Journal of Cosmology
and Astroparticle
Physics

PAPER • OPEN ACCESS

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Journal of Cosmology and Astroparticle Physics
An IOP and SISSA journal

Received: *May 20, 2024*
Revised: *July 9, 2025*
Accepted: *September 25, 2025*
Published: *November 5, 2025*

Scalar field dark matter with time-varying equation of state

Gaspard Poulot,[a] Elsa M. Teixeira,[a,b] Carsten van de Bruck[a] and Nelson J. Nunes[c]

[a] *School of Mathematical and Physical Sciences, University of Sheffield, Hounsfield Road, Sheffield S3 7RH, United Kingdom*

[b] *Laboratoire Univers & Particules de Montpellier, CNRS & Université de Montpellier (UMR-5299), 34095 Montpellier, France*

[c] *Instituto de Astrofísica e Ciências do Espaço, Faculdade de Ciências da Universidade de Lisboa, Campo Grande, Edifício C8, P-1749-016, Lisboa, Portugal*

E-mail: gmpoulot1@sheffield.ac.uk, elsa.teixeira@umontpellier.fr, c.vandebruck@sheffield.ac.uk, njnunes@ciencias.ulisboa.pt

Abstract: We propose a new model of scalar field dark matter interacting with dark energy. Adopting a fluid description of the dark matter field in the regime of rapid oscillations, we find that the equation of state for dark matter is non-zero and even becomes increasingly negative at late times during dark energy domination. Furthermore, the speed of sound of dark matter is non-vanishing at all length scales, and a non-adiabatic pressure contribution arises. The results indicate that there are still unexplored possible interactions within the dark sector that lead to novel background effects and can impact structure formation processes.

Keywords: dark matter theory, dark energy theory

ArXiv ePrint: 2404.10524

https://doi.org/10.1088/1475-7516/2025/11/005

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Contents

1 Introduction

The widely accepted standard cosmological model, known as ΛCDM, standing for "a cosmological constant (Λ) plus cold dark matter (CDM)", is in impressive agreement with various observational findings. This framework explains the accelerated expansion of the Universe at late times, structure formation in the Universe and the properties of the cosmic microwave background radiation (CMB) (see, e.g. [1] for a recent overview). However, despite its success in explaining phenomena at large scales, the ΛCDM model still faces significant observational challenges when applied to clustering on smaller scales and consistently reconstructing the Universe's expansion history. In particular, the so-called Hubble tension is still a persistent challenge, together with other statistically less significant tensions. We refer to [2] for an overview of these tensions and possible solutions. Moreover, from a theoretical perspective, the nature of the dark sector components, from which the model derives its name, remains a puzzle. Various candidates for dark matter (DM) have been proposed, but direct detection is lacking.

There is overwhelming observational evidence for dark matter from different cosmological and astrophysical probes, including galaxy rotation curves [3], the peculiar motion of clusters [4, 5], the Bullet cluster system [6], observations from the CMB and baryonic acoustic oscillations [7, 8]. CDM constitutes approximately 26% of the Universe's current energy density; therefore, illuminating its exact nature remains one of the most significant problems in cosmology and particle physics. A key factor is that dark matter must interact predominantly through gravity with the particles of the Standard Model of particle physics to assist structure formation in the Universe. The lack of direct detection of weakly-interacting massive particles (WIMPs) [9–16], the simplest and most popular CDM candidates, motivates the search for alternative proposals rooted in theories of high-energy particle physics. One such proposal of relevance for this work is the concept of *fuzzy dark matter* [17–21], which consists of modelling dark matter as an ultralight scalar field or axion-like particle with a mass around 10^{-22} eV. The ultra-light nature of these particles translates into a large *de*

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Broglie wavelength, effectively suppressing structure formation on small scales while retaining the successes of the CDM paradigm on larger scales [22–26]. Ultralight scalars often emerge in particle physics and string theory compactifications, where axion-like particles arise as Kaluza-Klein zero modes of anti-symmetric tensor fields [19, 20, 27]. Arguments based on measurements of CMB anisotropies constrain the mass of these ultralight fields to be $m_\phi \gtrsim 10^{-24}\,\mathrm{eV}$ [21, 28–31], while observations of the Lyman$-\alpha$ forest extend this lower bound to about $m_\phi \sim 10^{-21}\,\mathrm{eV}$ [32–36], assuming these particles account for over 30% of the total dark matter content. Albeit less widely agreed upon, studies on the kinematics of ultra-faint dwarf galaxies further bring the lower limit to around $m_\phi \sim 10^{-19}\,\mathrm{eV}$ [37–39]. Other probes include galaxy clustering [25, 40], weak lensing measurements [18, 41] and 21 cm observations [42–44]. Furthermore, recent pulsar timing array reports of a stochastic gravitational wave background impose significant constraints on ultralight axions [45, 46]. It has been shown that the travel time of pulsar radio beams is influenced by the gravitational potential induced by such ultralight DM particles [47–50]. The analysis of the second data release from the European Pulsar Timing Array [51] peaks at $m_\phi \sim 10^{-23}\,\mathrm{eV}$ and rules out particles with masses ranging from $10^{-24}\,\mathrm{eV}$ to $10^{-23.3}\,\mathrm{eV}$. It should be noted that the analyses mentioned above often rely on the assumption that the ultralight DM makes up the whole of DM and interacts solely through gravitational means.

The other dark ingredient of the standard model of cosmology is the cosmological constant Λ, the simplest realisation of dark energy in the form of a background energy component which accounts for the current accelerated expansion of the Universe. Given that the value of the cosmological constant has to be very small, alternative candidates for DE have been proposed, including slowly evolving scalar fields [52–56], three-form fields [57] and other more exotic proposals (see [58–60] for reviews on dark energy models). If DE is a dynamical degree of freedom, couplings to other matter forms are generally expected unless a particular symmetry forbids such interactions [61]. Constraints coming from the *Cassini* probe [62] suggest that the coupling of a slowly rolling DE scalar field to the standard model fields has to be much weaker than gravity, as the field's minute mass results in a very large interaction range. On the other hand, these constraints can be relaxed for couplings to DM only, derived based on model-dependent cosmological observations. Some interacting dark energy (IDE) models may address shortcomings of the ΛCDM model such as the Hubble tension [2, 63–67], providing a guiding direction and framework for further study. Of particular interest are a class of models in which the CDM-DE coupling is directly proportional to the energy density of dark energy, as studied for example in refs. [68–80].

Given our ignorance about the theoretical origins of DM and DE, it is essential to keep an open mind about their properties and whether they interact directly. In this investigation, we seek to implement such a coupling between DM and DE, starting from a fundamental field theory description from Lagrangian. In the framework studied here, both dark matter and dark energy are scalar fields. Our work aims to study *the properties of scalar field DM*. The novelty with respect to the previous literature is twofold. First, unlike what is the typical approach to coupled DM/DE models, where dark matter is a perfect fluid, here it is a fast oscillating field at the minimum of its potential (for similar discussions, see [81–83]). Secondly, the interaction is mediated not by dark energy but *by the fast oscillating dark*

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matter field, whereas dark energy is slowly varying. Still, we expect our results to hold even if DE is a cosmological constant (but coupled to DM) or a three-form field [57]. We will see in section 5 that in the specific case of a conformal transformation, the interaction is proportional to the averaged energy density of dark energy, which provides nice motivation for the phenomenologically studied models.

This article is organised as follows: we motivate and describe the proposed framework in section 2. In this section, we also collect the field equations and useful relations, which will be relevant to the subsequent calculations. In section 3, we derive an effective fluid-field description, a valid approximation from the onset of the DM field quick oscillations around the minimum of an effective potential. The treatment of the associated fluid-field cosmological perturbations is discussed in section 4. In section 5, we discuss concrete realisations of the model and their cosmological implications. We summarise our findings in section 6. Throughout this paper, we follow the $(-,+,+,+)$ metric signature and set the speed of light $c = 1$.

2 Model

The action we consider, inspired by effective field theories, contains two interacting scalar fields and is given by

$$\begin{aligned}\mathcal{S} = &\int d^4x\sqrt{-g}\left(\frac{M_{\rm Pl}^2}{2}\mathcal{R} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \mathrm{U}(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - V(\chi)\right) \\ &- \int d^4x\sqrt{-g}\left(\phi Q_0\left(\chi, X^{(\chi)}\right) + \frac{1}{2}\phi^2 Q_1\left(\chi, X^{(\chi)}\right) + \dots\right) + \mathcal{S}_{\rm SM}\,,\end{aligned} \tag{2.1}$$

where $\mathcal{R}$ is the Ricci-scalar and $M_{\rm Pl} \equiv 1/\sqrt{8\pi G}$ is the reduced Planck mass. In the action above, $\mathcal{S}_{\rm SM}$ denotes the Lagrangian containing the standard model fields. Q_0 and Q_1 are coupling functions, depending on the slowly varying DE field χ and its kinetic term $X^{(\chi)} = -\frac{1}{2}g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi$. The dots denote higher-order terms, which we assume are negligible in the following.

The equation of motion for the dark matter field ϕ reads

$$\nabla^\mu\nabla_\mu\phi = U_{,\phi} + Q_0 + \phi Q_1\,, \tag{2.2}$$

while the equation of motion for the dark energy field χ is

$$\begin{aligned}\left(1 - \phi Q_{0,X} - \frac{1}{2}\phi^2 Q_{1,X}\right)\nabla^\mu\nabla_\mu\chi = &- \nabla^\mu\chi\nabla_\mu\left(1 - \phi Q_{0,X} - \frac{1}{2}\phi^2 Q_{1,X}\right) \\ &+ V_{,\chi} + \phi Q_{0,\chi} + \frac{1}{2}\phi^2 Q_{1,\chi}\,.\end{aligned} \tag{2.3}$$

The Einstein equations for this system take the form

$$G_{\mu\nu} = \kappa^2\left(T^{(\phi)}_{\mu\nu} + T^{(\rm SM)}_{\mu\nu} + T^{(\chi)}_{\mu\nu} + T^{(\rm coup)}_{\mu\nu}\right)\,, \tag{2.4}$$

where $T_{\mu\nu}^{(\text{coup})}$ is the contribution of the coupling terms to the total energy-momentum tensor. We identify, as it is customary, $T_{\mu\nu}^{(\phi)}$ with the energy-momentum tensor of DM and $T_{\mu\nu}^{(\chi)} + T_{\mu\nu}^{(\text{coup})} \equiv T_{\mu\nu}^{\text{DE}}$ as the DE energy-momentum tensor.[1]

In the following, we specialise to the case in which the functions Q_0 and Q_1 are only functions of the field χ and do not depend on the kinetic term $X^{(\chi)}$. This is done to simplify the DE equations of motion, but it is important to note that our results regarding the DM dynamics do not change based on this choice.

Since ϕ plays the role of DM, we assume that ϕ has the potential $\text{U}(\phi) = \frac{1}{2}m^2\phi^2$. Concerning χ, we assume it is a DE field that evolves very slowly on time scales much larger than the oscillations of ϕ. Hence, $V(\chi)$ will be a typical quintessential potential, e.g. of exponential form. We assume a flat Friedmann-Lemaître-Robertson-Walker spacetime, for which

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2.$$

The Friedmann equation is given by

$$H^2 = \frac{1}{3M_{\text{Pl}}^2}\left(\rho_\phi + \rho_{\text{DE}} + \rho_{\text{SM}}\right), \tag{2.5}$$

where we have defined

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + \text{U}(\phi), \tag{2.6}$$

$$\rho_{\text{DE}} = \frac{1}{2}\dot{\chi}^2 + V(\chi) + \phi Q_0 + \frac{1}{2}\phi^2 Q_1, \tag{2.7}$$

and the pressure components pick up the appropriate minus signs in front of non-kinetic terms.

The modified Klein-Gordon equation for the ϕ and χ fields read

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = -Q(t) \equiv -Q_0(\chi) - Q_1(\chi)\phi, \tag{2.8}$$

and

$$\ddot{\chi} + 3H\dot{\chi} + V_\chi = -\phi Q_{0,\chi} - \frac{1}{2}\phi^2 Q_{1,\chi}. \tag{2.9}$$

We note that the modified Klein-Gordon equation in eq. (2.8) for the DM field ϕ can be written in the form

$$\dot{\rho_\phi} + 3H(\rho_\phi + P_\phi) = -(Q_0 + Q_1\phi)\dot{\phi}. \tag{2.10}$$

This concludes the derivation of the relevant equations for this theory. The next section will use them to derive a fluid-field description for the system.

[1] If we were to include the coupling terms into the definition of the energy-momentum tensor for DM, the effective potential has a minimum at negative values of the potential, a situation we want to avoid, as this can lead to negative energy density for DM. This problem does not exist with the split performed here.

3 Towards a fluid-field description

We now seek to solve the equations derived in the previous section to study the evolution of both the DM and DE species. The main difference and challenge of our model compared to interacting quintessence models is that the coupling described here by $Q(t)$ depends on ϕ, which must rapidly oscillate for ϕ to behave like DM. Therefore, we have to perform an analysis based on the time-averaged character of the field over a period of oscillation to extract its average evolution and make the equations more manageable.

From eq. (2.8), the equation of motion for ϕ can be recast into

$$\ddot{\phi} + 3H\dot{\phi} + m_{\rm eff}^2(t)\phi = -Q_0(t)\,, \tag{3.1}$$

where we have defined the effective mass

$$m_{\rm eff}^2(t) \equiv m^2 + Q_1(t)\,. \tag{3.2}$$

Before solving these equations, we want to stress that the following analysis hinges on the form of eq. (3.1) and that the functions Q_0 and Q_1 are slowly varying. The exact form of Q_0 and Q_1 does not play a role in the overall phenomenology of the DM.

We will be interested in the case in which $m > H$ and the scalar field is under-damped, allowing for the oscillations to begin before the DM-dominated epoch begins. To solve eq. (3.1) we employ an ansatz for ϕ:

$$\phi(t) = \phi_{\rm osc}(t) + A(t)\,, \tag{3.3}$$

where $\phi_{\rm osc}$ is the solution to the homogeneous version of the Klein-Gordon equation:

$$\ddot{\phi}_{\rm osc} + 3H\dot{\phi}_{\rm osc} + m_{\rm eff}^2(t)\phi_{\rm osc} = 0\,, \tag{3.4}$$

while $A(t)$ solves

$$\ddot{A} + 3H\dot{A} + m_{\rm eff}^2(t)A = -Q_0(t)\,. \tag{3.5}$$

On the basis that A is sourced by a slowly-evolving function, Q_0, we make the assumption that $A(t)$ itself is slowly varying such that $\ddot{A}, 3H\dot{A} \ll m_{\rm eff}^2 A$ and thus, we can neglect the first two terms of the above equation. This results in an approximated analytical solution for A:

$$A(t) \approx -\frac{Q_0(t)}{m_{\rm eff}^2(t)}\,. \tag{3.6}$$

Making use of the Wentzel-Kramer-Brillouin (WKB) approximation, we find the following solution for $\phi_{\rm osc}$:

$$\phi_{\rm osc}(t) = \left(\frac{a_0}{a}\right)^{3/2}\left(\frac{m_0}{m_{\rm eff}}\right)^{1/2}\left(\phi_+ \sin\left(m_{\rm eff}t\right) + \phi_- \cos\left(m_{\rm eff}t\right)\right)\,, \tag{3.7}$$

where m_0 is the effective mass taken at $t = t_0$, and ϕ_+ and ϕ_- are constants. We can then average ϕ over one oscillation period to get

$$\langle\phi\rangle = A\,, \tag{3.8}$$

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and for its velocity

$$\langle\dot{\phi}\rangle = \dot{A}\,. \tag{3.9}$$

Likewise, the variance of these quantities becomes,

$$\langle\phi^2\rangle = \frac{1}{2}(\phi_+^2+\phi_-^2)\left(\frac{m_0}{m_{\rm eff}}\right)\left(\frac{a_0}{a}\right)^3 + A^2\,, \tag{3.10}$$

$$\langle\dot{\phi}^2\rangle = \frac{1}{2}m_{\rm eff}^2(\phi_+^2+\phi_-^2)\left(\frac{m_0}{m_{\rm eff}}\right)\left(\frac{a_0}{a}\right)^3\,, \tag{3.11}$$

where in the last line we have ignored an $\dot{A}^2$ term, since under our assumptions $\dot{A} \ll m_{\rm eff}A$.

The above expressions are crucial to the final results shown in this section. For standard uncoupled scalar field DM, both the field value and its derivative average out to zero. These equations encode that, in our case, the field does not oscillate around zero; instead, its averaged value is shifted by A due to the interaction. Therefore, the average energy density and pressure of the field are

$$\langle\rho_\phi\rangle = \frac{1}{4}(\phi_+^2+\phi_-^2)\left(\frac{m_0}{m_{\rm eff}}\right)\left(\frac{a_0}{a}\right)^3(m_{\rm eff}^2+m^2) + \frac{m^2A^2}{2}\,, \tag{3.12}$$

and

$$\langle p_\phi\rangle = \frac{1}{4}(\phi_+^2+\phi_-^2)\left(\frac{m_0}{m_{\rm eff}}\right)\left(\frac{a_0}{a}\right)^3(m_{\rm eff}^2-m^2) - \frac{m^2A^2}{2}\,, \tag{3.13}$$

respectively. Here, we clearly see that the interaction between DM and DE results in a change in the DM pressure, causing its departure from zero. The pressure can also be expressed as

$$\langle p_\phi\rangle = \left(\langle\rho_\phi\rangle - \frac{m^2A^2}{2}\right)\frac{m_{\rm eff}^2-m^2}{m_{\rm eff}^2+m^2} - \frac{m^2A^2}{2}\,, \tag{3.14}$$

in which case the equation of state becomes

$$w_\phi \equiv \frac{\langle p_\phi\rangle}{\langle\rho_\phi\rangle} = \left(1 - \frac{m^2A^2}{2\langle\rho_\phi\rangle}\right)\frac{m_{\rm eff}^2-m^2}{m_{\rm eff}^2+m^2} - \frac{m^2A^2}{2\langle\rho_\phi\rangle}\,. \tag{3.15}$$

Provided the kinetic energy of the dark energy field is small comparatively to its potential, Q_1 is positive and $m_{\rm eff} > m$. This means that the equation of state of dark matter starts by being positive in its early stages. Around matter-dark energy equality, the equation of state begins to decrease and, in fact, becomes increasingly negative, as illustrated in example models in section 5. The very late-time occurrence of this transition is ascribed to the dependence of the coupling on the dark energy density, implying that the coupling only has a significant impact on w_ϕ when DE begins to dominate.

Differentiating the averaged density, eq. (3.12), we obtain

$$\frac{d\langle\rho_\phi\rangle}{dt} + 3H\left(\langle\rho_\phi\rangle + \langle p_\phi\rangle\right) = \frac{3H}{4}(\phi_+^2+\phi_-^2)\left(\frac{m_0}{m_{\rm eff}}\right)\left(\frac{a_0}{a}\right)^3(m_{\rm eff}^2-m^2) + m^2A\dot{A}\,, \tag{3.16}$$

or equivalently

$$\frac{d}{dt}\langle\rho_\phi\rangle + 3H\left(\langle\rho_\phi\rangle - \frac{m^2A^2}{2}\right) = m^2A\dot{A}\,. \tag{3.17}$$

Had we done the time-averaging of eq. (2.10) directly, we would conclude that, in fact

$$\frac{d\langle\rho_\phi\rangle}{dt} = \langle\dot{\rho}_\phi\rangle\,. \tag{3.18}$$

We now define an effective equation of state parameter for DM as

$$w_{\text{eff}} = -\frac{1}{\langle\rho_\phi\rangle}\left(\frac{m^2A^2}{2} + \frac{m^2A\dot{A}}{3H}\right), \tag{3.19}$$

such that eq. (3.17) becomes

$$\langle\dot{\rho}_\phi\rangle + 3H\langle\rho_\phi\rangle(1+w_{\text{eff}}) = 0\,. \tag{3.20}$$

Finally, and for completeness, the equation of motion for the dark energy field, obtained by substituting eqs. (3.8) and (3.9) into eq. (2.9), yields

$$\ddot{\chi} + 3H\dot{\chi} + V_{,\chi} = \frac{Q_0}{m_{\text{eff}}^2}Q_{0,\chi} - \frac{2\langle\rho_\phi\rangle + \frac{Q_0^2}{m_{\text{eff}}^2}}{2\left(m_{\text{eff}}^2 + m^2\right)}Q_{1,\chi}\,. \tag{3.21}$$

It is worth noting that the last term on the r.h.s. depends on the DM energy density, which, in principle, could be large. Therefore, it is a requirement that $\frac{Q_{1,\chi}}{m_{\text{eff}}^2+m^2}$ is small enough to avoid this coupling term driving the evolution of the χ field, which would invalidate our assumption that DE is slowly rolling.

4 Perturbations

We now turn to the study of linear perturbations for our model. This will lead up to the derivation of the sound speed, which is crucial to solving the fluid-scalar field equations of motion.

We follow the approach developed in ref. [84] to compute the sound speed, considering only scalar perturbations. We adopt the following conventions for the metric scalar perturbations in a general gauge:

$$\delta g_{00} = -2\Phi\,, \tag{4.1}$$

$$\delta g_{i0} = a\nabla_i B\,, \tag{4.2}$$

$$\delta g_{ij} = -2a^2(\delta_{ij}\Psi - \nabla_i\nabla_j E)\,, \tag{4.3}$$

where Φ, Ψ, B and E are the four scalar degrees of freedom which can be expanded and decomposed in independently evolving Fourier modes k. This results in the following equations of motion for the perturbations of the scalar fields $\delta\chi$ and $\delta\phi$:

$$\begin{aligned}\delta\ddot{\chi} \quad &+3H\delta\dot{\chi} + \left(\frac{k^2}{a^2} + V_{,\chi\chi}\right)\delta\chi = \dot{\chi}\left(\dot{\Phi} + 3\dot{\Psi} - \frac{k}{a}B + \dot{E}\right)\\ &+2(\ddot{\chi}+3H\dot{\chi})\Phi - \delta\phi Q_{0,\chi} - \phi Q_{0,\chi\chi}\delta\chi - \phi\delta\phi Q_{1,\chi} - \frac{1}{2}\phi^2 Q_{1,\chi\chi}\delta\chi\,,\end{aligned} \tag{4.4}$$

and

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} + \frac{k^2}{a^2}\delta\phi + m^2\delta\phi = \dot{\phi}\left(\dot{\Phi} + 3\dot{\Psi} - \frac{k}{a}B + \dot{E}\right) + 2(\ddot{\phi} + 3H\dot{\phi})\Phi - \delta Q\,. \tag{4.5}$$

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Here, we have defined δQ as the perturbation of the coupling function. As for the background evolution, we can expand it as

$$\delta Q = \delta Q_0 + \phi \delta Q_1 + \delta\phi Q_1 \,, \tag{4.6}$$

such that eq. (4.5) becomes

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} + \frac{k^2}{a^2}\delta\phi + m_{\rm eff}^2\delta\phi = \dot{\phi}\left(\dot{\Phi} + 3\dot{\Psi} - \frac{k}{a}B + \dot{E}\right) + 2(\ddot{\phi} + 3H\dot{\phi})\Phi - \delta Q_0 - \phi\delta Q_1 \,. \tag{4.7}$$

The exact form of δQ_0 and δQ_1 depends on the details of the model.

The perturbed perfect fluid quantities are defined in terms of the time-averaged oscillating DM scalar field and its perturbation [85]:

$$\delta\rho = \langle \dot{\phi}\delta\dot{\phi} - \dot{\phi}^2\Phi + m^2\phi\delta\phi \rangle \,, \tag{4.8}$$

$$\delta p = \langle \dot{\phi}\delta\dot{\phi} - \dot{\phi}^2\Phi - m^2\phi\delta\phi \rangle \,, \tag{4.9}$$

$$\frac{a}{k}(\rho + p)(v - B) = \langle \dot{\phi}\delta\phi \rangle \,, \tag{4.10}$$

where v is the scalar part of the DM fluid-velocity, which is related to the spacial component of the four-velocity by $u_i = av_{,i}$. It should be understood that the fluid variables are themselves time-averaged quantities. Bringing together eqs. (4.7)–(4.10), we arrive at the following expressions for the perturbed continuity equations:

$$\delta\dot{\rho} + 3H(\delta\rho + \delta p) = (\rho + p)\left(3\dot{\Psi} + \dot{E} - \frac{k}{a}v\right) - \langle \delta Q\dot{\phi} \rangle - \langle Q\delta\dot{\phi} \rangle \,, \tag{4.11}$$

$$\frac{1}{a^4(\rho + p)}\frac{d}{dt}[a^4(\rho + p)(v - B)] = \frac{k}{a}\Phi + \frac{k}{a(\rho + p)}[\delta p - \langle Q\delta\phi \rangle] \,. \tag{4.12}$$

For convenience and to simplify the calculations, we consider the axion-comoving gauge,[2] which amounts to setting $B = v$. Thus, in this context, eq. (4.12) reduces to

$$\Phi = -\frac{\delta p}{\rho + p} + \frac{\langle Q\delta\phi \rangle}{\rho + p} \,. \tag{4.13}$$

Analogous to background fluid-field treatment, we specify the following ansatz for the perturbation of the ϕ-field:

$$\delta\phi(k,t) = \delta\phi_+(k,t)\sin(m_{\rm eff}t) + \delta\phi_-(k,t)\cos(m_{\rm eff}t) + \delta\mathcal{A}(t) \,. \tag{4.14}$$

It is important to note that $\delta\mathcal{A}(t)$ is not just the perturbation of $A(t)$ defined in eq. (3.6), as will be shown explicitly below.

For the rest of the calculation, we will assume a quasi-static approximation following [84], and will also be discarding derivatives of A and $\delta\mathcal{A}$ on similar grounds. In comoving gauge we have $\langle \dot{\phi}\delta\phi \rangle = 0$ from eq. (4.10), which implies

$$\delta\phi_+\phi_- = \delta\phi_-\phi_+ \,. \tag{4.15}$$

[2]The name of the gauge is used in the literature [84, 86], but we note here that ϕ is not an axion field but a generic DM scalar field.

Now we can solve eq. (4.7) to leading order in H/m since the field oscillates if $m \gg H$. This implies assuming that metric perturbations vary only on cosmological time scales $t \sim H^{-1} \gg m^{-1}$ and then grouping the terms in powers of H/m,

$$\delta\ddot{\phi} + \frac{k^2}{a^2}\delta\phi + m_{\rm eff}^2\delta\phi = -2(m_{\rm eff}^2\phi + Q_0)\Phi - \delta Q_0 - \phi\delta Q_1 \,. \tag{4.16}$$

Once again, splitting the equation and matching the non-oscillating terms, we get:

$$\delta\ddot{\mathcal{A}} + \frac{k^2}{a^2}\delta\mathcal{A} + m_{\rm eff}^2\delta\mathcal{A} = -\delta Q_0 - \delta Q_1 A \,, \tag{4.17}$$

which, in the quasi-static approximation, yields

$$\delta\mathcal{A} = -\frac{\delta Q_0 + \delta Q_1 A}{\frac{k^2}{a^2} + m_{\rm eff}^2} = -\frac{\delta Q_0 - \frac{Q_0}{m_{\rm eff}^2}\delta Q_1}{\frac{k^2}{a^2} + m_{\rm eff}^2} \,. \tag{4.18}$$

Substituting the ansatz in eq. (4.14) into eq. (4.16) we obtain:

$$\Phi = -\frac{1}{2}\left(\frac{a}{a_0}\right)^{3/2}\left(\frac{m_0}{m_{\rm eff}}\right)^{-1/2}\frac{\delta\phi_+}{\phi_+}\frac{k^2}{m_{\rm eff}^2 a^2} - \frac{1}{2m_{\rm eff}^2}\delta Q_1 \,, \tag{4.19}$$

up to leading order in H/m. Replacing this expression into the perturbed fluid equations for δp and $\delta\rho$, eqs. (4.8) and (4.9) we arrive at

$$\begin{aligned}
\delta p =& \frac{a^{-\frac{3}{2}}}{2}\left(\frac{m_0}{m_{\rm eff}}\right)^{\frac{1}{2}}(\phi_+^2 + \phi_-^2)\frac{\delta\phi_+}{\phi_+}m_{\rm eff}^2\left[\frac{1}{2}\frac{k^2}{a^2 m_{\rm eff}^2} + \frac{Q_1}{m_{\rm eff}^2}\right] \\
&- m^2 A\delta\mathcal{A} + \frac{a^{-3}}{4}\left(\frac{m_0}{m_{\rm eff}}\right)(\phi_+^2 + \phi_-^2)\delta Q_1 \,, \qquad (4.20) \\
\delta \rho =& \frac{a^{-\frac{3}{2}}}{2}\left(\frac{m_0}{m_{\rm eff}}\right)^{\frac{1}{2}}(\phi_+^2 + \phi_-^2)\frac{\delta\phi_+}{\phi_+}m_{\rm eff}^2\left[\frac{1}{2}\frac{k^2}{a^2 m_{\rm eff}^2} + 2 - \frac{Q_1}{m_{\rm eff}^2}\right] \\
&+ m^2 A\delta\mathcal{A} + \frac{a^{-3}}{4}\left(\frac{m_0}{m_{\rm eff}}\right)(\phi_+^2 + \phi_-^2)\delta Q_1 \,. \qquad (4.21)
\end{aligned}$$

From here, and using eq. (4.19), we can compute the pressure perturbation, given by

$$\delta p = c_a^2\delta\rho - m^2 A\delta\mathcal{A}(1 + c_a^2) + \frac{1}{2}\frac{\rho + p}{m_{\rm eff}^2}\delta Q_1(1 - c_a^2) \,, \tag{4.22}$$

where we have defined the effective sound speed

$$c_a^2 = \frac{\frac{1}{2}\frac{k^2}{a^2 m_{\rm eff}^2} + \frac{Q_1}{m_{\rm eff}^2}}{\frac{1}{2}\frac{k^2}{a^2 m_{\rm eff}^2} + 2 - \frac{Q_1}{m_{\rm eff}^2}} = \frac{\frac{1}{2}\frac{k^2}{a^2 m^2} + \frac{Q_1}{m^2}}{\frac{1}{2}\frac{k^2}{a^2 m^2} + 2 + \frac{Q_1}{m^2}} \,, \tag{4.23}$$

and the second equality follows from the definition of $m_{\rm eff}^2$ in eq. (3.2). It is important to note that this effective sound speed is akin to that of the non-interacting axion, presented in refs. [84, 86]. Accordingly, and as expected, eq. (4.23) reduces to that case in the limit where Q_1 is zero.

Furthermore, according to eq. (4.22), in contrast to the non-interacting case, the pressure perturbation is not precisely proportional to the density perturbation. This can be interpreted as a non-adiabatic contribution to the pressure perturbation caused by the interaction with the dark energy scalar field [83, 87–89].

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5 Concrete model examples

The calculations performed in the last sections are valid for a theory which can be written in the form (2.1). What we have shown is that, as a proof of concept, interactions between DM and DE of the form displayed in (2.1) change the properties of DM. In this section, we provide concrete examples for obtaining equations of motion for the DM field of the same or similar form as eq. (3.1).

5.1 Conformal coupling

The first setup is inspired by that of field theories of dark energy, such as coupled quintessence [55, 81, 82, 90, 91], but with the role of the fields for DM and DE swapped. That is, we treat ϕ in the action below as a dark matter scalar field, and the dark energy sector is coupled conformally to the DM sector, resulting in the following effective action for this model:

$$\mathcal{S} = \int d^4x\sqrt{-g}\left(\frac{M_{\rm Pl}^2}{2}\mathcal{R} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \mathrm{U}(\phi)\right) + \mathcal{S}_{\rm SM} + \mathcal{S}_{\rm DE}. \tag{5.1}$$

In what follows, we assume that the standard model sector does not couple to DM or DE and that the DE sector can be described by a slowly evolving scalar field χ, described by the action

$$\mathcal{S}_{\rm DE} = \int d^4x\sqrt{-\tilde{g}}\left(\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - V(\chi)\right), \tag{5.2}$$

where the metric $\tilde{g}$ is related to the metric g via a conformal transformation of the form $\tilde{g}_{\mu\nu} = C(\phi)g_{\mu\nu}$. Rewriting $\mathcal{S}_{\rm DE}$ in terms of the metric $g_{\mu\nu}$ results in

$$\mathcal{S}_{\rm DE} = \int d^4x\sqrt{-g}\left(\frac{C(\phi)}{2}g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - C^2(\phi)V(\chi)\right). \tag{5.3}$$

In contrast to models such as coupled quintessence, in which C depends on the dark energy field, the function C in this framework is dependent on dark matter properties.

The equation of motion for the DE field χ is derived from the corresponding variation of the action in Equation (5.1) and reads

$$\nabla^\mu\nabla_\mu\chi - C\frac{dV}{d\chi} = -\frac{\nabla_\alpha C}{C}\nabla^\alpha\chi. \tag{5.4}$$

The equation for the DM field ϕ is derived in an analogous manner and yields

$$\nabla^\mu\nabla_\mu\phi - \frac{\partial U}{\partial\phi} = Q, \tag{5.5}$$

where the coupling Q was defined as

$$\begin{aligned} Q &= \nabla_\mu\left(\frac{\partial L_\chi}{\partial(\nabla_\mu\phi)}\right) - \frac{\partial L_\chi}{\partial\phi} \\ &= \frac{C_{,\phi}}{2C}\left[Cg^{\alpha\beta}\nabla_\alpha\chi\nabla_\beta\chi + 4C^2U\right]. \end{aligned} \tag{5.6}$$

The Einstein equations result from variation of the action with the gravitational metric $g_{\mu\nu}$, resulting in

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa^2\left(T^{(\phi)}_{\mu\nu} + T^{(\chi)}_{\mu\nu} + T^{(\rm SM)}_{\mu\nu}\right), \tag{5.7}$$

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where $R_{\mu\nu}$ is the Ricci tensor, $\kappa \equiv 1/M_{\rm Pl}$ and $T^{(i)}_{\mu\nu}$ are the energy-momentum tensors for each i-th fluid, which for the dark sector scalar fields ϕ and χ read

$$T^{(\phi)}_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi + V(\phi)\right), \tag{5.8}$$

$$T^{(\chi)}_{\mu\nu} = C\nabla_\mu\chi\nabla_\nu\chi - g_{\mu\nu}\left(\frac{C}{2}g^{\alpha\beta}\nabla_\alpha\chi\nabla_\beta\chi + C^2\mathrm{U}(\chi)\right), \tag{5.9}$$

respectively, and obey the following conservation equations:

$$\nabla^\mu T^{(\phi)}_{\mu\nu} = Q\nabla_\nu\phi, \quad \nabla^\mu T^{(\chi)}_{\mu\nu} = -Q\nabla_\nu\phi, \tag{5.10}$$

which are physically equivalent to the modified Klein-Gordon equations.

In the case of a flat Friedmann-Lemaître-Robertson-Walker universe the field equations read

$$\ddot{\phi} + 3H\dot{\phi} + U_{,\phi} = \frac{1}{2}C_{,\phi}\dot{\chi}^2 - 2CC_{,\phi}V, \tag{5.11}$$

$$\ddot{\chi} + \left(3H + \frac{C_{,\phi}}{C}\dot{\phi}\right)\dot{\chi} + CV_{,\chi} = 0. \tag{5.12}$$

The energy densities ρ_ϕ and ρ_χ and the corresponding pressures p_ϕ and p_χ are given by

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + \mathrm{U}(\phi), \quad p_\phi = \frac{1}{2}\dot{\phi}^2 - \mathrm{U}(\phi), \tag{5.13}$$

$$\rho_\chi = \frac{1}{2}C\dot{\chi}^2 + C^2V(\chi), \quad p_\chi = \frac{1}{2}C\dot{\chi}^2 - C^2V(\chi). \tag{5.14}$$

The Friedmann equations read

$$H^2 = \frac{\kappa^2}{3}\left[\frac{1}{2}\dot{\phi}^2 + \mathrm{U}(\phi) + \frac{1}{2}C(\phi)\dot{\chi}^2 + C(\phi)^2V(\chi)\right], \tag{5.15}$$

$$\dot{H} = -\frac{\kappa^2}{2}\left[\dot{\phi}^2 + C(\phi)\dot{\chi}^2\right]. \tag{5.16}$$

The modified Klein-Gordon equations for ϕ and χ, eqs. (5.11) and (5.12) can also be written as

$$\dot{\rho}_\phi + 3H\left(\rho_\phi + p_\phi\right) = -\frac{C_{,\phi}}{2C}\left(\rho_\chi - 3p_\chi\right)\dot{\phi}, \tag{5.17}$$

$$\dot{\rho}_\chi + 3H\left(\rho_\chi + p_\chi\right) = \frac{C_{,\phi}}{2C}\left(\rho_\chi - 3p_\chi\right)\dot{\phi}. \tag{5.18}$$

Since we wish for ϕ to behave as dark matter and χ as dark energy, in this analysis, we focus on the following self-interacting scalar field potentials:

$$V(\chi) = V_0 e^{-\kappa\lambda\chi}, \tag{5.19}$$

$$U(\phi) = \frac{1}{2}m^2\phi^2. \tag{5.20}$$

The function Q defined in eq. (5.6) can be written as

$$Q = \frac{C_{,\phi}}{2C}\left(\rho_\chi - 3p_\chi\right), \tag{5.21}$$

for a field-dependent conformal function C. We note that the form of this interaction term Q is proportional to the energy density of DE, providing a concrete theoretical realisation for a widely studied class of phenomenological interacting DE models (see [65, 69] and references therein). This means that the equation of motion for the ϕ field is of the form

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = -Q(t). \tag{5.22}$$

At linear perturbative level, the equation of motion for $\delta\phi$ is the same as that presented in section 4. For completeness, the corresponding equation for $\delta\chi$ is

$$\begin{aligned}\delta\ddot{\chi} \ &+ \left(3H + \frac{C_{,\phi}}{C}\dot{\phi}\right)\delta\dot{\chi} + \left(\frac{k^2}{a^2} + C(\phi)V_{,\chi\chi}\right)\delta\chi = \dot{\chi}\left(\dot{\Phi} + 3\dot{\Psi} - \frac{k}{a}B + \dot{E}\right)\\ &-2C(\phi)V_{,\chi}\Phi - C_{,\phi}V_{,\chi}\delta\phi - \frac{C_{,\phi}}{C}\dot{\chi}\delta\dot{\phi} - \left(\frac{C_{,\phi}}{C}\right)_{,\phi}\dot{\phi}\dot{\chi}\delta\phi.\end{aligned} \tag{5.23}$$

Choosing a polynomial form for $C(\phi)$ will yield a similar form to that described in eq. (2.1). From there, it is straightforward to read off the specific forms of Q_0 and Q_1, after which we can directly apply the results of sections 3 and 4. Concretely, we now choose

$$C(\phi) = 1 + 2\kappa\beta\phi, \tag{5.24}$$

for the conformal coupling where β is a dimensionless constant and we use $\kappa \equiv \frac{1}{M_{\rm Pl}}$. Note that the value of β is a *free* parameter, not predicted from theory, which needs to be constrained by comparing the theory to data, a task that is beyond the scope of this article. Below we will choose values for β which result in notable changes to the DM properties, but they act as illustration only.

With our choice of $C(\phi)$, we can decompose the source term above as

$$Q(t) = Q_0(t) + Q_1(t)\phi, \tag{5.25}$$

where the functions Q_0, Q_1 are of the specific form:

$$Q_0(\chi, X^{(\chi)}) = 4\kappa\beta V(\chi) - 2\kappa\beta X^{(\chi)} \tag{5.26}$$

$$Q_1(\chi) = 8\kappa^2\beta^2 V(\chi). \tag{5.27}$$

In an FLRW universe, the above coupling functions reduce to

$$Q_0(t) = -\kappa\beta(\dot{\chi}^2 - 4V), \tag{5.28}$$

$$Q_1(t) = 8\kappa^2\beta^2 V. \tag{5.29}$$

The equation of motion for ϕ can then be recast into

$$\ddot{\phi} + 3H\dot{\phi} + m^2_{\rm eff}(t)\phi = -Q_0(t), \tag{5.30}$$

as in section 3. After time-averaging the DM field, the DE equation of motion becomes

$$\ddot{\chi} + \left(3H + \frac{2\kappa\beta\dot{A}}{1 + 2\beta\kappa A}\right)\dot{\chi} + V_{,\chi}\,(1 + 2\beta\kappa A) = 0, \tag{5.31}$$

which is different to eq. (3.21) due to the extra kinetic contribution in Q_0. However, the differences in DE dynamics can easily be kept small whilst yielding the same changes in the DM dynamics.

Here we prove that the relation between Q_0 and Q_1 from eqs. (5.28), (5.29) implies that the EoS transits from positive to negative values around DM-DE equality. Indeed, we can see from eq. (3.15), that $w_\phi < 0$ when

$$\langle \rho_\phi \rangle < \frac{1+Z}{Z}\frac{m^2 A^2}{2}, \tag{5.32}$$

where $Z = (m_{\rm eff}^2 - m^2)/(m_{\rm eff}^2 + m^2) \approx Q_1/2m^2$. As $A \approx -Q_0/m^2$, it turns out that $w_\phi < 0$ when $\langle \rho_\phi \rangle$ drops below

$$\langle \rho_\phi \rangle < \frac{Q_0^2}{Q_1} \approx V \approx \rho_\chi \,. \tag{5.33}$$

Consequently, the DM pressure becomes appreciably negative only at very low redshift.

In contrast to our discussion in the previous sections, here Q_0 contains a dependency on the kinetic term $X^{(\chi)}$ which comes out of the conformal coupling. The difference brought by the kinetic dependence comes in the form of a modified equation of motion for the DE χ field. We have checked explicitly that this extra kinetic term does not spoil our initial assumption that χ is slow-rolling. We leave it for future work to study the modified dynamics of the DE in these models, and the effect of a kinetic coupling.

It is important to note that any model, such as the conformal coupling described above, that admits a Q_0 that depends on the kinetic term $X^{(\chi)}$ can be potentially problematic. Indeed, since the ϕ field is oscillating, the kinetic term for the χ field can become negative if $C(\phi)$ becomes negative, which can lead to instabilities in the $\delta\chi$ perturbations as the effective potential becomes tachyonic (see eq. (5.23)). This happens when $2\kappa\beta|\phi| > 1$. However, since the oscillations are on much smaller timescales than the evolution of the χ perturbations, we would expect that instabilities do not grow due to the kinetic term becoming negative only briefly every oscillation cycle. Since we do not study this in detail, we will keep to scenarios where strictly $2\kappa\beta\phi < 1$ in order to avoid possible instabilities. In this scenario, it is natural to think of the linear conformal coupling as a first order Taylor expansion of an exponential conformal coupling, which are common in the literature. This restriction on ϕ, it turns out, is quite limiting. To see this, let us first estimate the initial value of the ϕ field, ϕ_i when it starts rolling, at around $H \approx m$. To do this, we assume that the field will start rolling during the radiation epoch, which is required for ϕ to make up all of DM. Starting from

$$\rho_{\phi,0} \approx \frac{1}{2}\phi_i^2 m^2 a_i^3, \tag{5.34}$$

and writing $m \approx H$ as $m \approx H_{eq}\left(a_{eq}/a_i\right)^2$, we get the following approximation for the initial value:

$$\phi_i \approx \sqrt{2}\rho_{\phi,0}^{1/2} H_{eq}^{-3/4} a_{eq}^{-3/2} m^{-1/4}. \tag{5.35}$$

Setting the condition $2\kappa\beta\phi_i < 1$ ensures that the condition holds at all times, as ϕ only decays over time once it starts oscillating. Plugging in some numbers we are left with

$$\beta \lessapprox 10^{-19}\left(\frac{m}{\rm eV}\right)^{1/4} {\rm eV}^{-1}. \tag{5.36}$$

Now we can estimate the maximum value for the non-zero equation of state of DM today in this model using eqs. (3.15) and (5.26)

$$w_{\phi,0} \approx \frac{16\beta^2 V^2}{m^2 \rho_{\phi,0}}. \tag{5.37}$$

Plugging in numbers once more, we get this upper limit on $w_{\phi,0}$:

$$w_{\phi,0} \lessapprox 10^{-47} \left(\frac{m}{\mathrm{eV}}\right)^{-3/2}. \tag{5.38}$$

Even for the lightest possible scalar field DM currently allowed, we get a incredibly small value for $w_{\phi,0}$ which would be completely undistinguishable from ΛCDM. This means that unless there is an additional physical process which 'turns on' the conformal coupling at a later time — when ϕ has already decayed to a small enough number — this model cannot produce any different physics to ΛCDM. We will not study further any extensions of this model trying to address these limitations, but still highlight its usefulness in providing a concrete implementation of eq. (2.1) as well as a physical motivation for phenomenological interactive DE models. Instead, we will now study the simplest model described by eq. (2.1).

5.2 Minimal Λ scenario

The simpler model which we describe in the following will serve as an approximate toy-model to develop an intuition of the possible physical effects derived in sections 3 and 4.

We will take the 0^{th} order approximation of our framework. Let us consider a quasi-static χ field. This means that DE behaves almost as a cosmological constant, and that Q_0 and Q_1 are also very slowly varying. In this scenario, we have checked that the back reaction of the coupling onto the χ field dynamics is minimal, meaning that the DE stays slowly evolving. In this context, it is then sensible to approximate χ as a cosmological constant, and Q_0 and Q_1 as constants also. We can then follow our previous results and simply plug in constant Q_0 and Q_1. One can then solve for the background evolution assuming χ is non-dynamical (i.e. a cosmological constant).

To illustrate the physics of this minimal model, we produce a modified version of the `CLASS` code to solve for the dynamics of the present model. Figure 1 shows the evolution of the field's equation of state parameter w_ϕ from eq. (3.15). From this, we see that the DM equation of state parameter is positive at early time and becomes negative around DM-DE equality. It is important to note that the effective equation of state, from eq. (3.19), is strictly negative in this model, as $\dot{A} = 0$. Since it is $w_{\rm eff}$ that dictates the actual evolution of the DM component according to eq. (3.20), the DM in this model will behave like CDM (at background level) for most of the expansion history, until it starts diluting slower around DM-DE equality due to the negative effective EoS.

The effect of the coupling on the DM sound speed can be appreciated in figures 2 and 3. At early times, the k^2/a^2m^2 term dominates, and we get a similar behaviour to that of the non-interacting case. On the other hand, at late times, when $k^2/a^2m^2 \ll Q_1$, the sound speed becomes practically constant, according to $c_s^2 \approx \frac{Q_1}{2m^2}$. As expected, in the absence of the coupling, c_s^2 falls to zero at small redshifts. In future work we will constrain this model with cosmological data.

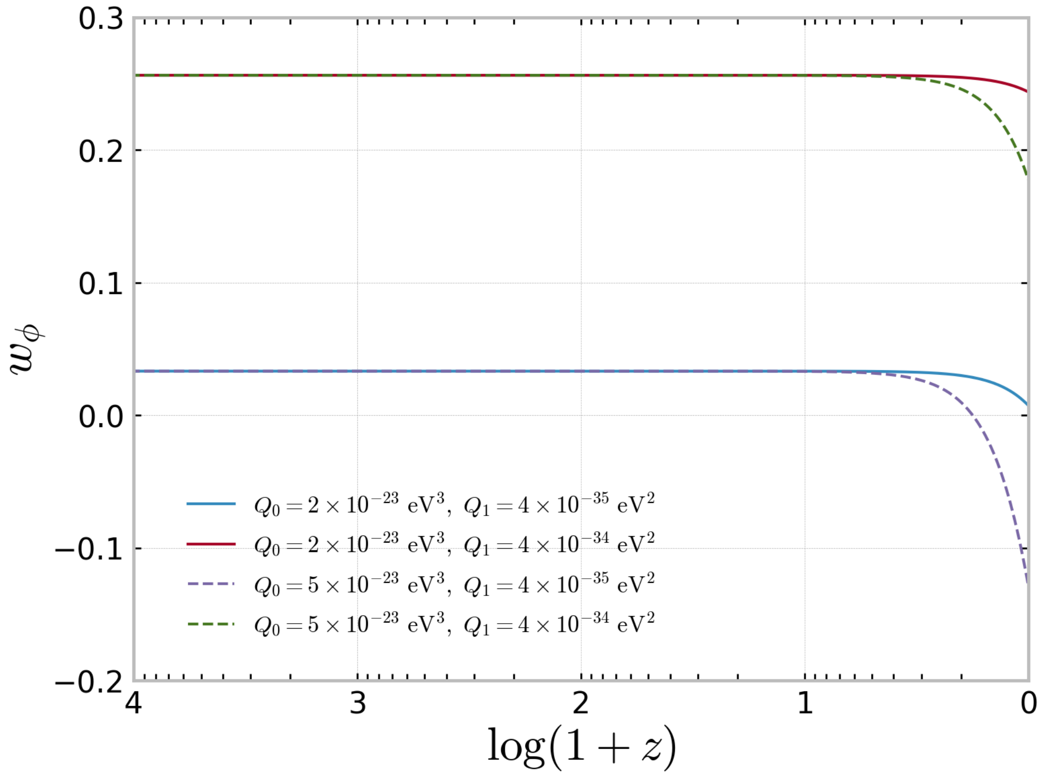

Figure 1. Evolution of the equation of state of dark matter w_ϕ defined in eq. (3.15), for the minimal Λ model. The model parameters used in this illustrative example are $m = 10^{-17}$ eV as well as the remaining Planck ΛCDM best-fit parameters. Different curves are shown for different values of Q_0 and Q_1.

6 Summary and conclusions

In this study, we have introduced a novel model of the dark sector consisting of scalar field dark matter interacting with quintessence dark energy. The couplings between the dark matter scalar field ϕ and the slowly evolving dark energy field χ is mediated by interactions of the form $\phi O(\chi, X^{(\chi)})$ and $\phi^2 O(\chi, X^{(\chi)})$. The aim of our work is to study whether the properties of the DM scalar field, such as the equation of state and the adiabatic sound speed, change in the presence of such couplings. Furthermore, as a result of these couplings, we have found an effective theory of interacting dark matter and dark energy in which the coupling term is linked to the energy density of DE rather than DM. Our framework not only offers a theoretical basis for many extensively studied models of interacting dark energy found in the existing literature, e.g. in refs. [65, 69, 73, 79, 80, 92–94], but also introduces significant differences. In our setup, the coupling between DM and DE offsets the oscillations of the scalar field, resulting in a non-zero average for its equation of state parameter. By averaging over these rapid oscillations, we have derived a fluid-field description of the system, revealing a non-zero average value for the field determined by the coupling. Consequently, the effective DM fluid exhibits a non-zero physical pressure, starkly contrasting with standard IDE models. This physical pressure transitions from slightly positive at early times to negative after DM-DE equality. The main observational effect most likely comes from the change in the

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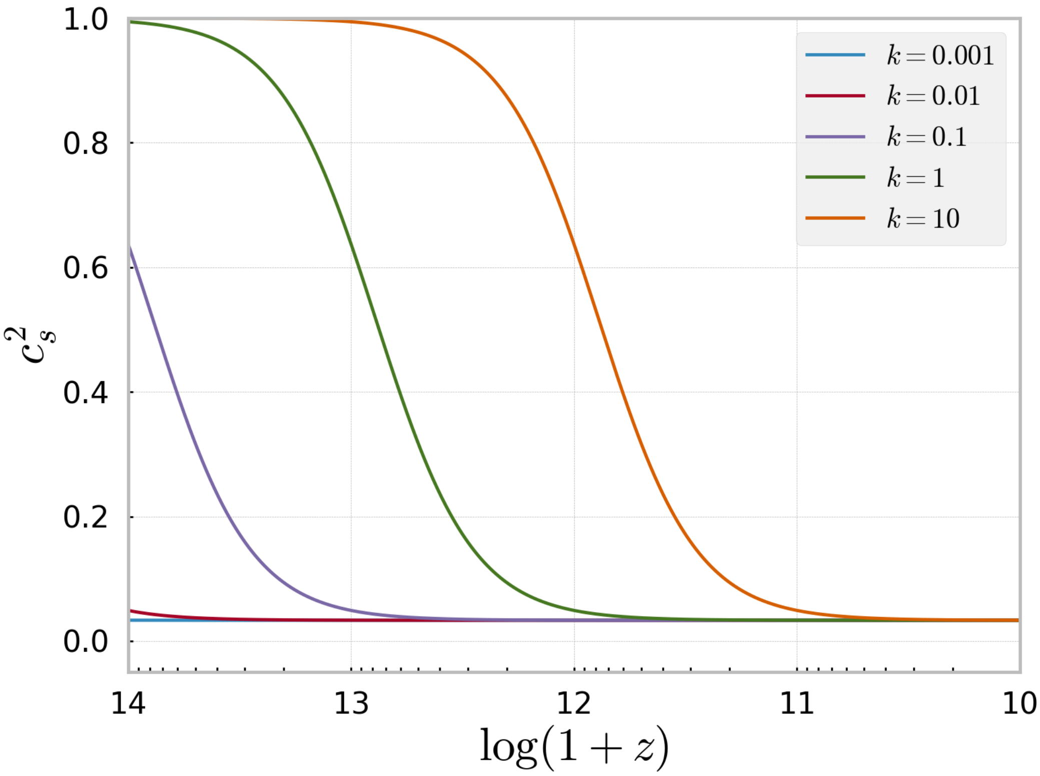

Figure 2. Evolution of the effective sound speed in terms of redshift z, as defined in eq. (4.23), for a set of k values $\{0.001, 0.01, 0.1, 1, 10\}$ (in Mpc^{-1}), and the following set of parameters: $m = 10^{-17}\mathrm{eV}$, $Q_0 = 0$, $Q_1 = 4 \times 10^{-35}\mathrm{eV}^2$ and all other parameters fixed to Planck ΛCDM best-fit.

DM equation of state at very late times, at a redshift smaller than the redshift of the DE-DM equality. Thus, our work carries implications in the context of searches for non-standard CDM physics and provides theoretical support for such models (see [95–98]). Notably, the non-standard equation of state of DM impacts the analysis and interpretation of cosmological observational data, such as Pantheon+ [99] or the recently published DESI data [100].

At the linear perturbation level, we have computed the pressure perturbation and sound speed of the averaged DM fluid. Our findings reveal terms proportional to the density perturbation $\delta\rho$ plus non-adiabatic pressure terms depending on the perturbation of the DE scalar field. Notably, due to the coupling, the adiabatic term deviates from the standard axion scalar field scenario. This results in an effective sound speed that remains non-zero across all scales, unlike the uncoupled case where it vanishes at small k. Consequently, we anticipate a slight power suppression in the matter power spectrum at all scales within the interacting scenario considered here. Furthermore, the influence of non-adiabatic pressure contributions on observables in this context warrants further investigation, as well as the changes to the properties of DE, such as the sound speed [101, 102] and the full evolution of perturbations in the DE field. A comprehensive study of these points requires a full implementation of the theory in publicly available Boltzmann codes such as `CLASS` or `CAMB`[3] [103], a task left for future work.

[3]https://github.com/cmbant/CAMB

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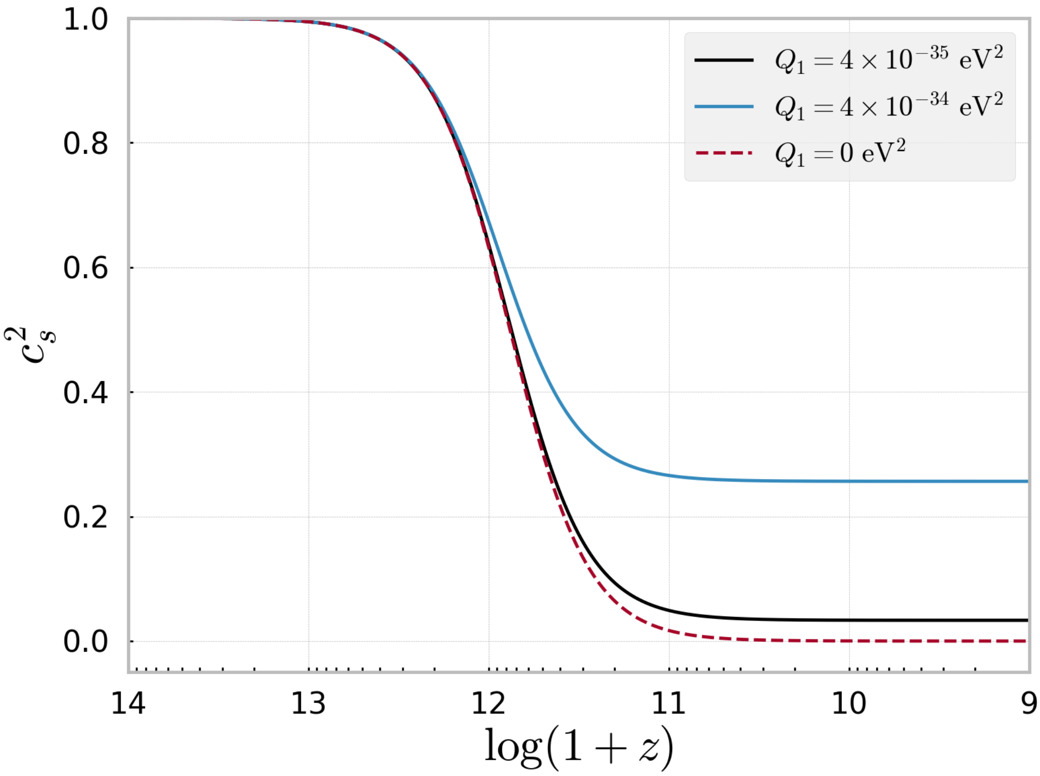

Figure 3. Evolution of the effective sound speed for two values of Q_1, defined in eq. (4.23), compared with a non-coupled standard axion, with the same parameters used as in figure 2 and $k = 10$ Mpc^{-1}.

Acknowledgments

We are grateful to Eleonora Di Valentino, William Giarè and Vivian Poulin for useful discussions and encouragement. EMT is supported by funding from the European Research Council (ERC) under the European Union's HORIZON-ERC-2022 (grant agreement no. 101076865). EMT was also supported by the grant SFRH/BD/143231/2019 from Fundação para a Ciência e a Tecnologia (FCT) during part of this work. CvdB is supported (in part) by the Lancaster–Sheffield Consortium for Fundamental Physics under STFC grant: ST/X000621/1. NJN acknowledges support from FCT via the following projects: UIDB/04434/2020 & UIDP/04434/2020 and PTDC/FIS-AST/0054/2021. This article is based upon work from COST Action CA21136 Addressing observational tensions in cosmology with systematics and fundamental physics (Cosmoverse) supported by COST (European Cooperation in Science and Technology).

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