A new take on permutation-invariant quantum codes

This is a Perspective on "A family of permutationally invariant quantum codes" by Arda Aydin, Max A. Alekseyev, and Alexander Barg, published in Quantum 8, 1321 (2024).

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A brief history

Quantum error-correction codes enable the correction of errors by encoding quantum information into a larger Hilbert space. A particularly intriguing set of states to consider for the encoding of quantum information is the set of permutation-invariant quantum states, which are invariant under any permutation of their underlying qubits. While it might seem strange to insist on using permutation-invariant systems for making quantum codes, this idea appears less strange considering that certain physical systems only allow permutation-invariant states.
For example, a system of indistinguishable bosons that must obey the symmetrization condition, and hence is necessarily permutation invariant.

Ruskai originally considered encoding quantum information into a permutation-invariant quantum code in 2000 [1]. In Ruskai's code, 9 qubits encode 1 logical qubit with a distance of 3. Ruskai and Pollatsek revisited this topic in 2004 [2], where they designed a 7-qubit permutation-invariant code that encodes 1 logical qubit and has a distance of 3. While proofs of these results appeal to the Knill-Laflamme quantum error-correction criterion [3], it was unclear how to generalize these codes. A decade would pass before any further progress was made on the topic of permutation-invariant quantum codes.

Starting in 2014, progress on research resumed on permutation-invariant quantum codes, beginning with the discovery of gnu codes [4] that generalized Ruskai's code. Gnu codes encode one logical qubit, and are parameterized by integers g, n, and real number $u \geq 1$ such that gnu is the total number of qubits. The distance of these codes is $d = \min\{g, n\}$, and can encode one logical qubit into d^2 qubits. Later papers explored variations [5,6] of gnu codes, along with applications in quantum storage [7], quantum communication [8], and quantum sensing [9,10]. Recently, permutation-invariant quantum codes were also recognized for their ability to correct untracked particle losses (deletion errors) [11,12,13] by virtue of their permutational symmetry and their code-distance property.

However there remained the open question of whether there are permutation-invariant quantum codes on fewer than d^2 qubits and with distance d.

The new result

In their new paper [14], Aydin *et al.* revisit constructing permutation-invariant codes that encode a single logical qubit. The authors restrict their attention to logical codewords that are linear combinations of Dicke states with real coefficients. Exploiting the relationship between correcting deletion errors and the distance of the code, the authors formulate the Knill-Laflamme conditions using the language of deletions. This new perspective gives four types of equality constraints (C1, C2, C3, and C4) that are equivalent to the Knill-Laflamme conditions.

In the code- construction problem, one wishes to know which real coefficients of the logical codewords to use. Treating these real coefficients as unknown variables in the equations C1-C4, we see that C1-C4 are quadratic in the variables and appear to be non-trivial to solve directly.

Fortunately, the authors propose a particular combinatorial form for the real coefficients of the logical codewords that allows C1-C4 to hold. This results in a code construction that depends on only four integers $g, m, \delta,$ and ϵ. The presented construction not only generalizes the gnu codes, but also introduces many new permutation-invariant quantum codes. In particular, when the distance is odd, the number of qubits required is $d^2 - d + 1$. When the distance is even, the number of qubits required is d^2, which is the same as the number of qubits required for gnu codes. As an example, when $d = 3$, this gives a seven-qubit permutation-invariant code that is distinct from the Pollatsek-Ruskai code.

This result solves the open problem of whether we can use fewer than d^2 qubits and with distance d for a permutation-invariant code in the affirmative. Namely, when d is odd, the construction uses $d - 1$ fewer qubits than the gnu codes.

The proof that this code construction satisfies C1-C4 is also extremely intriguing. If we substitute the chosen real coefficients into C1-C4, the most non-trivial equality constraint, C4, transforms into a particular combinatorial identity that seems difficult to prove at first sight. However, by leveraging the celebrated Lagrange–Bürmann inversion theorem in combinatorics, an aesthetically pleasing proof of this non-trivial combinatorial identity is obtained. In contrast, the proof of the quantum error-correction condition for gnu codes in [4] relies on a much simpler combinatorial identity.

After the authors prove that their code construction can correct deletions, they proceed to determine which codes in their code family can correct t amplitude-damping errors. When t is even, their codes that correct t amplitude-damping errors use t fewer qubits than the best corresponding gnu codes. When t is odd, their code uses 1 more qubit than the best gnu code.

The authors finally turn their attention to the Pollatsek-Ruskai code, which does not fall into their code family. The authors transform the equalities C1-C4 into an equivalent set of conditions D1-D3 for permutation-invariant codes. Remarkably, when $d = 3$, the conditions D1-D3 are precisely the conditions that Ruskai and Pollatsek found in [2, Thm.1]. In this sense, the conditions D1-D3 give a way to find generalizations of the Pollatsek-Ruskai code that can correct more errors. The authors illustrate this by numerically finding a 19-qubit permutation-invariant code with distance 5.

Conclusion

In summary, I think that this paper is quite a gem, because it achieves the following:

- It gives a new idea to construct permutation-invariant codes using the identities C1-C4.
- It gives an example of how considering the correction of deletion errors can give new insights into constructing new permutation-invariant codes.
- It gives new permutation-invariant codes with better parameters than previously known.

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