Diophantine approximations with restrained denominators. Balance condition on decay and growth rates.

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Abstract

We strengthen known results on Diophantine approximation with restricted denominators by presenting a new quantitative Schmidt-type theorem that applies to denominators growing much slower than in previous works. In particular, we can handle sequences of denominators with polynomial growth and Rajchmann measures exhibiting arbitrary slow decay, allowing several applications. For instance, our results yield non-trivial lower bounds on the Hausdorff dimensions of intersections of two sets of inhomogeneously well-approximable numbers (each with restricted denominators) and enable the construction of Salem subsets of well-approximable numbers of arbitrarily Hausdorff dimension.

1 Introduction

The famous Khintchine Theorem (see Theorem 1 in Subsection 2 below) was established more than a century ago and is now regarded as one of the central results in the metric theory of Diophantine approximation. It states that, for almost all real numbers, the precision of their best rational approximations is asymptotically essentially the same. In 1964, Schmidt [22] refined this theorem by showing that, for almost all real numbers, the best rational approximations appear at fairly regular intervals. Both of these results rely on the assumption that the approximating function ψ is monotonically non-increasing. The Duffin-Schaeffer conjecture, which allows for non-monotonic ψ, remained open for nearly eight decades and was ultimately proved by Koukoulopoulos and Maynard in 2021 [15]. The corresponding Schmidt-type refinement appeared even more recently, in 2022, by Aistleitner, Borda, and Hauke [1]. These latter two results concern the homogeneous case, whereas the inhomogeneous Duffin-Schaeffer conjecture remains unresolved.

A natural extension of these theorems involves restricting the set of possible denominators rather than allowing all integers; see, for example, [19]. Early work in this vein focused on rapidly growing (lacunary) denominator sequences q_n. Subsequent efforts aimed to slow down the growth of q_n. To the best of our knowledge, this is the first article to establish inhomogeneous results for polynomially growing sequences q_n.

Our main result, Theorem 3, gives asymptotic relationship as in Schmidt's theorem, or also in Aistleitner-Borda-Hauke theorem, for approximations with restrained denominators

*Research is supported by the British Academy in partnership with the Academy of Medical Sciences, the Royal Academy of Engineering, the Royal Society and Cara under the *Researchers at Risk Fellowships* Award RAR\100132

and this is a metric result applicable to a wide class of measure spaces, rather than only to Lebesgue's measure. The motivation for this expansion is explained further in this section. Our result does not need the assumption of the monotonicity of the function ψ and treats general inhomogeneous situation, as well as the homogeneous one. So it could be seen as a strong asymptotic version of inhomogeneous Duffin-Schaeffer conjecture with restrained denominators on M_0-sets.

This paper develops further metric theory of Diophantine approximations on so-called M_0-sets.

Definition 1. The set F is called an M_0*-set* if it supports a non-atomic probability Borel measure μ ($\mathrm{supp}(\mu) \subset F$) whose Fourier transform $\widehat{\mu}$ vanishes at infinity, i.e.

$$\lim_{|t|\to\infty} |\widehat{\mu}(t)| = 0,$$

where

$$\widehat{\mu}(t) := \int e^{-2\pi i t x}\, \mathrm{d}\mu(x), \qquad t \in \mathbb{Z}.$$

Such a measure μ is called a Rajchman measure.

The interest of studying M_0-sets is bifold. On the one hand, there are quite many classical (fractal) sets of interest that support a Rajchman measure (see, for example, [7, 8, 10, 13, 16, 21]). On another hand, it is quite natural to expect, at least heuristically, that such measures' behaviour have significant similarities with Lebesgue measure: indeed, one would naturally expect that, as $|\hat{\mu}(t)|$ tends to 0 as $t \to \infty$, the behaviour of μ is determined to a significant degree (whatever it means in concrete terms) by some first coefficients, $\hat{\mu}(t)$ for $t = -N, \ldots, N$. And then, of course, the measure with Fourier transform $\sum_{t=-N}^{N} \hat{\mu}(t) \exp(-2\pi i t x)$ is absolutely continuous with respect to Lebesgue[1]. See a more detailed discussion and some further references in [19], where some quantitative results that illustrate the heuristics above have been proven and some other results from the past literature have been discussed from this point of view.

In [19], a Khintchine type theorem (in fact, Schmidt-type counting theorem) is proven for sufficiently quickly growing sequences of denominators $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$. Loosely speaking, the growth rate imposed in that paper is such that $\log q_n \gg n^{\delta}$, for certain $\delta > 0$. At the same time, in [19] the convergence part of Khintchine theorem is established in quite a general situation: for arbitrary sequence of denominators $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ and under one of the following two conditions on the measure μ

$$\sum_{n=1}^{\infty} \max_{k\in\mathbb{Z}/\{0\}} |\widehat{\mu}(kq_n)| < \infty, \tag{1}$$

$$\sum_{n=1}^{\infty} \sum_{k\in\mathbb{Z}/\{0\}} \frac{|\widehat{\mu}(kq_n)|}{|k|} < \infty. \tag{2}$$

[1]Indeed, $Q(x) := \sum_{t=-N}^{N} \hat{\mu}(t) \exp(-2\pi i t x)$ is a trigonometric polynomial. So it is preserved by both Fourier and inverse Fourier transforms. Consequently, if we consider the measure ν on $[0, 1]$ with density defined by $Q(x)$, its Fourier transform is given by $Q(x)$ (and by uniqueness of Fourier transform, this is the only measure with this given Fourier transform). At the same time, ν is clearly absolutely continuous with respect to Lebesgue measure (we explicitly have its density).

In this paper, we make a step forward and prove quantitative Schmidt-type results for sequences of denominators of $\mathcal{A}$ of polynomial growth, that is now we impose a much milder condition $q_n \gg n^C$ for some $C > 0$. In fact, we establish general theorems under a balance condition between the growth rate of the sequence $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ and the decay rate of the Fourier transform $\widehat{\mu}$. This balance condition allows, on the one hand, to infer Khintchine Theorem for some well-known measures (for instance, Kaufmann measures on the sets on badly- or well-approximable numbers) with the sequences of denominators of polynomial growth. On another hand, this balance condition allows applications to Rajchman measures with a relatively slowly decaying Fourier transform, which is not possible with the previous results. For example, recently it was proven [16] that, under quite broad conditions, there exist Rajchman measures on self-similar sets associated to contractions, but the decay rate of the Fourier transforms of the measures constructed is $O(\log(|t|)^{-\delta})$ (when $t \to \infty$), where $\delta > 0$ is a constant not explicitly given in the paper, and the methods implied suggest that $\delta > 0$ is very small (much smaller than one, for instance). In such situation, our main result, Theorem 3, still allows to obtain a Khintchine type theorem for sufficiently sparse sequences of denominators.

It is important to remark that, to have any meaningful generalization of Khintchine Theorem to the case of sufficiently general μ, for instance to suitable Rajchman measures, we necessarily have to consider approximations with restrained sequences of denominators only, and this restriction has to impose non-trivial conditions on the growth rate of denominators in the sequence. Indeed, a classical construction by Kaufman [13], later updated by Queffelec-Ramaré [21], shows that the set of badly approximable numbers supports a Rajchman measure (indeed, an infinite family of such measures). So, it is well possible that, with respect to certain Rajchman measures μ, even with Fourier transform quickly converging to 0, μ-almost all numbers are badly approximable, contrary to what we should have had with the usual Khintchine Theorem. Similarly, it is known that Rajchmann measures are supported on the sets of well-approximable numbers. Moreover, it is even known that Rajchmann measures are supported on sets of well-approximable numbers with respect to a restreint sequence of denominators provided that this sequence grows not too quickly [10]. Hence in order to have Khintchine theorem one has to introduce lower bounds on the growth rate of the involved sequence of denominators. So in this aspect, our results of this paper are essentially optimal.

The plan of this paper is as follows.

In Section 2 we give some definitions and introduce some basic notation needed in our further discussion, and, in particular, to state our main result. In the same section we state Khintchine's theorem and also explain Schmidt's result, both of them are mentioned in the introduction.

In Section 3 we state the main result of this paper.

In Section 4 we provide nontrivial applications of our main result.

The research presented in this paper naturally builds upon various existing results. In our case, it turned out that many of the applications we had in mind required certain results—such as those from [19]—in modified forms. While these modified versions can be established using essentially the same arguments as in the original proofs, they do not follow directly from the stated results. To assist the reader and avoid vague references such as "a similar result can be proved by following the approach in that article," we provide the necessary adjusted proofs in the Appendix.

2 Khintchine-Szüsz, Duffin-Schaeffer and Schmidt theorems

We begin by giving a few of basic definitions.

Given a real number $\gamma \in \mathbb{I}$, approximation function $\psi : \mathbb{N} \to \mathbb{I}$ and a natural number $q \in \mathbb{N}$, let

$$E(q, \gamma, \psi) := \{x \in \mathbb{I} : \|qx - \gamma\| \leq \psi(q)\}, \tag{3}$$

where $\|\alpha\| := \min\{|\alpha - m| : m \in \mathbb{Z}\}$ denotes the distance from $\alpha \in \mathbb{R}$ to the nearest integer. For any sequence $\mathcal{A} = (q_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ of natural numbers, define, for any $N \in \mathbb{N}$, the counting function by

$$R(x, N) \; = \; R(x, N; \gamma, \psi, \mathcal{A}) := \#\big\{1 \leq n \leq N : x \in E(q_n, \gamma, \psi)\big\}. \tag{4}$$

Note that the definition above could be rewritten as

$$R(x, N) = \sum_{n=1}^{N} \chi_{E(q_n,\gamma,\psi)}(x), \tag{5}$$

where $\chi_{E(q_n,\gamma,\psi)}$ is the characteristic function of the set $E(q_n, \gamma, \psi)$.

Recall that the set of inhomogeneous ψ-well approximable real numbers

$$W_{\mathcal{A}}(\gamma; \psi) := \{x \in \mathbb{I} : \|q_n x - \gamma\| \leq \psi(q_n) \text{ for infinitely many } n \in \mathbb{N}\}. \tag{6}$$

Note that

$$W_{\mathcal{A}}(\gamma; \psi) = \limsup E(q_n, \gamma, \psi) = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E(q_k, \gamma, \psi). \tag{7}$$

We gathered in Proposition 1 below some basic properties of the function $R(x, N)$ and sets $W_{\mathcal{A}}(\gamma; \psi)$ that we will use in the proof of our main result.

Proposition 1. *Let $x \in \mathbb{I}$, $\gamma \in \mathbb{I}$ and let ψ, ψ_1, ψ_2 be auxiliary functions, that is $\psi, \psi_1, \psi_2 : \mathbb{N} \to \mathbb{I}$. Let $\mathcal{A} = (q_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ be a sequence of natural numbers. Then,*

(i) *$x \in W_{\mathcal{A}}(\gamma; \psi)$ if and only if*

$$\lim_{N\to\infty} R(x, N) = \infty;$$

(ii) *if $\psi_1(q_n) \leq \psi_2(q_n)$ for all $n \in \mathbb{N}$, then*

$$W_{\mathcal{A}}(\gamma; \psi_1) \subset W_{\mathcal{A}}(\gamma; \psi_2),$$

and

$$R(x, N; \gamma, \psi_1, \mathcal{A}) \leq R(x, N; \gamma, \psi_2, \mathcal{A});$$

(iii) *if for some auxiliary functions ψ_1, ψ_2 we have $x \notin W_{\mathcal{A}}(\gamma; \psi_2)$, then*

$$R(x, N; \gamma, \max\{\psi_1, \psi_2\}, \mathcal{A}) = R(x, N; \gamma, \psi_1, \mathcal{A}) + O(1).$$

Proof of Proposition 1. Statements (i) and (ii) of Proposition 1 follow directly from definitions (3), (4) and (6). In order to prove (iii), note that from (ii) of Proposition 1 we have, for any $N \in \mathbb{N}$,

$$R(x, N; \gamma, \psi_1, \mathcal{A}) \leq R(x, N; \gamma, \max\{\psi_1, \psi_2\}, \mathcal{A}) \leq R(x, N; \gamma, \psi_1, \mathcal{A}) + R(x, N; \gamma, \psi_2, \mathcal{A}). \tag{8}$$

Since $x \notin W_{\mathcal{A}}(\gamma; \psi_2)$, (i) of Proposition 1 implies that $R(x, N; \gamma, \psi_2, \mathcal{A})$ remains bounded as $N \to \infty$. This fact together with (8) completes the proof of Proposition 1. □

Khintchine-Szüsz Theorem provides the 0 and 1 law for the Lebesgue measure of the set $W_{\mathbb{N}}(\gamma;\psi)$. Khintchine [14] proved the homogeneous statement (the case $\gamma = 0$) in 1924, and later, in 1954, Szüsz [23] generalized Khintchine's result to the inhomogeneous case.

Theorem 1. Let $\gamma \in \mathbb{I}$ and $\psi : \mathbb{N} \to \mathbb{I}$ be a non-increasing function. Then

$$m\big(W_{\mathbb{N}}(\gamma;\psi)\big) = \begin{cases} 0 & \textit{if} \quad \sum\limits_{n=1}^{\infty} \psi(n) < \infty \ , \\ 1 & \textit{if} \quad \sum\limits_{n=1}^{\infty} \psi(n) = \infty \ , \end{cases}$$

where m is the Lebesgue measure.

In 1964 Schmidt [22] generalized Khintchine-Szüsz Theorem giving a quantitative result on the size of the counting function $R(x, N)$ given by (4) with decreasing auxiliary function ψ and $\mathcal{A} = \mathbb{N}$:

$$R(x, N) \;=\; 2\Psi(N) + O\Big(\Psi(N)^{1/2} \left(\log(\Psi(N))\right)^{2+\varepsilon}\Big), \quad N \in \mathbb{N}, \tag{9}$$

for every $\varepsilon > 0$ and for m-almost all x, where

$$\Psi(N) := \sum_{n=1}^{N} \psi(n).$$

If we drop the assumption on monotonicity of the function ψ, the result of Theorem 1 is no more true, as it was shown by an example by Duffin and Schaeffer. Instead, they conjectured the following result, which was recently proven by Koukoulopoulos and Maynard [15] after being an open problem for almost 80 years.

Theorem 2. Let $\gamma \in \mathbb{I}$ and $\psi : \mathbb{N} \to \mathbb{I}$. Then

$$m\big(W_{\mathbb{N}}(0;\psi)\big) = \begin{cases} 0 & \textit{if} \quad \sum\limits_{n=1}^{\infty} \frac{\phi(n)\psi(n)}{n} < \infty \ , \\ 1 & \textit{if} \quad \sum\limits_{n=1}^{\infty} \frac{\phi(n)\psi(n)}{n} = \infty \ , \end{cases}$$

where m is the Lebesgue measure and ϕ is Euler function.

In 2022, Aistleitner, Borda and Hauke [1] proved a refined result, providing the asymptotics of the Schmidt type (9) for the case of non-monotonic ψ.

Our main result, Theorem 3, presented in the next section, extends these results to the inhomogeneous setting with restrained denominators on M_0-sets.

3 Main result

The statement of our main theorem uses the following definition.

Definition 2. Let $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ be an increasing sequence of natural numbers and let $\alpha \in (0, 1)$ be a real number. We say that $\mathcal{A}$ is *α-separated* if there exists $m_0 \in \mathbb{N}$ such that, for all

$m, n \in \mathbb{N}$, $m_0 \leq m < n$, the set of solutions $(s,t) \in \mathbb{N}^2$ of the following system of Diophantine inequalities

$$\begin{cases} 1 \leq |sq_m - tq_n| < q_m^{\alpha}, \\ s \leq m^5, \end{cases}$$

is empty.

Remark 1. Note that Definition 2 is similar, but not identical, to the one in [19] (see [19, p. 12]). In Definition 2 we have optimized one exponent, so we have m^5 in place of m^{12} in [19]. As a result, if a sequence of denominators $\mathcal{A}$ is α-separated in the sense of [19], than it is necessarily α-separated in the sense of Definition 2 (but not necessarily vice versa).

Theorem 3. *Let μ be a non-atomic probability measure supported on a subset F of $\mathbb{I}$. Let $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ be an α-separated increasing sequence of natural numbers for some $\alpha \in (0,1)$. Suppose there exists a real constant $\rho > 2$ and a monotonically decreasing function $h : \mathbb{N} \to \mathbb{I}$ verifying*

$$h(q_n) \;=\; O\;(n^{-\rho}), \quad n \in \mathbb{N}. \tag{10}$$

such that

$$|\widehat{\mu}(t)| \;=\; O\;(h(|t|)), \quad t \in \mathbb{Z}. \tag{11}$$

Then, for all given $\gamma \in \mathbb{I}$, $\psi : \mathbb{N} \to \mathbb{I}$ and for any $\varepsilon > 0$ the counting function $R(x,N)$ satisfies

$$R(x,N) \;=\; 2\Psi(N) + O\Big(\left(\Psi(N) + E(N)\right)^{1/2} \left(\log(\Psi(N) + E(N) + 2)\,\right)^{2+\varepsilon}\Big) \tag{12}$$

for μ-almost all $x \in F$, where

$$\Psi(N) := \sum_{n=1}^{N} \psi(q_n) \tag{13}$$

and

$$E(N) \;:=\; \sum\sum_{1\leq m<n\leq N} (q_m, q_n)\;\min\left(\frac{\psi(q_m)}{q_m}, \frac{\psi(q_n)}{q_n}\right), \tag{14}$$

here (q_m, q_n) is the gcd of natural numbers q_m and q_n.

Remark 2. If, under all conditions of Theorem 3, for some $\varepsilon > 0$ the gcd term $E(N)$, given by (14), satisfies the equality

$$E(N) = O\left(\Psi^{2-\varepsilon}(N)\right), \quad N \in \mathbb{N}, \tag{15}$$

and $\Psi(N) \to \infty$ as $N \to \infty$, then for the counting function $R(x,N)$ we have the following asymptotic statement

$$\lim_{N\to\infty} \frac{R(x,N)}{2\Psi(N)} = 1, \tag{16}$$

for μ-almost all $x \in F$.

We postpone the proof of Theorem 3 until Section 4.3. In the following Section 4 we give some examples of applications of this theorem.

4 Applications of main result

In this section we give some examples of application of Theorem 3.

4.1 An example of a sequence $\mathcal{A}$ with polynomial growth and fitting all conditions of Theorem 3

In this part, we construct, as an example, a family of sequences of denominators $\mathcal{A}$ of polynomial growth for which Theorem 3 gives non-trivial results.

Example 1. *In this example, we provide a construction of an α-separated sequence $(q_m)_{m\in\mathbb{N}}$ of polynomial growth (more precisely, verifying (20) for parameters $\rho_1 < \rho_2$ as below).*

Let

$$1 < \rho_1 < 6\rho_1 < \rho_2 \tag{17}$$

and $c > 1$ be real parameters. Let us choose a sequence of integers $(n_k)_{k\in\mathbb{N}}$ as follows. First, choose n_1 to be a sufficiently large positive integer, so that $n_1^{\rho_1} < n_1^{\rho_2}/c$. Then, for every $k \in \mathbb{N}$, choose arbitrarily n_{k+1} to be an integer in the range

$$\lfloor n_k^{\rho_2} \rfloor / c \le n_{k+1} \le \lfloor n_k^{\rho_2} \rfloor. \tag{18}$$

For each $k \in \mathbb{N}$, define sets of integers

$$\mathcal{Q}_k := \left\{ s \cdot n_k \mid s = 1, \ldots, \lfloor n_k^{\rho_1 - 1} \rfloor \right\}. \tag{19}$$

Then, define $\mathcal{A} = (q_m)_{m\in\mathbb{N}}$ to be the set of numbers

$$\bigcup_{k\in\mathbb{N}} \mathcal{Q}_k$$

put in increasing order.

We claim that we have, for every $m \in \mathbb{N}$ sufficiently large (so that we have $m > n_1^{\rho_1 - 1}$),

$$\frac{\log q_m}{\log m} \le \frac{\rho_2}{\rho_1 - 1} + 1. \tag{20}$$

Indeed, every q_m has a form $s \cdot n_k$ for some $s, k \in \mathbb{N}$, where $s \le n_k^{\rho_1 - 1}$. The lower bound on m above means that $k \ge 2$. Then,

$$m \ge s + \lfloor n_{k-1}^{\rho_1 - 1} \rfloor \ge s + \lfloor n_k^{(\rho_1 - 1)/\rho_2} \rfloor.$$

Consider two cases: the first one when $s \le \lfloor n_k^{(\rho_1-1)/\rho_2} \rfloor$, and the second one when $s > \lfloor n_k^{(\rho_1-1)/\rho_2} \rfloor$.

Case 1. *Assume $s \le \lfloor n_k^{(\rho_1-1)/\rho_2} \rfloor$. Then, we have $m \ge n_k^{(\rho_1-1)/\rho_2}$ (we use here $s \ge 1$), hence*

$$\frac{\log q_m}{\log m} \le \frac{\log s + \log n_k}{\log n_k^{(\rho_1-1)/\rho_2}} \le \frac{\log n_k^{(\rho_1-1)/\rho_2} + \log n_k}{\log n_k^{(\rho_1-1)/\rho_2}} \le \frac{\rho_2}{\rho_1 - 1} + 1.$$

This proves (20) in the first case.

Case 2. *Assume $s > \lfloor n_k^{(\rho_1-1)/\rho_2} \rfloor$. As s is an integer, we also have then $s > n_k^{(\rho_1-1)/\rho_2}$, hence*

$$\frac{\log q_m}{\log m} \le \frac{\log s + \log n_k}{\log s} \le 1 + \frac{\log n_k}{\log s} < \frac{\rho_2}{\rho_1 - 1} + 1.$$

So we have verified (20) in both cases.

Furthermore, let's remark for a further use

$$q_m > m. \tag{21}$$

Indeed, it follows from our construction that, for every $m \in \mathbb{N}$,

$$q_{m+1} - q_m > 1,$$

and the claim follows by induction.

We proceed with proving that

$$\text{the sequence } (q_n) \text{ is } \alpha\text{-separated for } \alpha = 1/\rho_1. \tag{22}$$

So, let $m, n, s \in \mathbb{N}$ *verifying* $m < n$ *and* $s < m^5$. *In case if, for some* $k \in \mathbb{N}$, $q_m, q_n \in \mathcal{Q}_k$, *then* q_m *and* q_n *are both divisible by* n_k, *so in case if, for some* $t \in \mathbb{N}$, $|sq_m - tq_n| \geq 1$, *we necessarily have*

$$|sq_m - tq_n| \geq n_k,$$

hence

$$|sq_m - tq_n| \geq q_m^{1/\rho_1}.$$

To deal with the complementary case, if $q_m \in \mathcal{Q}_k$, $q_n \in \mathcal{Q}_l$ *for some* $k < l$, *note first that it could be deduced from the construction of the sequence* $(q_m)_{m\in\mathbb{N}}$, *that* $q_m \geq m^{\rho_1}$. *Then, we have*

$$|sq_m| < m^5 q_m < q_m^6 \leq n_k^{6\rho_1}.$$

So, for n_k *sufficiently large (otherwise speaking, for* m *sufficiently large) and taking into account* (17), *we have*

$$|sq_m| < \lfloor n_k^{\rho_2} \rfloor/(2c) \leq n_{k+1}/2 \leq q_n/2.$$

Thus in this case we have that either $s = t = 0$, *in which case, of course,* $sq_m - tq_n = 0$, *or*

$$|sq_m - tq_n| \geq q_m.$$

This completes the verification of (22).

Example 2. *We continue to work in the framework of Example 1, now assuming in addition* $c \geq 2$ *in* (18). *We want to update the construction from that example to ensure, apart from polynomial growth and* α*-separation, that also the term* $E(N)$, *defined in the statement of Theorem 3, has a size that allows optimal result in Theorem 3. More precisely, we are going to rectify Example 1 above, by constructing a sequence* $\widetilde{\mathcal{A}} = (\widetilde{q}_t)_{t\in\mathbb{N}}$, *so that we have*

$$E(N) = \sum\sum_{1\leq m<n\leq N} (\widetilde{q}_m, \widetilde{q}_n)\ \min\left(\frac{\psi(\widetilde{q}_m)}{\widetilde{q}_m}, \frac{\psi(\widetilde{q}_n)}{\widetilde{q}_n}\right) = O\left(\sum_{n=1}^{N} \psi(\widetilde{q}_n)\right). \tag{23}$$

Recall that we assume $c \geq 2$. *We choose all the numbers in the sequence* $(n_k)_{k\in\mathbb{N}}$ *to be prime, this is possible by Bertrand's postulate.*

Next, we modify the sets $\mathcal{Q}_k$, *defined in* (24), *as follows:*

$$\widetilde{\mathcal{Q}}_k := \left\{ s \cdot n_k \mid s = 1, \ldots, \lfloor n_k^{\rho_1 - 1} \rfloor,\ s \text{ is prime} \rfloor \right\}. \tag{24}$$

Then, similarly to Example 1, we define $\widetilde{\mathcal{A}} = (\widetilde{q}_t)_{t\in\mathbb{N}}$ *to be the set of numbers*

$$\bigcup_{k\in\mathbb{N}} \widetilde{\mathcal{Q}}_k$$

put in increasing order.

For the sake of comparison, let us denote by $\mathcal{A}$ *the set of denominators built as in Example 1 using the same sequence of primes* $(n_k)_{k\in\mathbb{N}}$ *that we have just constructed for* $\widetilde{\mathcal{A}}$*. Then, it follows from the definitions that the set of values of* $\widetilde{\mathcal{A}}$ *is a subset of values of* $\mathcal{A}$*, that is, for every index* $t \in \mathbb{N}$ *there exists an index* $m \in \mathbb{N}$ *such that* $\widetilde{q}_t = q_m$*. Note that by Prime Number Theorem, for all* $k \in \mathbb{N}$ *sufficiently large,*

$$\#\widetilde{\mathcal{Q}}_k \asymp \frac{\#\mathcal{Q}_k}{\log \#\mathcal{Q}_k}$$

Consequently, for all $t \in \mathbb{N}$ *large enough,* $\frac{m}{\log m} \ll t \ll m$*. Hence, for every* $\varepsilon > 0$*, we have, for all indices* $t \in \mathbb{N}$ *large enough,*

$$\frac{\log \widetilde{q}_t}{\log t} \leq \frac{\rho_2}{\rho_1 - 1} + 1 + \varepsilon,$$

so $(\widetilde{q}_t)_{t\in\mathbb{N}}$ *has a polynomial growth.*

We proceed with establishing (23)*. We claim that, for every* $n \in \mathbb{N}$*, we have*

$$\sum_{m=1}^{n-1} \frac{(\widetilde{q}_m, \widetilde{q}_n)}{\widetilde{q}_n} = O(1), \tag{25}$$

which clearly implies (23)*.*

By construction, $\widetilde{q}_n = s \cdot n_k$ *for some* $k \in \mathbb{N}$ *and prime* s *verifying* $s \leq n_k^{\rho_1 - 1}$*. Similarly, for all* $m \in \mathbb{N}$, $\widetilde{q}_m = s_1 \cdot n_l$*. Let us denote by* I_1 *the collection of indices* $m < n$ *such that* $l = k$*, and* I_2 *the complementary set of indices* $m < n$ *such that* $l < k$*. Naturally,* $\{1, \ldots, n-1\} = I_1 \sqcup I_2$*, hence*

$$\sum_{m=1}^{n} \frac{(\widetilde{q}_m, \widetilde{q}_n)}{\widetilde{q}_n} = \sum_{m\in I_1} \frac{(\widetilde{q}_m, \widetilde{q}_n)}{\widetilde{q}_n} + \sum_{m\in I_2} \frac{(\widetilde{q}_m, \widetilde{q}_n)}{\widetilde{q}_n}.$$

We are going to show that both sums in the right-hand side are $O(1)$*, which is equivalent to proving* (25)*.*

First, we consider the sum $\sum_{m\in I_1}$*. In this case, by definition of* I_1*,* $k = l$*. Then, we necessarily have* $s_1 < s$*, because* $m < n$*. Moreover, we have in this case*

$$(q_m, q_n) = (s_1 n_k, s n_k) = n_k,$$

because s_1 *and* s *are two distinct primes. So,*

$$\sum_{m\in I_1} \frac{(\widetilde{q}_m, \widetilde{q}_n)}{\widetilde{q}_n} = \sum_{\substack{2\leq s_1 < s \\ s_1 \text{ prime}}} \frac{1}{s} < 1.$$

Next, we consider the sum $\sum_{m\in I_2}$*. In this case, we have* $l < k$*, hence, by construction of sequences* $(n_t)_{t\in\mathbb{N}}$ *and* $\widetilde{\mathcal{A}}$*, we necessarily have* $\widetilde{q}_m \leq n_{k-1}^{\rho_1} \leq n_k^{\rho_1/\rho_2}$*, hence* $\widetilde{q}_n \geq n_k > \widetilde{q}_m^{\rho_2/\rho_1}$*. We find thus*

$$\sum_{m\in I_2} \frac{(\widetilde{q}_m, \widetilde{q}_n)}{\widetilde{q}_n} \leq \sum_{m} \frac{\widetilde{q}_m}{\widetilde{q}_m^{\rho_2/\rho_1}} < +\infty,$$

where at the last step we use (17)*. This proves* (25)*, hence* (23)*.*

4.2 Khintchine Theorem on the set of Liouville numbers

In this section, we establish the Khintchine-Szüsz theorem (for restrained denominators) on the set of Liouville numbers:

$$\mathbb{L} = \left\{ x \in \mathbb{R} \setminus \mathbb{Q} \;\mid\; \forall\, n \in \mathbb{N} \;\exists\; q \in \mathbb{N} : \|qx\| < q^{-n} \right\}.$$

In 2000 Bluhm [6] constructed a Rajchman measure $\mu_{\mathbb{L}}$ supported on $\mathbb{L}$ showing that the set of Liouville numbers is an M_0 set. In 2002 Bugeaud [9] calculated the exact decay rate of the Fourier transform $\widehat{\mu_{\mathbb{L}}}$:

$$|\widehat{\mu_{\mathbb{L}}}(t)| \le \exp\left\{ -c_2 \sqrt{\log(1+|t|)} \right\}, \quad t \in \mathbb{Z}, \tag{26}$$

where c_2 is a positive absolute constant appearing in [9]. We will use constant c_2 and measure $\widehat{\mu_{\mathbb{L}}}$ in the statement of Theorem 4 below).

This allows us to infer the following theorem.

Theorem 4. *Let $\psi : \mathbb{N} \to \mathbb{I}$, $\gamma \in \mathbb{I}$, $\alpha \in (0,1)$ and let $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ be an α-separated increasing sequence of natural numbers with growth rate*

$$\log q_n \ge \left(\frac{\rho}{c_2} \log n \right)^2, \quad n \ge n_0, \tag{27}$$

for some natural number n_0, $\rho > 2$ and c_2 from (26). Recall the measure $\widehat{\mu_{\mathbb{L}}}$ described before the statement of the theorem and appearing in (26).

Then, for any $\varepsilon > 0$, we have the following asymptotic counting result for $\widehat{\mu_{\mathbb{L}}}$:-almost every Liouville number x:

$$R(x,N) \;=\; 2\Psi(N) + O\Big((\Psi(N) + E(N))^{1/2} \left(\log(\Psi(N) + E(N) + 2)\,\right)^{2+\varepsilon} \Big),$$

where

$$\Psi(N) := \sum_{n=1}^{N} \psi(q_n)$$

and

$$E(N) \;:=\; \sum\sum_{1\le m<n\le N} (q_m, q_n)\; \min\left(\frac{\psi(q_m)}{q_m}, \frac{\psi(q_n)}{q_n} \right).$$

Proof of Theorem 4. We will deduce Theorem 4 from Theorem 3. In order to do this we put

$$h(t) = \exp\left\{ -c_2 \sqrt{\log(1+t)} \right\}, \quad t \in \mathbb{N}.$$

Conditions (10) and (11) of Theorem 3 now follow from (26) and (27), therefore all statements of Theorem 4 now follow from the corresponding statements of Theorem 3. □

Consider the following two examples of sequences $\mathcal{A}$. We will use these examples in Corollary 1 below.

1. Let $\mathcal{S}$ be a finite set of k distinct primes $p_1, \dots, p_k$. It is shown in [19] that any sequence of $\mathcal{S}$-smooth numbers $\mathcal{A}_1$ is an α-separated sequence, for any $\alpha \in (0,1)$. It is also shown there that $\mathcal{A}_1$ has a growth rate
$$\log q_n > \frac{\log 2}{2} n^{\frac{1}{k}}, \quad n \geq 2,$$
and has the gcd error term $E(N)$ of order $E(N) = O(\Psi(N))$ (see Theorem 5 in [19]).

2. Let $\mathcal{A}_2$ be a subsequence of the sequence $\widetilde{\mathcal{A}}$ verifying the growth condition (27). Since $\widetilde{\mathcal{A}}$ is an α-separated sequence, its subsequence $\mathcal{A}_2$ is also α-separated. Moreover, $\mathcal{A}_2$ verifies (23) because $\widetilde{\mathcal{A}}$ does.

For both of these examples, the gcd error term $E(N)$ is at most as large as the main term $2\Psi(N)$. This allows us to deduce the following Khintchine-type statement from Theorem 4.

Corollary 1. *Let $\mathbb{L}$ be the set of Liouville numbers with Rajchman measure $\mu_{\mathbb{L}}$ supported on it and verifying (26). Then, for any $\gamma \in \mathbb{I}$ and for the sequences $\mathcal{A}_1$, $\mathcal{A}_2$ constructed above, we have*
$$\mu_{\mathbb{L}}\big(W_{\mathcal{A}_i}(\gamma;\psi) \cap \mathbb{L}\big) = \begin{cases} 0 & \text{if} \quad \sum\limits_{n=1}^{\infty} \psi(q_n) < \infty\,, \\ 1 & \text{if} \quad \sum\limits_{n=1}^{\infty} \psi(q_n) = \infty\,, \qquad i = 1,2. \end{cases}$$

4.3 Hausdorff and Fourier dimensions of sets $W_{\mathcal{A}}(\gamma;\psi)$ and their intersections

In this section we deduce a nontrivial result on the Hausdorff and Fourier dimensions of some subsets of the sets $W_{\mathcal{A}}(\gamma;\psi)$.

Recall that the Fourier dimension of a set $A \subset \mathbb{R}$ is a number $\dim_F A$, that is a supremum of all $\eta \in [0,1]$ such that
$$|\widehat{\mu}(t)| = O\left(|t|^{-\frac{\eta}{2}}\right), \quad \text{as} \quad |t| \to \infty, \tag{28}$$
for some Borel non-atomic probability measure μ on $\mathbb{R}$ with $\mathrm{supp}(\mu) \subset A$.

It is known (see, for example, [17, 18]) that for any Borel set $A \subset \mathbb{R}$
$$\dim_F A \leq \dim_H A. \tag{29}$$
A set $A \subset \mathbb{R}$, for which $\dim_F A = \dim_H A$, is called a Salem set.

For the set of inhomogeneous ψ-well approximable real numbers $W_{\mathcal{A}}(\gamma;\psi)$ an upper bound of the Hausdorff dimension is known (see, for example, [10]):
$$\dim_H W_{\mathcal{A}}(\gamma;\psi) \leq \min\left\{\tau(\mathcal{A},\psi), 1\right\}, \tag{30}$$
where
$$\tau(\mathcal{A},\psi) = \inf\left\{\eta \geq 0 \;:\; \sum_{q\in\mathcal{A}} q\left(\frac{\psi(q)}{q}\right)^{\eta} < \infty\right\}. \tag{31}$$

The next theorem gives a nontrivial result about the Fourier dimension of the set of inhomogeneous ψ-well approximable real numbers lying inside some other set A and follows from Theorem 3.

Theorem 5. *Let $A \subset \mathbb{R}$ be a Borel set and let $\dim_F A = d \neq 0$. Let $\alpha \in (0,1)$, and let $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ be an increasing α-separated sequence of natural numbers. Assume that, for some $\delta > 0$ and $n_0 \in \mathbb{N}$, the sequence $\mathcal{A}$ has the growth rate*

$$q_n \geq n^{\frac{4}{d}+\delta}, \quad n \geq n_0. \tag{32}$$

Let $\psi : \mathbb{N} \to \mathbb{I}$ be an approximation function. Assume that

$$\Psi(N) := \sum_{n=1}^{N} \psi(q_n) \to \infty \quad \text{as} \quad N \to \infty. \tag{33}$$

Then, for any $\gamma \in \mathbb{I}$,

$$\dim_F (W_{\mathcal{A}}(\gamma;\psi) \cap A) = d.$$

Proof of Theorem 5. Firstly note that the definition of the Fourier dimension implies that for any $A_1, A_2 \subset \mathbb{R}$

$$\dim_F (A_1 \cap A_2) \leq \min \{\dim_F A_1, \dim_F A_2\}.$$

Therefore,

$$\dim_F (W_{\mathcal{A}}(\gamma;\psi) \cap A) \leq \dim_F A = d.$$

So, in order to prove Theorem 5, we need to show that

$$\dim_F (W_{\mathcal{A}}(\gamma;\psi) \cap A) \geq d, \tag{34}$$

which means that the intersection $W_{\mathcal{A}}(\gamma;\psi) \cap A$ supports some Borel non-atomic probability measure with a suitable decay rate of its Fourier transform.

Let us fix a positive number $\widetilde{\varepsilon} < \frac{d^2\delta}{8+2d\delta}$. Since $\dim_F A = d \neq 0$, then, for any $\varepsilon \in (0, \tilde{\varepsilon})$, there exists a non-atomic probability measure $\mu_A^{(\varepsilon)}$, supported on A and satisfying

$$|\widehat{\mu_A^{(\varepsilon)}}(t)| = O\left(|t|^{-\frac{d}{2}+\varepsilon}\right), \quad \text{as} \quad |t| \to \infty. \tag{35}$$

Put

$$h(t) = t^{-\frac{d}{2}+\widetilde{\varepsilon}}, \quad t \in \mathbb{N}.$$

Now, for any $\varepsilon \in (0,\tilde{\varepsilon})$ we can apply Theorem 3, choosing $\mu = \mu_A^{(\varepsilon)}$, $F = A$ and h as above. Conditions (10) and (11) of Theorem 3 follow from (35) and (32) respectively. By Theorem 3, we get that, for any $\varepsilon \in (0,\tilde{\varepsilon})$ and for $\mu_A^{(\varepsilon)}$-almost all $x \in A$, equality (12) holds true. Taking into account (33), from (12) we get that

$$R(x,N) \to \infty \ \text{ as } N \to \infty, \quad \text{for } \mu_A^{(\varepsilon)}\text{-almost all } x \in A,$$

so for $\mu_A^{(\varepsilon)}$-almost all $x \in A$ we have $x \in W_{\mathcal{A}}(\gamma;\psi)$. So, for any $\varepsilon \in (0,\tilde{\varepsilon})$, we have

$$\mu_A^{(\varepsilon)} (W_{\mathcal{A}}(\gamma;\psi) \cap A) = 1.$$

For any $\varepsilon \in (0,\tilde{\varepsilon})$ consider a measure space $(\mathbb{R}, \mathfrak{B}(\mathbb{R}), \mu_A^{(\varepsilon)})$. Put $B = W_{\mathcal{A}}(\gamma;\psi) \cap A$. Now we modify the measure $\mu_A^{(\varepsilon)}$ introducing a new Borel measure

$$\mu_B^{(\varepsilon)}(C) := \mu_A^{(\varepsilon)}(C \cap B), \quad \forall C \in \mathfrak{B}(\mathbb{R}).$$

Observing the full measure statement $\mu_A^{(\varepsilon)}(B) = 1$ and the fact that $\mu_A^{(\varepsilon)}$ is a non-atomic probability measure we deduce that the new measure space $(\mathbb{R}, \mathfrak{B}(\mathbb{R}), \mu_B^{(\varepsilon)})$ is a probability space with a non-atomic measure $\mu_B^{(\varepsilon)}$. From the definition of measures $\mu_B^{(\varepsilon)}$ it follows that

$$\operatorname{supp}(\mu_B^{(\varepsilon)}) \subset B, \quad \forall \varepsilon \in (0, \tilde{\varepsilon}).$$

Moreover, for any $t \in \mathbb{Z}$ and $\varepsilon \in (0, \tilde{\varepsilon})$

$$\int_{\mathbb{R}} e^{-2\pi i t x} \, \mathrm{d}\mu_A^{(\varepsilon)}(x) = \int_B + \int_{\mathbb{R}\setminus B} = \int_B e^{-2\pi i t x} \, \mathrm{d}\mu_A^{(\varepsilon)}(x) =$$

$$= \int_B e^{-2\pi i t x} \, \mathrm{d}\mu_B^{(\varepsilon)}(x) = \int_B + \int_{\mathbb{R}\setminus B} = \int_{\mathbb{R}} e^{-2\pi i t x} \, \mathrm{d}\mu_B^{(\varepsilon)}(x),$$

so $\mu_B^{(\varepsilon)}$ has the same decay rate of Fourier transform, as $\mu_A^{(\varepsilon)}$. Therefore, observing (35) with $\mu_B^{(\varepsilon)}$ instead of $\mu_A^{(\varepsilon)}$,

$$\dim_F \left(W_{\mathcal{A}}(\gamma; \psi) \cap A\right) \geq d,$$

so (34) holds. This finishes the proof of Theorem 5. □

The following two corollaries immediately follow from Theorem 5 and inequalities

$$\dim_F (A \cap W) \leq \dim_H (A \cap W) \leq \min \{\dim_H A, \dim_H W\} \leq \dim_H A. \tag{36}$$

In the above inequalities (36) the sets A and W are two Borel subsets of $\mathbb{R}$, these inequalities follow from (29) and definitions of Hausdorff and Fourier dimensions.

Corollary 2. *If, under all conditions of Theorem 5, we also have that A is a Salem set, then the intersection $W_{\mathcal{A}}(\gamma; \psi) \cap A$ is also a Salem set and*

$$\dim_H \left(W_{\mathcal{A}}(\gamma; \psi) \cap A\right) = \dim_F \left(W_{\mathcal{A}}(\gamma; \psi) \cap A\right) = d.$$

Remark 3. In view of Corollary 2, for any given $d \in (0, 1]$ we can build Salem subsets W of the set of inhomogeneous ψ-well approximable real numbers $W_{\mathcal{A}}(\gamma; \psi)$ with $\dim_H W = d$. For example, in [10] Hambrook proved that if the denominators sequence $\mathcal{A}_1$ satisfies

$$\sum_{q \in \mathcal{A}_1} \frac{1}{q} = \infty,$$

and for the approximation function ψ_1 we have

$$\liminf_{M\to\infty} \frac{-\log \psi_1(M)}{\log M} = \limsup_{M\to\infty} \frac{-\log \psi_1(M)}{\log M} = \lambda, \tag{37}$$

then for any $\gamma_1 \in \mathbb{I}$ the set $W_1 = W_{\mathcal{A}_1}(\gamma_1; \psi_1)$ is Salem with

$$\dim_H W_1 = \dim_F W_1 = \min\{\frac{2}{1+\lambda}, 1\}. \tag{38}$$

So, the subset $W = W_{\mathcal{A}}(\gamma; \psi) \cap W_{\mathcal{A}_1}(\gamma_1; \psi_1)$, whose elements are both ψ and ψ_1-well approximable (with different denominators sequences), is a Salem set. Of course, this readily extends to finite intersections of the sets of well-approximale numbers.

Corollary 3. *Let $\alpha \in (0,1)$, and let $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ be an increasing α-separated sequence of natural numbers with a growth rate*

$$q_n \geq n^{4+\delta} \tag{39}$$

for some $\delta > 0$. Assume that the approximation function ψ verifies (33). Then,

$$\dim_H W_{\mathcal{A}}(\gamma;\psi) = 1.$$

Proof. It is well-known that for every $d \in (0,1]$ there exists a Salem set with Fourier dimension d. For example, one could use the set W_1 mentioned in (38) with $\lambda = 2/d - 1$. Then, the result follows from Theorem 5 with $A = W_1$ (and the use of inequalities (36)). □

Remark 4. Corollary 3 provides an inhomogeneous Khintchine theorem for a wide range of sequences of restrained denominators (more precisely, it provides the divergence case of Khintchine's theorem, but the convergence case always readily follows from the classical Borel-Cantelli lemma). The homogeneous version of this result was previously established by Rynne [20]. He shows that

$$\dim_H W_{\mathcal{A}}(0;\psi) = \min\left\{\tau(\mathcal{A},\psi), 1\right\},$$

where $\tau(\mathcal{A},\psi)$ is defined by (31). Note that (33) implies $\tau(\mathcal{A},\psi) \geq 1$, so in homogeneous case, under (33), we have $\dim_H W_{\mathcal{A}}(0;\psi) = 1$. Inhomogeneous case is not so well studied. The existing results cover $\mathcal{A} = \mathbb{N}$ and, more generally, not too sparse subsets of $\mathbb{N}$ (verifying the condition $\sum_{q\in\mathcal{A}_1} \frac{1}{q} = \infty$). Corollary 3, complementarily, gives inhomogeneous results for sufficiently sparse denominators sequence (verifying (32)).

Of course, it would be very desirable to extend the results of Theorem 5 beyond the divergence condition (33). This could be done, at least partially, by using a powerful Mass Transference Principle. Theorem 6 below states a particular case of a general result that we use [3][Theorem 3].

Theorem 6. *Let $X \subset \mathbb{R}$ and let $\lambda, \mu > 0$. Let $(B_i)_{i\in\mathbb{N}}$ be a sequence of balls in X (considered as a metric space with the usual distance $d(x,y) = |x-y|$) such that radii of B_i tend to 0 as $i \to \infty$. For each ball $B_i = B_i(x_i, r_i)$ define $B_i^{\lambda/\mu} := B\left(x_i, r_i^{\lambda/\mu}\right)$.*

Assume that, for every ball $B \subset X$,

$$\dim_H \left(B \cap \limsup_{i\to\infty} B_i^{\lambda/\mu}\right) \geq \mu.$$

Then, for every ball $B \subset X$,

$$\dim_H \left(B \cap \limsup_{i\to\infty} B_i\right) \geq \min\left(\lambda, \dim_H X\right).$$

Theorem 7 below could be considered as a partial extension of Theorem 5. It provides a non-trivial lower bound for Hausdorff dimension of intersection of the set of well-approximable numbers with other sets.

Theorem 7. *Let $A \subset \mathbb{R}$ be a Borel set and let $\dim_F A = d \neq 0$. Let $\alpha \in (0,1)$ and let $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ be an increasing α-separated sequence of natural numbers. Assume that, for* *some $\delta > 0$ and $n_0 \in \mathbb{N}$, the sequence $\mathcal{A}$ has the growth rate (39). Let $\psi : \mathbb{N} \to \mathbb{I}$ be an approximation function. Then, for any $\gamma \in \mathbb{I}$,*

$$\dim_H \left(W_{\mathcal{A}}(\gamma;\psi) \cap A\right) \geq \tau(\mathcal{A},\psi) \cdot d,$$

where $\tau(\mathcal{A},\psi)$ is defined by (31).

Proof. The result follows from Theorem 5, inequalities (36) and Theorem 6 with $\lambda = d{\cdot}\tau(\mathcal{A},\psi)$ and $\mu = d$. □

Example 3. *Let $\lambda \geq 0$, $\gamma_1 \in \mathbb{I}$ and let ψ_1 and $\mathcal{A}_1$ be the same as in Remark 3 (for the sake of even further concretness, one could take $\psi_1 : t \to t^{-\lambda}$ and $\mathcal{A}_1 = \mathbb{Z}$). As it is discussed in Remark 3, in this case $W_{\mathcal{A}_1}(\gamma_1,\psi_1)$ is a Salem set verifying*

$$\dim_H W_{\mathcal{A}_1}(\gamma_1,\psi_1) = \dim_F W_{\mathcal{A}_1}(\gamma_1,\psi_1) = \frac{2}{1+\lambda}.$$

Further, let $a \in \mathbb{N}$, $a \geq 2$ and let $\mathcal{A} = (a^n)_{n\in\mathbb{N}}$. It is not hard to verify that the sequence $\mathcal{A}$ is $1/2$-separated. Then, it follows from Theorem 7 that, for any approximating function ψ,

$$\dim_H \left(W_{\mathcal{A}_1}(\gamma_1,\psi_1) \cap W_{\mathcal{A}}(\gamma,\psi)\right) \geq \frac{2\cdot\tau(\mathcal{A},\psi)}{1+\lambda}.$$

The following corollary extends Corollary 3, providing thus a Járnik-type theorem for a wide range on sequences of restrained denominators.

Corollary 4. *Let $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ be an increasing α-separated sequence of natural numbers, for some $\alpha \in (0,1)$, and assume that, for some $\delta > 0$ and $n_0 \in \mathbb{N}$, the sequence $\mathcal{A}$ has the growth rate (39). Let $\psi : \mathbb{N} \to \mathbb{I}$ be an approximation function. Then, for any $\gamma \in \mathbb{I}$,*

$$\dim_H \left(W_{\mathcal{A}}(\gamma;\psi)\right) = \tau(\mathcal{A},\psi),$$

where $\tau(\mathcal{A},\psi)$ is defined by (31).

Proof. Because of (30), we need to prove only the lower bound

$$\dim_H W_{\mathcal{A}}(\gamma;\psi) \geq \tau(\mathcal{A},\psi).$$

But this result follows from Theorem 7 with the remark that the interval $A = [0,1]$ has Fourier dimension 1. □

Appendix

The proof of Theorem 3 is based on the following lemma [11, Lemma 1.5].

Lemma 1. *Let $(X,\mathcal{B},\mu)$ be a probability space, let $(f_n(x))_{n\in\mathbb{N}}$ be a sequence of non-negative μ-measurable functions defined on X, and $(f_n)_{n\in\mathbb{N}}$, $(\phi_n)_{n\in\mathbb{N}}$ be sequences of real numbers such that*

$$0 \leq f_n \leq \phi_n \qquad (n = 1,2,\ldots).$$

Suppose that for arbitrary $a, b \in \mathbb{N}$ with $a < b$, we have

$$\int_X \left(\sum_{n=a}^{b} \left(f_n(x) - f_n\right)\right)^2 \mathrm{d}\mu(x) \;\leq\; C\sum_{n=a}^{b} \phi_n \tag{40}$$

for an absolute constant $C > 0$. Then, for any given $\varepsilon > 0$, we have

$$\sum_{n=1}^{N} f_n(x) \;=\; \sum_{n=1}^{N} f_n \;+\; O\left(\Phi(N)^{1/2}\log^{\frac{3}{2}+\varepsilon}\Phi(N) + \max_{1\le k\le N} f_k\right) \tag{41}$$

for μ-almost all $x \in X$, where $\Phi(N) := \sum\limits_{n=1}^{N} \phi_n$.

A mechanism of applying Lemma 1 for proving similar to our counting results can be found in [19]. We will move in parallel to their proof taking into account that, unlike [19], we do not have a growth conditions on the sequence $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ and decay rate for $\hat{\mu}(t)$.

Choosing $f_n(x)$ and f_n from Lemma 1

Let us consider Lemma 1 with

$$X := \mathbb{I}\,, \qquad f_n(x) := \chi_{E(q_n,\gamma,\psi)}(x) \qquad \text{and} \qquad f_n = 2\psi(q_n)\,. \tag{42}$$

It follows from (5) that, with this choice of $f_n(x)$,

$$\text{the l.h.s. of (41)} \;=\; R(x,N).$$

Note that $f_n(x)^2 = f_n(x)$, $x \in \mathbb{I}$, so for any $a, b \in \mathbb{N}$ with $a < b$,

$$\begin{aligned}\left(\sum_{n=a}^{b}(f_n(x) - f_n)\right)^2 \;&=\; \left(\sum_{n=a}^{b} f_n(x)\right)^2 \;+\; \left(\sum_{n=a}^{b} f_n\right)^2 \;-\; 2\sum_{n=a}^{b} f_n(x)\cdot\sum_{n=a}^{b} f_n \\ &=\; \sum_{n=a}^{b} f_n(x) \;+\; 2\sum\sum_{a\le m<n\le b} f_m(x)f_n(x) \;+\; \left(\sum_{n=a}^{b} f_n\right)^2 \;-\; 2\sum_{n=a}^{b} f_n\cdot\sum_{n=a}^{b} f_n(x)\,,\end{aligned}$$

and so it follows that

$$\begin{aligned}\text{the l.h.s. of (40)} \;=\; &\sum_{n=a}^{b}\mu(E_n) \;+\; 2\sum\sum_{a\le m<n\le b}\mu(E_m\cap E_n) - \\ &-\; 4\sum_{n=a}^{b}\psi(q_n)\left(\sum_{n=a}^{b}\mu(E_n) - \sum_{n=a}^{b}\psi(q_n)\right),\end{aligned} \tag{43}$$

here we used the following short notation $E_n := E(q_n,\gamma,\psi)$, $\quad n \in \mathbb{N}$.

Relation (43) shows that we need to obtain a 'good' estimates of the measure of sets $E_n = E(q_n,\gamma,\psi)$ and the measure of their intersections to satisfy the condition (40).

Estimating the measure of the sets E_n and their intersections

In this section we present different estimates that help us to prove the main result. We will use the Fourier analysis estimating the measure of sets E_n and their intersections.

Let ε and δ be real numbers such that $0 < \varepsilon \le 1$ and $0 < \delta < 1/4$. Let $\chi_\delta : \mathbb{I} \to \mathbb{R}$ be the characteristic function defined by

$$\chi_\delta(x) := \begin{cases} 1 & \text{if } \ \|x\| \le \delta, \\ 0 & \text{if } \ \|x\| > \delta\,, \end{cases}$$

and let $\chi^+_{\delta,\varepsilon} : \mathbb{I} \to \mathbb{R}$ and $\chi^-_{\delta,\varepsilon} : \mathbb{I} \to \mathbb{R}$ be the continuous upper and lower approximations of χ_δ given by

$$\chi^+_{\delta,\varepsilon}(x) := \begin{cases} 1 & \text{if } \ \|x\| \le \delta, \\ 1 + \dfrac{1}{\delta\varepsilon}(\delta - \|x\|) & \text{if } \ \delta < \|x\| \le (1+\varepsilon)\delta \\ 0 & \text{if } \ \|x\| > (1+\varepsilon)\delta\,, \end{cases}$$

and

$$\chi^-_{\delta,\varepsilon}(x) := \begin{cases} 1 & \text{if } \ \|x\| \le (1-\varepsilon)\delta \\ \dfrac{1}{\delta\varepsilon}(\delta - \|x\|) & \text{if } \ (1-\varepsilon)\delta < \|x\| \le \delta \\ 0 & \text{if } \ \|x\| > \delta\,. \end{cases}$$

Clearly, both $\chi^+_{\delta,\varepsilon}$ and $\chi^-_{\delta,\varepsilon}$ are periodic functions with period 1. Next, given a real positive function $\psi : \mathbb{N} \to \mathbb{I}$ and any integer $q \ge 4$, consider the functions $W^+_{q,\gamma,\varepsilon,\psi}$ and $W^-_{q,\gamma,\varepsilon,\psi}$ defined by

$$W^+_{q,\gamma,\varepsilon}(x) = W^+_{q,\gamma,\varepsilon,\psi}(x) \ := \ \left(\sum_{p=0}^{q-1} \delta_{\frac{p+\gamma}{q}}(x)\right) * \chi^+_{\frac{\psi(q)}{q},\varepsilon}(x) \tag{44}$$

and

$$W^-_{q,\gamma,\varepsilon}(x) = W^-_{q,\gamma,\varepsilon,\psi}(x) \ := \ \left(\sum_{p=0}^{q-1} \delta_{\frac{p+\gamma}{q}}(x)\right) * \chi^-_{\frac{\psi(q)}{q},\varepsilon}(x)\,,$$

where $*$ denotes convolution and δ_x denotes the Dirac delta-function at the point $x \in \mathbb{R}$.

It occurs that

$$W^+_{q,\gamma,\varepsilon}(x) = \sum_{p=0}^{q-1} \chi^+_{\frac{\psi(q)}{q},\varepsilon}\left(x - \tfrac{p+\gamma}{q}\right)$$

and

$$W^-_{q,\gamma,\varepsilon}(x) = \sum_{p=0}^{q-1} \chi^-_{\frac{\psi(q)}{q},\varepsilon}\left(x - \tfrac{p+\gamma}{q}\right)\,.$$

It thus follows that for any $0 < \varepsilon \le 1$ and any integer $q \ge 4$,

$$\int_0^1 W^-_{q,\gamma,\varepsilon}(x)\mathrm{d}\mu(x) \ \le \ \mu(E(q,\gamma,\psi)) \ \le \ \int_0^1 W^+_{q,\gamma,\varepsilon}(x)\mathrm{d}\mu(x). \tag{45}$$

Now we need to consider the Fourier series expansions

$$\sum_{k\in\mathbb{Z}} \widehat{W}^{\pm}_{q,\gamma,\varepsilon}(k)\exp(2\pi kix)$$

of $W^+_{q,\gamma,\varepsilon}$ and $W^-_{q,\gamma,\varepsilon}$.

The values and basic estimates of the Fourier coefficients $\widehat{W}^{\pm}_{q,\gamma,\varepsilon}(k)$ are presented in [19] (see Lemma 1 in [19]). For the convenience of the reader, we collect them in the following proposition.

Proposition 2. *Let $0<\varepsilon,\tilde{\varepsilon}\le 1$ and $\widehat{W}^{\pm}_{q,\gamma,\varepsilon}(k)$, $\quad k\in\mathbb{Z}$, are the Fourier coefficients of $W^+_{q,\gamma,\varepsilon}$ and $W^-_{q,\gamma,\varepsilon}$. Then, for any integers $q,r\ge 4$:*

(i) *for $k\neq 0$,*

$$\widehat{W}^{+}_{q,\gamma,\varepsilon}(k) \ = \begin{cases} \exp\left(-\dfrac{2\pi ik\gamma}{q}\right)\dfrac{q\left(\cos(2\pi k\psi(q)q^{-1})-\cos(2\pi k\psi(q)q^{-1}(1+\varepsilon))\right)}{2\pi^2k^2\psi(q)q^{-1}\varepsilon} & \text{if} \ \ q\mid k \\ 0 & \text{if} \ \ q\nmid k\,, \end{cases} \tag{46}$$

and for $k=0$,

$$\widehat{W}^{+}_{q,\gamma,\varepsilon}(0)=(2+\varepsilon)\,\psi(q)\,; \tag{47}$$

(ii) *for $k\neq 0$,*

$$\widehat{W}^{-}_{q,\gamma,\varepsilon}(k) \ = \begin{cases} \exp\left(-\dfrac{2\pi ik\gamma}{q}\right)\dfrac{q\left(\cos(2\pi k\psi(q)q^{-1}(1-\varepsilon))-\cos(2\pi k\psi(q)q^{-1})\right)}{2\pi^2k^2\psi(q)q^{-1}\varepsilon} & \text{if} \ \ q\mid k \\ 0 & \text{if} \ \ q\nmid k\,, \end{cases} \tag{48}$$

and for $k=0$,

$$\widehat{W}^{-}_{q,\gamma,\varepsilon}(0)=(2-\varepsilon)\,\psi(q)\,; \tag{49}$$

(iii) *for any $s\in\mathbb{Z}\setminus\{0\}$*

$$\left|\widehat{W}^{\pm}_{q,\gamma,\varepsilon}(sq)\right| \ \le \ (2+\varepsilon)\,\psi(q), \tag{50}$$

$$\left|\widehat{W}^{\pm}_{q,\gamma,\varepsilon}(sq)\right| \ \le \ \frac{1}{\pi^2s^2\psi(q)\varepsilon}. \tag{51}$$

(iv)

$$\sum_{s\in\mathbb{Z}}\left|\widehat{W}^{\pm}_{q,\gamma,\varepsilon}(sq)\right| \ < \ \frac{3}{\varepsilon^{1/2}} \tag{52}$$

and

$$\sum_{s\in\mathbb{Z}}\sum_{t\in\mathbb{Z}}\left|\widehat{W}^{\pm}_{q,\gamma,\varepsilon}(sq)\right|\left|\widehat{W}^{\pm}_{r,\gamma,\tilde{\varepsilon}}(tr)\right| \ \le \ \frac{9}{\varepsilon^{1/2}\,.\,\tilde{\varepsilon}^{1/2}}\,. \tag{53}$$

By (52), $\sum\limits_{k\in\mathbb{Z}} \left|\widehat{W}^{\pm}_{q,\gamma,\varepsilon}(k)\right| < \infty$, so the Fourier series

$$\sum_{k\in\mathbb{Z}} \widehat{W}^{\pm}_{q,\gamma,\varepsilon}(k)\exp(2\pi kix)$$

converges uniformly to $W^{\pm}_{q,\gamma,\varepsilon}(x)$ for all $x \in \mathbb{I}$. Hence, it follows that

$$\int_0^1 W^{\pm}_{q,\gamma,\varepsilon}(x)\,\mathrm{d}\mu(x) \;=\; \sum_{k\in\mathbb{Z}} \widehat{W}^{\pm}_{q,\gamma,\varepsilon}(k)\,\widehat{\mu}(-k)\,.$$

This together with (45), (47), (49) and the fact that $\widehat{\mu}(0) = 1$, implies that

$$\begin{aligned} \mu(E(q,\gamma,\psi)) \;&\le\; (2+\varepsilon)\,\psi(q) \;+ \sum_{k\in\mathbb{Z}\setminus\{0\}} \widehat{W}^{+}_{q,\gamma,\varepsilon}(k)\,\widehat{\mu}(-k) \\ \mu(E(q,\gamma,\psi)) \;&\ge\; (2-\varepsilon)\,\psi(q) \;+ \sum_{k\in\mathbb{Z}\setminus\{0\}} \widehat{W}^{-}_{q,\gamma,\varepsilon}(k)\,\widehat{\mu}(-k)\,. \end{aligned} \tag{54}$$

Estimating the sums from measures of E_n

Now we are ready to prove the following estimates for the sums of $\mu(E_n)$.

Lemma 2. *Under the conditions of Theorem 3, we have, for arbitrary $a, b \in \mathbb{N}$ with $a < b$,*

$$\sum_{n=a}^{b} \mu(E(q_n,\gamma,\psi)) = 2\sum_{n=a}^{b} \psi(q_n) \;+\; O\left(\min\left(1, \sum_{n=a}^{b} \psi(q_n)\right)\right). \tag{55}$$

We need the following well known statement in order to prove Lemma 2. We'll leave this statement without citation because, in our opinion, it belongs to the mathematical folklore.

Lemma 3. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers verifying $a_1 > 0$ and*

$$S_n = a_1 + a_2 + \ldots + a_n, \quad n \in \mathbb{N}.$$

Then for any real $\xi > 0$ the series

$$\sum_{n=1}^{\infty} \frac{a_n}{S_n^{1+\xi}}$$

converges.

Proof of Lemma 3. Firstly, note that, for any $n \in \mathbb{N}$,

$$\int_{S_{n-1}}^{S_n} \frac{dx}{x^{1+\xi}} \ge \int_{S_{n-1}}^{S_n} \frac{dx}{S_n^{1+\xi}} = \frac{a_n}{S_n^{1+\xi}}. \tag{56}$$

To prove the convergence of the series

$$\sum_{n=1}^{\infty} \frac{a_n}{S_n^{1+\xi}}$$

consider its partial sum

$$B_N = \sum_{n=1}^{N} \frac{a_n}{S_n^{1+\xi}}.$$

In view of (56), for any $N \in \mathbb{N}$,

$$B_N = \frac{a_1}{S_1^{1+\xi}} + \frac{a_2}{S_2^{1+\xi}} + \frac{a_3}{S_3^{1+\xi}} + \ldots + \frac{a_N}{S_N^{1+\xi}} \leq$$

$$\leq \frac{a_1}{S_1^{1+\xi}} + \int_{S_1}^{S_2} \frac{dx}{x^{1+\xi}} + \int_{S_2}^{S_3} \frac{dx}{x^{1+\xi}} + \ldots + \int_{S_{N-1}}^{S_N} \frac{dx}{x^{1+\xi}} = \frac{1}{a_1^{\xi}} + \int_{S_1}^{S_N} \frac{dx}{x^{1+\xi}} =$$

$$= \frac{1}{a_1^{\xi}} + \frac{1}{\xi}\left(\frac{1}{S_1^{\xi}} - \frac{1}{S_N^{\xi}}\right) \leq \frac{1}{a_1^{\xi}}\left(1 + \frac{1}{\xi}\right).$$

So, the sequence of partial sums $(B_N)_{N\in\mathbb{N}}$ for the series with nonnegative elements is bounded, which implies the convergence of this series. □

Now we can prove Lemma 2.

Proof of Lemma 2. We prove Lemma 2 under a weaker condition on the number ρ from (10), compared to Theorem 3, namely $\rho > \frac{3}{2}$. For any given sequence of real numbers $(\varepsilon_n)_{n=a}^{b}$ in $(0,1]$, it follows from (54) that

$$|\mu(E_n) - 2\psi(q_n)| \leq \psi(q_n)\varepsilon_n + \max_{\circ\in\{+,-\}} \left| \sum_{k\in\mathbb{Z}\setminus\{0\}} \widehat{W}^{\circ}_{q_n,\gamma,\varepsilon_n}(k)\widehat{\mu}(-k) \right|, \quad a \leq n \leq b. \tag{57}$$

It follows from (46) and (48) that $\widehat{W}^{\pm}_{q_n,\gamma,\varepsilon_n}(k) = 0$ unless $k = sq_n$ for some integer s. Also from (11) we have, that $|\widehat{\mu}(sq_n)| \ll n^{-\rho}$, where $\rho > \frac{3}{2}$. So, using (52),

$$\left| \sum_{k\in\mathbb{Z}\setminus\{0\}} \widehat{W}^{\pm}_{q_n,\gamma,\varepsilon_n}(k)\widehat{\mu}(-k) \right| \leq \sum_{s\in\mathbb{Z}\setminus\{0\}} \left|\widehat{W}^{\pm}_{q,\gamma,\varepsilon}(sq)\right|\left|\widehat{\mu}(sq_n)\right| \ll \frac{3}{n^{\rho}\varepsilon_n^{\frac{1}{2}}}. \tag{58}$$

Now, in view of (57) and (58),

$$\left| \sum_{n=a}^{b} \mu(E_n) - 2\sum_{n=a}^{b} \psi(q_n) \right| \ll \sum_{n=a}^{b} \psi(q_n)\varepsilon_n + \sum_{n=a}^{b} \frac{3}{n^{\rho}\varepsilon_n^{\frac{1}{2}}}, \tag{59}$$

therefore, for $\varepsilon_n = 1$, $a \leq n \leq b$,

$$\left| \sum_{n=a}^{b} \mu(E_n) - 2\sum_{n=a}^{b} \psi(q_n) \right| \ll \sum_{n=a}^{b} \psi(q_n)\,. \tag{60}$$

To complete the proof we need to get one more upper bound, namely

$$\left| \sum_{n=a}^{b} \mu(E_n) - 2\sum_{n=a}^{b} \psi(q_n) \right| \ll 1. \tag{61}$$

We can deduce the upper bound (61) again from (59) choosing suitable values of $(\varepsilon_n)_{n=a}^b$. Since $\rho > \frac{3}{2}$, let's choose a real number $\xi \in (0, 2\rho - 3)$. For any $n \in \mathbb{N}$ let

$$\Psi(n) := \sum_{k=1}^{n} \psi(q_k) \quad \text{and} \quad \varepsilon_n := \min\left(1, \Psi(n)^{-1-\xi}\right).$$

By definition, $|\psi(q_n)| \leq 1$ and so $\varepsilon_n^{-1} \leq n^{1+\xi}$. So, for the second term on the r.h.s. of (59) we have

$$\sum_{n=a}^{b} \frac{3}{n^{\rho}\varepsilon_n^{1/2}} \leq \sum_{n=1}^{\infty} \frac{3}{n^{\rho-\frac{\xi}{2}-\frac{1}{2}}} < \infty, \tag{62}$$

since $\rho - \frac{\xi}{2} - \frac{1}{2} > 1$ because of the choice of ξ.

It follows from Lemma 3 that, for $\xi > 0$, the series

$$\sum_{n=1}^{\infty} \frac{\psi(q_n)}{\Psi(n)^{1+\xi}}$$

converges, so

$$\sum_{n=1}^{\infty} \frac{\psi(q_n)}{\max\left(1, \Psi(n)^{1+\xi}\right)}$$

converges also. Therefore,

$$\sum_{n=a}^{b} \psi(q_n)\varepsilon_n < \sum_{n=1}^{\infty} \frac{\psi(q_n)}{\max\left(1, \Psi(n)^{1+\xi}\right)} < \infty. \tag{63}$$

The upper bound (61) now follows from the inequalities (59), (62) and (63). □

Estimating the sums from measures of intersections $E_n \cap E_m$

Estimating the sums from measures of intersections of the sets E_n is technically a more complicated work. Here we use the condition on α-separability (see Definition 2), which we don't use proving Lemma 2.

Before formulating the statement on the upper bound for the sums of measures of intersections we prove the following result, related to the case $\xi = 0$ of Lemma 3.

Lemma 4. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers, $a_1 \neq 0$ and*

$$S_n = a_1 + a_2 + \ldots + a_n, \quad n \in \mathbb{N}.$$

Then for any $N \in \mathbb{N}$ for the partial sums B_N of the series

$$\sum_{n=1}^{\infty} \frac{a_n}{S_n}$$

we have

$$B_N = \sum_{n=1}^{N} \frac{a_n}{S_n} \leq 1 + \log(S_N) - \log(a_1).$$

Proof of Lemma 4. Firstly note that for any $n \in \mathbb{N}$
$$\int_{S_{n-1}}^{S_n} \frac{dx}{x} \geq \int_{S_{n-1}}^{S_n} \frac{dx}{S_n} = \frac{a_n}{S_n}. \tag{64}$$
In view of (64), for any $N \in \mathbb{N}$,
$$B_N = \frac{a_1}{S_1} + \frac{a_2}{S_2} + \frac{a_3}{S_3} + \ldots + \frac{a_N}{S_N} \leq$$
$$\leq \frac{a_1}{S_1} + \int_{S_1}^{S_2} \frac{dx}{x} + \int_{S_2}^{S_3} \frac{dx}{x} + \ldots + \int_{S_{N-1}}^{S_N} \frac{dx}{x} = 1 + \int_{S_1}^{S_N} \frac{dx}{x} =$$
$$= 1 + \log(S_N) - \log(S_1) = 1 + \log(S_N) - \log(a_1).$$
□

Now we can formulate and prove the statement on the upper bound for the sums from measures of intersections.

Lemma 5. *Let for any* $\tau > 1$
$$\psi(q_n) \geq n^{-\tau}, \quad n \in \mathbb{N}. \tag{65}$$
Then, under all conditions of Theorem 3, for arbitrary $a, b \in \mathbb{N}$ *with* $a < b$,
$$2 \sum\sum_{a \leq m < n \leq b} \mu(E_m \cap E_n) \ \leq \ \left(\sum_{n=a}^{b} \mu(E_n)\right)^2 + O\left(\sum\sum_{a \leq m < n \leq b} (q_m, q_n) \min\left(\frac{\psi(q_m)}{q_m}, \frac{\psi(q_n)}{q_n}\right)\right)$$
$$+ O\left(\left(\sum_{n=a}^{b} \psi(q_n)\right) \log^+\left(\sum_{n=a}^{b} \psi(q_n)\right) + \sum_{n=a}^{b} \psi(q_n)\right), \tag{66}$$
where $\log^+(x) := \max(0, \log(x)), \quad x > 0.$

Proof of Lemma 5. In order to receive suitable estimates of the measures of intersections we start with some Fourier analysis (once again, we move here in parallel with the corresponding reasonings in [19]). So, let's fix a sequence of real numbers $(\varepsilon_n)_{n \in \mathbb{N}} \subset (0, 1]$ and consider the functions
$$W^+_{m,n}(x) := W^+_{q_m,\gamma,\varepsilon_m}(x) \cdot W^+_{q_n,\gamma,\varepsilon_n}(x), \quad x \in \mathbb{R},\ m, n \in \mathbb{N},\ m < n, \tag{67}$$
(recall that the function $W^+_{q,\gamma,\varepsilon}$ is defined by (44)). Then,
$$\begin{aligned} \mu(E_m \cap E_n) \ &\leq \ \int_0^1 W^+_{q_m,\gamma,\varepsilon_m}(x) W^+_{q_n,\gamma,\varepsilon_n}(x)\, \mathrm{d}\mu(x) \\ &= \ \int_0^1 W^+_{m,n}(x) \mathrm{d}\mu(x). \end{aligned}$$
The Fourier coefficients of the product $W^+_{m,n}$ are convolutions
$$\begin{aligned} \widehat{W}^+_{m,n}(k) \ &:= \ \int_0^1 W^+_{q_m,\gamma,\varepsilon_m}(x) W^+_{q_n,\gamma,\varepsilon_n}(x)\, \exp(-2\pi k i x) \mathrm{d}x \\ &= \ \sum_{j \in \mathbb{Z}} \widehat{W}^+_{q_m,\gamma,\varepsilon_m}(j)\, \widehat{W}^+_{q_n,\gamma,\varepsilon_n}(k - j), \quad k \in \mathbb{Z}. \end{aligned} \tag{68}$$

Moreover, $\sum_{k\in\mathbb{Z}} \left|\widehat{W}^{+}_{m,n}(k)\right| < \infty$, so the Fourier series

$$\sum_{k\in\mathbb{Z}} \widehat{W}^{+}_{m,n}(k)\exp(2\pi kix)$$

converges uniformly to $W^{+}_{m,n}(x)$ for all $x \in \mathbb{I}$. Hence, it follows that

$$\int_0^1 W^{+}_{m,n}(x)\,\mathrm{d}\mu(x) \;=\; \sum_{k\in\mathbb{Z}} \widehat{W}^{+}_{m,n}(k)\;\widehat{\mu}(-k)\,.$$

So,

$$\begin{aligned} \mu(E_m\cap E_n) = \mu(E(q_m,\gamma,\psi)\cap E(q_n,\gamma,\psi)) \quad &\leq \\ \leq \sum_{k\in\mathbb{Z}} \widehat{W}^{+}_{m,n}(k)\widehat{\mu}(-k) \quad &= \quad \widehat{W}^{+}_{m,n}(0) + \sum_{k\in\mathbb{Z}\setminus\{0\}} \widehat{W}^{+}_{m,n}(k)\widehat{\mu}(-k). \end{aligned} \tag{69}$$

We consider the two terms on the right hand side of (69) separately. From (68) with $k=0$, (and refer to (3)) we have

$$\begin{aligned} \widehat{W}^{+}_{m,n}(0) := \int_0^1 W^{+}_{q_m,\gamma,\varepsilon_m}(x)W^{+}_{q_n,\gamma,\varepsilon_n}(x)\mathrm{d}x \leq \\ \leq \left|E\left(q_m,\gamma,(1+\varepsilon_m)\cdot\psi\right)\cap E\left(q_m,\gamma,(1+\varepsilon_n)\cdot\psi\right)\right|, \end{aligned} \tag{70}$$

where $|\,.\,|$ is Lebesgue measure. It is relatively straightforward to verify (see [11, Equation 3.2.5][2] for the details) that for any $q, q' \in \mathbb{N}$

$$|E(q,\gamma,\psi)\cap E(q',\gamma,\psi)| \;=\; 4\psi(q)\psi(q') + O\left((q,q')\;\min\left(\frac{\psi(q)}{q},\frac{\psi(q')}{q'}\right)\right)\,.$$

Hence, it follows that

$$\begin{aligned} \left|E\left(q_m,\gamma,(1+\varepsilon_m)\cdot\psi\right)\cap E\left(q_m,\gamma,(1+\varepsilon_n)\cdot\psi\right)\right| \;&=\; 4(1+\varepsilon_m)(1+\varepsilon_n)\psi(q_m)\psi(q_n) \\ &+\;\; O\left((q_m,q_n)\min\left(\frac{\psi(q_m)}{q_m},\frac{\psi(q_n)}{q_n}\right)\right)\,. \end{aligned}$$

This together with (70) implies that

$$\widehat{W}^{+}_{m,n}(0) \;\leq\; 4(1+\varepsilon_m)(1+\varepsilon_n)\psi(q_m)\psi(q_n) + O\left((q_m,q_n)\min\left(\frac{\psi(q_m)}{q_m},\frac{\psi(q_n)}{q_n}\right)\right)\,, \tag{71}$$

which give us the estimate of the first term on the right hand side of (70).

We proceed with considering the second term, which we will denote by $S_{m,n}$. In view of (46) and (68), it follows that

$$\begin{aligned} S_{m,n} \quad &:= \quad \sum_{k\in\mathbb{Z}\setminus\{0\}} \widehat{W}^{+}_{m,n}(k)\widehat{\mu}(-k) = \\ &= \quad \sum\sum_{\substack{s,t\in\mathbb{Z}\\ sq_m - tq_n\neq 0}} \widehat{W}^{+}_{q_m,\gamma,\varepsilon_m}(sq_m)\widehat{W}^{+}_{q_n,\gamma,\varepsilon_n}(tq_n)\widehat{\mu}\left(-(sq_m+tq_n)\right). \end{aligned} \tag{72}$$

[2]Equation 3.2.5 in [11] as stated is not correct – the 'big O' error term is missing.

We decompose $S_{m,n}$ into three sums:

$$S_{m,n} = S_1(m,n) + S_2(m,n) + S_3(m,n),$$

where

$$\begin{aligned} S_1(m,n) &:= \sum_{t\in\mathbb{Z}\setminus\{0\}} \widehat{W}^+_{q_m,\gamma,\varepsilon_m}(0)\widehat{W}^+_{q_n,\gamma,\varepsilon_n}(tq_n)\widehat{\mu}(-tq_n), \\ S_2(m,n) &:= \sum_{s\in\mathbb{Z}\setminus\{0\}} \widehat{W}^+_{q_n,\gamma,\varepsilon_n}(0)\widehat{W}^+_{q_m,\gamma,\varepsilon_m}(sq_m)\widehat{\mu}(-sq_m), \\ S_3(m,n) &:= \sum\sum_{\substack{s,t\in\mathbb{Z}\setminus\{0\} \\ sq_m+tq_n\neq 0}} \widehat{W}^+_{q_m,\gamma,\varepsilon_m}(sq_m)\widehat{W}^+_{q_n,\gamma,\varepsilon_n}(tq_n)\widehat{\mu}\left(-(sq_m+tq_n)\right). \end{aligned}$$

Inequalities (47), (52) and balance condition (11) imply that, for any $m,n\in\mathbb{N}$, $m<n$,

$$|S_1(m,n)| \ll \frac{(2+\varepsilon_m)\psi(q_m)}{n^\rho}\sum_{t\in\mathbb{Z}}\widehat{W}^+_{q_n,\gamma,\varepsilon_n}(tq_n) \ll \frac{\psi(q_m)}{n^\rho\varepsilon_n^{1/2}}. \tag{73}$$

Symmetrically, for any $m,n\in\mathbb{N}$, $m<n$,

$$|S_2(m,n)| \ll \frac{\psi(q_n)}{m^\rho\varepsilon_m^{1/2}}. \tag{74}$$

Now we decompose S_3 into two sums:

$$S_3(m,n) = S_4(m,n) + S_5(m,n),$$

where

$$S_4(m,n) := \sum\sum_{\substack{s,t\in\mathbb{Z}\setminus\{0\} \\ |sq_m-tq_n|\geq q_m^\alpha}} \widehat{W}^+_{q_m,\gamma,\varepsilon_m}(-sq_m)\widehat{W}^+_{q_n,\gamma,\varepsilon_n}(tq_n)\widehat{\mu}\left(sq_m-tq_n\right)$$

and

$$S_5(m,n) := \sum\sum_{\substack{s,t\in\mathbb{Z}\setminus\{0\} \\ 1\leq|sq_m-tq_n|<q_m^\alpha}} \widehat{W}^+_{q_m,\gamma,\varepsilon_m}(-sq_m)\widehat{W}^+_{q_n,\gamma,\varepsilon_n}(tq_n)\widehat{\mu}\left(sq_m-tq_n\right), \tag{75}$$

here $\alpha\in(0,1)$ is the constant from α-separability condition of Theorem 3. Regarding S_4, by making use of the balance condition (11) with the restriction $|sq_m-tq_n|\geq q_m^\alpha$ and inequality (53), it follows that

$$|S_4(m,n)| \ll \frac{1}{m^\rho}\sum\sum_{s,t\in\mathbb{Z}\setminus\{0\}}\left|\widehat{W}^+_{q_m,\gamma,\varepsilon_m}(-sq_m)\right|\left|\widehat{W}^+_{q_n,\gamma,\varepsilon_n}(tq_n)\right| \ll \frac{1}{m^\rho\varepsilon_m^{1/2}\varepsilon_n^{1/2}}. \tag{76}$$

Finally, we decompose S_5 into two sums:

$$S_5(m,n) = S_6(m,n) + E(m,n),$$

where

$$S_6(m,n) := \sum_{\substack{s,t\in\mathbb{Z}\setminus\{0\}\\ s>m^3/\psi(q_m)\\ 1\le|sq_m-tq_n|<q_m^\alpha}}\sum \widehat{W}^+_{q_m,\gamma,\varepsilon_m}(-sq_m)\widehat{W}^+_{q_n,\gamma,\varepsilon_n}(tq_n)\widehat{\mu}\left(sq_m-tq_n\right)$$

and

$$E(m,n) := \sum_{\substack{s,t\in\mathbb{Z}\setminus\{0\}\\ 1\le s\le m^3/\psi(q_m)\\ 1\le|sq_m-tq_n|<q_m^\alpha}}\sum \widehat{W}^+_{q_m,\gamma,\varepsilon_m}(-sq_m)\widehat{W}^+_{q_n,\gamma,\varepsilon_n}(tq_n)\widehat{\mu}\left(sq_m-tq_n\right).$$

The restriction $1\le|sq_m-tq_n|<q_m^\alpha$ implies that

$$0<\left|s-t\frac{q_n}{q_m}\right|<1\,. \tag{77}$$

Hence, if nonzero s and t satisfy (77) then both necessarily must have the same sign and also for each fixed integer s there exists a set T_s of at most two non-zero integers t satisfying the restriction $1\le|sq_m-tq_n|<q_m^\alpha$. So, we can write S_6 as a single sum

$$S_6(m,n) = \sum_{\substack{s>m^3/\psi(q_m):\\ t\in T_s}} \widehat{W}^+_{q_m,\gamma,\varepsilon_m}(-sq_m)\widehat{W}^+_{q_n,\gamma,\varepsilon_n}(tq_n)\widehat{\mu}\left(sq_m-tq_n\right).$$

So, using the trivial bound $|\widehat{\mu}(t)|\le 1$ together with (50) to bound $|\widehat{W}^+_{q_n,\gamma,\varepsilon_n}(tq_n)|$ and (51) to bound $|\widehat{W}^+_{q_m,\gamma,\varepsilon_n}(sq_m)|$, we obtain that for any integers $1\le m<n$

$$\begin{aligned}|S_6(m,n)| \quad &\ll \sum_{\substack{s>m^3/\psi(q_m):\\ t\in T_s}} |\widehat{W}^+_{q_m,\gamma,\varepsilon_m}(sq_m)|\,|\widehat{W}^+_{q_n,\gamma,\varepsilon_n}(tq_n)| \\ &\ll \sum_{s>m^3/\psi(q_m)} \frac{1}{s^2\psi(q_m)\,\varepsilon_m}\psi(q_n) \;\ll\; \frac{\psi(q_n)}{m^3\,\varepsilon_m}\,.\end{aligned} \tag{78}$$

Working with $E(m,n)$ note, that in view of (65) the condition $s\le m^3/\psi(q_m)$ implies that $s\le m^5$. This, together with the fact that $(q_n)_{n\in\mathbb{N}}$ is α-separated implies that $E(m,n)$ is empty sum. Thus,

$$E(m,n)=0. \tag{79}$$

So, from upper bounds (73), (74), (76), (78) and equality (79), for any fixed $a,b\in\mathbb{N}$ with $a<b$, and all natural numbers m,n, with $a\le m<n\le b$:

$$|S_{m,n}|\ll \frac{\psi(q_m)}{n^\rho\varepsilon_n^{1/2}}+\frac{\psi(q_n)}{m^\rho\varepsilon_m^{1/2}}+\frac{1}{m^\rho\varepsilon_m^{1/2}\,\varepsilon_n^{1/2}}+\frac{\psi(q_n)}{m^3\,\varepsilon_m}\,. \tag{80}$$

Now, given $a\in\mathbb{N}$, define, for all $n\in\mathbb{N}$, $n\ge a$,

$$\varepsilon_n := \min\left(1,\ \left(\sum_{k=a}^{n}\psi(q_k)\right)^{-1}\right). \tag{81}$$

Then, since $\psi(k) \le 1$ for all $k \in \mathbb{N}$,

$$\varepsilon_n^{-1} \le \max(1, n) \le n, \quad n \ge a. \tag{82}$$

Therefore,

$$\sum_{n=a}^{\infty} \frac{1}{n^\rho \varepsilon_n^{1/2}} \le \sum_{n=a}^{\infty} \frac{1}{n^{\rho - \frac{1}{2}}} < \infty,$$

(recall that $\rho > 2$), and so, it follows from (80) that for any $a, b \in \mathbb{N}$, $a < b$,

$$\sum\sum_{a \le m < n \le b} |S_{m,n}| \ll \sum_{n=a}^{b} \psi(q_n) + \sum\sum_{a \le m < n \le b} \frac{1}{n^\rho \varepsilon_m^{1/2} \varepsilon_n^{1/2}} + \sum\sum_{a \le m < n \le b} \frac{\psi(q_n)}{m^3 \varepsilon_m}. \tag{83}$$

For the third sum in the right hand side of (83), from inequalities (82), we have

$$\sum\sum_{a \le m < n \le b} \frac{\psi(q_n)}{m^3 \varepsilon_m} \le \sum\sum_{a \le m < n \le b} \frac{\psi(q_n)}{m^2} \ll \sum_{n=a}^{b} \psi(q_n), \quad a, b \in \mathbb{N},\ a < b. \tag{84}$$

In order to estimate the second sum in the right hand side of (83) let's consider two cases.
Case 1: $\sum\limits_{k=a}^{b} \psi(q_k) > 1$. Then, by (81),

$$\frac{1}{n^\rho \varepsilon_m^{1/2} \varepsilon_n^{1/2}} \le \frac{1}{n^\rho} \sum_{k=a}^{b} \psi(q_k),$$

and so, since $\rho > 2$,

$$\sum\sum_{a \le m < n \le b} \frac{1}{n^\rho \varepsilon_m^{1/2} \varepsilon_n^{1/2}} \le \left(\sum\sum_{a \le m < n \le b} \frac{1}{n^\rho} \right) \cdot \left(\sum_{k=a}^{b} \psi(q_k) \right) \ll \sum_{k=a}^{b} \psi(q_n). \tag{85}$$

Case 2: $\sum\limits_{k=a}^{b} \psi(q_k) \le 1$. It follows, by (81), that $\varepsilon_n = 1$ for all $a \le n \le b$, therefore

$$\frac{1}{n^\rho \varepsilon_m^{1/2} \varepsilon_n^{1/2}} = \frac{1}{n^\rho}.$$

By using Lemma 5 with $\tau = \frac{\rho}{2}$, we get

$$\frac{1}{n^\rho \varepsilon_m^{1/2} \varepsilon_n^{1/2}} = \frac{1}{n^\rho} \le \frac{\psi(q_n)}{n^{\frac{\rho}{2}}}, \quad n \in \mathbb{N},$$

and so

$$\sum\sum_{a \le m < n \le b} \frac{1}{n^\rho \varepsilon_m^{1/2} \varepsilon_n^{1/2}} \le \sum\sum_{a \le m < n \le b} \frac{\psi(q_n)}{n^{\frac{\rho}{2}}} \le \sum\sum_{a \le m < n \le b} \frac{\psi(q_n)}{m^{\frac{\rho}{2}}} \ll \sum_{n=a}^{b} \psi(q_n). \tag{86}$$

Both cases give us the same upper bound. So, estimates (84), (85) and (86), together with (83), imply that, for all $a, b \in \mathbb{N}$ with $a < b$,

$$\sum\sum_{a \le m < n \le b} |S_{m,n}| \ll \sum_{n=a}^{b} \psi(q_n).$$

Therefore, by (69),

$$\sum\sum_{a\le m<n\le b}\mu(E_m\cap E_n)\le\sum\sum_{a\le m<n\le b}\widehat{W}^+_{m,n}(0)+O\left(\sum_{n=a}^b\psi(q_n)\right). \tag{87}$$

We now turn our attention to estimating the first term on the right hand of (87). Recall that according to (71)

$$\widehat{W}^+_{m,n}(0)\ \le\ 4(1+\varepsilon_m)(1+\varepsilon_n)\psi(q_m)\psi(q_n)+O\left((q_m,q_n)\min\Big(\frac{\psi(q_m)}{q_m},\frac{\psi(q_n)}{q_n}\Big)\right).$$

Since $(\varepsilon_n)_{n\ge a}$ is decreasing and $\varepsilon_n\le 1$, $n\ge a$, for all $m,n\in\mathbb{N}$ with $a\le m<n\le b$,

$$4(1+\varepsilon_m)(1+\varepsilon_n)\psi(q_m)\psi(q_n)\ \le\ 4\psi(q_m)\psi(q_n)+12\varepsilon_m\psi(q_m)\psi(q_n)\,. \tag{88}$$

Once again, we will consider two cases.

Case 1: $\sum\limits_{k=a}^{b}\psi(q_k)\le 1$. It follows, by (81), that $\varepsilon_n=1$ for all $a\le n\le b$, therefore

$$\sum\sum_{a\le m<n\le b}\varepsilon_m\psi(q_m)\psi(q_n)\le\sum\sum_{a\le m<n\le b}\psi(q_m)\psi(q_n)<\left(\sum_{n=a}^b\psi(q_n)\right)^2<\sum_{n=a}^b\psi(q_n). \tag{89}$$

Case 2: $\sum\limits_{k=a}^{b}\psi(q_k)>1$. Then, by (81),

$$\begin{aligned}\sum\sum_{a\le m<n\le b}\varepsilon_m\psi(q_m)\psi(q_n)&=\sum\sum_{a\le m<n\le b}\psi(q_m)\psi(q_n)\ \min\left(1,\left(\sum_{k=a}^m\psi(q_k)\right)^{-1}\right)=\\ &=\ \sum\sum_{a\le m<n\le b}\frac{\psi(q_m)\psi(q_n)}{\max\left(1,\sum_{k=a}^m\psi(q_k)\right)}\le\ \sum_{n=a}^b\psi(q_n)\sum_{m=a}^b\frac{\psi(q_m)}{\sum_{k=a}^m\psi(q_k)}\,.\end{aligned} \tag{90}$$

From Lemma 4 we have that, for all $a,b\in\mathbb{N}$ with $a<b$,

$$\sum_{m=a}^b\frac{\psi(q_m)}{\sum_{k=a}^m\psi(q_k)}\le 1-\log(\psi(q_a))+\log\left(\sum_{n=a}^b\psi(q_n)\right).$$

This together with (90) implies that

$$\begin{aligned}\sum\sum_{a\le m<n\le b}\varepsilon_m\psi(q_m)\psi(q_n)\ &\le\ \left(\sum_{n=a}^b\psi(q_n)\right)\left(1-\log(\psi(q_a))+\log\left(\sum_{n=a}^b\psi(q_n)\right)\right)\\ &\ll\ \sum_{n=a}^b\psi(q_n)\cdot\log\left(\sum_{n=a}^b\psi(q_n)\right).\end{aligned} \tag{91}$$

Combining the estimates (71), (88), (89) and (91) we find that

$$\sum\sum_{a\le m<n\le b} W^{+}_{m,n}(0) \le 2\left(\sum_{n=a}^{b}\psi(q_n)\right)^2 + O\left(\left(\sum_{n=a}^{b}\psi(q_n)\right)\log^{+}\left(\sum_{n=a}^{b}\psi(q_n)\right)\right)+$$

$$+\ O\left(\sum\sum_{a\le m<n\le b}(q_m,q_n)\min\left(\frac{\psi(q_m)}{q_m},\frac{\psi(q_n)}{q_n}\right)\right), \quad (92)$$

where we used the inequality

$$\left(\sum_{n=a}^{b}\psi(q_n)\right)^2 = \sum_{n=a}^{b}\psi^2(q_n) + 2\sum\sum_{a\le m<n\le b}\psi(q_m)\psi(q_n) \ge$$

$$\ge 2\sum\sum_{a\le m<n\le b}\psi(q_m)\psi(q_n).$$

On combining (87) and (92) we find that

$$2\sum\sum_{a\le m<n\le b}\mu(E_m\cap E_n)\ \le\ 4\left(\sum_{n=a}^{b}\psi(q_n)\right)^2 + O\left(\sum\sum_{a\le m<n\le b}(q_m,q_n)\min\left(\frac{\psi(q_m)}{q_m},\frac{\psi(q_n)}{q_n}\right)\right)$$

$$+O\left(\left(\sum_{n=a}^{b}\psi(q_n)\right)\log^{+}\left(\sum_{n=a}^{b}\psi(q_n)\right)+\sum_{n=a}^{b}\psi(q_n)\right).$$

Now we use Lemma 2 to complete the proof of Lemma 5. □

Applying Lemma 1 to prove Theorem 3

In this section we will prove our main result, namely Theorem 3. We deduce it from Lemma 1 using estimates from Lemma 2 and Lemma 5. In order to do this we will show that all conditions of mentioned above lemmas are satisfied under conditions of Theorem 3.

Proof of Theorem 3. Note that under balance condition (10) and (11) condition (1) (and, actually, condition (2) as well) is satisfied. Therefore, the convergence part of Theorem 3 follows from [19, Theorem 2]. Because of this, we need to prove only the divergence part. So in the rest of the proof we assume

$$\Psi(N) := \sum_{n=1}^{N}\psi(q_n)\to\infty, \qquad \text{when } N\to\infty. \quad (93)$$

Note that all conditions of Lemma 2 are satisfied under assumptions of Theorem 3 and condition (93).

We proceed by showing that it is enough to prove Theorem 3 with the extra assumption of (65). To this end, introduce two new auxiliary functions $\omega,\psi^{*}:\mathcal{A}\to[0,1]$,

$$\omega(q_n)=n^{-\tau}, \quad n\in\mathbb{N},$$

$$\psi^*(q_n) = \max\{\psi(q_n), \omega(q_n)\}, \quad n \in \mathbb{N}.$$

Note that

$$\sum_{n=1}^{\infty} \omega(q_n) = \sum_{n=1}^{\infty} n^{-\tau} < \infty,$$

so by convergence part of Theorem 3 already justified in the beginning of the proof (or see [19, Theorem 2]) we have that counting function $R(x, N; \gamma, \omega, \mathcal{A})$ remains bounded as $N \to \infty$, therefore, in view of Proposition 1 (i), $x \notin W_{\mathcal{A}}(\gamma; \omega)$. So, by Proposition 1 (iii), we have

$$R(x, N; \gamma, \psi^*, \mathcal{A}) = R(x, N; \gamma, \psi, \mathcal{A}) + O(1).$$

This implies that the conclusion of the theorem (12) for ψ^* is equivalent to (12) with original function ψ. In the meantime, ψ^* obviously satisfies condition (65). So, without loss of generality, we can assume that ψ satisfies condition (65).

So, we have checked that under the conditions of Theorem 3 and assumption (93) all conditions of Lemma 2 and Lemma 5 are fulfilled, and now we can start to apply Lemma 1. Using (55) and (66) on the right-hand side of (43), we find that, for $f_n(x)$ and f_n defined by (42) and for any $a, b \in \mathbb{N}$ with $a < b$,

$$\begin{aligned} \int_{\mathbb{I}} \left(\sum_{n=a}^{b} \big(f_n(x) - f_n\big) \right)^2 \mathrm{d}\mu(x) \ &= O\Bigg(\Big(\sum_{n=a}^{b} \psi(q_n) \Big) \Big(\log^+ \Big(\sum_{n=a}^{b} \psi(q_n) \Big) + 1 \Big) \\ &\qquad + \sum\sum_{a \le m < n \le b} (q_m, q_n) \min\left(\frac{\psi(q_m)}{q_m}, \frac{\psi(q_n)}{q_n} \right) \Bigg). \end{aligned} \tag{94}$$

Let $m \in \mathbb{N}$ be the smallest integer satisfying $a \le m \le b$ such that

$$\sum_{n=a}^{m} \psi(q_n) \ \ge \ \frac{1}{2} \sum_{n=a}^{b} \psi(q_n). \tag{95}$$

Note that by the definition of m, we have that

$$\sum_{n=m}^{b} \psi(q_n) \ \ge \ \frac{1}{2} \sum_{n=a}^{b} \psi(q_n) \tag{96}$$

and that for any integer n such that $m \le n \le b$

$$2\Psi(n) \ = 2 \sum_{k=1}^{n} \psi(q_k) \ \ge \ \sum_{k=a}^{b} \psi(q_k). \tag{97}$$

From inequalities (95), (96) and (97) we have

$$\left(\sum_{n=a}^{b} \psi(q_n) \right) \left(\log^+ \left(\sum_{n=a}^{b} \psi(q_n) \right) + 1 \right) \le \ 2 \left(\sum_{n=m}^{b} \psi(q_n) \right) \left(\log^+ \left(\sum_{n=a}^{b} \psi(q_n) \right) + 1 \right) \le$$

$$\le \ 2 \left(\sum_{n=m}^{b} \psi(q_n) \left(\log^+ (2\Psi(n)) + 1 \right) \right) \le \ 2 \left(\sum_{n=m}^{b} \psi(q_n) \left(\log^+ \Psi(n) + 2 \right) \right) \le$$

$$\leq \ 2\left(\sum_{n=a}^{b}\psi(q_n)\left(\log^+\Psi(n)+2\right)\right) .$$

Therefore,

$$\left(\sum_{n=a}^{b}\psi(q_n)\right)\left(\log^+\left(\sum_{n=a}^{b}\psi(q_n)\right)+1\right)+$$
$$+\sum\sum_{a\leq m<n\leq b}(q_m,q_n)\min\left(\frac{\psi(q_m)}{q_m},\frac{\psi(q_n)}{q_n}\right) \ \leq \ 2\sum_{n=a}^{b}\phi_n\,,$$

where

$$\phi_n:=\psi(q_n)\left(\log^+\Psi(n)+2\right)+\sum_{m=1}^{n-1}(q_m,q_n)\min\left(\frac{\psi(q_m)}{q_m},\frac{\psi(q_n)}{q_n}\right) . \tag{98}$$

This, together with (94), implies condition (40) of Lemma 1. Now we use Lemma 1 with $X, f_n(x)$ and f_n given by (42) and ϕ_n by (98). It is left to note that for any $n\in\mathbb{N}$, we have that $f_n\leq\phi_n$, $f_n\leq 2$ and

$$\Phi(N):=\sum_{n=1}^{N}\phi_n \ \leq \ \Psi(N)\left(\log^+\Psi(N)+2\right) \ + \ E(N)\,,$$

where $E(N)$ is given by (14). The counting statement (12) now follows from (41). □

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