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Dubrovin duality and mirror symmetry for ADE resolutions

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We show that, under Dubrovin's notion of 'almost' duality, the Frobenius manifold structure on the orbit spaces of the extended affine Weyl groups of type ADE is dual, for suitable choices of weight markings, to the equivariant quantum cohomology of the minimal resolution of the du Val singularity of the same Dynkin type. We also provide a uniform Lie-theoretic construction of Landau–Ginzburg (LG) mirrors for the quantum cohomology of ADE resolutions. The mirror B-model is described by a one-dimensional LG superpotential associated with the spectral curve of the $\widehat{\text{ADE}}$ affine relativistic Toda chain.

1. Introduction

Let $l \in \mathbb{Z}_{>0}$ and $\mathcal{R} = \text{ADE}_l$ be a rank- l simply-laced irreducible root system, and fix a choice of fundamental weight $\widehat{\omega}$ for the complex simple Lie algebra associated with $\mathcal{R}$ as follows:

- when $\mathcal{R} = A_l$, $\widehat{\omega}$ can be any fundamental weight;
- when $\mathcal{R} = D_l$ or $\mathcal{R} = E_l$, $\widehat{\omega}$ will be the highest weight of the fundamental representation of the highest dimension.

We will call the datum $(\mathcal{R}, \widehat{\omega})$ a *marked ADE pair*. The corresponding Dynkin diagrams, with node marking specified by $\widehat{\omega}$, are shown in [figure 1](#). In this paper, we will be concerned with three classes of Frobenius manifolds associated with a given pair $(\mathcal{R}, \widehat{\omega})$, arising, respectively, from representation theory, enumerative algebraic geometry and integrable systems.

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- For $\mathcal{R}$ the root system of any complex simple Lie algebra, and $\widehat{\omega} \in \Omega$ a particular choice of fundamental weight, Dubrovin and Zhang [1] famously constructed a canonical semi-simple Frobenius manifold structure $\text{EAW}(\mathcal{R}, \widehat{\omega})$ on the orbits of the reflection representation of the $\widehat{\omega}$ -extended affine Weyl group of $\mathcal{R}$, generalizing the classical construction of polynomial Frobenius manifolds on orbit spaces of Coxeter groups. The specialization to $(\mathcal{R}, \widehat{\omega})$ being a marked ADE pair will be the setup of sole concern to us in this paper, and we will use the shorthand notation,

$$\mathcal{M}_{\text{AW}} := \text{EAW}(\mathcal{R}, \widehat{\omega}).$$

- Let $G < \text{SL}(2, \mathbb{C})$, $|G| < \infty$ be the McKay group of type $\mathcal{R}$, and let $Z = \widetilde{\mathbb{C}^2/G}$ be the minimal resolution of the associated canonical surface singularity. The pair $(\mathcal{R}, \widehat{\omega})$ specifies a $\mathbb{C}^\times$ -action on Z , point-wise fixing the irreducible component of the exceptional locus of Z corresponding to the marked fundamental weight $\widehat{\omega}$ under the McKay correspondence [2]. The associated Frobenius manifold is the $\mathbb{C}^\times$ -equivariant quantum cohomology of Z ,

$$\mathcal{M}_{\text{GW}} := \text{QH}_{\mathbb{C}^\times}(Z).$$

- In [3], Dubrovin constructs a Frobenius manifold structure $\text{LG}(\lambda, \phi)$ on the Hurwitz moduli space of ramified covers of the projective line with given genus and ramification profile at infinity. Here, λ (the *Landau–Ginzburg (LG) superpotential*) denotes the universal map, and ϕ is the additional datum of a Saito form [4] on the fibres of the family. One can associate to the pair $(\mathcal{R}, \widehat{\omega})$ an algebraically completely integrable system—the type $(\mathcal{R}, \widehat{\omega})$ affine relativistic Toda chain [5] at vanishing Casimir—whose isospectral dynamics is encoded in a special Frobenius submanifold of a certain Hurwitz space [6]. In this context, λ is the spectral parameter of the relativistic Toda Lax matrix, and ϕ is the differential of the (logarithm of the) argument of its characteristic polynomial. We will denote this Frobenius manifold as

$$\mathcal{M}_{\text{LG}} := \text{LG}(\lambda, \phi).$$

(a) Dubrovin duality and mirror symmetry

Given a Frobenius manifold $\mathcal{M}$ with a linear Euler vector field E and semi-simple product

$$\circ : \Gamma(\mathcal{M}, \mathcal{T}\mathcal{M}) \otimes_{\mathcal{O}_{\mathcal{M}}} \Gamma(\mathcal{M}, \mathcal{T}\mathcal{M}) \longrightarrow \Gamma(\mathcal{M}, \mathcal{T}\mathcal{M}),$$

one can construct [7] a second family $\mathcal{M}^b$ of Frobenius rings on the locus where E is invertible in the $\circ$ -algebra. This is obtained by pre-composing the flat pairing on $\mathcal{M}$ with multiplication of each of its entries by $E^{-1/2}$,

$$\mathcal{M} \xrightarrow{E^{-1/2} \circ} \mathcal{M}^b. \quad (1.1)$$

In a canonical coordinate chart for $\mathcal{M}$, equation (1.1) corresponds to rescaling the coefficients of the (diagonal) Gram matrix of the Frobenius pairing by the inverse diagonal matrix of the canonical coordinates. It is further shown in [7] that the resulting rescaled pairing on $\mathcal{M}^b$ is flat and induced by a *dual* prepotential function to the original prepotential of $\mathcal{M}$. We will call $\mathcal{M}^b$ the *Dubrovin-dual*¹ Frobenius manifold of $\mathcal{M}$.

Dubrovin's duality acquires a particularly salient form when $\mathcal{M}$ admits a realization as a Landau–Ginzburg model on a family of algebraic curves. In this case, (1.1) amounts to replacing

¹The operation in equation (1.1) is often referred to as an ‘almost-duality’ of Frobenius manifolds, owing to the fact that $\mathcal{M}^b$ is a weak (or *almost*) Frobenius manifold: we review this in §2a. The terminology ‘duality’ is perhaps somewhat improper, as for once the operation in equation (1.1) is not involutive, but we abide by historical convention and refer to it as such (see also [8,9]).

$\mathcal{R}$	$\widehat{\omega}$	Marked ADE Dynkin diagram
A_l	$\omega_{\bar{k}}$	
D_l	ω_{l-2}	
E_6	ω_3	
E_7	ω_3	
E_8	ω_3	

Figure 1. Marked Dynkin diagrams of pairs $(\mathcal{R}, \widehat{\omega})$. The marked node corresponding to the weight $\widehat{\omega}$ is indicated in black.

the superpotential by its logarithm [3,7,10]:

$$\mathcal{M}^b \simeq \text{LG}(\log \lambda, \phi).$$

From the point of view of the bihamiltonian quasi-linear integrable hierarchy defined on the loop space of a Frobenius manifold [3], a mirror-symmetry presentation of $\mathcal{M}$ as a Landau–Ginzburg model is equivalent to a dispersionless Lax–Sato formulation of the integrable flows, with λ coinciding with the Lax symbol [11]. Dubrovin’s duality (1.1), on the other hand, corresponds to expressing the integrable flows on $\mathcal{M}$ in Darboux coordinates for the second Poisson bracket of the Principal Hierarchy (the limiting value of the Dubrovin–Novikov pencil at infinity), and accordingly to a different (‘dual’) notion of topological τ -function in the sense of [12,13].

(b) Main results

In this paper, we show that, for all marked ADE pairs $(\mathcal{R}, \widehat{\omega})$, the Frobenius manifolds $\mathcal{M}_{\text{AW}}$, $\mathcal{M}_{\text{GW}}$ and $\mathcal{M}_{\text{LG}}$ are either non-trivially isomorphic,² or Dubrovin-dual to each other.

²As all identifications here presuppose a choice of flat chart, we are really talking about local isomorphisms here. This only (trivially) acquires global nature for $\mathcal{M}_{\text{AW}}$ and $\mathcal{M}_{\text{GW}}$, as the underlying manifolds are vector spaces in both cases.

Theorem 1.1. For all marked ADE pairs $(\mathcal{R}, \widehat{\omega})$, we have

$$\begin{array}{ccc}
 \mathcal{M}_{\text{AW}} & \xrightarrow{\simeq} & \mathcal{M}_{\text{LG}} \\
 \downarrow E^{-1/2} \circ & \square & \downarrow E^{-1/2} \circ \\
 \mathcal{M}_{\text{AW}}^b & & \mathcal{M}_{\text{LG}}^b \\
 & \searrow \simeq \quad \swarrow \simeq & \\
 & \mathcal{M}_{\text{GW}} &
 \end{array}$$

The isomorphism in the top row,

$$\mathcal{M}_{\text{AW}} \simeq \mathcal{M}_{\text{LG}},$$

was proved in [6]. Therefore, the statement that the equivariant quantum cohomology of ADE resolutions is Dubrovin-dual to the corresponding extended affine Weyl Frobenius manifold,

$$\mathcal{M}_{\text{GW}} \simeq \mathcal{M}_{\text{AW}}^b, \quad (1.2)$$

is logically equivalent to proving its mirror realization as an LG model on a family of relativistic Toda spectral curves, upon replacing the spectral parameter as $\lambda \rightarrow \log \lambda$,

$$\mathcal{M}_{\text{GW}} \simeq \mathcal{M}_{\text{LG}}^b. \quad (1.3)$$

Our strategy will be to prove the mirror theorem (1.3) first, as a means to establish the Dubrovin duality (1.2) as a consequence, for the classical series $\mathcal{R} = A_l$ and $\mathcal{R} = D_l$; and vice-versa for the exceptional series $\mathcal{R} = E_l$.

Theorem 3.2 answers constructively, in all Dynkin types, a long-standing question about mirror symmetry in the fundamental setup of quantum cohomology of du Val resolutions. When $\mathcal{R} = A_l$, Z is a smooth two-dimensional toric Calabi–Yau surface: in this case a Landau–Ginzburg mirror has been known since Givental’s work on equivariant toric mirror symmetry [14,15]. The case where $\mathcal{R} \neq A_l$ and Z is not toric has been outstanding to-date, as the methods of [14] cannot be directly applied to this more general setup. An important consequence of equation (1.3) is that it automatically provides a global integral representation of the components of the J -function as univariate Laplace-type integrals: for type $\mathcal{R} = A_l$, this enhanced control on their analytic continuation was brought to fruition in [16] to prove Iritani’s integral K-theoretic and higher genus full-descendent Crepant Resolution Conjectures, using R-matrix quantization techniques. Theorem 3.2 opens the way for a similar analysis for all ADE types, which will be explored in future work. The isomorphism (1.2) further suggests a conjectural integrable hierarchy governing the higher genus Gromov–Witten theory of Z : this should coincide with the one constructed in [17] expressed in a suitable set of dual variables, given by Darboux coordinates for a second Hamiltonian structure. For the classical series $\mathcal{R} = A_l$ and $\mathcal{R} = D_l$, the hierarchy is a particular rational reduction of the 2-Toda hierarchy [18].

(c) Organization of the paper

This paper will be organized as follows. In §2a, we will start with a short, but self-contained review of basic notions from the theory of Frobenius manifolds and Dubrovin’s duality. We will follow this up in §2b with a detailed construction of the three Frobenius manifolds $\mathcal{M}_{\text{AW}}$, $\mathcal{M}_{\text{LG}}$ and $\mathcal{M}_{\text{GW}}$ specified by the datum of a marked ADE pair $(\mathcal{R}, \widehat{\omega})$. In §3a, we will explain how the embedding of the LG model (λ, ϕ) into a genus zero Hurwitz–Frobenius manifold for the classical series $\mathcal{R} = A_l$ and $\mathcal{R} = D_l$ allows us to systematically determine the structure constants of $\mathcal{M}_{\text{LG}}^b$. Armed with this, the mirror theorem in equation (1.3), and therefore the

Table 1. Notation used throughout the text. When working in local coordinates for $\mathcal{M}_{\text{AW}}$, we will consistently use lower-case Latin indices for a flat chart for the intersection form; upper-case Latin indices for a flat chart for the Saito metric; and Greek indices for a coordinate chart given by basic invariants.

$\mathcal{R}$, resp. $\mathcal{R}^+$	an ADE root system, resp. its set of positive roots
$\Pi = \{\alpha_1, \dots, \alpha_l\}$	the set of simple roots of $\mathcal{R}$
$\Omega = \{\omega_1, \dots, \omega_l\}$	the set of fundamental weights of $\mathcal{R}$
ρ_i , resp. Γ_i	irreducible representation, resp. weight system, with highest weight ω_i
$(\mathfrak{h}, \langle, \rangle)$	the Cartan subalgebra of $\mathfrak{g}$, together with its Cartan–Killing form
$\mathfrak{g}$	the complex simple Lie algebra associated with $\mathcal{R}$
$C_{ij} = \langle \alpha_i, \alpha_j \rangle$	the Cartan matrix of $\mathcal{R}$
$\widehat{\omega}$	a marked fundamental weight of $\mathcal{R}$ as in figure 1.
$\mathcal{W}/\widetilde{\mathcal{W}}/\widetilde{\mathcal{W}}$	the Weyl/affine Weyl/extended affine Weyl group of type $(\mathcal{R}, \widehat{\omega})$
X , resp. Z	the du Val singularity of type $\mathcal{R}$, resp. its minimal resolution
$\mathbb{T}$	the torus action on Z specified by $\widehat{\omega}$
$\mathcal{M}_{\text{AW}}$	the extended affine Weyl Frobenius manifold of type $(\mathcal{R}, \widehat{\omega})$
$\mathcal{M}_{\text{GW}}$	the equivariant quantum cohomology of Z of type $(\mathcal{R}, \widehat{\omega})$
$\mathcal{M}_{\text{LG}}$	the Landau–Ginzburg Frobenius manifold of the affine relativistic Toda chain of type $(\mathcal{R}, \widehat{\omega})$
$(\eta_\bullet, c_\bullet, \eta_\bullet^\flat, c_\bullet^\flat)$	the metric, product tensor and their duals on $\mathcal{M}_\bullet$ ($\bullet \in \{\text{AW}, \text{GW}, \text{LG}\}$)
$(Y_\alpha)_{\alpha=1}^l$	basic Weyl-invariant Fourier polynomials of $\mathfrak{h}$
$(W_\alpha)_{\alpha=1}^l$	the fundamental characters of $\exp(\mathfrak{g})$
$(y_\alpha)_{\alpha=1}^{l+1}$	basic extended affine Weyl-invariant Fourier polynomials of $\mathfrak{h} \oplus \mathbb{C}$
$(x_i)_{i=1}^{l+1}$, resp. $(p_i)_{i=1}^{l+1}$	linear coordinates for $\mathfrak{h} \oplus \mathbb{C}$ in the basis $\Pi^\vee$, resp. $\Omega^\vee$
$(t_A)_{A=1}^{l+1}$	a flat coordinate chart for $\mathcal{M}_{\text{AW}}$
$\mathfrak{I}$	a set of initial conditions for $\mathcal{M}_{\text{AW}}$

duality (1.2) with the extended affine Weyl orbit spaces, can be deduced upon comparison with the genus zero Gromov–Witten calculations of [16,19]. In §3b, we give a general representation-theoretic argument, applicable to all extended affine Weyl Frobenius manifolds, showing that the structure constants of the Frobenius product on both sides of equation (1.2) belong to a certain finite-dimensional vector space of quasi-homogeneous polynomials. As such, the corresponding (2,1)-tensors are determined by their values on a (small) finite set of points $\mathfrak{I}$ in the semi-simple locus of $\mathcal{M}_{\text{AW}}$, which we call a set of *initial conditions* for $\mathcal{M}_{\text{AW}}$. We perform this analysis specifically for $\mathcal{R} = E_l$, and show how the reduction to initial conditions drastically simplifies the verification of equation (1.2), and, therefore, the proof of the mirror theorem in (1.3), which we carry out for the entire exceptional series.

We shall never assume Einstein’s convention in this paper. For the reader’s convenience, we collect the notation employed throughout the text in table 1.

2. Setup

(a) Generalities on Frobenius manifolds

Let M be an n -dimensional complex manifold. We will write $\mathcal{T}_M$ (resp. Ω_M) for the sheaf of holomorphic sections of the holomorphic tangent (resp. cotangent) bundle $T^{1,0}M$ (resp. $T_{1,0}^*M$),

and $\mathfrak{X}(M) := H^0(M, \mathcal{T}_M)$ for the space of global holomorphic vector fields on M . A holomorphic Frobenius manifold structure³ on M is a 4-tuple $\mathcal{M} := (M, c, \eta, e)$ satisfying the following axioms:

FM1 the *metric* $\eta \in H^0(M, \text{Sym}^2 \Omega_M)$ is a flat perfect symmetric pairing on $\mathcal{T}_M$;

FM2 the *product* $c \in H^0(M, \text{Sym}^3 \Omega_M)$ is a totally symmetric $(0, 3)$ -tensor

$$\eta(X, Y \circ Z) = \eta(X \circ Y, Z) := c(X, Y, Z) \in \mathcal{O}_M(M), \quad X, Y, Z \in \mathfrak{X}(M),$$

inducing, holomorphically in $p \in M$, a structure of commutative, unital, associative Frobenius algebra on the tangent fibre $T_p^{1,0} M$;

FM3 $\nabla_W c(X, Y, Z)$ is totally symmetric in $W, X, Y, Z \in \mathfrak{X}(M)$, where ∇ denotes the Levi-Civita connection of η ;

FM4 the *identity vector field* $e \in \mathfrak{X}(M)$, defined by its fibrewise restriction to the identity of the algebra, is horizontal with respect to ∇ , $\nabla e = 0$.

Two supplementary conditions are often imposed on $\mathcal{M}$, the second of which may or may not be realized in the context of this paper.

FM5 $\mathcal{M}$ is *semi-simple* if the set

$$\text{Discr}(\mathcal{M}) := \{p \in M \mid \exists v \in T_p M \text{ with } v \circ v = 0\}$$

has positive complex co-dimension;

FM6 $\mathcal{M}$ is *conformal* if there exists a holomorphic vector field $E \in \mathfrak{X}(M)$ which is covariantly linear, $\nabla \nabla E = 0$, and is such that

$$\mathcal{L}_E \circ = \circ \quad \text{and} \quad \mathcal{L}_E \eta = (2 - d)\eta,$$

for some constant $d \in \mathbb{Q}$, known as the *charge* of $\mathcal{M}$.

All examples in this paper will be semi-simple, but not necessarily conformal. We will write

$$M^{\text{ss}} := M \setminus \text{Discr}(\mathcal{M})$$

to indicate the open semi-simple locus of $\mathcal{M}$, and the calligraphic notation $\mathcal{M}^{\text{ss}}$ for the Frobenius manifold structure induced by restriction of $\mathcal{M}$ to M^{ss} .

Since the metric η is flat, the $\text{GL}(n, \mathbb{C})$ -equivalence class of flat frames for η equips a Frobenius manifold with a canonical $\text{GL}(n, \mathbb{C}) \ltimes \mathbb{C}^n$ affine-linear equivalence class of flat charts, in which the Gram matrix of the pairing η is constant. The axioms FM1–FM6 in one such chart $(t_1, \dots, t_n)$ amount to the local existence of a holomorphic function F (the *prepotential*) with the following properties:

(i) the unit vector field is

$$e = \frac{\partial}{\partial t_1};$$

(ii) the Gram matrix $\eta_{AB} := \eta(\partial/\partial t_A, \partial/\partial t_B)$ is constant, non-degenerate and equal to the Hessian matrix of $\partial_{t_1} F$,

$$\eta_{AB} = \frac{\partial^3 F}{\partial t_1 \partial t_A \partial t_B};$$

(iii) the prepotential is weighted quasi-homogeneous in its arguments,

$$\sum_{A=1}^n \left(p_A t_A \frac{\partial}{\partial t_A} + r_A \frac{\partial}{\partial t_A} \right) F = (3 - d)F + \text{quadratic},$$

for some $p_A, q_A, d \in \mathbb{Q}$;

³We call *Frobenius manifold* here what is often elsewhere referred to as a *weak* (or *almost*) *Frobenius manifold*, since we do not require axiom ‘FM6’ (existence and covariant linearity of E) to hold in our definition. The classical notion of Frobenius manifold in [3] is what we call *conformal Frobenius manifold* in this paper.

(iv) for all $A, B, M, N \in \{1, \dots, n\}$, writing

$$\eta^{CD} := (\eta^{-1})_{CD},$$

the prepotential satisfies the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations,

$$\sum_{C,D=1}^n \left(\frac{\partial^3 F}{\partial t_A \partial t_B \partial t_C} \eta^{CD} \frac{\partial^3 F}{\partial t_D \partial t_M \partial t_N} - \frac{\partial^3 F}{\partial t_A \partial t_M \partial t_C} \eta^{CD} \frac{\partial^3 F}{\partial t_D \partial t_B \partial t_N} \right) = 0. \quad (2.1)$$

(i) Dubrovin duality

If $\mathcal{M}$ is semi-simple, and for all $p \in M$ such that $E|_p$ is in the group of units of the $\mathbb{C}$ -algebra $(T_p^{1,0}M, \circ)$, one may define a second metric η^b (the *intersection form*) by

$$\eta^b(X, Y) := \eta(E^{-1/2} \circ X, E^{-1/2} \circ Y) = \eta(E^{-1} \circ X, Y). \quad (2.2)$$

A consequence of FM5–FM6 [3, Lect. 3] is that the metric η^b is also flat; we will usually denote by $(x_1, \dots, x_n)$ a choice of a flat coordinate chart for it. Alongside η^b , we can define a second commutative, associative, $\mathcal{O}_{M^{\text{ss}}}$ -algebra structure on $\mathcal{X}(M^{\text{ss}})$ with unit E via

$$c^b(X, Y, Z) := \eta^b(X, Y \star Z) \quad \text{and} \quad X \star Y := E^{-1} \circ X \circ Y. \quad (2.3)$$

The $\star$ -product is compatible (in the sense of Frobenius algebras) with the pairing η^b : this is an immediate consequence of the $\circ$ -product being compatible with respect to the pairing η .

Proposition 2.1 ([7, 20]). *Let $\mathcal{M} = (M, c, \eta, e, E)$ be a semi-simple conformal Frobenius manifold of charge $d = 1$. Then $\mathcal{M}^b := (M^{\text{ss}}, c^b, \eta^b, E)$ is a semi-simple Frobenius manifold satisfying axioms FM1–5, with flat⁴ unit $E \in \mathcal{X}(M^{\text{ss}})$.*

In particular, if $d = 1$ and in a flat chart $(x_1, \dots, x_n)$ for the intersection form η^b , there exists a solution F^b of the WDVV equations (2.1) (with $\eta, (t_A)_A$ replaced by $\eta^b, (x_i)_i$) where furthermore

$$E = \frac{\partial}{\partial x_1} \quad \text{and} \quad \eta_{ij}^b = \frac{\partial^3 F^b}{\partial x_1 \partial x_i \partial x_j}.$$

The Frobenius manifold structure $\mathcal{M}^b = (M^{\text{ss}}, c^b, \eta^b, E)$ induced by F^b will be called the *Dubrovin-dual* structure to $\mathcal{M}$. The two perfect pairings η and η^b on T_M^{ss} locally induce isometries between tangent and cotangent fibres,

$$(T_p M^{\text{ss}}, \eta) \simeq (T_p^* M^{\text{ss}}, \eta^{-1}) \quad \text{and} \quad (T_p M^{\text{ss}}, \eta^b) \simeq (T_p^* M^{\text{ss}}, (\eta^b)^{-1}). \quad (2.4)$$

In Einstein's convention, this would be the familiar operation of 'raising the indices' with η^{AB} (resp. $(\eta^b)^{ij}$). The isomorphisms (2.4) define two, *a priori* distinct, $\mathcal{O}_{M^{\text{ss}}}$ -algebra structures on holomorphic 1-forms $\theta, \chi \in \Omega_{M^{\text{ss}}}$,

$$\theta \hat{\circ} \chi \quad \text{and} \quad \theta \hat{\star} \chi, \quad (2.5)$$

respectively, given by the dual of the $\circ$ -product under η^{-1} , and the dual of the $\star$ -product under $(\eta^b)^{-1}$ in equation (2.4). By equations (2.2) and (2.3), these two dual products on Ω_M are in fact identically isomorphic,

$$\theta \hat{\circ} \chi = \theta \hat{\star} \chi. \quad (2.6)$$

Spelling out equation (2.6) in the respective flat charts for η and η^b results in the following non-trivial relation between the prepotentials of $\mathcal{M}$ and $\mathcal{M}^b$:

$$\frac{\partial^3 F^b}{\partial x_i \partial x_j \partial x_k} = \sum_{a,b,A,B,C,M,N=1}^n \eta_{ia}^b \eta_{jb}^b \frac{\partial t_C}{\partial x_k} \frac{\partial x_a}{\partial t_C} \frac{\partial x_b}{\partial t_A} \frac{\partial x_b}{\partial t_B} \eta^{AM} \eta^{BN} \frac{\partial^3 F}{\partial t_C \partial t_M \partial t_N}. \quad (2.7)$$

⁴For charge $d \neq 1$, the unit E for M^b is not, in general, flat.

(b) Frobenius manifolds of type $(\mathcal{R}, \widehat{\omega})$

With these preliminaries, we shall consider three classes of semi-simple Frobenius manifolds associated with a marked ADE pair $(\mathcal{R}, \widehat{\omega})$.

(i) Extended affine Weyl Frobenius manifolds

Let $\mathfrak{g} = \mathfrak{ad}\mathfrak{e}_l$ be the rank- l complex, simple, simply-laced Lie algebra with root system $\mathcal{R}$. We shall denote by $\mathfrak{h}$ the associated Cartan subalgebra, and by $\mathcal{W}$ its Weyl group. The action of $\mathcal{W}$ on $\mathfrak{h}$ lifts to an action of the affine Weyl group $\widehat{\mathcal{W}} \cong \mathcal{W} \ltimes \Lambda_r^\vee$, with $\Lambda_r^\vee$ the lattice of co-roots:

$$\left. \begin{aligned} \widehat{\mathcal{W}} \times \mathfrak{h} &\longrightarrow \mathfrak{h}, \\ ((w, \alpha^\vee), h) &\longrightarrow w(h) + 2\pi i \alpha^\vee. \end{aligned} \right\} \quad (2.8)$$

For $(\mathcal{R}, \widehat{\omega})$ a marked ADE pair, the corresponding $\widehat{\omega}$ -extended affine Weyl group $\widetilde{\mathcal{W}}$ is defined as the semi-direct product $\widetilde{\mathcal{W}} := \widehat{\mathcal{W}} \ltimes \mathbb{Z}$ acting on $\mathfrak{h} \oplus \mathbb{C}$ by

$$\left. \begin{aligned} \widetilde{\mathcal{W}} \times \mathfrak{h} \oplus \mathbb{C} &\longrightarrow \mathfrak{h} \oplus \mathbb{C}, \\ ((w, \alpha^\vee, l), (h, v)) &\longrightarrow (w(h) + 2\pi i \alpha^\vee + 2\pi i l \widehat{\omega}, v - 2\pi i l). \end{aligned} \right\} \quad (2.9)$$

Let Σ denote the hyperplane arrangement associated with the root system $\mathcal{R}$, and $\mathfrak{h}^{\text{reg}} := \mathfrak{h} \setminus \Sigma$ be the set of regular elements in $\mathfrak{h}$. The restriction of equation (2.9) to $\mathfrak{h}^{\text{reg}} \oplus \mathbb{C}$ is a free affine action, whose quotient defines the regular orbit space of the extended affine Weyl group of $\mathcal{R}$ with marked weight $\widehat{\omega}$ as

$$M_{\text{AW}} := \frac{(\mathfrak{h}^{\text{reg}} \times \mathbb{C})}{\widetilde{\mathcal{W}}} \cong \frac{\mathcal{T}^{\text{reg}}}{\mathcal{W} \times \mathbb{C}^*}, \quad (2.10)$$

where $\mathcal{T}^{\text{reg}} = \exp(\mathfrak{h}^{\text{reg}})$ is the image of the set of regular elements of $\mathfrak{h}^{\text{reg}}$ under the exponential map to the maximal torus $\mathcal{T}$. Let $(x_1, \dots, x_l)$ be linear coordinates on $\mathfrak{h}$ with respect to the co-root basis $\{\alpha_1^\vee, \dots, \alpha_l^\vee\}$, and extend these to linear coordinates $(x_1, \dots, x_l; x_{l+1})$ on $\mathfrak{h} \oplus \mathbb{C}$, giving local coordinates on the regular orbit space. Denoting $\langle \alpha, \beta \rangle$ the pairing on $\mathfrak{h}^*$ induced by the restriction of the Killing form on the Cartan subalgebra, and writing

$$C_{ab} = \langle \alpha_a, \alpha_b^\vee \rangle, \quad d_a := \langle \omega_a, \widehat{\omega} \rangle, \quad \widehat{d} := \langle \widehat{\omega}, \widehat{\omega} \rangle,$$

we can define a non-degenerate pairing ξ on $\mathfrak{h} \times \mathbb{C}$ by orthogonal extension of minus the Cartan-Killing form on $\mathfrak{h}$ as

$$\eta_{\text{AW}}^b(\partial_{x_a}, \partial_{x_b}) := \begin{cases} -C_{ab} & \text{if } a, b < l+1, \\ \widehat{d} & \text{if } a = b = l+1, \\ 0 & \text{otherwise,} \end{cases} \quad (2.11)$$

with x_{l+1} parameterizing linearly the right summand in $\mathfrak{h} \oplus \mathbb{C}$. The quotient map $\mathfrak{K}: \mathcal{T}^{\text{reg}} \times \mathbb{C}^* \rightarrow M_{\text{AW}}$ from equation (2.10) defines a principal $\mathcal{W}$ -bundle on M_{AW} : a section $\tilde{\sigma}_i$ lifts a (sufficiently small) open $U \subset M_{\text{AW}}$ to the i th sheet of the cover $V_i \in \tilde{\sigma}_i^{-1}(U) \equiv V_1 \sqcup \dots \sqcup V_{|\mathcal{W}|}$. The invariant ring $\mathcal{I} := \mathbb{C}[\mathcal{T}]^{\mathcal{W}}$ is, from classical results about exponential Weyl invariants [21], a polynomial ring $\mathcal{I} \simeq \mathbb{C}[Y_1, \dots, Y_l]$, where

$$Y_i := S_{\mathcal{W}}(e^{(\omega_i, h)}), \quad (2.12)$$

and $S_{\mathcal{W}}$ is the average over the Weyl orbit. Equivalently, we have $\mathcal{I} \simeq \mathbb{C}[W_1, \dots, W_l]$, where

$$W_i := \sum_{\omega \in \Gamma_i} e^{(\omega, h)}. \quad (2.13)$$

Proposition 2.2 ([1, Theorem 1.1]). Let $\mathcal{A}$ be the ring of $\widetilde{\mathcal{W}}$ -invariant Fourier polynomials in the variables $x_1, \dots, x_l$ and x_{l+1}/f that are bounded in the limit

$$x = x^{[0]} + \widehat{\omega}\tau, \quad x_{l+1} = x_{l+1}^{[0]} - \tau, \quad \tau \rightarrow +\infty, \quad (2.14)$$

for any $(x^{[0]}, x_{l+1}^{[0]})$, where f is the determinant of the Cartan matrix. Then,

$$\mathcal{A} \simeq \mathbb{C}[y_1, \dots, y_l, e^{y_{l+1}}]$$

with

$$y_\alpha = \begin{cases} e^{d_\alpha x_{l+1}} Y_\alpha, & \alpha = 1, \dots, l, \\ x_{l+1}, & \alpha = l+1 \end{cases} \quad (2.15)$$

and $d_i := \langle \omega_i, \widehat{\omega} \rangle$.

In the following, for $k \in \{1, \dots, l\}$ such that $\omega_{\bar{k}} = \widehat{\omega}$, we will denote $\widehat{y} := y_{\bar{k}}$. We will define a grading on $\mathcal{A}$ by

$$\deg y_\alpha = d_\alpha (\alpha = 1, \dots, l), \quad \deg e^{y_{l+1}} = 1.$$

The following reconstruction theorem holds [1, Theorem 2.1].

Theorem 2.3. There exists a unique, up to isomorphism, semi-simple and conformal Frobenius manifold

$$\mathcal{M}_{\text{AW}} = (M_{\text{AW}}, c_{\text{AW}}, \eta_{\text{AW}}, e, E)$$

of charge $d = 1$ satisfying the following properties in flat coordinates $(t_1, \dots, t_{l+1})$ for η_{AW} :

$$\text{AW-I } \widehat{d}E = \partial_{x_{l+1}} = \sum_{j=1}^l dt_j \partial_{t_j} + \partial_{t_{l+1}};$$

$$\text{AW-II the intersection form is } \eta^b = \tilde{\sigma}_i^* \xi;$$

$$\text{AW-III the prepotential is polynomial in } t_1, \dots, t_{l+1}, e^{t_{l+1}}.$$

(ii) Quantum cohomology of ADE resolutions

Let $G < \text{SU}(2)$, $|G| < \infty$ be a finite subgroup of $\text{SU}(2)$. The classical McKay correspondence classifies G by root systems of type $\mathcal{R} = \text{ADE}_l$,

$$G \simeq \begin{cases} \frac{\mathbb{Z}}{(l+1)\mathbb{Z}'}, & \mathcal{R} = \text{A}_l, \\ BD_{4l-2} = \langle 2, 2, l \rangle, & \mathcal{R} = \text{D}_l, \\ BT = \langle 2, 3, 3 \rangle, & \mathcal{R} = \text{E}_6, \\ BO = \langle 2, 3, 4 \rangle, & \mathcal{R} = \text{E}_7, \\ BI = \langle 2, 3, 5 \rangle, & \mathcal{R} = \text{E}_8. \end{cases}$$

The corresponding du Val singularity is defined as the affine scheme

$$X := \text{Spec} \left(\mathbb{C}[x, y]^G \right),$$

where the action on $\mathbb{C}^2$ by G is induced by restriction of the fundamental representation of $\text{SU}(2)$. Klein's classical presentation of the ring of invariants,

$$\mathbb{C}[x, y]^G \simeq \frac{\mathbb{C}[u, v, w]}{\langle \mathcal{I}_G \rangle},$$

where

$$\mathcal{I}_G(u, v, w) = \begin{cases} w^2 + u^2 + v^{l+1}, & \mathcal{R} = \text{A}_l, \\ w^2 + v(u^2 + v^{l-2}), & \mathcal{R} = \text{D}_l, \\ w^2 + u^3 + v^4, & \mathcal{R} = \text{E}_6, \\ w^2 + u^3 + uv^3, & \mathcal{R} = \text{E}_7, \\ w^2 + u^3 + v^5, & \mathcal{R} = \text{E}_8, \end{cases}$$

realizes X as a hypersurface in $\mathbb{C}^3$ with an isolated singularity at the origin. There is a well-known canonical minimal resolution

$$Z \xrightarrow{\pi} X \quad (2.16)$$

obtained through a sequence of blowing-ups of the singularity [2]. The intersection diagram describing the configuration of irreducible rational curves in the exceptional locus is the Dynkin diagram of the corresponding ADE type, and the integral homology of the resolution,

$$H_\bullet(Z, \mathbb{Z}) = H_0(Z, \mathbb{Z}) \oplus H_2(Z, \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}^l,$$

is isomorphic to the affine root lattice of type $\mathcal{R}$. Writing

$$\Pi := \{\alpha_1, \dots, \alpha_l\} \quad \text{and} \quad \Omega := \{\omega_1, \dots, \omega_l\}$$

for, respectively, the set of simple roots and fundamental weights of the complex simple Lie algebra associated with $\mathcal{R}$. We will label the irreducible components $\{e_1, \dots, e_l\}$ of the exceptional locus accordingly, so that

$$e_i \longleftrightarrow \alpha_i \longleftrightarrow \omega_i. \quad (2.17)$$

Let $\mathbb{T} \simeq \mathbb{C}^\times$ and consider a $\mathbb{T}$ -representation on $\mathbb{C}^2$ commuting with the action of $\mathfrak{G}$. This induces $\mathbb{T}$ -actions on X and Z , respectively, by descent to the quotient and by $\mathbb{T}$ -equivariance of the resolution (2.16). When $\mathcal{R} = \text{DE}_l$, we have a unique possible choice for the action of $\mathbb{T}$: this is the scalar torus action on $\mathbb{C}^2$ with characters (t, t) on the affine coordinates (x, y) of $\mathbb{C}^2$. When $\mathcal{R} = A_l$, since $\mathfrak{G}$ is abelian, we have more generally that the full Cartan torus

$$\mathbb{T}' := (\mathbb{C}^\times)^2 < \text{GL}(2, \mathbb{C}),$$

with characters (t_x, t_y) will act effectively on X and Z . For $\bar{k} \in \{1, \dots, l\}$, we will be specially interested in the one-dimensional subtorus $\mathbb{T}$ acting with characters $(t_x, t_y) = (t^{l+1-\bar{k}}, t^{\bar{k}})$.

As the singularity is the only torus fixed point in X , the $\mathbb{T}$ -fixed locus of Z is fully contained in the exceptional set. We will write $[e_i]^\vee$ for the Poincaré dual classes to $[e_i]$ in the locally compact cohomology, φ_i for their canonical lifts to $\mathbb{T}$ -equivariant cohomology, and φ_{l+1} for the identity class in $H_{\mathbb{T}}(Z)$. The localized $\mathbb{T}$ -equivariant cohomology of Z is an $(l+1)$ -dimensional vector space over the field of fractions of $H(B\mathbb{T}, \mathbb{C}) \simeq \mathbb{C}[\nu]$, where ν is the first Chern class of the $\mathbb{T}$ -representation t . For $\mathcal{R} = A_l$, we can more generally consider the equivariant cohomology of Z with respect to the full two-dimensional torus $\mathbb{T}'$: this will now be an $(l+1)$ -dimensional vector space over $\mathbb{C}(\nu_1, \nu_2)$, with $\nu_i := c_1(t_i) \in H_{\mathbb{T}'}(\text{pt})$. We will slightly abuse notation and continue to write $\varphi_1, \dots, \varphi_l, \varphi_{l+1}$ for the lift of $e_i^\vee$ in the $\mathbb{T}'$ -equivariant cohomology of Z .

Remark 2.4. For $(\mathcal{R}, \widehat{\omega})$ a marked ADE pair, the marked fundamental weight $\widehat{\omega} \in \Omega$ corresponds to the (single) irreducible exceptional divisor $\widehat{e} \in H_2(Z, \mathbb{Z})$ which is point-wise fixed by $\mathbb{T}$

$$\widehat{e} \longleftrightarrow \widehat{\alpha} \longleftrightarrow \widehat{\omega}.$$

To see this, notice that each double point in the exceptional locus is $\mathbb{T}$ -fixed. For $\mathcal{R} = \text{DE}_l$, this entails that the $\mathbb{T}$ -action on the irreducible component $\widehat{e}$ labelled by the trivalent marked node in figure 1 must be trivial, since $\widehat{e}$ is a $\mathbb{P}^1$ with three $\mathbb{T} \simeq \mathbb{C}^\times$ -fixed points. When $\mathcal{R} = A_l$, the weights $w_i^\pm$ at the two $\mathbb{T}'$ -fixed points of e_i are [16, Appendix B]

$$(w_i^+, w_i^-) = (-i\nu_1 + (l+1-i)\nu_2, i\nu_1 - (l+1-i)\nu_2).$$

As discussed above, the restriction to a one-dimensional torus $\mathbb{T}$ acting with characters $(t^{l+1-\bar{k}}, t^{\bar{k}})$ sets

$$\nu_1 = (l+1-\bar{k})\nu \quad \text{and} \quad \nu_2 = \bar{k}\nu, \quad (2.18)$$

so that $w_k^\pm = 0$. Hence, in this case,

$$e_{\bar{k}} \longleftrightarrow \alpha_{\bar{k}} \longleftrightarrow \omega_{\bar{k}} = \widehat{\omega}.$$

The torus action on Z induces an action on its moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(Z, \beta)$. Since the $\mathbb{T}$ -action on Z is free away from the exceptional locus, its fixed locus in $\overline{\mathcal{M}}_{g,n}(Z, \beta)$ is proper, and it carries a torus-equivariant perfect obstruction theory and virtual fundamental class [22]. Gromov–Witten invariants of Z can then be defined by

$$\langle \varphi_{i_1}, \dots, \varphi_{i_n} \rangle_{g,n,d}^Z := \int_{[\overline{\mathcal{M}}_{g,n}(Z, \beta)]_{\mathbb{T}}^{\text{vir}}} \prod_{j=1}^n \text{ev}_j^*(\varphi_{i_j}) := \int_{[\overline{\mathcal{M}}_{g,n}(Z, \beta)]_{\mathbb{T}}^{\text{vir}}} \frac{\prod_{j=1}^n \iota^* \text{ev}_j^*(\varphi_{i_j})}{e_{\mathbb{T}}(N^{\text{vir}})} \in \mathbb{Q}(v),$$

with $\iota: \overline{\mathcal{M}}_{g,n}^{\mathbb{T}}(Z, \beta) \hookrightarrow \overline{\mathcal{M}}_{g,n}(Z, \beta)$ being the immersion of the substack of $\mathbb{T}$ -fixed points into the moduli space of stable maps and N^{vir} its $\mathbb{T}$ -equivariant normal bundle. Writing $x = \sum_{i=1}^{l+1} x_i \varphi_i \in H_{\mathbb{T}}(Z)$, the genus-zero invariants of Z define a Frobenius manifold satisfying the axioms FM1–FM5,

$$\mathcal{M}_{\text{GW}} := (H_{\mathbb{T}}(Z), c_{\text{GW}}, \eta_{\text{GW}}, \varphi_{l+1})$$

via

$$\left. \begin{aligned} c_{\text{GW}}(\varphi, \psi, \theta) &:= \int_{[Z]_{\mathbb{T}}} \varphi \cup \psi \cup \theta + \sum_{\beta \in H_2(Z, \mathbb{Z})} \langle \varphi, \psi, \theta \rangle_{0,3,\beta}^Z e^{(\beta, x)} \\ \eta_{\text{GW}}(\varphi, \psi) &:= c_{\text{GW}}(\varphi_{l+1}, \varphi, \psi) = \int_{[Z]_{\mathbb{T}}} \varphi \cup \psi. \end{aligned} \right\} \quad (2.19)$$

In [1], when the torus acts with characters (t, t) , the authors solved the genus zero $\mathbb{T}$ -equivariant Gromov–Witten theory of Z using degeneration arguments and the Aspinwall–Morrison formula for the super-rigid local curve. The formal power series (2.19) is the Fourier expansion of a trigonometric rational function given explicitly by

$$\left. \begin{aligned} c_{\text{GW}}(\varphi_{l+1}, \varphi_i, \varphi_j) &= \eta_{\text{GW}}(\varphi_i, \varphi_j) = \begin{cases} \frac{1}{v^2} \frac{1}{|\mathcal{G}|}, & i = j = l + 1, \\ -C_{ij}, & i, j \leq l, \\ 0, & \text{else} \end{cases} \\ c_{\text{GW}}(\varphi_i, \varphi_j, \varphi_k) &= -v \sum_{\beta \in \mathcal{R}^+} \langle \alpha_i, \beta \rangle \langle \alpha_j, \beta \rangle \langle \alpha_k, \beta \rangle \coth \left(\frac{\sum_{m=1}^l \langle \beta, \alpha_m \rangle x_m}{2} \right), \quad i, j, k \leq l. \end{aligned} \right\} \quad (2.20)$$

The corresponding prepotential is $F_{\text{GW}} = F_{\text{GW}}^0 + F_{\text{GW}}^+$, where the zero and positive degrees parts read

$$\left. \begin{aligned} F_{\text{GW}}^0 &= \frac{1}{6v^2} \frac{1}{|\mathcal{G}|} (x_{l+1})^3 - \frac{1}{2} x_{l+1} \sum_{i,j=1}^l C_{ij} x_i x_j - \frac{v}{6} \sum_{\beta \in \mathcal{R}^+} \sum_{i,j,k=1}^l \langle \alpha_i, \beta \rangle \langle \alpha_j, \beta \rangle \langle \alpha_k, \beta \rangle x_i x_j x_k \\ \text{and } F_{\text{GW}}^+ &= 2v \sum_{\beta \in \mathcal{R}^+} \text{Li}_3 \left(e^{-\langle \beta, h \rangle} \right). \end{aligned} \right\} \quad (2.21)$$

When $\mathcal{R} = A_l$, the same reasoning applied to the full two-dimensional $\mathbb{T}'$ -action on Z yields [15,23]

$$\left. \begin{aligned} F_{\text{GW}}^0 &= \frac{1}{6v_1 v_2} \frac{1}{|\mathcal{G}|} x_{l+1}^3 - \frac{1}{2} x_{l+1} \sum_{i,j=1}^l C_{ij} x_i x_j - \frac{1}{6} \sum_{i,j,k,i',j',k'=1}^l C_{ii'} C_{jj'} C_{kk'} \mathfrak{C}_{i'j'k'} x_i x_j x_k \\ \text{and } F_{\text{GW}}^+ &= (v_1 + v_2) \sum_{\beta \in \mathcal{R}^+} \text{Li}_3 \left(e^{-\langle \beta, h \rangle} \right), \end{aligned} \right\} \quad (2.22)$$

where the symmetric trilinear form $\mathfrak{C}_{ijk}$ is determined by its value at $i \leq j \leq k$ as

$$\mathfrak{C}_{ijk} = \frac{(jv_1 + (l+1-j)v_2)i(l+1-k)}{l+1}, \quad i \leq j \leq k.$$

(iii) Landau–Ginzburg mirrors from ADE spectral curves

Let $N \in \mathbb{Z}_{>0}$ and $H_{g,m}$ be the moduli space of smooth genus g -covers of $\mathbb{P}^1$ with ramification profile at infinity specified by a partition $m \vdash N$. We will write π , λ and Σ_i for, respectively, the universal family, the universal map, and the sections marking $\{\infty_i\}_i := \lambda^{-1}([1:0])$, as per the following commutative diagram:

$$\begin{array}{ccccc} C_g & \hookrightarrow & C_{g,m} & \xrightarrow{\lambda} & \mathbb{P}^1 \\ P_i \uparrow & & \downarrow \Sigma_i & & \downarrow \pi \\ & & [\lambda] & \xrightarrow{\text{pt}} & H_{g,m} \end{array} \quad (2.23)$$

We furthermore denote by $d = d_\pi$ the relative differential with respect to the universal family and $p_i^{\text{cr}} \in C_g \simeq \pi^{-1}([\lambda])$ the critical locus $d\lambda = 0$ of the universal map. By the Riemann existence theorem, the critical values of λ ,

$$(u_i)_{i=1,\dots,d_{g,m}},$$

serve as local coordinates away from the discriminant locus $u_i = u_j$ for $i \neq j$. On its complement, we can construct a family of semi-simple, commutative, $\mathbb{C}$ -algebra structures on the tangent fibres at $(u_1, \dots, u_{d_{g,m}})$ by stipulating that the coordinate vector fields in the u -chart are the idempotents of the algebra,

$$\partial_{u_i} \cdot \partial_{u_j} = \delta_{ij} \partial_{u_i}. \quad (2.24)$$

Let $\mu : C_{g,m} \rightarrow \mathbb{P}^1$ be a surjective morphism such that the relative one-form

$$d \log \mu \in \Omega_{C_{g,m}/H_{g,m}}^1(\infty_0 + \dots + \infty_m)$$

is an exact third-kind differential⁵ on the fibres of the universal curve with simple poles at ∞_i , with residues

$$\text{Res}_{\infty_i} d \log \mu = a_i \in \mathbb{Z}.$$

If μ does not factor through λ , this defines an Ehresmann connection on $TC_{g,m}$ where points in nearby fibres of the universal curve are identified if they have the same image under μ . This defines a meromorphic derivation

$$\delta_{\partial_{u_i}} : H^0(C_{g,m}, \mathcal{O}_{C_{g,m}}) \rightarrow H^0(C_{g,m}, \mathcal{K}_{C_{g,m}})$$

defined in local coordinates $p = (u_1, \dots, u_{d_{g,m}}; \mu)$ for $C_{g,m}$ as the partial derivative taken with respect to u_i while keeping μ constant. A Frobenius manifold structure $\mathcal{H}_{g,m}^{[\mu]} := (M_{\text{LG}}, c_{\text{LG}}, \eta_{\text{LG}}, e)$ can be defined on the Hurwitz space $M_{\text{LG}} := H_{g,m}$ by the residue formulas [3]

$$\eta_{\text{LG}}(X, Y) := \sum_i \text{Res}_{p_i^{\text{cr}}} \frac{\delta_X \lambda \delta_Y \lambda}{d\lambda} \phi^2, \quad (2.25)$$

$$c_{\text{LG}}(X, Y, Z) := \sum_i \text{Res}_{p_i^{\text{cr}}} \frac{\delta_X \lambda \delta_Y \lambda \delta_Z \lambda}{d\lambda} \phi^2, \quad (2.26)$$

$$\text{and} \quad \eta_{\text{LG}}^b(X, Y) := \sum_i \text{Res}_{p_i^{\text{cr}}} \frac{\delta_X \log \lambda \delta_Y \log \lambda}{d \log \lambda} \phi^2, \quad (2.27)$$

where

$$\phi := \frac{d\mu}{\mu}.$$

The universal map λ and the relative differential ϕ are referred to as the *superpotential* and the *primitive form* of $\mathcal{H}_{g,m}$.

A general mirror symmetry construction of Frobenius submanifolds of Hurwitz spaces isomorphic to $\mathcal{M}_{\text{AW}}$, and the associated Dubrovin-dual Frobenius manifolds was given in [6]

⁵This is a special case of a type III admissible differential, in the classification of [3, Lect. 5].

(see also [1,24]), as we review here. For $(\mathcal{R}, \widehat{\omega})$ as a marked ADE pair, let $\bar{k} \in \{1, \dots, l\}$ be such that $\omega_{\bar{k}} = \widehat{\omega}$. Fix $0 \neq \omega \in \Lambda_w^+$ a non-zero dominant weight. Starting from the characteristic polynomial of a regular element of $\mathcal{T}$ in the representation ρ_ω ,

$$\mathcal{Q} = \prod_{\omega' \in \mathcal{W}(\omega)} (e^{(\omega', x)} - \mu) \in \mathbb{Q}[Y_1, \dots, Y_l][[\mu]], \quad (2.28)$$

define

$$\mathcal{P}(y_1, \dots, y_{l+1}; \lambda, \mu) := \mathcal{Q}(Y_i = y_i e^{-d_i y_{l+1}} - \delta_{i\bar{k}} \lambda e^{-y_{l+1}}; \mu). \quad (2.29)$$

For fixed $y \in M_{AW}^{ss}$, equation (2.29) defines a plane algebraic curve $C_y := \mathbb{V}(\mathcal{P})$. Let $\overline{C}_y$ denote the normalization of the projective closure of the fibre at y , $g := h^{1,0}(\overline{C}_y)$, and let m be the ramification profile over infinity of the Cartesian projection $\lambda: \overline{C}_y \rightarrow \mathbb{P}^1$. The corresponding family is the pull-back of the universal curve to M_{AW}^{ss} , where the pull-back metric, product, and intersection tensors are given by equations (2.25)–(2.27). With $\{p_i^{cr}\}_m$, the ramification points of $\lambda: \overline{C}_y \rightarrow \mathbb{P}^1$. For convenience and later comparison with $\mathcal{M}_{GW}$, it will be helpful to rescale the primitive differential and the linear coordinate on the second factor of $\mathfrak{h}^{\text{reg}} \oplus \mathbb{C}$ as

$$\phi^2 \rightarrow \frac{2\nu \dim_{\mathbb{C}} \mathfrak{g}}{\langle \omega, \omega + 2w \rangle \dim_{\mathbb{C}} \rho_\omega} \left(\frac{d\mu}{\mu} \right)^2, \quad x_{l+1} \rightarrow \frac{1}{2\nu} x_{l+1}, \quad (2.30)$$

where

$$w := \frac{1}{2} \sum_{\beta \in \mathcal{R}^+} \beta = \sum_{i=1}^l \omega_i \quad (2.31)$$

is the Weyl vector. By equations (2.25)–(2.27), this just results in an overall rescaling of the prepotential for $\mathcal{M}_{LG}$.

Theorem 2.5 (Mirror symmetry for $\mathcal{M}_{AW}$). *For all marked pairs $(\mathcal{R}, \widehat{\omega})$, the Landau–Ginzburg formulas (2.25)–(2.27) define a semi-simple conformal Frobenius manifold*

$$\mathcal{M}_{LG} = (M_{LG}, \eta_{LG}, c_{LG}, e, E),$$

with $e = y_{l+1}^{-1} \partial_{\tilde{y}}$, $E = y_{l+1} \partial_{y_{l+1}}$. Furthermore,

$$\mathcal{M}_{LG} \simeq \mathcal{M}_{AW}.$$

The next proposition [7,10] shows that, for $\mathcal{M} \hookrightarrow \mathcal{H}_{g,m}$ a Frobenius submanifold of a Hurwitz space, the Dubrovin-dual Frobenius manifold $\mathcal{M}^b$ is obtained by replacing the superpotential by its logarithm

$$\lambda \longrightarrow \log \lambda.$$

Proposition 2.6. *Let $\iota: \mathcal{M} = (M, c, \eta, e, E) \hookrightarrow \mathcal{H}_{g,m}$ be a semi-simple conformal Frobenius submanifold of a Hurwitz space defined by a Landau–Ginzburg pair $(\tilde{\lambda}, \tilde{\phi})$. Let $\mathcal{M}^{\log}$ be the Frobenius submanifold structure on M^{ss} defined by the formulas (2.25)–(2.26) with $\lambda = \log \tilde{\lambda}$, $\phi = \tilde{\phi}$, and $X, Y, Z \in \iota_*(\mathcal{X}(M))$. Then,*

$$\mathcal{M}^{\log} \simeq \mathcal{M}^b.$$

From proposition 2.6, the Frobenius structure of $\mathcal{M}_{LG}^b \simeq \mathcal{M}_{AW}^b$ is given as

$$\eta_{AW}^b(X, Y) = \eta_{LG}^b(X, Y) = \sum_m \text{Res}_{p_m^{cr}} \frac{\delta_X \lambda \delta_Y \lambda}{\lambda d\lambda} \phi^2 \quad (2.32)$$

and

$$c_{AW}^b(X, Y, Z) = c_{LG}^b(X, Y, Z) = \sum_m \text{Res}_{p_m^{cr}} \frac{\delta_X \lambda \delta_Y \lambda \delta_Z \lambda}{\lambda^2 d\lambda} \phi^2. \quad (2.33)$$

3. Dubrovin duality and mirror symmetry

In this section, we will state and prove our two main theorems relating the Dubrovin-dual Frobenius structures of $\mathcal{M}_{\text{AW}}$ and $\mathcal{M}_{\text{LG}}$ to the $\mathbb{T}$ -equivariant quantum cohomology of ADE resolutions.

Theorem 3.1 (Dubrovin duality for $\mathcal{M}_{\text{AW}}$ and $\mathcal{M}_{\text{GW}}$). *For all $(\mathcal{R}, \hat{\omega})$, we have $\mathcal{M}_{\text{AW}}^b \simeq \mathcal{M}_{\text{GW}}$.*

Theorem 3.2 (Mirror symmetry for $\mathcal{M}_{\text{GW}}$). *For all $(\mathcal{R}, \hat{\omega})$, we have $\mathcal{M}_{\text{LG}}^b \simeq \mathcal{M}_{\text{GW}}$.*

Remark 3.3. In theorem 3.1, when comparing the Frobenius structures on $\mathcal{M}_{\text{AW}}^b$ and $\mathcal{M}_{\text{GW}}$, we shall need to formally set $\nu = 1$: by the Degree Axiom, the dependence on ν is reinstated on the Gromov–Witten side (or introduced on the extended affine Weyl side) upon rescaling

$$F_{\text{GW}} \rightarrow \nu F_{\text{GW}}, \quad F_{\text{AW}}^b \rightarrow \nu F_{\text{AW}}^b, \quad x_{l+1} \rightarrow \frac{x_{l+1}}{\nu}.$$

In the following, we will describe how theorem 3.2 for rational spectral curves translates into a comparison statement between the Gromov–Witten calculation in equations (2.21) and (2.22) on one hand, and an explicit summation of residues which localize to the fibre over zero of the λ -projection on the other: we will compute this in a closed-form for $\mathcal{R} = A_l$ and $\mathcal{R} = D_l$, thereby proving theorem 3.2, and therefore theorem 3.1, for the classical series. We also explain in general how to reduce theorem 3.1 to a comparison statement of the structure constants for the Frobenius algebras restricted to a finite set of points on the complement of the discriminant: computing this explicitly for $\mathcal{R} = E_l$ will provide a proof of theorem 3.1, and therefore theorem 3.2, for the three exceptional cases.

(a) Landau–Ginzburg mirror symmetry

We start by looking at the argument of the residues in equation (2.32),

$$\Upsilon_{i,j,k}(p) := \frac{\delta_{\partial_{x_i}}^{(\mu)} \lambda \delta_{\partial_{x_j}}^{(\mu)} \lambda \delta_{\partial_{x_k}}^{(\mu)} \lambda}{\lambda^2 \mu^2 \partial_\mu \lambda} d\mu(p), \quad (3.1)$$

so that

$$\eta_{\text{LG}}^b(\partial_{x_i}, \partial_{x_j}) = \sum_m \text{Res}_{p=p_m^{\text{cr}}} \Upsilon_{i,j,l+1}(p), \quad c_{\text{LG}}^b(\partial_{x_i}, \partial_{x_j}, \partial_{x_k}) = \sum_m \text{Res}_{p=p_m^{\text{cr}}} \Upsilon_{i,j,k}(p). \quad (3.2)$$

From equation (3.1), we deduce that the pole structure of $\Upsilon_{i,j,k}(p)$ is as follows:

- (i) it has at most simple poles at the critical points $\{p_l^{\text{cr}}\}$, for which $d\lambda(p_l^{\text{cr}}) = 0$;
- (ii) it has a pole of order at most

$$\max\{2 - \delta_{i,l+1} - \delta_{j,l+1} - \delta_{k,l+1}, 0\}$$

at $\lambda(p) = 0$: this follows from

$$\delta_{\partial_{x_{l+1}}}^{(\mu)} \lambda \propto \lambda,$$

which offsets a linear power in the vanishing of the denominator;

- (iii) it has at most simple poles at $\mu(p) = 0$ (when $\lambda(p) = \infty$) when $i = j = k = l + 1$;
- (iv) it has a pole of order at most

$$\max\{1 - \delta_{i,l+1} - \delta_{j,l+1} - \delta_{k,l+1}, 0\}$$

at the critical points $\{q_m^{\text{cr}}\}$ of the μ -projection, $d\mu(q_m^{\text{cr}}) = 0$. These are the loci where the Ehresmann connection induced by the μ -foliation is singular and $\delta_{\partial_{x_i}}^{(\mu)} \lambda$ possibly develops a pole. These singularities are partially offset by a vanishing of the same order of $d\mu/\partial_\mu \lambda$.

The residue sums (3.2) pick up the contributions from the residues of type (i) alone: the difficulty in writing the critical points of the superpotential as algebraic functions of $(x_1, \dots, x_l)$ makes, however, their individual computation unwieldy. To overcome this problem, we turn the contour

around and equate the sum of residues at the critical points in [equation \(2.27\)](#) to a much more manageable sum of residues at poles and zeros of μ and λ (type (ii) and (iii)) as well as a contribution from residues at critical points of μ (type (iv)).

Computing the individual contributions from type (iv) residues is as difficult, if not more so, than computing those arising from the critical points of λ . However, there are two scenarios when they can be shown to vanish identically.

— If any of i, j or k is equal to $l + 1$, we have

$$\text{ord}_{q_m^{\text{cr}}} \gamma_{i,j,k} = \max\{1 - \delta_{i,l+1} - \delta_{j,l+1} - \delta_{k,l+1}, 0\} = 0,$$

hence there is no pole at the ramification points of the μ -projection. The sum was calculated explicitly for all extended affine Weyl Frobenius manifolds with canonically marked node in [6, Theorem 3.5], showing that

$$\eta_{\text{LG}}^b(\partial_{x_i}, \partial_{x_j}) = \sum_m \text{Res}_{p=p_m^{\text{cr}}} \gamma_{i,j,l+1}(p) = \eta_{\text{GW}}(\partial_{x_i}, \partial_{x_j})$$

as expected.

- (2) If $\deg_{\lambda} \mathcal{P} = \deg_{\hat{\gamma}} \mathcal{Q} = 1$, implying that the spectral curve is rational, we have $\{q_m^{\text{cr}}\} = \emptyset$, so once again the sum over residues only picks up contributions from zeros and poles of μ and λ . For $i, j, k \neq l + 1$, the sole contribution arises from the zeroes of the rational function $\lambda(\mu)$.

When $\mathcal{R} = A_l$ or $\mathcal{R} = D_l$, it was shown in [6] that $\deg_{\lambda} \mathcal{P} = 1$, and for these two cases the summation over residues can be performed explicitly.⁶

(i) Proof of theorem 3.2 for the A_l series with arbitrary marked weight

Let ω be the highest weight of the defining representation ρ of $\mathfrak{g} = \mathfrak{sl}_{l+1}(\mathbb{C})$, and let $1 \leq \bar{k} \leq l$ be a choice of marked node. From [equation \(2.29\)](#), the corresponding superpotential reads

$$\lambda = \frac{y_{l+1} \prod_{\omega' \in \mathcal{W}(\omega)} (e^{(\omega', h)} - \mu)}{(-\mu)^{l+1-\bar{k}}} = (-1)^{l+1-\bar{k}} y_{l+1} \prod_{j=1}^l \kappa_j^{-l+1-\bar{k}/l+1} \frac{(1-q) \prod_{k=1}^l (1-\kappa_k q)}{q^{l+1-\bar{k}}}, \quad (3.3)$$

where

$$\kappa_j := \prod_{i=j}^l e^{-p_i}, \quad 1 \leq j \leq l, \quad q := e^{-(\omega_{\min}, h)} \mu, \quad \omega_{\min} := -\frac{1}{l+1} \sum_{i=1}^l i \alpha_i.$$

The structure constants of the Frobenius manifold structure on $\mathcal{M}_{\text{LG}}^b$ were computed using the method described in §3a in [10] and, in a slightly generalized fashion, in [16, Theorem 5.4], where more generally the three-point functions of the full $\mathbb{T}$ -equivariant Gromov–Witten theory of Z were seen to equate⁷ those arising from the Landau–Ginzburg pair $(\log \lambda, \phi)$, with

$$\lambda = e^{x_{l+1}/v_1+v_2} \prod_{j=1}^l \kappa_j^{-v_1/v_1+v_2} \frac{(1-q) \prod_{k=1}^l (1-\kappa_k q)}{q^{(l+1) \frac{v_1}{v_1+v_2}}}, \quad \phi^2 = (v_1 + v_2) \left(\frac{dq}{q} \right)^2.$$

Taking the restriction to the one-dimensional subtorus $\mathbb{T}$ that fixes the irreducible component $E_{\bar{k}}$ as in [equation \(2.18\)](#), the superpotential λ coincides with [equation \(3.3\)](#), hence

$$c_{\text{GW}}(\varphi_i, \varphi_j, \varphi_k) = c_{\text{LG}}^b(\partial_{x_i}, \partial_{x_j}, \partial_{x_k}),$$

which yields the statement of theorem 3.2 for $\mathcal{R} = A_l$ and $\widehat{\omega} = \omega_{\bar{k}}$.

⁶Using the superpotential derived in [24], the almost dual prepotentials for the BCD -cases with arbitrary marked node may easily be calculated. These are of the same form, corresponding to a BC_l -root system.

⁷More precisely, the comparison in [16] was performed for $Z \times \mathbb{C}$ with a 2-torus action acting with opposite weights on the canonical bundle of the two factors. The corresponding genus-zero GW potential differs from that of Z by an overall weight factor of $-(v_1 + v_2)$.

(ii) Proof of theorem 3.2 for the D_l series

Let ω be the highest weight of the defining representation of $\mathfrak{g} = \mathfrak{so}_{2l}(\mathbb{C})$. From equation (2.29), the corresponding superpotential reads

$$\lambda = \frac{\kappa_{l+1} \prod_{\omega' \in \mathcal{W}(\omega)} (e^{(\omega', x)} - \mu)}{\mu^{l-2}(\mu^2 - 1)^2} = \frac{\kappa_{l+1} \prod_{i=1}^l (\mu - \kappa_i)(\mu - \kappa_i^{-1})}{\mu^{l-2}(\mu^2 - 1)^2} \quad (3.4)$$

where

$$\log \kappa_i = \sum_{j=1}^l G_{ij} x_j, \quad 1 \leq i \leq l, \quad (3.5)$$

and $G_{ij} := \delta_{ij} - \delta_{i,j+1} + \delta_{i,l-1} \delta_{j,l}$; note that $(G^T G)_{ij} = C_{ij}$. As in equation (2.30), we shall normalize the coordinate x_{l+1} and the quadratic differential associated with the primitive form such that

$$\log \kappa_{l+1} = \frac{x_{l+1}}{2v} \quad \text{and} \quad \phi^2 = v \left(\frac{d\mu}{\mu} \right)^2. \quad (3.6)$$

Since the spectral curve is rational in this case and the μ -projection is unramified, to perform the sum of residues over critical points in equations (2.32)–(2.33) we take the contour of integration around on the fibres of the mirror family, and instead sum over residues at the other poles of the integrand. From equation (3.4), these are located at the support of the divisor (λ) : the locus of zeroes and poles of λ . We thus need to show that

$$c_{\text{GW}}(\varphi_i, \varphi_j, \varphi_k) = c_{\text{LG}}^\flat(\varphi_i, \varphi_j, \varphi_k) = - \sum_{p \in \text{supp}(\lambda)} \text{Res}_p \frac{\delta_{\partial x_i}(\lambda) \delta_{\partial x_j}(\lambda) \delta_{\partial x_k}(\lambda)}{\lambda^2 d\lambda} \phi^2. \quad (3.7)$$

Write

$$R_{ijk} := \frac{1}{2v} c_{\text{LG}}^\flat \left(\frac{\partial}{\partial \log \kappa_i}, \frac{\partial}{\partial \log \kappa_j}, \frac{\partial}{\partial \log \kappa_k} \right), \quad (3.8)$$

and $R_{ijk}^{[q]}$ for the contribution of the residue at $\mu = q$ in the sum (3.7). We start with the following:

Lemma 3.4. *We have*

$$\begin{aligned} R_{ijk}^{[1]} = R_{ijk}^{[-1]} = 0, \quad R_{ijk}^{[0]} = R_{ijk}^{[\infty]} = \frac{\delta_{i,l+1} \delta_{j,l+1} \delta_{k,l+1}}{2(l-2)}, \quad R_{ijk}^{[\kappa_m]} + R_{ijk}^{[1/\kappa_m]} = (\delta_{i,j,m} + \delta_{j,k,m} + \delta_{i,k,m}) \\ \times \left\{ (\delta_{i,j,k} - 1) [(\delta_{i,j,m} \delta_{k,l+1} + \delta_{j,k,m} \delta_{i,l+1} + \delta_{i,k,m} \delta_{j,l+1}) \right. \\ \left. - (1 - \delta_{i,j,m} \delta_{k,l+1} - \delta_{j,k,m} \delta_{i,l+1} - \delta_{i,k,m} \delta_{j,l+1}) (\delta_{i,j,m} p_{mk} + \delta_{k,i,m} p_{mj} + \delta_{j,k,m} p_{mi})] + \frac{\delta_{i,j,k} q_i}{3} \right\}, \end{aligned}$$

with

$$p_{ij} = \frac{\kappa_i(\kappa_j^2 - 1)}{(\kappa_i - \kappa_j)(\kappa_i \kappa_j - 1)} \quad \text{and} \quad q_k = \sum_{n \neq k} \frac{\kappa_n(1 - \kappa_k^2)}{(\kappa_k - \kappa_n)(\kappa_k \kappa_n - 1)}.$$

Proof. The statement follows from a lengthy but straightforward calculation of the rational residues in

$$R_{ijk}^{[q]} = - \text{Res}_{\mu=q} \frac{\prod_{m \in \{i,j,k\}} \left(\frac{-\kappa_m}{\mu - \kappa_m} + \frac{\kappa_m^{-1}}{\mu - \kappa_m^{-1}} \right)^{1 - \delta_{m,l+1}}}{\sum_{r=1}^l \left(\frac{1}{\mu - \kappa_r} + \frac{1}{\mu - \kappa_r^{-1}} \right) - \frac{l-2}{\mu} - \frac{4\mu}{\mu^2 - 1}} \frac{d\mu}{2\mu^2}, \quad (3.9)$$

for $q = 0, \infty, \pm 1, \kappa_m^{\pm}$; we omit the details here. ■

Corollary 3.5. *We have, for $1 \leq i, j \leq l+1$,*

$$c_{\text{LG}}^\flat \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_{l+1}} \right) = \begin{cases} \frac{1}{4v^2(l-2)}, & i = j = l+1, \\ -C_{ij}, & i, j \neq l+1, \\ 0 & \text{else.} \end{cases} \quad (3.10)$$

Proof. Since $\log \kappa_{l+1} = x_{l+1}/(2v)$, the l.h.s. for $i = j = l + 1$ is

$$\frac{1}{4v^2} R_{l+1,l+1,l+1} = \frac{R_{l+1,l+1,l+1}^{[0]} + R_{l+1,l+1,l+1}^{[\infty]}}{4v^2} = \frac{1}{4v^2(l-2)}.$$

The vanishing when $i \neq j$ and either $i = l + 1$ or $j = l + 1$ is immediate from lemma 3.4. For the case $1 \leq i, j \leq l$, we use equation (3.5) to get

$$c_{\text{LG}}^b \left(\frac{\partial}{\partial x_{l+1}}, \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = \sum_{k,m,n=1}^l G_{ki} (R_{k,m,l+1}^{[\kappa_n]} + R_{k,m,l+1}^{[1/\kappa_n]}) G_{mj} = - \sum_{k=1}^l G_{ki} G_{kj} = -C_{ij}.$$

Recall that with respect to the orthonormal basis $\{\epsilon_i\}_{i=1}^{l-1}$ of $\mathbb{R}^l$, the D_l root system is

$$\Pi = \{\epsilon_i - \epsilon_{i+1}\}_{i=1}^{l-1} \cup \{\epsilon_{l-1} + \epsilon_l\}, \quad \mathcal{R}^+ = \{\epsilon_i \pm \epsilon_j\}_{i < j}, \quad \langle \alpha_i, \epsilon_j \rangle = G_{ji}.$$

Definition 3.6. Choose a bijection

$$\sigma : \{(i, j) | 1 \leq i < j \leq l\} \longleftrightarrow \left\{ 1, \dots, \frac{l(l-1)}{2} \right\},$$

and for each pair (i, j) with $1 \leq i < j \leq l$, define $\Theta^\pm$ and Θ by

$$\Theta_{\sigma(i,j),k}^+ := \begin{cases} 1 & \text{if } k = i \text{ or } k = j, \\ 0 & \text{otherwise,} \end{cases} \quad \Theta_{\sigma(i,j),k}^- := \begin{cases} 1 & \text{if } k = i, \\ -1 & \text{if } k = j, \\ 0 & \text{otherwise,} \end{cases} \quad \Theta := \begin{pmatrix} \Theta^+ \\ \Theta^- \end{pmatrix}.$$

Corollary 3.7. Let ϵ be the column vector $(\epsilon_1, \dots, \epsilon_l)^T$. Then, the rows of $\Theta \cdot \epsilon$ give all the positive roots of D_l ,

$$(\beta_1, \dots, \beta_{l(l-1)}) = \Theta \epsilon. \quad (3.11)$$

Define now coordinates $(\tau_1, \dots, \tau_l)$ of $H^2(Z, \mathbb{C})$ via

$$(x_1, \dots, x_l)^T = C^{-1} G^T (\tau_1, \dots, \tau_l)^T. \quad (3.12)$$

By equation (3.5), we have

$$(\log \kappa_1, \dots, \log \kappa_l)^T = G(x_1, \dots, x_l)^T = G C^{-1} G^T (\tau_1, \dots, \tau_l)^T.$$

Using $G^T G = C$, we get $G C^{-1} G^T = I$, hence $\log \kappa_i = \tau_i$, $1 \leq i \leq l$.

Lemma 3.8. The positive degree part of the genus Gromov–Witten primary potential of Z is

$$F_{\text{GW}}^+ = 2v \sum_{i=1}^{l(l-1)} \text{Li}_3 \left(\exp \left(\sum_{j=1}^l -\Theta_{ij} \tau_j \right) \right). \quad (3.13)$$

Proof. In terms of the matrix $A_{ij} = \langle \beta_i, \alpha_j \rangle$ of coefficients of the positive roots $(\beta_1, \dots, \beta_{l(l-1)})$ in the Omega basis,

$$\mathcal{R}^+ = A \Omega, \quad (3.14)$$

we find, using equation (3.12),

$$F_{\text{GW}}^+ = 2v \sum_{\sigma=1}^{l(l-1)} \text{Li}_3 \left(\exp \left(\sum_{j=1}^l -(A C^{-1} G^T)_{\sigma j} \tau_j \right) \right).$$

On the other hand, expressing fundamental weights in the orthonormal basis,

$$(\beta_1, \dots, \beta_{l(l-1)})^T = A C^{-1} (\alpha_1, \dots, \alpha_l)^T = A C^{-1} G^T (\epsilon_1, \dots, \epsilon_l)^T.$$

From this, we deduce that $\Theta = A C^{-1} G^T$ by comparing to (3.11), thereby proving the claim. ■

Proof. By corollary 3.5, we need only consider the case $1 \leq i, j, k \leq l$. From proposition 3.8, we get

$$\frac{1}{2\nu} \frac{\partial^3 F_{\text{GW}}^+}{\partial \tau_i \partial \tau_j \partial \tau_k} = - \sum_{\sigma=1}^{l(l-1)} \Theta_{\sigma i} \Theta_{\sigma j} \Theta_{\sigma k} \frac{\exp(-\sum_{m=1}^l \Theta_{\sigma m} \tau_m)}{1 - \exp(-\sum_{m=1}^l \Theta_{\sigma m} \tau_m)}, \quad (3.15)$$

where σ is an element of the index set $\sigma \in \{(m, n) | 1 \leq m < n \leq l\}$, and consider the case $k = j = i$ for starters. From definition 3.6, this reads explicitly as

$$\frac{1}{2\nu} \frac{\partial^3 F_{\text{GW}}^+}{\partial \tau_i \partial \tau_i \partial \tau_i} = - \sum_{k \neq i} \frac{\exp(-\tau_i - \tau_k)}{1 - \exp(-\tau_i - \tau_k)} - \sum_{k > i} \frac{\exp(-\tau_i + \tau_k)}{1 - \exp(-\tau_i + \tau_k)} + \sum_{k < i} \frac{\exp(-\tau_k + \tau_i)}{1 - \exp(-\tau_k + \tau_i)}.$$

Likewise, for $k = j \neq i$, we spell out equation (3.15) to be

$$\frac{1}{2\nu} \frac{\partial^3 F_{\text{GW}}^+}{\partial \tau_i \partial \tau_i \partial \tau_j} = - \frac{\exp(-\tau_i - \tau_j)}{1 - \exp(-\tau_i - \tau_j)} + \frac{\exp(-\tau_i + \tau_j)}{1 - \exp(-\tau_i + \tau_j)} + \frac{1}{2} (\text{sgn}(i - j) + 1).$$

As, for fixed σ , $\Theta_{\sigma i}$ is only non-zero for exactly two values of i , we finally have that, for i, j, k all distinct,

$$\frac{\partial^3 F_{\text{GW}}^+}{\partial \tau_i \partial \tau_j \partial \tau_k} = 0.$$

Let us compare the above back to the structure constants of the Landau–Ginzburg product (3.8). From lemma 3.4 and for i, j, k all distinct, we find

$$R_{iii} = \frac{1}{2\nu} \frac{\partial^3 F_{\text{GW}}^+}{\partial \tau_i \partial \tau_i \partial \tau_i} - l + i, \quad R_{ijj} = \frac{1}{2\nu} \frac{\partial^3 F_{\text{GW}}^+}{\partial \tau_i \partial \tau_i \partial \tau_j} - \frac{1}{2} (\text{sgn}(i - j) + 1), \quad R_{ijk} = \frac{1}{2\nu} \frac{\partial^3 F_{\text{GW}}^+}{\partial \tau_i \partial \tau_j \partial \tau_k} = 0. \quad (3.16)$$

Hence, theorem 3.2 for type D_l reduces to verifying the following identities relating the additive terms in the r.h.s. of equation (3.16) to the Γ -equivariant triple intersection numbers of Z ,

$$\sum_{\sigma=1}^{l(l-1)} \Theta_{\sigma i}^2 \Theta_{\sigma j} = \begin{cases} 2(l - i), & 1 \leq i = j \leq l, \\ \text{sgn}(i - j) + 1, & 1 \leq i \neq j \leq l. \end{cases}$$

These are easily verified using the definition of Θ , concluding the proof. ■

(b) Dubrovin duality via initial conditions

A direct proof of theorem 3.1 would entail showing that the structure constants of $\mathcal{M}_{\text{AW}}^b$, expressed in terms of those of $\mathcal{M}_{\text{AW}}$ through the duality relation (2.7), coincide with the quantum cohomology product (2.20) of Z . Explicitly, in our case (2.7) reads

$$\frac{\partial^3 F_{\text{GW}}}{\partial x_i \partial x_j \partial x_k}(x) = \sum_{a,b,A,B,C} (\eta_{\text{AW}}^b)_{ia} (\eta_{\text{AW}}^b)_{jb} \frac{\partial t_C}{\partial x_k} \frac{\partial x_a}{\partial t_A} \frac{\partial x_b}{\partial t_B} (c_{\text{AW}})_{\text{C}}^{AB}(t(x)), \quad (3.17)$$

where

$$(c_{\text{AW}})_{\text{C}}^{AB} = \sum_{M,N=1}^{l+1} \eta_{\text{AW}}^{AM} \eta_{\text{AW}}^{BN} \frac{\partial^3 F_{\text{AW}}}{\partial t_M \partial t_N \partial t_C}. \quad (3.18)$$

For a given marked pair $(\mathcal{R}, \widehat{\omega})$, equation (3.17) could *a priori* be proved by brute-force, as both sides of the equality are calculable, at least in principle. For the r.h.s., closed-form expressions for the prepotential $F_{\text{AW}}(t)$ and the Saito flat coordinates $\{t_A(x)\}$ in linear coordinates on $\mathfrak{h} \times \mathbb{C}$ were found in [1,6,25]; for the l.h.s., the Gromov–Witten quantum prepotential is given by equation (2.20).

In practice, however, this approach is largely unfeasible. The entries of the Jacobian matrix $\partial_{x_k} t_C(x)$ are trigonometric polynomials with, e.g. up to billions of terms for $\mathcal{R} = \text{E}_8$; for the same reason, computing the inverse $\partial_{t_A} x_a$ relevant to equation (3.17) is well out of reach of symbolic computation packages, such as *Mathematica*, even for very low values of l .

As we will show, both sides of [equation \(3.17\)](#) are uniquely determined by their values at a (small) finite number of points in the semi-simple locus $\mathcal{M}_{\text{GW}}^{\text{ss}} = \mathfrak{h}^{\text{reg}} \oplus \mathbb{C}$: we will refer to this as a *set of initial conditions* for the Dubrovin duality. The functional equality [\(3.17\)](#) reduces then to a *numerical* equality over the set of initial conditions, which can, in turn, be verified very effectively. As explained in [remark 3.3](#), we will formally set $\nu = 1$ throughout this section.

(i) Reduction to initial conditions

We will start by establishing various homogeneity properties for the tensors appearing in [equation \(3.17\)](#). Following [\[1\]](#), we define $d_{l+1} := \deg(y_{l+1}) = 0$. On the complement of the discriminant, the coefficients $(\eta_{\text{AW}}^b)^{\alpha\beta}$ of [equation \(2.10\)](#) in the chart parametrized by the coordinates $(y_1, \dots, y_{l+1})$ satisfy [\[1, Lemma 2.1\]](#)

$$\deg(\eta_{\text{AW}}^b)^{\alpha\beta} = d_\alpha + d_\beta,$$

while the inverse Gram matrix of the Saito metric is, by definition,

$$\eta_{\text{AW}}^{\alpha\beta}(y) = \frac{\partial(\eta_{\text{AW}}^b)^{\alpha\beta}}{\partial \hat{y}}.$$

By [\[1, Corollary 2.5\]](#), the flat coordinates $\{t_A\}_{A=1, \dots, l}$ are polynomials in $y_1, \dots, y_l, e^{y_{l+1}}$ with $\deg t_A = d_A$. Furthermore, from [\[21\]](#), the determinant of the Jacobian matrix associated with the change-of-variables $Y_i \rightarrow y_i(x_1, \dots, x_l)$ is

$$\delta := \prod_{\beta \in R^+} (e^{\langle \beta, h \rangle / 2} - e^{-\langle \beta, h \rangle / 2}). \quad (3.19)$$

By definition, δ is anti-invariant under the Weyl group action,

$$\delta(wh) = (-1)^{l(w)} \delta(h).$$

By orthogonal extension to $\mathfrak{h} \times \mathbb{C}$, the determinant of the Jacobian matrix associated with the change-of-variables $y_i \rightarrow y_i(x_1, \dots, x_{l+1})$ is

$$\Delta := ce^{(d_1 + \dots + d_l)x_{l+1}} \delta, \quad (3.20)$$

and Δ is, therefore, for the same reason as δ , anti-invariant with respect to the Weyl group action.

Lemma 3.9. $\Delta^2 \in \mathcal{A}$. Moreover, Δ^2 is quasi-homogeneous of degree $2(d_1 + \dots + d_l)$.

Proof. We first show the invariance of Δ^2 under the extended affine Weyl group $\widetilde{\mathcal{W}}$. It suffices to check it on the generators, given by the Weyl reflections, the co-root lattice generators and an extra translation

$$(h, x_{l+1}) \rightarrow (h + 2\pi i \hat{\omega}, x_{l+1} - 2\pi i). \quad (3.21)$$

Obviously, since Δ is anti-invariant, Δ^2 is a Weyl group invariant. For the invariance under the other generators of $\widetilde{\mathcal{W}}$, we use

$$\delta = e^{\langle w, h \rangle} \prod_{\beta \in R^+} (1 - e^{-\langle \beta, h \rangle}), \quad (3.22)$$

where w is the Weyl vector [\(2.31\)](#). An affine translation by a co-root $\alpha^\vee$,

$$h \rightarrow h + 2\pi i \alpha^\vee,$$

will leave δ (and, therefore, Δ and Δ^2) invariant, since $\langle w, \alpha^\vee \rangle$ and $\langle \beta, \alpha^\vee \rangle$, $\beta \in R^+$ are integers. It only remains to consider the effect of the $\mathbb{Z}$ -action generated by [equation \(3.21\)](#): under this

translation, we have

$$\delta \longrightarrow e^{2\pi i \langle w, \widehat{w} \rangle} \delta, \quad e^{(d_1 + \dots + d_l)x_{l+1}} \longrightarrow e^{-2\pi i (d_1 + \dots + d_l)} e^{(d_1 + \dots + d_l)x_{l+1}}.$$

Hence, Δ is invariant, using [equation \(2.31\)](#) and the fact that

$$\langle w, \widehat{w} \rangle = \sum_{i=1}^l d_i, \quad (3.23)$$

since $d_i = \langle \omega_i, \widehat{w} \rangle$. To conclude that $\Delta^2 \in \mathcal{A}$, it remains to prove boundedness of Δ in the limit [\(2.14\)](#). From [equation \(3.23\)](#), the restriction of Δ to the locus in [equation \(2.14\)](#) is

$$ce^{(d_1 + \dots + d_l)x_{l+1}} e^{\langle w, h \rangle} \prod_{\beta \in \mathcal{R}^+} (1 - e^{-\langle \beta, h \rangle} e^{-\langle \beta, \widehat{w} \rangle \tau}).$$

This is bounded when $\tau \rightarrow +\infty$, since $\langle \beta, \widehat{w} \rangle \in \mathbb{Z}_{>0}$. It is finally immediate to see that $\Delta^2(x)$ is quasi-homogeneous of the claimed degree, since it is monomial in $e^{x_{l+1}}$ with exponent $2(d_1 + \dots + d_l)$. ■

Define now $(2, 1)$ -tensors

$$\ell_{\text{GW}} \in H^0(M_{\text{GW}}, \text{Sym}^2 \mathcal{T}_{M_{\text{GW}}} \otimes \mathcal{T}_{M_{\text{GW}}}^*) \quad \text{and} \quad \ell_{\text{AW}} \in H^0(M_{\text{AW}}, \text{Sym}^2 \mathcal{T}_{\mathcal{M}_{\text{AW}}} \otimes \mathcal{T}_{\mathcal{M}_{\text{AW}}}^*),$$

by the following expressions in the chart $(y_1, \dots, y_{l+1})$:

$$\left. \begin{aligned} (\ell_{\text{GW}})_\epsilon^{\alpha\beta} &:= \sum_{a,b,i,j,k} \frac{\partial^3 F_{\text{GW}}}{\partial x_i \partial x_j \partial x_k} (\eta_{\text{AW}}^\flat)^{ia} \frac{\partial y_\alpha}{\partial x_a} (\eta_{\text{AW}}^\flat)^{jb} \frac{\partial y_\beta}{\partial x_b} \frac{\partial x_k}{\partial y_\epsilon} \\ (\ell_{\text{AW}})_\epsilon^{\alpha\beta} &:= \sum_{L,M,N} \frac{\partial y_\alpha}{\partial t_L} \frac{\partial y_\beta}{\partial t_M} \frac{\partial t_N}{\partial y_\epsilon} (c_{\text{AW}})_N^{LM}. \end{aligned} \right\} \quad (3.24)$$

and

Proposition 3.10. *We have*

$$\Delta^2 (\ell_{\text{GW}})_\epsilon^{\alpha\beta} \in \mathcal{A} \quad \text{and} \quad (\ell_{\text{AW}})_\epsilon^{\alpha\beta} \in \mathcal{A}. \quad (3.25)$$

Proof. By [equations \(2.12\)](#) and [\(2.15\)](#), the entries of the Jacobian matrix (resp. its inverse),

$$\mathcal{J}_{\alpha i} := \frac{\partial y_\alpha}{\partial x_i}, \quad \text{resp.} \quad \mathcal{J}_{i\alpha}^{-1} = \frac{\partial x_i}{\partial y_\alpha},$$

are Laurent polynomials (resp. rational functions) in $(e^{x_1}, \dots, e^{x_{l+1}})$. Since Δ is the determinant of the Jacobian matrix, the expression

$$\Delta \frac{\partial x_k}{\partial y_\epsilon} \in \mathbb{C}[e^{\pm x_1}, \dots, e^{\pm x_{l+1}}]$$

is again a Fourier polynomial in $(x_1, \dots, x_{l+1})$. Moreover, from [equation \(2.21\)](#), the triple derivatives of the Gromov–Witten prepotential are rational functions in $e^{\pm x_1}, \dots, e^{\pm x_{l+1}}$ with at most first-order poles at $\delta = 0$. Hence,

$$\Delta^2 (\ell_{\text{GW}})_\epsilon^{\alpha\beta} \in \mathbb{C}[e^{\pm x_1}, \dots, e^{\pm x_{l+1}}]$$

is a Fourier polynomial in $(x_1, \dots, x_{l+1})$, as claimed. By lemma 3.9, it remains to check that the tensor components $(\ell_{\text{GW}})_\epsilon^{\alpha\beta}$ are $\widetilde{\mathcal{W}}$ invariant and bounded in the limit [\(2.14\)](#). Let

$$g^{\eta\alpha} := \sum_{a,b} \frac{\partial y_\eta}{\partial x_a} (\eta_{\text{AW}}^\flat)^{ab} \frac{\partial y_\alpha}{\partial x_b}. \quad (3.26)$$

By [1], $g^{\eta\alpha}$ are elements of $\mathcal{A}$. Therefore, noticing that

$$\begin{aligned} (\ell_{\text{GW}})_\epsilon^{\alpha\beta} &= \sum_{i,j,k,a,b} \frac{\partial^3 F_{\text{GW}}}{\partial x_i \partial x_j \partial x_k} (\eta_{\text{AW}}^b)^{ia} \frac{\partial y_\alpha}{\partial x_a} (\eta_{\text{AW}}^b)^{jb} \frac{\partial y_\beta}{\partial x_b} \frac{\partial x_k}{\partial y_\epsilon} \\ &= \sum_{i,j,k,\delta,\eta} \left(\frac{\partial^3 F_{\text{GW}}}{\partial x_i \partial x_j \partial x_k} \frac{\partial x_i}{\partial y_\eta} \frac{\partial x_j}{\partial y_\delta} \frac{\partial x_k}{\partial y_\epsilon} \right) g^{\eta\alpha} g^{\delta\beta}, \end{aligned} \quad (3.27)$$

it suffices to check that

$$\sum_{i,j,k} \frac{\partial^3 F_{\text{GW}}}{\partial x_i \partial x_j \partial x_k} \frac{\partial x_i}{\partial y_\eta} \frac{\partial x_j}{\partial y_\delta} \frac{\partial x_k}{\partial y_\epsilon}$$

is $\widetilde{\mathcal{W}}$ invariant. Since y_α is $\widetilde{\mathcal{W}}$ invariant for $1 \leq \alpha \leq l$, $y_{l+1} = x_{l+1}$, and the coordinates x_i which are linearly acted upon by $\widetilde{\mathcal{W}}$ are contracted, verifying the $\widetilde{\mathcal{W}}$ -invariance of the last expression amounts to checking the $\widetilde{\mathcal{W}}$ -invariance of the Gromov–Witten potential F_{GW} , up to an additive quadratic shift in x . Recall that $\text{Li}_3(e^z) = c_3(z) + \text{Li}_3(e^{-z})$, where $c_3(z)$ is a cubic polynomial in z , as can be seen by integrating both sides of the geometric series identity $\text{Li}_0(e^z) = 1 - \text{Li}_0(e^{-z})$. Using this to expand the sum over positive roots in equation (2.21) to a sum over all roots, we have

$$F_{\text{GW}} = D_3(x) + 2v \sum_{\beta \in \mathcal{R}} \text{Li}_3(e^{-(\beta, h)}),$$

for a cubic polynomial $D_3(x)$. It is straightforward to verify that $D_3(x)$ is Weyl invariant, hence F_{GW} is Weyl invariant. The invariance under the extended affine action is trivial, since for each β , $e^{(\beta, h)}$ is invariant under the corresponding translation in $\widetilde{\mathcal{W}}$.

It only remains to verify the boundedness property. Under the limit (2.14), since the entries of the Jacobian matrix $\mathcal{J}_{\alpha i}$ are bounded, so are the coefficients of its inverse $\mathcal{J}_{i\alpha}^{-1}$, and we have already shown that Δ is bounded. As for the triple derivatives of the Gromov–Witten potential, boundedness is trivial when either of $i, j, k = l+1$ by the string equation and (2.20). For $i, j, k \leq l$, we have

$$\frac{\partial^3 F_{\text{GW}}}{\partial x_i \partial x_j \partial x_k} = c_{ijk} + \sum_{\beta \in \mathcal{R}^+} d_{ijk}(\beta) \frac{e^{-(\beta, h)}}{1 - e^{-(\beta, h)}}, \quad (3.28)$$

for constants c_{ijk} , $d_{ijk}(\beta)$, with $\beta \in \mathcal{R}^+$. Since $\langle \beta, \widehat{\omega} \rangle > 0$, we have

$$\lim_{\tau \rightarrow +\infty} |e^{-(\beta, x_0 + \widehat{\omega}\tau)}| = 0,$$

hence equation (3.28) is bounded at infinity.

As for ℓ_{AW} , by [1,6], recall that when $1 \leq A \leq l$, $t_A(y)$ is a polynomial in $y_1, \dots, y_l$ and $e^{y_{l+1}}$ with polynomial inverse $y_i(t)$, and $t_{l+1} = y_{l+1}$. Moreover, $(c_{\text{AW}})_n^{lm}(t)$ are polynomials in $y_1, \dots, y_l$ and $e^{y_{l+1}}$. Therefore,

$$(\ell_{\text{AW}})_\epsilon^{\alpha\beta}(y) \in \mathbb{C}[y_1, \dots, y_l, e^{y_{l+1}}].$$

The claim then follows as $y_1(x), \dots, y_l(x)$ and $e^{y_{l+1}}$ are generators of $\mathcal{A}$. ■

The square Jacobian factor Δ^2 in $\Delta^2 \ell_{\text{GW}}$ entered our discussion so far only in order to offset the potential double poles of ℓ_{GW} along the discriminant. However, the equality (3.17) that we need to prove, $\ell_{\text{GW}} = \ell_{\text{AW}}$, would lead to the stronger expectation that ℓ_{GW} is in fact regular at $\delta = 0$. We now show that this is indeed the case. We start by proving the following:

Lemma 3.11. *Let*

$$\begin{aligned} m : \Lambda_w(\mathcal{R}) &\longrightarrow \mathbb{C} \\ \omega &\longrightarrow m_\omega \end{aligned}$$

be a complex-valued map on the weight lattice having finite support and Weyl-invariant fibres,

$$m_{w(\omega)} = m_\omega \quad \forall w \in \mathcal{W}, \quad |\{\omega \in \Lambda_w(\mathcal{R}) | m_\omega \neq 0\}| < \infty,$$

and consider the Weyl-invariant Fourier polynomial

$$p(x) := \sum_{\omega \in \Lambda_w(\mathcal{R})} m_\omega e^{\langle \omega, x \rangle} \in \mathbb{C}[e^{\pm x_1}, \dots, e^{\pm x_l}]^{\mathcal{W}}.$$

Then, for all $\beta \in \Lambda_r(\mathcal{R})$, the directional derivative of $p(x)$ along β is divisible by $(1 - e^{\pm \langle \beta, h \rangle})$ in the ring of Fourier polynomials,

$$\sum_{a=1}^l \langle \beta, \omega_a \rangle \frac{\partial p(x)}{\partial x_a} \in (1 - e^{\pm \langle \beta, h \rangle}) \mathbb{C}[e^{\pm x_1}, \dots, e^{\pm x_l}].$$

Proof. Since

$$\frac{\partial p(x)}{\partial x_a} = \sum_{\omega \in \Lambda_w(\mathcal{R})} m_\omega \langle \omega, \alpha_a^\vee \rangle e^{\langle \omega, h \rangle}, \quad \beta = \sum_a \langle \beta, \omega_a \rangle \alpha_a^\vee,$$

we have that

$$\sum_{a=1}^l \langle \beta, \omega_a \rangle \frac{\partial p(x)}{\partial x_a} = \sum_{\omega \in \Lambda_w(\mathcal{R})} m_\omega \langle \beta, \omega \rangle e^{\langle \omega, h \rangle}. \quad (3.29)$$

Let now $\beta \in \Lambda_r(\mathcal{R})$ and

$$\Gamma_\pm(\beta) := \{\omega \in \Lambda_w(\mathcal{R}) \mid \text{sgn}(\langle \beta, \omega \rangle) = \pm 1\}.$$

Since $\langle \beta, s_\beta(\omega) \rangle = -\langle \beta, \omega \rangle$, the Weyl reflection across $\beta^\perp$ gives a bijection

$$s_\beta : \Gamma_\pm(\beta) \rightarrow \Gamma_\mp(\beta).$$

Moreover, using $m_{s_\beta(\omega)} = m_\omega$, we can rewrite equation (3.29) as

$$\sum_{\omega \in \Gamma_+(\beta)} m_\omega \langle \beta, \omega \rangle (e^{\langle \omega, h \rangle} - e^{(s_\beta(\omega), h)}) = \sum_{\omega \in \Gamma_+(\beta)} m_\omega \langle \beta, \omega \rangle e^{\langle \omega, h \rangle} (1 - e^{-2\langle \omega, \beta \rangle \langle \beta, h \rangle / \langle \beta, \beta \rangle}).$$

As $\frac{2\langle \omega, \beta \rangle}{\langle \beta, \beta \rangle}$ is a (positive) integer, $(1 - e^{\pm \langle \beta, h \rangle})$ divides the r.h.s. in $\mathbb{C}[e^{\pm x_1}, \dots, e^{\pm x_l}]$. ■

Proposition 3.12. $(\ell_{\text{GW}})_\epsilon^{\alpha\beta} \in \mathcal{A}$.

Proof. From equations (2.15), (2.21) and (3.30), $(\ell_{\text{GW}})_\epsilon^{\alpha\beta}$ is a polynomial in $e^{x_{l+1}}$. Let then

$$L_\epsilon^{\alpha\beta} := (\ell_{\text{GW}})_\epsilon^{\alpha\beta} \big|_{x_{l+1}=0}.$$

The statement would follow from showing that $L_\epsilon^{\alpha\beta}$, which *a priori* has double poles along $\delta = 0$, is in fact a Fourier polynomial in $(x_1, \dots, x_l)$. By proposition 3.13, $\delta^2 L_\epsilon^{\alpha\beta}$ is Weyl invariant, and therefore $\delta L_\epsilon^{\alpha\beta}$ is Weyl anti-invariant. By Bourbaki [21, Ch. VI, Section 3, Proposition 2(iii)], the multiplication by δ is a bijection from the set of Weyl-invariant Fourier polynomials onto the set of Weyl anti-invariant Fourier polynomials. It then suffices to show that

$$\delta L_\epsilon^{\alpha\beta} \in \mathbb{C}[e^{\pm x_1}, \dots, e^{\pm x_l}].$$

From equation (3.24), we have

$$\delta L_\epsilon^{\alpha\beta} = \sum_{i,j,k,a,b} \frac{\partial^3 F_{\text{GW}}}{\partial x_i \partial x_j \partial x_k} (\eta_{\text{AW}}^\flat)^{ia} (\eta_{\text{AW}}^\flat)^{jb} \mathcal{J}_{\alpha a} \mathcal{J}_{\beta b} \mathcal{J}_{k\epsilon}^{-1} \delta \bigg|_{x_{l+1}=0},$$

where the r.h.s. has in principle simple poles at $\delta = 0$. Since $\Delta = \det \mathcal{J}$, $\delta \mathcal{J}_{k\epsilon}^{-1} \big|_{x_{l+1}=0}$ is regular at $\delta = 0$, therefore it would be sufficient to show that

$$\sum_{i,a} \frac{\partial^3 F_{\text{GW}}}{\partial x_i \partial x_j \partial x_k} (\eta_{\text{AW}}^\flat)^{ia} \mathcal{J}_{\alpha a} \bigg|_{x_{l+1}=0}$$

is pole-free at $\delta = 0$. By [equation \(2.21\)](#), the triple derivatives of the prepotential are pole-free along the discriminant unless $i, j, k \leq l$, in which case⁸

$$\frac{\partial^3 F_{\text{GW}}}{\partial x_i \partial x_j \partial x_k} = c_{ijk} + 2\nu \sum_{\beta \in \mathcal{R}^+} \langle \beta, \alpha_i^\vee \rangle \langle \beta, \alpha_j^\vee \rangle \langle \beta, \alpha_k^\vee \rangle \frac{1}{1 - e^{\langle \beta, h \rangle}}.$$

By lemma 3.11 with $p(x) = y_\alpha(x)$, we have that, for all $\beta \in \mathcal{R}^+$, $\sum_a \langle \beta, \omega_a \rangle \mathcal{J}_{\alpha a}$ has a simple zero at $\langle \beta, h \rangle = 0$. Hence, using $(\eta_{\text{AW}}^b)_{ab} = -C_{ab}$ for $a, b, \leq l$ by [equation \(2.11\)](#), the expression

$$\frac{1}{1 - e^{\langle \beta, h \rangle}} \sum_{i,a} \langle \beta, \alpha_i^\vee \rangle (C^{-1})_{ia} \mathcal{J}_{\alpha a}$$

is regular on the discriminant $\delta = 0$, concluding the proof. ■

Proposition 3.13. *For any α, β, ϵ , $(\ell_{\text{GW}})_\epsilon^{\alpha\beta}$ and $(\ell_{\text{AW}})_\epsilon^{\alpha\beta}$ are quasi-homogeneous polynomials in $y_1, \dots, y_l$ and $e^{y_{l+1}}$ of degree $d_\alpha + d_\beta - d_\epsilon$.*

Proof. The polynomiality statement follows from propositions 2.2, 3.10 and 3.12. The homogeneity property for ℓ_{GW} can be read off from [equation \(3.27\)](#): firstly, from (3.26), $g^{\eta\delta}$ and $g^{\delta\beta}$ are quasi-homogeneous of degree $d_\eta + d_\delta$ and $d_\delta + d_\beta$, respectively. The triple derivatives of F_{GW} are quasi-homogeneous of degree zero, and, therefore, each nonzero term of $(\ell_{\text{GW}})_\epsilon^{\alpha\beta}$ has the same degree $d_\alpha + d_\beta - d_\epsilon$. As for ℓ_{AW} , by [1], t_A is a quasi-homogeneous polynomial of degree d_A for $1 \leq A \leq l$, and $t_{l+1} = y_{l+1}$. Thus, recalling that $d_{l+1} = 0$, $\partial y_\alpha / \partial t_A$ is quasi-homogeneous of degree $d_\alpha - d_A$, and likewise $\partial t_B / \partial y_\beta$ has quasi-homogeneous degree $d_B - d_\beta$. Furthermore, by [1], $(c_{\text{AW}})_N^{LM}(t)$ is quasi-homogeneous of degree $d_L + d_M - d_N$. Therefore, each nonzero term of

$$(\ell_{\text{AW}})_\epsilon^{\alpha\beta} = \sum_{L,M,N} \frac{\partial y_\alpha}{\partial t_L} \frac{\partial y_\beta}{\partial t_M} \frac{\partial t_N}{\partial y_\epsilon} c_N^{LM}$$

will be of the same degree:

$$(d_\alpha - d_L) + (d_\beta - d_M) + (d_N - d_\epsilon) + (d_L + d_M - d_N) = d_\alpha + d_\beta - d_\epsilon,$$

concluding the proof. ■

Define a grading on $\mathbb{C}[Y_1, \dots, Y_l]$ by $\deg_Y Y_i = d_i$ for $1 \leq i \leq l$. Writing

$$D := \max\{(d_\eta + d_\delta - d_\epsilon) | 1 \leq \eta, \delta, \epsilon \leq l + 1\}, \quad (3.30)$$

we will write

$$\mathfrak{V}_{\text{adm}} := \{F \in \mathbb{C}[Y_1, \dots, Y_l] \mid \deg_Y F \leq D\}, \quad S_{\text{adm}} := \left\{ (n_1, \dots, n_l) \in \mathbb{Z}_{\geq 0}^l \mid \sum_{i=1}^l n_i d_i \leq D \right\},$$

for, respectively, the finite-dimensional vector subspace of polynomials in $(Y_1, \dots, Y_l)$ of degree less than or equal to D , and the set of monic monomials in $\mathfrak{V}_{\text{adm}}$. We will refer to S_{adm} as the set of *admissible exponents* of the root system $\mathcal{R}$, and to the corresponding monomial basis of $\mathfrak{V}_{\text{adm}}$ as the set of *admissible monomials*. In particular, for $N = (n_1, \dots, n_l) \in S_{\text{adm}}$, we will use the shorthand multi-index notation

$$Y^N := \prod_{i=1}^l Y_i^{n_i}.$$

Corollary 3.14. *We have*

$$(\ell_{\text{GW}})_\epsilon^{\eta\delta} |_{x_{l+1}=0} \in \mathfrak{V}_{\text{adm}}, \quad (\ell_{\text{AW}})_\epsilon^{\eta\delta} |_{x_{l+1}=0} \in \mathfrak{V}_{\text{adm}}.$$

Proof. Immediate from proposition 3.13. ■

⁸For A_j with a general 2-torus action, the expression is the same upon relacing $2\nu \rightarrow \nu_1 + \nu_2$.

Table 2. Degree bounds and size of the set of initial conditions for the exceptional series E_l , $l = 6, 7, 8$.

l	D	$ S_{\text{adm}} $
6	12	151
7	24	254
8	60	434

By corollary 3.14, the Weyl-invariant Fourier polynomials $(\ell_{\text{GW}})_\epsilon^{\eta^\delta}|_{x_{l+1}=0}$ and $(\ell_{\text{AW}})_\epsilon^{\eta^\delta}|_{x_{l+1}=0}$ are elements of the same finite-dimensional vector space $\mathfrak{V}_{\text{adm}}$, with $\dim_{\mathbb{C}} \mathfrak{V}_{\text{adm}} = |S_{\text{adm}}|$. This reduces the verification of the functional relation (3.17) to checking the *numerical* relation

$$(\ell_{\text{GW}})_\epsilon^{\eta^\delta}(x^{(K)}) = (\ell_{\text{AW}})_\epsilon^{\eta^\delta}(x^{(K)})$$

for *finitely many points* $\{x^{(1)}, \dots, x^{(|S_{\text{adm}}|)}\}$ in $\mathfrak{h}$, each providing a linear constraint on the coefficients of either side of equation (3.17) as an element of $\mathfrak{V}_{\text{adm}}$. If these points are chosen generically, the resulting linear system will have maximal rank and determine the $(2, 1)$ tensors ℓ_{GW} and ℓ_{AW} uniquely.

Definition 3.15. A set of $|S_{\text{adm}}|$ points $\{x^{(1)}, \dots, x^{(|S_{\text{adm}}|)}\} \subset M_{\text{AW}}^{\text{ss}}$ will be called a *set of initial conditions* for $\mathcal{M}_{\text{AW}}^b$ if the $|S_{\text{adm}}| \times |S_{\text{adm}}|$ generalized Vandermonde matrix minor

$$(Y^N(x^{(M)}))_{N, M \in S_{\text{adm}}}$$

is non-singular,

$$\det(Y^N(x^{(M)}))_{M, N} \neq 0. \quad (3.31)$$

Checking (3.17) on a set of initial conditions is poorly suited to a general proof of theorem 3.1 for all marked pairs $(\mathcal{R}, \widehat{\omega})$. On the other hand, it can be performed highly effectively on a case-by-case basis. As it just remains to prove theorem 1.1 for $\mathcal{R} = E_l$, we will construct sets of initial conditions to verify equation (3.17) directly in these three exceptional cases.

(ii) Proof of theorem 3.1 for the E_l series

It is straightforward to compute the degree bound D and the number $|S_{\text{adm}}|$ of admissible monomials for the exceptional series E_l . These are reproduced in table 2.

Configurations of $|S_{\text{adm}}|$ points that are not initial have measure zero in $\text{Sym}^{|S_{\text{adm}}|} \mathfrak{h}$, since (3.31) is an open condition. Due to the small size of S_{adm} for $\mathcal{R} = E_l$, it is straightforward to construct an initial set by picking a configuration of $|S_{\text{adm}}|$ points in the Cartan subalgebra and then checking *a posteriori* that the generalized Vandermonde minor (3.31) is indeed non-zero for them. Having constructed a set of initial conditions $\mathfrak{I}$, we are then just left with verifying directly the numerical identities

$$(\ell_{\text{GW}})_\gamma^{\alpha\beta}(x^{(K)}) = (\ell_{\text{AW}})_\gamma^{\alpha\beta}(x^{(K)}), \quad k = 1, \dots, |\mathfrak{I}|.$$

Proposition 3.16. For all $(\mathcal{R}, \widehat{\omega})$ with $\mathcal{R} = E_l$, there exists a set of initial conditions $\mathfrak{I}$ such that, $\forall x \in \mathfrak{I}$,

$$(\ell_{\text{GW}})_\gamma^{\alpha\beta}(x) = (\ell_{\text{AW}})_\gamma^{\alpha\beta}(x). \quad (3.32)$$

Proof. Direct calculation. ■

Example 3.17 ($\mathcal{R} = E_6$). We will sketch here the main elements entering the verification of equation (3.32) for $\mathcal{R} = E_6$. The l.h.s. of equation (3.17), as a function of $x = (x_1, \dots, x_7)$ is explicitly computed by equation (2.21). As for the r.h.s., $(\eta_{\text{AW}}^b)_{ia}$ is given by (2.11), and all we need to compute are the Jacobian matrix of the change-of-variables $t \rightarrow t(x)$, its inverse, and the $(2, 1)$ multiplication tensor c_{AW} in flat coordinates. For the latter, the prepotential of $\mathcal{M}_{\text{AW}}$ was

computed in [6] to be

$$F_{AW} = \frac{e^{12t_7}}{24} + \frac{1}{4}e^{8t_7}t_1t_5 + \frac{1}{12}e^{6t_7}(t_1^3 + t_5^3 + 3t_6^2) + \frac{1}{2}e^{5t_7}t_1t_6t_5 \\ + \frac{1}{72}e^{4t_7}(10t_1^2t_5^2 + t_1^2t_4 - t_2t_5^2 - t_2t_4) + \frac{e^{3t_7}}{12}t_6(t_1^3 - t_2t_1 + t_5^3 + 2t_6^2 + t_4t_5) \\ + \frac{e^{2t_7}}{144}(t_5t_1^4 - 2t_2t_5t_1^2 + (t_5^4 + 2t_4t_5^2 + 36t_6^2t_5 + t_4^2)t_1 + t_2^2t_5) + \frac{1}{72}e^{t_7}(t_1^2 - t_2)(t_5^2 + t_4)t_6 \\ - \frac{t_1^6 + t_5^6}{38880} - \frac{t_2t_1^4 - t_4t_5^4 + t_2^3 - t_4^3}{7776} - \frac{t_2^2t_1^2 + t_4^2t_5^2}{2592} - \frac{(t_2t_1 - t_4t_5)t_3}{36} - \frac{t_6^4}{192} + \frac{1}{8}t_3t_6^2 + \frac{1}{4}t_3^2t_7,$$

from which $(c_{AW})_C^{AB}$ in equation (3.18) is immediately computed as an explicit polynomial in $t_1, \dots, t_6$ and e^{t_7} . The flat coordinates are related to the fundamental traces in equation (2.13) as [6]

$$t_1 = W_0^{\frac{1}{3}}W_1, \quad t_2 = W_0^{\frac{2}{3}}(W_1^2 - 6W_2 - 12W_5), \quad t_3 = W_0(2W_1W_5 + W_3 + 3W_6 + 3), \\ t_4 = W_0^{\frac{2}{3}}(-W_5^2 + 12W_1 + 6W_4), \quad t_5 = W_0^{\frac{1}{3}}W_5, \quad t_6 = W_0^{\frac{1}{2}}(W_6 + 2), \quad t_7 = \frac{\log(W_0)}{6}, \quad (3.33)$$

where furthermore $W_0 = e^{1/2x_7}$. The fundamental traces W_1 and W_6 in, respectively, the ‘27’ and ‘78’ (adjoint) representation can be computed from the respective weight and root system, and their expressions are given in the Supplementary Online Material. The remaining traces can be computed from the following relations in the representation ring of E_6 ,

$$W_i(x) = W_{6-i}(-x), \quad i = 1, \dots, 5$$

and

$$W_2(x) = \frac{1}{2}(W_1^2(x) - W_1(2x)) \quad \text{and} \quad W_3(x) = \frac{1}{3}(W_2(x)W_1(x) - W_1(x)W_1(2x) + W_1(3x))$$

expressing, respectively, the fact that $\rho_i = \bar{\rho}_{6-i}$ for $1 \leq i \leq 5$ and $\rho_i = \wedge^i \rho_1$ for $i = 2, 3$. Plugging the resulting expressions into equation (3.33) gives the change-of-variables $t \rightarrow t(x)$, from which the Jacobian coefficients $\partial_{x_i} t_A$ could then in principle be computed as explicit, if cumbersome, Fourier polynomials with a few thousand terms. The resulting matrix inversion computing $\partial_{t_A} x_i$ as rational trigonometric functions of x is far out of reach of modern symbolic computation packages. On the other hand, evaluating on a (rational) set of initial conditions $\mathfrak{I}$ dramatically reduces the unwieldy expressions above in the field $\mathbb{Q}(e^{x_1}, \dots, e^{x_7})$ to eminently manageable manipulations of rational numbers. The exact inversion of the numerical 7×7 matrix $\partial_{t_A} x_i|_{x=x^{(k)}}$ over $\mathbb{Q}$ for $x^{(k)} \in \mathfrak{I}$ takes now a fraction of a second in *Mathematica* on an entry-level desktop computer⁹, as does the evaluation of all the other quantities entering equation (3.17). This allows for a straightforward and fast verification of equation (3.17) over $\mathfrak{I}$, and therefore, by propositions 3.13 and corollary 3.14, on the whole of M_{AW} .

The same *Mathematica* calculations take a couple of minutes for $\mathcal{R} = E_7$, and a few hours for $\mathcal{R} = E_8$ with the same setup. The Wolfram Language code used to verify proposition 3.16 is available to the reader in the Supplementary Online Material.

Corollary 3.18. *Theorems 3.1 and 3.2 hold for $(\mathcal{R}, \widehat{\omega})$ with $\mathcal{R} = E_l$.*

Proof. Theorem 3.1 follows from propositions 3.16, 3.13 and corollary 3.14, and it implies theorem 3.2 by theorem 2.5. ■

Example 3.19 (Landau–Ginzburg mirror symmetry for the E_6 du Val resolution). Let $\mathcal{R} = E_6$, so that $\widehat{\omega} = \omega_3$ is the highest weight of its 2925-dimensional fundamental representation. By theorem 3.2 and (2.30), the $\mathbb{T}$ -equivariant quantum cohomology of the du Val resolution of type E_6

⁹Absolute clock-times based on a setup with Intel Core i7-8700 @3.20 GHz processor and 16 GB RAM.

is mirror to a one-dimensional Landau–Ginzburg model with a log-meromorphic superpotential $\log \lambda$ and logarithmic primitive form

$$\phi^2 = \frac{\nu}{3} \left(\frac{d\mu}{\mu} \right)^2,$$

where λ and μ are the two Cartesian projections on the family of plane algebraic curves over $\mathfrak{h}^{\text{reg}} \oplus \mathbb{C}$ in equation (2.29) given by

$$\mathcal{P}(y_1, \dots, y_7; \lambda, \mu) = 0,$$

and we took $\omega := \omega_1$ in equation (2.29) to be the highest weight of the 27-dimensional fundamental representation ρ_1 . Expressing the characters of the exterior powers of ρ_1 in terms of fundamental characters $W_1, \dots, W_6$ (see [26, Eq. (6.21)]), and further relating the latter to the basic invariants as

$$W_1 = Y_1, \quad W_2 = Y_2 + 5Y_5, \quad W_3 = Y_3 + 4Y_1Y_5 - 9Y_6 - 63,$$

and

$$W_4 = Y_4 + 5Y_1, \quad W_5 = Y_5, \quad W_6 = Y_6 + 6,$$

as can be ascertained from a direct inspection of the fundamental weight systems Γ_i ($i = 1 \dots 6$), we compute from equations (2.29)–(2.30) that

$$\mathcal{P}(y; \lambda, \mu) = \mathcal{Q}(Y, \mu)|_{Y_\alpha = y_\alpha} e^{-d_\alpha y_7} + e^{-x_7/(2\nu)} \mathcal{Q}^{[1]}(Y, \mu) \lambda|_{Y_\alpha = y_\alpha} e^{-d_\alpha y_7} + e^{-x_7/\nu} \mu^9 (\mu^3 - 1)^3 \lambda^2,$$

where $\mathcal{Q}$ is the characteristic polynomial in equation (2.28), and

$$\begin{aligned} \mathcal{Q}^{[1]} = & 1 - 2\mu^2 Y_1 + \mu^3 (3Y_6 + 20) + \mu^4 (Y_1^2 - 2Y_2 - 13Y_5) - \mu^5 (Y_6 Y_1 + 9Y_1 + Y_4) \\ & + \mu^6 (2Y_3 + 12Y_1 Y_5 - 21Y_6 - 150) + \mu^7 (5Y_2 - 10Y_1^2 - Y_4 Y_1 + 24Y_5 - Y_5 Y_6) \\ & + \mu^8 (Y_5 Y_1^2 + 3Y_6 Y_1 + 26Y_1 - 13Y_5^2 + Y_4 - 2Y_2 Y_5) \\ & + \mu^9 (6Y_5 Y_1 - Y_1^3 + 3Y_2 Y_1 - 6Y_3 + 3Y_4 Y_5 + 48Y_6 + 343) \\ & + \mu^{10} (4Y_5 - 4Y_1^2 - 2Y_5^2 Y_1 - Y_4 Y_1 - 3Y_2 + 3Y_5 Y_6) \\ & + (\mu^k \longrightarrow (-\mu)^{21-k}, Y_1 \longleftrightarrow Y_5, Y_2 \longleftrightarrow Y_4). \end{aligned}$$

As a check, from equation (2.32), the dual pairing on $M_{\text{AW}} \simeq M_{\text{LG}}$ is

$$\eta_{\text{LG}}^b(\partial_{x_i}, \partial_{x_j}) = \frac{1}{6} \sum_m \text{Res}_{p_m^{\text{cr}}} \frac{\delta_{\partial_{x_i}} \lambda \delta_{\partial_{x_j}} \lambda}{\lambda \mu \partial_\mu \lambda} \frac{d\mu}{\mu} = -\frac{1}{6} \sum_{\lambda(p), \mu(p) \in \{0, \infty\}} \text{Res}_p \frac{\delta_{\partial_{x_i}} \lambda \delta_{\partial_{x_j}} \lambda}{\lambda \mu \partial_\mu \lambda} \frac{d\mu}{\mu}.$$

For $1 \leq i, j \leq l$, it is straightforward to check that the only non-vanishing residues arise from the zeroes of λ , i.e. when $\mu = e^{\langle \omega', x \rangle}$ for $\omega' \in \Gamma_1$. We find

$$\sum_{\omega' \in \Gamma_1} \text{Res}_{\mu = e^{\langle \omega', x \rangle}} \frac{\delta_{\partial_{x_i}} \lambda \delta_{\partial_{x_j}} \lambda}{\lambda \mu \partial_\mu \lambda} \frac{d\mu}{\mu} = \sum_{\omega' \in \Gamma_1} \langle \omega', \alpha_i \rangle \langle \omega', \alpha_j \rangle = \begin{pmatrix} 12 & -6 & 0 & 0 & 0 & 0 \\ -6 & 12 & -6 & 0 & 0 & 0 \\ 0 & -6 & 12 & -6 & 0 & -6 \\ 0 & 0 & -6 & 12 & -6 & 0 \\ 0 & 0 & 0 & -6 & 12 & 0 \\ 0 & 0 & -6 & 0 & 0 & 12 \end{pmatrix}_{ij},$$

and therefore

$$\eta_{\text{LG}}^b(\partial_{x_i}, \partial_{x_j}) = \eta_{\text{GW}}(\varphi_i, \varphi_j) = -C_{ij},$$

recovering the expression for the 2-point intersection pairing on Z in equation (2.20).

Data accessibility. The Wolfram Language code used to verify proposition 3.16 is available to the reader in the Supplementary Online Material.

The data are provided in electronic supplementary material [27].

Declaration of AI use. No, We have not used AI-assisted technologies in creating this article.

Authors' contributions. A.B.: conceptualization, funding acquisition, methodology, software, supervision, writing—original draft, writing—review and editing; J.M.: conceptualization, methodology, software, writing—original draft, writing—review and editing; I.S.: conceptualization, methodology, software, writing—original draft, writing—review and editing.

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