}_{\mathcal{P}}^*$.

The construction of the Hamiltonians $H_i : T^*\mathcal{M} \to \mathbb{C}$ satisfying (3.28) will be inspired by that of the Hitchin map [41]. Recall that the latter is constructed from a choice of

(i) G-invariant homogeneous polynomials $P_r : \mathfrak{g}^* \to \mathbb{C}$ for $r = 1, \ldots, \text{rk } \mathfrak{g}$ of degree $d_r + 1$, where $E = \{d_r\}_{r=1}^{\text{rk } \mathfrak{g}}$ is to the set of exponents of $\mathfrak{g}$.

Given such data, the Hitchin map

$$P := (P_1, \ldots, P_{\text{rk } \mathfrak{g}}) : H^0\Big(C, \bigwedge^{1,0} C \otimes \mathfrak{g}_{\mathcal{P}}^*\Big) \longrightarrow \bigoplus_{r=1}^{\text{rk } \mathfrak{g}} H^0\Big(C, \big(\bigwedge^{1,0} C\big)^{\otimes(d_r+1)}\Big),$$

$$(3.33)$$

takes as input a holomorphic section B of the bundle $\bigwedge^{1,0} C \otimes \mathfrak{g}_{\mathcal{P}}^*$. Recall that this is given by a family of $\mathfrak{g}^*$-valued $(1, 0)$-forms $B^I = B_{z_I}(z_I)dz_I \in \Omega^{1,0}(U_I, \mathfrak{g}^*)$ in the local trivialisation over the chart (U_I, z_I), where here the function B_{z_I} depends holomorphically on z_I, satisfying the second relation in (3.11), explicitly $B^I = \text{Ad}_{g_{IJ}}^* B^J$ on $U_I \cap U_J \neq \emptyset$. Since P_r is G-invariant, it follows that we have

$$P_r(B^I) = P_r\big(\text{Ad}_{g_{IJ}}^* B^J\big) = P_r(B^J)$$

on non-trivial overlaps $U_I \cap U_J \neq \emptyset$. In this way, we obtain a holomorphic section of the bundle $(\bigwedge^{1,0} C)^{\otimes(d_r+1)}$ for each $r = 1, \ldots, \text{rk } \mathfrak{g}$, which we denote by $P_r(B)$, and the Hitchin map returns a holomorphic section of $\bigoplus_{r=1}^{\text{rk } \mathfrak{g}} (\bigwedge^{1,0} C)^{\otimes(d_r+1)}$. To obtain individual complex-valued Hamiltonians one can then expand each component $P_r(B)$ of the Hitchin map in a basis of holomorphic $(d_r + 1, 0)$-differentials on C.

However, in our present setting, $B \in T_{A''}^*\mathcal{M}$ is only a *smooth* section of $\bigwedge^{1,0} C \otimes \mathfrak{g}_{\mathcal{P}}^*$. We can still form a smooth section $P_r(B)$ of $(\bigwedge^{1,0} C)^{\otimes(d_r+1)}$ for every $r = 1, \ldots, \text{rk } \mathfrak{g}$, however we can no longer expand it in a basis of holomorphic sections of $(\bigwedge^{1,0} C)^{\otimes(d_r+1)}$. Instead, in order to produce complex-valued Hamiltonians we will proceed along the lines of [71] by introducing a marked point on C for each Hamiltonian H_i (note that in [71] only one marked point was needed since a single Hamiltonian was considered). We therefore introduce the following additional data:

(ii)   Points $\mathsf{q}_{rl} \in C$ labelled by pairs (r, l) with $r = 1, \ldots, \text{rk } \mathfrak{g}$ and $l = 1, \ldots, m_r$, where

$$m_r := \dim \Big(H^0\Big(C, \big(\bigwedge^{1,0} C\big)^{\otimes(d_r+1)}\Big)\Big) = \begin{cases} (2d_r + 1)(g - 1), & \text{for } g \geq 2, \\ g, & \text{for } g = 0, 1 \end{cases}$$

and a set of holomorphic tangent vectors $V_{\mathsf{q}_{rl}} \in T_{\mathsf{q}_{rl}}^{1,0} C$.

We can now define $H_i : T^*\mathcal{M} \to \mathbb{C}$ by evaluating the smooth section $P_r(B)$ of $(\bigwedge^{1,0} C)^{\otimes(d_r+1)}$ at $\mathsf{q}_{rl} \in C$ and pairing the resulting element $P_r(B(\mathsf{q}_{rl})) \in (\bigwedge_{\mathsf{q}_{rl}}^{1,0} C)^{\otimes(d_r+1)}$ with $V_{\mathsf{q}_{rl}}^{d_r+1} \in (T_{\mathsf{q}_{rl}}^{1,0} C)^{\otimes(d_r+1)}$, that is

$$H_i(B) := \big\langle P_r\big(B(\mathsf{q}_{rl})\big), V_{\mathsf{q}_{rl}}^{d_r+1}\big\rangle. \tag{3.34}$$

Concretely, if (U_I, z_I) is a local chart around one of the points $\mathsf{q}_{rl} \in C$ then we can pick $V_{\mathsf{q}_{rl}} = \partial_{z_I}$ and the above geometric construction amounts to writing $P_r(B) = P_r\big(B_{z_I}(z_I, \bar{z}_I)\big)dz_I^{\otimes(d_r+1)}$ locally in the coordinate z_I and then evaluating its component at $\mathsf{q}_{rl} \in \mathbb{C}$, i.e.

$$H_i(B) = P_r\big(B_{z_I}(\mathsf{q}_{rl})\big). \tag{3.35}$$

The understanding in (3.34) and (3.35) is that the label i on the Hamiltonians runs over pairs (r, l) with $r = 1, \ldots, \operatorname{rk}\mathfrak{g}$ and $l = 1, \ldots, m_r$. Note that m_r being the dimension of the space of holomorphic sections of $(\bigwedge^{1,0} C)^{\otimes(d_r+1)}$ ensures that, when the points q_{rl} are generic, the number of Hamiltonians we produce coincides with the number of Hamiltonians obtained via the construction of the Hitchin map (3.33) when the smooth section B of the bundle $\bigwedge^{1,0} C \otimes \mathfrak{g}_{\mathcal{P}}^*$ becomes holomorphic. Indeed, our evaluation prescription (3.34) defines a bijection

$$H^0\Big(C, \big(\textstyle\bigwedge^{1,0}C\big)^{\otimes(d_r+1)}\Big) \xrightarrow{\cong} \mathbb{C}^{m_r},$$

for generic points $\mathsf{q}_{rl} \in C$, $l = 1, \ldots, m_r$. It is enough to show this is injective, which follows from the fact that divisors $D_r := \sum_{l=1}^{m_r} \mathsf{q}_{rl}$ on C for which $\deg(D_r) = m_r = \dim\big(H^0\big(C, (\bigwedge^{1,0}C)^{\otimes(d_r+1)}\big)\big)$ and $\dim\big(H^0\big(C, (\bigwedge^{1,0}C)^{\otimes(d_r+1)} \otimes \mathcal{O}(-D_r)\big)\big) = 0$ are generic.

Since the label i on the Hamiltonians runs over pairs (r, l) with $r = 1, \ldots, \operatorname{rk}\mathfrak{g}$ and $l = 1, \ldots, m_r$, it runs from 1 to

$$n := \sum_{r=1}^{\operatorname{rk}\mathfrak{g}} m_r = (g-1) \sum_{r=1}^{\operatorname{rk}\mathfrak{g}}(2d_r + 1) = (g-1)\dim\mathfrak{g} \tag{3.36}$$

when $g \geq 2$ and from 1 to $n := \operatorname{rk}\mathfrak{g}$ when $g = 1$. This number coincides with half the dimension of the phase space of Hitchin's integrable system when $g \geq 2$ and $g = 1$, respectively. When $g = 0$, however, there are no Hamiltonians since the set of points $\mathsf{q}_{rl} \in C$ introduced in condition (ii) above is empty. Producing non-trivial integrable systems in the case of genus $g = 0$ will require introducing additional marked points on C which we will turn to in Sect. 3.4 below.

Label notations: From now on, for notational convenience, we will simply use the common label i for the Hamiltonians H_i, the polynomials P_r, the points q_{rl} and the times t^i associated to the Hamiltonians. This amounts to relabelling P_r as P_{rl}, with the understanding that $P_{rl} = P_{rl'}$ for any $l, l' = 1, \ldots, m_r$, so that we can write simply $H_i(B) = P_i\big(B_{z_I}(\mathsf{q}_i)\big)$. Accordingly, we can keep denoting by t^i the time associated to H_i, rather than the cumbersome t^{rl} or $t^{(r,l)}$.

3.3. Lagrangian 1-form for the Hitchin System on $\mu^{-1}(0)/\mathcal{G}$.

So far we have introduced the action $S_\Gamma[\Sigma]$ in (3.27) for an immersion $\Sigma : \mathbb{R}^n \to T^*\mathcal{M} \times \mathbb{R}^n$ and an arbitrary curve $\Gamma : (0, 1) \to \mathbb{R}^n$, and showed in Proposition 3.2 that it is invariant under the left action of the group $\mathcal{G}$ provided that the Hamiltonians H_i themselves are $\mathcal{G}$-invariant in the sense that (3.28) holds. Note that the latter condition clearly holds for the Hamiltonians introduced in (3.35) by virtue of the G-invariance of the polynomials $P_i : \mathfrak{g}^* \to \mathbb{C}$. Moreover, we identified the moment map $\mu : T^*\mathcal{M} \to \mathfrak{G}^*$ associated with this symmetry as given by (3.29).

We now generalise the gauging procedure of Sect. 2.2 to the present infinite-dimensional setting. This requires introducing two elements: gauge transformations g and a gauge field $\mathcal{A}$. We begin with the gauge transformations.

For the Hitchin system, the group by which we wish to quotient is the group $\mathcal{G}$ of automorphisms of the bundle $\mathcal{P} \to C$. In the language of Sect. 2, the group $\mathcal{G}$ corresponds to the *global* symmetry group G. Thus here, in order to gauge $\mathcal{G}$, we consider the group of local transformations $g : \mathbb{R}^n \to \mathcal{G}$, *i.e.* automorphisms of the bundle $\mathcal{P} \to C$ that depend smoothly on $u \in \mathbb{R}^n$. In a local trivialisation of $\mathcal{P}$, this is represented by G-valued functions $g_I(z_I, \bar{z}_I, u)$. On overlaps $U_I \cap U_J \neq \emptyset$ they satisfy (3.3), in which g_{IJ} are the transition functions relative to the chosen trivialisation of $\mathcal{P} \to C$.

The action of a local gauge transformation g on the map $\Sigma : u \mapsto (A''(u), B(u), t(u))$ is exactly as in Sect. 3.1, except that g, A'', B and t depend on the parameters u (in addition to depending on local coordinates z_I). Explicitly, we write

$$g \cdot \Sigma : u \longmapsto \left({}^{g(u)}A''(u), \operatorname{Ad}^*_{g(u)} B(u), t(u) \right), \tag{3.37}$$

where ${}^{g(u)}A''(u)$ and $\operatorname{Ad}^*_{g(u)} B(u)$ are defined as in (3.9) and (3.18) for each u.

Next, we introduce the gauge field $\mathcal{A} = \mathcal{A}_i \, \mathrm{d}u^i$. This consists of functions $\mathcal{A}_i$ from $\mathbb{R}^n$ to the Lie algebra $\mathfrak{G}$ of $\mathcal{G}$. The Lie algebra $\mathfrak{G}$ is the space of sections of the bundle $\mathfrak{g}_{\mathcal{P}}$, so each $\mathcal{A}_i$ is a section of $\mathfrak{g}_{\mathcal{P}}$ that depends smoothly on $u \in \mathbb{R}^n$. In any local trivialisation, $\mathcal{A}_i$ is represented by functions $\mathcal{A}_i^I(z_I, \bar{z}_I, u)$ that take values in $\mathfrak{g}$, and on the overlap $U_I \cap U_J \neq \emptyset$ these are related by

$$\mathcal{A}_i^I = g_{IJ} \mathcal{A}_i^J g_{IJ}^{-1}. \tag{3.38}$$

Under a local transformation $g : \mathbb{R}^n \to \mathcal{G}$, these transform as

$$\mathcal{A}_i^I \longmapsto g_I \mathcal{A}_i^I g_I^{-1} - \frac{\partial g_I}{\partial u^i} g_I^{-1}. \tag{3.39}$$

Having introduced the local transformations and the gauge field, we are now able to write down the analogue of the gauged action (2.21) for the Hitchin system. Recall that this consists in adding to the ungauged action (3.27) a term that couples the moment map (3.29) to the gauge field as

$$S_\Gamma[\Sigma, \mathcal{A}, t] = \int_0^1 \left(-\frac{1}{2\pi i} \int_C \left\langle B(u), \frac{\partial A''}{\partial u^j} - \bar{\partial}^{A''} \mathcal{A}_j \right\rangle - H_i \big(B(u) \big) \frac{\partial t^i}{\partial u^j} \right) \frac{\mathrm{d}u^j}{\mathrm{d}s} \mathrm{d}s. \tag{3.40}$$

We now show that the first term in this action is exactly a multiform version of the 3d mixed BF Lagrangian. To do so, let

$$A := A'' + \mathcal{A}. \tag{3.41}$$

This is a partial connection on the pullback bundle $\pi_C^* \mathcal{P} = \mathcal{P} \times \mathbb{R}^n$ over $C \times \mathbb{R}^n$ along the projection $\pi_C : C \times \mathbb{R}^n \to C$. By definition, the bundle $\pi_C^* \mathcal{P}$ is trivialised over open sets $U_I \times \mathbb{R}^n$, and the transition functions between these sets are the transition functions g_{IJ} of $\mathcal{P}$, which obviously satisfy

$$\frac{\partial g_{IJ}}{\partial u^j} = 0. \tag{3.42}$$

In these local trivialisations, A takes the form

$$A^I = A''_{\bar{z}_I}(z_I, \bar{z}_I, u)\mathrm{d}\bar{z}_I + \mathcal{A}^I_i(z_I, \bar{z}_I, u)\mathrm{d}u^i. \tag{3.43}$$

This looks like the local expression for a connection, except that it is missing a $\mathrm{d}z_I$-component, which is why we refer to it as a "partial" connection. A partial connection is a connection that can only take derivatives in certain directions (the holomorphic structure A'' of $\mathcal{P}$ is also an example of a partial connection). To show that A is a well-defined partial connection we must check that it satisfies

$$A^I = g_{IJ}A^J g_{IJ}^{-1} - \bar{\partial}g_{IJ}g_{IJ}^{-1} - \mathrm{d}_{\mathbb{R}^n}g_{IJ}g_{IJ}^{-1} \tag{3.44}$$

on overlaps $U_I \cap U_J \neq \emptyset$, and that it is independent of the choice of local trivialisations of $\mathcal{P} \to C$. Equations (3.5), (3.38) and (3.42) show that (3.44) is satisfied. Independence of the choice of local trivialisation follows from (3.44): if we use different local trivialisations of $\mathcal{P} \to C$ over open sets $\{U_I\}_{I \in \mathcal{I}'}$ to construct a connection A', then A and A' will be related as in (3.44) on the overlaps $U_I \cap U_J$ for $I \in \mathcal{I}$ and $J \in \mathcal{I}'$. Therefore A' and A represent the same connection.

The curvature F_A of the partial connection A is defined in local trivialisations by

$$F^I_A = \left(\frac{\partial \mathcal{A}^I_i}{\partial \bar{z}_I} - \frac{\partial A''_{\bar{z}_I}}{\partial u^i} + \left[A''_{\bar{z}_I}, \mathcal{A}^I_i\right]\right)\mathrm{d}\bar{z}_I \wedge \mathrm{d}u^i + \frac{1}{2}\left(\frac{\partial \mathcal{A}^I_j}{\partial u^i} - \frac{\partial \mathcal{A}^I_i}{\partial u^j} + \left[\mathcal{A}^I_i, \mathcal{A}^I_j\right]\right)\mathrm{d}u^i \wedge \mathrm{d}u^j. \tag{3.45}$$

The family $B(u)$ of sections of $\bigwedge^{1,0} C \otimes \mathfrak{g}^*_{\mathcal{P}}$ naturally determines a section of $\pi^*_C \bigwedge^{1,0} C \otimes \mathfrak{g}^*_{\pi^*_C\mathcal{P}}$, which we will denote by the same symbol B. The following theorem is a direct consequence of our definitions of B and A, with corresponding curvature F_A.

Theorem 3.4. *The gauged multiform action* (3.40) *for Hitchin's system is equivalent to the multiform action for 3d mixed BF theory on $C \times \mathbb{R}^n$ for the pair (B, A) with a type B line defect along each coordinate t^i determined by the Hitchin Hamiltonian H_i given in* (3.35)*, namely*

$$S_\Gamma[B, A, t] = \frac{1}{2\pi i}\int_{C \times \Gamma}\langle B, F_A\rangle - \int_0^1 H_i\big(B(u(s))\big)\frac{\mathrm{d}t^i}{\mathrm{d}s}\mathrm{d}s. \tag{3.46}$$

for an arbitrary curve $\Gamma : (0, 1) \to \mathbb{R}^n$, $s \mapsto u(s)$.

The proof relies on the simple observation that, when pulled back along the one-dimensional curve Γ, only the first term in the expression (3.45) of the curvature is nonzero. That being said, the significance of Theorem 3.4 is that it shows how the multiform version of 3d mixed BF theory with type B defect is derived from our procedure of gauging a natural Lagrangian multiform on a cotangent bundle, applied to the Hitchin setup. As shown below, the similar derivation with the inclusion of marked points in the Hitchin picture corresponds to the inclusion of so-called type A defects in the 3d mixed BF (multiform) picture.

In this new interpretation of the action (3.40), local (gauge) transformations $g : \mathbb{R}^n \to \mathcal{G}$ play the role of bundle automorphisms of $\pi^*_C\mathcal{P}$, with local expressions $g_I(z_I, \bar{z}_I, u)$ in the trivialisations over $U_I \times \mathbb{R}^n$. From (3.9), (3.18), (3.39) and (3.43) it follows that these bundle automorphisms act on A and B in the expected way:

$$A^I \mapsto {}^{g_I}A^I = g_I A^I g_I^{-1} - \bar{\partial}g_I g_I^{-1} - \mathrm{d}_{\mathbb{R}^n}g_I g_I^{-1}, \tag{3.47}$$

$$B^I \mapsto \mathrm{Ad}_{g_I} B^I. \tag{3.48}$$

Remark 3.5. The partial connection A on $\pi_C^* \mathcal{P}$ was defined using the local trivialisations of $\pi_C^* \mathcal{P}$ obtained canonically from the local trivialisations of $\mathcal{P}$. In particular, in such a local trivialisation, the transition functions g_{IJ} are independent of u^j and A transforms as in (3.44) but without the last term. However, the resulting action (3.46) is well-defined in any local trivialisation of $\pi_C^* \mathcal{P}$, including those for which the transitions functions do depend on u^j, in which case A transforms as in (3.44) with the last term present. Indeed, with A transforming as in (3.44), the curvature F_A defines a section of $\bigwedge^2(C \times \mathbb{R}^n) \otimes \mathfrak{g}_{\pi_C^* \mathcal{P}}$ with $\mathfrak{g}_{\pi_C^* \mathcal{P}}$ the vector bundle associated to $\pi_C^* \mathcal{P}$ in the adjoint representation and hence (B, F_A) is a well-defined global 3-form on $C \times \mathbb{R}^n$. We will exploit such a change of local trivialisation leading to u^j-dependent transition functions later on in Sect. 4.2 to prove Theorem 4.1.

To derive the equations of motion for (3.46), we first introduce some notation. Let (U_I, z_I) be a coordinate chart containing the point $\mathsf{q} \in U_I$ with coordinate $w = z_I(\mathsf{q})$. We denote by $\delta(z_I - w)$ the 2-dimensional Dirac δ-distribution on U_I at the point $\mathsf{q} \in U_I$ in the local coordinate z_I. It has the defining property

$$\int_{U_I} f(z_I)\delta(z_I - w)\mathrm{d}z_I \wedge \mathrm{d}\bar{z}_I = f(w) \tag{3.49}$$

for any function $f : U_I \to \mathbb{C}$. Note that the distribution valued 2-form $\delta(z_I - w)\mathrm{d}z_I \wedge \mathrm{d}\bar{z}_I$ is invariant under coordinate transformations. Specifically, if (U_J, z_J) is another chart with $\mathsf{q} \in U_I \cap U_J \neq \emptyset$ and letting $w' := z_J(\mathsf{q})$, then we have $\delta(z_I - w)\mathrm{d}z_I \wedge \mathrm{d}\bar{z}_I = \delta(z_J - w')\mathrm{d}z_J \wedge \mathrm{d}\bar{z}_J$. It will therefore be convenient for later to introduce the notation

$$\delta_\mathsf{q} := \delta\big(z_I - z_I(\mathsf{q})\big)\mathrm{d}z_I \wedge \mathrm{d}\bar{z}_I \tag{3.50}$$

for any $\mathsf{q} \in C$, which is independent of the coordinate chart (U_I, z_I) used, so long as it contains the point q. We will also make use of the following useful fact

$$\partial_{\bar{z}_I} \frac{1}{z_I - w} = -2\pi i \, \delta(z_I - w). \tag{3.51}$$

Given any polynomial $P : \mathfrak{g}^* \to \mathbb{C}$, its gradient $\nabla P : \mathfrak{g}^* \to \mathfrak{g}$ is defined through the first order term in the variation of $P(\phi)$ at any $\phi \in \mathfrak{g}^*$, namely

$$P(\phi + \delta\phi) = P(\phi) + \big\langle \delta\phi, \nabla P(\phi)\big\rangle + O(\delta\phi^2). \tag{3.52}$$

Note, in particular, that for a coadjoint-invariant polynomial P, i.e. such that $P(\mathrm{Ad}_g^* \phi) = P(\phi)$ for any $g \in G$, it follows from considering $g = e^{hX}$ with $X \in \mathfrak{g}$ and h small that $\big\langle \mathrm{ad}_X^* \phi, \nabla P(\phi)\big\rangle = 0$.

Theorem 3.6. *The gauged univariational principle applied to the 3d mixed BF multiform action $S_\Gamma[B, A, t]$ in (3.46) gives rise to a set of equations for the fields B, A'' and $\mathcal{A}$. By working in any chart (U_I, z_I) of C and working in terms of the components $B_{z_I}(z_I, \bar{z}_I, u)$, $A''_{z_I}(z_I, \bar{z}_I, u)$ and $\mathcal{A}_j^I(z_I, \bar{z}_I, u)$ of these fields, these equations take the following form:*

$$\partial_{\bar{z}_I}\widetilde{\mathcal{A}}_i^I - \partial_{t^i} A''_{z_I} + [A''_{z_I}, \widetilde{\mathcal{A}}_i^I] = 2\pi i \, \nabla P_i\big(B_{z_I}(\mathsf{q}_i)\big)\delta\big(z_I - z_I(\mathsf{q}_i)\big), \tag{3.53a}$$

$$\partial_{\bar{z}_I} B_{z_I} + \mathrm{ad}_{A''_{z_I}}^* B_{z_I} = 0, \tag{3.53b}$$

$$\partial_{t^j} B_{z_I} + \text{ad}^*_{\widetilde{\mathcal{A}}^I_j} B_{z_I} = 0, \tag{3.53c}$$

$$\partial_{t^j} H_i(B) = 0 \tag{3.53d}$$

where we used the invertibility of the map $(u^j) \mapsto (t^i(u))$ *to define* $\widetilde{\mathcal{A}}^I_i = \frac{\partial u^j}{\partial t^i} \mathcal{A}^I_j$. *The equations associated to any pair of overlapping charts* (U_I, z_I) *and* (U_J, z_J) *are compatible on* $U_I \cap U_J \neq \emptyset$.

Finally, if B, A'' *satisfy the stability condition* (3.21) *then the following zero-curvature equations also hold*

$$\partial_{t^i} \widetilde{\mathcal{A}}^I_j - \partial_{t^j} \widetilde{\mathcal{A}}^I_i + [\widetilde{\mathcal{A}}^I_i, \widetilde{\mathcal{A}}^I_j] = 0. \tag{3.54}$$

Proof. Since the action (3.46) is local in all the fields, and in particular in the fields B and A, to work out the equations of motion for the latter at any point $\mathsf{p} \in C$ it is sufficient to restrict the integration over C in the action to any chart (U_I, z_I) for which $\mathsf{p} \in U_I$. In other words, to work out the equations of motion for the $\mathfrak{g}$-valued fields $B^I = B_{z_I}(z_I, \bar{z}_I, u)\mathrm{d}z_I$, $A''^I = A''_{\bar{z}_I}(z_I, \bar{z}_I, u)\mathrm{d}\bar{z}_I$ and $\mathcal{A}^I = \mathcal{A}^I_j(z_I, \bar{z}_I, u)\mathrm{d}u^j$ it is sufficient to consider the variation of the following action

$$S_{I,\Gamma}[B_{z_I}, A''_{\bar{z}_I}, \mathcal{A}^I_j, t] = \int_0^1 \left(\frac{1}{2\pi i} \int_{U_I} \langle B_{z_I}(z_I, \bar{z}_I, u(s)), F^I_{\bar{z}_I j}(z_I, \bar{z}_I, u(s)) \rangle \mathrm{d}z_I \wedge \mathrm{d}\bar{z}_I \right.$$
$$\left. - \sum_{\substack{i=1 \\ \mathsf{q}_i \in U_I}}^n P_i\left(B_{z_I}(\mathsf{q}_i, u(s)) \right) \frac{\partial t^i}{\partial u^j} \right) \frac{\mathrm{d}u^j}{\mathrm{d}s} \mathrm{d}s \tag{3.55}$$

where we used the fact that the pullback of the curvature $F^I \in \Omega^2(U_I \times \mathbb{R}^n, \mathfrak{g})$ along the curve $\Gamma : (0, 1) \to \mathbb{R}^n$ is given by

$$F^I(u(s)) = F^I_{\bar{z}_I j}(z_I, \bar{z}_I, u)\, \mathrm{d}\bar{z}_I \wedge \frac{\mathrm{d}u^j}{\mathrm{d}s} \mathrm{d}s \in \Omega^2\left(U_I \times (0, 1), \mathfrak{g} \right),$$

noting that the pullback of $F^I_{ij}(z_I, \bar{z}_I, u)\mathrm{d}u^i \wedge \mathrm{d}u^j$ vanishes since $\mathrm{d}s \wedge \mathrm{d}s = 0$.

We now work out the variation of (3.55) with respect to all four fields $B_{z_I}(z_I, \bar{z}_I, u)$, $A''_{\bar{z}_I}(z_I, \bar{z}_I, u)$, $\mathcal{A}^I_j(z_I, \bar{z}_I, u)$ and $t^i(u)$. Firstly, we have

$$\delta_{B_{z_I}} S_{I,\Gamma}[B_{z_I}, A''_{\bar{z}_I}, \mathcal{A}^I_j, t]$$
$$= \int_0^1 \int_{U_I} \left(\frac{1}{2\pi i} \langle \delta B_{z_I}(z_I, \bar{z}_I, u(s)), F^I_{\bar{z}_I j}(z_I, \bar{z}_I, u(s)) \rangle \right.$$
$$- \sum_{\substack{i=1 \\ \mathsf{q}_i \in U_I}}^n \left\langle \delta B_{z_I}(z_I, \bar{z}_I, u(s)), \nabla P_i\left(B_{z_I}(z_I, \bar{z}_I, u(s)) \right) \right\rangle \delta(z_I - z_I(\mathsf{q}_i)) \frac{\partial t^i}{\partial u^j} \right)$$
$$\times \frac{\mathrm{d}u^j}{\mathrm{d}s} \mathrm{d}z_I \wedge \mathrm{d}\bar{z}_I \wedge \mathrm{d}s$$

where in the second term on the right-hand side we used both the defining properties (3.49) of the Dirac δ-distribution and (3.52) of the gradient of P_i. Since the above

variation should vanish for all curves $\Gamma : (0, 1) \to \mathbb{R}^n$, $s \mapsto \big(u^j(s)\big)$ and for all variations $\delta^I B_{z_I}(z_I, \bar{z}_I, u)$ we deduce that

$$
\partial_{\bar{z}_I} \mathcal{A}_j^I(z_I, \bar{z}_I, u(s)) - \partial_{u^j} A_{\bar{z}_I}''(z_I, \bar{z}_I, u(s)) + \big[A_{\bar{z}_I}''(z_I, \bar{z}_I, u(s)), \mathcal{A}_j^I(z_I, \bar{z}_I, u(s)) \big]
$$
$$
= 2\pi i \sum_{\substack{i=1 \\ \mathsf{q}_i \in U_I}}^n \nabla P_i\big(B_{z_I}(z_I, \bar{z}_I, u(s))\big) \delta\big(z_I - z_I(\mathsf{q}_i)\big) \frac{\partial t^i}{\partial u^j} .
$$

(3.56a)

The first equation (3.53a) now follows by multiplying both sides by $\frac{\partial u^j}{\partial t^k}$, summing over $j = 1, \ldots, n$ and then relabelling k as j, and using the definition of $\widetilde{\mathcal{A}}_j^I$ given in the statement of the theorem. Next, for the variation of (3.55) with respect to $A_{\bar{z}_I}''(z_I, \bar{z}_I, u)$ we find

$$
\delta_{A_{\bar{z}_I}''} S_{I,\Gamma}[B_{z_I}, A_{\bar{z}_I}'', \mathcal{A}_j^I, t]
$$
$$
= - \int_0^1 \frac{1}{2\pi i} \int_{U_I} \Big\langle B_{z_I}(z_I, \bar{z}_I, u(s)),
$$
$$
\partial_{u^j} \delta A_{\bar{z}_I}''(z_I, \bar{z}_I, u(s)) + \big[\mathcal{A}_j^I(z_I, \bar{z}_I, u(s)), \delta A_{\bar{z}_I}''(z_I, \bar{z}_I, u(s)) \big] \Big\rangle \frac{du^j}{ds} dz_I \wedge d\bar{z}_I \wedge ds .
$$

Since this should vanish for all curves $\Gamma : (0, 1) \to \mathbb{R}^n$, $s \mapsto \big(u^j(s)\big)$ and all variations $\delta A_{\bar{z}_I}''(z_I, \bar{z}_I, u)$ we deduce that (3.53c) must hold, again after multiplying by $\frac{\partial u^j}{\partial t^k}$, summing over $j = 1, \ldots, n$ and relabelling k as j, and also using the definition of $\widetilde{\mathcal{A}}_j^I$. And by a very similar computation, varying (3.55) with respect to $\mathcal{A}_j^I(z_I, \bar{z}_I, u)$ we deduce that (3.53b) must hold. Finally, varying (3.55) with respect to $t^i(u)$ leads to

$$
\delta_{t^i(u)} S_{I,\Gamma}\big[B_{z_I}, A_{\bar{z}_I}'', \mathcal{A}_j^I, t\big] = - \int_0^1 \left(H_i\big(B(u(s))\big) \frac{\partial \delta t^i}{\partial u^j} \right) \frac{du^j}{ds} ds
$$

and since this should vanish for all curves $\Gamma : (0, 1) \to \mathbb{R}^n$, $s \mapsto \big(u^j(s)\big)$ and all variations $\delta t^i(u)$ we deduce that (3.53d) must hold.

The derivation of (3.54) follows the same reasoning as in the proof of Theorem 2.6 for (2.32). Using (3.53c), and with $\widetilde{F}_{ij}^I = \partial_{t^i} \widetilde{\mathcal{A}}_j^I - \partial_{t^j} \widetilde{\mathcal{A}}_i^I + \big[\widetilde{\mathcal{A}}_i^I, \widetilde{\mathcal{A}}_j^I \big]$, we deduce

$$
\big[\partial_{t^i}, \partial_{t^j} \big] B_{z_I} + \mathrm{ad}_{\widetilde{F}_{ij}^I}^* B_{z_I} = 0.
$$

(3.57)

The first term on the left-hand side vanishes since partial derivatives commute and hence

$$
\mathrm{ad}_{\widetilde{F}_{ij}^I}^* B_{z_I} = 0.
$$

(3.58)

Similarly, using (3.53a), we deduce

$$
\big[\partial_{t^i}, \partial_{t^j} \big] A_{\bar{z}_I}'' + \partial_{\bar{z}_I} \widetilde{F}_{ji}^I + \big[A_{\bar{z}_I}'', \widetilde{F}_{ji}^I \big]
$$
$$
= 2\pi i \left(\partial_{t^j} \nabla P_i\big(B_{z_I}(\mathsf{q}_i)\big) + \big[\widetilde{\mathcal{A}}_j^I, \nabla P_i\big(B_{z_I}(\mathsf{q}_i)\big) \big] \right) \delta\big(z_I - z_I(\mathsf{q}_i)\big)
$$
$$
- 2\pi i \left(\partial_{t^i} \nabla P_j\big(B_{z_I}(\mathsf{q}_j)\big) + \big[\widetilde{\mathcal{A}}_i^I, \nabla P_j\big(B_{z_I}(\mathsf{q}_j)\big) \big] \right) \delta\big(z_I - z_I(\mathsf{q}_j)\big). \quad (3.59)
$$

Now note that (3.53c) implies

$$\partial_{t^j} \nabla P_i\big(B_{z_I}(\mathsf{q}_i)\big) + \Big[\widetilde{\mathcal{A}}_j^I, \nabla P_i\big(B_{z_I}(\mathsf{q}_i)\big)\Big] = 0$$

so that both terms on the right-hand side of (3.59) vanish. The first term on the left-hand side of (3.59) also vanishes since partial derivatives commute and (3.59) reduces to hence

$$\partial_{\bar{z}_I} \widetilde{F}_{ji}^I + [A_{\bar{z}_I}'', \widetilde{F}_{ji}^I] = 0 \,. \tag{3.60}$$

Equation (3.54) now follows from (3.58) and (3.60) using the stability condition (3.21). □

Remark 3.7. In standard Lagrangian multiform terminology, the first three equations (3.53a)-(3.53c) are the multitime Euler–Lagrange equations and the last equation (3.53d) is (related to) the closure relation. Note that here the latter holds automatically as a consequence of (3.53c) and G-invariance of P_i, thus confirming that we indeed have a Lagrangian multiform (1-form here) for the Hitchin system. To see this, note that

$$\partial_{t^j} P_i\big(B_{z_I}\big) = \big\langle \partial_{t^j} B_{z_I}, \nabla P_i(B_{z_I}) \big\rangle = \Big\langle \partial_{t^j} B_{z_I} + \mathrm{ad}^*_{\mathcal{A}_j^I} B_{z_I}, \nabla P_i(B_{z_I}) \Big\rangle = 0, \tag{3.61}$$

where we used the infinitesimal coadjoint-invariance of P_i in the second equality, see the argument just after (3.52), and in the last equality we have used (3.53c).

3.4. Adding Type A Defects at Marked Points on C. We will now extend the construction from Sect. 3.2 and 3.3 in order to obtain a variational principle for Hitchin's system on a Riemann surface C with $N \in \mathbb{Z}_{\geq 1}$ distinct marked points $\{\mathsf{p}_\alpha\}_{\alpha=1}^N$, in particular to allow the Higgs field $B \in T_{A''}^*\mathcal{M}$ to have simple poles at these marked points. This requires introducing extra structure at the marked points p_α which will encode the residues of B.

To each marked point p_α we will associate a coadjoint orbit $\mathcal{O}_\alpha$ in the fibre of $\mathfrak{g}_\mathcal{P}^*$ over p_α. This coadjoint orbit $\mathcal{O}_\alpha$ is determined by an element $\Lambda_\alpha \in \mathfrak{g}^*$. In a local trivialisation of $\mathcal{P}$ over $U_I \ni \mathsf{p}_\alpha$, an element $X \in \mathcal{O}_\alpha$ is represented by

$$X^I = \mathrm{Ad}^*_{\varphi_\alpha^I} \Lambda_\alpha \quad \text{for some } \varphi_\alpha^I \in G. \tag{3.62}$$

If $\mathsf{p}_\alpha \in U_I \cap U_J$ then we must have that $X^I = \mathrm{Ad}^*_{g_{IJ}(\mathsf{p}_\alpha)} X^J$, so we require that $\varphi_\alpha^I = g_{IJ}(\mathsf{p}_\alpha)\varphi_\alpha^J$. This means that φ_α^I represents an element φ_α of the fibre of $\mathcal{P}$ over p_α.

The Kostant–Kirillov symplectic form on $\mathcal{O}_\alpha$ is given by $\omega_\alpha(X, Y) := \langle \Lambda_\alpha, [X, Y] \rangle$ for every $X, Y \in \mathfrak{g}$ representing tangent vectors in $T_{\Lambda_\alpha} \mathcal{O}_\alpha$. The Kostant-Kirillov form ω_α can equally be described as follows. Consider the map

$$G \longrightarrow \mathcal{O}_\alpha, \qquad \varphi_\alpha^I \longmapsto \mathrm{Ad}^*_{\varphi_\alpha^I} \Lambda_\alpha. \tag{3.63}$$

The pullback of ω_α under this map, which by abuse of notation we also denote by ω_α, is the 2-form given by $\omega_\alpha = \mathrm{d}\big\langle \Lambda_\alpha, (\varphi_\alpha^I)^{-1}\mathrm{d}\varphi_\alpha^I \big\rangle$. Note that this is exact, and that both ω_α and $\big\langle \Lambda_\alpha, (\varphi_\alpha^I)^{-1}\mathrm{d}\varphi_\alpha^I \big\rangle$ are invariant under changes of local trivialisation. We will denote the latter by $\big\langle \Lambda_\alpha, (\varphi_\alpha)^{-1}\mathrm{d}\varphi_\alpha \big\rangle$ to emphasise this.

Instead of working on the phase space $(T^*\mathcal{M}, \omega)$ as in Sects. 3.2 and 3.3, we will now work with the extended phase space

$$\left(T^*\mathcal{M} \times \prod_{\alpha=1}^N \mathcal{O}_\alpha, \ \omega + \sum_{\alpha=1}^N \omega_\alpha\right). \tag{3.64}$$

We extend the tautological 1-form α on $T^*\mathcal{M}$, given in (3.15), to a 1-form on the extended phase space (3.64), which we still denote by α and is given explicitly by

$$\alpha_{(B,A'',(\varphi_\alpha))} := \frac{1}{2\pi i} \int_C \langle B, \delta A'' \rangle + \sum_{\alpha=1}^N \langle \Lambda_\alpha, (\varphi_\alpha)^{-1} d\varphi_\alpha \rangle. \tag{3.65}$$

To evaluate the second term, we choose (for each α) an open set U_I containing p_α, and evaluate $\langle \Lambda_\alpha, (\varphi_\alpha^I)^{-1} d\varphi_\alpha^I \rangle$, where $\varphi_\alpha^I \in G$ is the representative of φ_α in our chosen trivialisation.

The action of the group $\mathcal{G}$ of bundle automorphisms on $T^*\mathcal{M} \times \prod_{\alpha=1}^N \mathcal{O}_\alpha$ is as follows, cf. (3.19),

$$\mathcal{G} \times T^*\mathcal{M} \times \prod_{\alpha=1}^N \mathcal{O}_\alpha \longrightarrow T^*\mathcal{M} \times \prod_{\alpha=1}^N \mathcal{O}_\alpha$$

$$\left(g, \left(B, A'', (\varphi_\alpha^I)\right)\right) \longmapsto g \cdot \left(B, A'', (\varphi_\alpha^I)\right) = \left(\mathrm{Ad}_g^* B, \, {}^g A'', \left(g^I(\mathsf{p}_\alpha)\varphi_\alpha^I\right)\right). \tag{3.66}$$

This induces the following infinitesimal left action

$$\delta_X B = \mathrm{ad}_X^* B, \quad \delta_X A'' = -\bar{\partial}^{A''} X, \quad \delta_X\left(\mathrm{Ad}_{\varphi_\alpha^I}^* \Lambda_\alpha\right) = \mathrm{ad}_{X^I(\mathsf{p}_\alpha)}^*\left(\mathrm{Ad}_{\varphi_\alpha^I}^* \Lambda_\alpha\right) \tag{3.67}$$

of $X \in \mathfrak{G}$ on $\left(B, A'', (\varphi_\alpha^I)\right) \in T^*\mathcal{M} \times \prod_{\alpha=1}^N \mathcal{O}_\alpha$.

Remark 3.8. The phase space (3.64) can be described more invariantly as follows. Let G_{Λ_α} be the stabiliser of $\Lambda_\alpha \in \mathfrak{g}^*$. Then $\mathrm{Ad}_{\varphi_\alpha^I}^* \Lambda_\alpha$ corresponds to a left coset $\varphi_\alpha^I G_{\Lambda_\alpha} \in G/G_{\Lambda_\alpha}$. This represents an element of the quotient $\mathcal{P}_{\mathsf{p}_\alpha}/G_{\Lambda_\alpha}$ in our chosen local trivialisation. An element of $\mathcal{P}_{\mathsf{p}_\alpha}/G_{\Lambda_\alpha}$ is usually called a reduction of the structure group of $\mathcal{P}$ at p_α to G_{Λ_α}. So our phase space is the product of the cotangent bundle of the space of holomorphic structures and the space of reductions of structure group to G_{Λ_α} at the subset of marked points $\{\mathsf{p}_\alpha\}_{\alpha=1}^N$.

We proceed with generalising the construction of Sect. 3.2. In the present setup, the Lagrangian 1-form (2.2) and action (2.9) take the form

$$\mathscr{L} = \frac{1}{2\pi i} \int_C \langle B, \delta A'' \rangle + \sum_{\alpha=1}^N \langle \Lambda_\alpha, \varphi_\alpha^{-1} d\varphi_\alpha \rangle - H_i\left(B, A'', (\varphi_\alpha)\right) dt^i,$$

for some Hamiltonians $H_i : T^*\mathcal{M} \times \prod_{\alpha=1}^N \mathcal{O}_\alpha \to \mathbb{C}$. Its pullback along an arbitrary immersion $\Sigma : \mathbb{R}^n \to T^*\mathcal{M} \times \prod_{\alpha=1}^N \mathcal{O}_\alpha \times \mathbb{R}^n$, $\Sigma(u) = \left(B(u), A''(u), (\varphi_\alpha(u)), t(u)\right)$ is then

$$\Sigma^*\mathscr{L} = \frac{1}{2\pi i} \int_C \langle B, d_{\mathbb{R}^n} A'' \rangle + \sum_{\alpha=1}^N \langle \Lambda_\alpha, \varphi_\alpha^{-1} d_{\mathbb{R}^n} \varphi_\alpha \rangle - H_i\left(B, A'', (\varphi_\alpha)\right) d_{\mathbb{R}^n} t^i,$$

so that for any parametrised curve $\Gamma : (0, 1) \to \mathbb{R}^n$, $s \mapsto (u^j(s))$ the corresponding action is given by $S_\Gamma[\Sigma] := \int_0^1 (\Sigma \circ \Gamma)^* \mathcal{L}$, which explicitly reads

$$
S_\Gamma[\Sigma] = \int_0^1 \left(- \frac{1}{2\pi i} \int_C \langle B(u), \partial_{u^j} A''(u) \rangle + \sum_{\alpha=1}^N \langle \Lambda_\alpha, \varphi_\alpha(u)^{-1} \partial_{u^j} \varphi_\alpha(u) \rangle \right.
$$
$$
\left. - H_i \Big(B(u(s)), A''(u(s)), (\varphi_\alpha(u(s))) \Big) \frac{\partial t^i}{\partial u^j} \right) \frac{du^j}{ds} \, ds . \tag{3.68}
$$

Note that this action, with the Hamiltonians H_i as in (3.35), bears some resemblance to the one in [56, §4.1.3]. However, the latter differs from (3.68) in two key respects. First, it is given by a sum of individual actions on $\mathbb{R} \times C$ for each time flow of the Hitchin system, rather than by a family of actions on $\mathbb{R}^n \times C$. As recalled in Sect. 2.1.1, the latter formulation is crucial to encode integrability through the Euler–Lagrange equation (2.10b). Second, the action in [56, §4.1.3] is treated as a gauge theory with "gauge group" $\mathcal{G}$. The invariance of the action (3.68) under the action of the group $\mathcal{G}$ on the extended phase space (3.64) is indeed the content of the next proposition, which is an extension of Proposition 3.2 to the case with type A defects. However, recall from Sect. 3.3, see also the second part of Remark 3.1, that unlike [56] we are treating $\mathcal{G}$ as a *global* symmetry group. The *gauging* of this global $\mathcal{G}$-symmetry is what led to the 3d mixed BF action with type B defects in Theorem 3.4; see also Theorem 3.10 below for the analogue in the presence of type A defects.

Proposition 3.9. *The action (3.68) is invariant under the action of $\mathcal{G}$ on $T^*\mathcal{M} \times \prod_{\alpha=1}^N \mathcal{O}_\alpha \times \mathbb{R}^n$ given by (3.66) and extended trivially to the $\mathbb{R}^n$ factor if and only if each H_i for $i = 1, \ldots, n$ is invariant under the group action, i.e.*

$$
H_i \Big(\mathrm{Ad}_g^* B, {}^g A'', (g(\mathsf{p}_\alpha)\varphi_\alpha) \Big) = H_i \big(B, A'', (\varphi_\alpha) \big) \tag{3.69}
$$

for any $(B, A'', (\varphi_\alpha)) \in T^*\mathcal{M} \times \prod_{\alpha=1}^N \mathcal{O}_\alpha$ and $g \in \mathcal{G}$. Moreover, the value of the associated moment map $\mu : T^*\mathcal{M} \times \prod_{\alpha=1}^N \mathcal{O}_\alpha \to \mathfrak{G}^*$ on an infinitesimal bundle morphism $X \in \mathfrak{G}$ is given by

$$
\mu_{(B, A'', (\varphi_\alpha))}(X) = \frac{1}{2\pi i} \int_C \langle B, \bar{\partial}^{A''} X \rangle + \sum_{\alpha=1}^N \int_C \langle \mathrm{Ad}_{\varphi_\alpha}^* \Lambda_\alpha, X \rangle \delta_{\mathsf{p}_\alpha}. \tag{3.70}
$$

Proof. For the invariance of the action under the group action (3.66), by Proposition 3.2 we only need to check the invariance of the new kinetic terms for the φ_α but this is immediate.

The extra term in the moment map (3.70) compared to the expression in (3.29) comes from generalising the argument in the last part of the proof of Proposition 3.2 to the present case and, again, the new contribution comes from the coadjoint orbit terms in the kinetic part. □

Following the discussion in Remark 3.3, we can use Stokes's theorem to rewrite the expression (3.70) for the moment map as

$$
\mu_{(B, A'', (\varphi_\alpha))}(X) = \int_C \left\langle \frac{1}{2\pi i} \bar{\partial}^{A''} B + \sum_{\alpha=1}^N \mathrm{Ad}_{\varphi_\alpha}^* \Lambda_\alpha \delta_{\mathsf{p}_\alpha}, X \right\rangle.
$$

In particular, we see that the vanishing of μ at any $(B, A'', (\varphi_\alpha)) \in T^*\mathcal{M} \times \prod_{\alpha=1}^N \mathcal{O}_\alpha$ corresponds to the condition

$$\bar\partial^{A''} B = -2\pi i \sum_{\alpha=1}^N \mathrm{Ad}^*_{\varphi_\alpha} \Lambda_\alpha \delta_{\mathsf{p}_\alpha}, \tag{3.71}$$

which reads locally as

$$\partial_{\bar z_I} B_{z_I} + \mathrm{ad}^*_{A''_{\bar z_I}} B_{z_I} = 2\pi i \sum_{\substack{\alpha=1 \\ \mathsf{p}_\alpha \in U_I}}^N \mathrm{Ad}^*_{\varphi_\alpha^I} \Lambda_\alpha \delta(z_I - z_I(\mathsf{p}_\alpha)). \tag{3.72}$$

To produce a (non trivial) dynamical theory we must choose Hamiltonian functions H_i on the symplectic manifold $T^*\mathcal{M} \times \prod_\alpha \mathcal{O}_\alpha$. We choose these in a similar way as in the case without marked points, with Hamiltonians given in (3.35) in terms of invariant polynomials P_i on $\mathfrak{g}^*$ and points q_i. This ensures in particular their invariance property as in Proposition 3.9. We will specify the number of points q_i more precisely when we look at specific examples in Sect. 5.

Finally, we proceed with generalising the construction of Sect. 3.3 to the case with marked points. Recall that we introduced the $\mathfrak{g}$-valued gauge field $\mathcal{A}$ along $\mathbb{R}^n$ and $\langle \Sigma^*\mu, \mathcal{A}\rangle$ was added to the pullback $\Sigma^*\mathscr{L}$ of the Lagrangian 1-form (3.22) to eventually produce the gauged action (3.40). In the present case we find the gauged action

$$S_\Gamma[\Sigma, \mathcal{A}, t] = \int_0^1 \left(-\frac{1}{2\pi i} \int_C \left\langle B(u), \frac{\partial A''}{\partial u^j} - \bar\partial^{A''}\mathcal{A}_j\right\rangle + \sum_{\alpha=1}^N \left\langle \Lambda_\alpha, \varphi_\alpha^{-1}\big(\partial_{u^j} + \mathcal{A}_j(\mathsf{p}_\alpha)\big)\varphi_\alpha\right\rangle \right.$$
$$\left. - H_i\big(B(u)\big)\frac{\partial t^i}{\partial u^j}\right)\frac{\mathrm{d}u^j}{\mathrm{d}s}\,\mathrm{d}s\,. \tag{3.73}$$

Rewriting this in terms of the connection $A = A'' + \mathcal{A}$ over $C \times \mathbb{R}^n$ yields the following analogue of Theorem 3.4.

Theorem 3.10. *The gauged multiform action* (3.73) *for Hitchin's system with marked points is given by the multiform action for 3d mixed BF theory on $C \times \mathbb{R}^n$ for the collection of fields $(B, A, (\varphi_\alpha))$ with a type B line defect along each coordinate t^i determined by the Hitchin Hamiltonian H_i defined as in (3.35) and a type A line defect at each marked point p_α, namely*

$$S_\Gamma[B, A, (\varphi_\alpha), t] = \frac{1}{2\pi i}\int_{C\times\Gamma}\langle B, F_A\rangle + \sum_{\alpha=1}^N \int_0^1 \left\langle \Lambda_\alpha, \varphi_\alpha^{-1}\big(\partial_{u^j} + \mathcal{A}_j(\mathsf{p}_\alpha)\big)\varphi_\alpha\right\rangle\frac{\mathrm{d}u^j}{\mathrm{d}s}\,\mathrm{d}s$$
$$- \int_0^1 H_i\big(B(u(s))\big)\frac{\mathrm{d}t^i}{\mathrm{d}s}\,\mathrm{d}s\,, \tag{3.74}$$

for an arbitrary curve $\Gamma : (0, 1) \to \mathbb{R}^n$, $s \mapsto u(s)$, where F_A is the curvature introduced in (3.45).

Finally, we also have the following analogue of Theorem 3.6.

Theorem 3.11. *The gauged univariational principle applied to the 3d mixed BF multi-form action $S_\Gamma[B, A, (\varphi_\alpha), t]$ in (3.74) yields a set of equations for the fields B, A and φ_α for $\alpha = 1, \ldots, N$. Working in any chart (U_I, z_I) of C and in terms of the components $B_{z_I}(z_I, \bar{z}_I, u)$, $A''_{\bar{z}_I}(z_I, \bar{z}_I, u)$, $\mathcal{A}^I_j(z_I, \bar{z}_I, u)$ of the various fields, these equations take the following form:*

$$\partial_{\bar{z}_I} \widetilde{\mathcal{A}}^I_i - \partial_{t^i} A''_{\bar{z}_I} + [A''_{\bar{z}_I}, \widetilde{\mathcal{A}}^I_i] = 2\pi i \, \nabla P_i\big(B_{z_I}(\mathsf{q}_i)\big)\delta\big(z_I - z_I(\mathsf{q}_i)\big), \quad (3.75a)$$

$$\partial_{\bar{z}_I} B_{z_I} + \mathrm{ad}^*_{A''_{\bar{z}_I}} B_{z_I} = 2\pi i \sum_{\substack{\alpha=1 \\ \mathsf{p}_\alpha \in U_I}}^{N} \mathrm{Ad}^*_{\varphi^I_\alpha} \Lambda_\alpha \delta\big(z_I - z_I(\mathsf{p}_\alpha)\big), \quad (3.75b)$$

$$\partial_{t^j} B_{z_I} + \mathrm{ad}^*_{\widetilde{\mathcal{A}}^I_j} B_{z_I} = 0, \quad (3.75c)$$

$$\partial_{t^j} H_i(B) = 0, \quad (3.75d)$$

$$\partial_{t^j}\big(\mathrm{Ad}^*_{\varphi^I_\alpha} \Lambda_\alpha\big) + \mathrm{ad}^*_{\widetilde{\mathcal{A}}^I_j(\mathsf{p}_\alpha)} \mathrm{Ad}^*_{\varphi^I_\alpha} \Lambda_\alpha = 0, \quad (3.75e)$$

where we have used the invertibility of the map $(u^j) \mapsto \big(t^i(u)\big)$ to define $\widetilde{\mathcal{A}}^I_i = \frac{\partial u^j}{\partial t^i} \mathcal{A}^I_j$. The equations associated to any pair of overlapping charts (U_I, z_I) and (U_J, z_J) are compatible on $U_I \cap U_J \neq \emptyset$. The following zero-curvature equations also hold

$$\partial_{t^i} \widetilde{\mathcal{A}}^I_j - \partial_{t^j} \widetilde{\mathcal{A}}^I_i + [\widetilde{\mathcal{A}}^I_i, \widetilde{\mathcal{A}}^I_j] = 0. \quad (3.76)$$

Proof. The computation for the variation of the action with respect to the fields B^I, A^I, and t^i is exactly as in the proof of Theorem 3.6 and so is the derivation of (3.76). The new terms in the action only modify the equation obtained by varying $\mathcal{A}^I$ and this gives the right-hand side of (3.75b). Finally, the variation with respect to φ_β gives

$$\delta_{\varphi_\beta} S_\Gamma[B, A, (\varphi_\alpha), t] = \delta_{\varphi_\beta} \int_0^1 \sum_{\alpha=1}^{N} \Big\langle \Lambda_\alpha, \varphi_\alpha^{-1}\big(\partial_{u^j} + \mathcal{A}_j(\mathsf{p}_\alpha)\big)\varphi_\alpha \Big\rangle \frac{\mathrm{d}u^j}{\mathrm{d}s} \mathrm{d}s$$

$$= \int_0^1 \Big\langle \Lambda_\beta, -(\varphi^I_\beta)^{-1}\delta\varphi^I_\beta(\varphi^I_\beta)^{-1}\big(\partial_{u^j} + \mathcal{A}^I_j(\mathsf{p}_\beta)\big)\varphi^I_\beta$$

$$+ (\varphi^I_\beta)^{-1}\big(\partial_{u^j} + \mathcal{A}^I_j(\mathsf{p}_\beta)\big)\delta\varphi^I_\beta \Big\rangle \frac{\mathrm{d}u^j}{\mathrm{d}s} \mathrm{d}s$$

$$= \int_0^1 \Big\langle \mathrm{Ad}^*_{\varphi^I_\beta} \Lambda_\beta, \big[\partial_{u^j} + \mathcal{A}^I_j(\mathsf{p}_\beta), \delta\varphi^I_\beta(\varphi^I_\beta)^{-1}\big] \Big\rangle \frac{\mathrm{d}u^j}{\mathrm{d}s} \mathrm{d}s$$

$$= -\int_0^1 \Big\langle \big(\partial_{u^j} + \mathrm{ad}^*_{\mathcal{A}^I_j(\mathsf{p}_\beta)}\big)\big(\mathrm{Ad}^*_{\varphi^I_\beta} \Lambda_\beta\big), \delta\varphi^I_\beta(\varphi^I_\beta)^{-1} \Big\rangle \frac{\mathrm{d}u^j}{\mathrm{d}s} \mathrm{d}s$$

so that we obtain (3.75e). $\square$

4. Hitchin System in Lax form and Its Variational Formulation

In Sect. 3 we showed that Hitchin's completely integrable system on the symplectic quotient $\mu^{-1}(0)/\mathcal{G}$ is described variationally by the action for 3d mixed BF theory with type B line defects associated with each of the Hamiltonians H_i and corresponding times t^i for $i = 1, \ldots, n$, see Theorem 3.4. We further extended this result to Hitchin's

integrable system with marked points in Sect. 3.4 by adding type A line defects at each marked point in C to the action, see Theorem 3.10.

The purpose of this section is twofold. First, in Sect. 4.1 we will show that the set of variational equations derived in Theorem 3.11 encode the hierarchy of equations of the Hitchin system in the standard Lax form, namely $\partial_{t^i} L = [M_i, L]$ for a meromorphic Lax matrix L and a collection of meromorphic Lax matrices M_i associated with each time t^i in the hierarchy for $i = 1, \ldots, n$. To our knowledge, the earliest connection between the Hitchin system and the Lax formalism appeared in [44,48]. Second, in Sect. 4.2 we will show that this hierarchy of Lax equations for the Hitchin system is itself variational and obtain the corresponding Lagrangian 1-form/action, which we refer to as the *unifying action*, directly from the 3d mixed BF multiform action (3.74). The key is to make use of the isomorphism (1.2) mentioned in the introduction in order to pass from the description of the Hitchin system on $\mu^{-1}(0)/\mathcal{G}$ in terms of smooth $\mathfrak{g}$-valued $(0, 1)$-connections $A'' \in \mathcal{M}$ and smooth Higgs fields $B \in T^*_{A''}\mathcal{M}$, as in Sect. 3.3 and 3.4, to a description of the Hitchin system on $T^*\mathrm{Bun}_G(C)$ in terms of holomorphic transition functions and holomorphic Higgs fields. We will thus establish the right-hand side of the following commutative diagram

$$
\begin{array}{ccc}
S_\Gamma[B, A, (\varphi_\alpha), t] & \xrightarrow[\text{description}]{\text{holomorphic}} & \text{Unifying action} \\
\scriptstyle\text{univariational}\ \downarrow\ \scriptstyle\text{principle} & & \scriptstyle\text{univariational}\ \downarrow\ \scriptstyle\text{principle} \\
\text{Variational equations (3.75)} & \xrightarrow[\text{description}]{\text{holomorphic}} & \partial_{t^i} L = [M_i, L]
\end{array}
$$

4.1. Hitchin System in Lax form. The key observation, following [71], is that (3.75c) takes the form of a hierarchy of Lax equations for Lax matrices B_{z_I} and $\widetilde{\mathcal{A}}^I_i$ defined on the coordinate patch (U_I, z_I) that become meromorphic *when solving* (3.75a)-(3.75b) *in a local trivialisation such that* $A''_{\bar{z}_I} = 0$. The compatibility of this hierarchy of Lax equations is ensured by (3.76). We now spell this out in detail.

As usual, in order to bring (3.75c) to the standard Lax form, we will identify $\mathfrak{g}^*$ with $\mathfrak{g}$ using a nondegenerate invariant bilinear form on $\mathfrak{g}$. We still denote the latter by $\langle\ ,\ \rangle$ (like the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$) hoping that it will not confuse the reader. This allows us to identify adjoint and coadjoint actions. Moreover, from now on we only consider matrix Lie groups and Lie algebras.

Next, we pick a point $\mathsf{p} \in C$ distinct from all the marked points p_α with $\alpha = 1, \ldots, N$ and q_i with $i = 1, \ldots, n$. Pick a neighbourhood U_0 of p not containing any of these other points and equipped with a local coordinate $z_0 : U_0 \to C$. We also let $U_1 := C\backslash\{\mathsf{p}\}$ so that $\{U_0, U_1\}$ forms an open cover of C. Let us stress that for $g \geq 1$ the open $U_1 \subset C$ is not a coordinate chart since in general it cannot be equipped with a holomorphic coordinate $U_1 \to \mathbb{C}$. However, we can further refine the cover by choosing a holomorphic atlas $\{(U_I, z_I)\}_{I \in \mathcal{I}}$ for U_1, where $\mathcal{I}$ is any indexing set not containing 0 and 1 so that we can also use the notation U_I for $I \in \{0, 1\}$. In other words, U_I for $I \in \mathcal{I}$ are open subsets of U_1 such that $\bigcup_{I \in \mathcal{I}} U_I = U_1$ and equipped with coordinates $z_I : U_I \to \mathbb{C}$.

For the purpose of describing the smooth principal G-bundle $\pi : \mathcal{P} \to C$ with a holomorphic structure specified by $A'' \in \mathcal{M}$, the open cover $\{U_0, U_1\}$ of C suffices. Indeed, since U_1 is a non-compact Riemann surface and G is semi-simple, we can trivialise the holomorphic principal G-bundle $(\mathcal{P}, A'')$ over U_1. Since we can also trivialise

$(\mathcal{P}, A'')$ over U_0, we obtain local trivialisations for the holomorphic principal G-bundle $(\mathcal{P}, A'')$ relative to the open cover $\{U_0, U_1\}$ of C. More importantly for us, we also have local trivialisations of the pullback bundle $\pi_C^* \mathcal{P} \cong \mathcal{P} \times \mathbb{R}^n$, see Sect. 3.3, which we write as

$$\psi_0 : \pi^{-1}(U_0 \times \mathbb{R}^n) \xrightarrow{\cong} U_0 \times \mathbb{R}^n \times G, \quad p \longmapsto (\pi(p), f_0(p)), \tag{4.1a}$$

$$\psi_1 : \pi^{-1}(U_1 \times \mathbb{R}^n) \xrightarrow{\cong} U_1 \times \mathbb{R}^n \times G, \quad p \longmapsto (\pi(p), f_1(p)) \tag{4.1b}$$

relative to the open cover $\{U_0 \times \mathbb{R}^n, U_1 \times \mathbb{R}^n\}$ of $C \times \mathbb{R}^n$. Let $g_{01} : (U_0 \cap U_1) \times \mathbb{R}^n \to G$ be the smooth transition function on the overlap $(U_0 \cap U_1) \times \mathbb{R}^n \neq \emptyset$ so that (with $(x, t) = \pi(p) \in (U_0 \cap U_1) \times \mathbb{R}^n$)

$$\psi_0 \circ \psi_1^{-1} : (U_0 \cap U_1) \times \mathbb{R}^n \times G \longrightarrow (U_0 \cap U_1) \times \mathbb{R}^n \times G, \quad (x, t, g) \longmapsto (x, t, g_{01}(x, t)g).$$

Since $U_I \subset U_1$ for each $I \in \mathcal{I}$, we will use the restriction of the local trivialisation (4.1b) on each $\pi^{-1}(U_I \times \mathbb{R}^n)$ so that the transition functions $g_{IJ} : (U_I \cap U_J) \times \mathbb{R}^n \to G$ and $g_{1I} : (U_1 \cap U_I) \times \mathbb{R}^n \to G$ are trivial for all $I, J \in \mathcal{I}$ and $g_{0I} : (U_0 \cap U_I) \times \mathbb{R}^n \to G$ is given by $g_{0I} = g_{01}$ for all $I \in \mathcal{I}$.

Consider the equations of motion (3.75c) derived in Theorem 3.11. Since we are now identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ and the coadjoint action with the adjoint action, we can rewrite this set of equations as

$$\partial_{t^i} B_{z_I} = \left[-\widetilde{\mathcal{A}}_i^I, B_{z_I} \right] \tag{4.2}$$

on the open chart $U_I \subset C$ for each $I \in \mathcal{I} \cup \{0\}$. We note following [71] that these clearly resemble a hierarchy of Lax equations. We can rewrite them without using coordinates by expressing them in terms of the $\mathfrak{g}$-valued $(1, 0)$-forms $B^I = B_{z_I}(z_I, \bar{z}_I, t)\mathrm{d}z_I \in \Omega^{1,0}(U_I \times \mathbb{R}^n, \mathfrak{g})$ for $I \in \mathcal{I} \cup \{0\}$ as

$$\partial_{t^i} B^I = \left[-\widetilde{\mathcal{A}}_i^I, B^I \right]. \tag{4.3a}$$

Since these equations for $I \in \mathcal{I}$ are compatible on overlaps $U_I \cap U_J \neq \emptyset$ and in fact $B^I = B^J$ and $\widetilde{\mathcal{A}}_i^I = \widetilde{\mathcal{A}}_i^J$ since the transition functions on these overlaps are trivial, we may consider the equations (4.3a) simply for $I \in \{0, 1\}$. In that case, $B^I \in \Omega^{1,0}(U_I \times \mathbb{R}^n, \mathfrak{g})$ and $\widetilde{\mathcal{A}}_i^I$ is a $\mathfrak{g}$-valued function on $U_I \times \mathbb{R}^n$ for $I \in \{0, 1\}$ which on the overlap $U_0 \cap U_1$ are related by

$$B^0 = g_{01} B^1 g_{01}^{-1}, \qquad \widetilde{\mathcal{A}}_i^0 = g_{01}\widetilde{\mathcal{A}}_i^1 g_{01}^{-1} - \partial_{t^i} g_{01} g_{01}^{-1}. \tag{4.3b}$$

We could also further rewrite (4.3a) more compactly as

$$\mathrm{d}_{\mathbb{R}^n} B^I = \left[-\mathcal{A}^I, B^I \right] \tag{4.4}$$

recalling that $\mathcal{A}^I = \mathcal{A}_j^I \mathrm{d}u^j = \widetilde{\mathcal{A}}_i^I \mathrm{d}t^i$, see Theorem 3.11, and correspondingly the second equation in (4.3b) would read

$$\mathcal{A}^0 = g_{01}\mathcal{A}^1 g_{01}^{-1} - \mathrm{d}_{\mathbb{R}^n} g_{01} g_{01}^{-1}. \tag{4.5}$$

In order for (4.3) to describe Lax equations on the Riemann surface C for the hierarchy of the Hitchin system, however, we need B^I and $\widetilde{\mathcal{A}}_i^I$ to be solutions of (3.75a) and (3.75b) (recall that the latter represent the moment map condition $\mu = 0$). In addition, to ensure

that the resulting Lax matrices are meromorphic in the spectral parameter, we make a change of local trivialisation on $\mathcal{P}$, moving from (4.1) to new local trivialisations $\tilde{\psi}_0$ and $\tilde{\psi}_1$ with respect to which $A''^I = 0$ for $I \in \{0, 1\}$, or equivalently $A''_{\bar{z}_I} = 0$ in each local chart (U_I, z_I) for $I \in \mathcal{I} \cup \{0\}$, see Sect. 3.1.1 for details. Let $h = (h_I) \in \check{C}^0(C, G)$ be the Čech 0-cochain implementing this change of trivialisation and let

$$L^I := h_I B^I h_I^{-1}, \qquad -M_i^I := h_I \widetilde{\mathcal{A}}_i^I h_I - \partial_{t^i} h_I h_I^{-1} \tag{4.6}$$

be the local expressions for B^I and $\widetilde{\mathcal{A}}_i^I$ with respect to this new local trivialisation. The additional minus sign in the definition of M_i^I is introduced so that the equations (4.3a) take on the standard Lax form in this local trivialisation, namely

$$\partial_{t^i} L^I = \left[M_i^I, L^I \right] \tag{4.7a}$$

on U_I for $I \in \{0, 1\}$. On the overlap $(U_0 \cap U_1) \times \mathbb{R}^n$ we have the same relations as in (4.3b), namely

$$L^0 = \gamma L^1 \gamma^{-1}, \qquad M_i^0 = \gamma M_i^1 \gamma^{-1} + \partial_{t^i} \gamma \gamma^{-1}, \tag{4.7b}$$

where the new transition function

$$\gamma := h_0 g_{01} h_1^{-1} : (U_0 \cap U_1) \times \mathbb{R}^n \to G \tag{4.8}$$

is now holomorphic on $U_0 \cap U_1$, see Sect. 3.1.1. We note here that the plus sign in the last term of the second equation in (4.7b) stems from the minus sign introduced in the second definition in (4.6). In other words, $-M_i^0$ and $-M_i^1$ are related by an ordinary gauge transformation by γ. If we introduce also the $\mathfrak{g}$-valued 1-forms $M^I := M_i^I \, \mathrm{d}_{\mathbb{R}^n} t^i \in \Omega^1(U_I \times \mathbb{R}^n, \mathfrak{g})$ then the Lax hierarchy (4.7a) can be rewritten more compactly as in (4.4), namely

$$\mathrm{d}_{\mathbb{R}^n} L^I = [M^I, L^I] \tag{4.9a}$$

and also the relations (4.7b) become

$$L^0 = \gamma L^1 \gamma^{-1}, \qquad M^0 = \gamma M^1 \gamma^{-1} + \mathrm{d}_{\mathbb{R}^n} \gamma \gamma^{-1}. \tag{4.9b}$$

In this adapted local trivialisation, the equations of motion (3.75a) and (3.75b) become

$$\partial_{\bar{z}_I} M_i^I = -2\pi i \nabla P_i \left(L_{z_I}(\mathsf{q}_i) \right) \delta \left(z_I - z_I(\mathsf{q}_i) \right), \tag{4.10a}$$

$$\partial_{\bar{z}_I} L_{z_I} = 2\pi i \sum_{\substack{\alpha=1 \\ \mathsf{p}_\alpha \in U_I}}^{N} \varphi_\alpha \Lambda_\alpha \varphi_\alpha^{-1} \delta \left(z_I - z_I(\mathsf{p}_\alpha) \right) \tag{4.10b}$$

for each $I \in \mathcal{I} \cup \{0\}$, where $L^I = L_{z_I}(z_I, \bar{z}_I, t) \mathrm{d} z_I$ is the local expression of the Lax matrix L^I in the coordinate chart (U_I, z_I). Equivalently, using the identity (3.51), we have

$$\partial_{\bar{z}_I} \left(M_i^I - \frac{\nabla P_i \left(L_{z_I}(\mathsf{q}_i) \right)}{z_I - z_I(\mathsf{q}_i)} \right) = 0, \tag{4.11a}$$

$$\partial_{\bar{z}_I}\left(L_{z_I} + \sum_{\substack{\alpha=1 \\ \mathsf{p}_\alpha \in U_I}}^{N} \frac{\varphi_\alpha \Lambda_\alpha \varphi_\alpha^{-1}}{z_I - z_I(\mathsf{p}_\alpha)}\right) = 0. \tag{4.11b}$$

So M_i^I and L_{z_I} are meromorphic in the chosen local trivialisation, with a specific pole structure.

First, equation (4.11b) tells us that L^1 is a $\mathfrak{g}$-valued meromorphic $(1,0)$-form on U_1 with simple poles at each p_α with residue $-\varphi_\alpha \Lambda_\alpha \varphi_\alpha^{-1}$ there. Indeed, note that the residue of a meromorphic $(1,0)$-form is independent of the local coordinate used. Locally around the point p_α for any $\alpha = 1, \ldots, N$, if $\mathsf{p}_\alpha \in U_I$ for some $I \in \mathcal{I}$ then in the coordinate chart (U_I, z_I) we can write

$$L^1 = L^I = \left(-\frac{\varphi_\alpha \Lambda_\alpha \varphi_\alpha^{-1}}{z_I - z_I(\mathsf{p}_\alpha)} + J_\alpha^I\right)\mathrm{d}z_I \tag{4.12a}$$

where the first equality follows from the fact that the transition functions of the principal G-bundle $\mathcal{P}$ between the opens U_1 and $U_I \subset U_1$ were, by definition, trivial. In the last expression, J_α^I denotes the holomorphic part of L_{z_I} in the neighbourhood of p_α. Also, since $\mathsf{p}_\alpha \notin U_0$ by assumption, it follows that $L^0 = L_{z_0}\mathrm{d}z_0$ is holomorphic on U_0. Therefore, using the relation (4.7b) we have

$$\gamma L^1 \gamma^{-1} = L^0 \tag{4.12b}$$

on $U_0 \cap U_1$, which expresses the fact that the $\mathfrak{g}$-valued $(1,0)$-form $\gamma L^1 \gamma^{-1} \in \Omega^{1,0}\big((U_1 \cap U_0) \times \mathbb{R}^n, \mathfrak{g}\big)$ extends holomorphically to $U_0 \times \mathbb{R}^n$. We will refer to L as the *Lax matrix* of the Hitchin system.

Next, equation (4.11a) tells us that M_i^1 is a $\mathfrak{g}$-valued meromorphic function on U_1 with a simple pole at the point q_i, with the local expression

$$M_i^1 = M_i^I = \frac{\nabla P_i\big(L_{z_I}(\mathsf{q}_i)\big)}{z_I - z_I(\mathsf{q}_i)} + K_i^I \tag{4.13}$$

in the chart (U_I, z_I) which is such that $\mathsf{q}_i \in U_I$. Here K_i^I denotes the holomorphic part of M_i^I in the neighbourhood of q_i. Moreover, since we have $\mathsf{q}_i \notin U_0$ by assumption, M_i^0 is a holomorphic $\mathfrak{g}$-valued function on U_0 which is related to M_i^1 on the overlap $(U_0 \cap U_1) \times \mathbb{R}^n$ by the second relation in (4.7b). That is, $\gamma M_i^1 \gamma^{-1} + \partial_{t^i}\gamma \gamma^{-1} = M_i^0$ extends holomorphically from $U_0 \cap U_1$ to U_0.

To summarise the above discussion, the Hitchin system can indeed be presented as the hierarchy of Lax equations (4.7a), as a direct consequence expressing our variational equations (3.75) in a local trivialisation with $A''^I = 0$, in terms of:

(1) the Lax matrix L which is a meromorphic section of $\pi_C^* \bigwedge^{1,0} C \otimes \mathfrak{g}_{\pi_C^* \mathcal{P}}$ fixed in terms of the following degrees of freedom:
  (a) Maps $\varphi_\alpha : \mathbb{R}^n \to G/G_{\Lambda_\alpha}$ into the coadjoint orbits $\mathcal{O}_\alpha \cong G/G_{\Lambda_\alpha}$ for each $\alpha = 1, \ldots, N$,
  (b) The transition function $\gamma : (U_0 \cap U_1) \times \mathbb{R}^n \to G$ holomorphic in $U_0 \cap U_1$ and encoding the holomorphic structure on the principal G-bundle $\mathcal{P}$. In particular L satisfies (4.12b).
(2) The $\mathfrak{g}$-valued meromorphic functions M_i^0 and M_i^1 for each $i = 1, \ldots, n$ with pole structure dictated by (4.13) and satisfying the second equation in (4.9b).

4.2. Unifying Lagrangian 1-form. The description of the Hitchin system as a hierarchy of Lax equations (4.7a) was obtained in Sect. 4.1 from the set of equations of Theorem 3.11 by moving to a local trivialisation in which $A''^I = 0$ and solving the moment map condition (3.75b). The main point is that these equations are *variational*, deriving from the multiform 3d BF action (3.74), so it is natural to ask whether we can also derive (4.7a) directly as Euler–Lagrange equations of an appropriate action.

In Theorem 4.1 below we construct the *unifying action* and associated Lagrangian 1-form by writing the 3d mixed BF action (3.74) in a local trivialisation with $A''^I = 0$ and explicitly solving the moment map condition (3.75b). In Theorem 4.3 we then prove that this unifying action does indeed reproduce the Lax equations (4.7a) variationally.

Theorem 4.1. *The 3d mixed BF action* (3.74) *written in a local trivialisation where* $A''^0 = A''^1 = 0$ *and with the moment map condition* (3.75b) *explicitly solved, leads to the unifying action*

$$S_{H,\Gamma}\big[L, \gamma, (\varphi_\alpha), t\big] = \int_\Gamma \mathscr{L}_H, \tag{4.14a}$$

for any parametrised curve $\Gamma : (0, 1) \to \mathbb{R}^n$, *where* $\mathscr{L}_H \in \Omega^1(\mathbb{R}^n)$ *is the Hitchin Lagrangian 1-form defined using a small counter-clockwise oriented loop* c_p *in* $U_0 \cap U_1$ *around the point* $p \in C$ *by*

$$\mathscr{L}_H := \frac{1}{2\pi i} \int_{c_p} \langle L^0, d_{\mathbb{R}^n}\gamma\gamma^{-1}\rangle + \sum_{\alpha=1}^N \langle \Lambda_\alpha, \varphi_\alpha^{-1}d_{\mathbb{R}^n}\varphi_\alpha\rangle - H_i d_{\mathbb{R}^n}t^i, \qquad H_i = P_i\big(L_{z_I}^1(q_i)\big). \tag{4.14b}$$

In particular, under a change of 'residual' local trivialisation $h = (h_0, h_1) \in \check{C}^0(C, G)$ *with* h_I *holomorphic on* U_I *for* $I \in \{0, 1\}$ *so that the condition* $A''^0 = A''^1 = 0$ *is preserved, the action* (4.14a) *is invariant in the sense that*

$$S_{H,\Gamma}\big[hLh^{-1}, h_0\gamma h_1^{-1}, (h_1\varphi_\alpha)\big] = S_{H,\Gamma}\big[L^0, \gamma, (\varphi_\alpha)\big], \tag{4.15}$$

where hLh^{-1} *stands for the Lax matrix given by* $h_0 L^0 h_0^{-1}$ *on* U_0 *and* $h_1 L^1 h_1^{-1}$ *on* U_1.

Proof. We choose local trivialisations over $U_0 \times \mathbb{R}^n$ and $U_1 \times \mathbb{R}^n$ in which the components $A''_{\bar{z}_0}$, $A''_{\bar{z}_1}$ of the partial connection A vanish. Then the local expression for the curvature (3.45) becomes

$$F_A^I = \partial_{\bar{z}_I}\mathcal{A}_j^I d\bar{z}_I \wedge du^j + \frac{1}{2}\big(\partial_{u^i}\mathcal{A}_j^I - \partial_{u^j}\mathcal{A}_i^I + [\mathcal{A}_i^I, \mathcal{A}_j^I]\big)du^i \wedge du^j. \tag{4.16}$$

Unlike earlier trivialisations used in Sect. 3.3 for which we had (3.42), these trivialisations will have a transition function $\gamma = g_{01} : U_0 \cap U_1 \times \mathbb{R}^n \to G$ that depends on $u \in \mathbb{R}^n$ (because our choice of trivialisation depends on A'', and A'' depends on u). On the overlaps of coordinate charts we have

$$\mathcal{A}_j^0 = \gamma\mathcal{A}_j^1\gamma^{-1} - \partial_{u^j}\gamma\gamma^{-1}, \qquad \bar{\partial}\gamma = 0. \tag{4.17}$$

Recall from Sect. 4.1 that to emphasise that B solves the moment map condition (3.75b) we denote it by L and refer to it as a Lax matrix. Now the moment map condition (4.11b) can be written concisely as

$$\bar{\partial}L^0 = 0, \qquad \bar{\partial}L^1 = -2\pi i\sum_{\alpha=1}^N \varphi_\alpha\Lambda_\alpha\varphi_\alpha^{-1}\delta_{p_\alpha}. \tag{4.18}$$

In order to evaluate the action (3.74) in these trivialisations, we choose a circle c_p as described in the statement of the theorem, and let $R_0 \subset U_0$, $R_1 \subset U_1$ be the closure of the interior and exterior of this circle. Then $R_0 \cap R_1 = c_\mathsf{p}$ and $R_0 \cup R_1 = C$. The first term in the action (3.74) is

$$\frac{1}{2\pi i} \int_{C \times \Gamma} \langle B, F_A \rangle = \sum_{I \in \{0,1\}} \frac{1}{2\pi i} \int_{R_I \times \Gamma} \langle L^I, \bar{\partial} \mathcal{A}_j^I \rangle \wedge \mathrm{d}u^j. \qquad (4.19)$$

We evaluate these terms separately. For $I = 1$ we find

$$\frac{1}{2\pi i} \int_{R_1} \langle L^1, \bar{\partial} \mathcal{A}_j^1 \rangle = \frac{1}{2\pi i} \int_{R_1} \left(-\bar{\partial} \langle L^1, \mathcal{A}_j^1 \rangle + \langle \bar{\partial} L^1, \mathcal{A}_j^1 \rangle \right)$$

$$= -\frac{1}{2\pi i} \int_{\partial R_1} \langle L^1, \mathcal{A}_j^1 \rangle - \sum_{\alpha=1}^{N} \langle \varphi_\alpha \Lambda_\alpha \varphi_\alpha^{-1}, \mathcal{A}_j^1(\mathsf{p}_\alpha) \rangle$$

$$= \frac{1}{2\pi i} \int_{c_\mathsf{p}} \langle L^0, \gamma \mathcal{A}_j^1 \gamma^{-1} \rangle - \sum_{\alpha=1}^{N} \langle \varphi_\alpha \Lambda_\alpha \varphi_\alpha^{-1}, \mathcal{A}_j^1(\mathsf{p}_\alpha) \rangle,$$

where in the second equality we used the second relation in (4.18) and in the last step we used the first relation in (4.9b) and the fact that ∂R_1 is equal to c_p with reversed orientation. For $I = 0$,

$$\frac{1}{2\pi i} \int_{R_0} \langle L^0, \bar{\partial} \mathcal{A}_j^0 \rangle = -\frac{1}{2\pi i} \int_{R_0} \bar{\partial} \langle L^0, \mathcal{A}_j^0 \rangle = -\frac{1}{2\pi i} \int_{\partial R_0} \langle L^0, \mathcal{A}_j^0 \rangle$$

$$= -\frac{1}{2\pi i} \int_{c_\mathsf{p}} \langle L^0, \gamma \mathcal{A}_j^1 \gamma^{-1} \rangle + \frac{1}{2\pi i} \int_{c_\mathsf{p}} \langle L^0, \partial_{u^j} \gamma \gamma^{-1} \rangle,$$

where the first equality uses the first relation in (4.18) and in the third step we used (4.17) and the fact that ∂R_0 coincides with c_p and has the same orientation. It now follows from the above computation of both integrals on the right-hand side of (4.19) that

$$\frac{1}{2\pi i} \int_{C \times \Gamma} \langle B, F_A \rangle = \frac{1}{2\pi i} \int_{c_\mathsf{p} \times \Gamma} \langle L^0, \mathrm{d}_{\mathbb{R}^n} \gamma \gamma^{-1} \rangle - \sum_{\alpha=1}^{N} \int_\Gamma \langle \varphi_\alpha \Lambda_\alpha \varphi_\alpha^{-1}, \mathcal{A}_j^1(\mathsf{p}_\alpha) \rangle \mathrm{d}u^j. \qquad (4.20)$$

Substituting this into (3.74) and setting $B = L$ in the remaining terms gives the desired result.

To prove the last 'in particular' statement, consider a change of 'residual' local trivialisation $h = (h_0, h_1) \in \check{C}^0(C, G)$, with h_I holomorphic on U_I so that the condition $A''_{\bar{z}_I} = 0$ is preserved. We have $\gamma \mapsto \tilde{\gamma} = h_0 \gamma h_1^{-1}$, $\tilde{L}^0 = h_0 L^0 h_0^{-1}$ and $\tilde{L}^1 = h_1 L^1 h_1^{-1}$ so that $\tilde{L}^0 = \tilde{\gamma} \tilde{L}^1 \tilde{\gamma}^{-1}$. Then the integral in the first term on the right-hand side of (4.14b) transforms to

$$\int_{c_\mathsf{p}} \langle \tilde{L}^0, \mathrm{d}_{\mathbb{R}^n} \tilde{\gamma} \tilde{\gamma}^{-1} \rangle = \int_{c_\mathsf{p}} \langle L^0, \mathrm{d}_{\mathbb{R}^n} \gamma \gamma^{-1} \rangle + \int_{c_\mathsf{p}} \langle L^0, h_0^{-1} \mathrm{d}_{\mathbb{R}^n} h_0 \rangle - \int_{c_\mathsf{p}} \langle L^1, h_1^{-1} \mathrm{d}_{\mathbb{R}^n} h_1 \rangle. \qquad (4.21)$$

Note that the second term on the right-hand side vanishes by Cauchy's theorem since h_0 and L_0 are both holomorphic on U_0 and c_p is a small contour around $\mathsf{p} \in U_0$. To evaluate

the last term on the right-hand side of (4.21), recall that the section L^1 has poles at the marked points p_α given by (4.12a). Since $h_1 : U_1 \to G$ is holomorphic on $U_1 = C\backslash\{\mathsf{p}\}$ we then deduce using the residue theorem that

$$\frac{1}{2\pi i} \int_{c_\mathsf{p}} \langle L^1, h_1^{-1} \mathrm{d}_{\mathbb{R}^n} h_1 \rangle = -\sum_{\alpha=1}^{N} \langle -\varphi_\alpha \Lambda_\alpha \varphi_\alpha^{-1}, h_1(\mathsf{p}_\alpha)^{-1} \mathrm{d}_{\mathbb{R}^n} h_1(\mathsf{p}_\alpha) \rangle$$

where the first sign comes from noting that c_p can be contracted down to a sum of small *clockwise* circles around each point p_α. Substituting this into the right-hand side of (4.21) we deduce the transformation property

$$\frac{1}{2\pi i} \int_{c_\mathsf{p}} \langle \tilde{L}^0, \mathrm{d}_{\mathbb{R}^n} \tilde{\gamma}\tilde{\gamma}^{-1} \rangle = \frac{1}{2\pi i} \int_{c_\mathsf{p}} \langle L^0, \mathrm{d}_{\mathbb{R}^n} \gamma\gamma^{-1} \rangle - \sum_{\alpha=1}^{N} \langle \Lambda_\alpha, \varphi_\alpha^{-1} \big(h_1(\mathsf{p}_\alpha)^{-1} \mathrm{d}_{\mathbb{R}^n} h_1(\mathsf{p}_\alpha) \big) \varphi_\alpha \rangle.$$
$$(4.22)$$

Consider now the second term in the Lagrangian 1-form (4.14b). Since φ_α transforms under the change of local trivialisation as $\varphi_\alpha \mapsto \tilde{\varphi}_\alpha = h_1(\mathsf{p}_\alpha)\varphi_\alpha$, see the discussion at the start of Sect. 3.4, this term in the Lagrangian transforms as

$$\sum_{\alpha=1}^{N} \langle \Lambda_\alpha, \tilde{\varphi}_\alpha^{-1} \mathrm{d}_{\mathbb{R}^n} \tilde{\varphi}_\alpha \rangle = \sum_{\alpha=1}^{N} \langle \Lambda_\alpha, \varphi_\alpha^{-1} \mathrm{d}_{\mathbb{R}^n} \varphi_\alpha \rangle + \sum_{\alpha=1}^{N} \langle \Lambda_\alpha, \varphi_\alpha^{-1} \big(h_1(\mathsf{p}_\alpha)^{-1} \mathrm{d}_{\mathbb{R}^n} h_1(\mathsf{p}_\alpha) \big) \varphi_\alpha \rangle.$$
$$(4.23)$$

We see now from (4.22) and (4.23) that the Lagrangian 1-form (4.14b) is invariant under residual changes of local trivialisations $h = (h_0, h_1) \in \check{C}^0(C, G)$ with h_I holomorphic on U_I. $\square$

Remark 4.2. The term $\int_{c_\mathsf{p}} \langle L^0, \mathrm{d}_{\mathbb{R}^n} \gamma\gamma^{-1} \rangle$ in the kinetic part of the unifying Lagrangian (4.14b) arises from a subtle mechanism. In the proof of Theorem 4.1, we work on the pullback bundle $\pi_C^* \mathcal{P}$ but in a local trivialisation in which $\mathcal{A}''^I = 0$ for each $I \in \mathcal{I}$. This implies, in particular, that the transition function $g_{IJ} : (U_I \cap U_J) \times \mathbb{R}^n \to G$ is holomorphic in $U_I \cap U_J$ and dependent on the coordinates u^j of $\mathbb{R}^n$. Thus, specialising the $\mathcal{A}$ component of the identity (3.44) to this setting, we find that

$$\bar{\partial}\mathcal{A}^I = g_{IJ} \, \bar{\partial}\mathcal{A}^J \, g_{IJ}^{-1}.$$

In particular, it follows that $\langle B^I, \bar{\partial}\mathcal{A}^I \rangle = \langle B^J, \bar{\partial}\mathcal{A}^J \rangle$ on overlaps $U_I \cap U_J \neq \emptyset$ so that we obtain a $\mathfrak{g}$-valued 3-form $\langle B, \bar{\partial}\mathcal{A} \rangle \in \Omega^3(C \times \mathbb{R}^n)$ which, in particular, is a global $(1, 1)$-form along C so that it can be integrated over C (strictly speaking this is a fibre integration along the fibres of the projection $\pi_{\mathbb{R}^n} : C \times \mathbb{R}^n \to \mathbb{R}^n$). Now, by contrast, in this local trivialisation where the transition functions of $\pi_C^* \mathcal{P}$ explicitly depend on $\mathbb{R}^n$, we find using the transformation property (3.44) that

$$\langle \bar{\partial} B^I, \mathcal{A}^I \rangle = \langle \bar{\partial} B^J, \mathcal{A}^J \rangle - \langle \bar{\partial} B^J, g_{IJ}^{-1} \mathrm{d}_{\mathbb{R}^n} g_{IJ} \rangle. \tag{4.24}$$

In other words, the local expressions $\langle \bar{\partial} B^I, \mathcal{A}^I \rangle$ do not define a global $\mathfrak{g}$-valued 3-form on $C \times \mathbb{R}^n$. Likewise, the local expressions $\langle B^I, \mathcal{A}^I \rangle$ transform in a similar way to (4.24) and therefore do not define a global $\mathfrak{g}$-valued 2-form on $C \times \mathbb{R}^n$ either. This means that in the present context, "naive" integration by parts in an expression like $\int_C \langle B, \bar{\partial}\mathcal{A} \rangle$ is not possible, cf. Remark 3.3. This is the source of the term $\int_{c_\mathsf{p}} \langle L^0, \mathrm{d}_{\mathbb{R}^n} \gamma\gamma^{-1} \rangle$.

We now derive the univariational equations for the unifying 1d action in (4.14). To do so, it is helpful to recall the constraints on the various degrees of freedom. We recall that L^0 and L^1 are holomorphic $\mathfrak{g}$-valued (1,0)-forms on U_0 and $U_1 \setminus \{\mathsf{p}_\alpha\}$, γ is a holomorphic G-valued function on $U_0 \cap U_1$, and φ_α are elements of G. All of these depend on $t \in \mathbb{R}^n$, and are constrained by

$$L^1 = \gamma^{-1} L^0 \gamma, \qquad \operatorname*{Res}_{\mathsf{p}_\alpha} L^1_{z_I} = -\varphi_\alpha \Lambda_\alpha \varphi_\alpha^{-1}. \tag{4.25}$$

We continue to assume that the Higgs bundle determined by γ and L satisfies the stability condition (3.21). This means that there are no non-zero holomorphic functions $X^I : U_I \to \mathfrak{g}$ for $I \in \{0, 1\}$ with the property that $[L^I, X^I] = 0$ on U_I and $X^1 = \gamma^{-1} X^0 \gamma$ on $U_0 \cap U_1$.

Theorem 4.3. *The equations of motion of the unifying 1d action $S_{\mathrm{H},\Gamma}\big[L, \gamma, (\varphi_\alpha), t\big]$ in (4.14) with respect to the variables L, γ and φ_α take the form*

$$\partial_{t^i} L^0 = \big[M_i^0, L^0\big], \qquad \partial_{t^i} L^1 = \big[M_i^1, L^1\big]. \tag{4.26}$$

Here M_i^0, M_i^1 are holomorphic $\mathfrak{g}$-valued functions on U_0 and $U_1 \setminus \{\mathsf{q}_i\}$ that are uniquely determined by the constraints

$$\gamma^{-1} \partial_{t^i} \gamma = \gamma^{-1} M_i^0 \gamma - M_i^1, \qquad \operatorname*{Res}_{\mathsf{q}_I} M_i^1 = \nabla P_i\big(L^1_{z_I}(\mathsf{q}_i)\big). \tag{4.27}$$

Moreover, they satisfy the zero-curvature equations

$$\partial_{t^i} M_j^0 - \partial_{t^j} M_i^0 - \big[M_i^0, M_j^0\big] = 0, \qquad \partial_{t^i} M_j^1 - \partial_{t^j} M_i^1 - \big[M_i^1, M_j^1\big] = 0. \tag{4.28}$$

The equation of motion with respect to the set of times t^i for $i = 1, \dots, n$ describe the conservation equations $\partial_{t^i} H_j = 0$ for all $i, j = 1, \dots, n$.

Proof. The last statement for the variation with respect to t^i for $i = 1, \dots, n$ is straightforward, so we focus on deriving the equations of motions associated to variations of L, γ and φ_α.

It follows from the condition (4.25) that a variation of L, γ is described by a meromorphic $\mathfrak{g}$-valued $(1, 0)$-form δL^1 on U_1, a holomorphic $\mathfrak{g}$-valued $(1, 0)$-form δL^0 on U_0, and a holomorphic $\mathfrak{g}$-valued function $\gamma^{-1}\delta\gamma$ satisfying

$$\delta L^1 = \gamma^{-1} \delta L^0 \gamma + [L^1, \gamma^{-1}\delta\gamma] \tag{4.29}$$

on the intersection $U_0 \cap U_1$. The action should be stationary with respect to all such variations of L and to arbitrary variations of φ_α. In particular, it should be stationary when $\gamma^{-1}\delta\gamma = 0$, so that

$$\delta L^1 = \gamma^{-1} \delta L^0 \gamma, \tag{4.30}$$

and when $\delta\varphi_\alpha = 0$, in which case the second constraint in (4.25) gives $\operatorname{Res}_{\mathsf{p}_\alpha} \delta L^1_{z_I} = 0$. Thus, in that case δL^1 is holomorphic on U_1. The corresponding variation of $S_{\mathrm{H},\Gamma}$ is

$$\delta S_{\mathrm{H},\Gamma} = \int_0^1 \left(\frac{1}{2\pi i} \oint_{c_\mathsf{p}} \big\langle \delta L^1, \gamma^{-1}\partial_{u^j}\gamma \big\rangle - \big\langle \delta L^1_{z_I}(\mathsf{q}_i), \nabla P_i(L^1_{z_I}(\mathsf{q}_i)) \big\rangle \frac{\partial t^i}{\partial u^j} \right) \frac{\mathrm{d}u^j}{\mathrm{d}s}\,\mathrm{d}s. \tag{4.31}$$

This must vanish for all curves $\Gamma : (0, 1) \to \mathbb{R}^n$ so that for $i = 1, \ldots, n$ we have

$$0 = \frac{1}{2\pi i} \oint_{c_p} \langle \delta L^1, \gamma^{-1}\partial_{t^i}\gamma \rangle - \langle \delta L^1_{z_I}(q_i), \nabla P_i(L^1_{z_I}(q_i)) \rangle. \tag{4.32}$$

Consider, to begin with, a variation such that $\delta L^1_{z_I}(q_i) = 0$ for some fixed i. We find that

$$0 = \frac{1}{2\pi i} \oint_{c_p} \langle \delta L^1, \gamma^{-1}\partial_{t^i}\gamma \rangle = \frac{1}{2\pi i} \oint_{c_p} \langle \delta L^0, \partial_{t^i}\gamma\gamma^{-1} \rangle, \tag{4.33}$$

where the second equality just follows from the invariance of the bilinear form $\langle \cdot, \cdot \rangle :$ $\mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ and the constraint (4.30). Let us interpret the equation (4.33) using sheaf cohomology. For each $t \in \mathbb{R}^n$ we denote by $E_t \to C$ the holomorphic vector bundle with transition function $\mathrm{Ad}_{\gamma(t)}$ between the two local trivialisations over U_1 and U_0. The pair $\delta L = (\delta L^0, \delta L^1)$ of $\mathfrak{g}$-valud $(1, 0)$-forms on U_0 and U_1 related by the constraint (4.30) describes a holomorphic section of $\bigwedge^{1,0} C \otimes E_t$ that vanishes at q_i and hence it determines an element of the Čech cohomology group $H^0(C, \Lambda^{1,0}C \otimes E_t(-q_i))$, in which $(-q_i)$ indicates that the section vanishes at q_i. On the other hand, the pair $(\partial_{t^i}\gamma\gamma^{-1}, \gamma^{-1}\partial_{t^i}\gamma)$ of $\mathfrak{g}$-valued functions on $U_0 \cap U_1$ describe a holomorphic section of E_t over $U_0 \cap U_1$ defined relative to the local trivialisations over U_0 and U_1. It thus determines an element of the Čech cohomology group $H^1(C, E_t(q_i))$. The two integral expressions in (4.33) then correspond to the Serre duality pairing between $H^0(C, \Lambda^{1,0}C \otimes E_t(-q_i))$ and $H^1(C, E_t(q_i))$ described using the local trivialisation of E_t over U_1 and U_0, respectively. This pairing is nondegenerate, so if (4.33) holds for all variation δL then $(\partial_{t^i}\gamma\gamma^{-1}, \gamma^{-1}\partial_{t^i}\gamma)$ is zero in cohomology. This means that

$$\gamma^{-1}\partial_{t^i}\gamma = \gamma^{-1}M_i^0\gamma - M_i^1, \tag{4.34}$$

where M_i^0 and M_i^1 are sections of $E_t(q_i)$ over U_0 and U_1, respectively, and the right-hand side of (4.34) is the coboundary map in sheaf cohomology, expressed in the local trivialisation over U_1. More precisely, M_i^0 is a $\mathfrak{g}$-valued holomorphic function on U_0 and M_i^1 is a $\mathfrak{g}$-valued holomorphic on $U_1 \setminus \{q_i\}$ with a simple pole at q_i. There is some freedom in the choice of M_i^I solving (4.34): if N_i^0 and N_i^1 are sections of E_t over U_0 and $U_1 \setminus \{q_i\}$, respectively, such that N_i^1 has a simple pole at q_i and $\gamma^{-1}N_i^0\gamma = N_i^1$, then adding N_i^I to M_i^I produces a new solution of (4.34). Thus M_i^0, M_i^1 are unique up to the addition of elements of $H^0(C, E_t(q_i))$.

Now we insert (4.34) into (4.32), in which the variation δL^1 is no longer constrained to vanish at q_i. We find that

$$0 = \frac{1}{2\pi i} \oint_{c_p} \langle \delta L^1, \gamma^{-1}M_i^0\gamma - M_i^1 \rangle - \langle \delta L^1_{z_I}(q_i), \nabla P_i(L^1_{z_I}(q_i)) \rangle$$

$$= \frac{1}{2\pi i} \oint_{c_p} \langle \delta L^0, M_i^0 \rangle - \frac{1}{2\pi i} \oint_{c_p} \langle \delta L^1, M_i^1 \rangle - \langle \delta L^1_{z_I}(q_i), \nabla P_i(L^1_{z_I}(q_i)) \rangle. \tag{4.35}$$

The contour integral of $\langle \delta L^0, M_i^0 \rangle$ on the right-hand side vanishes because its integrand is holomorphic on U_0. Deforming the contour integral of $\langle \delta L^1, M_i^1 \rangle$ from the anticlockwise contour c_p to a clockwise small circle around q_i yields

$$\frac{1}{2\pi i} \oint_{c_p} \langle \delta L^1, M_i^1 \rangle = -\langle \delta L^1_{z_I}(q_i), \operatorname*{Res}_{q_i} M_i^1 \rangle.$$

We may then rewrite (4.35) as

$$0 = \Big\langle \delta L^1_{z_I}(\mathsf{q}_i), \operatorname*{Res}_{\mathsf{q}_i} M^1_i - \nabla P_i\big(L^1_{z_I}(\mathsf{q}_i)\big)\Big\rangle. \tag{4.36}$$

We would like to conclude from this that

$$\operatorname*{Res}_{\mathsf{q}_i} M^1_i = \nabla P_i\big(L^1_{z_I}(\mathsf{q}_i)\big). \tag{4.37}$$

However, this does not follow immediately. The variation δL^1 is constrained by (4.30), and this constraint may mean that $\delta L^1_{z_I}(\mathsf{q}_i)$ takes values in a proper subspace of the fibre $E_t|_{\mathsf{q}_i}$ of E_t at q_i. If so, (4.36) does not constrain all of the components of $\operatorname{Res}_{\mathsf{q}_i} M^1_i$. So to find a solution M^0_i, M^1_i of (4.36) we use more sophisticated methods.

Consider the exact sequence of sheaves on C given by

$$0 \longrightarrow E_t \longrightarrow E_t(\mathsf{q}_i) \longrightarrow E_t|_{\mathsf{q}_i} \longrightarrow 0. \tag{4.38}$$

Here, by abuse of notation, E_t denotes the sheaf of holomorphic sections of the vector bundle E_t, and $E_t(\mathsf{q}_i)$ denotes the sheaf of sections that are holomorphic on $C \setminus \{\mathsf{q}_i\}$ with a simple pole at q_i. Moreover, $E_t|_{\mathsf{q}_i}$ is the fibre of E_t at q_i, which we regard as a skyscraper sheaf on C supported at the point q_i. This exact sequence of sheaves induces a long exact sequence in Čech cohomology:

$$0 \longrightarrow H^0(C, E_t) \overset{\iota}{\longrightarrow} H^0(C, E_t(\mathsf{q}_i)) \overset{\operatorname{Res}_{\mathsf{q}_i}}{\longrightarrow} E_t|_{\mathsf{q}_i} \overset{h}{\longrightarrow} H^1(C, E_t) \longrightarrow \dots . \tag{4.39}$$

The map ι is the inclusion of the space of holomorphic sections in the space of sections with a simple pole. The map $\operatorname{Res}_{\mathsf{q}_i}$ is given by evaluating the residue of a section at the point q_i. The so-called connecting homomorphism h acts as follows: given $X_i \in E_t|_{\mathsf{q}_i}$, we use the exactness of (4.38) at $E_t|_{\mathsf{q}_i}$ to choose a holomorphic section $h(X_i)$ of E_t over $U_1 \setminus \{\mathsf{q}_i\}$ with a simple pole at q_i with residue X_i there. The restriction of this holomorphic section to $U_0 \cap U_1$ determines a cohomology class in $H^1(C, E_t)$ which is independent of the choice of holomorphic section.

With this notation and letting $X_i := \operatorname{Res}_{\mathsf{q}_i} M^1_i - \nabla P_i\big(L^1_{z_I}(\mathsf{q}_i)\big)$, we can rewrite (4.36) as follows:

$$0 = \big\langle \delta L^1_{z_I}(\mathsf{q}_i), X_i \big\rangle = -\frac{1}{2\pi i} \oint_{c_\mathsf{p}} \big\langle \delta L^1, h(X_i) \big\rangle = -\frac{1}{2\pi i} \oint_{c_\mathsf{p}} \big\langle \delta L^0, \gamma h(X_i) \gamma^{-1} \big\rangle, \tag{4.40}$$

where the last step is as in (4.33). These two integral expressions describe the Serre duality pairing between $\delta L \in H^0(C, \Lambda^{1,0} \otimes E^*_t)$ and $\big(\gamma h(X_i)\gamma^{-1}, h(X_i)\big) \in H^1(C, E_t)$ in the local trivialisation of E_t over U_1 and U_0, respectively. Since (4.40) holds for all $\delta L \in H^0(C, \Lambda^{1,0} \otimes E_t)$, and since the pairing is nondegenerate, we conclude that $h(X_i) = 0 \in H^1(C, E_t)$. Since the sequence (4.39) is exact, there exists an $N_i \in H^0\big(C, E_t(\mathsf{q}_i)\big)$ such that $X_i = \operatorname{Res}_{\mathsf{q}_i} N_i$. We then have that $\tilde{M}_i := M_i - N_i$ solves both (4.34) and (4.37), as required.

There is still some freedom in the choice of M_i solving (4.34), (4.37). The solution of (4.34) is unique up to the addition of $\mathfrak{g}$-valued functions N^0_i, N^1_i on U_0, U_1 satisfying $N^1_i = \gamma^{-1} N^0_i \gamma$. Since the pole of M^1_i is fixed by (4.37), these functions N^I_i are holomorphic, so they determine a holomorphic section of E_t. Therefore M_i is unique up to addition of elements of $H^0(C, E_t)$.

The analysis so far was based on variations such that $\delta\gamma = 0$ and $\delta\varphi_\alpha = 0$, which produced necessary conditions for the stationarity of the action. We now consider general variations of L, φ_α and γ. We assume that M_i has been chosen solving the necessary conditions (4.34) and (4.37). From (4.25), δL^1 is now meromorphic and the variations satisfy

$$[L^0, \delta\gamma\gamma^{-1}] = \gamma\delta L^1\gamma^{-1} - \delta L^0, \tag{4.41}$$

$$\operatorname*{Res}_{\mathsf{p}_\alpha} \delta L^1_{z_I} = -[\delta\varphi_\alpha\varphi_\alpha^{-1}, \varphi_\alpha\Lambda_\alpha\varphi_\alpha^{-1}]. \tag{4.42}$$

The variation of $S_{H,\Gamma}$ is

$$\delta S_{H,\Gamma} = \int_0^1 \left(\frac{1}{2\pi i} \oint_{c_\mathsf{p}} \langle\delta L^1, \gamma^{-1}\partial_{u^j}\gamma\rangle + \frac{1}{2\pi i}\oint_{c_\mathsf{p}}\langle L^1, \gamma^{-1}\partial_{u^j}(\delta\gamma\gamma^{-1})\gamma\rangle \right.$$
$$\left. + \sum_{\alpha=1}^N \langle\Lambda_\alpha, \varphi_\alpha^{-1}\partial_{u^j}(\delta\varphi_\alpha\varphi_\alpha^{-1})\varphi_\alpha\rangle - \langle\delta L^1_{z_I}(\mathsf{q}_i), \nabla P_i(L^1_{z_I}(\mathsf{q}_i))\rangle \frac{\partial t^i}{\partial u^j} \right) \frac{\mathrm{d}u^j}{\mathrm{d}s} \mathrm{d}s. \tag{4.43}$$

We will rewrite this by substituting (4.34) in the first term and by using the following contour integral, which is evaluated using (4.37) and (4.42):

$$\frac{1}{2\pi i}\oint_{c_\mathsf{p}}\langle\delta L^1, M_i^1\rangle = -\langle\delta L^1_{z_I}(\mathsf{q}_i), \nabla P_i(L^1_{z_I}(\mathsf{q}_i))\rangle + \sum_{\alpha=1}^N \langle[\delta\varphi_\alpha\varphi_\alpha^{-1}, \varphi_\alpha\Lambda_\alpha\varphi_\alpha^{-1}], M_i^1(\mathsf{p}_\alpha)\rangle. \tag{4.44}$$

We obtain

$$\delta S_{H,\Gamma} = \int_0^1 \left(\frac{1}{2\pi i}\oint_{c_\mathsf{p}}\langle\gamma\,\delta L^1\,\gamma^{-1}, M_i^0\rangle\frac{\partial t^i}{\partial u^j} + \frac{1}{2\pi i}\oint_{c_\mathsf{p}}\langle\gamma\,L^1\,\gamma^{-1}, \partial_{u^j}(\delta\gamma\gamma^{-1})\rangle \right.$$
$$\left. + \sum_{\alpha=1}^N\langle\Lambda_\alpha, \varphi_\alpha^{-1}\partial_{u^j}(\delta\varphi_\alpha\varphi_\alpha^{-1})\varphi_\alpha\rangle - \sum_{\alpha=1}^N\langle[\delta\varphi_\alpha\varphi_\alpha^{-1}, \varphi_\alpha\Lambda_\alpha\varphi_\alpha^{-1}], M_i^1(\mathsf{p}_\alpha)\rangle\frac{\partial t^i}{\partial u^j} \right) \frac{\mathrm{d}u^j}{\mathrm{d}s}\mathrm{d}s. \tag{4.45}$$

We now rewrite this using (4.25) and (4.41):

$$\delta S_{H,\Gamma} = \int_0^1 \left(\frac{1}{2\pi i}\oint_{c_\mathsf{p}}\langle\delta L^0 + [L^0, \delta\gamma\gamma^{-1}], M_i^0\rangle\frac{\partial t^i}{\partial u^j} + \frac{1}{2\pi i}\oint_{c_\mathsf{p}}\langle L^0, \partial_{u^j}(\delta\gamma\gamma^{-1})\rangle \right.$$
$$\left. + \sum_\alpha\langle\varphi_\alpha\Lambda_\alpha\varphi_\alpha^{-1}, \partial_{u^j}(\delta\varphi_\alpha\varphi_\alpha^{-1})\rangle - \sum_\alpha\langle[\delta\varphi_\alpha\varphi_\alpha^{-1}, \varphi_\alpha\Lambda_\alpha\varphi_\alpha^{-1}], M_i^1(\mathsf{p}_\alpha)\rangle\frac{\partial t^i}{\partial u^j} \right) \frac{\mathrm{d}u^j}{\mathrm{d}s}\mathrm{d}s. \tag{4.46}$$

The contour integral of $\langle\delta L^0, M_i^0\rangle$ vanishes because it is holomorphic on U_0. After integration by parts, the variation becomes

$$\delta S_{H,\Gamma} = \int_0^1 \left(\frac{1}{2\pi i}\oint_{c_\mathsf{p}}\langle[M_i^0, L^0], \delta\gamma\gamma^{-1}\rangle\frac{\partial t^i}{\partial u^j} - \frac{1}{2\pi i}\oint_{c_\mathsf{p}}\langle\partial_{u^j}L^0, \delta\gamma\gamma^{-1}\rangle \right.$$

$$-\sum_\alpha \big\langle \partial_{u^j}(\varphi_\alpha \wedge_\alpha \varphi_\alpha^{-1}), \delta\varphi_\alpha \varphi_\alpha^{-1}\big\rangle + \sum_\alpha \Big\langle [M_i^1(\mathsf{p}_\alpha), \varphi_\alpha \wedge_\alpha \varphi_\alpha^{-1}], \delta\varphi_\alpha \varphi_\alpha^{-1}\Big\rangle \frac{\partial t^i}{\partial u^j}\Big)\frac{du^j}{ds}\,ds\,.$$

$$(4.47)$$

Let us restrict attention to variations for which $\delta\varphi_\alpha = 0$. Then $\delta S_{\mathrm{H},\Gamma}$ vanishes for all curves Γ if and only if

$$0 = \oint_{c_\mathsf{p}} \big\langle \partial_{t^i}L^0 - [M_i^0, L^0], \delta\gamma\,\gamma^{-1}\big\rangle = \oint_{c_\mathsf{p}} \big\langle \partial_{t^i}L^1 - [M_i^1, L^1], \gamma^{-1}\delta\gamma\big\rangle \quad (4.48)$$

for all variations $\delta\gamma\,\gamma^{-1}$ satisfying (4.41), where the second equality follows from noting that

$$\gamma^{-1}\big(\partial_{t^i}L^0 - [M_i^0, L^0]\big)\gamma = \partial_{t^i}L^1 - [M_i^1, L^1] \qquad (4.49)$$

as a consequence of (4.34). In particular, it follows that $\partial_{t^i}L^I - [M_i^I, L^I]$ can be interpreted as a section of E_t. It could have poles at the point q_i and any of the points p_α, because M_i^1 and L^1 have a simple poles there. However, from (4.37) the residue at q_i is given by

$$-\Big[\operatorname*{Res}_{\mathsf{q}_i} M_i^1, L_{z_I}^1(\mathsf{q}_i)\Big] = -\Big[\nabla P_i\big(L_{z_I}^1(\mathsf{q}_i)\big), L_{z_I}^1(\mathsf{q}_i)\Big] \qquad (4.50)$$

and this vanishes because P_i is an invariant polynomial. So $\partial_{t^i}L^1 - [M_i^1, L^1]$ only has poles at the points p_α, and it determines a holomorphic section in $H^0\big(C, \Lambda^{1,0}C \otimes E_t^*(\sum_\alpha \mathsf{p}_\alpha)\big)$.

To interpret (4.48) using Serre duality, we note using the constraint (4.41) that the components of the holomorphic section $Y := (\delta\gamma\,\gamma^{-1}, \gamma^{-1}\delta\gamma)$ of E_t over $U_0 \cap U_1$ defined relative to the local trivialisations over U_0 and U_1, respectively, satisfy

$$\big[L^0, \delta\gamma\,\gamma^{-1}\big] = \gamma\delta L^1 \gamma^{-1} - \delta L^0, \qquad \big[L^1, \gamma^{-1}\delta\gamma\big] = \delta L^1 - \gamma^{-1}\delta L^0 \gamma.$$

Since $\delta L^0, \delta L^1$ are holomorphic on U_0, U_1 (because we are assuming that $\delta\varphi_\alpha = 0$), these equations say that $[L, Y]$ lies in the image of the Čech coboundary map, so determines a trivial element of $H^1\big(C, \Lambda^{1,0}C \otimes E_t\big)$. In other words, Y is in the kernel of the map

$$[L, \cdot] : H^1\big(C, E_t\big(-\textstyle\sum_\alpha \mathsf{p}_\alpha\big)\big) \longrightarrow H^1\big(C, \Lambda^{1,0}C \otimes E_t\big). \qquad (4.51)$$

(The notation $(-\sum_\alpha \mathsf{p}_\alpha)$ in the domain of this map denotes the sheaf of sections vanishing at the points p_α, and appears because L has poles at these points). Thus we may interpret the integrals in (4.48) as the Serre duality pairing between $\partial_{t^i}L - [M_i, L] \in H^0\big(C, \Lambda^{1,0}C \otimes E_t^*(\sum_\alpha \mathsf{p}_\alpha)\big)$ and $Y = (\delta\gamma\,\gamma^{-1}, \gamma^{-1}\delta\gamma) \in H^1\big(C, E_t(-\sum_\alpha \mathsf{p}_\alpha)\big)$. The pairing is required to vanish for all Y in the kernel of the map (4.51). This happens if and only if $\partial_{t^i}L - [M_i, L]$ lies in the image of the adjoint of (4.51). The adjoint of (4.51) is the map

$$-[L, \cdot] : H^0\big(C, E_t\big) \longrightarrow H^0\big(C, \Lambda^{1,0}C \otimes E_t(\textstyle\sum_\alpha \mathsf{p}_\alpha)\big) \qquad (4.52)$$

because, by Serre duality, the dual space of $H^1(C, V)$ is $H^0(C, \Lambda^{1,0}C \otimes V^*)$, and

$$\oint_{c_\mathsf{p}} \langle [L, u], v\rangle = -\oint_{c_\mathsf{p}} \langle u, [L, v]\rangle \qquad (4.53)$$

for every $u \in H^1\big(C, E_t(-\sum_\alpha \mathsf{p}_\alpha)\big)$ and $v \in H^0(C, E_t)$. Since $\partial_{t^i} L - [M_i, L]$ is in the image of the map (4.52), there exist holomorphic functions $N_i^I : U_I \to \mathfrak{g}$ for $I = 0, 1$ satisfying

$$\partial_{t^i} L^I - \big[M_i^I, L^I\big] = \big[N_i^I, L^I\big], \qquad \gamma^{-1} N_i^0 \gamma = N_i^1. \tag{4.54}$$

Recall from earlier that the solution of (4.34) and (4.37) was unique up to addition of elements of $H^0(C, E_t)$. Therefore we are free to define $\tilde{M}_i^I := M_i^I + N_i^I$, and this choice of M_i satisfies the Lax equations (4.26) and the constraints (4.27). The choice of M_i is now unique up to addition of elements of $\ker[L, \cdot] \subset H^0(C, E_t)$. But the stability condition (3.21) means that there are no holomorphic sections of C that commute with L. So M_i is unique, as claimed.

We have shown that $S_{\mathrm{H},\Gamma}$ is critical with respect to variations of L, γ with φ_α fixed if and only if L solves the Lax equation (4.26). It remains to check that solutions of these equations are also critical points of $S_{\mathrm{H},\Gamma}$ with respect to variations of φ_α. The variation of $S_{\mathrm{H},\Gamma}$ is given in (4.47). The term involving $\delta\gamma$ vanishes because L satisfies the Lax equations (4.26). The term involving $\delta\varphi_\alpha$ also vanishes because taking residues of (4.26) gives

$$\partial_{t^i}\big(\varphi_\alpha \Lambda_\alpha \varphi_\alpha^{-1}\big) = \big[M_i^1(\mathsf{p}_\alpha), \varphi_\alpha \Lambda_\alpha \varphi_\alpha^{-1}\big]. \tag{4.55}$$

So the variation vanishes and $S_{\mathrm{H},\Gamma}$ is critical with respect to variations of $L, \gamma, \varphi_\alpha$.

Finally, to show the zero-curvature equation (4.28) we note that the Lax equations (4.26) imply

$$\big[\partial_{t^i} M_j^I - \partial_{t^j} M_i^I - \big[M_i^I, M_j^I\big], L^I\big] = 0 \tag{4.56}$$

for $I \in \{0, 1\}$. On the other hand, it also follows from the first equation in (4.27) that

$$\gamma^{-1}\big(\partial_{t^i} M_j^0 - \partial_{t^j} M_i^0 - \big[M_i^0, M_j^0\big]\big)\gamma = \partial_{t^i} M_j^1 - \partial_{t^j} M_i^1 - \big[M_i^1, M_j^1\big] \tag{4.57}$$

on $U_0 \cap U_1$. The zero-curvature equation (4.28) now follows by the stability condition. $\qquad\square$

5. Examples

The unifying Lagrangian 1-form (4.14b) has the typical structure of a Lagrangian 1-form in phase space coordinates as first used in [12] for finite-dimensional integrable systems on coadjoint orbits, and further developed in [13] in relation to the univariational principle. It is the difference of a kinetic part and a potential part involving all the invariant Hamiltonians H_i. The essential novelty compared to all Lagrangian 1-forms considered so far is that the kinetic part involves not only the group coordinates φ_α of the coadjoint orbits at the marked points p_α for $\alpha = 1, \ldots, N$, but also the transition function $\gamma : (U_0 \cap U_1) \times \mathbb{R}^n \to G$ of the holomorphic principal G-bundle. This introduces a new type of kinetic term in the action which is crucial for generating the complete set of kinetic contributions. We will see explicit examples of this in the genus 1 case considered in Sect. 5.2, where we derive a Lagrangian 1-form for the elliptic Gaudin model and also the elliptic spin Calogero–Moser model as a special case. In the genus 0 case considered in Sect. 5.1, this new kinetic term is absent and we recover the familiar Lagrangian 1-form first introduced in [12,15].

Since this section deals with explicit examples and we only work with a matrix Lie algebra $\mathfrak{g}$, we fix the nondegenerate invariant bilinear pairing $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{C}$ to be the usual one given in terms of the trace by $\langle A, B \rangle = \mathrm{Tr}(AB)$. Also, throughout this section, we identify $\mathfrak{g}^*$ with $\mathfrak{g}$ and the coadjoint action with the adjoint action. Further, we fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ with basis $\{\mathsf{H}_\mu\}_{\mu=1}^{\mathrm{rk}\,\mathfrak{g}}$ and the corresponding root generators $\{\mathsf{E}_\varrho\}$ of $\mathfrak{g}$.

5.1. Rational Gaudin Hierarchy.

We begin with the simplest case of genus 0 so that $C = \mathbb{C}P^1$ is the Riemann sphere. We will show that when the unifying Lagrangian 1-form (4.14b) is specialised to the case $\gamma = 1$ we obtain the rational Gaudin hierarchy. Gaudin models are a broad class of integrable systems associated with Lie algebras equipped with a nondegenerate invariant bilinear form. They were first introduced for the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ by Gaudin in [35] as quantum integrable spin chains with long-range interactions, and later extended to arbitrary semisimple Lie algebras in [36]. In the rational case, the classical Gaudin model provides the simplest example of a Hitchin system with marked points and has been extensively studied from this perspective; see, for instance, [28,55].

We choose $N \in \mathbb{Z}_{\geq 1}$ distinct marked points $\{\mathsf{p}_\alpha\}_{\alpha=1}^N$. We also need sufficiently many points $\{\mathsf{q}_i\}_{i=1}^n$ which are distinct from the $\{\mathsf{p}_\alpha\}_{\alpha=1}^N$, where n will be half the dimension of the phase space, to obtain a sufficient number of Hamiltonians. Let U_0 be a neighbourhood of a point $\mathsf{p} \in C$ distinct from $\{\mathsf{p}_\alpha\}_{\alpha=1}^N$ and $\{\mathsf{q}_i\}_{i=1}^n$, such that U_0 does not contain any of these other points. Further, let $U_1 := C\backslash\{\mathsf{p}\}$ which we equip with a holomorphic coordinate $z\colon U_1 \rightarrow \mathbb{C}$, where, in contrast to the general case of §4, we have dropped the index on z for notational simplicity.

Lax matrix: Following the general setup in §4.1, we obtain a Lax matrix for the rational Gaudin model as the solution of the moment map condition $\mu = 0$ (4.11b) in a local trivialisation where $A''^I = 0$. Using (4.12a), we can write

$$L^1 = L_z \mathrm{d}z = \left(\frac{L_\alpha}{z - z(\mathsf{p}_\alpha)} + J_\alpha^1 \right) \mathrm{d}z. \tag{5.1}$$

locally near the point p_α. Here $L_\alpha = -\varphi_\alpha \Lambda_\alpha \varphi_\alpha^{-1}$ denotes the residue at p_α and J_α^1 the holomorphic part of L_z in the neighbourhood of p_α. Since $\gamma = 1$ we have $L^0 = L^1$ on $U_0 \cap U_1$ so that L^1 extends holomorphically to the point $\mathsf{p} \in C = \mathbb{C}P^1$ at infinity in the coordinate z. We can therefore write the Lax matrix L^1 as a global $\mathfrak{g}$-valued meromorphic $(1, 0)$-form

$$L^1 = L_z \mathrm{d}z = \sum_{\alpha=1}^N \frac{L_\alpha}{z - z(\mathsf{p}_\alpha)} \mathrm{d}z \tag{5.2}$$

together with the constraint $\sum_\alpha L_\alpha = 0$ coming from the fact that L^1 is regular at $z(\mathsf{p}) = \infty$.

Lagrangian 1-form: Let us look at the unifying Lagrangian 1-form (4.14b) in the present case. Since $\gamma = 1$, the kinetic term in the Lagrangian 1-form (4.14b) involving the transition function drops out, and we are left with

$$\mathscr{L}_{\mathrm{RG}} = \sum_{\alpha=1}^N \mathrm{Tr}\left(\Lambda_\alpha \varphi_\alpha^{-1} \mathrm{d}_{\mathbb{R}^n} \varphi_\alpha \right) - H_i \, \mathrm{d}_{\mathbb{R}^n} t^i \tag{5.3}$$

where

$$H_i = P_i(L_z(\mathsf{q}_i)) \tag{5.4}$$

with G-invariant polynomials P_i acting on L_z given by (5.2). The Lagrangian 1-form (5.3) describes the rational Gaudin hierarchy. A Lagrangian 1-form for the rational Gaudin hierarchy was first obtained in [12, Section 7] using an algebraic approach. The Lagrangian 1-form describing the dynamics for a single Hamiltonian of the Gaudin model was also obtained previously in [71]. By the same derivation as in those references, one shows that the Euler–Lagrange equations associated to (5.3) give the following collection of equations

$$\partial_{t^i} L_\alpha = \left[\frac{\nabla P_i(L_z(\mathsf{q}_i))}{z(\mathsf{p}_\alpha) - z(\mathsf{q}_i)}, L_\alpha \right], \tag{5.5}$$

for all i. One can check that one recovers a hierarchy of Lax equations from the above collection of equations using (4.13).

5.2. Elliptic Gaudin and Elliptic spin Calogero–Moser Hierarchies.

Let us now consider the genus 1 case so that the Riemann surface C is a torus $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ with $\mathrm{Im}(\tau) > 0$. Following the general setup from Sect. 4 we fix a point $\mathsf{p} \in C$ and take U_0 to be a small neighbourhood of p, and we let $U_1 := C\backslash\{\mathsf{p}\}$. We fix a coordinate z on C with the identifications $z \sim z + 1$ and $z \sim z + \tau$ such that $z(\mathsf{p}) = 0$. Our starting point is the Lagrangian 1-form (4.14b) but now specialised to a holomorphic transition function γ on the annulus $U_0 \cap U_1$ given by

$$\gamma = \exp\left(\frac{\mathcal{Q}}{z}\right) \quad \text{with} \quad \mathcal{Q} = q^\mu \mathsf{H}_\mu \tag{5.6}$$

where $q^\mu \in \mathbb{C}$, for each $\mu \in \{1, \ldots, \mathrm{rk}\,\mathfrak{g}\}$, is constant in z. We will show that the resulting Lagrangian 1-form describes the elliptic Gaudin hierarchy, and in the case with $N = 1$ marked point the elliptic spin Calogero–Moser hierarchy.

The elliptic Gaudin model we construct was obtained from a Hamiltonian reduction procedure in [28,55]. To avoid possible confusion, it is worth noting in passing that for $\mathfrak{g} = \mathfrak{sl}_m$ there is another integrable system that also goes by the name elliptic Gaudin model which was originally derived as a limit of the XYZ spin chain in [62,63]. It is not clear if and how these two models are related, especially since the elliptic Gaudin model we consider is known to have a dynamical r-matrix [28] while the one of [62,63] can be built from Belavin's elliptic solution [6] of the classical Yang–Baxter equation by following the procedure of [43]. A spin generalisation of the Calogero–Moser model was first defined in [37], and its case with elliptic potential was realised as a Hitchin system in [28,38,52,55]. The main goal of this section is to use the unifying Lagrangian 1-form (4.14b) to obtain variational descriptions of these two hierarchies.[3]

We choose $N \in \mathbb{Z}_{\geq 1}$ distinct marked points $\{\mathsf{p}_\alpha\}_{\alpha=1}^N$ in $C\backslash\overline{U_0}$. As in the general case, we need sufficiently many additional points $\mathsf{q}_i \in C \setminus \overline{U_0}$ distinct from $\{\mathsf{p}_\alpha\}_{\alpha=1}^N$ to obtain a sufficient number of Hamiltonians. We will later specialise to the case of a single marked point p_1, i.e. taking $N = 1$.

---

[3] Recently, the Lagrangian 1-form structure of certain Calogero–Moser-type systems was studied in [45]. Starting from a general ansatz for the kinetic term of the Lagrangian 1-form, the authors used the resulting generalised Euler–Lagrange equations to derive the integrable cases of these systems.

Lax matrix: From (4.12a), we have

$$L^1 = L_z dz = \left(\frac{L_\alpha}{z - z(\mathsf{p}_\alpha)} + J_\alpha^1 \right) dz \tag{5.7}$$

locally around the point p_α in our choice of local trivialisation. Here $L_\alpha = -\varphi_\alpha \Lambda_\alpha \varphi_\alpha^{-1}$ denotes the residue at p_α and J_α^1 the holomorphic part of L_z in the neighbourhood of p_α. We can express L_α and $L^1 = L_z dz$ in the basis $(\mathsf{H}_\mu, \mathsf{E}_\varrho)$ as

$$L_\alpha = (L_\alpha)^\mu \mathsf{H}_\mu + (L_\alpha)^\varrho \mathsf{E}_\varrho \quad \text{and} \quad L_z = L^\mu \mathsf{H}_\mu + L^\varrho \mathsf{E}_\varrho. \tag{5.8}$$

Since $L^1 = L_z dz$ is a meromorphic 1-form on the punctured torus U_1, it satisfies

$$L^1(z+1) = L^1(z+\tau) = L^1(z). \tag{5.9}$$

Also, recall from §4.1 that L^0 is holomorphic on U_0 and we have the relation $L^0 = \gamma L^1 \gamma^{-1}$ on $U_0 \cap U_1$. Therefore, we can write

$$L^0 = \left(L^\mu \mathsf{H}_\mu + L^\varrho \exp\left(\frac{\varrho(\mathcal{Q})}{z} \right) \mathsf{E}_\varrho \right) dz. \tag{5.10}$$

We can express L^μ and L^ϱ in (5.8) and (5.10) in terms of the Weierstrass ζ-function and the Weierstrass σ-function which are respectively defined by the relations

$$\frac{d\zeta(z)}{dz} = -\wp(z), \quad \lim_{z \to 0}\left(\zeta(z) - \frac{1}{z} \right) = 0 \tag{5.11}$$

and

$$\frac{d\log(\sigma(z))}{dz} = \zeta(z), \quad \lim_{z \to 0} \frac{\sigma(z)}{z} = 1 \tag{5.12}$$

where $\wp$ is the Weierstrass $\wp$-function. While the Weierstrass $\wp$-function is doubly periodic, the σ-function and the ζ-function satisfy

$$\zeta(z + 2\omega_l) = \zeta(z) + 2\eta_l, \tag{5.13}$$

$$\sigma(z + 2\omega_l) = -\sigma(z)e^{2\eta_l(z+2\omega_l)}, \tag{5.14}$$

where $2\omega_1$ and $2\omega_2$ are the periods of a generic torus and $\eta_l = \zeta(\omega_l)$, for $l = 1, 2$. Here we have $2\omega_1 = 1$ and $2\omega_2 = \tau$.

It follows from (5.7) that the elliptic function L^μ is meromorphic with simple poles at $\{\mathsf{p}_\alpha\}_{\alpha=1}^N$ with residues $(L_\alpha)^\mu$, so it can be expressed as

$$L^\mu = \pi^\mu + \sum_{\alpha=1}^N (L_\alpha)^\mu \zeta(z - z(\mathsf{p}_\alpha)) \tag{5.15a}$$

where π^μ is constant in z, and we have

$$\sum_{\alpha=1}^N (L_\alpha)^\mu = 0 \tag{5.15b}$$

since the sum of residues over an irreducible set of poles of an elliptic function vanishes. Next, for the function L^ϱ, (5.7) tells us that it has poles at $\{\mathsf{p}_\alpha\}_{\alpha=1}^N$ with residues $(L_\alpha)^\varrho$ and (5.10) implies that the function $L^\varrho e^{\frac{\varrho(\mathcal{Q})}{z}}$ is holomorphic near $z = 0$. It then follows that

$$L^\varrho = \sum_{\alpha=1}^N (L_\alpha)^\varrho \frac{\sigma(\varrho(\mathcal{Q}) + z - z(\mathsf{p}_\alpha))}{\sigma(\varrho(\mathcal{Q}))\sigma(z - z(\mathsf{p}_\alpha))} e^{-\varrho(\mathcal{Q})\zeta(z)}. \tag{5.16}$$

Then, we get $L^1 = (L^\mu \mathsf{H}_\mu + L^\varrho \mathsf{E}_\varrho)dz$ with L^μ and L^ϱ given by (5.15) and (5.16) as the Lax matrix of the elliptic Gaudin model.

In the special case when $N = 1$ and $\mathsf{p}_1 = 0$, the components (5.15) simply reduce to $L^\mu = \pi^\mu$, while the components (5.16) read

$$L^\varrho = \sum_{\alpha=1}^N (L_\alpha)^\varrho \frac{\sigma(\varrho(\mathcal{Q}) + z)}{\sigma(\varrho(\mathcal{Q}))\sigma(z)} e^{-\varrho(\mathcal{Q})\zeta(z)}. \tag{5.17}$$

When $\mathfrak{g} = \mathfrak{sl}_m(\mathbb{C})$, coincides with the Lax matrix of the elliptic spin Calogero–Moser model, see, e.g. [3, Chapter 7].

Lagrangian 1-form: We now specialise the unifying Lagrangian 1-form (4.14b) to the present case. Recalling that $\mathfrak{g}$ is taken to be a matrix Lie algebra we take the nondegenerate bilinear form on $\mathfrak{g}$ is given by the trace, this takes the form

$$\mathscr{L}_{\mathrm{EG}} = \frac{1}{2\pi i} \int_{c_\mathsf{p}} \mathrm{Tr}\left(L^1 \wedge \gamma^{-1}\mathrm{d}_{\mathbb{R}^n}\gamma\right) + \sum_{\alpha=1}^N \mathrm{Tr}\left(\Lambda_\alpha\varphi_\alpha^{-1}\mathrm{d}_{\mathbb{R}^n}\varphi_\alpha\right) - H_i\,\mathrm{d}_{\mathbb{R}^n}t^i \tag{5.18}$$

where the transition function γ is given by (5.6) and

$$H_i = P_i(L_z(\mathsf{q}_i)). \tag{5.19}$$

Let us evaluate the integral in the first term on the right-hand side of (5.18). We have

$$\frac{1}{2\pi i} \int_{c_\mathsf{p}} \mathrm{Tr}\left(L_z\mathrm{d}z \wedge \gamma^{-1}\mathrm{d}_{\mathbb{R}^n}\gamma\right) = \frac{1}{2\pi i} \int_{c_\mathsf{p}} \mathrm{Tr}\left(L_z\mathrm{d}z \wedge \frac{\mathrm{d}_{\mathbb{R}^n}\mathcal{Q}}{z}\right) = p_\mu\mathrm{d}_{\mathbb{R}^n}q^\mu \tag{5.20}$$

where the coordinates q^μ were introduced in (5.6) and we defined

$$p_\mu := \mathrm{Tr}\left(L_z(z(\mathsf{p}))\mathsf{H}_\mu\right), \quad \mu \in \{1, \ldots, \mathrm{rk}\,\mathfrak{g}\}. \tag{5.21}$$

Plugging back (5.20) into (5.18), we get a Lagrangian 1-form for the elliptic Gaudin hierarchy

$$\mathscr{L}_{\mathrm{EG}} = p_\mu\mathrm{d}_{\mathbb{R}^n}q^\mu + \sum_{\alpha=1}^N \mathrm{Tr}\left(\Lambda_\alpha\varphi_\alpha^{-1}\mathrm{d}_{\mathbb{R}^n}\varphi_\alpha\right) - H_i\,\mathrm{d}_{\mathbb{R}^n}t^i, \tag{5.22}$$

with H_i defined by (5.19) depending on the elliptic Gaudin Lax matrix given by (5.15) and (5.16). Note that in contrast with the Lagrangian 1-form (5.3) obtained in the genus 0 case, the Lagrangian 1-form above has an additional kinetic term corresponding to

the cotangent bundle degrees of freedom p_μ, q^μ arising from the non-triviality of the principal G-bundle in genus 1.

Writing a Lagrangian 1-form for the elliptic spin Calogero–Moser hierarchy only requires us to set $N = 1$ in the Lagrangian 1-form for the elliptic Gaudin hierarchy. We then get

$$\mathscr{L}_{\text{ESCM}} = p_\mu \, \mathrm{d}_{\mathbb{R}^n} q^\mu + \mathrm{Tr}\left(\Lambda_1 \varphi_1^{-1} \mathrm{d}_{\mathbb{R}^n} \varphi_1\right) - H_i \, \mathrm{d}_{\mathbb{R}^n} t^i \tag{5.23}$$

for this special case.

Alternative description: Lax matrices for the elliptic Gaudin and elliptic spin Calogero–Moser models have appeared before in the literature and we now explain how they are related to the ones we used above to formulate our Lagrangian 1-form, hence justifying that we have indeed built a variational description for the elliptic Gaudin hierarchy [28,55]. The two descriptions are related by a change of local trivialisation, as we now show.

Recall that under a change of local trivialisation implemented through smooth functions f_0 on U_0 and f_1 on U_1, the $(0, 1)$-connection A''^I and the $(1, 0)$-form B^I transform as

$$A''^I \longmapsto \widetilde{A}''^I = f_I A''^I f_I^{-1} - \bar{\partial} f_I f_I^{-1}, \tag{5.24a}$$

$$L^I \longmapsto \widetilde{L}^I = f_I L^I f_I^{-1} \tag{5.24b}$$

locally in U_I for each $I \in \{0, 1\}$, while the new transition function $\widetilde{\gamma}$ reads

$$\widetilde{\gamma} = f_0 \gamma f_1^{-1}, \tag{5.25}$$

and $\widetilde{\varphi}_\alpha = f_1(z(p_\alpha)) \, \varphi_\alpha$ so that

$$\widetilde{L}_\alpha = f_1(z(p_\alpha)) \, L_\alpha \, f_1(z(p_\alpha))^{-1}. \tag{5.26}$$

In our choice of local trivialisation with respect to which $A''^I = 0$ for $I \in \{0, 1\}$, we obtain a Lax matrix L for the elliptic Gaudin model described by (5.15) and (5.16). The alternative description of the elliptic Gaudin model, which can be found in [55], for instance, corresponds to a local trivialisation with respect to which

$$\widetilde{A}''^I = \mathcal{V} = v^\mu \mathsf{H}_\mu, \tag{5.27}$$

where v^μ is constant in $(z, \bar{z})$ for each $I \in \{0, 1\}$, and with trivial transition function. To implement the change of local trivialisation from the one with respect to which $A''^I = 0$ to the one defined by (5.27), we need functions f_0, f_1 to satisfy

$$-\bar{\partial} f_I f_I^{-1} = \widetilde{A}''^I = \mathcal{V}, \quad I \in \{0, 1\}. \tag{5.28}$$

The new transition function is given by

$$\widetilde{\gamma} = f_0 \gamma f_1^{-1}. \tag{5.29}$$

Then choosing f_0, f_1 such that $f_0 = f_1 \exp\left(-\frac{\mathcal{Q}}{z}\right)$ we obtain $\widetilde{\gamma} = 1$. Finally, since f_1 is a smooth function on the punctured torus U_1, it must satisfy:

$$f_1(z + 1) = f_1(z + \tau) = f_1(z). \tag{5.30}$$

From (5.28) and (5.30), we get

$$f_1 = \exp\left(\mathcal{Q}\left(\zeta(z) + 2\zeta(\tfrac{1}{2})\frac{z\bar{\tau} - \bar{z}\tau}{\tau - \bar{\tau}} - 2\zeta(\tfrac{\tau}{2})\frac{z - \bar{z}}{\tau - \bar{\tau}}\right)\right) \tag{5.31}$$

together with the relation

$$\mathcal{V} = -\mathcal{Q}\frac{2}{\tau - \bar{\tau}}\left(\zeta(\tfrac{\tau}{2}) - \tau\zeta(\tfrac{1}{2})\right) = \frac{2\pi i}{\tau - \bar{\tau}}\mathcal{Q}, \tag{5.32}$$

where in the final step we used Legendre's relation between the periods and quasiperiods of $\zeta(z)$. Expressing $\widetilde{L}^1 = f_1 L^1 f_1^{-1} = \widetilde{L}_z \mathrm{d}z$ in terms of the basis $(\mathsf{H}_\mu, \mathsf{E}_\varrho)$ of $\mathfrak{g}$ as

$$\widetilde{L}_z = \widetilde{L}^\mu \mathsf{H}_\mu + \widetilde{L}^\varrho \mathsf{E}_\varrho = L^\mu \mathsf{H}_\mu + L^\varrho f_1 \mathsf{E}_\varrho f_1^{-1}, \tag{5.33}$$

we find

$$\widetilde{L}^\mu = L^\mu = \pi^\mu + \sum_{\alpha=1}^{N}(L_\alpha)^\mu \zeta(z - z(\mathsf{p}_\alpha)) = \pi^\mu + \sum_{\alpha=1}^{N}(\widetilde{L}_\alpha)^\mu \zeta(z - z(\mathsf{p}_\alpha)) \tag{5.34}$$

and

$$\begin{aligned}
\widetilde{L}^\varrho &= \sum_{\alpha=1}^{N}(L_\alpha)^\varrho \frac{\sigma(\varrho(\mathcal{Q}) + z - z(\mathsf{p}_\alpha))}{\sigma(\varrho(\mathcal{Q}))\sigma(z - z(\mathsf{p}_\alpha))} e^{\varrho(\mathcal{Q})\left(2\zeta(\frac{1}{2})\frac{z\bar{\tau}-\bar{z}\tau}{\tau-\bar{\tau}} - 2\zeta(\frac{\tau}{2})\frac{z-\bar{z}}{\tau-\bar{\tau}}\right)} \\
&= \sum_{\alpha=1}^{N}(\widetilde{L}_\alpha)^\varrho \frac{\sigma(\varrho(\mathcal{Q}) + z - z(\mathsf{p}_\alpha))}{\sigma(\varrho(\mathcal{Q}))\sigma(z - z(\mathsf{p}_\alpha))} e^{\varrho(\mathcal{Q})\left(2\zeta(\frac{1}{2})\frac{(z-z(\mathsf{p}_\alpha))\bar{\tau}-(\bar{z}-\bar{z}(\mathsf{p}_\alpha))\tau}{\tau-\bar{\tau}} - 2\zeta(\frac{\tau}{2})\frac{(z-z(\mathsf{p}_\alpha))-(\bar{z}-\bar{z}(\mathsf{p}_\alpha))}{\tau-\bar{\tau}}\right)}.
\end{aligned} \tag{5.35}$$

Note that, using again Legendre's relation,

$$2\zeta(\tfrac{1}{2})\frac{z\bar{\tau} - \bar{z}\tau}{\tau - \bar{\tau}} - 2\zeta(\tfrac{\tau}{2})\frac{z - \bar{z}}{\tau - \bar{\tau}} = -2\zeta(\tfrac{1}{2})z + \frac{2\pi i}{\tau - \bar{\tau}}(z - \bar{z}) \tag{5.36}$$

so that we can write

$$\begin{aligned}
\widetilde{L}^\varrho &= \sum_{\alpha=1}^{N}(\widetilde{L}_\alpha)^\varrho \frac{\sigma(\varrho(\mathcal{Q}) + z - z(\mathsf{p}_\alpha))}{\sigma(\varrho(\mathcal{Q}))\sigma(z - z(\mathsf{p}_\alpha))} e^{\varrho(\mathcal{Q})\left(\frac{2\pi i}{\tau-\bar{\tau}}(z-\bar{z})\right)} \\
&\quad \times e^{-\varrho(\mathcal{Q})\left(\frac{2\pi i}{\tau-\bar{\tau}}(z(\mathsf{p}_\alpha)-\bar{z}(\mathsf{p}_\alpha))\right)} e^{-2\zeta(\frac{1}{2})\varrho(\mathcal{Q})(z-z(\mathsf{p}_\alpha))}.
\end{aligned} \tag{5.37}$$

This last expression allows us to compare the Lax matrix $\widetilde{L}^1 = (\widetilde{L}^\mu \mathsf{H}_\mu + \widetilde{L}^\varrho \mathsf{E}_\varrho)\mathrm{d}z$ with $\widetilde{L}^\mu$ and $\widetilde{L}^\varrho$ given by (5.34) and (5.37), with the elliptic Gaudin Lax matrix obtained in [55, (4.5)] (when specialising to the case $\mathfrak{g} = \mathfrak{gl}_m(\mathbb{C})$ so that writing (5.6) as $\mathcal{Q} = \sum_{k=1}^{m} x_k \mathsf{E}_{kk}$ in the standard basis $\{\mathsf{E}_{ij}\}_{i,j=1}^{m}$ for $\mathfrak{gl}_m(\mathbb{C})$, we have $\varrho(\mathcal{Q}) = x_i - x_j$ for the $\mathfrak{gl}_m(\mathbb{C})$ root $\varrho = \epsilon_i - \epsilon_j$). They coincide up to the factor $e^{-2\varrho(\mathcal{Q})\zeta(\frac{1}{2})(z-z(\mathsf{p}_\alpha))}$ which possibly arises from the use of a different convention for the σ-function and the ζ-function in [55] relative to the standard (5.13)–(5.14).

6. Conclusion and Outlook

In this paper, we derived a variational formulation of Hitchin's completely integrable system associated with a compact Riemann surface of arbitrary genus by extending the theory of Lagrangian multiforms to the setting of gauge theories. As an application, in the genus one case we obtained an explicit phase-space Lagrangian multiform for the elliptic Gaudin hierarchy (and also the elliptic spin Calogero–Moser hierarchy as a special case) for the first time. The present work thus extends the results of [12,15], where explicit Lagrangian multiforms were constructed for both non-cyclotomic and cyclotomic rational Gaudin models using the algebraic framework of Lie dialgebras. In that respect, a natural question is how to generalise the well-understood connection between classical r-matrices and Lagrangian multiforms in the non-dynamical case [12,15,19] to the present setting. Since the Hitchin system associated with an elliptic curve is known to have a dynamical r-matrix [28], the work [10] appears particularly promising in bridging this gap.

One of the virtues of our construction is that it makes manifest the explicit connection between classical 3d mixed holomorphic–topological BF theory and Hitchin's completely integrable system. The relevance of this 3d holomorphic–topological gauge theory for the study of the Hitchin system has been noted previously in the literature [34,73], see also [33], in fact, already at the quantum level. In those works, however, its appearance is less direct: 3d mixed BF theory can be obtained as a certain level-zero limit of 3d Chern–Simons theory which upon quantisation undergoes the familiar shift to the critical level. The latter is in turn deeply related to the representation theory of affine Kac–Moody algebras at the critical level, via conformal blocks of the WZW model living on the boundary, which underlies the geometric Langlands correspondence and the quantisation of the Hitchin system through the work of Beilinson-Drinfel'd [5].

As emphasised in the introduction, the fruitful merging of the framework of Lagrangian 1-forms with that of 3d mixed holomorphic-topological BF theory was achieved through the study of Hitchin's completely integrable system in the Lagrangian framework. A natural generalisation of the present work will be to similarly bring together the framework of Lagrangian 2-forms with that of the celebrated 4d semi-holomorphic Chern–Simons theory [26] with the aim of obtaining a gauge-theoretic origin for hierarchies of 2d integrable field theories. In connection with these ideas, it is important to note also the work [49] on the construction of a 2d field-theoretic generalisation of Hitchin's integrable system based on affine Higgs bundles, which was shown to be closely related to the 4d Chern–Simons setup. Examples of affine Hitchin systems constructed in this way include 2d field theoretic analogues of the elliptic Gaudin and Calogero–Moser models, both of which we have cast in the Lagrangian multiform framework in the present work (see also, for instance, [1] and references therein concerning the so-called field analogue of elliptic (spin) Calogero–Moser model). Naturally, a related goal is then to use a higher-dimensional generalisation of our present work to obtain a variational analogue of the affine Higgs bundle setup.

Finally, a central motivation behind the Lagrangian multiform programme is to provide a path integral quantisation framework for integrable hierarchies. By constructing Lagrangian multiforms for Hitchin systems and demonstrating their connection to 3d classical mixed BF theory in the present work, we have opened the way for studying the quantisation of Hitchin systems via path integral methods. This approach also offers the potential to uncover new insights into their relations with gauge theories at the quantum level.

Data Availability There is no data associated to this work.

Declarations

Conflict of Interest The authors have no Conflict of interest to disclose.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

A. Stability for $G = SL_m(\mathbb{C})$

In this appendix we show that stable bundles with $G = SL_m(\mathbb{C})$ satisfy our differential stability condition (3.21). A stable $SL_m(\mathbb{C})$ principal Higgs bundle $\mathcal{P}$ is equivalent to a stable vector Higgs bundle $E = \mathcal{P} \times_{SL_m(\mathbb{C})} \mathbb{C}^n$ for which the holomorphic line bundle $\det(E)$ is trivial. We recall that a vector Higgs bundle consists of a holomorphic vector bundle $E \to C$ and a holomorphic section B of $\Lambda^{1,0}C \otimes \text{End}(E)$. Such a bundle is called *stable* if, for all holomorphic sub-bundles $F \subset E$ satisfying $B(F) \subset \Lambda^{1,0}C \otimes F$, we have

$$\frac{\deg(F)}{\text{rank}(F)} < \frac{\deg(E)}{\text{rank}(E)}. \tag{A.1}$$

Let h be any Hermitian metric on the holomorphic vector bundle E. Then h, together with E, determines a Chern connection A. This is the unique unitary connection such that the holomorphic structure $\bar{\partial}$ of E coincides with the $(0, 1)$-part $\bar{\partial}^A$ of the covariant derivative d^A. We let $B^\dagger$ denote the Hermitian-adjoint of B; this is a smooth section of $\Lambda^{0,1}C \otimes \text{End}(E)$. Then we can form two operators

$$D' = \bar{\partial}^A + B, \tag{A.2}$$

$$D'' = \partial^A + B^\dagger. \tag{A.3}$$

Simpson showed [61] that every stable bundle admits a unique Hermitian metric such that

$$D'D'' + D''D' = \lambda\, \omega \otimes \text{id}_E, \tag{A.4}$$

in which ω is a chosen area form for the Riemann surface and $\lambda \in \mathbb{C}$ is a constant.

We will prove that a stable vector Higgs bundle satisfies the property (3.21), following a standard argument. To do so, suppose for contradiction that X is a non-zero traceless section of $\text{End}(E)$ satisfying $\bar{\partial} X = 0$ and $[B, X] = 0$. Let h be the Hermitian metric on E satisfying (A.4). We note that

$$i \int_C \text{tr}\big(D'' X^\dagger \wedge D' X - D' X^\dagger \wedge D'' X \big) = \int_C \text{tr}\big(\star \mathrm{d}^A X^\dagger \wedge \mathrm{d}^A X + \star[\Phi, X]^\dagger \wedge [\Phi, X] \big),$$

$$\text{(A.5)}$$

in which $\star$ is the Hodge star, $\mathrm{d}^A = \partial^A + \bar{\partial}^A$ and $\Phi = B - B^\dagger$. On the other hand, (A.4) and integration by parts show that

$$\int_C \text{tr}(D''X^\dagger \wedge D'X + D'X^\dagger \wedge D''X) = -\int_C \lambda \text{tr}(X^\dagger[\text{id}_E, X])\omega = 0. \quad \text{(A.6)}$$

Our assumptions on X mean that $D''X = 0$. So, (A.6) implies that the left-hand side of (A.5) vanishes. Since the right-hand side of (A.5) is the integral of a non-negative function, its integrand must vanish, meaning that $\mathrm{d}^A X = 0$ and $[\Phi, X] = 0$. In other words, X is parallel and commutes with Φ.

Since X is parallel, its eigenvalues are constant. Let F_1 be the bundle of eigenspaces for one of its eigenvalues. Since X is traceless, $F_1 \neq E$. Let F_2 be the orthogonal complement of F_1 obtained using the Hermitian metric h. Since $\mathrm{d}^A X = 0$ and A is unitary, A restricts to unitary connections on F_1 and F_2. Similarly, since $[\Phi, X] = 0$ and Φ is anti-Hermitian, F_1 and F_2 are Φ-invariant, *i.e.* $\Phi(F_i) \subset \Lambda^{1,0}C \otimes F_i$. Since A determines the holomorphic structure, F_1 and F_2 are both holomorphic sub-bundles of E. Since B is the $(1, 0)$-part of Φ, $B(F_i) \subset \Lambda^{1,0} \otimes F_i$ for $i = 1, 2$. Since $E = F_1 \oplus F_2$,

$$\deg(E) = \text{rank}(F_1)\deg(F_2) + \text{rank}(F_2)\deg(F_1). \quad \text{(A.7)}$$

In terms of the slope μ, this means that

$$\mu(F_1) + \mu(F_2) = \frac{\text{rank}(F_1) + \text{rank}(F_2)}{\text{rank}(F_1)\text{rank}(F_2)}\mu(E) \geq 2\mu(E). \quad \text{(A.8)}$$

This means that either $\mu(F_1) \geq \mu(E)$ or $\mu(F_2) \geq \mu(E)$. Either way, we have a contradiction to the stability of E.

References

1. Atalikov, K., Zotov, A.: Field generalization of elliptic Calogero–Moser system in the form of higher rank Landau–Lifshitz model. arXiv:2506.00938 [math-ph]
2. Atiyah, M.F., Bott, R.H.: The Yang–Mills equations over Riemann surfaces. Philos. Trans. Roy. Soc. London Ser. A **308**(1505), 523–615 (1983)
3. Babelon, O., Bernard, D., Talon, M.: Introduction to classical integrable systems. Cambridge University Press (2003)
4. Bassi, C., Lacroix, S.: Integrable deformations of coupled σ-models. J. High Energy Phys. **5**, 1–59 (2020)
5. Beilinson, A., Drinfeld, V.: Quantization of Hitchin's integrable system and Hecke eigensheaves. http://www.math.uchicago.edu/drinfeld/langlands/QuantizationHitchin.pdf
6. Belavin, A.A.: Discrete groups and the integrability of quantum systems. Funct. Anal. Its Appl. **14**(4), 260–267 (1981)
7. Berkovits, N., Pitombo, R.S.: 4D Chern–Simons and the pure spinor $AdS_5 \times S^5$ superstring. Phys. Rev. D **109**(10), 106015 (2024)
8. Bittleston, R., Skinner, D.: Twistors, the ASD Yang–Mills equations and 4d Chern–Simons theory. J. High Energy Phys. **2023**(2), 1–51 (2023)
9. Bottacin, F.: Symplectic geometry on moduli spaces of stable pairs. Ann. scient. Éc. Norm. Sup. **28**(4), 391–433 (1995)
10. Braden, H.W., Dolgushev, V.A., Olshanetsky, M.A., Zotov, A.V.: Classical r-matrices and the Feigin–Odesskii algebra via Hamiltonian and Poisson reductions. J. Phys. A **36**, 697 (2003)
11. Bruzzo, U., Otero, B.G.: Semistable and numerically effective (Higgs) bundles. Adv. Math. **226**(4), 3655–3676 (2011)
12. Caudrelier, V., Dell'Atti, M., Singh, A.A.: Lagrangian multiforms on coadjoint orbits for finite-dimensional integrable systems. Lett. Math. Phys. **114**, 34 (2024)
13. Caudrelier, V., Harland, D.: On the geometry of Lagrangian one-forms. Lett. Math. Phys. **115**, 38 (2025)

14. Caudrelier, V., Nijhoff, F.W., Sleigh, D., Vermeeren, M.: Lagrangian multiforms on Lie groups and non-commuting flows. J. Geom. Phys. **187**, 104807 (2023). arXiv:2204.09663
15. Caudrelier, V., Singh, A.A., Vicedo, B.: Lagrangian multiforms for cyclotomic Gaudin models. SIGMA **20**, 100 (2024)
16. Caudrelier, V., Stoppato, M.: Hamiltonian multiform description of an integrable hierarchy. J. Math. Phys. **61**, 123506 (2020). arXiv:2004.01164
17. Caudrelier, V., Stoppato, M.: Multiform description of the AKNS hierarchy and classical r-matrix. J. Phys. A: Math. Theor. **54**, 235204 (2021). arXiv:2010.07163
18. Caudrelier, V., Stoppato, M., Vicedo, B.: On the Zakharov–Mikhailov action: 4d Chern–Simons origin and covariant Poisson algebra of the Lax connection. Lett. Math. Phys. **111**, 82 (2021). arXiv:2012.04431
19. Caudrelier, V., Stoppato, M., Vicedo, B.: Classical Yang–Baxter equation, Lagrangian multiforms and ultralocal integrable hierarchies. Commun. Math. Phys. **405**, 12 (2024). arXiv:2201.08286
20. Cole, L.T., Cullinan, R.A., Hoare, B., Liniado, J., Thompson, D.C.: Integrable deformations from twistor space. SciPost Phys. **17**(1), 008 (2024)
21. Cole, L.T., Cullinan, R.A., Hoare, B., Liniado, J., Thompson, D.C.: Gauging the diamond: integrable coset models from twistor space. JHEP **12**, 202 (2024)
22. Costello, K.: Supersymmetric gauge theory and the Yangian. arXiv:1303.2632
23. Costello, K.: Integrable lattice models from four-dimensional field theories. Proc. Symp. Pure Math. **88**, 3–24 (2014). arXiv:1308.0370
24. Costello, K., Witten, E., Yamazaki, M.: Gauge theory and integrability. I. ICCM Not. **6**, 46–119 (2018). arXiv:1709.09993
25. Costello, K., Witten, E., Yamazaki, M.: Gauge theory and integrability, II. ICCM Not. **6**, 120–149 (2018). arXiv:1802.01579
26. Costello, K., Yamazaki, M.: Gauge theory and integrability, III. arXiv:1908.02289
27. Delduc, F., Lacroix, S., Magro, M., Vicedo, B.: A unifying 2D action for integrable σ-models from 4D Chern–Simons theory. Lett. Math. Phys. **110**(7), 1645–1687 (2020)
28. Enriquez, B., Rubtsov, V.: Hitchin systems, higher Gaudin operators and r-matrices. Math. Res. Lett. **3**, 343–357 (1996)
29. Fukushima, O., Sakamoto, J.I., Yoshida, K.: Comments on η-deformed principal chiral model from 4D Chern–Simons theory. Nucl. Phys. B **957**, 115080 (2020)
30. Fukushima, O., Sakamoto, J.I., Yoshida, K.: Faddeev–Reshetikhin model from a 4D Chern–Simons theory. JHEP **02**, 115 (2021)
31. Fukushima, O., Sakamoto, J.I., Yoshida, K.: Integrable deformed $T^{1,1}$ sigma models from 4D Chern–Simons theory. JHEP **09**, 037 (2021)
32. Fukushima, O., Sakamoto, J.I., Yoshida, K.: Non-Abelian Toda field theories from a 4D Chern–Simons theory. JHEP **03**, 158 (2022)
33. Gaiotto, D., Teschner, J.: Quantum analytic Langlands correspondence. SciPost Phys. **18**, 144 (2025)
34. Gaiotto, D., Witten, E.: Gauge theory and the analytic form of the geometric Langlands program. Ann. Henri Poincare **25**(1), 557–671 (2024)
35. Gaudin, M.: Diagonalisation d'une classe d'Hamiltoniens de spin. J. Phys. **37**, 1087–1098 (1976)
36. Gaudin, M.: La fonction d'onde de Bethe. Collection du Commissariat à l'Énergie Atomique, Série Scientifique, Masson, Paris (1983)
37. Gibbons, J., Hermsen, T.: A generalisation of the Calogero–Moser system. Physica D **11**, 337–348 (1984)
38. Gorsky, A., Nekrasov, N.: Elliptic Calogero–Moser system from two dimensional current algebra. arXiv:hep-th/9401021
39. Gwilliam, O., Rabinovich, E., Williams, B.R.: Quantization of topological-holomorphic field theories: local aspects. Commun. Anal. Geom. **32**(4), 1157–1231 (2024)
40. He, Y.J., Tian, J., Chen, B.: Deformed integrable models from holomorphic Chern–Simons theory. Sci. China Phys. Mech. Astron. **65**(10), 100413 (2022)
41. Hitchin, N.J.: Stable bundles and integrable systems. Duke Math. J. **54**, 91–114 (1987)
42. Hoare, B., Lacroix, S.: Yang-Baxter deformations of the principal chiral model plus Wess–Zumino term. J. Phys. A **53**(50), 505401 (2020)
43. Jurčo, B.: Classical Yang–Baxter equations and quantum integrable systems. J. Math. Phys. **30**, 1289–1293 (1989)
44. Krichever, I.: Vector bundles and Lax equations on algebraic curves. Comm. Math. Phys. **229**, 229–269 (2002)
45. Kongkoom, T., Nijhoff, F.W., Yoo-Kong, S.: Lagrangian 1-form structure of Calogero–Moser type systems. arXiv:2410.15773
46. Lacroix, S., Vicedo, B.: Integrable $\mathcal{E}$-models, 4d Chern–Simons theory and affine Gaudin models I. Lagrangian aspects. SIGMA **17**, 058 (2021)
47. Lacroix, S., Wallberg, A.: Geometry of the spectral parameter and renormalisation of integrable σ-models. arXiv:2401.13741

48. Levin, A., Olshanetsky, M., Zotov, A.: Hitchin systems: symplectic Hecke correspondence and two-dimensional version. Comm. Math. Phys. **236**, 93–133 (2003)
49. Levin, A., Olshanetsky, M., Zotov, A.: 2d integrable systems, 4d Chern–Simons theory and affine Higgs bundles. Eur. Phys. J. C **82**, 635 (2022)
50. Liniado, J., Vicedo, B.: Integrable degenerate $\mathcal{E}$-models from 4d Chern–Simons theory. Ann. Henri Poincaré **24**(10), 3421–3459 (2023)
51. Lobb, S., Nijhoff, F.W.: Lagrangian multiforms and multidimensional consistency. J. Phys. A: Math. Theor. **42**, 454013 (2009). arXiv:0903.4086
52. Markman, E.: Spectral curves and integrable systems. Compos. Math. **93**, 255–290 (1994)
53. Faria Martins, J., Nijhoff, F.W., Riccombeni, D.: Darboux–Kadomtsev–Petviashvili system as an integrable Chern–Simons multiform theory in infinite dimensional space. Phys. Rev. D **109**, L021701 (2024)
54. Nijhoff, F.W.: Lagrangian 3-form structure for the Darboux system and the KP hierarchy. Lett. Math. Phys. **113**, 27 (2023). arXiv:2206.14338
55. Nekrasov, N.: Holomorphic bundles and many-body systems. Comm. Math. Phys. **180**, 587–604 (1996). arXiv:hep-th/9503157
56. Olshanetsky, M.: Classical integrable systems and gauge field theories. Phys. Part. Nuclei **40**, 93–114 (2009)
57. Petrera, M., Suris, Y.B.: Variational symmetries and pluri-Lagrangian systems in classical mechanics. J. Nonlin. Math. Phys. **24**, 121–145 (2021). arXiv:1710.01526
58. Petrera, M., Vermeeren, M.: Variational symmetries and pluri-Lagrangian structures for integrable hierarchies of PDEs. Eur. J. Math. **7**, 741–765 (2021). arXiv:1906.04535
59. Ramanathan, A.: Stable principal bundles on a compact Riemann surface. Math. Ann. **213**(217), 129–152 (1975)
60. Schmidt, D.M.: Holomorphic Chern–Simons theory and lambda models: PCM case. JHEP **04**, 060 (2020)
61. Simpson, C.T.: Constructing variations of Hodge structure using Yang–Mills theory and applications to uniformization. J. A.M.S. **1**, 867–918 (1988)
62. Sklyanin, E., Takebe, T.: Algebraic Bethe ansatz for the XYZ Gaudin model. Phys. Lett. A **219**, 217–225 (1996)
63. Sklyanin, E., Takebe, T.: Separation of variables in the elliptic Gaudin model. Comm. Math. Phys. **204**, 17–38 (1999)
64. Sleigh, D., Nijhoff, F.W., Caudrelier, V.: A variational approach to Lax representations. J. Geom. Phys. **142**, 66–79 (2019). arXiv:1812.08648
65. Sleigh, D., Nijhoff, F.W., Caudrelier, V.: Variational symmetries and Lagrangian multiforms. Lett. Math. Phys. **110**, 805–826 (2020). arXiv:1906.05084
66. Sleigh, D., Nijhoff, F.W., Caudrelier, V.: Lagrangian multiforms for Kadomtsev–Petviashvili (KP) and the Gelfand–Dickey hierarchy. Int. Math. Res. Not. **2**, 1420–1460 (2023). arXiv:2011.04543
67. Sleigh, D., Vermeeren, M.: Semi-discrete Lagrangian 2-forms and the Toda hierarchy. J. Phys. A: Math. Theor. **55**, 475204 (2022). arXiv:2204.13063
68. Souriau, J.-M.: Structure des systèmes dynamiques, Dunod Paris (1969)
69. Suris, Y.B.: Variational formulation of commuting Hamiltonian flows: multi-time Lagrangian 1-forms. J. Geom. Mech. **5**, 365–379 (2013). arXiv:1212.3314
70. Suris, Y.B., Vermeeren, M.: On the Lagrangian structure of integrable hierarchies. In: Bobenko, A.I. (ed.) Advances in Discrete Differential Geometry. Springer, Berlin, Heidelberg (2016)
71. Vicedo, B., Winstone, J.: 3-Dimensional mixed BF theory and Hitchin's integrable system. Lett. Math. Phys. **112**(4), 79 (2022)
72. Yoo-Kong, S., Lobb, S., Nijhoff, F.W.: Discrete-time Calogero–Moser system and Lagrangian 1-form structure. J. Phys. A: Math. Theor. **44**, 365203 (2011). arXiv:1102.0663
73. Zeng, K.: Monopole operators and bulk-boundary relation in holomorphic topological theories. SciPost Phys. **14**(6), 153 (2023)