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Unitary rigid supersymmetry for the chiral graviton and chiral gravitino in de Sitter spacetime

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Abstract: It is commonly believed that a unitary supersymmetric quantum field theory (QFT) involving graviton and gravitino fields on fixed 4-dimensional de Sitter spacetime (dS_4) cannot exist due to known challenges associated with supersymmetry (SUSY) in dS_4. In this paper, we contradict this expectation by presenting a new unitary supersymmetric QFT on dS_4: the free supersymmetric theory of the chiral graviton and chiral gravitino fields. By chiral we mean that the corresponding field strengths are anti-self-dual, and the gauge potentials are complex, each carrying a single complex propagating degree of freedom. The global SUSY transformations are generated by the standard Dirac Killing spinors of dS_4. The theory overcomes the known obstacles to unitary global SUSY on dS_4 by closing the commutator between two SUSY transformations on $so(4,2) \oplus u(1)$ rather than the de Sitter algebra $so(4,1)$. Crucially, the $so(4,2)$ symmetry is realised through unconventional conformal-like transformations. This free theory cannot become interacting while preserving SUSY in a way that makes the spin-2 sector the true graviton sector of General Relativity, as the three-graviton coupling cannot be $u(1)$-invariant.

We establish the unitarity of the free supersymmetric theory in two complementary ways. First, by studying the action of the superalgebra generators on the space of physical gravitino and graviton mode solutions. In particular, we introduce positive-definite, invariant inner products and demonstrate that the SUSY representation is unitary, forming a direct sum of two unitary irreducible representations — one with negative-helicity modes and the other with positive-helicity modes. Second, by quantising the fields and explicitly constructing the complex quantum supercharges Q_A and $Q^{A\dagger}$, we show that the trace $\sum_A \{Q_A, Q^{A\dagger}\}$ is positive-definite.

Before constructing the supersymmetric theory, we examine the free graviton and gravitino fields on dS_4, where the gravitino is known to have an imaginary mass parameter. We introduce a Hermitian, gauge-invariant, and local Lagrangian for the free gravitino field and explain why the requirement of unitarity forces the field to be chiral, removing half of the propagating helicity states.

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Contents

1 Introduction

Apart from its significance in inflationary cosmology [1], de Sitter (dS) spacetime is also relevant to the physics of our present Universe, as suggested by recent observational data

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supporting an accelerated spatial expansion [2–5]. Both these eras require a quantum understanding [6, 7]. It is thus important to develop tools for a deeper theoretical understanding of quantum de Sitter spacetime.

In recent times, the attempts towards a deeper theoretical understanding of dS spacetime have manifested themselves in (at least) two main approaches. The first approach concerns the study of lower-dimensional models in order to develop a more complete quantum understanding [8–26]. The second approach concerns the study of a large class of quantum fields in four-dimensional (or higher-dimensional) dS spacetime [27–54]. In this approach, the Unitary Irreducible Representations of the de Sitter algebra $so(D,1)$ [55–58] play a central role because they are identified with elementary particles on D-dimensional dS spacetime, as a generalisation of Wigner's classification for Minkowski spacetime. In this paper, we take the latter approach, and we uncover new features of supersymmetric quantum field theory on four-dimensional de Sitter spacetime, placing special emphasis on group-theoretic aspects.

Four-dimensional dS spacetime (dS_4), is the maximally symmetric solution of the vacuum Einstein equations with positive cosmological constant [59],

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0, \tag{1.1}$$

where $\Lambda = 3\,\mathcal{R}_{dS}^{-2}$ is the cosmological constant, $\mathcal{R}_{dS}$ is the dS radius, $g_{\mu\nu}$ is the metric tensor, $R_{\mu\nu} = 3\mathcal{R}_{dS}^{-2}\, g_{\mu\nu}$ is the Ricci tensor and R is the Ricci scalar. We will work in units where $\mathcal{R}_{dS} = 1$. Unlike in anti-de Sitter and Minkowski spacetimes, formulating supersymmetric theories in de Sitter spacetime presents fundamental challenges. The main obstacles to the existence of unitary, unbroken de Sitter supersymmetry (SUSY), which differ depending on whether one considers global or local SUSY, can be summarised as follows (see for instance the discussion in section 4 of [9]):

- **Problems concerning the unitarity of global SUSY on a fixed de Sitter background.** This can be understood already at the level of abstract representation theory [60, 61]. It is possible to supersymmetrise the dS algebra $so(4,1)$ by introducing spinorial supercharges $Q_A^{(i)}$, i.e. odd generators, which we take to be Dirac spinors for the sake of the discussion. The index A is a spinor index referring to the fundamental spinor representation of $so(4,1)$, and i is an extended SUSY index keeping track of the number of supercharges. Alternatively, one can double the number of supercharges and introduce a symplectic Majorana reality condition, as in [61], but this will lead to the same representation-theoretic results. As shown in [60, 61], from the structure of the algebra it follows that $\sum_{A,i}\{Q_A^{(i)}, Q^{(i)A\,\dagger}\} = 0$, and thus, all non-trivial representations of the dS superalgebra on a Hilbert space must be non-unitary (i.e. positive-norm and negative-norm states must exist). On the other hand, requiring that negative-norm states do not appear implies that all the $Q_A^{(i)}$'s, as well as all the dS generators, must annihilate all states in the Hilbert space, i.e. only the trivial representation is possible.[1] Super-extensions of $so(2,1)$, $so(3,1)$ and $so(5,1)$ also exist but unitary representations are allowed only in the case of $so(2,1)$ [60].

[1]De Sitter supersymmetry has also been studied in the ambient space formalism in ref. [62]. However, since the anti-commutator between two supercharges closes on $so(4,1)$, the theory is most likely non-unitary.

- **Problems concerning the unitarity of dS_4 Supergravity.** The explicit construction of the $N = 2$ Supergravity action with a positive cosmological constant was carried out in [61], involving a real vierbein, a real photon, and two symplectic Majorana gravitini. The number of gravitini was doubled in order to apply the symplectic Majorana condition because the conventional Majorana condition for the gravitino on dS_4 **cannot** be used — see also section 3. This difficulty is related to the fact that the mass parameter of the gravitino on dS_4 is imaginary. Similarly, the conventional Majorana condition is **not** consistent with the Killing spinor equation on dS_4, but the sympelctic Majorana condition is. According to ref. [61], although the $N = 2$ dS Supergravity action is invariant under local SUSY, the photon kinetic term has the wrong sign, i.e. it is a ghost.

  On the other hand, by relaxing the requirement of unbroken SUSY, certain solutions are known. For example, an explicit dS_4 Supergravity action invariant under spontaneously broken local $N = 1$ SUSY was given in [63].[2] This model includes a massive gravitino (this has a real mass parameter), and is also consistent with unitarity. Another interesting example of stable dS vacuua corresponds to the matter-coupled Supergravity theories with $N = 2$ SUSY as described in ref. [66]. The main ingredients of the construction include non-Abelian non-compact gaugings, de Roo-Wagemans rotation angles and Fayet-Iliopoulos terms. However, the question of whether these vacuua can be lifted to string theory remains open.[3,4] Interestingly, as anticipated by [60], and shown in [9], the problems related to unitary dS_4 Supergravity with unbroken SUSY can be bypassed in two dimensions.

A way out for global SUSY in dS_4. In the case of global SUSY, it is possible to have unitary representations by enlarging the even symmetry algebra to the conformal algebra $so(4, 2) \supset so(4, 1)$. Now the anti-commutator of two supercharges closes on $so(4, 2)$ instead of $so(4, 1)$, and the trace $\sum_{A,i}\{Q_A^{(i)},\ Q^{(i)A\,\dagger}\}$ does **not** have to vanish. Thus, unitary representations exist [73]. Such unitary representations are realised in the case of superconformal field theories on a fixed dS_4 background spacetime, like the ones constructed in [74].

1.1 New results

In this paper, we present a new unitary supersymmetric quantum field theory (QFT) on a fixed dS_4 background that includes (a version of) the fields of the supergravity multiplet. In particular, we present:

The free *supersymmetric theory of the chiral graviton and chiral gravitino fields.*

[2]See also [64, 65].

[3]This is not an easy task, as the no-go theorem of ref. [67] presents serious obstacles for obtaining dS_4 vacuua from smooth, classical compactifications of higher-dimensional Supergravity. One possible way to circumvent the no-go theorem of [67] is to include orientifolds in the construction. De Sitter solutions of 10-dimensional supergravity have been obtained in this way; see e.g., [68] for a review and [69] for a recent example. Another approach is to consider time-dependent compactifications — see [70] for recent examples.

[4]Another important question concerns the non-perturbative existence of dS vacua in string theory — see, e.g., [71, 72].

By 'chiral' we mean that the corresponding field strengths are self-dual or anti-self-dual, and thus complex. The corresponding chiral graviton and chiral gravitino gauge potentials are complex and carry one complex propagating degree of freedom each. In this paper, we choose to work with the anti-self-dual case, without loss of generality. The global SUSY transformations of the chiral graviton and chiral gravitino are generated by the standard complex (Dirac) Killing spinors of dS_4.[5] However, the theory is not associated with a local action functional as the splitting of helicities needed for the theory to become chiral can be achieved only on-shell.

We show that the new supersymmetric theory of the chiral graviton and chiral gravitino avoids the obstacles [60, 61] to unitary global SUSY on dS_4 mentioned above because the commutator between two SUSY variations closes on the even algebra $so(4,2) \bigoplus u(1)$. Interestingly, unlike in the superconformal theory of ref. [74], the $so(4,2)$ symmetry in our theory is realised in an unconventional way that does not correspond to standard infinitesimal conformal transformations [75, 76].[6]

Another point worth emphasising is that, although the non-closure of the superalgebra on $so(4,1)$ is a necessary condition for unitarity, it is **not** sufficient. This is demonstrated with the following example. As we discuss in detail, the theory of a standard (i.e. non-chiral) complex graviton and a standard Dirac gravitino also carries a representation of the same superalgebra as in the case of the chiral supermultiplet, but the representation is **non-unitary**, despite the closure of the commutator between two SUSY transformations on $so(4,2) \bigoplus u(1)$. Interestingly, the unitarity of the theory is achieved by imposing the anti-self-duality constraint on the field strengths which removes all negative-norm states from the Hilbert space, i.e. it is the supermultiplet of the chiral graviton and chiral gravitino that carries a unitary representation of SUSY. The appearance of negative-norm states for helicity degrees of freedom that one would expect to be physical according to Minkowskian intuition [33, 34, 76], and the necessity for the anti-self-duality (or self-duality) constraint on the field strength to remove the negative-norm states, appears already in the quantum theory of the free Dirac gravitino field on dS_4, and we will discuss it in detail.

The unitarity of the chiral graviton-chiral gravitino supermultiplet is demonstrated in detail in two different ways:

- **Unitary SUSY on the space of mode solutions.** We study the action of our superalgebra on the space of standard gravitino [33, 34, 76] and graviton [77] physical mode solutions on global dS_4, which furnish discrete series UIRs of the dS algebra $so(4,1)$. The Minkowskian short-distance behaviour of the modes allows us to distinguish between generalised positive-frequency and negative-frequency solutions, as is customary for field theories on global dS_4 [33, 34, 76, 77]. We also recall that the discrete series

[5]The idea of dropping the reality conditions as an attempt to construct supersymmetric theories on dS_4 was first mentioned as a speculation by Deser and Waldron in ref. [43].

[6]Such conformal-like symmetries were first known to exist in the case of strictly massless gauge potentials of any spin on AdS_4 [75]. Recently, conformal-like symmetries — the ones used in the present paper — were found for strictly massless fermionic gauge potentials on dS_4 [76]. Moreover, it was shown that the strictly massless tensor-spinor mode solutions that form the fermionic discrete series UIRs of $so(4,1)$, also form UIRs of $so(4,2)$. This result is generalised to the case of graviton modes on dS_4 in the present paper, and plays a central role in the final unitary supersymmetric theory of the chiral graviton and chiral gravitino.

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UIRs of $so(4,1)$ formed by gravitino modes extend to UIRs of $so(4,2)$ [76] with the help of the conformal-like transformations [75, 76]. In addition, we show, for the first time, that the same happens for the graviton modes, i.e. the graviton modes furnishing discrete series UIRs of $so(4,1)$ also furnish $so(4,2)$ UIRs. Once we clarify how the spaces of fixed-helicity graviton and gravitino modes furnish UIRs of $so(4,2)(\bigoplus u(1))$, we show that SUSY is represented irreducibly on these spaces. That is, there is a direct sum of two irreducible SUSY representations: a negative-helicity representation with helicities $(-2,-3/2)$, and a positive-helicity representation with helicities $(+2,+3/2)$. We show that each of these irreducible SUSY representations is a UIR according to the group-theoretic definition of unitarity: we introduce positive-definite scalar products that are invariant under even generators $\in so(4,2)\bigoplus u(1)$, as well as under SUSY transformations.

Each of the two afore-mentioned SUSY UIRs can be formed by either positive-frequency or negative-frequency modes. However, the unitary supersymmetric QFT of the chiral graviton and chiral gravitino, discussed in section 6, includes: a positive-frequency single-particle Hilbert space furnishing only the SUSY UIR with helicities $(-2,-3/2)$, and a negative-frequency single-particle Hilbert space furnishing only the SUSY UIR with helicities $(+2,+3/2)$. Although allowed at the abstract representation theory level, the SUSY UIR with helicities $(+2,+3/2)$ is omitted from the positive-frequency sector, and the SUSY UIR with helicities $(-2,-3/2)$ is omitted from the negative-frequency sector. These states, which are removed from the physical state space with the help of the anti-self-duality constraint, have negative norms because of the curious features of the quantum gravitino field on dS_4. This phenomenon is discussed in detail in subsection 3.4.

- **Unitary SUSY on the QFT Fock space.** We quantise the chiral graviton and chiral gravitino fields by fully fixing the gauge, and then, we construct the quantum operators corresponding to the four complex SUSY Noether charges $Q[\epsilon]=\overline{\eta}^A Q_A$ — one for each Dirac Killing spinor ϵ (5.21) of dS_4. Unitarity is demonstrated by showing that these quantum charges generate the afore-mentioned SUSY UIRs by acting on single-particle states: a negative-helicity UIR in the positive-frequency sector, and a positive-helicity UIR in the negative-frequency sector. We also demonstrate the desired positivity of the anti-commutator of spinorial supercharges $\sum_{A=1}^4\{Q_A, Q^{A\,\dagger}\}$.

1.2 Key ingredients, new results as by-products, and outline

Before presenting the new unitary supersymmetric theory, we will discuss its key ingredients in detail: the free graviton and gravitino fields on global dS_4, their $so(4,1)$ and $so(4,2)$ representation-theoretic properties, their quantisation, and the properties of the (Dirac) Killing spinors on dS_4. In the process of discussing these ingredients, we will present various new results as by-products which will play a significant role in our unitary supersymmetric theory. Let us give the outline of the paper with emphasis on the new results that appear as by-products:

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- In section 2, we review the basics about the geometry of global dS_4, and we give our notation and conventions.

- In section 3, we study the gravitino field on dS_4. We introduce an alternative local action functional (3.8) for the Dirac gravitino that is **hermitian**, unlike the naive conventional Rarita-Schwinger action which is non-hermitian because of the imaginary mass parameter. In subsection 3.1, we review how the gravitino modes with helicities $-3/2$ and $+3/2$ on global dS_4 form a direct sum of two discrete series UIRs of $so(4,1)$ [33, 34]. Then, in subsection 3.2, we review how the gravitino modes with helicities $-3/2$ and $+3/2$ on global dS_4 form a direct sum of two UIRs of the conformal-like algebra $so(4,2)$ [76]. In subsection 3.3, we show, *for the first time*, that the hermitian action (3.8) is not only dS-invariant but also invariant under conformal-like transformations. In subsection 3.4, we study the quantisation of the Dirac gravitino on *global* dS_4, *for the first time* — a preliminary study of this question was initiated in [78]. We explain why unitarity requires the quantum gravitino field to be chiral. In particular, we show that the gravitino QFT associated with the hermitian local action functional (3.8) has a curious feature: half of the propagating helicities have negative norm and the other half have positive norm, as was already suggested by the mode analysis in refs. [34, 76]. We thus introduce the anti-self-duality constraint on the gravitino field strength, rendering the gravitino chiral, and this restricts the theory to its positive-norm sector.[7]

- In section 4, we study the graviton field on dS_4, with special emphasis on the chiral graviton, as this is the superpartner of the chiral gravitino needed for our unitary supersymmetric theory. In subsection 4.1, we review how the standard graviton modes with helicities -2 and $+2$ on global dS_4 form a direct sum of two discrete series UIRs of $so(4,1)$ [77]. In subsection 4.2, we discuss the conformal-like symmetry of the graviton on dS_4. In particular, in subsection 4.2.1, we present the expressions for the conformal-like transformations of the real graviton field on dS_4 generated by the five non-Killing conformal Killing vectors. We show that these are symmetries of the field equations. We also show that the symmetry algebra closes on $so(5,1)$ up to gauge transformations. Interestingly, the conformal-like transformations preserve neither the linearised Einstein-Hilbert action nor the Klein-Gordon inner product. In subsection 4.2.2, we discuss the conformal-like symmetry of the complex graviton (complex strictly massless spin-2 field). Redefining the conformal-like transformations of the real graviton by introducing a factor of $i = \sqrt{-1}$, we show that the symmetry algebra for the complex graviton closes on $so(4,2)$ up to gauge transformations. The complex graviton field equations are shown to be invariant under the conformal-like symmetries. We also show, for the first time, that the hermitian action functional for the complex graviton (4.49) is invariant under the conformal-like symmetries, and so is the Klein-Gordon inner product. In subsection 4.2.3, we show, for the first time, that the graviton modes of helicity -2 and $+2$ on global dS_4 furnish a direct sum of two $so(4,2)$ UIRs. In subsection 4.3, we quantise the chiral graviton field.

[7]This comes in contrast with the gravitino in Minkowski and AdS spacetimes, where choosing a chiral gravitino field is optional rather than necessary.

- In section 5, we review the basics about Dirac Killing spinors, and their bilinears, on dS_4. We explain how explicit expressions for Killing spinors on dS_4 can be obtained by analytically continuing Killing spinors on S^4. We also explain, for the first time, how the conformal-like $so(4,2)$ symmetry acts on dS Killing spinors.

- Section 6 focuses on our main result: the supersymmetric QFT of the chiral graviton and chiral gravitino on dS_4 is unitary. This is discussed in subsection 6.2. However, before presenting the unitary theory, in subsection 6.1 we begin by discussing the non-chiral supersymmetric theory of a complex graviton and a complex gravitino on dS_4, each with two complex propagating degrees of freedom. Although we show that this theory is non-unitary, many of its features will be inherited by its unitary chiral counterpart. Therefore, in subsection 6.1, we begin by presenting the global SUSY transformations for the non-chiral theory. We show that the field equations are SUSY-invariant, and so is the hermitian action functional of the theory. Then, we find the Noether charges and currents associated with SUSY invariance. We also calculate the commutator of two SUSY transformations, and we show that the SUSY algebra closes on $so(4,2) \bigoplus u(1)$. We also find the SUSY transformations of the gauge-invariant field strengths. We show that duality transformations commute with SUSY transformations. Then, the non-unitarity of the non-chiral theory is discussed. Finally, in subsection 6.2, we present our unitary supersymmetric theory of the chiral graviton and chiral gravitino, and we clarify which features are inherited from the non-chiral theory of subsection 6.1. The unitarity of our chiral supersymmetric theory is demonstrated explicitly at the level of mode solutions in subsection 6.2.1. The unitarity of the theory in the supersymmetric QFT Fock space is demonstrated in subsection 6.2.2.

- In section 7, we discuss possible future directions.

There are five appendices. In appendix A, we review the classification of the $so(4,1)$ UIRs. The rest of the appendices focus on technical details that have been omitted from the main text.

2 Background material on global dS geometry, notation, and conventions

The solutions of the field equations used in this paper will be expressed in the global slicing of dS_4. In these coordinates, the line element of dS_4 is expressed as [7]

$$ds^2 = -dt^2 + \cosh^2 t \, d\Omega^2. \tag{2.1}$$

We have denoted the line element of S^3 as $d\Omega^2$, which can be parameterised as

$$d\Omega^2 = d\theta_3^2 + \sin^2\theta_3 \left(d\theta_2^2 + \sin^2\theta_2 \, d\theta_1^2\right), \tag{2.2}$$

where $0 \leq \theta_j \leq \pi$ (for $j = 2,3$) and $0 \leq \theta_1 < 2\pi$. We will also use the following notation for a point on S^3: $\boldsymbol{\theta_3} \equiv (\theta_3, \theta_2, \theta_1)$. The conformal time τ is defined by $\tan\tau = \sinh t$ $(-\pi/2 < \tau < \pi/2)$, and the metric (2.1) can also be given as

$$ds^2 = \sec^2\tau \, (-d\tau^2 + d\Omega^2). \tag{2.3}$$

The 'curved space gamma matrices', $\gamma^\mu(x)$, are defined with the use of the vierbein fields as $\gamma^\mu(x) = e^\mu{}_b(x)\gamma^b$, where γ^b $(b = 0,1,2,3)$ are the flat-space gamma matrices. The gamma matrices $\gamma^\mu(x)$ satisfy the anti-commutation relations

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\,\mathbf{1}, \tag{2.4}$$

where $\mathbf{1}$ is the 4-dimensional spinorial identity matrix. The vierbein and co-vierbein fields satisfy

$$e_\mu{}^a\, e_\nu{}^b\eta_{ab} = g_{\mu\nu}, \quad e^\mu{}_a\, e_\mu{}^b = \delta^b_a, \tag{2.5}$$

where $\eta_{ab} = diag(-1,1,1,1)$. The fifth gamma matrix γ^5 is determined as [79]

$$\gamma^{[a}\gamma^b\gamma^c\gamma^{d]} = -i\varepsilon^{abcd}\gamma^5, \tag{2.6}$$

where $\varepsilon_{\mu\nu\rho\sigma}$ are the components of the dS_4 volume element. In the vierbein basis, we have $\varepsilon^{0123} = -1$, while in the coordinate basis we have $\varepsilon^{t\theta_1\theta_2\theta_3} = -\frac{1}{\sqrt{-g}}$, where g is the determinant of the dS metric. Equivalently

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{2.7}$$

The matrix γ^5 anti-commutes with the other four gamma matrices, and, hence, with the Dirac operator on dS_4.

Our sign convention for the 'vierbein postulate' is:

$$\partial_\mu e^\rho{}_b + \Gamma^\rho_{\mu\sigma}e^\sigma{}_b - \omega_\mu{}^c{}_b\, e^\rho{}_c = 0. \tag{2.8}$$

The covariant derivative acts on vector-spinors as

$$\nabla_\nu\Psi_\mu = \left(\partial_\nu + \frac{1}{4}\omega_{\nu bc}\gamma^{bc}\right)\Psi_\mu - \Gamma^\lambda{}_{\nu\mu}\Psi_\lambda, \tag{2.9}$$

where $\omega_{\nu bc} = \omega_{\nu[bc]} = e_\nu{}^a\omega_{abc}$ are the components of the spin connection. The gamma matrices are covariantly constant, $\nabla_\nu\gamma_\mu = 0$. This can be easily checked by computing their covariant derivative as

$$\nabla_\nu\gamma_\mu = \partial_\nu\gamma_\mu + \frac{1}{4}\omega_{\nu bc}[\gamma^{bc},\gamma_\mu] - \Gamma^\lambda{}_{\nu\mu}\gamma_\lambda.$$

Details on the Christoffel symbols, spin connection, and vierbein on global de Sitter, as well as our representation of gamma matrices, can be found in appendix B.

De Sitter spacetime has ten Killing vectors, ξ^μ,

$$\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu = 0, \tag{2.10}$$

generating the dS algebra, $so(4,1)$, and five genuine conformal Killing vectors, V^μ, satisfying

$$\nabla_\mu V_\nu + \nabla_\nu V_\mu = g_{\mu\nu}\frac{\nabla^\alpha V_\alpha}{2} \tag{2.11}$$

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with $\nabla^\alpha V_\alpha \neq 0$. The 15-dimensional Lie algebra generated by the dS Killing vectors and the genuine conformal Killing vectors is isomorphic to the conformal algebra $so(4,2)$. The $so(4,2)$ Lie brackets are given by

$$\begin{aligned} [\xi, \xi']^\mu &= \pounds_\xi \xi'^\mu, \\ [\xi, V]^\mu &= \pounds_\xi V^\mu, \\ [V, V']^\mu &= \pounds_V V'^\mu, \end{aligned} \tag{2.12}$$

where $\pounds$ is the Lie derivative, ξ^μ and ξ'^μ are any two Killing vectors, V^μ and V'^μ are any two genuine conformal Killing vectors, $\pounds_\xi V^\mu$ is a genuine conformal Killing vector, while $\pounds_V V'^\mu$ is a Killing vector. Note that each of the five genuine conformal Killing vectors of dS_4 can be expressed as:

$$V_\mu = \nabla_\mu \phi_V, \tag{2.13}$$

where the scalar function ϕ_V satisfies[8]

$$\nabla_\mu V_\nu = \nabla_\mu \nabla_\nu \phi_V = -g_{\mu\nu} \phi_V. \tag{2.14}$$

There are five such independent functions: $\phi_{V^{(0)}}, \phi_{V^{(1)}}, \ldots, \phi_{V^{(4)}}$. These functions are related to the embedding space coordinates for dS_4. Specifically, by embedding dS_4 as a hyperboloid in 5-dimensional Minkowski space,

$$-(X^0)^2 + \sum_{A=1}^{4} (X^A)^2 = 1,$$

we have $\phi_{V^{(0)}} = X^0$, and $\phi_{V^{(A)}} = X^A$ (for $A = 1, \ldots, 4$).

Notation and conventions. We use the mostly plus metric sign convention for dS_4. Lowercase Greek tensor indices refer to components with respect to the 'coordinate basis'. Coordinate basis tensor indices on S^3 are denoted as $\tilde{\mu}, \tilde{\nu}, \ldots$. Lowercase Latin tensor indices refer to components with respect to the vielbein basis. Repeated indices are summed over. Spinor indices are suppressed, except in the case of spinorial supercharges. We denote the symmetrisation of indices with the use of round brackets, e.g., $A_{(\mu\nu)} = (A_{\mu\nu} + A_{\nu\mu})/2$, and the anti-symmetrisation with the use of square brackets, e.g., $A_{[\mu\nu]} = (A_{\mu\nu} - A_{\nu\mu})/2$. Complex conjugation is denoted using the symbol $*$ and hermitian conjugation using $\dagger$. Totally anti-symmetrised products of gamma matrices are denoted as: $\gamma^{bc} = \gamma^{[b}\gamma^{c]}$, $\gamma^{bcd} = \gamma^{[b}\gamma^c\gamma^{d]}$ and $\gamma^{abcd} = \gamma^{[a}\gamma^b\gamma^c\gamma^{d]}$. For a tensor (or tensor-spinor with suppressed spinor indices) $B_{\mu_1\nu_1\ldots}$ that is anti-symmetric under the exchange of the tensor indices $\mu_1 \leftrightarrow \nu_1$, the duality operation is denoted using the 'wide tilde' symbol: $\widetilde{B}_{\mu_1\nu_1\ldots} = \frac{1}{2}\varepsilon_{\mu_1\nu_1}{}^{\alpha\beta} B_{\alpha\beta\ldots}$, with $\widetilde{\widetilde{B}}_{\mu_1\nu_1\ldots} = -B_{\mu_1\nu_1\ldots}$. For quantities that depend on two spacetime points, x and x', primed tensor indices are associated with point x' and unprimed indices with point x. The real graviton gauge potential is denoted as $h_{\mu\nu}$. The symbol h does **not** stand for the trace of $h_{\mu\nu}$ — see, e.g., eq. (4.2). The complex graviton gauge potential is denoted with the symbol $\mathfrak{h}$, as $\mathfrak{h}_{\mu\nu}$. The symbol $\mathfrak{h}$

[8] See, e.g., ref. [80].

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in this paper does **not** stand for the trace of $\mathfrak{h}_{\mu\nu}$ — see, e.g., eq. (4.36). The superscript '(TT)' will be used to indicate that the graviton or gravitino gauge potential is in the transverse-traceless gauge — 'TT gauge' for short [see, e.g., eqs. (3.13) and (4.39)]. TT graviton mode solutions are denoted as $\varphi_{\mu\nu}$, where labels indicating particular solutions will be also introduced — see, e.g., eq. (4.10).

3 Free gravitino gauge potential on dS_4, UIRs of $so(4,1)$ and $so(4,2)$, quantisation and (anti-)self-duality

Background material for the gravitino on $\mathbf{dS_4}$. Let us start with some useful observations, some familiar and others less commonly recognised, concerning the massless Rarita-Schwinger (RS) field (gauge potential), also known as gravitino, on a fixed dS spacetime.

The free gravitino field on dS_4 is described by a vector-spinor gauge potential satisfying the Rarita-Schwinger (RS) equation with an imaginary mass parameter[9] [43]

$$\gamma^{\mu\rho\sigma}\left(\nabla_\rho + \frac{i}{2}\gamma_\rho\right)\Psi_\sigma = 0, \tag{3.1}$$

where [79]

$$\gamma^{\mu\rho\sigma} = \gamma^{[\mu}\gamma^{\rho}\gamma^{\sigma]} = \gamma^{\mu}\gamma^{\rho}\gamma^{\sigma} - g^{\mu\rho}\gamma^{\sigma} - g^{\rho\sigma}\gamma^{\mu} + g^{\mu\sigma}\gamma^{\rho}, \tag{3.2}$$

and hence,

$$\gamma^{\mu}\left(\nabla\!\!\!/ - i\right)\gamma^{\beta}\Psi_\beta - \gamma^{\mu}\nabla^{\beta}\Psi_\beta - \nabla^{\mu}\gamma^{\beta}\Psi_\beta + \left(\nabla\!\!\!/ + i\right)\Psi^{\mu} = 0.$$

Let $\overline{\Psi}_\mu = i\Psi^{\dagger}_{\mu}\gamma^0$ be the Dirac conjugate of Ψ_μ. The field equation for $\overline{\Psi}_\mu$ can be found by taking the hermitian conjugate of eq. (3.1) as

$$\left(\nabla_\rho\overline{\Psi}_\sigma + \frac{i}{2}\overline{\Psi}_\sigma\gamma_\rho\right)\gamma^{\mu\rho\sigma} = 0, \tag{3.3}$$

where we have used $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$. The 'strict masslessness' of the gravitino manifests itself by the fact that the field equation (3.1) is invariant under infinitesimal gauge transformations of the form

$$\delta^{\text{gauge}}(\lambda)\,\Psi_\mu = \left(\nabla_\mu + \frac{i}{2}\,\gamma_\mu\right)\lambda, \tag{3.4}$$

where λ are spinor gauge functions. Similarly, the equation for $\overline{\Psi}_\mu$ (3.3) is invariant under the gauge transformations

$$\delta^{\text{gauge}}(\lambda)\,\overline{\Psi}_\mu \equiv \left(\delta^{\text{gauge}}(\lambda)\,\Psi_\mu\right)^\dagger i\gamma^0 = \nabla_\mu\overline{\lambda} + \frac{i}{2}\,\overline{\lambda}\gamma_\mu, \tag{3.5}$$

where $\overline{\lambda} = i\lambda^\dagger\gamma^0$. For later convenience let us define

$$\mathcal{R}^{\mu}(\Psi) \equiv \gamma^{\mu\rho\sigma}\left(\nabla_\rho + \frac{i}{2}\gamma_\rho\right)\Psi_\sigma, \tag{3.6}$$

which will be understood to be non-zero off-shell and zero on-shell. This is gauge invariant off-shell: $\mathcal{R}^{\mu}(\delta^{\text{gauge}}(\lambda)\,\Psi) = 0$.

[9] A massive RS field satisfies eq. (3.1) with i replaced by a real mass parameter $\mathcal{M}$: $\gamma^{\mu\rho\sigma}\left(\nabla_\rho + \frac{\mathcal{M}}{2}\gamma_\rho\right)\Psi_\sigma = 0$.

Problems of the conventional RS action functional and an alternative. The conventional RS action [79]

$$S_{RS} = -\int d^4x\,\sqrt{-g}\,\overline{\Psi}_\mu\,\mathcal{R}^\mu = -\int d^4x\,\sqrt{-g}\,\overline{\Psi}_\mu\gamma^{\mu\rho\sigma}\left(\nabla_\rho + \frac{i}{2}\gamma_\rho\right)\Psi_\sigma \tag{3.7}$$

is not hermitian in de Sitter spacetime because of the imaginary mass term. This non-hermiticity leads to some problematic consequences: although the Euler-Lagrange equation for Ψ_μ derived from the action (3.7) is the desired RS equation (3.1), the Euler-Lagrange equation for $\overline{\Psi}_\mu$ is

$$\left(\nabla_\rho\overline{\Psi}_\sigma - \frac{i}{2}\overline{\Psi}_\sigma\gamma_\rho\right)\gamma^{\mu\rho\sigma} = 0,$$

which does **not** correspond to the hermitian conjugate of (3.1) as it has the wrong sign for the mass term [compare with eq. (3.3)]. However, the alternative action [78]

$$S_{\frac{3}{2}} = -\int d^4x\,\sqrt{-g}\,\overline{\Psi}_\mu\,\gamma^5\,\mathcal{R}^\mu(\Psi) \tag{3.8}$$

is hermitian and its Euler-Lagrange equations for Ψ_μ and $\overline{\Psi}_\mu$ are eqs. (3.1) and (3.3), respectively, as consistency requires. Moreover, interestingly, the conventional RS action (3.7) is **not** invariant under the gauge transformation of $\overline{\Psi}_\mu$ (3.5), but the alternative action (3.8) is.

Incompatibility with the Majorana condition. It is known that the gravitino equation (3.1) is not consistent with the Majorana condition because of the imaginary mass parameter (however, a symplectic Majorana condition is possible — see e.g., ref. [61]). This can be easily verified by recalling the definition of the charge conjugate of Ψ_μ that preserves the RS equation (3.1):

$$\Psi^C_\mu \equiv B_-^{-1}\Psi^*_\mu, \tag{3.9}$$

where the matrix B_- satisfies

$$-\left(\gamma^\mu\right)^* = B_-\gamma^\mu B_-^{-1}, \tag{3.10}$$

and, in our conventions $B_- = \gamma^0\gamma^2\gamma^3 = i\gamma^5\gamma^1$ (defined up to a phase) — see also appendix B. Although Ψ^C_μ satisfies the same equation as Ψ_μ, i.e.

$$\gamma^{\mu\rho\sigma}\left(\nabla_\rho + \frac{i}{2}\gamma_\rho\right)\Psi^C_\sigma = 0, \tag{3.11}$$

it is easy to check that one cannot use the matrix B_- to define a consistent reality condition. In particular, by applying charge conjugation twice we find

$$\left(\Psi^C_\mu\right)^C = -\Psi_\mu. \tag{3.12}$$

Because of this, Ψ_μ cannot be Majorana. In this paper, we are considering only Dirac vector-spinors.[10]

[10]On the other hand, one can define charge conjugation using the matrix $B_+ = \gamma^5 B_-$, where $(\gamma^\mu)^* = B_+\gamma^\mu B_+^{-1}$. In this case, charge conjugation is defined as: $\Psi^{C+}_\mu \equiv B_+^{-1}\Psi^*_\mu$. Although the property $\left(\Psi^{C+}_\mu\right)^{C+} = \Psi_\mu$ holds and a Majorana condition can be introduced, the charge conjugate, Ψ^{C+}_μ, does not preserve the field equation. To be precise, Ψ^{C+}_μ satisfies eq. (3.11) with the opposite sign for the mass parameter.

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3.1 Discrete series UIRs of $so(4,1)$ in the space of gravitino modes

Let us review how the gravitino positive frequency mode functions on global dS_4 form a direct sum of discrete series UIRs of the dS algebra, $so(4,1)$ [33, 34, 76]. As is well known, if the space of (physical) positive frequency mode solutions forms a UIR, it can be identified with the single-particle Hilbert space of the corresponding free quantum field theory. Thus, this subsection sets the stage for the quantisation of the gravitino field, which is carried out in subsection 3.4.

The gravitino mode solutions that form the $so(4,1)$ UIRs are solutions of the RS equation (3.1) in the transverse-traceless (TT) gauge ($\nabla^\alpha \Psi^{(\mathrm{TT})}_\alpha = \gamma^\alpha \Psi^{(\mathrm{TT})}_\alpha = 0$). The field equations and the TT gauge conditions read [33, 43, 81]

$$\begin{aligned} (\nabla\!\!\!\!/ + i)\, \Psi^{(\mathrm{TT})}_\mu &= 0, \\ \nabla^\alpha \Psi^{(\mathrm{TT})}_\alpha &= 0, \qquad\qquad \gamma^\alpha \Psi^{(\mathrm{TT})}_\alpha = 0. \end{aligned} \tag{3.13}$$

Only a subset of the initial gauge transformations (3.4) preserve eqs. (3.13). These are the restricted gauge transformations:

$$\delta^{\mathrm{gauge}}_{\mathrm{res}}(X)\, \Psi^{(\mathrm{TT})}_\mu = \left(\nabla_\mu + \frac{i}{2}\, \gamma_\mu \right) X, \tag{3.14}$$

where the spinor gauge functions satisfy

$$\nabla\!\!\!\!/ X = -2i\, X. \tag{3.15}$$

The generators of $so(4,1)$ (i.e. the Killing vectors of dS_4) act on vector-spinors Ψ_μ via the Lie-Lorentz derivative [82]

$$\mathbb{L}_\xi \Psi_\mu = \xi^\nu \nabla_\nu \Psi_\mu + (\nabla_\mu \xi^\nu)\, \Psi_\nu + \frac{1}{4} (\nabla_\kappa \xi_\lambda) \gamma^{\kappa\lambda} \Psi_\mu, \tag{3.16}$$

where ξ^μ is any Killing vector of dS_4. If Ψ_μ is a solution of eq. (3.13), then so is $\mathbb{L}_\xi \Psi_\mu$. Moreover, the Lie-Lorentz derivative preserves the Lie bracket between any two Killing vectors ξ^μ and ξ'^μ [82]

$$[\mathbb{L}_\xi, \mathbb{L}_{\xi'}] \Psi_\mu = \mathbb{L}_{[\xi,\xi']} \Psi_\mu. \tag{3.17}$$

This means that the space of gravitino mode solutions of eq. (3.13) is a representation space for the dS algebra $so(4,1)$.

Equations (3.13) admit **physical** and TT **pure-gauge** mode solutions. The pure-gauge modes can be identified with zero in the solution space, while the physical modes are the ones forming the direct sum of discrete series UIRs of $so(4,1)$ [33, 34]. Some details are in order.

TT pure-gauge gravitino modes. The TT pure-gauge modes are expressed in the form[11]

$$\psi^{(pg)}_\mu = \nabla_\mu X + \frac{i}{2} \gamma_\mu X, \tag{3.18}$$

[11] We have omitted the quantum number labels from the pure-gauge modes for convenience. Details about these labels can be found in refs. [33, 34, 76].

where

$$(\not\nabla + 2i)\, X = 0, \tag{3.19}$$

in agreement with eqs. (3.14) and (3.15). Explicit expressions for the spinors X can be found in ref. [34].

Physical gravitino modes. The physical modes come in two helicities: negative $(-3/2)$ and positive $(+3/2)$ helicity modes [33, 34, 76]. In global coordinates (2.1), the physical modes with negative and positive helicity are given by [33, 34, 76][12]

$$\psi_t^{(phys,\,-\ell;\,m;k)}(t,\boldsymbol{\theta}_3) = 0, \qquad \psi_{\tilde{\mu}}^{(phys,\,-\ell;\,m;k)}(t,\boldsymbol{\theta}_3) = \left(\frac{\ell+2}{2(\ell+1)}\right)^{1/2} \begin{pmatrix} \alpha_\ell(t)\, \tilde{\psi}_{-\tilde{\mu}}^{(\ell;m;k)}(\boldsymbol{\theta_3}) \\ -i\beta_\ell(t)\, \tilde{\psi}_{-\tilde{\mu}}^{(\ell;m;k)}(\boldsymbol{\theta_3}) \end{pmatrix}, \tag{3.20}$$

and

$$\psi_t^{(phys,\,+\ell;\,m;k)}(t,\boldsymbol{\theta}_3) = 0, \qquad \psi_{\tilde{\mu}}^{(phys,\,+\ell;\,m;k)}(t,\boldsymbol{\theta}_3) = \left(\frac{\ell+2}{2(\ell+1)}\right)^{1/2} \begin{pmatrix} i\beta_\ell(t)\, \tilde{\psi}_{+\tilde{\mu}}^{(\ell;m;k)}(\boldsymbol{\theta_3}) \\ -\alpha_\ell(t)\, \tilde{\psi}_{+\tilde{\mu}}^{(\ell;m;k)}(\boldsymbol{\theta_3}) \end{pmatrix}, \tag{3.21}$$

respectively, where $\tilde{\mu}$ is a vector index on S^3, while ℓ, m and k are angular momentum quantum numbers corresponding to the chain of subalgebras $so(4) \supset so(3) \supset so(2)$ with $\ell \in \{1,2,\ldots\}$, $m \in \{1,2,\ldots,\ell\}$ and $k \in \{-m-1,-m,\ldots,0,\ldots,m\}$. The functions describing the time dependence are conveniently expressed in terms of the variable

$$x(t) = \frac{\pi}{2} - it \tag{3.22}$$

as

$$\alpha_\ell(t) = \left(\sin\frac{x(t)}{2}\right)^{\ell+1} \left(\cos\frac{x(t)}{2}\right)^{-\ell-2} \left(1 - \frac{\sin^2\frac{x(t)}{2}}{\ell+2}\right), \tag{3.23}$$

$$\beta_\ell(t) = \frac{1}{\ell+2} \left(\sin\frac{x(t)}{2}\right)^{\ell+2} \left(\cos\frac{x(t)}{2}\right)^{-\ell-1}, \tag{3.24}$$

where

$$\cos\frac{x(t)}{2} = \left(\sin\frac{x(t)}{2}\right)^* = \frac{\sqrt{2}}{2}\left(\cosh\frac{t}{2} + i\sinh\frac{t}{2}\right), \tag{3.25}$$

$$\sin^2\frac{x(t)}{2} = \frac{1 - i\sinh t}{2}. \tag{3.26}$$

We note that

$$\frac{\partial}{\partial x}\beta_\ell(t) = i\frac{\partial}{\partial t}\beta_\ell(t) = \frac{1}{2}\alpha_\ell(t). \tag{3.27}$$

[12] In ref. [76], the functions $\alpha_\ell(t)$ and $\beta_\ell(t)$ are denoted as $\alpha_\ell^{(1)}(t)$ and $\beta_\ell^{(1)}(t)$, respectively, while in ref. [33] they are denoted as $\Phi_{M\ell}^{(-1)}(t)$ and $\Psi_{M\ell}^{(-1)}(t)$ (with $M = i$), respectively. In [33, 76], these functions are expressed in terms of the Gauss hypergeometric functions. However, in the present paper, as the hypergeometric series terminates, we have chosen to express the functions in a simpler form in our eqs. (3.23) and (3.24).

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It is also useful to note that with the conformal time τ defined by $\tan\tau = \sinh t$ we have

$$\cos\frac{x(t)}{2} = \frac{e^{i\tau/2}}{\sqrt{2\cos\tau}}, \tag{3.28}$$

$$\sin\frac{x(t)}{2} = \frac{e^{-i\tau/2}}{\sqrt{2\cos\tau}}, \tag{3.29}$$

so that

$$\alpha_\ell(t) = \frac{e^{-i(\ell+\frac{3}{2})\tau}}{\sqrt{2\cos\tau}}\left(2\cos\tau - \frac{e^{-i\tau}}{\ell+2}\right), \tag{3.30}$$

$$\beta_\ell(t) = \frac{e^{-i(\ell+\frac{3}{2})\tau}}{(\ell+2)\sqrt{2\cos\tau}}. \tag{3.31}$$

Transverse-traceless vector-spinor spherical harmonics on S^3. The $\boldsymbol{\theta_3}$-dependence of the physical modes in eqs. (3.20) and (3.21) is given by the transverse-traceless vector-spinor spherical harmonics on S^3, $\tilde{\psi}^{(\ell;m;k)}_{\pm\tilde{\mu}}(\boldsymbol{\theta_3})$. These satisfy [34, 83, 84]

$$\begin{aligned}\tilde{\nabla\!\!\!/}\,\tilde{\psi}^{(\ell;m;k)}_{\pm\tilde{\mu}}(\boldsymbol{\theta_3}) &= \pm i\left(\ell+\frac{3}{2}\right)\tilde{\psi}^{(\ell;m;k)}_{\pm\tilde{\mu}}(\boldsymbol{\theta_3}), \qquad \ell\in\{1,2,\dots\}\\ \tilde{\gamma}^{\tilde{\mu}}\tilde{\psi}^{(\ell;m;k)}_{\pm\tilde{\mu}}(\boldsymbol{\theta_3}) &= \tilde{\nabla}^{\tilde{\mu}}\tilde{\psi}^{(\ell;m;k)}_{\pm\tilde{\mu}}(\boldsymbol{\theta_3}) = 0,\end{aligned} \tag{3.32}$$

where the tildes have been used to denote quantities on S^3. They are normalised with the standard inner product on S^3 [34]:

$$\begin{aligned}\int_{S^3}\sqrt{\tilde{g}}\,d\boldsymbol{\theta_3}\;\tilde{g}^{\tilde{\mu}\tilde{\nu}}\;\tilde{\psi}^{(\ell';m';k')}_{\sigma'\,\tilde{\mu}}(\boldsymbol{\theta_3})^\dagger\;\tilde{\psi}^{(\ell;m;k)}_{\sigma\,\tilde{\nu}}(\boldsymbol{\theta_3})\\ = \delta_{\sigma\sigma'}\,\delta_{\ell\ell'}\,\delta_{m\,m'}\delta_{kk'},\end{aligned} \tag{3.33}$$

where $\sigma,\sigma'\in\{+,-\}$ and $d\boldsymbol{\theta_3}\equiv d\theta_3 d\theta_2 d\theta_1$. For each value of $\ell\in\{1,2,\dots\}$, the set $\{\tilde{\psi}^{(\ell;m;k)}_{+\tilde{\mu}}\}$ forms a $so(4)$ representation with highest weight given by [83]:

$$\vec{f}^{(+3/2)}_\ell = \left(\ell+\frac{1}{2},\frac{3}{2}\right). \tag{3.34}$$

The set $\{\tilde{\psi}^{(\ell;m;k)}_{-\tilde{\mu}}\}$ forms a $so(4)$ representation with highest weight given by [83]:

$$\vec{f}^{(-3/2)}_\ell = \left(\ell+\frac{1}{2},-\frac{3}{2}\right). \tag{3.35}$$

Let $\tilde{\varepsilon}_{\tilde{\mu}\tilde{\nu}\tilde{\alpha}}$ denote the invariant 3-form on S^3 with $\tilde{\varepsilon}_{\theta_1\theta_2\theta_3} = \sqrt{\tilde{g}}$, where $\tilde{g}$ is the determinant of the S^3 metric (2.2). Let us also introduce the duality operator (helicity operator) acting on the vector-spinor spherical harmonics (3.32) as [77]:

$$\frac{1}{\ell+3/2}\,\tilde{\varepsilon}_{\tilde{\mu}\tilde{\nu}\tilde{\alpha}}\tilde{\nabla}^{\tilde{\nu}}\tilde{\psi}^{(\ell;m;k)\tilde{\alpha}}_{\pm}. \tag{3.36}$$

This is the analogue of the flat-space helicity operator. Using $\tilde{\varepsilon}_{\tilde{\mu}\tilde{\nu}\tilde{\alpha}} = -i\tilde{\gamma}_{[\tilde{\mu}}\tilde{\gamma}_{\tilde{\nu}}\tilde{\gamma}_{\tilde{\alpha}]}$ [79], we find that the duality operator is proportional to the Dirac operator on S^3, as:

$$\frac{1}{\ell+3/2}\tilde{\varepsilon}_{\tilde{\mu}\tilde{\nu}\tilde{\alpha}}\tilde{\nabla}^{\tilde{\nu}}\tilde{\psi}^{(\ell;m;k)\tilde{\alpha}}_{\pm} = -\frac{i}{\ell+3/2}\,\tilde{\nabla\!\!\!/}\,\tilde{\psi}^{(\ell;m;k)}_{\pm\tilde{\mu}} = \pm\tilde{\psi}^{(\ell;m;k)}_{\pm\tilde{\mu}}. \tag{3.37}$$

Thus, the modes $\tilde{\psi}^{(\ell;m;k)}_{+\tilde{\mu}}$ are self-dual, while the modes $\tilde{\psi}^{(\ell;m;k)}_{-\tilde{\mu}}$ are anti-self-dual. This notion of (anti-)self-duality should not be confused with the notion of (anti-)self-duality defined using $\varepsilon_{\mu\nu\rho\sigma}$ on dS_4 — see e.g., eqs. (3.64) and (4.58).

Positive and negative frequency. The mode functions (3.20) and (3.21) are the analogues of positive frequency modes, as for short wavelengths, $\ell \gg 1$, they satisfy [76]

$$\frac{\partial}{\partial t}\psi^{(phys,\,\pm\ell;\,m;k)}_{\mu}(t,\boldsymbol{\theta}_3) \sim -i\frac{\ell}{\cosh t}\psi^{(phys,\,\pm\ell;\,m;k)}_{\mu}(t,\boldsymbol{\theta}_3). \tag{3.38}$$

Eq. (3.13) also admits physical transverse-traceless solutions that are the analogues of negative frequency modes given by [76]

$$v^{(phys,\,-\ell;\,m;k)}_{t}(t,\boldsymbol{\theta}_3)=0,\qquad v^{(phys,\,-\ell;\,m;k)}_{\tilde{\mu}}(t,\boldsymbol{\theta}_3)=\left(\frac{\ell+2}{2(\ell+1)}\right)^{1/2}\begin{pmatrix} i\beta^{*}_{\ell}(t)\,\tilde{\psi}^{(\ell;m;k)}_{-\tilde{\mu}}(\boldsymbol{\theta_3})\\ \alpha^{*}_{\ell}(t)\,\tilde{\psi}^{(\ell;m;k)}_{-\tilde{\mu}}(\boldsymbol{\theta_3})\end{pmatrix}, \tag{3.39}$$

and

$$v^{(phys,\,+\ell;\,m;k)}_{t}(t,\boldsymbol{\theta}_3)=0,\qquad v^{(phys,\,+\ell;\,m;k)}_{\tilde{\mu}}(t,\boldsymbol{\theta}_3)=\left(\frac{\ell+2}{2(\ell+1)}\right)^{1/2}\begin{pmatrix} \alpha^{*}_{\ell}(t)\,\tilde{\psi}^{(\ell;m;k)}_{+\tilde{\mu}}(\boldsymbol{\theta_3})\\ i\beta^{*}_{\ell}(t)\,\tilde{\psi}^{(\ell;m;k)}_{+\tilde{\mu}}(\boldsymbol{\theta_3})\end{pmatrix}. \tag{3.40}$$

The negative frequency modes can be obtained by applying charge conjugation (3.9) to the positive frequency modes. For short wavelengths, $\ell \gg 1$, they satisfy the generalised negative frequency condition

$$\frac{\partial}{\partial t}v^{(phys,\,\pm\ell;\,m;k)}_{\mu}(t,\boldsymbol{\theta}_3) \sim +i\frac{\ell}{\cosh t}v^{(phys,\,\pm\ell;\,m;k)}_{\mu}(t,\boldsymbol{\theta}_3). \tag{3.41}$$

Note. The field strength (3.65) calculated for the positive frequency modes of helicity $-3/2$, $\psi^{(phys,\,-\ell;\,m;k)}_{\mu}$, is anti-self-dual, and so is the field strength for the negative frequency modes of helicity $+3/2$, $v^{(phys,\,+\ell;\,m;k)}_{\mu}$. Similarly, the field strength (3.65) calculated for the positive frequency modes of helicity $+3/2$, $\psi^{(phys,\,+\ell;\,m;k)}_{\mu}$, is self-dual, and so is the field strength for the negative frequency modes of helicity $-3/2$, $v^{(phys,\,-\ell;\,m;k)}_{\mu}$. The mode expansion of the field strength and (anti)-self-duality are discussed further in subsection 3.4.

Discrete series UIRs of $so(4,1)$. The two sets of (positive frequency) physical modes $\{\psi^{(phys,\,-\ell;\,m;k)}_{\mu}\}$ and $\{\psi^{(phys,\,+\ell;\,m;k)}_{\mu}\}$ separately form two irreducible representations of $so(4,1)$ (and, thus, a different choice for a scalar product is allowed for each set) [33, 34, 76]. This can be understood as follows. First, it is clear that the modes $\{\psi^{(phys,\,+\ell;\,m;k)}_{\mu}\}$ do not mix with the modes $\{\psi^{(phys,\,-\ell;\,m;k)}_{\mu}\}$ under any $so(4)$ transformation as they belong to different $so(4)$ representations — the former correspond to the $so(4)$ highest weights in (3.34), while the latter to the ones in (3.35). Moreover, under the infinitesimal isometry generated by the boost Killing vector

$$B=B^{\mu}\partial_{\mu}=\cos\theta_3\,\frac{\partial}{\partial t}-\tanh t\sin\theta_3\,\frac{\partial}{\partial\theta_3}, \tag{3.42}$$

physical modes of a given helicity transform only among themselves. To be specific, they transform as [34, 76]:

$$\begin{aligned}\mathbb{L}_B \psi_\mu^{(phys,\,\pm\ell;m;k)} = &-\frac{i}{2}\sqrt{(\ell-m+1)(\ell+m+3)}\,\psi_\mu^{(phys,\,\pm(\ell+1)\,;m;k)} \\ &-\frac{i}{2}\sqrt{(\ell-m)(\ell+m+2)}\,\psi_\mu^{(phys,\,\pm(\ell-1)\,;m;k)} + (\text{pure-gauge}), \end{aligned} \tag{3.43}$$

where the term '(pure-gauge)' is a TT pure-gauge mode (3.18). As the $so(4,1)$ algebra can be generated using only the $so(4)$ generators and just one dS boost, we conclude that the modes $\{\psi_\mu^{(phys,\,-\ell;\,m;k)}\}$ and $\{\psi_\mu^{(phys,\,+\ell;\,m;k)}\}$ separately form irreducible representations of $so(4,1)$ with the equivalence relation: if for any two physical modes, $\psi_\mu^{(1)}$ and $\psi_\mu^{(2)}$, the difference $\psi_\mu^{(1)} - \psi_\mu^{(2)}$ is a linear combination of TT pure-gauge modes, then $\psi_\mu^{(1)}$ and $\psi_\mu^{(2)}$ belong to the same equivalence class. This equivalence relation is introduced because the pure-gauge modes can be identified with zero, as will become clear shortly. Note that eq. (3.43) agrees with the expression for the infinitesimal boost matrix elements in the discrete series UIRs of $so(4,1)$ with $\Delta = 5/2$ and $s = 3/2$ [55, 56] in the 'modern notation' for labels — see appendix A and refs. [33, 85] for the translation between the old and modern notation for the labels of the UIRs.

The unitarity of the afore-mentioned irreducible representations formed by $\{\psi_\mu^{(phys,\,-\ell;\,m;k)}\}$ and $\{\psi_\mu^{(phys,\,+\ell;\,m;k)}\}$ can be demonstrated as follows [33, 34, 76]. Let $\langle\psi^{(1)}|\psi^{(2)}\rangle_{ax}$ be the following dS invariant[13] and time-independent scalar product [33, 34, 76]:

$$\langle\psi^{(1)}|\psi^{(2)}\rangle_{ax} = \int_{S^3}\sqrt{-g}\,d\boldsymbol{\theta_3}\,g^{\mu\nu}\,\psi_\mu^{(1)}(t,\boldsymbol{\theta_3})^\dagger\,\gamma^5\,\psi_\nu^{(2)}(t,\boldsymbol{\theta_3}), \tag{3.44}$$

where $\psi_\mu^{(1)}$ and $\psi_\nu^{(2)}$ are any two solutions of the field equations in the TT gauge (3.13). The scalar product (3.44) is the time-independent Noether charge associated with the axial current[14] [34, 76]

$$J^\mu_{ax}\left(\psi^{(1)},\psi^{(2)}\right) = i\,\overline{\psi^{(1)}}_\nu\gamma^\mu\gamma^5\psi^{(2)\nu}, \quad \nabla_\mu J^\mu_{ax}\left(\psi^{(1)},\psi^{(2)}\right) = 0, \tag{3.45}$$

specifically,

$$\langle\psi^{(1)}|\psi^{(2)}\rangle_{ax} = \int_{S^3}\sqrt{-g}\,d\boldsymbol{\theta_3}\;J^t_{ax}\left(\psi^{(1)},\psi^{(2)}\right). \tag{3.46}$$

The physical modes (3.20), (3.21), (3.39) and (3.40) are normalised as:

$$\langle\psi^{(phys,\,\sigma\ell;\,m;k)}|\psi^{(phys,\,\sigma'\ell';\,m';k')}\rangle_{ax} = (-\sigma)\times\delta_{\sigma\sigma'}\delta_{\ell\ell'}\delta_{mm'}\delta_{kk'}, \tag{3.47}$$

$$\begin{aligned}\langle\upsilon^{(phys,\,\sigma\ell;\,m;k)}|\upsilon^{(phys,\,\sigma'\ell';\,m';k')}\rangle_{ax} &= (+\sigma)\times\delta_{\sigma\sigma'}\delta_{\ell\ell'}\delta_{mm'}\delta_{kk'} \\ \langle\upsilon^{(phys,\,\sigma\ell;\,m;k)}|\psi^{(phys,\,\sigma'\ell';\,m';k')}\rangle_{ax} &= 0,\end{aligned} \tag{3.48}$$

with $\sigma,\sigma' \in \{+,-\}$. Also,

$$\langle\psi^{(1)}|\psi^{(pg)}\rangle_{ax} = 0, \tag{3.49}$$

[13] By 'dS-invariant scalar product' we mean that the $so(4,1)$ generators/Lie derivatives are realised as anti-hermitian operators with respect to the scalar product under consideration [28].

[14] The axial current (3.45) is covariantly conserved because of the imaginary mass parameter of the gravitino [34, 76]. It is easy to check that in the case of a real-mass spin-3/2 field the axial current is not conserved.

where $\psi_\mu^{(1)}$ is any physical or TT pure-gauge mode, and thus, the pure-gauge modes can be identified with zero. It is interesting that, with respect to the scalar product (3.44), there is indefiniteness of the norm among the positive frequency physical modes, as well as among the negative frequency physical modes [34, 76]. Moreover, we observe that the sign of the norm depends on the helicity $\sigma \in \{+,-\}$ [see eqs. (3.47) and (3.48)]. Unitarity requires a positive-definite inner product that is invariant under dS transformations, i.e. the dS generators (Lie-Lorentz derivatives) are realised as anti-hermitian operators. Indeed the scalar product (3.44) is dS invariant as, for any dS Killing vector ξ^μ, we have [34, 76]

$$\langle \mathbb{L}_\xi \psi^{(1)} | \psi^{(2)} \rangle_{ax} + \langle \psi^{(1)} | \mathbb{L}_\xi \psi^{(2)} \rangle_{ax} = 0. \tag{3.50}$$

As we mentioned earlier, according to eq. (3.47) the scalar product (3.44) is positive definite for the physical modes $\{\psi_\mu^{(phys,\,-\ell;\,m;k)}\}$ and negative definite for the physical modes $\{\psi_\mu^{(phys,\,+\ell;\,m;k)}\}$.[15] As the two sets do not mix with each other under dS transformations, we conclude:

- The positive frequency physical gravitino modes with positive helicity, $\{\psi_\mu^{(phys,\,+\ell;m;k)}\}$, form the discrete series UIR $D^+(\Delta, s) = D^+(5/2, 3/2)$ of $so(4,1)$ — see appendix A. The $so(4)$ content corresponds to the $so(4)$ highest weights (3.34). The $so(4,1)$-invariant inner product that is positive definite is given by the negative of eq. (3.44).
- The positive frequency physical gravitino modes with negative helicity, $\{\psi_\mu^{(phys,\,-\ell;m;k)}\}$, form the discrete series UIR $D^-(\Delta, s) = D^-(5/2, 3/2)$ of $so(4,1)$ — see appendix A. The $so(4)$ content corresponds to the $so(4)$ highest weights (3.35). The $so(4,1)$-invariant inner product that is positive definite is given by eq. (3.44).

Thus, the two sets of positive frequency modes, $\{\psi_\mu^{(phys,\,+\ell;m;k)}\}$ and $\{\psi_\mu^{(phys,\,-\ell;m;k)}\}$, with the afore-mentioned choice of positive-definite scalar products, form the direct sum $D^+(5/2,3/2) \bigoplus D^-(5/2,3/2)$. The negative frequency modes, $\{v_\mu^{(phys,\,+\ell;m;k)}\}$ [eq. (3.40)] and $\{v_\mu^{(phys,\,-\ell;m;k)}\}$ [eq. (3.39)], form the same direct sum of UIRs. The transformation $\mathbb{L}_B v_\mu^{(phys,\,\pm\ell;m;k)}$ is found from (3.43) by replacing $\psi_\mu^{(phys,\,\pm(\ell\pm1);m;k)}$ with $v_\mu^{(phys,\,\pm(\ell\pm1);m;k)}$, while the coefficients in the linear combination on the right-hand side must be replaced by the complex conjugates of the ones in (3.43).

3.2 Conformal-like symmetry and UIRs of $so(4,2)$

It was recently found that the two sets of mode functions, $\{\psi_\mu^{(phys,\,+\ell;m;k)}\}$ and $\{\psi_\mu^{(phys,\,-\ell;m;k)}\}$, form not only a direct sum of $so(4,1)$ UIRs but also a direct sum of $so(4,2)$ UIRs [76]. Let us review the basic findings of [76], as these will be useful in our discussions on SUSY later on.

Conformal-like symmetries of the field equations. The $so(4,2)$ symmetry that preserves the solution space of eq. (3.13) is generated by the ten familiar infinitesimal dS transformations (3.16) [generating the dS subalgebra of $so(4,2)$], as well as by five infinitesimal

[15] The conventional inner product, $\int_{S^3} \sqrt{-g}\, d\boldsymbol{\theta_3}\, g^{\mu\nu}\, \psi_\mu^{(1)\dagger}(t,\boldsymbol{\theta_3})\, \psi_\nu^{(2)}(t,\boldsymbol{\theta_3})$, despite its positive definiteness, is neither dS invariant nor time-independent. Therefore, it is not a 'good' choice for a representation-theoretic analysis [33, 34, 76].

conformal-like transformations [76]:

$$\mathbb{T}_V \Psi_\mu \equiv \gamma^5 \Big(V^\rho \nabla_\rho \Psi_\mu + i\, V^\rho \gamma_\rho \Psi_\mu - i\, V^\rho \gamma_\mu \Psi_\rho - \frac{3}{2} \phi_V\, \Psi_\mu \Big) \\ - \frac{2}{3} \left(\nabla_\mu + \frac{i}{2} \gamma_\mu \right) \gamma^5 \Psi_\rho V^\rho, \tag{3.51}$$

where V^μ is any genuine conformal Killing vector (2.13). If Ψ_μ is a solution of (3.13), i.e. $\Psi_\mu = \Psi_\mu^{(\mathrm{TT})}$, then so is $\mathbb{T}_V \Psi_\mu^{(\mathrm{TT})}$.[16]

Note. The conformal-like symmetry transformation (3.51) is also a symmetry of the non-gauge-fixed RS equation (3.1) [76]. In this case, the last term in eq. (3.51) can be omitted as it corresponds to an off-shell gauge transformation (3.4) that leaves the RS equation (3.1) invariant. However, this gauge transformation cannot be omitted when working in the TT gauge, as it ensures that if Ψ_μ is in the TT gauge, then so is $\mathbb{T}_V \Psi_\mu$ [76].

The full symmetry algebra (10 dS isometries plus 5 conformal-like symmetries) closes on $so(4,2)$ up to field-dependent gauge transformations. In particular, we have [76],:

$$[\mathbb{L}_\xi, \mathbb{L}_{\xi'}] \Psi_\mu^{(\mathrm{TT})} = \mathbb{L}_{[\xi,\xi']} \Psi_\mu^{(\mathrm{TT})}, \tag{3.52a}$$

$$[\mathbb{L}_\xi, \mathbb{T}_V] \Psi_\mu^{(\mathrm{TT})} = \mathbb{T}_{[\xi,V]} \Psi_\mu^{(\mathrm{TT})}, \tag{3.52b}$$

$$[\mathbb{T}_{V'}, \mathbb{T}_V] \Psi_\mu^{(\mathrm{TT})} = \mathbb{L}_{[V',V]} \Psi_\mu^{(\mathrm{TT})} + \left(\nabla_\mu + \frac{i}{2} \gamma_\mu \right) K_{[V',V]}, \tag{3.52c}$$

where

$$K_{[V',V]} = \frac{4}{9} \left(\left(\nabla^\lambda - \frac{i}{2}\, \gamma^\lambda \right) \Psi^{(\mathrm{TT})\rho}\, \nabla_\lambda [V', V]_\rho - 2\, \Psi^{(\mathrm{TT})\rho}\, [V', V]_\rho \right). \tag{3.53}$$

Here, $[\mathbb{L}_\xi, \mathbb{T}_V] \equiv \mathbb{L}_\xi \mathbb{T}_V - \mathbb{T}_V \mathbb{L}_\xi$, $[\mathbb{T}_{V'}, \mathbb{T}_V] \equiv \mathbb{T}_{V'} \mathbb{T}_V - \mathbb{T}_V \mathbb{T}_{V'}$, and so forth. It is clear that the algebra (3.52a)–(3.52c) has the structure of the conformal algebra $so(4,2)$ [eq. (2.12)] up to the gauge transformation in (3.52c). This $so(4,2)$ symmetry is the dS analogue of the $so(4,2)$ symmetry found for strictly massless gauge potentials on AdS_4 in the unfolded formalism by Vasiliev [75].

UIRs of $so(4,2)$ formed by gravitino modes. Each of the two positive frequency single-helicity sets of modes, $\{\psi_\mu^{(phys,\,+\ell;m;k)}\}$ and $\{\psi_\mu^{(phys,\,-\ell;m;k)}\}$, forms a UIR of $so(4,2)$ [76]. This fact is readily demonstrated by specialising to the following genuine conformal Killing vector

$$V_\mu^{(0)} = \nabla_\mu \sinh t, \tag{3.54}$$

i.e. $(V_t^{(0)}, V_{\theta_3}^{(0)}, V_{\theta_2}^{(0)}, V_{\theta_1}^{(0)}) = (\cosh t, 0, 0, 0)$. The conformal-like transformations (3.51) generated by $V^{(0)}$ act on the physical modes as:

$$\mathbb{T}_{V^{(0)}} \psi_\mu^{(phys,\,-\ell;\,m;k)} = +i \left(\ell + \frac{3}{2} \right) \psi_\mu^{(phys,\,-\ell;\,m;k)} \tag{3.55}$$

[16] Note that the field-dependent gauge transformation on the second line of (3.51) is **not** a restricted gauge transformation (3.14), but it still is an off-shell gauge transformation (3.4). It is needed to preserve the TT gauge conditions [76].

and

$$\mathbb{T}_{V^{(0)}}\psi_\mu^{(phys,\,+\ell;\,m;k)} = -i\left(\ell+\frac{3}{2}\right)\psi_\mu^{(phys,\,+\ell;\,m;k)}. \tag{3.56}$$

Thus, from eqs. (2.12) and (3.52a)–(3.52c) it follows that $\{\psi_\mu^{(phys,\,+\ell;m;k)}\}$ and $\{\psi_\mu^{(phys,\,-\ell;m;k)}\}$ separately form irreducible representations of $so(4,2)$. These representations are unitary because the conformal-like generators (3.51) are anti-hermitian with respect to the scalar product (3.44) [76]:

$$\langle \mathbb{T}_V\psi^{(1)}|\psi^{(2)}\rangle_{ax} + \langle \psi^{(1)}|\mathbb{T}_V\psi^{(2)}\rangle_{ax} = 0, \tag{3.57}$$

for any two solutions $\psi_\mu^{(1)}, \psi_\nu^{(2)}$ of (3.13). However, as in the $so(4,1)$ case, a different choice of a positive-definite norm is needed for each $so(4,2)$ UIR of single helicity — see the discussion below (3.50).

As in the $so(4,1)$ case, the negative frequency modes (3.39) and (3.40) form the same $so(4,2)$ UIRs as the positive frequency ones; their conformal-like transformations under $V^{(0)\mu}$ (3.54) are

$$\mathbb{T}_{V^{(0)}}v_\mu^{(phys,\,-\ell;\,m;k)} = -i\left(\ell+\frac{3}{2}\right)v_\mu^{(phys,\,-\ell;\,m;k)}, \tag{3.58}$$

and

$$\mathbb{T}_{V^{(0)}}v_\mu^{(phys,\,+\ell;\,m;k)} = +i\left(\ell+\frac{3}{2}\right)v_\mu^{(phys,\,+\ell;\,m;k)}. \tag{3.59}$$

3.3 Conformal-like symmetry of the hermitian action (3.8)

Interestingly, as we will present here for the first time, the conformal-like symmetry transformation (3.51) is also an off-shell symmetry of the hermitian action (3.8). To prove this, let us consider the variation of the action (3.8),

$$\delta S_{\frac{3}{2}} = -\int d^4x\,\sqrt{-g}\,\left(\delta\overline{\Psi}_\mu\,\gamma^5\,\mathcal{R}^\mu(\Psi) + \overline{\Psi}_\mu\,\gamma^5\,\delta\mathcal{R}^\mu(\Psi)\right), \tag{3.60}$$

under $\delta\Psi_\mu = \mathbb{T}_V\Psi_\mu$, where now Ψ_μ is an off-shell field configuration with no gauge conditions imposed. After a straightforward off-shell calculation, we find the following useful quantities:

$$\begin{aligned}\delta\overline{\Psi}_\mu &= (\mathbb{T}_V\Psi_\mu)^\dagger\, i\gamma^0 &(3.61)\\ &= -\left(V^\rho\nabla_\rho\overline{\Psi}_\mu + i\,V^\rho\overline{\Psi}_\mu\gamma_\rho - i\,V^\rho\overline{\Psi}_\rho\,\gamma_\mu - \frac{3}{2}\phi_V\,\overline{\Psi}_\mu\right)\gamma^5 + \frac{2}{3}\overline{\Psi}_\rho V^\rho\gamma^5\left(\overleftarrow{\nabla}_\mu + \frac{i}{2}\gamma_\mu\right),\end{aligned}$$

and

$$\begin{aligned}\delta\mathcal{R}^\mu(\Psi) &= \gamma^{\mu\rho\sigma}\left(\nabla_\rho + \frac{i}{2}\gamma_\rho\right)\delta\Psi_\sigma\\ &= \gamma^5\phi_V\frac{5}{2}\mathcal{R}^\mu(\Psi) + \gamma^5\left(-V^\rho\nabla_\rho + iV^\rho\gamma_\rho\right)\mathcal{R}^\mu(\Psi) - i\gamma^5V^\mu\gamma^\sigma\mathcal{R}_\sigma(\Psi), &(3.62)\end{aligned}$$

where $\mathcal{R}^\mu(\Psi)$ is defined in (3.6).[17] Then, we easily find

$$\delta S_{\frac{3}{2}} = -\int d^4x\,\sqrt{-g}\,\nabla_\rho\left(-V^\rho\overline{\Psi}_\mu\mathcal{R}^\mu(\Psi)\right), \tag{3.63}$$

and thus, the conformal-like transformation (3.51) is an off-shell symmetry of the hermitian action (3.8).

[17] For off-shell fields we have $\gamma^\sigma\mathcal{R}_\sigma(\Psi) = 2\left(\nabla\!\!\!/\gamma^\sigma\Psi_\sigma - \frac{3}{2}i\,\gamma^\sigma\Psi_\sigma - \nabla^\sigma\Psi_\sigma\right)$.

3.4 Quantisation of the gravitino field, the necessity for the anti-self-duality constraint, and UIRs in the fermionic Fock space

Why does the (anti-)self-duality constraint have to be imposed? So far, we have explained how the gravitino positive frequency modes furnish a direct sum of discrete series UIRs of $so(4,1)$ and a direct sum of UIRs of $so(4,2)$. Our next task is to realise these UIRs in the single-particle Hilbert space associated with the QFT of the (free) quantum gravitino field on global dS_4. In this task we encounter a problem related to the discussions in subsections 3.1 and 3.2. On the one hand, the single-particle Hilbert space of the QFT must be equipped with a positive-definite and dS-invariant scalar product. On the other hand, this space is identified with the space of physical positive frequency solutions, and, in our case, there is **no** dS-invariant inner product that remains **positive-definite for mode functions of both helicities**. One could argue that since positive-helicity and negative-helicity modes separately form UIRs, two different positive-definite inner products can be used for each fixed-helicity subspace for the quantisation of the theory, as discussed in the passage below eq. (3.50). However, although this approach works at the level of the classical mode solutions, it does not seem to work in a quantum field-theoretic setting if one insists on the locality of the action functional of the theory. In particular, one can see that locality requires the indefiniteness of the norm, in the sense that the negative- or positive-definiteness of the norm depends on the helicity of each state, as follows. If one decides to include both helicities in the positive frequency, as well as in the negative frequency, sectors of the quantum gravitino field (as one usually does in Minkowski spacetime, for example), then they can follow the canonical quantisation procedure using the hermitian and local Lagrangian density in (3.8) to define the conjugate momentum, and impose equal-time anti-commutation relations. Then, by expanding the field in modes, one finds that the equal-time anti-commutator (3.74) between the field and its conjugate momentum requires the anti-commutators between creation and annihilation operators to have helicity-dependent signs [eq. (3.77)], leading to the indefiniteness of the norm in the Fock space of the local QFT. This is explained in more detail in the passages below eq. (3.71). Moreover, note that insisting on keeping all propagating helicity degrees of freedom while simultaneously enforcing the positivity of the norm, by using a different positive-definite scalar product for each fixed-helicity subspace, leads to a non-local theory. This becomes evident upon observing that the two distinct scalar products — namely, the axial scalar product (3.44) and its negative — can essentially be re-expressed in terms of a single scalar product. This modified scalar product arises from the initial axial scalar product (3.44) with the insertion of the 'helicity' operator (3.36), which is non-local.

To avoid the appearance of negative norms, and achieve positive definiteness in the QFT Fock space, we will quantise the 'chiral' gravitino field. The positive frequency sector of this field will furnish a UIR with helicity $-3/2$ (or $+3/2$), corresponding to the $so(4,1)$ discrete series $D^{-}(5/2,3/2)$ (or $D^{+}(5/2,3/2)$), while the negative frequency sector will furnish a UIR with helicity $+3/2$ (or $-3/2$), corresponding to $D^{+}(5/2,3/2)$ (or $D^{-}(5/2,3/2)$) — see appendix A for details on UIRs of $so(4,1)$. Note that, for the chiral field under consideration, the helicity of the positive frequency sector is opposite from the helicity of the negative frequency sector.[18] In order to restrict to a theory with particles of a single helicity $-3/2$ (or $+3/2$), we will impose a chirality constraint, i.e. anti-self-duality (or self-duality) constraint, on the gravitino field strength. Without loss of generality, we will work with a quantum field whose positive and negative frequency sectors contain states with helicities $-3/2$ and $+3/2$, respectively, corresponding to an anti-self-dual field strength. The anti-self-duality constraint ensures the positivity of the norm without violating the dS invariance of the theory.

Takahashi's quantisation method. The conventional way to quantise the gravitino field is the canonical quantisation procedure, in which one makes use of a local hermitian action functional, such as (3.8), in order to define the conjugate momentum, and then imposes equal-time anti-commutation relations. Here, we will follow a different method discussed in detail by Takahashi [88],[19] which aligns well with the emphasis we have put on the group-theoretic properties of the mode solutions, as well as with the fact that we are considering a theory with an on-shell chirality constraint — the two helicities of the gravitino cannot be split locally at the level of the action, but such a split is possible on-shell by imposing the (anti-)self-duality condition. The starting point in Takahashi's method is the field equation. In our case, the field equation is (3.13) accompanied by the anti-self-duality constraint on the field strength[20]

$$\widetilde{F}_{\mu\nu} = -iF_{\mu\nu}, \tag{3.64}$$

where $\widetilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$.[21] The field strength is defined as [76]

$$F_{\mu\nu} = \left(\nabla_{[\mu} + \frac{i}{2}\gamma_{[\mu}\right)\,\Psi_{\nu]}. \tag{3.65}$$

It is divergence-free, gamma-traceless, and satisfies [76]

$$\widetilde{F}_{\mu\nu} = -i\gamma^5 F_{\mu\nu}, \tag{3.66}$$

by virtue of eqs. (2.6), (3.13) and (E.10), i.e. the duality operation on $F_{\mu\nu}$ is equivalent to an infinitesimal chiral rotation — see also appendix E.[22] Later we will use the superscript '$-$' to denote the field strength that satisfies the anti-self-duality constraint (3.64), as $F^-_{\mu\nu}$ in eq. (3.78). The field strength (3.65) is gauge invariant, and thus, the field Ψ_ν in (3.65) can be replaced by the gravitino gauge potential in any gauge without affecting the form of $F_{\mu\nu}$. Here we will impose the TT gauge condition.[23] Then, the quantum gravitino field operator $\Psi^{(\mathrm{TT})}_\mu(t, \boldsymbol{\theta_3})$ that is required to satisfy eqs. (3.13) and (3.64) is expressed as a mode sum in terms of our previously obtained mode functions, where the expansion coefficients are promoted to creation and annihilation operators. The main objective of Takahashi's

[18]See refs. [86, 87] for further discussions on self-dual and anti-self-dual field strengths of massless fields and their quantum theory.

[19]Takahashi's method refers to free quantum fields in Minkowski spacetime, but the generalisation to dS spacetime is straightforward.

[20]For the reader who is interested in the quantisation of chiral theories using Takahashi's method, the treatment of the chiral (massless) Majorana spin-1/2 field can be found in Takahashi's book [88].

[21]The need to impose a (anti-)self-duality constraint was explained at the beginning of this subsection.

[22]Using (3.66), it follows that the anti-self-duality constraint (3.64) is equivalent to $\gamma^5 F_{\mu\nu} = +F_{\mu\nu}$.

[23]Imposing the TT gauge condition allows residual gauge symmetry. To quantise the gravitino field we will fix the gauge completely — see eq. (3.68).

method for the theory at hand is to determine the operators $\Psi^{(\mathrm{TT})}_\mu$ and $Q^{dS}_{\frac{3}{2}}[\xi]$ such that the Heisenberg equations of motion are satisfied:[24]

$$-i\,\mathbb{L}_\xi \Psi^{(\mathrm{TT})}_\mu(t,\boldsymbol{\theta_3}) = \left[\Psi^{(\mathrm{TT})}_\mu(t,\boldsymbol{\theta_3}), Q^{dS}_{\frac{3}{2}}[\xi]\right], \tag{3.67}$$

where ξ^μ is any Killing vector of dS spacetime, and $Q^{dS}_{\frac{3}{2}}[\xi]$ is the hermitian quantum operator (dS charge) that generates the infinitesimal dS transformation on the QFT Fock space. The subscript '$\frac{3}{2}$' in $Q^{dS}_{\frac{3}{2}}[\xi]$ has been used to distinguish between the quantum generators of the chiral gravitino and of the chiral graviton — see eq. (4.61). In addition, quantum operators representing physical quantities must (anti-)commute with one another for spacelike separations (microcausality).

Mode expansion. Let us now quantise the chiral gravitino field following the steps outlined in the previous paragraph. From our discussions in subsection 3.1 it follows that the t-component of the gravitino field in the TT gauge is pure-gauge, i.e. it can be gauged away. Thus, to isolate the propagating degrees of freedom, we consider the completely gauge-fixed field, $\Psi^{(\mathrm{TT})}_t = g^{\tilde\mu\tilde\nu}\gamma_{\tilde\mu}\Psi^{(\mathrm{TT})}_{\tilde\nu} = 0$,[25] and we expand it in modes as follows

$$\Psi^{(\mathrm{TT})-}_{\tilde\mu}(t,\boldsymbol{\theta_3}) = \sum_{\ell=1}^{\infty}\sum_{m=1}^{\ell}\sum_{k=-m-1}^{m}\left(a^{(-)}_{\ell mk}\psi^{(phys,\,-\ell\,;m;k)}_{\tilde\mu}(t,\boldsymbol{\theta_3}) + b^{(+)\dagger}_{\ell mk}\,v^{(phys,\,+\ell\,;m;k)}_{\tilde\mu}(t,\boldsymbol{\theta_3})\right), \tag{3.68}$$

where $\tilde\mu$ is a vector index on S^3. The superscript '$-$' in $\Psi^{(\mathrm{TT})-}_\mu$ refers to the fact that the corresponding field strength is anti-self-dual (this will be verified below). The non-zero anti-commutators between creation and annihilation operators are

$$\{a^{(-)}_{\ell mk}, a^{(-)\dagger}_{\ell' m' k'}\} = \delta_{\ell\ell'}\delta_{mm'}\delta_{kk'}, \qquad \{b^{(+)}_{\ell mk}, b^{(+)\dagger}_{\ell' m' k'}\} = \delta_{\ell\ell'}\delta_{mm'}\delta_{kk'}. \tag{3.69}$$

The vacuum is defined as the state, $|0\rangle_{\frac{3}{2}}$, in the Fock space that satisfies:

$$a^{(-)}_{\ell mk}\,|0\rangle_{\frac{3}{2}} = b^{(+)}_{\ell mk}\,|0\rangle_{\frac{3}{2}} = 0, \tag{3.70}$$

for all ℓ, m, k. Using the dS-invariant inner product (3.44) [see eqs. (3.47) and (3.48)], we find

$$a^{(-)}_{\ell mk} = \langle\psi^{(phys,\,-\ell\,;m;k)}|\Psi^{(\mathrm{TT})-}\rangle_{ax}, \qquad b^{(+)\dagger}_{\ell mk} = \langle v^{(phys,\,+\ell\,;m;k)}|\Psi^{(\mathrm{TT})-}\rangle_{ax}. \tag{3.71}$$

The dS invariance of the vacuum follows from the dS invariance of the positive frequency solution space, and it will be further verified below by showing that the quantum dS generators annihilate the vacuum.

What goes wrong if we include both helicities? To demonstrate the appearance of negative norms in the Fock space for a non-chiral gravitino, let us consider a completely gauge-fixed quantum gravitino field Ψ_μ, as in (3.68), which now includes all helicities in the mode sum as follows:

$$\Psi^{(\mathrm{TT})}_{\tilde\mu}(t,\boldsymbol{\theta_3}) = \sum_{\sigma\in\{+,-\}}\sum_{\ell,m,k}\left(a^{(\sigma)}_{\ell mk}\psi^{(phys,\,\sigma\ell\,;m;k)}_{\tilde\mu}(t,\boldsymbol{\theta_3}) + b^{(\sigma)\dagger}_{\ell mk}\,v^{(phys,\,\sigma\ell\,;m;k)}_{\tilde\mu}(t,\boldsymbol{\theta_3})\right), \tag{3.72}$$

[24]The equality here is modulo pure-gauge TT solutions.

[25]This gauge is the analogue of the Coulomb gauge for the Maxwell gauge potential.

where from eqs. (3.47) and (3.48) we have

$$a^{(\mp)}_{\ell mk} = \pm \langle \psi^{(phys,\, \mp\ell\,;m;k)}|\Psi^{(\mathrm{TT})}\rangle_{ax}\,, \qquad b^{(\pm)\dagger}_{\ell mk} = \pm \langle v^{(phys,\, \pm\ell\,;m;k)}|\Psi^{(\mathrm{TT})}\rangle_{ax}\,. \tag{3.73}$$

Let us also denote the vacuum annihilated by all annihilation operators as $|\Omega\rangle$. In this case we can proceed with the canonical quantisation procedure using the hermitian Lagrangian in (3.8). The standard equal-time anti-commutation relations (expressed in the form of a 4×4 spinorial matrix) are

$$\begin{aligned}\Big\{\Psi^{(\mathrm{TT})}_{\tilde\mu}(t,\boldsymbol{\theta_3}),\,\Psi^{(\mathrm{TT})}_{\tilde\nu'}(t,\boldsymbol{\theta_3'})^{\dagger}\gamma^5\Big\} &= \frac{e_{\tilde\mu}{}^{m} e_{\tilde\nu'}{}^{n'}}{\sqrt{-g}}\begin{pmatrix}\sqrt{\tilde g}\;\Delta^{TT}_{mn'}(\boldsymbol{\theta_3},\boldsymbol{\theta_3'}) & \mathbf{0}\\ \mathbf{0} & \sqrt{\tilde g}\;\Delta^{TT}_{mn'}(\boldsymbol{\theta_3},\boldsymbol{\theta_3'})\end{pmatrix}\\ &= \frac{1}{\cosh t}\begin{pmatrix}\Delta^{TT}_{\tilde\mu\tilde\nu'}(\boldsymbol{\theta_3},\boldsymbol{\theta_3'}) & \mathbf{0}\\ \mathbf{0} & \Delta^{TT}_{\tilde\mu\tilde\nu'}(\boldsymbol{\theta_3},\boldsymbol{\theta_3'})\end{pmatrix},\end{aligned} \tag{3.74}$$

where

$$\Delta^{TT}_{\tilde\mu\tilde\nu'}(\boldsymbol{\theta_3},\boldsymbol{\theta_3'}) = \cosh^{-2}t\; e_{\tilde\mu}{}^{m} e_{\tilde\nu'}{}^{n'}\,\Delta^{TT}_{mn'}(\boldsymbol{\theta_3},\boldsymbol{\theta_3'})$$

is the transverse and $\tilde\gamma$-traceless delta function for vector-spinors on S^3 defined by

$$\Delta^{TT}_{\tilde\mu\tilde\nu'}(\boldsymbol{\theta_3},\boldsymbol{\theta_3'}) = \sum_{\sigma\in\{+,-\}}\sum_{\ell=1}^{\infty}\sum_{m,k}\tilde\psi^{(\ell;m;k)}_{\sigma\tilde\mu}(\boldsymbol{\theta_3})\otimes\tilde\psi^{(\ell;m;k)}_{\sigma\tilde\nu'}(\boldsymbol{\theta_3'})^{\dagger}. \tag{3.75}$$

In particular, if $\tilde\psi_{\tilde\mu}(\boldsymbol{\theta_3})$ is a vector-spinor on S^3, and $\tilde\psi'_{\tilde\mu}(\boldsymbol{\theta_3})$ is its divergence-free and $\tilde\gamma$-traceless part, then

$$\tilde\psi'_{\tilde\mu}(\boldsymbol{\theta_3}) = \int_{S^3} d\boldsymbol{\theta_3}\sqrt{\tilde g}\,\Delta^{TT}_{\tilde\mu\tilde\nu'}(\boldsymbol{\theta_3},\boldsymbol{\theta_3'})\;\tilde\psi^{\tilde\nu'}(\boldsymbol{\theta_3'}). \tag{3.76}$$

Using the expressions (3.73), as well as the anticommutation relations (3.74), we find

$$\{a^{(\sigma)}_{\ell mk}, a^{(\sigma')\dagger}_{\ell' m' k'}\} = (-\sigma)\,\delta_{\sigma\sigma'}\delta_{\ell\ell'}\delta_{mm'}\delta_{kk'}\,, \qquad \{b^{(\sigma)}_{\ell mk}, b^{(\sigma')\dagger}_{\ell' m' k'}\} = \sigma\,\delta_{\sigma\sigma'}\delta_{\ell\ell'}\delta_{mm'}\delta_{kk'}\,, \tag{3.77}$$

$\sigma,\sigma'\in\{+,-\}$, while the rest of the anti-commutators are zero. It is clear that the states $a^{(+)\dagger}_{\ell mk}\,|\Omega\rangle$ and $b^{(-)\dagger}_{\ell mk}\,|\Omega\rangle$ have negative norm, i.e. the theory is non-unitary. As a verification of this fact, we can expand in modes the fields on the left-hand side of the equal-time anti-commutator (3.74). After some algebra, one can arrive at the expression in terms of the transverse-traceless delta function on the right-hand side of (3.74) only if the anti-commutators (3.77) are used. In other words, the equal-time anti-commutation relations for a gravitino field that contains both helicities require the appearance of negative-norm states in the QFT Fock space. This justifies our choice to exclude half of the helicities, i.e. quantise the chiral gravitino field (3.68), to achieve unitarity in the QFT Fock space. We will now continue the quantisation of the chiral gravitino field.

Anti-self-duality constraint. Let us demonstrate that our choice for the mode expansion for the chiral gravitino (3.68) is consistent with the anti-self-duality constraint (3.64). To be specific, we will show that the following field strength:

$$F^{-}_{\mu\nu} = \left(\nabla_{[\mu} + \frac{i}{2}\gamma_{[\mu}\right)\,\Psi^{(\mathrm{TT})-}_{\nu]}, \tag{3.78}$$

is anti-self-dual. Substituting the mode expansion (3.68) into (3.78), we find the mode expansion for the field strength

$$F^{-}_{\mu\nu}(t,\boldsymbol{\theta_3}) = \sum_{\ell=1}^{\infty}\sum_{m,k}\left(a^{(-)}_{\ell mk}\, f^{(-\ell;m;k)}_{\mu\nu}(t,\boldsymbol{\theta_3}) + b^{(+)\dagger}_{\ell mk}\, f^{(+\ell;m;k)C}_{\mu\nu}(t,\boldsymbol{\theta_3})\right), \tag{3.79}$$

where

$$\begin{aligned} f^{(-\ell;m;k)}_{\mu\nu}(t,\boldsymbol{\theta_3}) &\equiv \left(\nabla_{[\mu} + \frac{i}{2}\gamma_{[\mu}\right)\psi^{(phys,-\ell\,;m;k)}_{\nu]}(t,\boldsymbol{\theta_3}),\\ f^{(+\ell;m;k)C}_{\mu\nu}(t,\boldsymbol{\theta_3}) &\equiv \left(\nabla_{[\mu} + \frac{i}{2}\gamma_{[\mu}\right)v^{(phys,+\ell\,;m;k)}_{\nu]}(t,\boldsymbol{\theta_3}). \end{aligned}$$

For convenience, let us start by showing that the $t\tilde{\nu}$-component, $F^{-}_{t\tilde{\nu}}$, satisfies the anti-self-duality constraint. Substituting the expressions of the mode functions (3.20) and (3.40) into (3.79), we find after a straightforward calculation

$$\begin{aligned} F^{-}_{t\tilde{\nu}}(t,\boldsymbol{\theta_3}) = \frac{1}{2}\sum_{\ell=1}^{\infty}\sum_{m,k}\left(\frac{\ell+2}{2(\ell+1)}\right)^{1/2}&\left[a^{(-)}_{\ell mk}\begin{pmatrix}\left(\partial_t\alpha_\ell(t) - \frac{i}{2}\beta_\ell(t)\right)\,\tilde{\psi}^{(\ell;m;k)}_{-\tilde{\nu}}(\boldsymbol{\theta_3})\\ 0\end{pmatrix}\right.\\ &\left.+\,b^{(+)\dagger}_{\ell mk}\begin{pmatrix}\left(\partial_t\alpha^{*}_\ell(t) + \frac{i}{2}\beta^{*}_\ell(t)\right)\,\tilde{\psi}^{(\ell;m;k)}_{+\tilde{\nu}}(\boldsymbol{\theta_3})\\ 0\end{pmatrix}\right], \end{aligned}\tag{3.80}$$

where we have also used (3.27). It is clear that $F^{-}_{t\tilde{\nu}}$ is an eigenfunction of γ^5 [eq. (B.5)], $\gamma^5 F^{-}_{t\tilde{\nu}} = +F^{-}_{t\tilde{\nu}}$. Using (3.66), it follows that, for all the components of the field strength, we have $\gamma^5 F^{-}_{\mu\nu} = +F^{-}_{\mu\nu}$. This means that the anti-self-duality constraint (3.64) is satisfied.

Quantum symmetry generators. The hermitian dS generators can be constructed in the standard way [85, 88], using the dS-invariant inner product (3.44):

$$Q^{dS}_{\frac{3}{2}}[\xi] = -i : \langle \Psi^{(\mathrm{TT})-}|\mathbb{L}_\xi \Psi^{(\mathrm{TT})-}\rangle_{ax} :, \tag{3.81}$$

where the symbols $:\ldots:$ denote normal ordering. Here we will give the explicit expression only for the generator corresponding to the dS boost $\xi^\mu = B^\mu$ [eq. (3.42)]; the other dS generators can be constructed similarly. Expanding the field in modes (3.68), and using eqs. (3.43), (3.47) and (3.48), we find:

$$\begin{aligned} Q^{dS}_{\frac{3}{2}}[B] = &-\frac{1}{2}\sum_{\ell=1}^{\infty}\sum_{m,k}\left(\sqrt{(\ell-m+1)(\ell+m+3)}\,a^{(-)\dagger}_{(\ell+1)mk}a^{(-)}_{\ell mk} + \sqrt{(\ell-m)(\ell+m+2)}\,a^{(-)\dagger}_{(\ell-1)mk}a^{(-)}_{\ell mk}\right)\\ &-\frac{1}{2}\sum_{\ell=1}^{\infty}\sum_{m,k}\left(\sqrt{(\ell-m+1)(\ell+m+3)}\,b^{(+)\dagger}_{(\ell+1)mk}b^{(+)}_{\ell mk} + \sqrt{(\ell-m)(\ell+m+2)}\,b^{(+)\dagger}_{(\ell-1)mk}b^{(+)}_{\ell mk}\right). \end{aligned}\tag{3.82}$$

This can be clearly expressed as a sum of two independent hermitian generators,

$$Q^{dS}_{\frac{3}{2}}[B] = Q^{dS-}_{\frac{3}{2}}[B] + Q^{dS+}_{\frac{3}{2}}[B],$$

where $Q^{dS-}_{\frac{3}{2}}[B]$ is given by the expression in the first line of (3.82), while $Q^{dS+}_{\frac{3}{2}}[B]$ is given by the expression in the second line. The two generators $Q^{dS-}_{\frac{3}{2}}[B]$ and $Q^{dS+}_{\frac{3}{2}}[B]$ act on the negative- and positive-helicity sectors, respectively (i.e. positive-frequency and negative-frequency sectors, respectively). In particular, the two charges generate the two discrete series UIRs of $so(4,1)$, $D^-(\Delta = 5/2, s = 3/2)$ and $D^+(\Delta = 5/2, s = 3/2)$, respectively. The corresponding infinitesimal dS transformations of the creation operators are

$$\begin{aligned}\delta_B a^{(-)\dagger}_{\ell m k} \equiv \left[a^{(-)\dagger}_{\ell m k},\, Q^{dS}_{\frac{3}{2}}[B]\right] = \left[a^{(-)\dagger}_{\ell m k},\, Q^{dS-}_{\frac{3}{2}}[B]\right] &= \frac{1}{2}\sqrt{(\ell - m + 1)(\ell + m + 3)}\, a^{(-)\dagger}_{(\ell+1)mk} \\ &+ \frac{1}{2}\sqrt{(\ell - m)(\ell + m + 2)}\, a^{(-)\dagger}_{(\ell-1)mk} \end{aligned} \quad (3.83)$$

and

$$\begin{aligned}\delta_B b^{(+)\dagger}_{\ell m k} \equiv \left[b^{(+)\dagger}_{\ell m k},\, Q^{dS}_{\frac{3}{2}}[B]\right] = \left[b^{(+)\dagger}_{\ell m k},\, Q^{dS+}_{\frac{3}{2}}[B]\right] &= \frac{1}{2}\sqrt{(\ell - m + 1)(\ell + m + 3)}\, b^{(+)\dagger}_{(\ell+1)mk} \\ &+ \frac{1}{2}\sqrt{(\ell - m)(\ell + m + 2)}\, b^{(+)\dagger}_{(\ell-1)mk}. \end{aligned} \quad (3.84)$$

Using these expressions, it is clear that single-particle states $a^{(-)\dagger}_{\ell m k}\, |0\rangle_{\frac{3}{2}}$ transform as the corresponding positive frequency modes (3.43), i.e. they furnish the $so(4,1)$ discrete series UIR $D^-(\Delta = 5/2, s = 3/2)$. Similarly, single-particle states $b^{(+)\dagger}_{\ell m k}\, |0\rangle_{\frac{3}{2}}$ furnish the $so(4,1)$ discrete series UIR $D^+(\Delta = 5/2, s = 3/2)$ — see appendix A. Finally, it is straightforward to find that the quantum field operator transforms as

$$\left[\Psi^{(\mathrm{TT})-}_{\mu}, Q^{dS}_{\frac{3}{2}}[B]\right] = -i\, \mathbb{L}_B \Psi^{(\mathrm{TT})-}_{\mu}, \quad (3.85)$$

modulo pure-gauge TT solutions, where $\mathbb{L}_B$ is the Lie-Lorentz derivative (3.16) with respect to the dS boost Killing vector B [eq. (3.42)], in agreement with the Heisenberg equations of motion (3.67).

It is also interesting to note that we can construct the five hermitian generators of the conformal-like symmetry (3.51) in the same way,

$$Q^{\mathrm{conf}}_{\frac{3}{2}}[V] = -i :\langle \Psi^{(\mathrm{TT})-} | \mathbb{T}_V \Psi^{(\mathrm{TT})-}\rangle_{ax} :, \quad (3.86)$$

such that the Heisenberg equations of motion are again satisfied

$$-i\, \mathbb{T}_V \Psi^{(\mathrm{TT})-}_{\mu}(t, \boldsymbol{\theta_3}) = [\ \Psi^{(\mathrm{TT})-}_{\mu}(t, \boldsymbol{\theta_3}), Q^{\mathrm{conf}}_{\frac{3}{2}}[V]\ ], \quad (3.87)$$

modulo pure-gauge TT solutions, where $\mathbb{T}_V$ is the conformal-like transformation (3.51). This can be easily checked for the conformal-like symmetry generated by the genuine conformal Killing vector $V^{(0)\mu}$ [eq. (3.54)], for which the quantum generator is found to be

$$Q^{\mathrm{conf}}_{\frac{3}{2}}[V^{(0)}] = \sum_{\ell=1}^{\infty} \sum_{m,k} \left(\ell + \frac{3}{2}\right) \left(a^{(-)\dagger}_{\ell m k} a^{(-)}_{\ell m k} - b^{(+)\dagger}_{\ell m k} b^{(+)}_{\ell m k}\right). \quad (3.88)$$

This conformal-like quantum charge is expressed as a sum of two independent conformal-like charges,

$$\begin{aligned} Q^{\text{conf}}_{\frac{3}{2}}[V^{(0)}] &= Q^{\text{conf}-}_{\frac{3}{2}}[V^{(0)}] + Q^{\text{conf}+}_{\frac{3}{2}}[V^{(0)}], \\ Q^{\text{conf}-}_{\frac{3}{2}}[V^{(0)}] &= \sum_{\ell=1}^{\infty}\sum_{m,k}\left(\ell+\frac{3}{2}\right) a^{(-)\dagger}_{\ell m k} a^{(-)}_{\ell m k}, \\ Q^{\text{conf}+}_{\frac{3}{2}}[V^{(0)}] &= -\sum_{\ell=1}^{\infty}\sum_{m,k}\left(\ell+\frac{3}{2}\right) b^{(+)\dagger}_{\ell m k} b^{(+)}_{\ell m k}. \end{aligned} \tag{3.89}$$

The charges $Q^{\text{conf}-}_{\frac{3}{2}}[V^{(0)}]$ and $Q^{\text{conf}+}_{\frac{3}{2}}[V^{(0)}]$ generate $so(4,2)$ UIRs on the spaces of negative-helicity and positive-helicity states, respectively. Note that the vacuum of the chiral gravitino, $|0\rangle_{\frac{3}{2}}$, is invariant under the whole conformal-like symmetry, $so(4,2)$.

Microcausality. Finally, we will demonstrate the microcausality of the theory by computing the anti-commutator between two gauge invariant quantities, the anti-self-dual field strength and its hermitian conjugate, at two spacelike separated points:

$$\left\{F^{-}_{\mu\nu}(t,\boldsymbol{\theta_3}), F^{-}_{\alpha'\beta'}(t',\boldsymbol{\theta'_3})^{\dagger}\right\}.$$

For convenience, let us start by choosing two equal-time points $(t,\boldsymbol{\theta_3})$ and $(t,\boldsymbol{\theta'_3})$, and compute the equal-time anti-commutator for the following components of the field strengths:

$$\left\{F^{-}_{t\tilde{\mu}}(t,\boldsymbol{\theta_3}), F^{-}_{t\tilde{\nu}'}(t,\boldsymbol{\theta'_3})^{\dagger}\right\}. \tag{3.90}$$

Expanding the field strengths in modes [as in (3.79)], we find[26]

$$\left\{F^{-}_{t\tilde{\mu}}(t,\boldsymbol{\theta_3}), F^{-}_{t\tilde{\nu}'}(t,\boldsymbol{\theta'_3})^{\dagger}\right\} = -\frac{1}{4\cosh^3 t}\begin{pmatrix} \left(\tilde{\nabla}^2+\frac{1}{4}\right)\Delta^{TT}_{\tilde{\mu}\tilde{\nu}'}(\boldsymbol{\theta_3},\boldsymbol{\theta'_3}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{pmatrix}, \tag{3.91}$$

where $\Delta^{TT}_{\tilde{\mu}\tilde{\nu}'}(\boldsymbol{\theta_3},\boldsymbol{\theta'_3})$ is the transverse and $\tilde{\gamma}$-traceless delta function in the space of vector-spinors on S^3 defined by eq. (3.75). In appendix C we show that, while $\Delta^{TT}_{\tilde{\mu}\tilde{\nu}'}(\boldsymbol{\theta_3},\boldsymbol{\theta'_3})$ is non-local, the quantity $\left(\tilde{\nabla}^2+\frac{1}{4}\right)\Delta^{TT}_{\tilde{\mu}\tilde{\nu}'}(\boldsymbol{\theta_3},\boldsymbol{\theta'_3})$ in (3.91) is local, i.e. it vanishes for $\boldsymbol{\theta_3}\neq\boldsymbol{\theta'_3}$ [see eq. (C.21)]. This means that the anti-commutator (3.91) is also local. Then, taking the dual of (3.91) and using (3.66), it is easy to conclude that all equal-time anti-commutators of the form

$$\left\{F^{-}_{\mu\nu}(t,\boldsymbol{\theta_3}), F^{-}_{\alpha'\beta'}(t,\boldsymbol{\theta'_3})^{\dagger}\right\} \tag{3.92}$$

are also local, for any value of the tensor indices μ,ν, α',β'. Finally, the locality of the anti-commutator for any two causally disconnected (not necessarily equal-time) points can be easily demonstrated by exploiting the following observation: any two points $(t,\boldsymbol{\theta_3})$ and $(t',\boldsymbol{\theta'_3})$ that

[26]The anti-commutator (3.91) is a 4-dimensional bi-spinorial matrix and each of its components are bi-vectors on S^3. The vector index $\tilde{\mu}$ refers to the tangent space at the point $(t,\boldsymbol{\theta_3})$, while $\tilde{\nu}'$ to the tangent space at $(t,\boldsymbol{\theta'_3})$.

are causally disconnected can be moved to the same equal-time Cauchy surface by a suitable dS transformation. Thus, the locality of (3.92) implies that all anti-commutators of the form

$$\left\{ F^{-}_{\mu\nu}(t, \boldsymbol{\theta_3}), F^{-}_{\alpha'\beta'}(t', \boldsymbol{\theta'_3})^{\dagger} \right\}$$

vanish for any two points $(t, \boldsymbol{\theta_3})$ and $(t', \boldsymbol{\theta'_3})$ that are spacelike separated. Then, it follows that Grassmann-even observables, such as the currents $J^{\rho}(t, \boldsymbol{\theta_3}) = \overline{F}^{-}_{\mu\nu}(t, \boldsymbol{\theta_3})\gamma^{\rho} F^{-\mu\nu}(t, \boldsymbol{\theta_3})$, commute for spacelike separations. This shows the microcausality of the theory. This concludes the discussion of the quantisation of the chiral gravitino field.

4 Free graviton gauge potential on dS_4, UIRs of $so(4,1)$ and $so(4,2)$, quantisation and (anti-)self-duality

The graviton and its unitarity in de Sitter spacetime have been studied more extensively than the gravitino [27, 29, 77, 85]. The linearised Einstein-Hilbert action around a dS_4 background describes a real massless spin-2 field, $h_{\mu\nu} = h_{(\mu\nu)}$, propagating on a fixed dS spacetime. The linearised Einstein-Hilbert action (after some integrations by parts) can be expressed as [27]

$$S_{EH} = -\frac{1}{4} \int d^4x \sqrt{-g}\, h^{\mu\nu}\, H_{\mu\nu}(h), \tag{4.1}$$

with

$$\begin{aligned} H_{\mu\nu}(h) \equiv \nabla_{\mu}\nabla_{\alpha} h^{\alpha}_{\nu} + \nabla_{\nu}\nabla_{\alpha} h^{\alpha}_{\mu} - \Box h_{\mu\nu} + g_{\mu\nu}\,\Box h^{\alpha}{}_{\alpha} - \nabla_{\mu}\nabla_{\nu} h^{\alpha}{}_{\alpha} \\ - g_{\mu\nu}\,\nabla^{\alpha}\nabla^{\beta} h_{\alpha\beta} + 2\,h_{\mu\nu} + g_{\mu\nu} h^{\alpha}{}_{\alpha}, \end{aligned} \tag{4.2}$$

where $\Box = g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}$ is the Laplace-Beltrami operator on dS_4. The symbol h in $H^{\mu\nu}(h)$ does **not** stand for the trace of $h_{\alpha\beta}$. The equations of motion for linearised gravity on dS_4 are

$$H_{\mu\nu}(h) = 0. \tag{4.3}$$

The action (4.1) is invariant under gauge transformations of the form

$$\delta^{\text{gauge}}(Z)\, h_{\mu\nu} = \nabla_{(\mu} Z_{\nu)}, \tag{4.4}$$

where Z_{ν} is an arbitrary real vector gauge function. Note that, for any Z_{ν}, we have

$$H^{\mu\nu}(\delta^{\text{gauge}}(Z) h) = 0,$$

corresponding to the well-known gauge invariance of the linearised Einstein equations. As is well known (see, e.g., [77]), one can impose the transverse-traceless (TT) gauge condition, in which the field equations are

$$\begin{aligned} \Box h^{(\text{TT})}_{\mu\nu} &= 2 h^{(\text{TT})}_{\mu\nu}, \\ \nabla^{\mu} h^{(\text{TT})}_{\mu\nu} &= 0, \quad h^{(\text{TT})\alpha}_{\alpha} = 0. \end{aligned} \tag{4.5}$$

These equations still enjoy invariance under restricted gauge transformations

$$\delta^{\text{gauge}}_{\text{res}}(A)\, h^{(\text{TT})}_{\mu\nu} = \nabla_{(\mu} A_{\nu)}, \tag{4.6}$$

with

$$\Box A_\nu = -3A_\nu, \\ \nabla^\nu A_\nu = 0. \tag{4.7}$$

The dS Killing vectors ξ^μ act on tensors $h_{\mu\nu}$ via the Lie derivative

$$\pounds_\xi h_{\mu\nu} = \xi^\rho \nabla_\rho h_{\mu\nu} + (\nabla_\mu \xi^\rho)\, h_{\rho\nu} + (\nabla_\nu \xi^\rho)\, h_{\mu\rho}. \tag{4.8}$$

If $h_{\mu\nu}$ is a solution of eq. (4.5) (or (4.3)), then so is $\pounds_\xi h_{\mu\nu}$. Since Lie derivatives preserve the Lie brackets between Killing vectors, $[\pounds_\xi, \pounds_{\xi'}] h_{\mu\nu} = \pounds_{[\xi,\xi']} h_{\mu\nu}$, the space of solutions of eq. (4.5) is a representation space for the dS algebra $so(4,1)$. As in the case of the gravitino discussed earlier, a key aspect of our analysis will be how the mode solutions of (4.5) on global dS_4 form discrete series UIRs of $so(4,1)$. This has been discussed in detail in refs. [77, 85]. In the following subsection, we briefly review the main results from these references.

4.1 Discrete series UIRs of $so(4,1)$ in the space of graviton modes

A general classical TT solution $h^{(\mathrm{TT})}_{\mu\nu}$ of (4.5) can be expressed as a linear combination of physical modes, $\varphi^{(phys,\pm L;\, M;K)}_{\mu\nu}$, and pure-gauge modes, $\varphi^{(pg)}_{\mu\nu}$ [77]. Let us present the form of these modes.

TT pure-gauge graviton modes. The pure-gauge solutions of (4.5) are expressed in the form

$$\varphi^{(pg)}_{\mu\nu} = \nabla_{(\mu} A_{\nu)}, \tag{4.9}$$

where A_ν satisfies (4.7).

Physical graviton modes. In global coordinates (2.1), the physical modes of eq. (4.5), with negative (-2) and positive $(+2)$ helicity, are given by [77]

$$\varphi^{(phys,\,-L;\,M;K)}_{t\mu}(t,\boldsymbol{\theta}_3) = 0, \qquad \mu \in \{t,\theta_3,\theta_2,\theta_1\} \\ \varphi^{(phys,\,-L;\,M;K)}_{\tilde{\mu}\tilde{\nu}}(t,\boldsymbol{\theta}_3) = \left(\frac{2(L+1)}{L(L+2)}\right)^{1/2} \kappa_L(t)\, \tilde{T}^{(-;LM;K)}_{\tilde{\mu}\tilde{\nu}}(\boldsymbol{\theta_3}), \tag{4.10}$$

and

$$\varphi^{(phys,\,+L;\,M;K)}_{t\mu}(t,\boldsymbol{\theta}_3) = 0, \qquad \mu \in \{t,\theta_3,\theta_2,\theta_1\} \\ \varphi^{(phys,\,+L;\,M;K)}_{\tilde{\mu}\tilde{\nu}}(t,\boldsymbol{\theta}_3) = \left(\frac{2(L+1)}{L(L+2)}\right)^{1/2} \kappa_L(t)\, \tilde{T}^{(+;LM;K)}_{\tilde{\mu}\tilde{\nu}}(\boldsymbol{\theta_3}), \tag{4.11}$$

respectively,[27] where $\tilde{\mu}, \tilde{\nu}$ are tensor indices on S^3. The labels L, M and K are angular momentum quantum numbers corresponding to the chain of subalgebras $so(4) \supset so(3) \supset$

[27] There is an extra factor of $2^{1/2}$ in the normalisation factor of the mode functions (4.10) and (4.11) relative to the mode functions in [77] because of our different convention for the Klein-Gordon scalar product (4.23).

$so(2)$ with $L \in \{2,3,\ldots\}$, $M \in \{2,3,\ldots,L\}$ and $K \in \{-M,\ldots,0,\ldots,M\}$. The function $\kappa_L(t)$ is given by

$$\begin{aligned}\kappa_L(t) =& \; 2\left(\sin\frac{x(t)}{2}\right)^{L+2}\left(\cos\frac{x(t)}{2}\right)^{-L}\left(1+\frac{\cos\left(x(t)\right)}{L+1}\right)\\ &= \cosh t\left(1+\frac{i\sinh t}{L+1}\right)\left(\frac{1-i\sinh t}{1+i\sinh t}\right)^{(L+1)/2}, \end{aligned} \tag{4.12}$$

where the variable $x(t)$ is defined in (3.22). With the conformal time τ this can be given as

$$\kappa_L(t) = \frac{1}{\cos\tau}\left(1+\frac{i\tan\tau}{L+1}\right)e^{-i(L+1)\tau}. \tag{4.13}$$

Symmetric rank-2 tensor spherical harmonics on S^3. The $\boldsymbol{\theta_3}$-dependence of the physical modes in eqs. (4.10) and (4.11) is given by the rank-2 tensor spherical harmonics on S^3, $\tilde{T}^{(\pm;L;M;K)}_{\tilde{\mu}\tilde{\nu}}(\boldsymbol{\theta_3}) = \tilde{T}^{(\pm;L;M;K)}_{(\tilde{\mu}\tilde{\nu})}(\boldsymbol{\theta_3})$. These satisfy

$$\begin{aligned}\tilde{\Box}\tilde{T}^{(\pm;L;M;K)}_{\tilde{\mu}\tilde{\nu}} &= (-L(L+2)+2)\,\tilde{T}^{(\pm;L;M;K)}_{\tilde{\mu}\tilde{\nu}},\\ \tilde{\nabla}^{\tilde{\mu}}\tilde{T}^{(\pm;L;M;K)}_{\tilde{\mu}\tilde{\nu}} &= 0, \quad \tilde{g}^{\tilde{\mu}\tilde{\nu}}\tilde{T}^{(\pm;L;M;K)}_{\tilde{\mu}\tilde{\nu}} = 0, \end{aligned} \tag{4.14}$$

where $\tilde{\Box} = \tilde{g}^{\tilde{\alpha}\tilde{\beta}}\tilde{\nabla}_{\tilde{\alpha}}\tilde{\nabla}_{\tilde{\beta}}$ is the Laplace-Beltrami operator on S^3. The spherical harmonics $\tilde{T}^{(+;L;M;K)}_{\tilde{\mu}\tilde{\nu}}$ and $\tilde{T}^{(-;L;M;K)}_{\tilde{\mu}\tilde{\nu}}$ are self-dual and anti-self-dual,[28] respectively, in the sense that they are eigenfunctions of the duality operator (helicity operator), as [77]

$$\frac{1}{L+1}\tilde{\varepsilon}_{\tilde{\mu}}{}^{\tilde{\alpha}\tilde{\beta}}\tilde{\nabla}_{\tilde{\alpha}}\tilde{T}^{(\pm;L;M;K)}_{\tilde{\beta}\tilde{\nu}} = \pm\tilde{T}^{(\pm;L;M;K)}_{\tilde{\mu}\tilde{\nu}}. \tag{4.15}$$

(The anti-symmetric part of the left-hand side vanishes because $\tilde{T}^{(\pm;L;M;K)}_{\tilde{\mu}\tilde{\nu}}$ are transverse and traceless.) We note that

$$\tilde{\varepsilon}_{\tilde{\mu}}{}^{\tilde{\alpha}\tilde{\beta}}\tilde{\nabla}_{\tilde{\alpha}}\left(\tilde{\varepsilon}_{\tilde{\beta}}{}^{\tilde{\lambda}\tilde{\kappa}}\tilde{\nabla}_{\tilde{\lambda}}\tilde{T}^{(\sigma;L;M;K)}_{\tilde{\kappa}\tilde{\nu}}\right) = (-\tilde{\nabla}_{\tilde{\alpha}}\tilde{\nabla}^{\tilde{\alpha}}+3)T^{(\sigma;L;M;K)}_{\tilde{\mu}\tilde{\nu}}, \quad \sigma = \pm. \tag{4.16}$$

Thus, the duality operator is a square-root of the operator $-\tilde{\nabla}_{\tilde{\alpha}}\tilde{\nabla}^{\tilde{\alpha}}+3$ on the TT spin-2 tensors on S^3. The TT spin-2 tensor spherical harmonics are normalised with respect to the standard inner product on S^3 [28]:

$$\begin{aligned}\int_{S^3}\sqrt{\tilde{g}}\,d\boldsymbol{\theta_3}\;\tilde{g}^{\tilde{\mu}\tilde{\nu}}\tilde{g}^{\tilde{\alpha}\tilde{\beta}}\;\tilde{T}^{(\sigma;L;M;K)*}_{\tilde{\mu}\tilde{\alpha}}(\boldsymbol{\theta_3})\;\tilde{T}^{(\sigma';L';M';K')}_{\tilde{\nu}\tilde{\beta}}(\boldsymbol{\theta_3})&\\ = \delta_{\sigma\sigma'}\,\delta_{LL'}\,\delta_{M\,M'}\delta_{KK'},& \end{aligned} \tag{4.17}$$

where $\sigma,\sigma' \in \{+,-\}$. For each value of $L \in \{2,3,\ldots\}$, the set $\{\tilde{T}^{(+;L;M;K)}_{\tilde{\mu}\tilde{\nu}}\}$ forms a $so(4)$ representation with highest weight given by [83, 85]:

$$\vec{f}^{(+2)}_L = (L,+2), \tag{4.18}$$

while the set $\{\tilde{T}^{(-;L;M;K)}_{\tilde{\mu}\tilde{\nu}}\}$ forms a $so(4)$ representation with highest weight given by [83, 85]:

$$\vec{f}^{(-2)}_L = (L,-2). \tag{4.19}$$

[28] This notion of (anti-)self-duality should not be confused with the notion of (anti-)self-duality defined using $\varepsilon_{\mu\nu\rho\sigma}$ on dS_4 — see e.g., eqs. (3.64) and (4.58).

Positive and negative frequency. The physical graviton mode functions (4.10) and (4.11) are the analogues of positive frequency modes, as for short wavelengths, $L \gg 1$, they satisfy [77]

$$\frac{\partial}{\partial t}\varphi^{(phys,\,\pm L;\,M;K)}_{\mu\nu}(t,\boldsymbol{\theta}_3) \sim -i\frac{L}{\cosh t}\varphi^{(phys,\,\pm L;\,M;K)}_{\mu\nu}(t,\boldsymbol{\theta}_3). \tag{4.20}$$

Eqs. (4.5) also admits physical TT solutions that are the analogues of negative frequency modes. The negative frequency graviton modes $\varphi^{(phys,\pm L;\,M;K)\star}_{\mu\nu}$ are obtained from the positive frequency graviton modes $\varphi^{(phys,\,\pm L;\,M;K)}_{\mu\nu}$ given by eqs. (4.10) and (4.11) by replacing $\kappa_L(t)$ with its complex conjugate. That is,

$$\varphi^{(phys,\,\pm L;\,M;K)\star}_{\tilde{\mu}\tilde{\nu}}(t,\boldsymbol{\theta}_3) = \left(\frac{2(L+1)}{L(L+2)}\right)^{1/2}\kappa^*_L(t)\,\tilde{T}^{(\pm;LM;K)}_{\tilde{\mu}\tilde{\nu}}(\boldsymbol{\theta_3}). \tag{4.21}$$

Note. The field strength (4.59) calculated for the positive frequency modes of helicity -2, $\varphi^{(phys,\,-L;\,M;K)}_{\mu\nu}$, is anti-self-dual, and so is the field strength for the negative frequency modes of helicity $+2$, $\varphi^{(phys,\,+L;\,M;K)\star}_{\mu\nu}$. Similarly, the field strength (4.59) calculated for the positive frequency modes of helicity $+2$, $\varphi^{(phys,\,+L;\,M;K)}_{\mu\nu}$, is self-dual, and so is the field strength for the negative frequency modes of helicity -2, $\varphi^{(phys,\,-L;\,M;K)\star}_{\mu\nu}$. See subsection 4.3 for more details on the mode expansion of the field strength and (anti-)self-duality.

Graviton discrete series UIRs of $\boldsymbol{so(4,1)}$. The two sets of (positive frequency) physical modes $\{\varphi^{(phys,\,-L;\,M;K)}_{\mu\nu}\}$ and $\{\varphi^{(phys,\,+L;\,M;K)}_{\mu\nu}\}$ separately form two discrete series UIRs of $so(4,1)$ [77, 85]. In particular, they form the direct sum: $D^+(3,2)\bigoplus D^-(3,2)$ — see appendix A for details on our notation of the UIRs. It can be seen that each of these two sets of modes forms an UIR as follows. The two sets $\{\varphi^{(phys,\,-L;\,M;K)}_{\mu\nu}\}$ and $\{\varphi^{(phys,\,+L;\,M;K)}_{\mu\nu}\}$ do not mix with each other under any $so(4)$ transformation as they belong to different $so(4)$ representations [eqs. (4.18) and (4.19), respectively]. Also, they do not mix with each other under any dS boost, as under (3.42) they transform as [77]:

$$\begin{aligned}\pounds_B\varphi^{(phys,\,\pm L;\,M;K)}_{\mu\nu} = &-\frac{i}{2}\sqrt{(L-M+1)(L+M+2)}\,\varphi^{(phys,\,\pm(L+1);\,M;K)}_{\mu\nu}\\ &-\frac{i}{2}\sqrt{(L-M)(L+M+1)}\,\varphi^{(phys,\,\pm(L-1);\,M;K)}_{\mu\nu}+(\text{pure-gauge}),\end{aligned} \tag{4.22}$$

where the term '(pure-gauge)' is a TT pure-gauge mode (4.9). One can thus conclude that $\{\varphi^{(phys,\,-L;\,M;K)}_{\mu\nu}\}$ and $\{\varphi^{(phys,\,+L;\,M;K)}_{\mu\nu}\}$ separately form irreducible representations. As the pure-gauge modes are orthogonal to themselves and to all physical modes with respect to the Klein-Gordon inner product (this inner product will be introduced shortly) [77], the physical modes form representations with the following equivalence relation: if for any two physical modes, $\varphi^{(1)}_{\mu\nu}$ and $\varphi^{(2)}_{\mu\nu}$, the difference $\varphi^{(1)}_{\mu\nu}-\varphi^{(2)}_{\mu\nu}$ is a linear combination of pure-gauge modes, then $\varphi^{(1)}_{\mu\nu}$ and $\varphi^{(2)}_{\mu\nu}$ belong to the same equivalence class.[29] These irreducible representations are unitary because the Klein-Gordon inner product:

$$\langle\varphi^{(1)}|\varphi^{(2)}\rangle_{KG} = \frac{i}{4}\int_{S^3}d\boldsymbol{\theta_3}\sqrt{-g}\left(\varphi^{(1)\mu\nu *}\,\frac{\partial}{\partial t}\varphi^{(2)}_{\mu\nu}-\varphi^{(2)}_{\mu\nu}\,\frac{\partial}{\partial t}\varphi^{(1)\mu\nu *}\right), \tag{4.23}$$

[29] Eq. (4.22) agrees with the expression for the infinitesimal boost matrix elements in the discrete series UIRs of $so(4,1)$ with $\Delta = 3$ and $s = 2$ [55, 56]. See appendix A and refs. [33, 85] for the translation between the old and modern notation for the labels of the $so(4,1)$ UIRs.

is both positive definite (for physical positive frequency modes) and dS invariant, where $\varphi^{(1)}$ and $\varphi^{(2)}$ are any two classical solutions of eqs. (4.5). The Klein-Gordon inner product is related to the Klein-Gordon current,

$$J^{\mu}_{KG}\left(\varphi^{(1)},\varphi^{(2)}\right)=-\frac{i}{4}\left(\varphi^{(1)\alpha\beta*}\ \nabla^{\mu}\varphi^{(2)}_{\alpha\beta}-\varphi^{(2)}_{\alpha\beta}\ \nabla^{\mu}\varphi^{(1)\alpha\beta*}\right),\quad \nabla_{\mu}J^{\mu}_{KG}\left(\varphi^{(1)},\varphi^{(2)}\right)=0, \tag{4.24}$$

as

$$\langle\varphi^{(1)}|\varphi^{(2)}\rangle_{KG}=\int_{S^3}d\boldsymbol{\theta_3}\sqrt{-g}\ J^{t}_{KG}\left(\varphi^{(1)},\varphi^{(2)}\right). \tag{4.25}$$

The positive definiteness of the Klein-Gordon inner product in the positive frequency sector — and negative definiteness in the negative frequency sector — has been explicitly verified in refs. [28, 77], as:

$$\langle\varphi^{(phys,\,\sigma L;\,M;K)}|\varphi^{(phys,\,\sigma' L';\,M';K')}\rangle_{KG}=\delta_{\sigma\sigma'}\delta_{LL'}\delta_{MM'}\delta_{KK'}, \tag{4.26}$$

$$\langle\varphi^{(phys,\,\sigma L;\,M;K)\star}|\varphi^{(phys,\,\sigma' L';\,M';K')\star}\rangle_{KG}=-\delta_{\sigma\sigma'}\delta_{LL'}\delta_{MM'}\delta_{KK'},$$
$$\langle\varphi^{(phys,\,\sigma L;\,M;K)\star}|\varphi^{(phys,\,\sigma' L';\,M';K')}\rangle_{KG}=0 \tag{4.27}$$

with $\sigma,\sigma'\in\{+,-\}$. Also,

$$\langle\varphi^{(1)}|\varphi^{(pg)}\rangle_{KG}=0, \tag{4.28}$$

where $\varphi^{(1)}_{\mu\nu}$ is any physical or pure-gauge mode, and thus, the pure-gauge modes can be identified with zero as they are orthogonal to all modes, including themselves. Moreover, the anti-hermiticity of the generators (Lie derivatives) is known [28, 77], as:

$$\langle\pounds_{\xi}\varphi^{(1)}|\varphi^{(2)}\rangle_{KG}+\langle\varphi^{(1)}|\pounds_{\xi}\varphi^{(2)}\rangle_{KG}=0, \tag{4.29}$$

for any two solutions $\varphi^{(1)},\varphi^{(2)}$ of eqs. (4.5) and any Killing vector ξ^{μ}. To conclude:

- The positive frequency physical graviton modes with positive helicity, $\{\varphi^{(phys,\,+L;\,M;K)}_{\mu\nu}\}$, form the discrete series UIR $D^{+}(\Delta,s)=D^{+}(3,2)$ of $so(4,1)$ — see appendix A. The $so(4)$ content corresponds to the $so(4)$ highest weights (4.18). The $so(4,1)$-invariant inner product that is positive definite is given by (4.23).

- The positive frequency physical graviton modes with negative helicity, $\{\varphi^{(phys,\,-L;\,M;K)}_{\mu\nu}\}$, form the discrete series UIR $D^{-}(\Delta,s)=D^{-}(3,2)$ of $so(4,1)$ — see appendix A. The $so(4)$ content corresponds to the $so(4)$ highest weights (4.19). The $so(4,1)$-invariant inner product that is positive definite is again given by eq. (4.23).

The negative frequency modes, $\{\varphi^{(phys,\,-L;\,M;K)\star}_{\mu\nu}\}$ and $\{\varphi^{(phys,\,+L;\,M;K)\star}_{\mu\nu}\}$, also form the direct sum $D^{+}(3,2)\bigoplus D^{-}(3,2)$ of discrete series UIRs of $so(4,1)$, where the positive-definite inner product is given by the negative of the Klein-Gordon inner product. The transformation of the negative frequency modes under the dS boost (3.42) is found by replacing the coefficients on the right-hand side of (4.22) with their complex conjugates.

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4.2 Conformal-like symmetry for the (real and complex) graviton and UIRs of $so(4,2)$

In this subsection, we discuss a conformal-like symmetry of the graviton gauge potential generated by the genuine conformal Killing vectors (2.13) of dS_4, akin to the conformal-like symmetry of the gravitino discussed in subsections 3.2 and 3.3. This symmetry is the dS analogue of the symmetry found for strictly massless gauge potentials on AdS_4 in the unfolded formalism by Vasiliev [75]. We will present new details on how the conformal-like symmetry acts on graviton mode functions on dS_4 and how these form UIRs of $so(4,2)$. We will also investigate the invariance of the action functional (4.49) under conformal-like transformations. Before proceeding to the technical details and mathematical expressions, let us give some details on the outline of this subsection since there are certain subtleties concerning the reality properties of the graviton — see also [75].

Outline and subtleties concerning the conformal-like symmetries of mode solutions, and of graviton field theory. We will start by discussing the conformal-like transformation for the graviton, $T_V h_{\mu\nu}$ [eq. (4.31)], which is a symmetry (a map from solutions to other solutions) for both the full linearised Einstein equations (4.3) and the graviton equations in the TT gauge (4.5). We will show that the conformal-like transformations T_V enlarge the symmetry of the field equations from $so(4,1)$ to $so(5,1)$, but the $so(5,1)$ algebra closes up to gauge transformations of the graviton. However, when the transformation T_V acts on TT mode solutions it fails to preserve the Klein-Gordon inner product (4.23), and thus, the graviton mode solutions cannot form UIRs of the enlarged algebra $so(5,1)$. Moreover, the conformal-like transformation fails to be a symmetry of the linearised Einstein-Hilbert action (4.1). Interestingly, introducing a modified version of the conformal-like transformation by inserting a factor of $i=\sqrt{-1}$ as $\mathcal{T}_V \equiv iT_V$ [eq. (4.35)], we will show the modified transformation is a symmetry of not only the field equations (4.3) and (4.5) but also of the Klein-Gordon inner product. The Lie brackets will also be modified so that the full algebra closes on $so(4,2)$ (up to gauge transformations), instead of $so(5,1)$. Once this modification of the conformal-like transformation has been made, we will show that the positive frequency mode functions, $\{\varphi_{\mu\nu}^{(phys,\,-L;\,M;K)}\}$ and $\{\varphi_{\mu\nu}^{(phys,\,+L;\,M;K)}\}$, separately form UIRs of the conformal-like algebra $so(4,2)$, as in the case of the gravitino modes discussed in subsection 3.2. We will also show that there is a hermitian action (4.49) functional for the complex graviton $\mathfrak{h}_{\mu\nu}$ which enjoys invariance under the conformal-like transformations $\mathcal{T}_V\mathfrak{h}_{\mu\nu}$.

Subtleties concerning the conformal-like symmetries and the reality of the graviton. At this point, certain subtleties need to be discussed concerning the reality properties of the graviton related to the afore-mentioned introduction of a factor of i. The transformation $\mathcal{T}_V$ cannot act on the real field $h_{\mu\nu}$ because of the factor of i. Thus, the field $h_{\mu\nu}$ needs to be replaced by a complex field, which we denote by $\mathfrak{h}_{\mu\nu}$. The negative frequency part of $\mathfrak{h}_{\mu\nu}$ describes the 'anti-particle' whereas the positive frequency part describes the 'particle'. Both parts acquire the same phase factor under the transformation $\mathcal{T}_V$. This means that the phase factor for the positive frequency modes for the 'anti-particle' is the complex conjugate of the phase factor of the positive frequency modes for the 'particle'. Some basic properties of the transformations T_V and $\mathcal{T}_V$ are summarised in tables 1 and 2.

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Conformal-like transformation on modes	Is (✓)/ Is not (×) a symmetry of	Algebra
$T_V\varphi^{(phys,\,\pm L;\,M;K)}_{\mu\nu}$ (4.31)	Field equation (4.5) ✓. Inner product (4.23) ×	$so(5,1)$
$\mathcal{T}_V\varphi^{(phys,\,\pm L;\,M;K)}_{\mu\nu}$ (4.35)	Field equation (4.5) ✓. Inner product (4.23) ✓	$so(4,2)$

Table 1. Conformal-like symmetry and graviton mode solutions.

Conformal-like transformation of the:	Is (✓)/ Is not (×) a symmetry of
Real graviton, $T_V h_{\mu\nu}$ (4.31)	Field equations (4.3), (4.5) ✓. Action (4.1) ×
Complex graviton, $\mathcal{T}_V\mathfrak{h}_{\mu\nu}$ (4.35)	Field equations (4.36), (4.39) ✓. Action (4.49) ✓

Table 2. Conformal-like symmetry: real vs. complex graviton field theory.

Note. The complex graviton will be relevant in our discussion on SUSY in section 6. In particular, in our unitary supersymmetric model in subsection 6.2, the super-partner of a chiral gravitino is a chiral graviton. Both of these fields have anti-self-dual field strengths, and thus, must be complex [87].

4.2.1 Real graviton field theory, conformal-like transformation and $so(5,1)$

The differential operator underlying the conformal-like symmetry transformation is

$$ {}_V D_{\mu\nu}{}^{\alpha\beta} = \frac{1}{2} V^\rho \varepsilon_{\rho\sigma\lambda(\mu}\left(\delta^\beta_{\nu)}\, g^{\lambda\alpha}\,\nabla^\sigma + \delta^\alpha_{\nu)}\, g^{\lambda\beta}\,\nabla^\sigma\right), \tag{4.30} $$

where V^ρ is any genuine conformal Killing vector (2.13). The conformal-like transformation acts on generic symmetric spin-2 fields, $B_{\mu\nu}$, as[30]

$$ T_V B_{\mu\nu} \equiv {}_V D_{\mu\nu}{}^{\alpha\beta} B_{\alpha\beta} = V^\rho \varepsilon_{\rho\sigma\lambda(\mu}\nabla^\sigma B^\lambda{}_{\nu)}. \tag{4.31} $$

Conformal-like invariance of real graviton field equations. We will first show that the conformal-like transformation $T_V h_{\mu\nu}$ for the real graviton is a symmetry of the standard linearised Einstein equations $H_{\mu\nu}(h) = 0$ [see eq. (4.3)]. For the sake of generality, let us work with a symmetric spin-2 field $B_{\mu\nu}$ which may not obey the linearised Einstein equations. Using the expressions (D.2), it is easy to show that $H_{\mu\nu}(T_V B)$ can be expressed as

$$ H_{\mu\nu}(T_V B) = V^\rho \varepsilon_{\rho\sigma\lambda(\mu}\nabla^\sigma H^\lambda{}_{\nu)}(B) = T_V H_{\mu\nu}(B), \tag{4.32} $$

for any symmetric spin-2 field $B_{\mu\nu}$. This means that if $B_{\mu\nu} = h_{\mu\nu}$ satisfies the linearised Einstein equations, $H_{\mu\nu}(h) = 0$ [eq. (4.3)], then $T_V h_{\mu\nu}$ also satisfies the same equations, i.e. $H_{\mu\nu}(T_V h) = 0$. In other words, the conformal-like transformation (4.31) of the real graviton is a symmetry of the linearised Einstein equations (4.3). Furthermore, as eqs. (D.2) hold for any symmetric spin-2 field, we can also apply them to the case of the real graviton in the TT gauge $B_{\mu\nu} = h^{(\mathrm{TT})}_{\mu\nu}$. We thus find that if $h^{(\mathrm{TT})}_{\mu\nu}$ satisfies eqs. (4.5), then so does $T_V h^{(\mathrm{TT})}_{\mu\nu}$, i.e. $T_V h^{(\mathrm{TT})}_{\mu\nu}$ is a TT solution to the linearised Einstein equations. Thus, the conformal-like transformation (4.31) is a symmetry of the field equations of the real graviton in the TT gauge [eqs. (4.5)] — this is also true, of course, for the case of the TT graviton mode solutions which are complex.

[30] Note the similarity of the expression (4.31) with the hidden symmetry transformation of the Maxwell gauge potential in Minkowski spacetime, given by equation (41) in [89].

***so*(5, 1) symmetry for real graviton.** Consider a TT solution of (4.5), $h_{\mu\nu}^{(\mathrm{TT})}$. The structure of the full symmetry algebra, generated by the ten dS isometries (4.8) and the five conformal-like symmetries (4.31), is described by the following commutation relations:

$$[\pounds_\xi, \pounds_{\xi'}]h_{\mu\nu}^{(\mathrm{TT})} = \pounds_{[\xi,\xi']}h_{\mu\nu}^{(\mathrm{TT})}, \tag{4.33a}$$

$$[\pounds_\xi, T_V]h_{\mu\nu}^{(\mathrm{TT})} = T_{[\xi,V]}h_{\mu\nu}^{(\mathrm{TT})}, \tag{4.33b}$$

$$[T_{V'}, T_V]h_{\mu\nu}^{(\mathrm{TT})} = -\pounds_{[V',V]}h_{\mu\nu}^{(\mathrm{TT})} + \nabla_{(\mu}\left[-\frac{1}{2}\left(\nabla^\kappa h^{(\mathrm{TT})}_{\ \ \nu)\sigma}\right)\nabla_\kappa[V',V]^\sigma + [V',V]^\sigma h^{(\mathrm{TT})}_{\nu)\sigma}\right], \tag{4.33c}$$

where ξ^μ, ξ'^μ are any two dS Killing vectors, while V^μ, V'^μ are any two genuine conformal Killing vectors (2.13). The commutators (4.33a)–(4.33c) coincide with the commutation relations of $so(5,1)$ up the field-dependent gauge transformation in (4.33c). If the minus sign in front of $\pounds_{[V',V]}$ in (4.33c) gets flipped, then the commutation relations will be the ones of $so(4,2)$. This will be the case when we consider the modified conformal-like transformation acting on complex gravitons later.

Non-invariance of Klein-Gordon inner product. Let $\varphi_{\mu\nu}^{(1)}$ and $\varphi_{\mu\nu}^{(2)}$ be any two TT graviton mode solutions of (4.5). A straightforward calculation shows that the infinitesimal change of the Klein-Gordon inner product under T_V [(4.31)] is **not** zero

$$\langle T_V\varphi^{(1)}|\varphi^{(2)}\rangle_{KG} + \langle\varphi^{(1)}|T_V\varphi^{(2)}\rangle_{KG} \neq 0.$$

In other words, the conformal-like transformations T_V are not anti-hermitian, and thus, the corresponding $so(5,1)$ representation cannot be unitary. In fact, the conformal-like transformations $T_V h_{\mu\nu}$ are hermitian.

Non-invariance of the linearised Einstein-Hilbert action (4.1). Using (4.32), it is easy to show that the variation of the action (4.1) under $\delta h_{\mu\nu} = T_V h_{\mu\nu}$, does **not** vanish,

$$\delta S_{EH} = -\frac{1}{4}\int d^4x\,\sqrt{-g}\,\left(T_V h^{\mu\nu}\,H_{\mu\nu}(h) + h^{\mu\nu}\,H_{\mu\nu}(T_V h)\right) \neq 0. \tag{4.34}$$

Also, δS_{EH} is **not** equal to the integral of a total divergence. We conclude that $T_V h_{\mu\nu}$ is **not** a symmetry of the Einstein-Hilbert action.

4.2.2 Complex graviton field theory, conformal-like transformation and *so*(4, 2)

Consider a modified version of the conformal-like transformation of the real graviton by introducing a factor of i, as $\mathcal{T}_V \equiv iT_V$. We will also refer to $\mathcal{T}_V$ as a conformal-like transformation. As we explained earlier, although T_V [eq. (4.31)] can act on both real and complex graviton fields, which we denote as $h_{\mu\nu}$ and $\mathfrak{h}_{\mu\nu}$, respectively, the transformation $\mathcal{T}_V$ acts only on the complex graviton field, as

$$\mathcal{T}_V\mathfrak{h}_{\mu\nu} \equiv iT_V\mathfrak{h}_{\mu\nu} = iV^\rho\varepsilon_{\rho\sigma\lambda(\mu}\nabla^\sigma\mathfrak{h}^\lambda_{\ \nu)}, \tag{4.35}$$

where V^μ is any genuine conformal Killing vector (2.13). Let us first give some details for the complex graviton theory.

The complex graviton field. The on-shell complex graviton field satisfies the linearised Einstein equations (4.3) — with $h_{\mu\nu}$ replaced by $\mathfrak{h}_{\mu\nu}$ — as

$$H_{\mu\nu}(\mathfrak{h}) = 0, \tag{4.36}$$

where

$$\begin{aligned} H_{\mu\nu}(\mathfrak{h}) \equiv \nabla_\mu \nabla_\alpha \mathfrak{h}^\alpha_\nu + \nabla^\nu \nabla_\alpha \mathfrak{h}^\alpha_\mu - \Box \mathfrak{h}_{\mu\nu} + g_{\mu\nu} \Box \mathfrak{h}^\alpha{}_\alpha - \nabla_\mu \nabla_\nu \mathfrak{h}^\alpha{}_\alpha \\ - g_{\mu\nu} \nabla^\alpha \nabla^\beta \mathfrak{h}_{\alpha\beta} + 2\, \mathfrak{h}_{\mu\nu} + g_{\mu\nu} \mathfrak{h}^\alpha{}_\alpha. \end{aligned} \tag{4.37}$$

Eq. (4.36) is invariant under complex gauge transformations of the form

$$\delta^{\text{gauge}}(\mathcal{Z})\, \mathfrak{h}_{\mu\nu} = \nabla_{(\mu} \mathcal{Z}_{\nu)}, \tag{4.38}$$

where $\mathcal{Z}_\nu$ is an arbitrary complex vector gauge function. In fact the gauge invariance of the field equation follows from the off-shell property: $H_{\mu\nu}(\delta^{\text{gauge}}(\mathcal{Z})\mathfrak{h}) = 0$. In the TT gauge, the field equations for the complex graviton are

$$\begin{aligned} \Box \mathfrak{h}^{(\text{TT})}_{\mu\nu} &= 2\, \mathfrak{h}^{(\text{TT})}_{\mu\nu}, \\ \nabla^\mu \mathfrak{h}^{(\text{TT})}_{\mu\nu} &= 0, \quad \mathfrak{h}^{(\text{TT})\alpha}_{\ \alpha} = 0, \end{aligned} \tag{4.39}$$

and they enjoy invariance under restricted gauge transformations

$$\delta^{\text{gauge}}_{\text{res}}(\mathfrak{A})\, \mathfrak{h}^{(\text{TT})}_{\mu\nu} = \nabla_{(\mu} \mathfrak{A}_{\nu)}, \tag{4.40}$$

where the complex gauge function satisfies

$$\begin{aligned} \Box \mathfrak{A}_\nu &= -3\, \mathfrak{A}_\nu, \\ \nabla^\nu \mathfrak{A}_\nu &= 0. \end{aligned} \tag{4.41}$$

Conformal-like invariance of complex graviton field equations. As eq. (4.32) holds for any (complex or real) symmetric spin-2 field, it follows that if $\mathfrak{h}_{\mu\nu}$ is a solution of the field equation (4.36), then so is $\mathcal{T}_V \mathfrak{h}_{\mu\nu} = i\, T_V \mathfrak{h}_{\mu\nu}$. In the TT gauge it is easy to show by using (D.2) that if $\mathfrak{h}^{(\text{TT})}_{\mu\nu}$ is a solution of eq. (4.39), then so is $\mathcal{T}_V \mathfrak{h}^{(\text{TT})}_{\mu\nu}$. Thus, $\mathcal{T}_V$ is a symmetry of the complex graviton field equations both in their non-gauge-fixed form (4.36) and in the TT gauge (4.39).

***so*(4, 2) symmetry for complex graviton.** Consider a TT solution of (4.39), $\mathfrak{h}^{(\text{TT})}_{\mu\nu}$. The commutators for the full symmetry algebra, generated by the ten dS isometries (4.8) and the five conformal-like symmetries (4.35), can be found by multiplying V and V' by i and replacing $h_{\mu\nu}$ by $\mathfrak{h}_{\mu\nu}$ in eqs. (4.33a)–(4.33c). We find in this manner,

$$[\mathcal{L}_\xi, \mathcal{L}_{\xi'}] \mathfrak{h}^{(\text{TT})}_{\mu\nu} = \mathcal{L}_{[\xi,\xi']} \mathfrak{h}^{(\text{TT})}_{\mu\nu}, \tag{4.42a}$$

$$[\mathcal{L}_\xi, \mathcal{T}_V] \mathfrak{h}^{(\text{TT})}_{\mu\nu} = \mathcal{T}_{[\xi,V]} \mathfrak{h}^{(\text{TT})}_{\mu\nu}, \tag{4.42b}$$

$$[\mathcal{T}_{V'}, \mathcal{T}_V] \mathfrak{h}^{(\text{TT})}_{\mu\nu} = +\mathcal{L}_{[V',V]} \mathfrak{h}^{(\text{TT})}_{\mu\nu} - \nabla_{(\mu} \left[-\frac{1}{2} \left(\nabla^\kappa \mathfrak{h}^{(\text{TT})}_{\nu)\sigma} \right) \nabla_\kappa [V', V]^\sigma + [V', V]^\sigma \mathfrak{h}^{(\text{TT})}_{\nu)\sigma} \right], \tag{4.42c}$$

where ξ^μ, ξ'^μ are any two dS Killing vectors, and V^μ, V'^μ are any two genuine conformal Killing vectors (2.13). The commutators (4.42a)–(4.42c) coincide with the commutation relations of $so(4, 2)$ up the field-dependent gauge transformation in (4.42c), as in the gravitino case in (3.52a)–(3.52c). The sign difference between eqs. (4.33c) and (4.42c), corresponds to the difference between $so(5, 1)$ and $so(4, 2)$.

Conformal-like invariance of Klein-Gordon inner product. Let $\varphi^{(1)}_{\mu\nu}$ and $\varphi^{(2)}_{\mu\nu}$ be any two TT graviton mode solutions — these are solutions of both (4.5) and (4.39). We will show that the Klein-Gordon inner product (4.23) is invariant under the (complex) conformal-like transformations [eq. (4.35)]

$$\mathcal{T}_V \varphi^{(1,2)}_{\mu\nu} = i\, V^\rho \varepsilon_{\rho\sigma\lambda(\mu} \nabla^\sigma \varphi^{(1,2)\lambda}{}_{\nu)}.$$

Let us start by considering the Klein-Gordon current $J^\mu_{KG}\left(\varphi^{(1)}, \varphi^{(2)}\right)$ [eq. (4.24)]. After a straightforward calculation, the infinitesimal change $\delta_V J^\mu_{KG}\left(\varphi^{(1)}, \varphi^{(2)}\right)$ under $\mathcal{T}_V$ is found to be equal to the divergence of an rank-2 antisymmetric tensor as follows:

$$\begin{aligned}\delta_V J^\mu_{KG}\left(\varphi^{(1)}, \varphi^{(2)}\right) &= J^\mu_{KG}\left(\mathcal{T}_V\varphi^{(1)}, \varphi^{(2)}\right) + J^\mu_{KG}\left(\varphi^{(1)}, \mathcal{T}_V\varphi^{(2)}\right) \\ &= -\frac{1}{2}\nabla_\sigma\left(V^\rho\ \varphi^{(1)\lambda*}_{\ \ \beta}\ \varepsilon_{\rho\lambda\alpha}{}^{[\sigma}\ \nabla^{\mu]}\varphi^{(2)\alpha\beta} + V^\rho\ \varphi^{(2)\lambda}_{\ \ \beta}\ \varepsilon_{\rho\lambda\alpha}{}^{[\sigma}\ \nabla^{\mu]}\varphi^{(1)\alpha\beta*}\right).\end{aligned} \tag{4.43}$$

It immediately follows that the Klein-Gordon inner product (4.23) remains invariant under infinitesimal conformal-like transformations, as

$$\delta_V\, \langle\varphi^{(1)}|\varphi^{(2)}\rangle_{KG} = \int_{S^3} d\boldsymbol{\theta_3}\sqrt{-g}\ \ \delta_V J^t_{KG}\left(\varphi^{(1)}, \varphi^{(2)}\right) = 0, \tag{4.44}$$

for any genuine conformal Killing vector V^μ (2.13). This implies the anti-hermiticity of all five conformal-like generators

$$\langle\mathcal{T}_V\varphi^{(1)}|\varphi^{(2)}\rangle_{KG} + \langle\varphi^{(1)}|\mathcal{T}_V\varphi^{(2)}\rangle_{KG} = 0. \tag{4.45}$$

Since the requirements of positive-definiteness of the Klein-Gordon inner product and anti-hermiticity of all 15 $so(4,2)$ generators (10 isometries+5 conformal-like symmetries) are satisfied, the physical graviton modes must form UIRs of not only $so(4,1)$ but also $so(4,2)$, as in the case of gravitino modes discussed in subsection 3.2. Let us elaborate on this further.

4.2.3 UIRs of $so(4,2)$ formed by graviton modes

In the previous subsections we showed that the space of graviton mode solutions is a representation space for $so(4,2)$ — see the commutation relations (4.42a)–(4.42c). Here we will show, for the first time, that each of the two positive frequency single-helicity sets of physical graviton modes, $\{\varphi^{(phys,\,-L;\,M;K)}_{\mu\nu}\}$ and $\{\varphi^{(phys,\,+L;\,M;K)}_{\mu\nu}\}$, forms a UIR of $so(4,2)$.

As in the gravitino case discussed in subsection 3.2, it is sufficient to study the conformal-like transformation generated by one (out of five) genuine conformal Killing vectors, specifically the genuine conformal Killing vector $V^{(0)\mu}$ (3.54). From eq. (4.35), we have:

$$\mathcal{T}_{V^{(0)}}\varphi^{(phys,\,\pm L;\,M;K)}_{\mu\nu} = -i\,\cosh t\ \ \varepsilon_{t\sigma\lambda(\mu}\nabla^\sigma\varphi^{(phys,\,\pm L;\,M;K)\lambda}{}_{\nu)}. \tag{4.46}$$

Using the explicit expressions of the physical modes (4.10) and (4.11), as well as $\varepsilon_{t\tilde{\alpha}\tilde{\beta}\tilde{\gamma}} = \cosh^3 t\ \tilde{\varepsilon}_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}$ and (4.15), we find

$$\mathcal{T}_{V^{(0)}}\varphi^{(phys,\,-L;\,M;K)}_{\mu\nu} = +i(L+1)\,\varphi^{(phys,\,-L;\,M;K)}_{\mu\nu} \tag{4.47}$$

and

$$\mathcal{T}_{V^{(0)}}\varphi^{(phys,+L;M;K)}_{\mu\nu} = -i(L+1)\,\varphi^{(phys,+L;M;K)}_{\mu\nu}. \tag{4.48}$$

Thus, from eqs. (4.47) and (4.48), as well as (4.42a)–(4.42c), it follows that $\{\varphi^{(phys,+L;M;K)}_{\mu\nu}\}$ and $\{\varphi^{(phys,-L;M;K)}_{\mu\nu}\}$ separately form irreducible representations of $so(4,2)$. These representations are unitary because the Klein-Gordon inner product (4.23) is positive definite and all $so(4,2)$ generators are anti-hermitian (4.45). Similarly, one can show that the negative frequency modes $\{\varphi^{(phys,+L;M;K)\star}_{\mu\nu}\}$ and $\{\varphi^{(phys,-L;M;K)\star}_{\mu\nu}\}$ separately form UIRs of $so(4,2)$ with positive-definite inner product given by the negative of the Klein-Gordon inner product.

4.2.4 A hermitian action for the complex graviton and its conformal-like invariance

A hermitian action for the complex graviton, which gives rise to the desired Euler-Lagrange equations (4.36), is

$$S_2 = -\frac{1}{4}\int d^4x\,\sqrt{-g}\,\mathfrak{h}^\dagger_{\mu\nu}\,H^{\mu\nu}(\mathfrak{h}). \tag{4.49}$$

This action is invariant under the gauge transformations in (4.38) and their complex conjugates

$$\delta^{\text{gauge}}(\mathcal{Z})\,\mathfrak{h}^\dagger_{\mu\nu} = \nabla_{(\mu}\mathcal{Z}^\dagger_{\nu)}. \tag{4.50}$$

For later convenience, let us introduce the conserved symplectic current [90, 91] of the theory as follows. The covariant conjugate momentum current density for $\mathfrak{h}^\dagger_{\lambda\nu}$ is defined as

$$\begin{aligned}
p^{\mu\nu\lambda} &= \frac{1}{\sqrt{-g}}\frac{\delta S_2}{\delta\nabla_\mu\mathfrak{h}^\dagger_{\nu\lambda}} \\
&= -\frac{1}{4}\nabla^\mu\mathfrak{h}^{\nu\lambda} + \frac{1}{4}(2g^{\mu(\nu}\nabla_\alpha\mathfrak{h}^{\lambda)\alpha} - g^{\nu\lambda}\nabla_\alpha\mathfrak{h}^{\alpha\mu}) \\
&\quad -\frac{1}{4}g^{\mu(\nu}\nabla^{\lambda)}\mathfrak{h}^\alpha_\alpha + \frac{1}{4}g^{\nu\lambda}\nabla^\mu\mathfrak{h}^\alpha_\alpha\,.
\end{aligned} \tag{4.51}$$

Thus, the conserved symplectic current between two complex classical solutions $\mathfrak{h}^{(1)}_{\nu\lambda}$ and $\mathfrak{h}^{(2)}_{\nu\lambda}$ of eq. (4.36) is

$$\begin{aligned}
J^\mu_{\text{symp}}(\mathfrak{h}^{(1)},\mathfrak{h}^{(2)}) &= i\left(\mathfrak{h}^{(1)*}_{\nu\lambda}\;p^{(2)\mu\nu\lambda} - p^{(1)\mu\nu\lambda*}\;\mathfrak{h}^{(2)}_{\nu\lambda}\right) \\
&= -\frac{i}{4}\Big(\mathfrak{h}^{(1)*}_{\nu\lambda}\nabla^\mu\mathfrak{h}^{(2)\nu\lambda} - 2\mathfrak{h}^{(1)\mu*}{}_\lambda\nabla_\alpha\mathfrak{h}^{(2)\alpha\lambda} + \mathfrak{h}^{(1)\beta*}_\beta\nabla_\alpha\mathfrak{h}^{(2)\alpha\mu} \\
&\qquad + \mathfrak{h}^{(1)\mu*}{}_\lambda\nabla^\lambda\mathfrak{h}^{(2)\alpha}_\alpha - \mathfrak{h}^{(1)\beta*}_\beta\nabla^\mu\mathfrak{h}^{(2)\alpha}_\alpha \\
&\qquad - \mathfrak{h}^{(2)}_{\nu\lambda}\nabla^\mu\mathfrak{h}^{(1)\nu\lambda*} + 2\mathfrak{h}^{(2)\mu}{}_\lambda\nabla_\alpha\mathfrak{h}^{(1)\alpha\lambda*} - \mathfrak{h}^{(2)\beta}_\beta\nabla_\alpha\mathfrak{h}^{(1)\alpha\mu*} \\
&\qquad - \mathfrak{h}^{(2)\mu}{}_\lambda\nabla^\lambda\mathfrak{h}^{(1)\alpha*}_\alpha + \mathfrak{h}^{(2)\beta}_\beta\nabla^\mu\mathfrak{h}^{(1)\alpha*}_\alpha\Big)\,,
\end{aligned} \tag{4.52}$$

see, e.g., refs. [92, 93]. The time-independent (pre-)symplectic scalar product between $\mathfrak{h}^{(1)}_{\nu\lambda}$ and $\mathfrak{h}^{(2)}_{\nu\lambda}$ is

$$\langle \mathfrak{h}^{(1)} | \mathfrak{h}^{(2)} \rangle_{\text{symp}} = \int_{S^3} d\boldsymbol{\theta_3} \sqrt{-g}\; J^t_{\text{symp}} \left(\mathfrak{h}^{(1)}, \mathfrak{h}^{(2)} \right). \tag{4.53}$$

Importantly, the scalar product (4.53) is gauge-independent [92–95], and thus invariant under gauge transformations (4.38), as

$$\begin{aligned} \langle \delta^{\text{gauge}}(\mathcal{Z}) \mathfrak{h}^{(1)} | \mathfrak{h}^{(2)} \rangle_{\text{symp}} &= \int_{S^3} d\boldsymbol{\theta_3} \sqrt{-g}\; J^t_{\text{symp}} \left(\delta^{\text{gauge}}(\mathcal{Z}) \mathfrak{h}^{(1)}, \mathfrak{h}^{(2)} \right) = 0 \\ \langle \mathfrak{h}^{(1)} | \delta^{\text{gauge}}(\mathcal{Z}) \mathfrak{h}^{(2)} \rangle_{\text{symp}} &= \int_{S^3} d\boldsymbol{\theta_3} \sqrt{-g}\; J^t_{\text{symp}} \left(\mathfrak{h}^{(1)}, \delta^{\text{gauge}}(\mathcal{Z}) \mathfrak{h}^{(2)} \right) = 0. \end{aligned} \tag{4.54}$$

Indeed it is straightforward to show that

$$J^\mu_{\text{symp}} \left(\mathfrak{h}^{(1)}, \delta^{\text{gauge}}(\mathcal{Z}) \mathfrak{h}^{(2)} \right) = \nabla_\rho A^{\rho\mu}, \tag{4.55}$$

where $A^{\rho\mu} = A^{[\rho\mu]}$ is an anti-symmetric rank-2 tensor depending on the gauge parameter $\mathcal{Z}$ and on $\mathfrak{h}^{(1)*}_{\mu\nu}$ (and their derivatives) — clearly, a similar statement also holds for $J^\mu_{\text{symp}} \left(\delta^{\text{gauge}}(\mathcal{Z}) \mathfrak{h}^{(1)}, \mathfrak{h}^{(2)} \right)$. Thus, pure-gauge solutions, i.e. complex solutions of the form $\mathfrak{h}_{\mu\nu} = \nabla_{(\mu} \mathcal{Z}_{\nu)}$ for any $\mathcal{Z}_\nu$, are orthogonal to themselves and to all other solutions, with respect to the (pre-)symplectic scalar product (4.53). Note that, if one imposes the TT gauge condition (4.39) for **both** arguments of the symplectic current (4.52), then the symplectic current coincides with the Klein-Gordon current (4.24). Thus, the (pre-)symplectic scalar product (4.53) coincides with the Klein-Gordon inner product (4.25) on the space of TT solutions.

Conformal-like invariance of the action (4.49). The hermitian action in (4.49) is not only invariant under dS transformations but also under conformal-like transformations (4.35), unlike the linearised Einstein-Hilbert action (4.1) for the real graviton which is **not** invariant under the corresponding conformal-like transformations (4.34). The conformal-like invariance of the action (4.49) can be readily checked as follows. Computing the variation

$$\delta_V S_2 = -\frac{1}{4} \int d^4x \sqrt{-g}\, \left(\delta_V \mathfrak{h}^\dagger_{\mu\nu}\, H^{\mu\nu}(\mathfrak{h}) + \mathfrak{h}^\dagger_{\mu\nu}\, H^{\mu\nu}(\delta_V \mathfrak{h}) \right), \tag{4.56}$$

under $\delta_V \mathfrak{h}_{\mu\nu} = \mathcal{T}_V \mathfrak{h}_{\mu\nu} = i T_V \mathfrak{h}_{\mu\nu}$ and $\delta_V \mathfrak{h}^\dagger_{\mu\nu} = (\mathcal{T}_V \mathfrak{h}_{\mu\nu})^\dagger$, and using (4.32) with $B_{\mu\nu} = \mathfrak{h}_{\mu\nu}$, we find

$$\delta_V S_2 = \int d^4x \sqrt{-g}\, \nabla^\sigma \left(\frac{i}{4}\, V^\rho \varepsilon_{\rho\sigma\lambda\gamma}\, h^{\dagger\lambda}_{\nu}\, H^{\gamma\nu}(\mathfrak{h}) \right), \tag{4.57}$$

which demonstrates the conformal-like invariance of the action S_2. Notice that $(\mathcal{T}_V \mathfrak{h}_{\mu\nu})^\dagger$ has an extra minus sign relative to $\mathcal{T}_V \mathfrak{h}_{\mu\nu}$ (see (4.35)), which plays a crucial role in the calculation.

4.3 Quantisation of the chiral graviton field, anti-self-duality constraint, and UIRs in the bosonic Fock space

Here we will discuss a particular case of a chiral graviton field, the graviton with anti-self-dual field strength as this will be the superpartner of the chiral gravitino, as discussed in section 6.[31] The chiral graviton is described by a complex symmetric spin-2 field $\mathfrak{h}_{\mu\nu}$, as the one discussed in 4.2.2, but with the extra restriction of anti-self-duality on the field strength. In the present subsection, we will quantise the chiral graviton following Takahashi's method, as we also did for the chiral gravitino in subsection 3.4.

Following Takahashi's method [88], we take as our starting point the field equation (4.39) accompanied by the anti-self-duality constraint on the (complex) graviton field strength

$$\widetilde{U}_{\alpha\beta\mu\nu} = -i\, U_{\alpha\beta\mu\nu}, \tag{4.58}$$

where $\widetilde{U}_{\alpha\beta\mu\nu} = \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}U^{\gamma\delta}{}_{\mu\nu}$. The complex graviton field strength, which we also call 'complex linearised Weyl tensor', is defined as in the case of the real graviton [98]:

$$U_{\alpha\beta\mu\nu} = \Big(-\nabla_\mu \nabla_{[\alpha}\mathfrak{h}_{\beta]\nu} - g_{\mu[\alpha}\mathfrak{h}_{\beta]\nu}\Big) - (\mu \leftrightarrow \nu), \tag{4.59}$$

and is invariant under the gauge transformations (4.38). The field strength (4.59) has the symmetries of the Riemann tensor. The anti-self-dual linearised Weyl tensor, i.e. the linearised Weyl tensor field that satisfies the anti-self-duality constraint (4.58), will be later denoted as $U^{-}_{\alpha\beta\mu\nu}$ — see eq. (4.66). The complex graviton field $\mathfrak{h}_{\mu\nu}$ in (4.59) can, of course, be chosen to be in any gauge without affecting $U_{\alpha\beta\mu\nu}$. To proceed with the quantisation we will choose to work in the TT gauge. If $\mathfrak{h}_{\mu\nu}$ satisfies the field equations (4.36) or (4.39), then it is easy to show that

$$g^{\alpha\mu}U_{\alpha\beta\mu\nu} = g^{\beta\nu}U_{\alpha\beta\mu\nu} = 0, \qquad \nabla^\alpha U_{\alpha\beta\mu\nu} = \nabla^\mu U_{\alpha\beta\mu\nu} = 0. \tag{4.60}$$

As we mentioned earlier, the main objective of Takahashi's method applied here is to determine the quantum field operator $\mathfrak{h}^{(\mathrm{TT})}_{\mu\nu}$ and the hermitian quantum dS generators $Q_2^{dS}[\xi]$ such that the Heisenberg equations of motion are satisfied:

$$-i\, \pounds_\xi \mathfrak{h}^{(\mathrm{TT})}_{\mu\nu}(t,\boldsymbol{\theta_3}) = \Big[\mathfrak{h}^{(\mathrm{TT})}_{\mu\nu}(t,\boldsymbol{\theta_3}), Q_2^{dS}[\xi]\Big]\,, \tag{4.61}$$

up to pure-gauge TT solutions, for any dS Killing vector ξ^μ. The subscript '2' in $Q_2^{dS}[\xi]$ has been used to distinguish between the quantum dS generators of the chiral graviton and of the chiral gravitino — see eq. (3.67). As an additional requirement, quantum operators representing physical quantities must commute with one another for spacelike separations (microcausality).

Mode expansion. Let us now quantise the chiral graviton field.[32] From subsection 4.1, it follows that the components $\mathfrak{h}^{(\mathrm{TT})}_{t\nu}$ are nonzero only for pure-gauge modes (see also ref. [77]).

[31] Chiral gravitational tensor perturbations around de Sitter spacetime in terms of Ashtekar variables have been discussed in refs. [96, 97].

[32] The real graviton field on global dS_4 has been quantised in, e.g., [29].

To isolate the physical degrees of freedom, we fix the gauge completely by imposing the gauge conditions: $\mathfrak{h}^{(\mathrm{TT})}_{t\mu}=0$, and $g^{\tilde{\alpha}\tilde{\mu}}\nabla_{\tilde{\alpha}}\mathfrak{h}^{(\mathrm{TT})}_{\tilde{\mu}\tilde{\nu}}=0$. We then expand the chiral graviton gauge potential in modes as

$$\begin{aligned}
&\mathfrak{h}^{(\mathrm{TT})-}_{t\mu}(t,\boldsymbol{\theta_3})=0, \\
&\mathfrak{h}^{(\mathrm{TT})-}_{\tilde{\mu}\tilde{\nu}}(t,\boldsymbol{\theta_3})=\sum_{L=2}^{\infty}\sum_{M=2}^{L}\sum_{K=-M}^{M}\Big(c^{(-)}_{LMK}\varphi^{(phys,-L;M;K)}_{\tilde{\mu}\tilde{\nu}}(t,\boldsymbol{\theta_3})+d^{(+)\dagger}_{LMK}\varphi^{(phys,+L;M;K)\star}_{\tilde{\mu}\tilde{\nu}}(t,\boldsymbol{\theta_3})\Big),
\end{aligned} \tag{4.62}$$

where μ is a tensor index on dS_4, while $\tilde{\mu}$ and $\tilde{\nu}$ are tensor indices on S^3. The superscript '$-$' in $\mathfrak{h}^{(\mathrm{TT})-}_{\mu\nu}$ refers to the fact that the field strength is anti-self-dual (anti-self-duality is demonstrated below). The non-zero commutators are

$$[c^{(-)}_{LMK},c^{(-)\dagger}_{L'M'K'}]=\delta_{LL'}\delta_{MM'}\delta_{KK'}, \qquad [d^{(+)}_{LMK},d^{(+)\dagger}_{L'M'K'}]=\delta_{LL'}\delta_{MM'}\delta_{KK'}. \tag{4.63}$$

The dS-invariant vacuum is the state $|0\rangle_2$ in the Fock space that satisfies:

$$c^{(-)}_{LMK}\,|0\rangle_2=d^{(+)}_{LMK}\,|0\rangle_2=0, \tag{4.64}$$

for all L,M,K. Using the Klein-Gordon scalar product (4.23), we find

$$c^{(-)}_{LMK}=\langle\varphi^{(phys,-L;M;K)}|\mathfrak{h}^{(\mathrm{TT})-}\rangle_{KG}\,, \qquad d^{(+)\dagger}_{LMK}=-\,\langle\varphi^{(phys,+L;M;K)\star}|\mathfrak{h}^{(\mathrm{TT})-}\rangle_{KG} \tag{4.65}$$

[see eqs. (4.26) and (4.27)].

Anti-self-duality constraint. Let us verify that the mode expansion for the chiral graviton (4.62) is consistent with the anti-self-duality constraint (4.58). In other words, we will show that the following field strength

$$U^{-}_{\alpha\beta\mu\nu}=\Big(-\nabla_\mu\nabla_{[\alpha}\mathfrak{h}^{(\mathrm{TT})-}_{\beta]\nu}-g_{\mu[\alpha}\mathfrak{h}^{(\mathrm{TT})-}_{\beta]\nu}\Big)-(\mu\leftrightarrow\nu), \tag{4.66}$$

is anti-self-dual. For convenience, let us demonstrate this for the component $U^{-}_{\tilde{\rho}\tilde{\gamma}t\tilde{\nu}}$, where $\tilde{\rho},\tilde{\gamma}$ and $\tilde{\nu}$ are spatial indices — the calculation for the rest of the components is similar. In the global coordinates (2.1), we have

$$U^{-}_{\tilde{\rho}\tilde{\gamma}t\tilde{\nu}}=\Big(-\frac{\partial}{\partial t}+2\tanh t\Big)\tilde{\nabla}_{[\tilde{\rho}}\mathfrak{h}^{(\mathrm{TT})-}_{\tilde{\gamma}]\tilde{\nu}}, \tag{4.67}$$

where $\tilde{\nabla}_{\tilde{\rho}}$ is the covariant derivative on S^3. On the other hand, for the $\tilde{\rho}\tilde{\gamma}t\tilde{\nu}$-component of the dual field strength, we have

$$\begin{aligned}
\frac{1}{2}\varepsilon_{\tilde{\rho}\tilde{\gamma}\alpha\beta}\,U^{-\alpha\beta}{}_{t\tilde{\nu}}=\varepsilon_{\tilde{\rho}\tilde{\gamma}t\tilde{\delta}}\,U^{-t\tilde{\delta}}{}_{t\tilde{\nu}}&=-\cosh t\;\tilde{\varepsilon}_{\tilde{\rho}\tilde{\gamma}\tilde{\delta}}\;\tilde{g}^{\tilde{\delta}\tilde{\mu}}\;U^{-}_{t\tilde{\mu}t\tilde{\nu}} \\
&=-\cosh t\left(-\frac{\partial^2}{\partial t^2}+2\tanh t+\frac{2}{\cosh^2 t}\right)\frac{\tilde{\varepsilon}_{\tilde{\rho}\tilde{\gamma}\tilde{\delta}}}{2}\;\tilde{g}^{\tilde{\delta}\tilde{\mu}}\;\mathfrak{h}^{(\mathrm{TT})-}_{\tilde{\mu}\tilde{\nu}}.
\end{aligned} \tag{4.68}$$

We want to show that eq. (4.68) is equal to $-i\times$ (4.67). Indeed, substituting the mode expansion (4.62) into eqs. (4.67) and (4.68), and making use of (4.15), one finds that the anti-self-duality constraint (4.58) is satisfied, as

$$\frac{1}{2}\varepsilon_{\tilde{\rho}\tilde{\gamma}\alpha\beta}\,U^{-\alpha\beta}{}_{t\tilde{\nu}}=-i\,U^{-}_{\tilde{\rho}\tilde{\gamma}t\tilde{\nu}}. \tag{4.69}$$

This can be similarly verified for the rest of the components of the field strength (4.66).

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Quantum symmetry generators. The hermitian dS generators for the chiral graviton can be constructed in the standard way [85, 88]:

$$Q_2^{dS}[\xi] = -i : \langle \mathfrak{h}^{(\mathrm{TT})-} | \pounds_\xi \mathfrak{h}^{(\mathrm{TT})-} \rangle_{KG} : , \tag{4.70}$$

where ξ^μ is any dS Killing vector. Here we will give the explicit expression only for the dS boost $\xi^\mu = B^\mu$ [eq. (3.42)]. Expanding the field in modes (4.62), and using eqs. (4.22), (4.26) and (4.27), we find:

$$Q_2^{dS}[B] = Q_2^{dS-}[B] + Q_2^{dS+}[B], \tag{4.71}$$

where

$$Q_2^{dS-}[B] = -\frac{1}{2}\sum_{L=2}^{\infty}\sum_{M.K}\Bigg(\sqrt{(L-M+1)(L+M+2)}\, c^{(-)\dagger}_{(L+1)MK} c^{(-)}_{LMK} + \sqrt{(L-M)(L+M+1)}\, c^{(-)\dagger}_{(L-1)MK} c^{(-)}_{LMK}\Bigg) \tag{4.72}$$

and

$$Q_2^{dS+}[B] = -\frac{1}{2}\sum_{L=2}^{\infty}\sum_{M,K}\Bigg(\sqrt{(L-M+1)(L+M+2)}\, d^{(+)\dagger}_{(L+1)MK} d^{(+)}_{LMK} + \sqrt{(L-M)(L+M+2)}\, d^{(+)\dagger}_{(L-1)MK} d^{(+)}_{LMK}\Bigg). \tag{4.73}$$

As in the gravitino case, the dS charge has two independent parts: $Q_2^{dS}[B] = Q_2^{dS-}[B] + Q_2^{dS+}[B]$, where $Q_2^{dS-}[B]$ and $Q_2^{dS+}[B]$ act on the negative- and positive-helicity sectors, respectively. In other words, they generate the two discrete series UIRs of $so(4,1)$, $D^-(\Delta = 3, s = 2)$ and $D^+(\Delta = 3, s = 2)$, respectively. The charge $Q_2^{dS}[B]$ generates the following dS transformations of creation operators:

$$\delta_B c^{(-)\dagger}_{LMK} \equiv \Big[c^{(-)\dagger}_{LMK}, Q_2^{dS}[B]\Big] = \Big[c^{(-)\dagger}_{LMK}, Q_2^{dS-}[B]\Big] = \frac{1}{2}\sqrt{(L-M+1)(L+M+2)}\, c^{(-)\dagger}_{(L+1)MK} + \frac{1}{2}\sqrt{(L-M)(L+M+1)}\, c^{(-)\dagger}_{(L-1)MK} \tag{4.74}$$

and

$$\delta_B d^{(+)\dagger}_{LMK} \equiv \Big[d^{(+)\dagger}_{LMK}, Q_2^{dS}[B]\Big] = \Big[d^{(+)\dagger}_{LMK}, Q_2^{dS+}[B]\Big] = \frac{1}{2}\sqrt{(L-M+1)(L+M+2)}\, d^{(+)\dagger}_{(L+1)MK} + \frac{1}{2}\sqrt{(L-M)(L+M+1)}\, d^{(+)\dagger}_{(L-1)MK}. \tag{4.75}$$

Using these expressions, it is clear that negative-helicity single-particle states $c^{(-)\dagger}_{LMK}|0\rangle_2$ transform as the corresponding positive frequency modes (4.22), thus furnishing the $so(4,1)$ discrete series UIR $D^-(\Delta = 3, s = 2)$. Similarly, positive-helicity single-particle states $d^{(+)\dagger}_{LMK}|0\rangle_2$ furnish the $so(4,1)$ discrete series UIR $D^+(\Delta = 3, s = 2)$ — see appendix A. It is now straightforward to show that the Heisenberg equation of motion (4.61) is satisfied

$$\Big[\mathfrak{h}^{(\mathrm{TT})-}_{\mu\nu}, Q_2^{dS}[B]\Big] = -i\, \pounds_B \mathfrak{h}^{(\mathrm{TT})-}_{\mu\nu}, \tag{4.76}$$

module a pure-gauge TT solution, where $\pounds_B$ is the Lie derivative with respect to the dS boost Killing vector B^μ (3.42).

As in the gravitino case, apart from the ten dS charges we can also construct the five hermitian charges corresponding to the conformal-like symmetry (4.35),

$$Q_2^{\rm conf}[V] = -i : \langle \mathfrak{h}^{(\rm TT)-} | \mathcal{T}_V \mathfrak{h}^{(\rm TT)-} \rangle_{KG} :, \tag{4.77}$$

such that the Heisenberg equations of motion are again satisfied

$$-i\, \mathcal{T}_V \mathfrak{h}_{\mu\nu}^{(\rm TT)-}(t, \boldsymbol{\theta_3}) = [\, \mathfrak{h}_{\mu\nu}^{(\rm TT)-}(t, \boldsymbol{\theta_3}), Q_2^{\rm conf}[V]\,], \tag{4.78}$$

module a pure-gauge TT solution, where $\mathcal{T}_V$ is the conformal-like transformation (4.35) with respect to any genuine conformal Killing vector V^μ (2.13). An easy way to verify the Heisenberg equations of motion is to focus on the conformal-like symmetry generated by the genuine conformal Killing vector $V^{(0)\mu}$ [eq. (3.54)], for which the quantum generator is readily found to be

$$Q_2^{\rm conf}[V^{(0)}] = \sum_{L=2}^{\infty} \sum_{M,K} (L+1) \left(c_{LMK}^{(-)\dagger} c_{LMK}^{(-)} - d_{LMK}^{(+)\dagger} d_{LMK}^{(+)} \right). \tag{4.79}$$

This quantum generator consists of two independent conformal-like charges, as

$$\begin{aligned} Q_2^{\rm conf}[V^{(0)}] &= Q_2^{\rm conf-}[V^{(0)}] + Q_2^{\rm conf+}[V^{(0)}], \\ Q_2^{\rm conf-}[V^{(0)}] &= \sum_{L=2}^{\infty} \sum_{M,K} (L+1)\, c_{LMK}^{(-)\dagger} c_{LMK}^{(-)}, \\ Q_2^{\rm conf+}[V^{(0)}] &= -\sum_{L=2}^{\infty} \sum_{M,K} (L+1)\, d_{LMK}^{(+)\dagger} d_{LMK}^{(+)}, \end{aligned} \tag{4.80}$$

acting on negative-helicity and positive-helicity states, respectively. Using this expression for $Q_2^{\rm conf}[V^{(0)}]$, as well as the mode expansion for $\mathfrak{h}_{\mu\nu}^{(\rm TT)-}$ (4.62), one can readily verify the Heisenberg equations of motion. Finally, it is also easy to verify that, as in the gravitino case, the chiral graviton vacuum $|0\rangle_2$ is also invariant under the whole $so(4,2)$ symmetry.

Microcausality. We will show the microcausality of the chiral graviton theory by demonstrating that, for any two spacelike separated points, $(t, \boldsymbol{\theta_3})$ and $(t', \boldsymbol{\theta'_3})$, the commutator

$$\left[U^-_{\mu\nu\rho\sigma}(t, \boldsymbol{\theta_3}), U^-_{\alpha'\beta'\gamma'\delta'}(t', \boldsymbol{\theta'_3})^\dagger \right] \tag{4.81}$$

vanishes. This can be inferred from the standard theory of the real graviton [29, 77], as follows. Let $h_{\mu\nu}^{(\rm TT)}$ be the real graviton gauge potential that has been completely gauge-fixed as $h_{t\mu}^{(\rm TT)} = 0$, and $g^{\tilde\alpha\tilde\mu} \nabla_{\tilde\alpha} h_{\tilde\mu\tilde\nu}^{(\rm TT)} = 0$. This field can be expanded in terms of the Bunch-Davies mode functions of both helicities (4.10) and (4.11) [29, 77]. The real graviton is related to our chiral graviton potential as

$$h_{\mu\nu}^{(\rm TT)}(t, \boldsymbol{\theta_3}) = \mathfrak{h}_{\mu\nu}^{(\rm TT)-}(t, \boldsymbol{\theta_3}) + \left(\mathfrak{h}_{\mu\nu}^{(\rm TT)-}(t, \boldsymbol{\theta_3}) \right)^\dagger.$$

Let also $U^{(real)}_{\mu\nu\rho\sigma}$ be the field strength of $h^{(\mathrm{TT})}_{\mu\nu}$, i.e. real linearised Weyl tensor, which has the symmetries of the Riemann tensor and satisfies eq. (4.60). The real linearised Weyl tensor can be expressed in terms of the field strength of our chiral graviton as

$$U^{(real)}_{\mu\nu\rho\sigma}(t,\boldsymbol{\theta_3}) = U^{-}_{\mu\nu\rho\sigma}(t,\boldsymbol{\theta_3}) + U^{-}_{\mu\nu\rho\sigma}(t,\boldsymbol{\theta_3})^{\dagger}, \tag{4.82}$$

where the anti-self-dual part of $U^{(real)}_{\mu\nu\rho\sigma}$ is

$$U^{-}_{\mu\nu\rho\sigma}(t,\boldsymbol{\theta_3}) = \frac{1}{2}\left(U^{(real)}_{\mu\nu\rho\sigma}(t,\boldsymbol{\theta_3}) + i\widetilde{U}^{(real)}_{\mu\nu\rho\sigma}(t,\boldsymbol{\theta_3})\right), \tag{4.83}$$

while its self-dual part is

$$U^{-}_{\mu\nu\rho\sigma}(t,\boldsymbol{\theta_3})^{\dagger} = \frac{1}{2}\left(U^{(real)}_{\mu\nu\rho\sigma}(t,\boldsymbol{\theta_3}) - i\widetilde{U}^{(real)}_{\mu\nu\rho\sigma}(t,\boldsymbol{\theta_3})\right) \equiv U^{+}_{\mu\nu\rho\sigma}(t,\boldsymbol{\theta_3}). \tag{4.84}$$

Because of the microcausality of the real graviton field on dS_4, the Weyl-Weyl commutator between any two causally disconnected points vanishes:[33]

$$\left[U^{(real)}_{\mu\nu\rho\sigma}(t,\boldsymbol{\theta_3}), U^{(real)}_{\alpha'\beta'\gamma'\delta'}(t',\boldsymbol{\theta_3'})\right] = 0, \text{ for spacelike separated points } \ (t,\boldsymbol{\theta_3}),\ (t',\boldsymbol{\theta_3'}). \tag{4.85}$$

It is now easy to explain how this implies the locality of the commutator (4.81) that is relevant to the chiral graviton theory. By taking the dual of the Weyl tensor on the left slot of the commutator in (4.85), we have

$$\left[\widetilde{U}^{(real)}_{\mu\nu\rho\sigma}(t,\boldsymbol{\theta_3}), U^{(real)}_{\alpha'\beta'\gamma'\delta'}(t',\boldsymbol{\theta_3'})\right] = 0, \text{ for spacelike separated points } \ (t,\boldsymbol{\theta_3}),\ (t',\boldsymbol{\theta_3'}). \tag{4.86}$$

Then, by adding (4.85) $+i\times$(4.86), and with the use of eq. (4.83), we find

$$\left[U^{-}_{\mu\nu\rho\sigma}(t,\boldsymbol{\theta_3}), U^{(real)}_{\alpha'\beta'\gamma'\delta'}(t',\boldsymbol{\theta_3'})\right] = 0, \text{ for spacelike separated points } \ (t,\boldsymbol{\theta_3}),\ (t',\boldsymbol{\theta_3'}). \tag{4.87}$$

Then, by taking the dual of the Weyl tensor on the right slot of the commutator in (4.87), and working similarly, we find

$$\left[U^{-}_{\mu\nu\rho\sigma}(t,\boldsymbol{\theta_3}), U^{-}_{\alpha'\beta'\gamma'\delta'}(t',\boldsymbol{\theta_3'})^{\dagger}\right] = 0, \text{ for spacelike separated points } \ (t,\boldsymbol{\theta_3}),\ (t',\boldsymbol{\theta_3'}). \tag{4.88}$$

This demonstrates the microcausality of the chiral graviton theory.

In the quantisation presented above, we started from a complex graviton field and restricted it to its chiral (anti-self-dual) part, which might seem puzzling at first. However, we could have started from the linearised Einstein-Hilbert action with a *real* graviton field and defined its anti-self-dual part, which is a complex field, to construct the chiral graviton field. Then, the completely gauge-fixed real field and the corresponding linearised Weyl tensor would be identical to, respectively, $h^{(\mathrm{TT})}_{\mu\nu}(t,\boldsymbol{\theta_3})$ and $U^{(real)}_{\mu\nu\rho\sigma}(t,\boldsymbol{\theta_3})$ discussed above. That is, there would be no need to start from a complex graviton field if our only purpose was to define the chiral graviton field. However, as we shall see, the SUSY transformation on the real graviton field would be highly non-local: it is a complex (i.e., non-real) transformation and the anti-self-dual part and its complex conjugate, the self-dual part, transform differently. In particular, the U(1) transformation, which is part of the superalgebra, assigns the opposite charges to the self-dual and anti-self-dual gravitons. In contrast, as we shall see, it is possible to define a simple SUSY transformation on a non-chiral complex graviton field, which can be restricted to a chiral graviton field. For this reason we started from a complex graviton field to construct the chiral graviton field.

[33] See refs. [92, 98] for related discussions.

5 Complex Killing spinors on dS_4 and their conformal-like symmetry

Let us review the basics about Killing spinors on dS_4 — see also, e.g., ref. [74]. Killing spinors, ϵ_+ and ϵ_-, on dS_4 satisfy

$$\nabla_\mu \epsilon_\pm = \pm \frac{i}{2} \gamma_\mu \epsilon_\pm . \tag{5.1}$$

The Killing spinors with the two different signs in eq. (5.1) are related to each other as $\epsilon_- = \gamma^5 \epsilon_+$.[34] There are no Majorana Killing spinors[35] satisfying eq. (5.1) — the explanation is similar to the one for the absence of a Majorana condition in the case of the gravitino, see the passage below eq. (3.9). There are four independent complex (Dirac) Killing spinors ϵ_+, and four independent ϵ_-. The Killing spinors ϵ_+ and ϵ_- form equivalent finite-dimensional (non-unitary) representations of the dS algebra. In what follows, we will **only use the Killing spinors** ϵ_-, and therefore, we will omit the subscript '$-$', denoting them as ϵ.

The Killing spinors ϵ form a 4-dimensional non-unitary representation of $so(4,1)$. The dS generators act on Killing spinors in terms of the Lie-Lorentz derivative

$$\mathbb{L}_\xi \epsilon = \xi^\nu \nabla_\nu \epsilon + \frac{1}{4} (\nabla_\kappa \xi_\lambda) \gamma^{\kappa\lambda} \epsilon, \tag{5.2}$$

where ξ^μ is a dS Killing vector. For later convenience, note that using any two Killing spinors, ϵ_1 and ϵ_2, satisfying eq. (5.1) with the '$-$' sign, one can construct the following bilinears:

- the real Killing vectors

  $$\xi^\mu_{(\epsilon)} = \frac{1}{4} \bar{\epsilon}_2 \gamma^5 \gamma^\mu \epsilon_1 - \frac{1}{4} \bar{\epsilon}_1 \gamma^5 \gamma^\mu \epsilon_2 = \frac{1}{4} \bar{\epsilon}_2 \gamma^5 \gamma^\mu \epsilon_1 + \frac{1}{4} \left(\bar{\epsilon}_2 \gamma^5 \gamma^\mu \epsilon_1 \right)^\dagger , \tag{5.3}$$

  where $\nabla_\mu \xi^\mu_{(\epsilon)} = 0$ and $\nabla_\mu \xi_{(\epsilon)\nu} + (\mu \leftrightarrow \nu) = 0$. The factors of $\frac{1}{4}$ in (5.3) have been inserted for later convenience.

- the real genuine conformal Killing vectors [see eq. (2.13)]

  $$\begin{aligned} V^\mu_{(\epsilon)} &= \frac{1}{4} \bar{\epsilon}_2 \gamma^\mu \epsilon_1 - \frac{1}{4} \bar{\epsilon}_1 \gamma^\mu \epsilon_2 = \frac{1}{4} \bar{\epsilon}_2 \gamma^\mu \epsilon_1 + \frac{1}{4} \left(\bar{\epsilon}_2 \gamma^\mu \epsilon_1 \right)^\dagger \\ &= \frac{1}{4} \nabla^\mu \left(i\, \bar{\epsilon}_2 \epsilon_1 - i\, \bar{\epsilon}_1 \epsilon_2 \right) = \nabla^\mu \phi_{V_{(\epsilon)}}, \qquad \phi_{V_{(\epsilon)}} \equiv \frac{i}{4}\, \bar{\epsilon}_2 \epsilon_1 - \frac{i}{4}\, \bar{\epsilon}_1 \epsilon_2 , \end{aligned} \tag{5.4}$$

  where $\nabla_\mu V_{(\epsilon)\nu} = -g_{\mu\nu} \phi_{V_{(\epsilon)}} = \frac{1}{4} g_{\mu\nu} \nabla^\alpha V_{(\epsilon)\alpha}$. The factors of $\frac{1}{4}$ in (5.4) have been inserted for later convenience.

The afore-mentioned real Killing spinor bilinears will appear in the commutators of SUSY transformations [eqs. (6.16) and (6.17)] in the following subsections. Complex Killing vectors and complex genuine conformal Killing vectors are given by

$$\xi^{(2,1)\mu}_{\mathbb{C}} = \frac{1}{4} \bar{\epsilon}_2 \gamma^5 \gamma^\mu \epsilon_1, \qquad \xi^{(1,2)\mu}_{\mathbb{C}} = \frac{1}{4} \bar{\epsilon}_1 \gamma^5 \gamma^\mu \epsilon_2 = - \left(\xi^{(2,1)\mu}_{\mathbb{C}} \right)^* \tag{5.5}$$

and

$$V^{(2,1)\mu}_{\mathbb{C}} = \frac{1}{4} \bar{\epsilon}_2 \gamma^\mu \epsilon_1 \equiv \nabla^\mu \phi_{V^{(2,1)}_{\mathbb{C}}}, \qquad V^{(1,2)\mu}_{\mathbb{C}} = \frac{1}{4} \bar{\epsilon}_1 \gamma^\mu \epsilon_2 \equiv \nabla^\mu \phi_{V^{(1,2)}_{\mathbb{C}}} = - \left(V^{(2,1)\mu}_{\mathbb{C}} \right)^* , \tag{5.6}$$

respectively. However, as we will show below, only their real parts will appear in the commutators (6.16) and (6.17) of two SUSY transformations.

[34] The Dirac adjoint $\bar{\epsilon}_\pm$ of a Killing spinor $\epsilon_\pm$ satisfies $\nabla_\mu \bar{\epsilon}_\pm \mp \frac{i}{2}\, \bar{\epsilon}_\pm\, \gamma_\mu = 0$.

[35] However, eq. (5.1) admits symplectic Majorana Killing spinor solutions.

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Killing spinors and their conformal-like symmetry. Something that is not widely known, and to the best of our knowledge will be presented here for the first time, is that dS Killing spinors enjoy a conformal-like $so(4,2)$ symmetry akin to the conformal-like symmetry for the graviton and gravitino discussed earlier. In particular, the Killing spinor equation (5.1) is invariant under the following conformal-like transformations:

$$\mathbb{T}_V\,\epsilon = \gamma^5\left(V^\rho\nabla_\rho\epsilon + \frac{1}{2}\phi_V\,\epsilon\right), \tag{5.7}$$

where $V_\mu = \nabla_\mu\phi_V$ is any genuine conformal Killing vector (2.13). It can be readily verified that if ϵ satisfies the Killing spinor equation (5.1), then $\mathbb{T}_V\,\epsilon$ satisfies the same equation. The $so(4,2)$ commutation relations are given by

$$[\mathbb{L}_\xi, \mathbb{L}_{\xi'}]\epsilon = \mathbb{L}_{[\xi,\xi']}\epsilon, \tag{5.8a}$$
$$[\mathbb{L}_\xi, \mathbb{T}_V]\epsilon = \mathbb{T}_{[\xi,V]}\epsilon, \tag{5.8b}$$
$$[\mathbb{T}_{V'}, \mathbb{T}_V]\epsilon = \mathbb{L}_{[V',V]}\epsilon, \tag{5.8c}$$

where ξ^μ and ξ'^μ are any two dS Killing vectors, while V^μ and V'^μ are any two genuine conformal Killing vectors (2.13). Note that the Dirac adjoint of $\mathbb{T}_V\epsilon$ (5.7) is

$$\overline{\mathbb{T}_V\,\epsilon} = -\left(V^\rho\nabla_\rho\bar{\epsilon} + \frac{1}{2}\phi_V\bar{\epsilon}\right)\gamma^5. \tag{5.9}$$

Explicit expressions for Killing spinors on dS_4. Explicit expressions for the Killing spinors $\epsilon(t,\boldsymbol{\theta_3})$ on global dS_4 can be found by analytically continuing the Killing spinors on S^4 — see ref. [32] for details on the analytic continuation of spinor eigenfunctions of the Dirac operator from S^4 to dS_4. The line element on the unit S^4 is

$$d\Omega^2_{(4)} = d\theta_4^2 + \sin^2\theta_4\; d\Omega^2, \tag{5.10}$$

where $\pi \geq \theta_4 \geq 0$, and $d\Omega^2$ is the line element (2.2) of S^3. It is well-known that one can analytically continue the line element of S^4 to obtain the line element of global dS_4 (2.1) by making the replacement [28]

$$\theta_4 \to \frac{\pi}{2} - it. \tag{5.11}$$

It is also known that Killing spinors on S^4 are eigenfunctions of the Dirac operator with the lowest allowed eigenvalue[36]

$$\nabla\!\!\!/\;\psi(\theta_4,\boldsymbol{\theta_3}) = -\,2i\,\psi(\theta_4,\boldsymbol{\theta_3}). \tag{5.12}$$

There are four such independent spinor eigenfunctions [99, 100] forming a 4-dimensional representation of $so(5)$ with highest weight given by $\tau = (\frac{1}{2},\frac{1}{2})$ [99]. It can be easily verified

[36]The spinor eigenfunctions of the Dirac operator on S^4 have two different signs for their eigenvalues: $\nabla\!\!\!/\psi_n = -i(n+2)\psi_n$ and $\nabla\!\!\!/\psi'_n = +i(n+2)\psi'_n$, where $n = 0,1,2,\ldots$ [99]. The two families of eigenfunctions, ψ_n and ψ'_n, form equivalent representations of $so(5)$ for each fixed n. The two families are related to each other as $\psi'_n = \gamma^5\psi_n$. For $n=0$, the spinors $\psi_0 \equiv \psi$ are Killing spinors on S^4 satisfying $\nabla_\mu\psi = -\frac{i}{2}\gamma_\mu\psi$, as we show in the main text. The spinors $\psi'_0 = \gamma^5\psi_0$ are also Killing spinors that satisfy $\nabla_\mu\psi'_0 = +\frac{i}{2}\gamma_\mu\psi'_0$.

that the spinor eigenfunctions ψ of the Dirac operator satisfy the Killing spinor equation on S^4, as follows. Define the following vector-spinors on S^4:

$$\zeta_\mu \equiv \left(\nabla_\mu + \frac{i}{2}\gamma_\mu\right)\psi. \tag{5.13}$$

It can be shown that these vector-spinors are identically zero by computing their norm using the standard inner product on S^4 [34]

$$\int_{S^4} \sin^3\theta_4 \,\sqrt{\tilde{g}}\; d\theta_4 \; d\boldsymbol{\theta_3} \;\; \zeta_\mu(\theta_4, \boldsymbol{\theta_3})^\dagger \;\; \zeta^\mu(\theta_4, \boldsymbol{\theta_3}), \tag{5.14}$$

where $\sin^3\theta_4\,\sqrt{\tilde{g}}$ is the square root of the determinant of the S^4 metric, $\tilde{g}$ is the determinant of the S^3 metric (2.2), and $d\boldsymbol{\theta_3} \equiv d\theta_3\, d\theta_2\, d\theta_1$. The computation of the norm is straightforward and it involves some integration by parts, and one also has to use [99]

$$\Box\,\psi = \left(\nabla\!\!\!/^2 + \frac{R}{4}\right)\psi, \tag{5.15}$$

where $R = 12$ is the scalar curvature of the unit S^4. As the inner product on S^4 is positive definite, the vanishing of the norm implies $\zeta_\mu = 0$, and thus

$$\nabla_\mu \psi = -\frac{i}{2}\gamma_\mu \psi, \tag{5.16}$$

which is the Killing spinor equation on S^4. In other words, the eigenfunctions ψ of the Dirac operator with the lowest eigenvalue on S^4 are Killing spinors — their explicit expressions can be found in [99, 100].

Now, one can use the replacement (5.11) to analytically continue the Killing spinor equation (5.16) on S^4 to the Killing spinor equation (5.1) (with the '$-$' sign) on dS_4. In particular, making the replacement (5.11), the S^4 Killing spinors $\psi(\theta_4, \boldsymbol{\theta_3})$ are analytically continued to dS_4 Killing spinors $\epsilon(t, \boldsymbol{\theta_3})$. In this manner, we find that there are four Killing spinors on dS_4. Two of them have 'positive helicity' and the other two have 'negative helicity'. The four Killing spinors on global dS_4 are given by

$$\epsilon^{(-;q)}(t, \boldsymbol{\theta_3}) = \begin{pmatrix} \cos\left(\frac{\pi/2 - it}{2}\right)\; \tilde{\epsilon}_{-,q}(\boldsymbol{\theta_3}) \\ -i\sin\left(\frac{\pi/2 - it}{2}\right)\; \tilde{\epsilon}_{-,q}(\boldsymbol{\theta_3}) \end{pmatrix}, \qquad \epsilon^{(+;q)}(t, \boldsymbol{\theta_3}) = \begin{pmatrix} i\,\sin\left(\frac{\pi/2 - it}{2}\right)\; \tilde{\epsilon}_{+,q}(\boldsymbol{\theta_3}) \\ -\cos\left(\frac{\pi/2 - it}{2}\right)\; \tilde{\epsilon}_{+,q}(\boldsymbol{\theta_3}) \end{pmatrix}, \tag{5.17}$$

where $\tilde{\epsilon}_{\pm,q}(\boldsymbol{\theta_3})$ are Killing spinors on the unit S^3 satisfying

$$\tilde{\nabla}_{\tilde{\mu}} \tilde{\epsilon}_{\pm,q}(\boldsymbol{\theta_3}) = \pm\frac{i}{2}\tilde{\gamma}_{\tilde{\mu}}\; \tilde{\epsilon}_{\pm,q}(\boldsymbol{\theta_3}), \tag{5.18}$$

and the meaning of the label q will be explained shortly. The Killing spinors $\epsilon^{(\sigma;q)}(t, \boldsymbol{\theta_3})$ and $\tilde{\epsilon}_{\sigma,q}(\boldsymbol{\theta_3})$ ($\sigma = \pm$) will be treated as commuting; Grassmann-odd Killing spinors will be discussed below. The two 'helicity' labels $\pm$ in the dS Killing spinors (5.17) stem from the Killing spinors on S^3 and their behaviour under $so(4)$ rotations. In particular, on S^3, there are two independent 'positive-helicity' Killing spinors $\tilde{\epsilon}_{+,q}(\boldsymbol{\theta_3})$ and two independent 'negative-helicity' Killing spinors $\tilde{\epsilon}_{-,q}(\boldsymbol{\theta_3})$. The Killing spinors on S^3 coincide with the spinor eigenfunctions of the Dirac operator on S^3 with the lowest eigenvalue

$$\tilde{\nabla\!\!\!/}\,\tilde{\epsilon}_{\pm,q}(\boldsymbol{\theta_3}) = \pm\,\frac{3}{2}i\,\tilde{\epsilon}_{\pm,q}(\boldsymbol{\theta_3}),$$

and their explicit expressions can be found from [99]. The label $q=0,-1$ is a S^1 angular momentum quantum number, related to $so(2)$ rotations generated by ∂_{θ_1} in the coordinates (2.2) — see also eq. (C.6). Specifically, $\partial_{\theta_1}\tilde{\epsilon}_{\pm,q} = i(q+1/2)\tilde{\epsilon}_{\pm,q}$ since, according to the construction of [99], the label $q\in\{0,-1\}$ determines the θ_1-dependence for the Killing spinors in the coordinates (2.2), as $\tilde{\epsilon}_{\pm,q}(\boldsymbol{\theta_3}) \equiv \tilde{\epsilon}_{\pm,q}(\theta_3,\theta_2,\theta_1) \propto e^{i(q+1/2)\theta_1}$. The two Killing spinors $\{\tilde{\epsilon}_{+,q}\}_{q=0,-1}$ on S^3 form the 2-dimensional representation of $so(4)$ with highest weight $\tilde{\tau}^+ = (\frac{1}{2},\frac{1}{2})$ [99]. Similarly, the two Killing spinors $\{\tilde{\epsilon}_{-,q}\}_{q=0,-1}$ on S^3 form the 2-dimensional representation of $so(4)$ with highest weight $\tilde{\tau}^- = (\frac{1}{2},-\frac{1}{2})$. For later convenience, note that the scalar quantities $\tilde{\epsilon}^\dagger_{\pm,q}(\boldsymbol{\theta_3})\tilde{\epsilon}_{\pm,q'}(\boldsymbol{\theta_3})$ are constant on S^3 for any $q,q'\in\{0,-1\}$ — this is easy to check. Here, we normalise the Killing spinors on S^3 such that

$$\tilde{\epsilon}^\dagger_{\pm,q}(\boldsymbol{\theta_3})\tilde{\epsilon}_{\pm,q'}(\boldsymbol{\theta_3}) = \delta_{qq'}\frac{1}{2\,\pi^2}, \tag{5.19}$$

and thus,

$$\int_{S^3}\sqrt{\tilde{g}}d\boldsymbol{\theta_3}\;\tilde{\epsilon}^\dagger_{\pm,q}(\boldsymbol{\theta_3})\tilde{\epsilon}_{\pm,q'}(\boldsymbol{\theta_3}) = \delta_{qq'}\frac{1}{2\pi^2}\times\left(\int_{S^3}\sqrt{\tilde{g}}d\boldsymbol{\theta_3}\right) = \delta_{qq'}. \tag{5.20}$$

To sum up, in total, there are four independent Killing spinors (5.17) on dS_4: $\epsilon^{(+;-1)}(t,\boldsymbol{\theta_3})$. $\epsilon^{(+;0)}(t,\boldsymbol{\theta_3})$, $\epsilon^{(-;-1)}(t,\boldsymbol{\theta_3})$ and $\epsilon^{(-;0)}(t,\boldsymbol{\theta_3})$. Each of these Killing spinors can be re-expressed in the form of a spacetime-dependent spinorial matrix acting on a constant spinor $\eta^{(\sigma;q)}$, as in ref. [100]. To be specific,

$$\epsilon^{(\sigma;q)}(t,\boldsymbol{\theta_3}) = S(t,\boldsymbol{\theta_3})\,\eta^{(\sigma;q)}, \tag{5.21}$$

where the spinorial matrix $S(t,\boldsymbol{\theta_3})$ is given by

$$S(t,\boldsymbol{\theta_3}) = e^{-\frac{\pi/2-it}{2}\gamma^0}\;e^{-\frac{i\,\theta_3}{2}\gamma^{03}}\;e^{\frac{\theta_2}{2}\gamma^{32}}\;e^{\frac{\theta_1}{2}\gamma^{21}}, \tag{5.22}$$

and the constant spinors are:

$$\eta^{(-;-1)} = \frac{1+i}{2\pi}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \qquad \eta^{(-;0)} = \frac{-1+i}{2\pi}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \tag{5.23}$$

$$\eta^{(+;-1)} = \frac{-1-i}{2\pi}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \qquad \eta^{(+;0)} = \frac{1-i}{2\pi}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \tag{5.24}$$

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As we mentioned earlier, the Killing spinors denoted as $\epsilon^{(\sigma;q)}$ will be treated as commuting, and thus, the constant spinors $\eta^{(\sigma;q)}$ in eq. (5.21) are also commuting. Grassmann-odd Killing spinors are also expressed as

$$\epsilon(t, \boldsymbol{\theta_3}) = S(t, \boldsymbol{\theta_3})\, \eta, \tag{5.25}$$

but now η is a Grassmann-odd constant spinor parameter.

6 Unitary rigid SUSY for the supermultiplet of the chiral graviton and chiral gravitino

6.1 Non-unitary SUSY representation for complex (non-chiral) graviton and gravitino

In this subsection, we will start by demonstrating that the multiplet consisting of the complex graviton and the complex gravitino on dS_4, each with 2 complex propagating degrees of freedom, carries a non-unitary representation of global SUSY. Then, in subsection 6.2, we will specialise to the case where both the graviton and the gravitino are chiral — i.e. their corresponding field strengths are anti-self-dual — and we will show that the supermultiplet consisting of these two fields carries a representation of global SUSY which is unitary.

As we show below, the SUSY transformations for the supermultiplet of the complex graviton $\mathfrak{h}_{\mu\nu}$ and the Dirac gravitino Ψ_μ on dS_4 are:

$$\delta^{\text{susy}}(\epsilon)\Psi_\mu = \frac{1}{4}\left(i\, \mathfrak{h}_{\mu\sigma}\gamma^\sigma + \nabla_\lambda \mathfrak{h}_{\mu\sigma}\gamma^{\sigma\lambda}\right)\epsilon, \tag{6.1}$$

$$\delta^{\text{susy}}(\epsilon)\mathfrak{h}_{\mu\nu} = \frac{\overline{\epsilon}}{2}\gamma^5\left(\gamma_\mu\Psi_\nu + \gamma_\nu\Psi_\mu\right), \tag{6.2}$$

where ϵ is an anti-commuting complex Killing spinor satisfying eq. (5.1) with the '$-$' sign. SUSY transformations with commuting Killing spinors will be also used when we consider their action on mode solutions in subsections 6.2.1 and 6.2.2. These transformations are gauge invariant. That is, if we consider pure-gauge solutions,

$$\Psi^{(pg)}_\mu = \left(\nabla_\mu + \frac{i}{2}\gamma_\mu\right)X, \tag{6.3}$$

$$\mathfrak{h}^{(pg)}_{\mu\nu} = \nabla_{(\mu}\mathcal{Z}_{\nu)}, \tag{6.4}$$

then

$$(\delta^{\text{susy}}(\epsilon)\Psi_\mu)^{(pg)} = \frac{1}{8}\left(\nabla_\mu + \frac{i}{2}\gamma_\mu\right)\{[2i\mathcal{Z}_\sigma\gamma^\sigma + (\nabla_\lambda\mathcal{Z}_\sigma)\gamma^{\sigma\lambda}]\epsilon\}, \tag{6.5}$$

$$(\delta^{\text{susy}}(\epsilon)\mathfrak{h}_{\mu\nu})^{(pg)} = \nabla_{(\mu}(\overline{\epsilon}\gamma^5\gamma_{\nu)}X). \tag{6.6}$$

It is easy to find the Dirac conjugate of $\delta^{\text{susy}}(\epsilon)\Psi_\mu$ and the hermitian conjugate of $\delta^{\text{susy}}(\epsilon)\mathfrak{h}_{\mu\nu}$, as:

$$\overline{\delta^{\text{susy}}(\epsilon)\Psi}_\mu = -\frac{\overline{\epsilon}}{4}\left(-i\, \mathfrak{h}^\dagger_{\mu\sigma}\gamma^\sigma + \nabla_\lambda \mathfrak{h}^\dagger_{\mu\sigma}\gamma^{\sigma\lambda}\right),$$

$$(\delta^{\text{susy}}(\epsilon)\mathfrak{h}_{\mu\nu})^\dagger = \frac{1}{2}\left(\overline{\Psi}_\nu\gamma_\mu + \overline{\Psi}_\mu\gamma_\nu\right)\gamma^5\epsilon.$$

Let us emphasise that the SUSY transformations (6.1) and (6.2) are relevant to two different supersymmetric theories:

- The first corresponds to the theory of a complex graviton and a complex gravitino, each with two complex propagating helicity degrees of freedom. This theory is non-unitary as it involves a gravitino field containing all of its propagating helicity degrees of freedom which leads to the appearance of negative-norm states, as explained in subsection 3.4.

- The second theory is obtained from the first by a simple projection, and is our theory of interest. It consists of a chiral graviton and a chiral gravitino, each with one complex propagating helicity degree of freedom (the field strength of each gauge potential is anti-self-dual). For the unitarity of this theory it is crucial to demonstrate that the SUSY transformations are consistent with the anti-self-duality constraint [eqs. (3.64) and (4.58)]. In other words, we have to show that the anti-self-dual gravitino field strength (3.79) transforms only into the anti-self-dual linearised Weyl tensor (4.66) and vice versa — see subsection 6.1.4. This means that gravitons with helicity -2 $(+2)$ transform into gravitini with helicity $-\frac{3}{2}$ $(+\frac{3}{2})$ and vice versa. Once the compatibility of the SUSY transformations with the anti-self-duality constraint has been verified (this can be done only on-shell), one can rewrite the SUSY transformations (6.1) and (6.2) in a form that refers explicitly to the supermultiplet of a chiral graviton and a chiral gravitino, as

$$\delta^{\text{susy}}(\epsilon)\Psi^-_\mu = \frac{1}{4}\left(i\,\mathfrak{h}^-_{\mu\sigma}\gamma^\sigma + \nabla_\lambda \mathfrak{h}^-_{\mu\sigma}\gamma^{\sigma\lambda}\right)\epsilon, \tag{6.7}$$

$$\delta^{\text{susy}}(\epsilon)\mathfrak{h}^-_{\mu\nu} = \frac{\overline{\epsilon}}{2}\gamma^5\left(\gamma_\mu\Psi^-_\nu + \gamma_\nu\Psi^-_\mu\right), \tag{6.8}$$

where $\mathfrak{h}^-_{\mu\nu}$ and Ψ^-_μ are the chiral graviton and gravitino gauge potentials.[37] The **main result** of this paper is that the supermultiplet consisting of the chiral graviton and the chiral gravitino $\left(\mathfrak{h}^-_{\mu\nu}, \Psi^-_\mu\right)$, with their corresponding field strengths satisfying the anti-self-duality constraints (4.58) and (3.64), respectively, forms a unitary representation of global SUSY that is also unitarily realised on the QFT Fock space.

Note. One can instead consider the chiral supermutliplet $\left(\mathfrak{h}^+_{\mu\nu}, \Psi^+_\mu\right)$, with corresponding field strengths being self-dual instead of anti-self-dual. This theory also realises a unitary representation of global SUSY where the SUSY transformations are given by (6.7) and (6.8) with $\mathfrak{h}^-_{\mu\nu}$ and Ψ^-_μ replaced by $\mathfrak{h}^+_{\mu\nu}$ and Ψ^+_μ, respectively. The two supermutliplets $\left(\mathfrak{h}^-_{\mu\nu}, \Psi^-_\mu\right)$ and $\left(\mathfrak{h}^+_{\mu\nu}, \Psi^+_\mu\right)$ separately form unitary representations of global SUSY in dS_4. Although in this paper we show the unitarity of the supermultiplet $\left(\mathfrak{h}^-_{\mu\nu}, \Psi^-_\mu\right)$, the unitarity of the supermultiplet $\left(\mathfrak{h}^+_{\mu\nu}, \Psi^+_\mu\right)$ can be shown in the same way. However, if the two theories are combined together to form the supermultiplet $(\mathfrak{h}_{\mu\nu}, \Psi_\mu)$ with $\mathfrak{h}_{\mu\nu} = \mathfrak{h}^-_{\mu\nu} + \mathfrak{h}^+_{\mu\nu}$ and $\Psi_\mu = \Psi^-_\mu + \Psi^+_\mu$, then the resulting theory would be non-unitary because, in this case, the gravitino field contains all of its helicities, giving rise to negative norms in the QFT Fock space — see subsection 4.3.

The 'chiral' SUSY transformations (6.7) and (6.8), which are the transformations relevant to our theory of interest, are a special case of the initial non-chiral SUSY transformations (6.1)

[37] See eqs. (4.62) and (3.68), respectively, for the mode expansion of the completely gauge-fixed version of the chiral gauge potentials.

and (6.2). We will show that the latter non-chiral transformations are symmetries at the level of both the hermitian action (6.11) of the theory and the field equations. On the other hand, the theory that contains only a chiral graviton and a chiral gravitino has no local action principle, as it is not possible to split the helicities in a local way at the level of the action. However, the 'chiral' SUSY transformations (6.7) and (6.8) are symmetries at the level of the equations of motion — we will show that this follows from the invariance of the equations of motion under the (non-chiral) SUSY transformations (6.1) and (6.2). In other words, we will show that the 'chiral graviton-chiral gravitino' supermultiplet $(\mathfrak{h}^{-}_{\mu\nu}, \Psi^{-}_{\mu})$ carries a representation of SUSY. The unitarity of this representation will be demonstrated in the next subsection.

Let us start discussing the general theory of a complex graviton and gravitino, each with two complex propagating helicity degrees of freedom, and specialise to our chiral theory later.

6.1.1 SUSY invariance of non-gauge-fixed field equations

Let $\mathfrak{h}_{\mu\nu}$ and Ψ_μ be complex off-shell field configurations, and let us consider the differential operators appearing in the field equations, $H_{\mu\nu}(\mathfrak{h})$ [eq. (4.37)] and $\mathcal{R}(\Psi)$ [eq. (3.6)], respectively, acting on the off-shell fields. After a straightforward, but lengthy, off-shell calculation, one can show that $H_{\mu\nu}(\mathfrak{h})$ and $\mathcal{R}(\Psi)$ transform into each other under the SUSY transformations (6.1) and (6.2), as

$$\begin{aligned}\delta^{\rm susy}(\epsilon)H^{\mu\nu}(\mathfrak{h}) \equiv H^{\mu\nu}(\delta^{\rm susy}(\epsilon)\mathfrak{h}) =&\ \bar{\epsilon}\gamma^5\left(\frac{5}{2}i\,\gamma^{(\mu} + \nabla^{(\mu} - \gamma^{(\mu}\not{\nabla}\right)\mathcal{R}^{\nu)}(\Psi)\\ =&\ \bar{\epsilon}\gamma^5\left(\frac{5}{2}i\,\gamma^{(\mu} + \gamma^{\lambda(\mu}\nabla_\lambda\right)\mathcal{R}^{\nu)}(\Psi),\end{aligned} \tag{6.9}$$

and

$$\delta^{\rm susy}(\epsilon)\mathcal{R}^\mu(\Psi) \equiv \mathcal{R}^\mu(\delta^{\rm susy}(\epsilon)\Psi) = \frac{1}{4}\gamma_\alpha\epsilon\, H^{\mu\alpha}(\mathfrak{h}). \tag{6.10}$$

(These equations hold for both commuting and anti-commuting Killing spinors.) This shows that the solution spaces of equations $\mathcal{R}^\mu(\Psi) = 0$ [eq. (3.1)] and $H^{\mu\nu}(\mathfrak{h}) = 0$ [eq. (4.36)] transform into each other under the SUSY transformations (6.1) and (6.2). Thus, the supermultiplet of the complex graviton and the complex gravitino carries a representation of global SUSY. As we mentioned earlier, this SUSY representation is bound to be non-unitary, but unitarity will be achieved by restricting to the 'chiral graviton-chiral gravitino' supermultiplet in subsection 6.2.

6.1.2 SUSY invariance of the hermitian action and supercurrents

The hermitian action for the theory consisting of the complex graviton and Dirac gravitino is given by the sum of the free actions (3.8) and (4.49):

$$S = S_2 + S_{\frac{3}{2}} = \int d^4x\,\sqrt{-g}\,\left(-\frac{1}{4}\mathfrak{h}^{\dagger}_{\mu\nu}\,H^{\mu\nu}(\mathfrak{h}) - \overline{\Psi}_\mu\gamma^5\mathcal{R}^\mu(\Psi)\right). \tag{6.11}$$

It is useful to show that the action (6.11) is SUSY-invariant, as this will allow us to find the conserved Noether currents associated with SUSY. Varying the action (6.11) under

$\delta^{\rm susy}(\epsilon)\Psi_\mu$ and $(\delta^{\rm susy}(\epsilon)\mathfrak{h}_{\mu\nu})^\dagger$, we find

$$\delta S = \int d^4x\,\sqrt{-g}\,\left(-\frac{1}{4}\,(\delta^{\rm susy}(\epsilon)\mathfrak{h}_{\mu\nu})^\dagger\; H^{\mu\nu}(\mathfrak{h}) - \overline{\Psi}_\mu\gamma^5\;\mathcal{R}^\mu(\delta^{\rm susy}(\epsilon)\Psi)\right) = 0, \tag{6.12}$$

where $\mathcal{R}^\mu(\delta^{\rm susy}(\epsilon)\Psi)$ is given by eq. (6.10). Also, varying the action under $\overline{\delta^{\rm susy}(\epsilon)\Psi}_\mu$ and $\delta^{\rm susy}(\epsilon)\mathfrak{h}_{\mu\nu}$, we find that δS is equal to the integral of a total divergence, as

$$\begin{aligned}\delta S &= \int d^4x\,\sqrt{-g}\,\left(-\frac{1}{4}\mathfrak{h}^\dagger_{\mu\nu}\,H^{\mu\nu}(\delta^{\rm susy}(\epsilon)\mathfrak{h}) - \overline{\delta^{\rm susy}(\epsilon)\Psi}_\mu\,\gamma^5\mathcal{R}^\mu(\Psi)\right)\\ &= \int d^4x\,\sqrt{-g}\;\nabla_\lambda\left(\frac{\overline{\epsilon}}{4}\gamma^5\gamma^{\sigma\lambda}\mathcal{R}^\nu(\Psi)\;\mathfrak{h}^\dagger_{\nu\sigma}\right),\end{aligned} \tag{6.13}$$

where $H^{\mu\nu}(\delta^{\rm susy}(\epsilon)\mathfrak{h})$ is given by eq. (6.9).

The covariantly conserved Noether vector currents arising from the SUSY invariance of the action are easily found as

$$\begin{aligned}\left(\mathcal{J}^\mu_{(\epsilon)}(\mathfrak{h},\Psi)\right)^\dagger &= \frac{i}{4}\overline{\Psi}_\kappa\gamma^{\kappa\mu\sigma}\gamma^5\,(i\mathfrak{h}_{\sigma\nu}\gamma^\nu + \gamma^{\nu\rho}\nabla_\rho\mathfrak{h}_{\sigma\nu})\,\epsilon = i\;\overline{\Psi}_\kappa\gamma^{\kappa\mu\sigma}\gamma^5\;\delta^{\rm susy}(\epsilon)\Psi_\sigma,\\ \mathcal{J}^\mu_{(\epsilon)}(\mathfrak{h},\Psi) &= \frac{i}{4}\overline{\epsilon}\left(i\mathfrak{h}^\dagger_{\sigma\nu}\gamma^\nu - \gamma^{\nu\rho}\nabla_\rho\mathfrak{h}^\dagger_{\sigma\nu}\right)\gamma^{\sigma\mu\kappa}\gamma^5\Psi_\kappa = i\;\;\overline{\delta^{\rm susy}(\epsilon)\Psi}_\sigma\;\gamma^{\sigma\mu\kappa}\gamma^5\Psi_\kappa.\end{aligned} \tag{6.14}$$

The Grassmann-odd fermionic supercurrents $\mathfrak{J}^\mu_A$ and $\overline{\mathfrak{J}}^{\mu\,A}$ are related to the SUSY Noether currents as

$$\mathcal{J}^\mu_{(\epsilon)} = \overline{\epsilon}^A\;\mathfrak{J}^\mu_A, \qquad \mathcal{J}^{\mu\dagger}_{(\epsilon)} = \overline{\mathfrak{J}}^{\mu\,A}\epsilon_A,$$

where $A = 1,\ldots,4$ is a spinor index. The time-independent (complex) Noether charges associated to the vector currents (6.14) are defined as [79]

$$Q^{\rm susy}[\epsilon] = \int_{S^3} d\boldsymbol{\theta_3}\sqrt{-g}\;\mathcal{J}^t_{(\epsilon)}(\mathfrak{h},\Psi), \qquad (Q^{\rm susy}[\epsilon])^\dagger = \int_{S^3} d\boldsymbol{\theta_3}\sqrt{-g}\;\left(\mathcal{J}^t_{(\epsilon)}(\mathfrak{h},\Psi)\right)^\dagger. \tag{6.15}$$

We refer to $\mathcal{J}^\mu_{(\epsilon)}$ and $\mathcal{J}^{\mu\dagger}_{(\epsilon)}$ as SUSY Noether currents, and to $Q^{\rm susy}[\epsilon]$, $Q^{\rm susy}[\epsilon]^\dagger$ as SUSY Noether charges. Since the Killing spinors ϵ are Grassmann-odd, then $\mathcal{J}^\mu_{(\epsilon)}$ and $Q^{\rm susy}[\epsilon]$ are Grassmann-even. If we use commuting Killing spinors $\epsilon^{(\sigma;q)}$ [eq. (5.21)], then $\mathcal{J}^\mu_{(\epsilon^{(\sigma;q)})}$ and $Q^{\rm susy}[\epsilon^{(\sigma;q)}]$ are also conserved, and they are Grassmann-odd. In subsection 6.2.2, it will be convenient to work with these Grassmann-odd SUSY Noether currents and charges.

6.1.3 SUSY algebra with complex Killing spinors

After a straightforward calculation, the commutator of two SUSY transformations [eqs. (6.1) and (6.2)] on the complex graviton is found to be

$$\begin{aligned}\left[\,\delta^{\rm susy}(\epsilon_2),\delta^{\rm susy}(\epsilon_1)\,\right]\,\mathfrak{h}_{\mu\nu} = &-\mathcal{L}_{\xi_{(\epsilon)}}\mathfrak{h}_{\mu\nu} + \mathcal{T}_{V_{(\epsilon)}}\,\mathfrak{h}_{\mu\nu} - i\left(\frac{1}{4}\overline{\epsilon}_2\gamma^5\epsilon_1 - \frac{1}{4}\overline{\epsilon}_1\gamma^5\epsilon_2\right)\mathfrak{h}_{\mu\nu}\\ &+\nabla_{(\mu}\left[\mathfrak{h}_{\nu)\sigma}\,\xi^\sigma_{(\epsilon)}\right],\end{aligned} \tag{6.16}$$

where no use of the equations of motion was made. The first term, $-\mathcal{L}_{\xi_{(\epsilon)}}\mathfrak{h}_{\mu\nu}$, on the right-hand side of eq. (6.16) is an infinitesimal dS transformation (Lie derivative) generated by the Killing vector $\xi^\mu_{(\epsilon)}$ defined in eq. (5.3). The second term, $\mathcal{T}_{V_{(\epsilon)}}\mathfrak{h}_{\mu\nu}$, is a conformal-like transformation (4.35) generated by the genuine conformal Killing vector $V^\mu_{(\epsilon)}$ defined in eq. (5.4). The third term is an infinitesimal $u(1)$ transformation [the phase factor $\frac{1}{4}\overline{\epsilon}_2\gamma^5\epsilon_1 - \frac{1}{4}\overline{\epsilon}_1\gamma^5\epsilon_2$ is real as $-\overline{\epsilon}_1\gamma^5\epsilon_2 = (\overline{\epsilon}_2\gamma^5\epsilon_1)^\dagger$, and constant, as $\nabla_\mu\left(\overline{\epsilon}_2\gamma^5\epsilon_1\right) = \nabla_\mu\left(\overline{\epsilon}_1\gamma^5\epsilon_2\right) = 0$, as expected for $u(1)$ transformations]. The last term is a field-dependent gauge transformation akin to the gauge transformation appearing in linearised Supergravity in Minkowski spacetime — see e.g., ref. [101]. We conclude that the even subalgebra of the SUSY algebra closes on $so(4,2)\bigoplus u(1)$ up to gauge transformations.[38]

Calculating the commutator of two SUSY transformations on the Dirac gravitino Ψ_μ is much more tedious than the complex graviton case above. Moreover, one has to make use of the equations of motion, as well as of the Fierz rearrangement identities [79]. The result is[39]

$$\begin{aligned}\left[\,\delta^{\rm susy}(\epsilon_2), \delta^{\rm susy}(\epsilon_1)\,\right]\,\Psi_\mu = &-\mathbb{L}_{\xi_{(\epsilon)}}\Psi_\mu + \mathbb{T}'_{V_{(\epsilon)}}\Psi_\mu - \frac{5i}{2}\left(\frac{1}{4}\overline{\epsilon}_2\gamma^5\epsilon_1 - \frac{1}{4}\overline{\epsilon}_1\gamma^5\epsilon_2\right)\Psi_\mu \\ &+\left(\nabla_\mu + \frac{i}{2}\gamma_\mu\right)\frac{A_{(\epsilon)}}{2},\end{aligned} \tag{6.17}$$

where $A_{(\epsilon)}$ is a field-dependent spinor gauge function given by

$$\begin{aligned}A_{(\epsilon)} = &\frac{3}{4}\left(\frac{1}{4}\overline{\epsilon_2}\gamma^5\epsilon_1 - \frac{1}{4}\overline{\epsilon_1}\gamma^5\epsilon_2\right)\gamma^\alpha\Psi_\alpha + \xi^\alpha_{(\epsilon)}\Psi_\alpha + \left(\frac{1}{4}\xi^\rho_{(\epsilon)}\gamma_\rho - \frac{i}{8}\nabla^\lambda\xi^\rho_{(\epsilon)}\gamma_{\lambda\rho}\right)\gamma^\alpha\Psi_\alpha \\ &-\gamma^5 V^\alpha_{(\epsilon)}\Psi_\alpha - \frac{1}{4}\gamma^5 V^\rho_{(\epsilon)}\gamma_\rho\gamma^\alpha\Psi_\alpha - \frac{3i}{4}\phi_{V_{(\epsilon)}}\gamma^\alpha\Psi_\alpha.\end{aligned} \tag{6.18}$$

The terms that appear on the right-hand side of eq. (6.17) are similar to the terms appearing in the complex graviton case (6.16). In particular, the first term, $-\mathbb{L}_{\xi_{(\epsilon)}}\Psi_\mu$, is an infinitesimal dS transformation (3.16) generated by the Killing vector $\xi^\mu_{(\epsilon)}$ defined in eq. (5.3). The second term, $\mathbb{T}'_{V_{(\epsilon)}}\Psi_\mu$, is given by a conformal-like transformation $\mathbb{T}_{V_{(\epsilon)}}\Psi_\mu$ (3.51) plus a gauge transformation that cancels the last term of the conformal-like transformation (3.51), where $V^\mu_{(\epsilon)}$ is the genuine conformal Killing vector defined in eq. (5.4). To be specific, $\mathbb{T}'_{V_{(\epsilon)}}\Psi_\mu = \mathbb{T}_{V_{(\epsilon)}}\Psi_\mu + \frac{2}{3}\left(\nabla_\mu + \frac{i}{2}\gamma_\mu\right)\gamma^5\Psi_\rho V^\rho_{(\epsilon)}$. The third term on the right-hand side of (6.17) is a $u(1)$ transformation, while the last term is a field-dependent gauge transformation. It is thus clear that the even subalgebra of the SUSY algebra closes on $so(4,2)\bigoplus u(1)$ up to gauge transformations.

6.1.4 SUSY transformations of the field strengths and of their duals

Let us give here again the expressions of the field strengths for the complex graviton [eq. (4.59)] and complex gravitino [eq. (3.65)]:

$$U_{\alpha\beta\mu\nu} = \Big(-\nabla_\mu\nabla_{[\alpha}\mathfrak{h}_{\beta]\nu} - g_{\mu[\alpha}\mathfrak{h}_{\beta]\nu}\Big) - (\mu\leftrightarrow\nu), \qquad F_{\mu\nu} = \left(\nabla_{[\mu} + \frac{i}{2}\gamma_{[\mu}\right)\Psi_{\nu]}.$$

Their properties are summarised in appendix E.

[38] The $so(4,2)$ algebra generated by infinitesimal dS transformations and conformal-like transformations was studied in subsections 3.2 and 4.2.2, for the gravitino and the graviton, respectively.

[39] We made use of the Mathematica tensor computer algebra package FieldsX [102] to simplify certain parts of the calculation that involved products of (generalised) gamma matrices.

SUSY transformations of field strengths. The SUSY transformations of the field strengths can be obtained by direct calculation using the SUSY transformations of the gauge potentials [eqs. (6.1) and (6.2)], as

$$\delta^{\rm susy}(\epsilon)F_{\mu\nu} = \left(\nabla_{[\mu} + \frac{i}{2}\gamma_{[\mu}\right)\,\delta^{\rm susy}(\epsilon)\Psi_{\nu]},$$

and

$$\delta^{\rm susy}(\epsilon)U_{\alpha\beta\mu\nu} = \Big(-\nabla_\mu\nabla_{[\alpha}\delta^{\rm susy}(\epsilon)\mathfrak{h}_{\beta]\nu} - g_{\mu[\alpha}\delta^{\rm susy}(\epsilon)\mathfrak{h}_{\beta]\nu}\Big) - (\mu\leftrightarrow\nu).$$

The result is

$$\delta^{\rm susy}(\epsilon)F_{\mu\nu} = \frac{1}{8}\gamma^{\kappa\lambda}\epsilon\,U_{\kappa\lambda\mu\nu}, \tag{6.19}$$

$$\delta^{\rm susy}(\epsilon)U^{\alpha\beta}{}_{\mu\nu} = \bar\epsilon\gamma^5\Big(\left(\gamma^{[\alpha}\nabla^{\beta]} - \frac{i}{2}\gamma^{\alpha\beta}\right)F_{\mu\nu} + \left(\gamma_{[\mu}\nabla_{\nu]} - \frac{i}{2}\gamma_{\mu\nu}\right)F^{\alpha\beta} + 2i\gamma_{[\mu}{}^{[\alpha}F^{\beta]}{}_{\nu]}\Big), \tag{6.20}$$

where no use of the equations of motion was made.

Duality commutes with SUSY transformations. It is convenient to re-write the SUSY transformation (6.20) as

$$\delta^{\rm susy}(\epsilon)U_{\alpha\beta\mu\nu} = \bar\epsilon\gamma^5\Big((\gamma_{[\alpha}\nabla_{\beta]} - i\gamma_{\alpha\beta})F_{\mu\nu} + (\gamma_{[\mu}\nabla_{\nu]} - i\gamma_{\mu\nu})F_{\alpha\beta}\Big). \tag{6.21}$$

To derive eq. (6.21) from eq. (6.20), we have used

$$2\gamma_{[\mu}{}^{[\alpha}F^{\beta]}{}_{\nu]} = -\frac{1}{2}\gamma^{\alpha\beta}F_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}F^{\alpha\beta}, \tag{6.22}$$

which can be proved by using the on-shell properties of the spin-3/2 field strength, as well as properties of products of (generalised) gamma matrices — see appendix E. Given the SUSY transformations (6.19) and (6.21), a straightforward calculation using some formulae in appendix E shows that the duality operation commutes with them, as

$$\delta^{\rm susy}(\epsilon)\widetilde{F}_{\mu\nu} = \frac{1}{8}\gamma^{\kappa\lambda}\epsilon\,\widetilde{U}_{\kappa\lambda\mu\nu}, \tag{6.23}$$

$$\delta^{\rm susy}(\epsilon)\widetilde{U}_{\alpha\beta\mu\nu} = \bar\epsilon\gamma^5\Big((\gamma_{[\alpha}\nabla_{\beta]} - i\gamma_{\alpha\beta})\widetilde{F}_{\mu\nu} + (\gamma_{[\mu}\nabla_{\nu]} - i\gamma_{\mu\nu})\widetilde{F}_{\alpha\beta}\Big). \tag{6.24}$$

SUSY algebra for the field strengths. The commutator of two SUSY transformations acting on $U_{\alpha\beta\mu\nu}$ is

$$[\,\delta^{\rm susy}(\epsilon_2),\delta^{\rm susy}(\epsilon_1)\,]\,U_{\alpha\beta\mu\nu} = -\,\pounds_{\xi_{(\epsilon)}}U_{\alpha\beta\mu\nu} + \mathcal{T}_{V_{(\epsilon)}}\,U_{\alpha\beta\mu\nu} - i\left(\frac{1}{4}\bar\epsilon_2\gamma^5\epsilon_1 - \frac{1}{4}\bar\epsilon_1\gamma^5\epsilon_2\right)U_{\alpha\beta\mu\nu}, \tag{6.25}$$

where the Killing vector $\xi^\mu_{(\epsilon)}$ and the genuine conformal Killing vector $V^\mu_{(\epsilon)}$ are defined in eqs. (5.3) and (5.4), respectively. The interpretation of the terms on the right-hand side of eq. (6.25) is the same as in the case of the complex graviton gauge potential (6.16), except, of course, for the gauge transformation term in (6.16), which drops out. The conformal-like transformation $\mathcal{T}_V\,U_{\alpha\beta\mu\nu}$, generated by genuine conformal Killing vectors,

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is given by the product of a conventional infinitesimal conformal transformation times a duality transformation (times i), as

$$\mathcal{T}_V U_{\alpha\beta\mu\nu} = i\left(\pounds_V - \frac{1}{4}\nabla^\rho V_\rho\right)\widetilde{U}_{\alpha\beta\mu\nu} = i\left(V^\rho\nabla_\rho - 3\phi_V\right)\widetilde{U}_{\alpha\beta\mu\nu}. \tag{6.26}$$

For the sake of completeness, let us also compute the commutator of a SUSY variation and a conformal-like variation: $[\delta^{\text{susy}}(\epsilon), \delta_V]\, U_{\alpha\beta\mu\nu}$, where $\delta_V U_{\alpha\beta\mu\nu} = \mathcal{T}_V U_{\alpha\beta\mu\nu}$ and V^μ is any genuine conformal Killing vector (2.13). We find

$$\begin{aligned}[\delta^{\text{susy}}(\epsilon), \delta_V]\, U_{\alpha\beta\mu\nu} &= \overline{\mathbb{T}_V\epsilon}\,\gamma^5\Big((\gamma_{[\alpha}\nabla_{\beta]} - i\gamma_{\alpha\beta})F_{\mu\nu} + (\gamma_{[\mu}\nabla_{\nu]} - i\gamma_{\mu\nu})F_{\alpha\beta}\Big),\\ &= \delta^{\text{susy}}\Big(\mathbb{T}_V\epsilon\Big)\, U_{\alpha\beta\mu\nu},\end{aligned} \tag{6.27}$$

which is a SUSY variation of $U_{\alpha\beta\mu\nu}$ (6.21), but with the Killing spinor ϵ replaced by its conformal-like-transformed version, $\mathbb{T}_V\epsilon$ [eq. (5.7)]. One similarly finds the commutator between a SUSY variation and a dS variation,

$$[\delta^{\text{susy}}(\epsilon), \delta_\xi]\, U_{\alpha\beta\mu\nu} = \delta^{\text{susy}}(\mathbb{L}_\xi\epsilon) U_{\alpha\beta\mu\nu}, \tag{6.28}$$

where $\delta_\xi U_{\alpha\beta\mu\nu} = \pounds_\xi U_{\alpha\beta\mu\nu}$ and ξ^μ is any Killing vector.

The commutator of two SUSY transformations for $F_{\mu\nu}$ is

$$[\,\delta^{\text{susy}}(\epsilon_2), \delta^{\text{susy}}(\epsilon_1)\,]\; F_{\mu\nu} = -\,\mathbb{L}_{\xi_{(\epsilon)}}F_{\mu\nu} + \mathbb{T}_{V_{(\epsilon)}}\,F_{\mu\nu} - \frac{5i}{2}\left(\frac{1}{4}\overline{\epsilon}_2\gamma^5\epsilon_1 - \frac{1}{4}\overline{\epsilon}_1\gamma^5\epsilon_2\right)F_{\mu\nu}, \tag{6.29}$$

where the interpretation of the terms on the right-hand side is as in the case of the complex gravitino gauge potential (6.17). The conformal-like transformation of the gravitino field strength, generated by any genuine conformal Killing vector V^μ (2.13), is given by [76]

$$\mathbb{T}_V F_{\mu\nu} = i\left(\mathbb{L}_V + \frac{1}{8}\nabla_\rho V^\rho\right)\widetilde{F}_{\mu\nu} = \gamma^5\left(V^\rho\nabla_\rho - \frac{5}{2}\phi_V\right)F_{\mu\nu}. \tag{6.30}$$

The commutator between a SUSY variation and a conformal-like variation, as well as the commutator between a SUSY variation and a dS variation, are given by expressions similar to (6.27) and (6.28), respectively.

6.1.5 SUSY representation on the TT solution spaces and non-unitarity of the non-chiral theory

The TT gauge is a particularly convenient gauge as the field equations have a simple form. In addition, the TT mode solutions and their representation-theoretic properties are known — see subsections 3.1 and 4.1. For convenience, let us write here again the field equations for the complex graviton and the Dirac gravitino in the TT gauge [eqs. (4.39) and (3.13)]

$$\begin{aligned}\Box\mathfrak{h}^{(\mathrm{TT})}_{\mu\nu} &= 2\,\mathfrak{h}^{(\mathrm{TT})}_{\mu\nu}, &&\\ \nabla^\mu\mathfrak{h}^{(\mathrm{TT})}_{\mu\nu} &= 0, && \mathfrak{h}^{(\mathrm{TT})\alpha}_{\ \alpha} = 0,\\ (\nabla\!\!\!/ + i)\,\Psi^{(\mathrm{TT})}_\mu &= 0, &&\\ \nabla^\alpha\Psi^{(\mathrm{TT})}_\alpha &= 0, && \gamma^\alpha\Psi^{(\mathrm{TT})}_\alpha = 0.\end{aligned}$$

To achieve the compatibility of the SUSY transformations (6.1) and (6.2) with the TT conditions — $\nabla^\mu \mathfrak{h}^{(\mathrm{TT})}_{\mu\nu} = g^{\alpha\beta}\mathfrak{h}^{(\mathrm{TT})}_{\alpha\beta} = 0$ and $\nabla^\mu \Psi^{(\mathrm{TT})}_\mu = \gamma^\mu \Psi^{(\mathrm{TT})}_\mu = 0$ — we have to modify the SUSY transformation of the graviton (6.2) by introducing a gauge transformation. To be specific, we modify the SUSY transformation $\delta^{\mathrm{susy}}(\epsilon)\mathfrak{h}_{\mu\nu}$ given by eq. (6.2) by adding a gauge transformation term with the field-dependent gauge parameter $-\frac{i}{3}\bar{\epsilon}\gamma^5\Psi_\nu$ to ensure that if the fields $\mathfrak{h}_{\mu\nu}$ and Ψ_μ are in the TT gauge, then the SUSY-transformed graviton remains in the TT gauge (no such gauge correction is required for the gravitino). Thus, the SUSY transformations that preserve the solution space of the TT field equations (4.39) and (3.13) are:

$$\delta^{\mathrm{susy}}(\epsilon)\Psi^{(\mathrm{TT})}_\mu = \frac{1}{4}\left(i\,\mathfrak{h}^{(\mathrm{TT})}_{\mu\sigma}\gamma^\sigma + \nabla_\lambda \mathfrak{h}^{(\mathrm{TT})}_{\mu\sigma}\gamma^{\sigma\lambda}\right)\epsilon, \tag{6.31}$$

$$\begin{aligned}\delta^{\mathrm{susy}}(\epsilon)'\mathfrak{h}^{(\mathrm{TT})}_{\mu\nu} &= \delta^{\mathrm{susy}}(\epsilon)\mathfrak{h}^{(\mathrm{TT})}_{\mu\nu} + \delta^{\mathrm{gauge}}\left(-\frac{i}{3}\bar{\epsilon}\gamma^5\Psi^{(\mathrm{TT})}\right)\mathfrak{h}^{(\mathrm{TT})}_{\mu\nu} \\ &= \frac{\bar{\epsilon}}{2}\gamma^5\left(\gamma_\mu\Psi^{(\mathrm{TT})}_\nu + \gamma_\nu\Psi^{(\mathrm{TT})}_\mu\right) + \nabla_{(\mu}\left(-\frac{i}{3}\bar{\epsilon}\gamma^5\Psi^{(\mathrm{TT})}_{\nu)}\right) \\ &= \frac{7}{6}\bar{\epsilon}\gamma^5\gamma_{(\mu}\Psi^{(\mathrm{TT})}_{\nu)} - \frac{i}{3}\bar{\epsilon}\gamma^5\nabla_{(\mu}\Psi^{(\mathrm{TT})}_{\nu)},\end{aligned} \tag{6.32}$$

where ϵ is a Grassmann-odd Killing spinor satisfying eq. (5.1) with the '$-$' sign. Note that the complex graviton gauge transformation in (6.32) is **not** a restricted gauge transformation (4.40) as it is not divergence-free. In particular,

$$\nabla^\mu\nabla_{(\mu}\left(-\frac{i}{3}\bar{\epsilon}\gamma^5\Psi^{(\mathrm{TT})}_{\nu)}\right) = -\nabla^\mu\delta^{\mathrm{susy}}(\epsilon)\mathfrak{h}^{(\mathrm{TT})}_{\mu\nu} = -i\,\bar{\epsilon}\gamma^5\Psi^{(\mathrm{TT})}_\nu,$$

leading to $\nabla^\mu\delta^{\mathrm{susy}}(\epsilon)'\mathfrak{h}^{(\mathrm{TT})}_{\mu\nu} = 0$. It follows from the gauge invariance of the SUSY transformations that the commutators of the SUSY transformations in the TT gauge (6.31) and (6.32) takes the following form:

$$\begin{aligned}&\left(\delta^{\mathrm{susy}}(\epsilon_2)\delta^{\mathrm{susy}}(\epsilon_1)' - \delta^{\mathrm{susy}}(\epsilon_1)\delta^{\mathrm{susy}}(\epsilon_2)'\right)\mathfrak{h}^{(\mathrm{TT})}_{\mu\nu} \\ &= -\mathcal{L}_{\xi_{(\epsilon)}}\mathfrak{h}^{(\mathrm{TT})}_{\mu\nu} + \mathcal{T}_{V_{(\epsilon)}}\mathfrak{h}^{(\mathrm{TT})}_{\mu\nu} - i\left(\frac{1}{4}\bar{\epsilon}_2\gamma^5\epsilon_1 - \frac{1}{4}\bar{\epsilon}_1\gamma^5\epsilon_2\right)\mathfrak{h}^{(\mathrm{TT})}_{\mu\nu} + (\text{pure-gauge term})\,,\end{aligned} \tag{6.33}$$

$$\begin{aligned}&\left(\delta^{\mathrm{susy}}(\epsilon_2)'\delta^{\mathrm{susy}}(\epsilon_1) - \delta^{\mathrm{susy}}(\epsilon_1)'\delta^{\mathrm{susy}}(\epsilon_2)\right)\Psi^{(\mathrm{TT})}_\mu \\ &= -\mathbb{L}_{\xi_{(\epsilon)}}\Psi^{(\mathrm{TT})}_\mu + \mathbb{T}_{V_{(\epsilon)}}\Psi^{(\mathrm{TT})}_\mu - \frac{5i}{2}\left(\frac{1}{4}\bar{\epsilon}_2\gamma^5\epsilon_1 - \frac{1}{4}\bar{\epsilon}_1\gamma^5\epsilon_2\right)\Psi^{(\mathrm{TT})}_\mu + (\text{pure-gauge term})\,.\end{aligned} \tag{6.34}$$

This is the same algebra structure as in the non-gauged-fixed case [eqs. (6.16) and (6.17)], with the only difference being that each of the field transformations in eqs. (6.33) and (6.34) preserves the TT gauge conditions.

Note. The TT SUSY transformations (6.31) and (6.32) also describe symmetries of the TT field equations if one uses commuting Killing spinors $\epsilon^{(\sigma;q)}$ [eq. (5.21)] instead of Grassmann-odd Killing spinors. Moreover, if one uses commuting Killing spinors, the commutator of two SUSY transformations on the TT graviton is the same as in (6.33). However, the commutator of two SUSY transformations on the TT gravitino will be given by eq. (6.34) with opposite signs on the right-hand side. These comments also apply to the case of the SUSY transformations of the non-gauge-fixed fields [eqs. (6.1) and (6.2)].

The question of unitarity of SUSY and its failure in the QFT Fock space of the non-chiral theory. Until now, we have demonstrated the existence of a representation of our SUSY algebra on the solution space of the classical field equations of the complex graviton and Dirac gravitino, for both their non-gauge-fixed version and in the TT gauge. However, we have not addressed the question of unitarity. Let us specialise to the TT gauge. Our representation space is the direct sum of a bosonic and a fermionic solution space, i.e. the direct sum of the Hilbert spaces of the graviton and gravitino mode solutions $\mathcal{H}_2 \bigoplus \mathcal{H}_{\frac{3}{2}}$. The solution spaces of positive frequency mode solutions are:

$$\begin{aligned}\mathcal{H}_2 &= \mathcal{H}_2^+ \bigoplus \mathcal{H}_2^- = \{\varphi_{\mu\nu}^{(phys,\,+L;\,M;K)}\} \bigoplus \{\varphi_{\mu\nu}^{(phys,\,-L;\,M;K)}\},\\ \mathcal{H}_{\frac{3}{2}} &= \mathcal{H}_{\frac{3}{2}}^+ \bigoplus \mathcal{H}_{\frac{3}{2}}^- = \{\psi_{\mu}^{(phys,\,+\ell;\,m;k)}\} \bigoplus \{\psi_{\mu}^{(phys,\,-\ell;\,m;k)}\},\end{aligned}$$

where the TT pure-gauge modes are identified with zero, as discussed in sections 4 and 3, respectively. As an equivalent representation space one can choose the space of negative frequency mode solutions, denoted as $\mathcal{H}_2^*$ and $\mathcal{H}_{\frac{3}{2}}^*$. The bosonic and fermionic solution spaces, $\mathcal{H}_2$ and $\mathcal{H}_{\frac{3}{2}}$, are equipped with $so(4,2)$-invariant scalar products, $\langle\cdot|\cdot\rangle_{KG}$ (4.23) and $\langle\cdot|\cdot\rangle_{ax}$ (3.44), respectively, and we have already explained how the mode solutions form UIRs of $so(4,2)$ in subsections 3.2 and 4.2. A interesting feature, discussed in section 3, is that the positive-definite scalar product in $\mathcal{H}_{\frac{3}{2}}^-$ is the axial scalar product (3.44), while the positive-definite scalar product in $\mathcal{H}_{\frac{3}{2}}^+$ is the **negative** of the axial scalar product. One has the freedom to choose a different positive-definite scalar product for each solution space, $\mathcal{H}_{\frac{3}{2}}^-$ and $\mathcal{H}_{\frac{3}{2}}^+$, as each of these two spaces separately forms a UIR. However, when we quantised the gravitino theory in subsection 3.4, it became clear that the requirement of the positivity of the norm in the QFT Fock space forces the gravitino to be chiral, and thus, not both $\mathcal{H}_{\frac{3}{2}}^-$ and $\mathcal{H}_{\frac{3}{2}}^+$ can be part of the positive frequency sector of the QFT — one has to use only one of them. Thus, as the notion of unitarity is tied to the positivity of the norm, it is clear that the SUSY representation realised on the QFT Fock space of the complex non-chiral graviton and gravitino is non-unitary because the positive frequency sector of the gravitino contains both $\mathcal{H}_{\frac{3}{2}}^-$ and $\mathcal{H}_{\frac{3}{2}}^+$.

Note. One can construct two different SUSY UIRs at the level of classical mode solutions: one UIR formed by $\mathcal{H}_2^- \oplus \mathcal{H}_{\frac{3}{2}}^-$ and another one formed by $\mathcal{H}_2^+ \oplus \mathcal{H}_{\frac{3}{2}}^+$. However, for the same reasons as in the gravitino case in subsection 3.4, the unitary supersymmetric QFT of a chiral graviton and a chiral gravitino in subsection 6.2 will have a positive frequency sector consisting only of $\mathcal{H}_2^- \oplus \mathcal{H}_{\frac{3}{2}}^-$. The space $\mathcal{H}_2^+ \oplus \mathcal{H}_{\frac{3}{2}}^+$ will be excluded from the Hilbert space with the help of the anti-self-duality constraints.

6.2 Unitary SUSY for the chiral graviton and chiral gravitino

6.2.1 Unitarity of SUSY in the space of chiral mode solutions

In this subsection, we demonstrate how the mode solutions that are relevant to the supersymmetric theory of the chiral graviton and chiral gravitino form UIRs of SUSY. In particular, we work in the TT gauge and show that the SUSY transformations (6.31) and (6.32) generate a

UIR of SUSY that is realised on the space of classical TT mode solutions of positive frequency with helicities -2 and $-3/2$, $\mathcal{H}_2^- \bigoplus \mathcal{H}_{\frac{3}{2}}^-$. We also show that another UIR of SUSY is formed in the space of negative frequency modes $\mathcal{H}_2^{*+} \bigoplus \mathcal{H}_{\frac{3}{2}}^{*+}$. We focus on these representation spaces, i.e. spaces on which the field strengths are anti-self-dual, because the axial scalar product (3.44) is positive definite in both of them [see eqs. (3.47) and (3.48)].[40] To proceed with the representation-theoretic discussion, we recall that the bosonic and fermionic solution spaces that form the SUSY representation are equipped with the Klein-Gordon, $\langle\cdot|\cdot\rangle_{KG}$ (4.23), and axial, $\langle\cdot|\cdot\rangle_{ax}$ (3.44), scalar products, respectively.

Let us give the definition of unitarity for representations of our superalgebra — the structure of the superalgebra is determined by eqs. (6.33) and (6.34). A unitary representation of SUSY must satisfy the following three conditions simultaneously [103]:

1. Positivity of the norm in both the bosonic and fermionic solution spaces.

2. Invariance of the inner products under the generators of the even subalgebra [$so(4,2) \bigoplus u(1)$ in our case], i.e. anti-hermiticity of even generators.

3. SUSY-invariance of the inner products, in the sense that, for any TT solution ψ_μ of eq. (3.13), and any TT solution $\varphi_{\mu\nu}$ of eq. (4.39), the following equation holds:
$$\langle \delta^{\text{susy}}(\epsilon)\psi|\psi\rangle_{ax} = \langle \varphi|\delta^{\text{susy}}(\epsilon)'\varphi\rangle_{KG}\,. \tag{6.35}$$
According to eqs. (6.31) and (6.32), the SUSY transformations of the TT solutions are
$$\delta^{\text{susy}}(\epsilon)\psi_\mu = \frac{1}{4}\left(i\,\varphi_{\mu\sigma}\gamma^\sigma + \nabla_\lambda\varphi_{\mu\sigma}\gamma^{\sigma\lambda}\right)\epsilon, \tag{6.36}$$
$$\begin{aligned}\delta^{\text{susy}}(\epsilon)'\varphi_{\mu\nu} &= \delta^{\text{susy}}(\epsilon)\varphi_{\mu\nu} + \delta^{\text{gauge}}\left(-\frac{i}{3}\bar{\epsilon}\gamma^5\psi\right)\varphi_{\mu\nu}\\ &= \frac{\bar{\epsilon}}{2}\gamma^5\left(\gamma_\mu\psi_\nu + \gamma_\nu\psi_\mu\right) + \nabla_{(\mu}\left(-\frac{i}{3}\bar{\epsilon}\gamma^5\psi_{\nu)}\right).\end{aligned} \tag{6.37}$$
Condition 3 can be proved for both commuting and Grassmann-odd Killing spinors. However, in this subsection, we will focus on the commuting Killing spinors $\epsilon^{(\sigma,q)}$ (5.21).

Let us now demonstrate that each of the conditions 1, 2 and 3 is satisfied for the SUSY representation furnished by the positive frequency solution space $\mathcal{H}_2^- \bigoplus \mathcal{H}_{\frac{3}{2}}^-$. The following analysis can be straightforwardly generalised to the case of the SUSY UIR furnished by the negative frequency solution space $\mathcal{H}_2^{*+} \bigoplus \mathcal{H}_{\frac{3}{2}}^{*+}$, as will be discussed briefly later.

1. Positivity of the norm. The positivity of the norm for the physical gravitino modes of helicity $-3/2$, $\mathcal{H}_{\frac{3}{2}}^- = \{\psi_\mu^{(phys,\,-\ell;\,m;k)}\}$, has been demonstrated in eq. (3.47). Also, the

[40] One can similarly show that two SUSY UIRs are separately formed by the TT solution spaces $\mathcal{H}_2^+ \bigoplus \mathcal{H}_{\frac{3}{2}}^+$ and $\mathcal{H}_2^{*-} \bigoplus \mathcal{H}_{\frac{3}{2}}^{*-}$, in which the axial scalar product (3.44) is negative definite, i.e. the **negative** of the axial scalar product is positive definite. We do not give the corresponding representation-theoretic details because they are similar to the ones presented in the main text, as well as because the solution spaces $\mathcal{H}_2^+ \bigoplus \mathcal{H}_{\frac{3}{2}}^+$ and $\mathcal{H}_2^{*-} \bigoplus \mathcal{H}_{\frac{3}{2}}^{*-}$ are omitted in the unitary quantum theory of the chiral graviton and chiral gravitino.

positivity of the norm for the physical graviton modes of helicity -2, $\mathcal{H}_2^- = \{\varphi_{\mu\nu}^{(phys,\,-L;\,M;K)}\}$, has been demonstrated in eq. (4.26). Thus, it is clear that the norm is positive in the direct sum of spaces $\mathcal{H}_2^- \bigoplus \mathcal{H}_{\frac{3}{2}}^-$.

Let us also verify that the SUSY transformations (6.31) and (6.32) preserve the space $\mathcal{H}_2^- \bigoplus \mathcal{H}_{\frac{3}{2}}^-$. This is important because if there are SUSY transformations acting on positive frequency graviton modes of helicity -2 by transforming them into positive frequency gravitino modes of helicity $+3/2$, then negative norms will appear, as the axial scalar product is negative definite in $\mathcal{H}_{\frac{3}{2}}^+$ [eq. (3.47)]. Thus, we want to ensure that graviton modes in $\mathcal{H}_2^-$ transform under SUSY only into gravitino modes in $\mathcal{H}_{\frac{3}{2}}^-$, and vice versa. It can be seen that SUSY transformations do not mix the spaces $\mathcal{H}_2^- \bigoplus \mathcal{H}_{\frac{3}{2}}^-$ and $\mathcal{H}_2^+ \bigoplus \mathcal{H}_{\frac{3}{2}}^+$ because the SUSY transformations of the field strengths commute with duality transformations, as shown in subsection 6.1.4. To be specific, eqs. (6.23) and (6.24) imply that the anti-self-dual spin-3/2 field strength transforms into the anti-self-dual spin-2 field strength and vice versa. However, this observation does not rule out the possibility that some SUSY transformations mix the space $\mathcal{H}_2^- \bigoplus \mathcal{H}_{\frac{3}{2}}^-$ with the space $\mathcal{H}_2^{*+} \bigoplus \mathcal{H}_{\frac{3}{2}}^{*+}$ since both spaces consist of anti-self-dual mode solutions. We prove that the space $\mathcal{H}_2^- \bigoplus \mathcal{H}_{\frac{3}{2}}^-$ is indeed fixed under the SUSY transformations by investigating how the individual TT graviton and gravitino modes transform.

First, we determine the gravitino SUSY transformation (6.31) generated by the commuting Killing spinors $\epsilon^{(\sigma;q)}$ [eq. (5.21)], by working at the level of mode solutions in our representation space, $\mathcal{H}_2^- \bigoplus \mathcal{H}_{\frac{3}{2}}^-$. More specifically, we will substitute the modes $\varphi_{\mu\nu}^{(phys,\,-L;\,M;K)}(t,\boldsymbol{\theta_3})$ [eq. (4.10)] into the right-hand side of the SUSY transformation (6.36), and then we will re-express the transformed modes in terms of gravitino modes.

As both gravitino and graviton modes are expressed in terms of TT spherical harmonics on S^3 of spin 3/2 (3.32) and spin 2 (4.14), respectively, it is useful first to clarify how SUSY acts on them. In particular, given a TT spin-2 spherical harmonic $\tilde{T}_{\tilde{\mu}\tilde{\nu}}^{(\sigma;L;M;K)}(\boldsymbol{\theta_3})$ ($\sigma = \pm$) and Killing spinors $\tilde{\epsilon}_{\pm,q}(\boldsymbol{\theta_3})$ (5.18) on S^3, we can construct TT spin-3/2 spherical harmonics on S^3 as

$$\tilde{T}_{\tilde{\mu}\tilde{\nu}}^{(\sigma;L;M;K)}\tilde{\gamma}^{\tilde{\nu}}\tilde{\epsilon}_{+,q}, \quad \text{and} \quad \tilde{T}_{\tilde{\mu}\tilde{\nu}}^{(\sigma;L;M;K)}\tilde{\gamma}^{\tilde{\nu}}\tilde{\epsilon}_{-,q}. \tag{6.38}$$

These can be viewed as SUSY transforms of TT spin-3/2 spherical harmonics on S^3. Indeed it can be readily verified that these are eigenfunctions of the Dirac operator on S^3 as

$$\tilde{\nabla\!\!\!/}\left(\tilde{T}_{\tilde{\mu}\tilde{\nu}}^{(\sigma;L;M;K)}\tilde{\gamma}^{\tilde{\nu}}\tilde{\epsilon}_{\mp,q}\right) = i\,\sigma\left(L - \delta_{\sigma,\mp} + \frac{3}{2}\right)\tilde{T}_{\tilde{\mu}\tilde{\nu}}^{(\sigma;L;M;K)}\tilde{\gamma}^{\tilde{\nu}}\tilde{\epsilon}_{\mp,q}, \tag{6.39}$$

where $\delta_{+,+} = \delta_{-,-} = 1$ and $\delta_{+,-} = \delta_{-,+} = 0$. We can thus identify $\tilde{T}_{\tilde{\mu}\tilde{\nu}}^{(\sigma;L;M;K)}\tilde{\gamma}^{\tilde{\nu}}\tilde{\epsilon}_{+,q}$ and $\tilde{T}_{\tilde{\mu}\tilde{\nu}}^{(\sigma;L;M;K)}\tilde{\gamma}^{\tilde{\nu}}\tilde{\epsilon}_{-,q}$ with linear combinations for TT spin-3/2 spherical harmonics on S^3 (3.32). Thus,

$$\begin{aligned}
\tilde{T}_{\tilde{\mu}\tilde{\nu}}^{(\pm;L;M;K)}(\boldsymbol{\theta_3})\tilde{\gamma}^{\tilde{\nu}}\tilde{\epsilon}_{\pm,q}(\boldsymbol{\theta_3}) &= \sum_{m'=M-1}^{M}\tilde{\beta}_{\pm,q}^{(\pm,\ell',m',k';M)}\;\tilde{\psi}_{\pm\tilde{\mu}}^{(\ell';\,m';k')}(\boldsymbol{\theta_3}), \quad \ell'=L-1, \quad k'=K+q,\\
\tilde{T}_{\tilde{\mu}\tilde{\nu}}^{(\pm;L;M;K)}(\boldsymbol{\theta_3})\tilde{\gamma}^{\tilde{\nu}}\tilde{\epsilon}_{\mp,q}(\boldsymbol{\theta_3}) &= \sum_{m'=M-1}^{M}\tilde{\beta}_{\mp,q}^{(\pm,\ell',m',k';M)}\;\tilde{\psi}_{\pm\tilde{\mu}}^{(\ell';\,m';k')}(\boldsymbol{\theta_3}), \quad \ell'=L, \quad k'=K+q,
\end{aligned} \tag{6.40}$$

where

$$\tilde{\beta}^{(\sigma,\ell',m',k';M)}_{\pm,q} = \int_{S^3} \sqrt{\tilde{g}}\ d\boldsymbol{\theta_3}\ \left(\tilde{\psi}^{(\ell';\ m';k')\tilde{\mu}}_{\sigma}(\boldsymbol{\theta_3})\right)^{\dagger}\ \tilde{T}^{(\sigma;L;M;K)}_{\tilde{\mu}\tilde{\nu}}(\boldsymbol{\theta_3})\tilde{\gamma}^{\tilde{\nu}}\tilde{\epsilon}_{\pm,q}(\boldsymbol{\theta_3}). \tag{6.41}$$

We find $k' = K + q$ by considering how both sides depend on θ_1. The range of m' is found by noting that the tensors $\tilde{T}^{(\pm;L;M;K)}_{\tilde{\mu}\tilde{\nu}}(\boldsymbol{\theta_3})$ and each of the sets of Killing spinors, $\{\tilde{\epsilon}_{+,q}\}_{q=-1,0}$ and $\{\tilde{\epsilon}_{-,q}\}_{q=-1,0}$ form $so(3)$ representations with highest weights M and $1/2$, respectively, and that the vector-spinors $\tilde{\psi}^{(\ell';\ m';k')}_{\pm\ \tilde{\mu}}(\boldsymbol{\theta_3})$ form a representation with highest weight $m' + 1/2$. If the vector-spinor with the label (ℓ', m', k') does not exist, then the corresponding coefficient is set to 0. We do not need the explicit form of the nonzero coefficients $\tilde{\beta}^{(\sigma,\ell',m',k';M)}_{\pm,q}$.

Similarly, we can construct TT spin-2 spherical harmonics from TT spin-3/2 spherical harmonics and Killing spinors on S^3, as

$$\mathcal{S}^{(\ell;\ m;k)}_{(\pm,q;\sigma)\tilde{\mu}\tilde{\nu}} = \tilde{\epsilon}^{\dagger}_{\pm,q}\left(\sigma\ \tilde{\gamma}_{\tilde{\mu}\tilde{\lambda}}\tilde{\nabla}^{\tilde{\lambda}}\tilde{\psi}^{(\ell;\ m;k)}_{\sigma\ \tilde{\nu}} \mp 2i\ \tilde{\gamma}_{\tilde{\mu}}\tilde{\psi}^{(\ell;\ m;k)}_{\sigma\ \tilde{\nu}}\right) + (\tilde{\mu} \leftrightarrow \tilde{\nu}). \tag{6.42}$$

These can be viewed as SUSY transforms for TT spin-2 spherical harmonics on S^3. They are eigenfunctions of the duality operator defined by eq. (4.15):

$$\tilde{\varepsilon}_{\tilde{\mu}}^{\ \tilde{\alpha}\tilde{\beta}}\tilde{\nabla}_{\tilde{\alpha}}\ \mathcal{S}^{(\ell;\ m;k)}_{(\pm,q;\sigma)\tilde{\beta}\tilde{\nu}} = \sigma(\ell + \delta_{\sigma,\pm} + 1)\mathcal{S}^{(\ell;\ m;k)}_{(\pm,q;\sigma)\tilde{\mu}\tilde{\nu}}. \tag{6.43}$$

As a result, we have that $\mathcal{S}^{(\ell;\ m;k)}_{(\pm,q;\sigma)\tilde{\beta}\tilde{\nu}}$ are eigenfunctions of the Laplace-Beltrami operator on S^3, as

$$(-\tilde{\nabla}_{\tilde{\alpha}}\tilde{\nabla}^{\tilde{\alpha}} + 3)\mathcal{S}^{(\ell;\ m;k)}_{(\pm,q;\sigma)\tilde{\mu}\tilde{\nu}} = (\ell + \delta_{\sigma,\pm} + 1)^2\mathcal{S}^{(\ell;\ m;k)}_{(\pm,q;\sigma)\tilde{\mu}\tilde{\nu}}, \tag{6.44}$$

where $\delta_{+,+} = \delta_{-,-} = 1$ and $\delta_{+,-} = \delta_{-,+} = 0$ as defined after eq. (6.39). (For $\ell = 1$ one has $\pm(\ell + \delta_{\pm,\mp} + 1) = \pm 2$. There are no TT spin-2 spherical harmonics with these eigenvalues for the duality operator. This implies that $\mathcal{S}^{(\ell=1;\ m;k)}_{(\pm,q;\mp)\tilde{\mu}\tilde{\nu}} = 0$.) Hence, one can express $\mathcal{S}^{(\ell;\ m;k)}_{(\pm,q;\sigma)\tilde{\mu}\tilde{\nu}}$ as linear combinations of TT spin-2 spherical harmonics, as

$$\begin{aligned}
\mathcal{S}^{(\ell;\ m;k)}_{(\pm,q;\,\mp)\tilde{\mu}\tilde{\nu}}(\boldsymbol{\theta_3}) &= \sum_{M'=m}^{m+1} \breve{\beta}^{(\mp,L',M',K';m)}_{\pm,q}\ T^{(\mp;L';M';K')}_{\tilde{\mu}\tilde{\nu}}(\boldsymbol{\theta_3}), \quad L' = \ell, \quad K' = k - q, \\
\mathcal{S}^{(\ell;\ m;k)}_{(\pm,q;\,\pm)\tilde{\mu}\tilde{\nu}}(\boldsymbol{\theta_3}) &= \sum_{M'=m}^{m+1} \breve{\beta}^{(\pm,L',M',K';m)}_{\pm,q}\ T^{(\pm;L';M';K')}_{\tilde{\mu}\tilde{\nu}}(\boldsymbol{\theta_3}), \quad L' = \ell + 1, \quad K' = k - q,
\end{aligned} \tag{6.45}$$

where

$$\breve{\beta}^{(\sigma,L',M',K';m)}_{\pm,q} = \int_{S^3} \sqrt{\tilde{g}}\ d\boldsymbol{\theta_3}\ T^{(\sigma;L';M';K')*}_{\tilde{\mu}\tilde{\nu}}(\boldsymbol{\theta_3})\ \mathcal{S}^{(\ell;\ m;k)\tilde{\mu}\tilde{\nu}}_{(\pm,q;\sigma)}(\boldsymbol{\theta_3})\,. \tag{6.46}$$

The relation $K' = k - q$ and the range of M' have been determined as before. Again, if there is no TT spin-2 spherical harmonic with the label (L', M', K'), we let $\breve{\beta}^{(\sigma,L',M',K';m)}_{\pm,q} = 0$. From eqs. (6.41) and (6.46) we find the following relations:

$$\breve{\beta}^{(\pm,L,M,K;m)}_{\mp,q} = 2iL\tilde{\beta}^{(\pm,\ell,m,k;M)*}_{\mp,q}, \qquad L = \ell, \qquad K = k - q, \tag{6.47}$$

$$\breve{\beta}^{(\pm,L,M,K;m)}_{\pm,q} = 2i(L+2)\tilde{\beta}^{(\pm,\ell,m,k;M)*}_{\pm,q}, \qquad L = \ell + 1, \qquad K = k - q. \tag{6.48}$$

Having sketched how SUSY acts on spherical harmonics on S^3, we can readily compute the SUSY transformation [eqs. (6.36) and (6.37)] using the explicit expressions of the physical spin-3/2 and spin-2 modes, $\psi_\nu^{(phys,-\ell;m;k)}$ and $\varphi_{\mu\nu}^{(phys,\,-L;\,M;K)}$ [eqs. (3.20) and (4.10)] and the expressions of the Killing spinors (5.17) on dS_4. The result is

$$\begin{aligned}\left(\delta^{\rm susy}(\epsilon^{(-;q)})\psi\right)_\mu^{(-L;\,M;K)} &\equiv \frac{1}{4}\left(i\,\varphi_{\mu\sigma}^{(phys,\,-L;\,M;K)}\,\gamma^\sigma+\nabla_\lambda\varphi_{\mu\sigma}^{(phys,\,-L;\,M;K)}\,\gamma^{\sigma\lambda}\right)\epsilon^{(-;q)}\\ &=\frac{i}{2}\sqrt{L+2}\sum_{m'=M-1}^{M}\tilde{\beta}_{-,q}^{(-,\ell',m',k';M)}\;\psi_\mu^{(phys,\,-\ell';\,m';k')},\\ \ell'&=L-1,\quad k'=K+q,\end{aligned}\tag{6.49}$$

and

$$\begin{aligned}\left(\delta^{\rm susy}(\epsilon^{(+;q)})\psi\right)_\mu^{(-L;\,M;K)} &\equiv \frac{1}{4}\left(i\,\varphi_{\mu\sigma}^{(phys,\,-L;\,M;K)}\,\gamma^\sigma+\nabla_\lambda\varphi_{\mu\sigma}^{(phys,\,-L;\,M;K)}\,\gamma^{\sigma\lambda}\right)\epsilon^{(+;q)}\\ &=-\frac{1}{2}\sqrt{L}\sum_{m'=M-1}^{M}\tilde{\beta}_{+,q}^{(-,\ell',m',k';M)}\;\psi_\mu^{(phys,\,-\ell';\,m';k')},\\ \ell'&=L,\quad k'=K+q,\end{aligned}\tag{6.50}$$

where $q\in\{-1,0\}$. The coefficients $\tilde{\beta}_{\pm,q}^{(-,\ell',m',k';M)}$, and the angular momentum quantum numbers m' and k', have been introduced in eq. (6.40). Equations (6.49) and (6.50) describe the transformation rules for the gravitino modes under SUSY generated by the four Killing spinors (5.17) of dS_4.

One can similarly obtain the SUSY transformation rules for the graviton modes using eq. (6.37), as

$$\begin{aligned}&\left(\delta^{\rm susy}(\epsilon^{(-;q)})'\varphi\right)_{\mu\nu}^{(-\ell;m;k)}\\ &\equiv\frac{\overline{\epsilon^{(-;q)}}}{2}\gamma^5\left(\gamma_\mu\psi_\nu^{(phys,-\ell;m;k)}+\gamma_\nu\psi_\mu^{(phys,-\ell;m;k)}\right)+\nabla_{(\mu}\left(-\frac{i}{3}\,\overline{\epsilon^{(-;q)}}\gamma^5\psi_{\nu)}^{(phys,-\ell;m;k)}\right)\\ &=-\frac{i}{2}\sqrt{L'+2}\sum_{M'=m}^{m+1}\tilde{\beta}_{-,q}^{(-,\ell,m,k;M')*}\;\varphi_{\mu\nu}^{(phys,-L';M';K')}\\ &\quad+(\text{TT pure-gauge graviton mode}),\qquad L'=\ell+1,\quad K'=k-q,\end{aligned}\tag{6.51}$$

and

$$\begin{aligned}&\left(\delta^{\rm susy}(\epsilon^{(+;q)})'\varphi\right)_{\mu\nu}^{(-\ell;m;k)}\\ &\equiv\frac{\overline{\epsilon^{(+;q)}}}{2}\gamma^5\left(\gamma_\mu\psi_\nu^{(phys,-\ell;m;k)}+\gamma_\nu\psi_\mu^{(phys,-\ell;m;k)}\right)+\nabla_{(\mu}\left(-\frac{i}{3}\,\overline{\epsilon^{(+;q)}}\gamma^5\psi_{\nu)}^{(phys,-\ell;m;k)}\right)\\ &=-\frac{1}{2}\sqrt{L'}\sum_{M'=m}^{m+1}\tilde{\beta}_{+,q}^{(-,\ell,m,k;M')*}\;\varphi_{\mu\nu}^{(phys,-L';M';K')}\\ &\quad+(\text{TT pure-gauge graviton mode}),\qquad L'=\ell,\quad K'=k-q,\end{aligned}\tag{6.52}$$

with $q\in\{-1,0\}$, where eqs. (6.47) and (6.48) have been used. Equations (6.51) and (6.52) describe the transformation rules for the graviton modes under SUSY generated by the four Killing spinors (5.17) of dS_4.

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The transformation rules (6.49)–(6.52) prove that positive frequency gravitino modes of helicity $-3/2$ and positive frequency graviton modes of helicity -2 transform among themselves. Thus, condition 1 is satisfied for the representation space $\mathcal{H}_2^- \bigoplus \mathcal{H}_{\frac{3}{2}}^-$.

Note on negative frequency modes. Our results concerning the positivity of the norm and the irreducibility of the SUSY representation formed by the positive frequency solution space $\mathcal{H}_2^- \bigoplus \mathcal{H}_{\frac{3}{2}}^-$ can be readily adapted to the case of the negative frequency space $\mathcal{H}_2^{*+} \bigoplus \mathcal{H}_{\frac{3}{2}}^{*+} = \{\varphi_{\mu\nu}^{(phys,\,+L;\,M;K)\star}\} \bigoplus \{v_\mu^{(phys,\,+\ell;\,m;k)}\}$. In other words, the SUSY representation formed by $\mathcal{H}_2^{*+} \bigoplus \mathcal{H}_{\frac{3}{2}}^{*+}$ satisfies condition 1. Let us also write the SUSY gravitino transformation rules at the level of the corresponding negative frequency mode solutions:

$$\begin{aligned}
\left(\delta^{\text{susy}}(\epsilon^{(+;q)})v\right)_\mu^{(+L;M;K)} &\equiv \frac{1}{4}\left(i\,\varphi_{\mu\sigma}^{(phys,\,+L;\,M;K)\star}\,\gamma^\sigma + \nabla_\lambda \varphi_{\mu\sigma}^{(phys,\,+L;\,M;K)\star}\,\gamma^{\sigma\lambda}\right)\epsilon^{(+;q)} \\
&= \frac{1}{2}\sqrt{L+2}\sum_{m'=M-1}^{M} \tilde{\beta}_{+,q}^{(+,\ell',m',k';M)}\; v_\mu^{(phys,\,+\ell';\,m';k')}, \\
\ell' &= L-1,\ k' = K+q
\end{aligned} \tag{6.53}$$

$$\begin{aligned}
\left(\delta^{\text{susy}}(\epsilon^{(-;q)})v\right)_\mu^{(+L;M;K)} &\equiv \frac{1}{4}\left(i\,\varphi_{\mu\sigma}^{(phys,\,+L;\,M;K)\star}\,\gamma^\sigma + \nabla_\lambda \varphi_{\mu\sigma}^{(phys,\,+L;\,M;K)\star}\,\gamma^{\sigma\lambda}\right)\epsilon^{(-;q)} \\
&= -\frac{i}{2}\sqrt{L}\sum_{m'=M-1}^{M} \tilde{\beta}_{-,q}^{(+,\ell',m',k';M)}\; v_\mu^{(phys,\,+\ell';\,m';k')}, \\
\ell' &= L,\ k' = K+q,
\end{aligned} \tag{6.54}$$

where the coefficients $\tilde{\beta}_{\pm,q}^{(+,\ell',m',k';M)}$ have been introduced in (6.40). Similarly, we find the graviton SUSY transformation for the negative frequency modes

$$\begin{aligned}
&\left(\delta^{\text{susy}}(\epsilon^{(+;q)})'\varphi\right)_{\mu\nu}^{(+\ell;m;k)\star} \\
&\equiv \frac{\overline{\epsilon^{(+;q)}}}{2}\gamma^5\left(\gamma_\mu v_\nu^{(phys,+\ell;m;k)} + \gamma_\nu v_\mu^{(phys,+\ell;m;k)}\right) + \nabla_{(\mu}\left(-\frac{i}{3}\,\overline{\epsilon^{(+;q)}}\gamma^5 v_{\nu)}^{(phys,+\ell;m;k)}\right) \\
&= -\frac{1}{2}\sqrt{L'+2}\sum_{M'=m}^{m+1} \tilde{\beta}_{+,q}^{(+,\ell,m,k;M')*}\; \varphi_{\mu\nu}^{(phys,+L';M';K')\star} \\
&\quad + (\text{TT pure-gauge graviton mode}), \qquad L' = \ell+1, \quad K' = k-q
\end{aligned} \tag{6.55}$$

and

$$\begin{aligned}
&\left(\delta^{\text{susy}}(\epsilon^{(-;q)})'\varphi\right)_{\mu\nu}^{(+\ell;m;k)\star} \\
&\equiv \frac{\overline{\epsilon^{(-;q)}}}{2}\gamma^5\left(\gamma_\mu v_\nu^{(phys,+\ell;m;k)} + \gamma_\nu v_\mu^{(phys,+\ell;m;k)}\right) + \nabla_{(\mu}\left(-\frac{i}{3}\,\overline{\epsilon^{(-;q)}}\gamma^5 v_{\nu)}^{(phys,+\ell;m;k)}\right) \\
&= -\frac{i}{2}\sqrt{L'}\sum_{M'=m}^{m+1} \tilde{\beta}_{-,q}^{(+,\ell,m,k;M')*}\; \varphi_{\mu\nu}^{(phys,+L';M';K')\star} \\
&\quad + (\text{TT pure-gauge graviton mode}), \qquad L' = \ell, \quad K' = k-q
\end{aligned} \tag{6.56}$$

where $q \in \{-1, 0\}$. Here, we have used eqs. (6.47) and (6.48) again.

2. Anti-hermiticity of even generators. The even generators of our SUSY algebra are those of $so(4,2) \bigoplus u(1)$. In the case of gravitino modes, the anti-hermiticity of all $so(4,2)$ generators with respect to the axial scalar product has been demonstrated in eqs. (3.50) and (3.57). In the case of graviton modes, the anti-hermiticity of all $so(4,2)$ generators with respect to the Klein-Gordon scalar product has been demonstrated in eqs. (4.29) and (4.45). In particular, we have already established that each of the solution spaces $\mathcal{H}^-_{\frac{3}{2}}$ and $\mathcal{H}^-_2$ furnishes a UIR of $so(4,2)$ in subsections 3.2 and 4.2, respectively. In these subsections, it was also shown that each of the negative frequency spaces $\mathcal{H}^{*+}_{\frac{3}{2}}$ and $\mathcal{H}^{*+}_2$ furnishes a UIR of $so(4,2)$. Finally, it is easy to check that both the axial scalar product and the Klein-Gordon scalar product are $u(1)$-invariant. Thus, condition 2 is satisfied for the SUSY representation formed by $\mathcal{H}^-_2 \bigoplus \mathcal{H}^-_{\frac{3}{2}}$, as well as for the SUSY representation formed by $\mathcal{H}^{*+}_2 \bigoplus \mathcal{H}^{*+}_{\frac{3}{2}}$.

3. SUSY-invariance of inner products. Our aim is to show that condition 3 is satisfied, which means that we have to prove eq. (6.35). Equivalently we can show that the axial (3.45) and Klein-Gordon (4.24) currents satisfy

$$\begin{aligned} J^\mu_{ax}\left(\delta^{\text{susy}}(\epsilon)\psi, \psi\right) = &J^\mu_{KG}\left(\varphi, \delta^{\text{susy}}(\epsilon)'\varphi\right) \\ &+ (\text{total divergence of rank-2 anti-symmetric tensor}), \end{aligned} \tag{6.57}$$

as by integrating the t-component of eq. (6.57) over S^3 we find the desired eq. (6.35). Here, $\varphi_{\mu\nu}$ is any TT graviton solution and ψ_μ is any TT gravitino solution, where $\delta^{\text{susy}}(\epsilon)\psi_\mu$ and $\delta^{\text{susy}}(\epsilon)'\varphi_{\mu\nu} = \delta^{\text{susy}}(\epsilon)\varphi_{\mu\nu} + \delta^{\text{gauge}}(-\frac{i}{3}\bar{\epsilon}\gamma^5\psi)\varphi_{\mu\nu}$ are given by eqs. (6.36) and (6.37), respectively. Let us first observe that, since both $\delta^{\text{susy}}(\epsilon)\psi_\mu$ and ψ_μ are TT gravitino solutions, the SUSY Noether current (6.14) is directly expressed as

$$\mathcal{J}^\mu_{(\epsilon)}(\varphi, \psi) = J^\mu_{ax}\left(\delta^{\text{susy}}(\epsilon)\psi, \psi\right). \tag{6.58}$$

This expression relates the SUSY Noether current with the axial current. The next step is to re-write the SUSY Noether current $\mathcal{J}^\mu_{(\epsilon)}(\varphi, \psi)$ in terms of the Klein-Gordon current. A straightforward calculation gives

$$\mathcal{J}^\mu_{(\epsilon)}(\varphi, \psi) = J^\mu_{\text{symp}}\left(\varphi, \delta^{\text{susy}}(\epsilon)\varphi\right) - \nabla_\rho\left(\frac{i}{4}\bar{\epsilon}\gamma^5\gamma^{\nu\mu\rho}\psi^\sigma\;\varphi^*_{\sigma\nu} + \frac{i}{2}\bar{\epsilon}\gamma^5\gamma^{[\rho}\psi_\sigma\;\varphi^{\mu]\,\sigma*}\right), \tag{6.59}$$

where the second term is the divergence of an anti-symmetric tensor, and J^μ_{symp} is the symplectic current (4.52) with

$$\begin{aligned} &J^\mu_{\text{symp}}\left(\varphi, \delta^{\text{susy}}(\epsilon)\varphi\right) \\ &\quad= -\frac{i}{4}\left(\varphi^*_{\nu\lambda}\;\nabla^\mu\delta^{\text{susy}}(\epsilon)\varphi^{\nu\lambda} - \delta^{\text{susy}}(\epsilon)\varphi^{\nu\lambda}\;\nabla^\mu\varphi^*_{\nu\lambda} - 2\varphi^{\mu\,*}_{\;\;\lambda}\;\nabla_\alpha\delta^{\text{susy}}(\epsilon)\varphi^{\alpha\lambda}\right). \end{aligned} \tag{6.60}$$

Note that the graviton SUSY transformation $\delta^{\text{susy}}(\epsilon)\varphi_{\mu\nu}$ appearing in eqs. (6.59) and (6.60) is **not** a TT graviton solution, but it is related to the TT SUSY transformation $\delta^{\text{susy}}(\epsilon)'\varphi_{\mu\nu}$ through eq. (6.37). Then, using eq. (6.37) to express $\delta^{\text{susy}}(\epsilon)\varphi_{\mu\nu}$ in terms of $\delta^{\text{susy}}(\epsilon)'\varphi_{\mu\nu}$, and recalling that the Klein-Gordon current coincides with the symplectic current when both arguments are TT solutions,[41] we can re-express $J^{\mu}_{\rm symp}\left(\varphi,\delta^{\rm susy}(\epsilon)\varphi\right)$ as

$$J^{\mu}_{\rm symp}\left(\varphi,\delta^{\rm susy}(\epsilon)\varphi\right)=J^{\mu}_{KG}\left(\varphi,\delta^{\rm susy}(\epsilon)'\varphi\right)-J^{\mu}_{\rm symp}\left(\varphi,\ \delta^{\rm gauge}\left(-\frac{i}{3}\bar{\epsilon}\gamma^5\psi\right)\varphi\right). \tag{6.61}$$

Then, comparing eqs. (6.58) and (6.59), and making use of eq. (6.61), we find

$$\begin{aligned}J^{\mu}_{ax}\left(\delta^{\rm susy}(\epsilon)\psi,\psi\right)=\ &J^{\mu}_{KG}\left(\varphi,\delta^{\rm susy}(\epsilon)'\varphi\right)-J^{\mu}_{\rm symp}\left(\varphi,\ \delta^{\rm gauge}\left(-\frac{i}{3}\bar{\epsilon}\gamma^5\psi\right)\varphi\right)\\&-\nabla_{\rho}\left(\frac{i}{4}\bar{\epsilon}\gamma^5\gamma^{\nu\mu\rho}\psi^{\sigma}\ \varphi^{*}_{\sigma\nu}+\frac{i}{2}\bar{\epsilon}\gamma^5\gamma^{[\rho}\psi_{\sigma}\ \varphi^{\mu]\,\sigma *}\right).\end{aligned} \tag{6.62}$$

Finally, we find that this equation takes the desired form (6.57) by using eq. (4.55). We have thus shown that condition 3 is satisfied. This condition can also be verified directly by noting that the coefficients in eqs. (6.51) and (6.52) are the complex conjugates of those in eqs. (6.49) and (6.50), respectively.

6.2.2 Unitary SUSY in the QFT Fock space of the chiral graviton and chiral gravitino

In the previous subsection we showed that the space of TT positive frequency modes $\mathcal{H}_2^-\bigoplus\mathcal{H}_{\frac{3}{2}}^-=\{\varphi^{(phys,\,-L;\,M;K)}_{\mu\nu}\}\bigoplus\{\psi^{(phys,\,-\ell;\,m;k)}_{\mu}\}$ forms a UIR of SUSY with SUSY transformations given by eqs. (6.36) and (6.37) — or equivalently by eqs. (6.31) and (6.32). The commutator of two SUSY transformations is given in eqs. (6.33) and (6.34) and the even part of the superalgebra is isomorphic to $so(4,2)\bigoplus u(1)$. We also showed that the space of TT negative frequency modes $\mathcal{H}_2^{*+}\bigoplus\mathcal{H}_{\frac{3}{2}}^{*+}=\{\varphi^{(phys,\,+L;\,M;K)\star}_{\mu\nu}\}\bigoplus\{v^{(phys,\,+\ell;\,m;k)}_{\mu}\}$ forms a UIR of the same SUSY algebra but with opposite helicities relative to the positive frequency modes. Now we will study the realisation of unitary SUSY in the QFT Fock space of the chiral graviton and chiral gravitino. Recall that by 'chiral' we mean that the corresponding field strengths are anti-self-dual. Let us start by reviewing the main features of the chiral graviton and chiral gravitino from the previous sections:

- **Chiral gravitino.** The completely gauge-fixed chiral gravitino field $\Psi^{(\rm TT)-}_{\mu}$ was quantised in subsection 3.4. For convenience let us give here again the mode expansion (3.68):

$$\begin{aligned}&\Psi^{(\rm TT)-}_{t}(t,\boldsymbol{\theta_3})=0,\\&\Psi^{(\rm TT)-}_{\tilde{\mu}}(t,\boldsymbol{\theta_3})=\sum_{\ell=1}^{\infty}\sum_{m,k}\left(a^{(-)}_{\ell mk}\psi^{(phys,\,-\ell\,;m;k)}_{\tilde{\mu}}(t,\boldsymbol{\theta_3})+b^{(+)\dagger}_{\ell mk}\,v^{(phys,\,+\ell\,;m;k)}_{\tilde{\mu}}(t,\boldsymbol{\theta_3})\right),\end{aligned}$$

  where

$$\{a^{(-)}_{\ell mk},a^{(-)\dagger}_{\ell' m' k'}\}=\delta_{\ell\ell'}\delta_{mm'}\delta_{kk'},\qquad \{b^{(+)}_{\ell mk},b^{(+)\dagger}_{\ell' m' k'}\}=\delta_{\ell\ell'}\delta_{mm'}\delta_{kk'}.$$

  The field strength of the chiral gravitino [eq. (3.78)] satisfies the anti-self-duality constraint (3.64). The chiral gravitino vacuum is denoted as $|0\rangle_{\frac{3}{2}}$ and satisfies $a^{(-)}_{\ell mk}\,|0\rangle_{\frac{3}{2}}=$

[41] See the passage below eq. (4.55).

$b^{(+)}_{\ell mk}\,|0\rangle_{\frac{3}{2}} = 0$, for all allowed values of ℓ, m, k. The vacuum is invariant under $so(4,1)$ and the single-particle Hilbert spaces of the QFT furnish a direct sum of two $\Delta = 5/2$ discrete series UIRs of $so(4,1)$ with opposite helicities. The vacuum is also invariant under $so(4,2)$ and the single-particle UIRs of $so(4,1)$ extend to a direct sum of $so(4,2)$ UIRs with opposite helicities — see subsection 3.4. It is easy to show that these statements also extend from $so(4,2)$ to $so(4,2) \bigoplus u(1)$.

- **Chiral graviton.** The completely gauge-fixed chiral graviton field $\mathfrak{h}^{(\mathrm{TT})-}_{\mu\nu}$ has been quantised in subsection 4.3. Let us present here again the mode expansion for the chiral graviton (4.62):

$$\mathfrak{h}^{(\mathrm{TT})-}_{t\mu}(t, \boldsymbol{\theta_3}) = 0,$$
$$\mathfrak{h}^{(\mathrm{TT})-}_{\tilde{\mu}\tilde{\nu}}(t, \boldsymbol{\theta_3}) = \sum_{L=2}^{\infty} \sum_{M,K} \left(c^{(-)}_{LMK} \varphi^{(phys,\,-L;\,M;K)}_{\tilde{\mu}\tilde{\nu}}(t, \boldsymbol{\theta_3}) + d^{(+)\dagger}_{LMK}\, \varphi^{(phys,\,+L;\,M;K)\star}_{\tilde{\mu}\tilde{\nu}}(t, \boldsymbol{\theta_3}) \right),$$

with

$$[c^{(-)}_{LMK}, c^{(-)\dagger}_{L'M'K'}] = \delta_{LL'}\delta_{MM'}\delta_{KK'}, \qquad [d^{(+)}_{LMK}, d^{(+)\dagger}_{L'M'K'}] = \delta_{LL'}\delta_{MM'}\delta_{KK'}.$$

The field strength of the chiral graviton [eq. (4.66)] satisfies the anti-self-duality constraint (4.58). The chiral graviton vacuum $|0\rangle_2$ satisfies $c^{(-)}_{LMK}\,|0\rangle_2 = d^{(+)}_{LMK}\,|0\rangle_2 = 0$, for all allowed values of L, M, K. The vacuum is invariant under $so(4,1)$ and the single-particle Hilbert spaces of the QFT furnish a direct sum of two $\Delta = 3$ discrete series UIRs of $so(4,1)$ with opposite helicities. The vacuum is also invariant under $so(4,2)$ and the single-particle UIRs of $so(4,1)$ extend to a direct sum of $so(4,2)$ UIRs with opposite helicities — see subsection 4.3. Again, it is easy to show that these statements also extend from $so(4,2)$ to $so(4,2) \bigoplus u(1)$.

Let us also recall what we know so far about the SUSY representation carried by the chiral graviton and chiral gravitino. First, as the SUSY transformations of the field strengths commute with duality transformations (see subsections 6.1.4 and 6.2.1), it is clear that the chiral graviton and chiral gravitino gauge potentials $(\mathfrak{h}^{(\mathrm{TT})-}_{\mu\nu}, \Psi^{(\mathrm{TT})-}_{\mu})$ form a supermultiplet. The SUSY transformations of the chiral gauge potentials are given by eqs. (6.31) and (6.32) with $\mathfrak{h}^{(\mathrm{TT})}_{\mu\nu}$ and $\Psi^{(\mathrm{TT})}_{\mu}$ replaced by $\mathfrak{h}^{(\mathrm{TT})-}_{\mu\nu}$ and $\Psi^{(\mathrm{TT})-}_{\mu}$, respectively. The commutators of two SUSY variations are given again by eqs. (6.33) and (6.34), but with $\mathfrak{h}^{(\mathrm{TT})}_{\mu\nu}$ and $\Psi^{(\mathrm{TT})}_{\mu}$ replaced by $\mathfrak{h}^{(\mathrm{TT})-}_{\mu\nu}$ and $\Psi^{(\mathrm{TT})-}_{\mu}$, respectively. The TT gauge transformations in eqs. (6.33) and (6.34) are identified with zero (recall that the UIRs formed by mode solutions were defined in terms of equivalence classes of mode solutions). We are allowed to do this because the transformations of quantum fields are attributed to transformations of the creation and annihilation operators and these have gauge-invariant definitions.[42] We also know from the results of subsection 6.2.1 that the space of positive frequency modes $\mathcal{H}^-_2 \bigoplus \mathcal{H}^-_{\frac{3}{2}}$ and the space of negative frequency modes $\mathcal{H}^{*+}_2 \bigoplus \mathcal{H}^{*+}_{\frac{3}{2}}$ separately form UIRs of our superalgebra.

[42] In particular, eq. (3.71) implies the gauge independence of the gravitino creation and annihilation operators as the axial scalar product is invariant under TT gauge transformations (3.14). Similarly, eq. (4.65) implies the invariance of the graviton creation and annihilation operators under TT gauge transformation (4.40).

Let us now show that SUSY is realised unitarily in the QFT Fock space of the chiral graviton and chiral gravitino. In fact, unitarity follows from the analysis we have already presented, as we have explicitly constructed the QFT Fock space and we have shown that the norm is positive. In addition, we have shown that the single-particle Hilbert space ($\cong$ Hilbert space of TT mode solutions) carries a direct sum of UIRs of our superalgebra — see subsection 6.2.1. Nevertheless, for the sake of completeness, we will construct the quantum operators corresponding to the SUSY Noether charges (6.15) and we will show that they generate unitary representations of our superalgebra in the QFT Fock space.

Quantum SUSY generators. The quantum SUSY Noether charges that are relevant to our theory, $Q^{\text{susy}}[\epsilon]$, are found by replacing $\mathfrak{h}^{\dagger}_{\mu\nu}$ and Ψ_μ in eq. (6.15) with the chiral quantum fields $\mathfrak{h}^{(\text{TT})-\dagger}_{\mu\nu}$ and $\Psi^{(\text{TT})-}_{\mu}$, respectively. The standard approach is to use Grassmann-odd Killing spinors, rendering the SUSY Noether charges $Q^{\text{susy}}[\epsilon] = \overline{\eta}^A\, Q_A$ Grassmann-even, where η is the constant Grassmann-odd spinor parameter in eq. (5.25). The anti-commutators of the spinorial supercharges, $\left\{Q_A, Q^{B\,\dagger}\right\}$, are then encoded in the commutators $\left[Q^{\text{susy}}[\epsilon], Q^{\text{susy}}[\epsilon']^{\dagger}\right]$. Here, we will adopt an alternative approach where we will use the commuting Killing spinors $\epsilon^{(\sigma;q)}(t,\boldsymbol{\theta_3}) = S(t,\boldsymbol{\theta_3})\eta^{(\sigma;q)}$ [eq. (5.21)]. We will thus work with the *Grassmann-odd SUSY Noether charges* $Q^{\text{susy}}[\epsilon^{(\sigma;q)}]$ — see the discussion below eq. (6.15). Now, the SUSY algebra is determined by anti-commutators

$$\left\{Q^{\text{susy}}[\epsilon^{(\sigma;q)}], Q^{\text{susy}}[\epsilon^{(\sigma';q')}]^{\dagger}\right\}.$$

Let us now re-express the Grassmann-odd SUSY Noether charges $Q^{\text{susy}}[\epsilon^{(\sigma\,;q)}]$ in a more convenient form. Using eqs. (6.58) and (6.57), we re-express eq. (6.15) as:

$$Q^{\text{susy}}[\epsilon^{(\sigma\,;q)}] = \langle \delta^{\text{susy}}(\epsilon^{(\sigma\,;q)})\Psi^{(\text{TT})-}|\Psi^{(\text{TT})-}\rangle_{ax} = \langle \mathfrak{h}^{(\text{TT})-}|\; \delta^{\text{susy}}(\epsilon^{(\sigma;q)})'\; \mathfrak{h}^{(\text{TT})-}\rangle_{KG}. \tag{6.63}$$

The quantum charge $Q^{\text{susy}}[\epsilon^{(\sigma;q)}]^{\dagger}$ is given by the hermitian conjugate of this expression. There are four independent SUSY Noether charges, one charge for each Killing spinor $\epsilon^{(+;-1)}(t,\boldsymbol{\theta_3})$, $\epsilon^{(+;0)}(t,\boldsymbol{\theta_3})$, $\epsilon^{(-;-1)}(t,\boldsymbol{\theta_3})$, and $\epsilon^{(-;0)}(t,\boldsymbol{\theta_3})$ — see eq. (5.17). Below we express $Q^{\text{susy}}[\epsilon^{(\sigma;q)}]$ in terms of creation and annihilation operators.

Let us first recall that the transformations of the field operators are attributed to transformations of their creation and annihilation operators. By expanding the fields in modes, eq. (6.63) gives

$$Q^{\text{susy}}[\epsilon^{(\sigma;q)}] = \sum_{L=2}^{\infty}\sum_{M,K}\left(c^{(-)\dagger}_{LMK}\;\delta^{\text{susy}}(\epsilon^{(\sigma;q)})' c^{(-)}_{LMK} - d^{(+)}_{LMK}\;\delta^{\text{susy}}(\epsilon^{(\sigma;q)})' d^{(+)\dagger}_{LMK}\right) \tag{6.64}$$

$$= \sum_{\ell=1}^{\infty}\sum_{m,k}\left(\delta^{\text{susy}}(\epsilon^{(\sigma;q)}) a^{(-)\dagger}_{\ell mk}\; a^{(-)}_{\ell mk} + \delta^{\text{susy}}(\epsilon^{(\sigma;q)}) b^{(+)}_{\ell mk}\; b^{(+)\dagger}_{\ell mk}\right). \tag{6.65}$$

By construction, the quantum SUSY Noether charges generate the desired SUSY transformations [eqs. (6.31) and (6.32)] on our chiral quantum fields, as

$$[\;\mathfrak{h}^{(\text{TT})-}_{\mu\nu}, Q^{\text{susy}}[\epsilon^{(\sigma;q)}]\;] = \delta^{\text{susy}}(\epsilon^{(\sigma;q)})'\mathfrak{h}^{(\text{TT})-}_{\mu\nu}, \tag{6.66}$$

$$\left\{\Psi^{(\text{TT})-}_{\mu}, Q^{\text{susy}}[\epsilon^{(\sigma;q)}]^{\dagger}\right\} = \delta^{\text{susy}}(\epsilon^{(\sigma;q)})\Psi^{(\text{TT})-}_{\mu}. \tag{6.67}$$

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Let us now obtain explicit expressions for the SUSY transformations of the creation and annihilation operators of the chiral graviton.[43] From eq. (4.65) we find

$$\delta^{\rm susy}(\epsilon^{(\sigma;q)})' c^{(-)}_{LMK} = \langle \varphi^{(phys,\,-L;\,M;K)} | \delta^{\rm susy}(\epsilon^{(\sigma;q)})' \mathfrak{h}^{(\rm TT)-} \rangle_{KG}$$
$$= \langle \Big(\delta^{\rm susy}(\epsilon^{(\sigma;q)}) \psi \Big)^{(-L;\,M;K)} | \Psi^{(\rm TT)-} \rangle_{ax} , \qquad (6.68)$$

and

$$\delta^{\rm susy}(\epsilon^{(\sigma;q)})' d^{(+)\dagger}_{LMK} = - \langle \varphi^{(phys,\,+L;\,M;K)\star} | \delta^{\rm susy}(\epsilon^{(\sigma;q)})' \mathfrak{h}^{(\rm TT)-} \rangle_{KG}$$
$$= - \langle \Big(\delta^{\rm susy}(\epsilon^{(\sigma;q)}) v \Big)^{(+L;\,M;K)} | \Psi^{(\rm TT)-} \rangle_{ax} , \qquad (6.69)$$

where we have made use of the SUSY-invariance of the axial and Klein-Gordon inner products [eq. (6.35)]. The SUSY transformation of the positive frequency gravitino mode $\Big(\delta^{\rm susy}(\epsilon^{(\sigma;q)}) \psi \Big)^{(-L;\,M;K)}_{\mu}$ is given in eqs. (6.49) and (6.50). The SUSY transformation of the negative frequency gravitino mode $\Big(\delta^{\rm susy}(\epsilon^{(\sigma;q)}) v \Big)^{(+L;\,M;K)}_{\mu}$ is given in eqs. (6.53) and (6.54). We expect to find that $\delta^{\rm susy}(\epsilon^{(\sigma;q)})' c^{(-)}_{LMK}$ and $\delta^{\rm susy}(\epsilon^{(\sigma;q)})' d^{(+)\dagger}_{LMK}$ are proportional to a negative-helicity gravitino annihilation operator and a positive-helicity gravitino creation operator, respectively.

Indeed, using eqs. (6.49), (6.50), (6.53) and (6.54), as well as eq. (3.71), we find the SUSY transformation formulae for the creation and annihilation operators of the chiral graviton, as

$$\delta^{\rm susy}(\epsilon^{(-;q)})' c^{(-)}_{LMK} = -\frac{i}{2}\sqrt{L+2} \sum_{m'=M-1}^{M} \tilde{\beta}^{(-,\ell',m',k';M)*}_{-,q} \, a^{(-)}_{\ell' m' k'}, \quad \ell' = L-1, \; k' = K+q,$$
$$\delta^{\rm susy}(\epsilon^{(-;q)})' d^{(+)\dagger}_{LMK} = -\frac{i}{2}\sqrt{L} \sum_{m'=M-1}^{M} \tilde{\beta}^{(+,\ell',m',k';M)*}_{-,q} \, b^{(+)\dagger}_{\ell' m' k'}, \quad \ell' = L, \; k' = K+q, \quad (6.70)$$

and

$$\delta^{\rm susy}(\epsilon^{(+;q)})' c^{(-)}_{LMK} = -\frac{1}{2}\sqrt{L} \sum_{m'=M-1}^{M} \tilde{\beta}^{(-,\ell',m',k';M)*}_{+,q} \, a^{(-)}_{\ell' m' k'}, \quad \ell' = L, \; k' = K+q,$$
$$\delta^{\rm susy}(\epsilon^{(+;q)})' d^{(+)\dagger}_{LMK} = -\frac{1}{2}\sqrt{L+2} \sum_{m'=M-1}^{M} \tilde{\beta}^{(+,\ell',m',k';M)*}_{+,q} \, b^{(+)\dagger}_{\ell' m' k'}, \quad \ell' = L-1, \; k' = K+q. \quad (6.71)$$

The coefficients $\tilde{\beta}^{(\sigma,\ell',m',k';M)}_{\pm,q}$ $(\sigma = \pm)$ and the angular momentum quantum numbers m' and k' have been introduced in eq. (6.40). The label $q \in \{0, -1\}$ is a $so(2)$ quantum number labelling the Killing spinors of dS_4 — see eq. (5.17). It is clear that annihilation/creation operators of the chiral graviton transform into annihilation/creation operators of the chiral gravitino, demonstrating the SUSY invariance of the vacuum $|0\rangle_2 \otimes |0\rangle_{\frac{3}{2}}$.

Now that we have determined the SUSY transformation formulae for $\delta^{\rm susy}(\epsilon^{(\pm;q)})' c^{(-)}_{LMK}$ and $\delta^{\rm susy}(\epsilon^{(\pm;q)})' d^{(+)\dagger}_{LMK}$, let us substitute them into eq. (6.64). The quantum SUSY Noether

[43] The SUSY transformations of gravitino creation and annihilation operators can be obtained similarly.

charges are then found to be:

$$Q^{\rm susy}[\epsilon^{(-;q)}] = -\frac{i}{2}\sum_{L=2}^{\infty}\sum_{M,K}\Bigg(\sqrt{L+2}\;\; c^{(-)\dagger}_{LMK}\sum_{m'=M-1}^{M}\tilde{\beta}^{(-,L-1,m',K+q;M)*}_{-,q}\,a^{(-)}_{L-1,\,m',\,K+q}$$
$$-\sqrt{L}\;\; d^{(+)}_{LMK}\sum_{m'=M-1}^{M}\tilde{\beta}^{(+,L,m',K+q;M)*}_{-,q}\;\; b^{(+)\dagger}_{L,\,m',\,K+q}\Bigg), \tag{6.72}$$

and

$$Q^{\rm susy}[\epsilon^{(+;q)}] = -\frac{1}{2}\sum_{L=2}^{\infty}\sum_{M,K}\Bigg(\sqrt{L}\;\; c^{(-)\dagger}_{LMK}\sum_{m'=M-1}^{M}\tilde{\beta}^{(-,L,m',K+q;M)*}_{+,q}\;\; a^{(-)}_{L,\,m',\,K+q} \tag{6.73}$$
$$-\sqrt{L+2}\;\; d^{(+)}_{LMK}\sum_{m'=M-1}^{M}\tilde{\beta}^{(+,L-1,m',K+q;M)*}_{+,q}\;\; b^{(+)\dagger}_{L-1,\,m',\,K+q}\Bigg),$$

$q\in\{-1,0\}$, where it is clear that they annihilate the vacuum $|0\rangle_2\otimes|0\rangle_{\frac{3}{2}}$. Note that the quantum SUSY Noether charges (6.72) and (6.73) can be expressed as a sum of two independent charges that anti-commute with each other; one charge generates a SUSY UIR in the positive-frequency sector, and the other generates a SUSY UIR (of opposite helicity) in the negative-frequency sector.

The unitary realisation of SUSY in our QFT Fock space is now manifest as it is easy to check that single-particle states furnish the UIRs of our superalgebra presented in subsection 6.2.1. For example, graviton single-particle states, such as $c^{(-)\dagger}_{LMK}\,|0\rangle_2\otimes|0\rangle_{\frac{3}{2}}$, transform under SUSY as

$$Q^{\rm susy}[\epsilon^{(\sigma;q)}]^{\dagger}\;\Big(c^{(-)\dagger}_{LMK}\,|0\rangle_2\otimes|0\rangle_{\frac{3}{2}}\Big) = \Big[\;Q^{\rm susy}[\epsilon^{(\sigma;q)}]^{\dagger}\,,c^{(-)\dagger}_{LMK}\Big]\,\Big(|0\rangle_2\otimes|0\rangle_{\frac{3}{2}}\Big)$$
$$= |0\rangle_2\otimes\delta^{\rm susy}(\epsilon^{(\sigma;q)})'c^{(-)\dagger}_{LMK}\,|0\rangle_{\frac{3}{2}}\,. \tag{6.74}$$

According to our analysis in the previous paragraphs, this gives

$$Q^{\rm susy}[\epsilon^{(-;q)}]^{\dagger}\Big(c^{(-)\dagger}_{LMK}|0\rangle_2\otimes|0\rangle_{\frac{3}{2}}\Big)=\frac{i}{2}\sqrt{L+2}\sum_{m'=M-1}^{M}\tilde{\beta}^{(-,L-1,m',K+q;M)}_{-,q}\;|0\rangle_2\otimes a^{(-)\dagger}_{L-1,m',K+q}|0\rangle_{\frac{3}{2}}\,, \tag{6.75}$$

and

$$Q^{\rm susy}[\epsilon^{(+;q)}]^{\dagger}\;\Big(c^{(-)\dagger}_{LMK}\,|0\rangle_2\otimes|0\rangle_{\frac{3}{2}}\Big) = -\frac{1}{2}\sqrt{L}\sum_{m'=M-1}^{M}\tilde{\beta}^{(-,L,m',K+q;M)}_{+,q}\;|0\rangle_2\otimes a^{(-)\dagger}_{L,m',K+q}\,|0\rangle_{\frac{3}{2}}\,, \tag{6.76}$$

$q\in\{-1,0\}$, in agreement with the transformation rules (6.49) and (6.50), respectively, of the mode functions forming the SUSY UIRs. One can similarly find the SUSY transformations of gravitino single-particle states, such as $|0\rangle_2\otimes a^{(-)\dagger}_{\ell mk}\,|0\rangle_{\frac{3}{2}}$, by using the following:

$$Q^{\rm susy}[\epsilon^{(\sigma;q)}]\;\Big(|0\rangle_2\otimes a^{(-)\dagger}_{\ell mk}\,|0\rangle_{\frac{3}{2}}\Big) = \{Q^{\rm susy}[\epsilon^{(\sigma;q)}],a^{(-)\dagger}_{\ell mk}\}\,\Big(|0\rangle_2\otimes|0\rangle_{\frac{3}{2}}\Big)$$
$$= \delta^{\rm susy}(\epsilon^{(\sigma;q)})a^{(-)\dagger}_{\ell mk}\,|0\rangle_2\otimes|0\rangle_{\frac{3}{2}}\,. \tag{6.77}$$

Extra check for unitarity. As mentioned in the Introduction, in the cases where the unitarity of global SUSY on a fixed dS_4 background fails [60, 61], the main obstacle is that the sum of anti-commutators of spinorial supercharges Q_A,

$$\sum_A \{Q_A, Q^{A\dagger}\}, \tag{6.78}$$

which must be positive-definite, is shown to vanish identically using the de Sitter superalgebra. If this anti-commutator vanishes in a theory that carries a non-trivial representation of global SUSY, then negative-norm states must exist, rendering the theory non-unitary. We will show that, in our supersymmetric theory of the chiral graviton and gravitino, this anti-commutator is positive, as required by unitarity.

Let us start by introducing the Grassmann-odd spinorial supercharges, Q_A, of our theory. Using the expression (5.21) for the commuting Killing spinors, and the definition (6.15) of the Grassmann-odd SUSY Noether charges, we have

$$\begin{aligned} Q^{\text{susy}}[\epsilon^{(\sigma;q)}] &= \int_{S^3} d\boldsymbol{\theta_3}\sqrt{-g}\;\; \overline{\epsilon}^{(\sigma;q)\,B}\;\; \mathfrak{J}^t_B = \overline{\eta}^{(\sigma;q)\,A} \int_{S^3} d\boldsymbol{\theta_3}\sqrt{-g}\;\; \left(-\gamma^0 S(t,\boldsymbol{\theta_3})^\dagger\gamma^0\right)_A^{\;\;B}\;\; \mathfrak{J}^t_B \\ &\equiv \overline{\eta}^{(\sigma;q)\,A}\; Q_A. \end{aligned} \tag{6.79}$$

Now, let us show that the operator given in eq. (6.78) is proportional to the following sum of anti-commutators between SUSY Noether charges:

$$\sum_{\sigma\in\{+,-\}}\sum_{q\in\{0,-1\}} \left\{Q^{\text{susy}}[\epsilon^{(\sigma;q)}],\; Q^{\text{susy}}[\epsilon^{(\sigma;q)}]^\dagger\right\}. \tag{6.80}$$

We straightforwardly have

$$\begin{aligned} \sum_{\sigma\in\{+,-\}}\sum_{q\in\{0,-1\}} \left\{Q^{\text{susy}}[\epsilon^{(\sigma;q)}],\; Q^{\text{susy}}[\epsilon^{(\sigma;q)}]^\dagger\right\} &= \sum_{\sigma\in\{+,-\}}\sum_{q\in\{0,-1\}} \left\{\overline{\eta}^{(\sigma;q)A}Q_A,\; \overline{Q}^B\eta^{(\sigma;q)}_B\right\} \\ &= \sum_{\sigma\in\{+,-\}}\sum_{q\in\{0,-1\}} \left\{\overline{\eta}^{(\sigma;q)A}Q_A,\; Q^{B\dagger}\left(\overline{\eta}^{(\sigma;q)}\right)^\dagger_B\right\}. \end{aligned} \tag{6.81}$$

Using the explicit expressions for the Killing spinors (5.21), we find

$$\sum_{\sigma\in\{+,-\}}\sum_{q\in\{0,-1\}} \left\{Q^{\text{susy}}[\epsilon^{(\sigma;q)}],\; Q^{\text{susy}}[\epsilon^{(\sigma;q)}]^\dagger\right\} = \frac{1}{2\pi^2}\sum_{A=1}^{4} \left\{Q_A,\; Q^{A\dagger}\right\}. \tag{6.82}$$

To determine $\sum_{\sigma,q}\left\{Q^{\text{susy}}[\epsilon^{(\sigma;q)}],\; Q^{\text{susy}}[\epsilon^{(\sigma;q)}]^\dagger\right\}$, it is convenient to study the action of two consecutive SUSY variations on our chiral quantum fields. These are expressed as

$$\begin{aligned} \delta^{\text{susy}}(\epsilon^{(\sigma';q')})\delta^{\text{susy}}(\epsilon^{(\sigma;q)})'\mathfrak{h}^{(\text{TT})-}_{\mu\nu} &= \left\{\;\left[\mathfrak{h}^{(\text{TT})-}_{\mu\nu}, Q^{\text{susy}}[\epsilon^{(\sigma;q)}]\right], Q^{\text{susy}}[\epsilon^{(\sigma';q')}]^\dagger\;\right\} \\ &= \left[\;\mathfrak{h}^{(\text{TT})-}_{\mu\nu}, \left\{Q^{\text{susy}}[\epsilon^{(\sigma;q)}], Q^{\text{susy}}[\epsilon^{(\sigma';q')}]^\dagger\right\}\;\right], \end{aligned} \tag{6.83}$$

$$\begin{aligned} \delta^{\text{susy}}(\epsilon^{(\sigma';q')})'\,\delta^{\text{susy}}(\epsilon^{(\sigma;q)})\Psi^{(\text{TT})-}_{\mu} &= \left[\;\left\{\Psi^{(\text{TT})-}_{\mu}, Q^{\text{susy}}[\epsilon^{(\sigma;q)}]^\dagger\right\}, Q^{\text{susy}}[\epsilon^{(\sigma';q')}]\;\right] \\ &= \left[\;\Psi^{(\text{TT})-}_{\mu}, \left\{Q^{\text{susy}}[\epsilon^{(\sigma;q)}]^\dagger, Q^{\text{susy}}[\epsilon^{(\sigma';q')}]\right\}\;\right], \end{aligned} \tag{6.84}$$

where we have used $\left\{\Psi_\mu^{(\mathrm{TT})-}, Q^{\mathrm{susy}}[\epsilon^{(\sigma;q)}]\right\} = \left[\mathfrak{h}_{\mu\nu}^{(\mathrm{TT})-}, Q^{\mathrm{susy}}[\epsilon^{(\sigma;q)}]^\dagger\right] = 0$ for any σ and q. Then, using the explicit expressions for the SUSY transformations in eqs. (6.31) and (6.32), we can also express the consecutive SUSY transformations in the following form:

$$\begin{aligned}
\delta^{\mathrm{susy}}(\epsilon^{(\sigma';q')})\delta^{\mathrm{susy}}(\epsilon^{(\sigma;q)})'\,\mathfrak{h}_{\mu\nu}^{(\mathrm{TT})-} &= \pounds_{\xi_{\mathbb{C}}^{((\sigma;q),(\sigma';q'))}}\mathfrak{h}_{\mu\nu}^{(\mathrm{TT})-} - \mathcal{T}_{V_{\mathbb{C}}^{((\sigma;q),(\sigma';q'))}}\,\mathfrak{h}_{\mu\nu}^{(\mathrm{TT})-} \\
&\quad + i\,\frac{\overline{\epsilon}^{(\sigma;q)}\gamma^5\epsilon^{(\sigma';q')}}{4}\mathfrak{h}_{\mu\nu}^{(\mathrm{TT})-}, \qquad (6.85)\\
\delta^{\mathrm{susy}}(\epsilon^{(\sigma';q')})'\delta^{\mathrm{susy}}(\epsilon^{(\sigma;q)})\,\Psi_\mu^{(\mathrm{TT})-} &= \mathbb{L}_{\xi_{\mathbb{C}}^{((\sigma';q'),(\sigma;q))}}\Psi_\mu^{(\mathrm{TT})-} - \mathbb{T}_{V_{\mathbb{C}}^{((\sigma';q'),(\sigma;q))}}\,\Psi_\mu^{(\mathrm{TT})-} \\
&\quad + \frac{5i}{2}\,\frac{\overline{\epsilon}^{(\sigma';q')}\gamma^5\epsilon^{(\sigma;q)}}{4}\Psi_\mu^{(\mathrm{TT})-}. \qquad (6.86)
\end{aligned}$$

Here, the complex Killing vector $\xi_{\mathbb{C}}^{((\sigma';q'),(\sigma;q))}$ and genuine conformal Killing vector $V_{\mathbb{C}}^{((\sigma';q'),(\sigma;q))}$ are given by complex Killing spinor bilinears as follows:

$$\begin{aligned}
\xi_{\mathbb{C}}^{((\sigma;q),(\sigma';q'))\mu} &= \frac{1}{4}\overline{\epsilon}^{(\sigma;q)}\gamma^5\gamma^\mu\epsilon^{(\sigma';q')} = -(\xi_{\mathbb{C}}^{((\sigma';q'),(\sigma;q))\mu})^*,\\
V_{\mathbb{C}}^{((\sigma;q),(\sigma';q'))\mu} &= \frac{1}{4}\overline{\epsilon}^{(\sigma;q)}\gamma^\mu\epsilon^{(\sigma';q')} = -(V_{\mathbb{C}}^{((\sigma';q'),(\sigma;q))\mu})^*. \qquad (6.87)
\end{aligned}$$

Note also

$$\overline{\epsilon}^{(\sigma;q)}\gamma^5\epsilon^{(\sigma';q')} = -(\overline{\epsilon}^{(\sigma';q')}\gamma^5\epsilon^{(\sigma;q)})^\dagger. \qquad (6.88)$$

The complex Killing vectors $\xi_{\mathbb{C}}^{((\sigma;q),(\sigma';q'))\mu}$ were first introduced in eq. (5.5), in a slightly different notation. Similarly, the complex genuine conformal Killing vectors $V_{\mathbb{C}}^{((\sigma;q),(\sigma';q'))\mu}$ were first introduced in eq. (5.6). The scalars $\overline{\epsilon}^{(\sigma';q')}\gamma^5\epsilon^{(\sigma;q)}$ are constant and, in general, they are complex.

From eqs. (6.85), (6.86) and eqs. (6.83), (6.84), we see that we can determine the 'traced' anti-commutator in eq. (6.82) by summing over all dS Killing spinors (5.21), as

$$\sum_{\sigma\in\{+,-\}}\sum_{q\in\{0,-1\}}\delta^{\mathrm{susy}}(\epsilon^{(\sigma;q)})\delta^{\mathrm{susy}}(\epsilon^{(\sigma;q)})'\mathfrak{h}_{\mu\nu}^{(\mathrm{TT})-} = \frac{1}{2\pi^2}\left[\mathfrak{h}_{\mu\nu}^{(\mathrm{TT})-}, \sum_{A=1}^{4}\left\{Q_A, Q^{A\dagger}\right\}\right], \qquad (6.89)$$

$$\sum_{\sigma\in\{+,-\}}\sum_{q\in\{0,-1\}}\delta^{\mathrm{susy}}(\epsilon^{(\sigma;q)})'\,\delta^{\mathrm{susy}}(\epsilon^{(\sigma;q)})\Psi_\mu^{(\mathrm{TT})-} = \frac{1}{2\pi^2}\left[\Psi_\mu^{(\mathrm{TT})-}, \sum_{A=1}^{4}\left\{Q_A, Q^{A\dagger}\right\}\right]. \qquad (6.90)$$

Thus, we are interested in the case where the two Killing spinors are equal to each other, $\epsilon^{(\sigma;q)} = \epsilon^{(\sigma';q')}$ (i.e. $\sigma = \sigma'$ and $q = q'$), in eqs. (6.85) and (6.86). The Killing spinor bilinears in eqs. (6.85) and (6.86) are imaginary for $\epsilon^{(\sigma;q)} = \epsilon^{(\sigma';q')}$. This means that the complex Killing vector $\xi_{\mathbb{C}}^{((\sigma;q),(\sigma;q))\mu}$ can be expressed as $i = \sqrt{-1}$ times a real Killing vector. Similarly, the complex genuine conformal Killing vector $V_{\mathbb{C}}^{((\sigma;q),(\sigma;q))\mu}$ can be expressed as i times a real genuine conformal Killing vector. Explicit expressions for these complex Killing spinor bilinears can be found by using the explicit expressions for the Killing spinors (5.21):

- For the complex Killing vectors $\xi_{\mathbb{C}}^{((\sigma;q),(\sigma;q))\mu}$, we find

$$\begin{aligned}
&\xi_{\mathbb{C}}^{((\sigma;-1),(\sigma;-1))t} = \xi_{\mathbb{C}}^{((\sigma;0),(\sigma;0))t} = 0, \qquad (6.91)\\
&\xi_{\mathbb{C}}^{((\sigma;-1),(\sigma;-1))\tilde{\mu}}\partial_{\tilde{\mu}} = -\xi_{\mathbb{C}}^{((\sigma;0),(\sigma;0))\tilde{\mu}}\partial_{\tilde{\mu}} = -\frac{i}{4}\frac{1}{2\pi^2}\left(\cos\theta_2\frac{\partial}{\partial\theta_3} - \cot\theta_3\sin\theta_2\frac{\partial}{\partial\theta_2} - \sigma\frac{\partial}{\partial\theta_1}\right),
\end{aligned}$$

for $\sigma = \pm$. We conclude that $\xi_{\mathbb{C}}^{((\sigma;-1),(\sigma;-1))\,\mu} = -\xi_{\mathbb{C}}^{((\sigma;0),(\sigma;0))\,\mu}$ is equal to i times a linear combination of Killing vectors of S^3, and we observe that

$$\sum_{\sigma\in\{+,-\}}\sum_{q\in\{0,-1\}} \xi_{\mathbb{C}}^{((\sigma;q),(\sigma;q))\,\mu} = 0.$$

- For the complex genuine conformal Killing vectors $V_{\mathbb{C}}^{((\sigma;q),(\sigma;q))\mu}$, we find

$$V_{\mathbb{C}}^{((\sigma;-1),(\sigma;-1))\,\mu} = V_{\mathbb{C}}^{((\sigma;0),(\sigma;0))\,\mu} = \frac{i}{4}\frac{1}{2\pi^2}\,\partial^\mu \sinh t = \frac{i}{4}\frac{1}{2\pi^2}V^{(0)\mu}, \tag{6.92}$$

where $V^{(0)\mu}$ is the real genuine conformal Killing vector in eq. (3.54). We observe that

$$\sum_{\sigma\in\{+,-\}}\sum_{q\in\{0,-1\}} V_{\mathbb{C}}^{((\sigma;q),(\sigma;q))\,\mu} = \frac{i}{2\pi^2}V^{(0)\mu}.$$

- For the constant scalars $\overline{\epsilon}^{(\sigma;q)}\gamma^5\epsilon^{(\sigma;q)}$ we have

$$\overline{\epsilon}^{(\sigma;-1)}\gamma^5\epsilon^{(\sigma;-1)} = \overline{\epsilon}^{(\sigma;0)}\gamma^5\epsilon^{(\sigma;0)} = \sigma\frac{i}{2\pi^2}, \quad \text{for} \quad \sigma=\pm, \tag{6.93}$$

and thus,

$$\sum_{\sigma\in\{+,-\}}\sum_{q\in\{0,-1\}} \overline{\epsilon}^{(\sigma;q)}\gamma^5\epsilon^{(\sigma;q)} = 0.$$

We use these properties of the complex Killing spinor bilinears and eqs. (6.85) and (6.86) to evaluate the left-hand side of eqs. (6.89) and (6.90). Thus, we find

$$\left[\,\mathfrak{h}^{(\mathrm{TT})-}_{\mu\nu},\sum_{A=1}^{4}\left\{Q_A,Q^{A\dagger}\right\}\right] = -\,\mathcal{T}_{i\,V^{(0)}}\mathfrak{h}^{(\mathrm{TT})-}_{\mu\nu}, \tag{6.94}$$

$$\left[\,\Psi^{(\mathrm{TT})-}_{\mu},\sum_{A=1}^{4}\left\{Q_A,Q^{A\dagger}\right\}\right] = -\,\mathbb{T}_{i\,V^{(0)}}\Psi^{(\mathrm{TT})-}_{\mu}. \tag{6.95}$$

It is straightforward to find explicit expressions for $\mathbb{T}_{i\,V^{(0)}}\Psi^{(\mathrm{TT})-}_{\mu}$ and $\mathcal{T}_{i\,V^{(0)}}\mathfrak{h}^{(\mathrm{TT})-}_{\mu\nu}$ by expanding the field in modes and calculating the action of the transformations on the mode functions. In practice, the factor of i in $iV^{(0)\mu}$ results in an imaginary phase rotation of the mode functions. Working as in subsections 3.2 and 4.2.3, we find

$$\begin{aligned}
&-\mathbb{T}_{i\,V^{(0)}}\Psi^{(\mathrm{TT})-}_{\mu}(t,\boldsymbol{\theta_3})\\
&=\sum_{\ell=1}^{\infty}\sum_{m,k}\left(\ell+\frac{3}{2}\right)\left(a^{(-)}_{\ell mk}\psi_\mu^{(phys,\,-\ell\,;m;k)}(t,\boldsymbol{\theta_3}) - b^{(+)\dagger}_{\ell mk}\,v_\mu^{(phys,\,+\ell\,;m;k)}(t,\boldsymbol{\theta_3})\right) &(6.96)\\
&=\left[\Psi^{(\mathrm{TT})-}_{\mu}(t,\boldsymbol{\theta_3}),\; Q^{\mathrm{conf}-}_{\frac{3}{2}}[V^{(0)}] - Q^{\mathrm{conf}+}_{\frac{3}{2}}[V^{(0)}]\,\right], &(6.97)
\end{aligned}$$

and

$$\begin{aligned}
&-\mathcal{T}_{i\,V^{(0)}}\mathfrak{h}^{(\mathrm{TT})-}_{\mu\nu}(t,\boldsymbol{\theta_3})\\
&=\sum_{L=2}^{\infty}\sum_{M,K}(L+1)\left(c^{(-)}_{LMK}\varphi_{\mu\nu}^{(phys,\,-L;\,M;K)}(t,\boldsymbol{\theta_3}) - d^{(+)\dagger}_{LMK}\,\varphi_{\mu\nu}^{(phys,\,+L;\,M;K)\star}(t,\boldsymbol{\theta_3})\right) &(6.98)\\
&=\left[\mathfrak{h}^{(\mathrm{TT})-}_{\mu\nu}(t,\boldsymbol{\theta_3}),\; Q^{\mathrm{conf}-}_{2}[V^{(0)}] - Q^{\mathrm{conf}+}_{2}[V^{(0)}]\,\right], &(6.99)
\end{aligned}$$

where $Q^{\text{conf}\pm}_{\frac{3}{2}}[V^{(0)}]$ are the quantum conformal-like charges (3.89) of the chiral gravitino acting on states of helicity $\pm 3/2$, while $Q^{\text{conf}\pm}_{2}[V^{(0)}]$ are the quantum conformal-like charges (4.80) of the chiral graviton acting on states of helicity ± 2. Thus, we identify the trace of the supercharge anti-commutator as

$$\sum_{A=1}^{4} \left\{ Q_A, Q^{A\dagger} \right\} = Q^{\text{conf}-}_{\frac{3}{2}}[V^{(0)}] - Q^{\text{conf}+}_{\frac{3}{2}}[V^{(0)}] + Q^{\text{conf}-}_{2}[V^{(0)}] - Q^{\text{conf}+}_{2}[V^{(0)}], \tag{6.100}$$

which is clearly positive, i.e. its expectation values are always greater than or equal to zero — see eqs. (3.89) and (4.80).

Note. From eqs. (6.72) and (6.73) it follows that the spinorial supercharges consist of two independent parts, $Q_A = Q^-_A + Q^+_A$, with $\{Q^-_A, Q^{+B}\} = \{Q^-_A, Q^{+B\dagger}\} = \{Q^+_A, Q^{-B\dagger}\} = 0$, which separately generate the two SUSY UIRs with helicities $(-2, -3/2)$ and $(+2, +3/2)$, respectively. We thus have

$$\sum_{A=1}^{4} \left\{ Q_A, Q^{A\dagger} \right\} = \sum_{A=1}^{4} \left\{ Q^-_A, Q^{-A\dagger} \right\} + \sum_{A=1}^{4} \left\{ Q^+_A, Q^{+A\dagger} \right\}, \tag{6.101}$$

where each of the two traced anti-commutators on the right-hand side is separately positive.

SUSY algebra in terms of quantum charges. Using the expressions for the consecutive SUSY transformations (6.85) and (6.86), as well as eqs. (6.83) and (6.84), one can re-express the commutators (6.33) and (6.34) of two SUSY variations in terms of anti-commutators of quantum SUSY Noether charges (6.72) and (6.73), as

$$\begin{aligned} &\left\{ Q^{\text{susy}}[\epsilon^{(\sigma;q)}], Q^{\text{susy}}[\epsilon^{(\sigma';q')}]^{\dagger} \right\} - ((\sigma;q) \leftrightarrow (\sigma';q')) \\ &= -2\, i Q^{dS} \left[Re(\xi_{\mathbb{C}}^{((\sigma';q'),(\sigma;q))}) \right] + 2\, i Q^{\text{conf}} \left[Re(V_{\mathbb{C}}^{((\sigma';q'),(\sigma;q))}) \right] \\ &\quad - i Q^{u(1)} \left[\frac{1}{4} \overline{\epsilon}^{(\sigma';q')} \gamma^5 \epsilon^{(\sigma;q)} - \frac{1}{4} \overline{\epsilon}^{(\sigma;q)} \gamma^5 \epsilon^{(\sigma';q')} \right]. \end{aligned} \tag{6.102}$$

Here, the even hermitian generators

$$\begin{aligned} Q^{dS} \left[Re(\xi_{\mathbb{C}}^{((\sigma';q'),(\sigma;q))}) \right] &\equiv Q^{dS-} \left[Re(\xi_{\mathbb{C}}^{((\sigma';q'),(\sigma;q))}) \right] + Q^{dS+} \left[Re(\xi_{\mathbb{C}}^{((\sigma';q'),(\sigma;q))}) \right], \\ \text{with} \quad Q^{dS\pm} \left[Re(\xi_{\mathbb{C}}^{((\sigma';q'),(\sigma;q))}) \right] &\equiv \sum_{s \in \{2, \frac{3}{2}\}} Q^{dS\pm}_{s} \left[Re(\xi_{\mathbb{C}}^{((\sigma';q'),(\sigma;q))}) \right], \end{aligned} \tag{6.103}$$

are the quantum dS charges, (3.81) and (4.70), associated with the real Killing vector $Re(\xi_{\mathbb{C}}^{((\sigma';q'),(\sigma;q))})^{\mu} = [\overline{\epsilon}^{(\sigma';q')} \gamma^5 \gamma^{\mu} \epsilon^{(\sigma;q)} - \overline{\epsilon}^{(\sigma;q)} \gamma^5 \gamma^{\mu} \epsilon^{(\sigma';q')}]/8$ [see eq. (6.87)]. The '−' dS charges and the '+' dS charges commute with each other, and they generate discrete series UIRs of $so(4,1)$ with negative and positive helicity, respectively. The even hermitian generators

$$\begin{aligned} Q^{\text{conf}} \left[Re(V_{\mathbb{C}}^{((\sigma';q'),(\sigma;q))}) \right] &\equiv Q^{\text{conf}-} \left[Re(V_{\mathbb{C}}^{((\sigma';q'),(\sigma;q))}) \right] + Q^{\text{conf}+} \left[Re(V_{\mathbb{C}}^{((\sigma';q'),(\sigma;q))}) \right], \\ \text{with} \quad Q^{\text{conf}\pm} \left[Re(V_{\mathbb{C}}^{((\sigma';q'),(\sigma;q))}) \right] &= \sum_{s \in \{2, \frac{3}{2}\}} Q^{\text{conf}\pm}_{s} \left[Re(V_{\mathbb{C}}^{((\sigma';q'),(\sigma;q))}) \right] \end{aligned} \tag{6.104}$$

are the quantum conformal-like charges, (3.86) and (4.77), associated with the real genuine conformal Killing vector $Re(V_{\mathbb{C}}^{((\sigma';q'),(\sigma;q))})^\mu = [\overline{\epsilon}^{(\sigma';q')}\gamma^\mu\epsilon^{(\sigma;q)} - \overline{\epsilon}^{(\sigma;q)}\gamma^\mu\epsilon^{(\sigma';q')}]/8$ [see eq. (6.87)]. Again, the '$-$' conformal-like charges commute with the '$+$' conformal-like charges. The charges ($Q^{dS\mp}$, $Q^{\text{conf}\mp}$) generate UIRs of $so(4,2)$ with $\mp$ helicities. As mentioned in the previous sections, given a real Killing vector ξ^μ, and a real genuine conformal Killing vector V^μ, these hermitian charges generate the following transformations:

$$[\ \mathfrak{h}_{\mu\nu}^{(\text{TT})-}, Q^{dS}[\xi]\ ] = -i\,\pounds_\xi \mathfrak{h}_{\mu\nu}^{(\text{TT})-},\quad [\ \Psi_\mu^{(\text{TT})-}, Q^{dS}[\xi]\ ] = -i\,\mathbb{L}_\xi \Psi_\mu^{(\text{TT})-}, \tag{6.105}$$

$$[\ \mathfrak{h}_{\mu\nu}^{(\text{TT})-}, Q^{\text{conf}}[V]\ ] = -i\,\mathcal{T}_V \mathfrak{h}_{\mu\nu}^{(\text{TT})-},\quad [\ \Psi_\mu^{(\text{TT})-}, Q^{\text{conf}}[V]\ ] = -i\,\mathbb{T}_V \Psi_\mu^{(\text{TT})-}. \tag{6.106}$$

Finally, we have denoted the hermitian $u(1)$ quantum charges as $Q^{u(1)}$. These also consist of two independent $u(1)$ charges $Q^{u(1)} = Q^{u(1)-} + Q^{u(1)+}$, acting on negative-helicity and positive-helicity states respectively. For real constant parameters α, they act on our quantum fields as

$$[\ \mathfrak{h}_{\mu\nu}^{(\text{TT})-}, Q^{u(1)}[\alpha]\ ] = -i\,\delta_\alpha^{phase} \mathfrak{h}_{\mu\nu}^{(\text{TT})-},\quad [\ \Psi_\mu^{(\text{TT})-}, Q^{u(1)}[\alpha]\ ] = -i\,\delta_\alpha^{phase} \Psi_\mu^{(\text{TT})-}, \tag{6.107}$$

where

$$\delta_\alpha^{phase} \mathfrak{h}_{\mu\nu}^{(\text{TT})-} = i\,\alpha\,\mathfrak{h}_{\mu\nu}^{(\text{TT})-},\quad \delta_\alpha^{phase} \Psi_\mu^{(\text{TT})-} = \frac{5i}{2}\,\alpha\,\Psi_\mu^{(\text{TT})-}. \tag{6.108}$$

We similarly find the following anti-commutator of quantum SUSY Noether charges:

$$\begin{aligned}
&\left\{Q^{\text{susy}}[\epsilon^{(\sigma;q)}], Q^{\text{susy}}[\epsilon^{(\sigma';q')}]^\dagger\right\}\\
&= -i\sum_{p=\pm} Q^{dSp}[Re(\xi_{\mathbb{C}}^{((\sigma';q'),(\sigma;q))})] + i\sum_{p=\pm} Q^{\text{conf p}}[Re(V_{\mathbb{C}}^{((\sigma';q'),(\sigma;q))})]\\
&\quad - i\sum_{p=\pm} Q^{u(1)p}\left[Re\left(\frac{1}{4}\overline{\epsilon}^{(\sigma';q')}\gamma^5\epsilon^{(\sigma;q)}\right)\right]\\
&\quad + \sum_{p=\pm} p\,Q^{dSp}[Im(\xi_{\mathbb{C}}^{((\sigma';q'),(\sigma;q))})] - \sum_{p=\pm} p\,Q^{\text{conf p}}[Im(V_{\mathbb{C}}^{((\sigma';q'),(\sigma;q))})]\\
&\quad + \sum_{p=\pm} pQ^{u(1)p}\left[Im\left(\frac{1}{4}\overline{\epsilon}^{(\sigma';q')}\gamma^5\epsilon^{(\sigma;q)}\right)\right].
\end{aligned} \tag{6.109}$$

Note that the even quantum charges that depend on the imaginary parts of the Killing spinor bilinears are multiplied by factors of $p=\pm$, as was already evident from the traced anti-commutator in eq. (6.100). Given a real Killing vector ξ^μ, a real genuine conformal Killing vector V^μ, and a real constant parameter α, these quantum charges generate transformations parametrised by the 'imaginary counterparts' of ξ^μ, V^μ and α, as:

$$\begin{aligned}
&[\ \mathfrak{h}_{\mu\nu}^{(\text{TT})-}, \sum_{p=\pm}(-p)\,Q^{dSp}[\xi]\ ] = -\,\pounds_{i\,\xi}\mathfrak{h}_{\mu\nu}^{(\text{TT})-},\\
&[\ \Psi_\mu^{(\text{TT})-}, \sum_{p=\pm}(-p)\,Q^{dSp}[\xi]\ ] = -\,\mathbb{L}_{i\,\xi}\Psi_\mu^{(\text{TT})-},
\end{aligned} \tag{6.110}$$

$$\begin{aligned}
&[\ \mathfrak{h}_{\mu\nu}^{(\text{TT})-}, \sum_{p=\pm}(-p)\,Q^{\text{conf p}}[V]\ ] = -\,\mathcal{T}_{i\,V}\mathfrak{h}_{\mu\nu}^{(\text{TT})-},\\
&[\ \Psi_\mu^{(\text{TT})-}, \sum_{p=\pm}(-p)\,Q^{\text{conf p}}[V]\ ] = -\,\mathbb{T}_{i\,V}\Psi_\mu^{(\text{TT})-},
\end{aligned} \tag{6.111}$$

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and,

$$
\begin{aligned}
&[\ \mathfrak{h}_{\mu\nu}^{(\mathrm{TT})-}, \sum_{p=\pm}(-p)\, Q^{u(1)\,p}[\alpha]\ ] = -\,\delta_{i\,\alpha}^{phase}\mathfrak{h}_{\mu\nu}^{(\mathrm{TT})-}, \\
&[\ \Psi_{\mu}^{(\mathrm{TT})-}, \sum_{p=\pm}(-p)\, Q^{u(1)\,p}[\alpha]\ ] = -\,\delta_{i\,\alpha}^{phase}\Psi_{\mu}^{(\mathrm{TT})-},
\end{aligned}
\tag{6.112}
$$

where $\delta_{i\,\alpha}^{phase}$ describes infinitesimal scale transformations [compare with the real-parameter case in eq. (6.108)]. Recalling that the quantum SUSY Noether charges can be expressed as a sum of two independent charges, $Q^{\mathrm{susy}}[\epsilon^{(\sigma;q)}] = Q^{\mathrm{susy}-}[\epsilon^{(\sigma;q)}] + Q^{\mathrm{susy}+}[\epsilon^{(\sigma;q)}]$, generating separately SUSY UIRs with negative and positive helicity, respectively, we may re-express the anti-commutator (6.109) as

$$
\begin{aligned}
&\left\{Q^{\mathrm{susy}\,-}[\epsilon^{(\sigma;q)}], Q^{\mathrm{susy}\,-}[\epsilon^{(\sigma';q')}]^{\dagger}\right\} \\
&= -i\, Q^{dS-}[Re(\xi_{\mathbb{C}}^{((\sigma';q'),(\sigma;q))})] + i\, Q^{\mathrm{conf}\,-}[Re(V_{\mathbb{C}}^{((\sigma';q'),(\sigma;q))})] \\
&\quad - i\, Q^{u(1)-}\left[Re\left(\frac{1}{4}\overline{\epsilon}^{(\sigma';q')}\gamma^5\epsilon^{(\sigma;q)}\right)\right] \\
&\quad - \Bigg(Q^{dS-}[Im(\xi_{\mathbb{C}}^{((\sigma';q'),(\sigma;q))})] - Q^{\mathrm{conf}\,-}[Im(\, V_{\mathbb{C}}^{((\sigma';q'),(\sigma;q))}\,)] \\
&\qquad\quad + Q^{u(1)-}\left[Im\left(\frac{1}{4}\overline{\epsilon}^{(\sigma';q')}\gamma^5\epsilon^{(\sigma;q)}\right)\right]\Bigg),
\end{aligned}
\tag{6.113}
$$

and

$$
\begin{aligned}
&\left\{Q^{\mathrm{susy}\,+}[\epsilon^{(\sigma';q')}]^{\dagger}, Q^{\mathrm{susy}\,+}[\epsilon^{(\sigma;q)}]\right\} \\
&= i\, Q^{dS+}[Re(\xi_{\mathbb{C}}^{((\sigma;q),(\sigma';q'))})] - i\, Q^{\mathrm{conf}\,+}[Re(V_{\mathbb{C}}^{((\sigma;q),(\sigma';q'))})] \\
&\quad + i\, Q^{u(1)+}\left[Re\left(\frac{1}{4}\overline{\epsilon}^{(\sigma;q)}\gamma^5\epsilon^{(\sigma';q')}\right)\right] \\
&\quad + Q^{dS+}[Im(\, \xi_{\mathbb{C}}^{((\sigma;q),(\sigma';q'))}\,)] - Q^{\mathrm{conf}\,+}[Im(\, V_{\mathbb{C}}^{((\sigma;q),(\sigma';q'))}\,)] \\
&\quad + Q^{u(1)+}\left[Im\left(\frac{1}{4}\overline{\epsilon}^{(\sigma;q)}\gamma^5\epsilon^{(\sigma';q')}\right)\right].
\end{aligned}
\tag{6.114}
$$

In the second equation, we have used the fact that the real parts of Killing spinor bilinears change sign under the exchange $(\sigma;q) \leftrightarrow (\sigma';q')$, while their imaginary parts remain the same — see eq. (6.87). We thus have two independent sets of generators that form superalgebras separately. However, the superalgebra formed by the generators

$$
Q^{\mathrm{susy}-}, Q^{\mathrm{susy}-\,\dagger}, Q^{dS-}, Q^{\mathrm{conf}-}, Q^{u(1)-}
$$

is isomorphic to the superalgebra formed by the generators

$$
-Q^{\mathrm{susy}+\dagger}, -Q^{\mathrm{susy}+}, -Q^{dS+}, -Q^{\mathrm{conf}+}, -Q^{u(1)+}.
$$

In particular, the role of $Q^{\mathrm{susy}-}$ is played by $-Q^{\mathrm{susy}+\,\dagger}$. Moreover, the even generators $-Q^{dS+}$ and $-Q^{\mathrm{conf}+}$ generate the algebra $so(2,4) \cong so(4,2)$, which is isomorphic to the algebra generated by Q^{dS-} and $Q^{\mathrm{conf}-}$.

7 Discussions and open questions

In this paper, we showed that the free supersymmetric theory of the chiral graviton and chiral gravitino fields on fixed dS_4 is unitary. This free unitary theory cannot become interacting while preserving SUSY in a way that makes the spin-2 sector the true graviton sector of General Relativity, as the three-graviton coupling cannot be $u(1)$-invariant. Nevertheless, it remains worthwhile to investigate whether a non-linear version of the theory exists. If such a supergravity-like theory were to exist, SUSY would have to be locally realised.

As a step toward exploring possible consistent interactions, it would be mathematically interesting, and perhaps natural, to reformulate the free chiral graviton-chiral gravitino theory in terms of spin-tensors belonging to "unbalanced" representations of the Lorentz group, as in ref. [42]. Then, the question of finding possible interactions can be investigated using, for example, methods based on the presymplectic BV-AKSZ formulation [104–106]. Interestingly, a framework for the study of consistent interactions of local gauge theories in this formulation has been recently proposed in [104], and it simplifies significantly the analysis of consistent interactions.

We note in passing that if an interacting theory involving our supermultiplet of chiral graviton and gravitino were to exist, it would require the gauging of global symmetries and might also require the inclusion of additional fields, as suggested by recent work [107], where consistent interactions were studied for a real (non-unitary) partially massless graviton and two Majorana gravitini on AdS_4. This field content forms the basis of what one would call 'linearised partially massless supergravity around AdS_4' [107, 108]. The point of resemblance with our theory lies in the fact that in [107], it was found that the global symmetries of linearised partially massless supergravity around AdS_4 include conformal-like symmetries for the gravitini, similar to those in our equation (3.51). However, it was also shown that there are obstructions to the Jacobi identity of the gauge algebra, i.e. the global symmetries cannot be gauged, unless the field content is modified. Interestingly, it was suggested that by adding extra fields to the theory, so that the field content matches that of $N = 1$ pure conformal supergravity around AdS_4, the global algebra (including the conformal-like symmetries) can be gauged, and consistent interactions might be constructed. We also speculate that a non-linear version of the theory presented in this paper could be related to a complex, chiral version of conformal supergravity admitting dS_4 solutions.

Another interesting future direction is to investigate possible relations between our linear supersymmetric theory and 'chiral Supergravity', as discussed in refs. [109–112].

Finally, we note that it is likely that an analogue of our chiral graviton-gravitino supersymmetric theory exists on dS_2. In such a two-dimensional theory the question of consistent interactions would be easier to tackle. In particular, the $\Delta = 2$ and $\Delta = 3/2$ discrete series UIRs of $so(2,1)$, corresponding to a shift-symmetric 'tachyonic' scalar [8] and a shift-symmetric imaginary-mass spinor [10], respectively, on dS_2, can be viewed as the two-dimensional analogues of the graviton and gravitino, respectively [9, 10]. Each of these two fields on dS_2 corresponds to a direct sum of two $so(2,1)$ UIRs with opposite 'chirality', akin to the four-dimensional case. Moreover, the $\Delta = 2$ discrete series scalar field on dS_2 (as well as the ones with $\Delta > 2$) was recently shown to enjoy a hidden global conformal symmetry [14] (akin to the conformal-like symmetry for the four-dimensional graviton that we discussed in this paper). We expect that the fermionic counterparts [10] of the discrete series scalar fields on dS_2 will also enjoy such a conformal symmetry. Then, it would be interesting to investigate whether one can construct a unitary supersymmetric theory on dS_2 using a chiral $\Delta = 2$ scalar field and a chiral $\Delta = 3/2$ spinor field. If this theory resembles the four-dimensional theory presented in this paper, then the commutator between two SUSY transformations will close on the hidden conformal symmetries. We leave the investigation of this model for future work.

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A Classification of the UIRs of the dS algebra

The dS algebra $so(4,1)$ has 10 generators $J_{AB} = -J_{BA}$, with $A, B \in \{0,1,2,3,4\}$. These satisfy the commutation relations:

$$[J_{AB}, J_{CD}] = (\eta_{BC} J_{AD} + \eta_{AD} J_{BC}) - (A \leftrightarrow B), \tag{A.1}$$

where $\eta_{AB} = diag(-1,1,1,1,1)$. In the case of unitary representations, each of the generators J_{AB} must be realised as an anti-hermitian operator with respect to a positive-definite scalar product.

Let us review the classification of the $so(4,1)$ UIRs under the decomposition $so(4,1) \supset so(4)$ [55, 56]. An irreducible representation of $so(4)$ appears with multiplicity one in a UIR of $so(4,1)$ or it does not appear at all [113]. An irreducible representation of $so(4)$ is specified by the highest weight [83, 114, 115]

$$\vec{f} = (f_1, f_2), \tag{A.2}$$

where

$$f_1 \geq |f_2|. \tag{A.3}$$

The numbers f_1 and f_2 are both integers or half-odd-integers, and f_2 can be negative.

UIRs of $so(4,1)$. A UIR of $so(4,1)$ is specified by two numbers, the scaling dimension Δ and the spin s, denoted collectively as $\vec{\mathcal{F}} = (\Delta, s)$. The number $s \geq 0$ is an integer or half-odd integer. For the $so(4)$ representations $\vec{f} = (f_1, f_2)$ contained in the UIR $\vec{\mathcal{F}} = (\Delta, s)$ we have:

$$f_1 \geq s \geq |f_2|. \tag{A.4}$$

The representation-theoretic labels in refs. [33, 85] are related to the labels of the present paper as: $\Delta = F_0 + 3$ and $s = F_1$. The UIRs of $so(4,1)$ are listed below [55, 56]:

- **Principal Series $D_{\text{prin}}(\vec{\mathcal{F}})$:**

$$\Delta = \frac{3}{2} + iy, \qquad (y > 0). \tag{A.5}$$

  s is an integer or half-odd integer.

- **Complementary Series $D_{\text{comp}}(\vec{\mathcal{F}})$:**

$$\frac{3}{2} \leq \Delta < 3 - \tilde{n}, \qquad \tilde{n} \in \{0, 1\}. \tag{A.6}$$

  If $\tilde{n} = 0$, then $s = 0$, and for the $so(4)$ content we have $f_2 = 0$. If $\tilde{n} = 1$, then s is a positive integer.

- **Exceptional Series $D_{\text{ex}}(\vec{\mathcal{F}})$:**

$$\Delta = 2. \tag{A.7}$$

  s is a positive integer and $f_2 = 0$.

- **Discrete Series $D^{\pm}(\vec{\mathcal{F}})$:** Δ is real. The representation-theoretic labels Δ and s are both integers or half-odd integers. There are two different cases of discrete series UIRs depending on the $so(4)$ content:

$$s \geq f_2 \geq \Delta - 1 \geq \frac{1}{2} \qquad \text{for } D^{+}(\vec{\mathcal{F}}), \tag{A.8}$$

$$-s \leq f_2 \leq -\Delta + 1 \leq -\frac{1}{2} \qquad \text{for } D^{-}(\vec{\mathcal{F}}). \tag{A.9}$$

  Form eq. (A.8), it is clear that the $so(4)$ content of D^{+} UIRs corresponds to $so(4)$ irreps with positive last component, f_2, of the highest weight (A.3). Similarly, according to eq. (A.9), only $so(4)$ irreps with negative f_2 are contained in D^{-} UIRs.

The graviton on dS_4 corresponds to $\Delta = 3$ and $s = 2$. In particular, the positive frequency modes of the graviton on dS_4 form the direct sum of discrete series UIRs $D^{-}(3,2) \bigoplus D^{+}(3,2)$ [77, 85]. The gravitino (i.e. strictly massless spin-3/2 field) on dS_4 corresponds to $\Delta = 5/2$ and $s = 3/2$. The positive-frequency modes of the gravitino on dS_4 form the direct sum of discrete series UIRs $D^{-}(5/2, 3/2) \bigoplus D^{+}(5/2, 3/2)$ [33, 34]. In general, the positive-frequency modes for any strictly massless boson or fermion of any spin $s \geq 1/2$ on dS_4 correspond to the direct sum $D^{-}(s+1, s) \bigoplus D^{+}(s+1, s)$ [33, 76, 85]. For further discussions on representation-theoretic aspects of fields on dS spacetime see refs. [8, 10, 14, 28, 33, 34, 44, 51, 76, 85, 116–121].

The quadratic Casimir of $so(4,1)$ is defined as

$$C_2 \equiv \sum_{A=1}^{4} (J_{0A})^2 - \frac{1}{2} \delta^{IK} \delta^{JL} J_{IJ} J_{KL} \qquad (I, J, K, L \in \{1, 2, 3, 4\}). \tag{A.10}$$

For a $so(4,1)$ UIR labelled by $\vec{\mathcal{F}} = (\Delta, s)$ the quadratic Casimir has the (real) eigenvalue:

$$c_2(\vec{\mathcal{F}}) = (\Delta - 3)\,\Delta + s(s+1). \tag{A.11}$$

B Global dS geometry (Christoffel symbols, spin connection and all that)

In global coordinates (2.1), the non-zero Christoffel symbols are

$$\begin{aligned} \Gamma^{t}_{\tilde\mu\tilde\nu} &= \cosh t \sinh t\, \tilde g_{\tilde\mu\tilde\nu}, & \Gamma^{\tilde\mu}_{\tilde\nu t} &= \tanh t\, \tilde g^{\tilde\mu}_{\tilde\nu}, \\ \Gamma^{\tilde\kappa}_{\tilde\mu\tilde\nu} &= \tilde\Gamma^{\tilde\kappa}_{\tilde\mu\tilde\nu}, & \tilde\mu,\tilde\nu,\tilde\kappa &\in \{\theta_1,\theta_2,\theta_3\}, \end{aligned} \tag{B.1}$$

where $\tilde g_{\tilde\mu\tilde\nu}$ and $\tilde\Gamma^{\tilde\kappa}_{\tilde\mu\tilde\nu}$ are the metric tensor and the Christoffel symbols, respectively, on S^3.

We work with the following representation of gamma matrices:

$$\gamma^0 = i\begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} 0 & i\tilde\gamma^j \\ -i\tilde\gamma^j & 0 \end{pmatrix}, \tag{B.2}$$

($j = 1,2,3$) where $\mathbf{1}$ is the 2-dimensional spinorial identity matrix. The timelike gamma matrix is anti-hermitian, while the spacelike ones are hermitian. The lower-dimensional gamma matrices, $\tilde\gamma^j$, satisfy the Euclidean Clifford algebra in 3 dimensions:

$$\{\tilde\gamma^j, \tilde\gamma^k\} = 2\delta^{jk}\mathbf{1}, \qquad j,k = 1,2,3. \tag{B.3}$$

As in refs. [33, 34, 99], the representation of the lower-dimensional gamma matrices we use is:

$$\tilde\gamma^1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \tilde\gamma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{B.4}$$

In our representation for the four-dimensional gamma matrices, the fifth gamma matrix (2.7) is given by

$$\gamma^5 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \tag{B.5}$$

and we note that (2.7) can be re-written as $\varepsilon_{\mu\nu\rho\sigma} = i\gamma^5\gamma_{\mu\nu\rho\sigma}$. Also, under hermitian conjugation we have: $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$, $(\gamma^{\rho\sigma})^\dagger = \gamma^0\gamma^{\rho\sigma}\gamma^0$ and $(\gamma^{\mu\rho\sigma})^\dagger = -\gamma^0\gamma^{\mu\rho\sigma}\gamma^0$. Note also the following useful properties [79]:

$$\begin{aligned} \varepsilon_{\alpha\beta\rho\gamma}\varepsilon^{\alpha\beta\mu\nu} &= -4\delta^{[\mu}_{\rho}\delta^{\nu]}_{\gamma}, \\ \varepsilon_{\alpha\eta\rho\gamma}\varepsilon^{\alpha\kappa\mu\nu} &= (-3!)\delta^{[\kappa}_{\eta}\delta^{\mu}_{\rho}\delta^{\nu]}_{\gamma}, \\ \varepsilon_{\alpha\beta\sigma\delta}\varepsilon^{\kappa\lambda\mu\nu} &= (-4!)\delta^{[\kappa}_{\alpha}\delta^{\lambda}_{\beta}\delta^{\mu}_{\sigma}\delta^{\nu]}_{\delta}. \end{aligned} \tag{B.6}$$

For the vierbein fields on global dS_4, we choose the expressions:

$$e^{t}{}_{0} = 1, \qquad e^{\tilde\mu}{}_{i} = \frac{1}{\cosh t}\tilde e^{\tilde\mu}{}_{i}, \qquad i = 1,2,3, \tag{B.7}$$

where $\tilde e^{\tilde\mu}{}_{i}$ are the dreibein fields on S^3. The non-zero components of the dS spin connection are given by

$$\omega_{ijk} = \frac{\tilde\omega_{ijk}}{\cosh t}, \qquad \omega_{i0k} = -\omega_{ik0} = -\tanh t\, \delta_{ik}, \qquad i,j,k \in \{1,2,3\}, \tag{B.8}$$

where $\tilde\omega_{ijk}$ is the spin connection on S^3.

C Transverse, $\tilde{\gamma}$-traceless delta function (3.75) and locality of the equal-time anti-commutator (3.91)

In this appendix, we will explicitly demonstrate the locality of the equal-time anti-commutator (3.91). To achieve this, we will first show that the transverse and gamma-traceless delta function on S^3 which appears in the anti-commutator (3.91), and is defined in (3.75), can be re-expressed as

$$\Delta^{TT}_{\tilde{\mu}\tilde{\nu}'}(\boldsymbol{\theta_3},\boldsymbol{\theta'_3}) = \left(\tilde{g}_{\tilde{\mu}\tilde{\nu}'}\mathbb{U}(\boldsymbol{\theta_3},\boldsymbol{\theta'_3}) - \frac{1}{3}\tilde{\gamma}_{\tilde{\mu}}\mathbb{U}(\boldsymbol{\theta_3},\boldsymbol{\theta'_3})\tilde{\gamma}_{\tilde{\nu}'}\right)\frac{\delta(\boldsymbol{\theta_3}-\boldsymbol{\theta'_3})}{\sqrt{\tilde{g}}} + \frac{3}{2}\tilde{\nabla}^T_{\tilde{\mu}}\left(\frac{1}{\tilde{\nabla\!\!\!/}^2+9/4}\sum_{\sigma\in\{+,-\}}\sum_{n=1}^{\infty}\sum_{l,q}\chi^{(n;l;q)}_{\sigma}(\boldsymbol{\theta_3})\otimes\chi^{(n;l;q)}_{\sigma}(\boldsymbol{\theta'_3})^\dagger\right)\overleftarrow{\tilde{\nabla}}^T{}_{\tilde{\nu}'}, \tag{C.1}$$

where $\mathbb{U}(\boldsymbol{\theta_3},\boldsymbol{\theta'_3})$ is the spinor parallel propagator on S^3 with $\mathbb{U}(\boldsymbol{\theta_3},\boldsymbol{\theta_3})=\mathbf{1}$, and $\tilde{g}_{\tilde{\mu}\tilde{\nu}'}$, $\tilde{g}$, $\tilde{\gamma}_{\tilde{\mu}}$ are the bi-vector of parallel transport, determinant of the metric and gamma matrices, respectively, on S^3. The superscript T on the covariant derivatives denotes their gamma-traceless part:

$$\tilde{\nabla}^T_{\tilde{\mu}} = \tilde{\nabla}_{\tilde{\mu}} - \frac{1}{3}\tilde{\gamma}_{\tilde{\mu}}\tilde{\gamma}^{\tilde{\alpha}}\tilde{\nabla}_{\tilde{\alpha}} \quad \text{and} \quad \overleftarrow{\tilde{\nabla}}^T{}_{\tilde{\nu}'} = \overleftarrow{\tilde{\nabla}}_{\tilde{\nu}'} - \frac{1}{3}\overleftarrow{\tilde{\nabla}}_{\tilde{\alpha}'}\tilde{\gamma}^{\tilde{\alpha}'}\tilde{\gamma}_{\nu'}. \tag{C.2}$$

The spinors $\chi^{(n;l;q)}_{\pm}$ in (C.1) are the spinor spherical harmonics on the unit S^3 which are eigenfunctions of the Dirac operator, $\tilde{\nabla\!\!\!/} = \tilde{\gamma}^{\tilde{\alpha}}\tilde{\nabla}_{\tilde{\alpha}}$, [99]

$$\tilde{\nabla\!\!\!/}\chi^{(n;l;q)}_{\pm}(\boldsymbol{\theta_3}) = \pm i\left(n+\frac{3}{2}\right)\chi^{(n;l;q)}_{\pm}(\boldsymbol{\theta_3}), \quad n\in\{0,1,2,\ldots\}, \tag{C.3}$$

where the quantum numbers n,l,q correspond to the chain of subalgebras $so(4)\supset so(3)\supset so(2)$ with $n+\frac{1}{2}\geq l+\frac{1}{2}\geq |q+\frac{1}{2}|\geq\frac{1}{2}$. The spinor spherical harmonics are normalised on S^3 as

$$\int_{S^3} d\boldsymbol{\theta_3}\sqrt{\tilde{g}}\;\chi^{(n;l;q)}_{\sigma}(\boldsymbol{\theta_3})^\dagger\chi^{(n';l';q')}_{\sigma'}(\boldsymbol{\theta_3}) = \delta_{\sigma\sigma'}\delta_{nn'}\delta_{ll'}\delta_{qq'}. \tag{C.4}$$

They also satisfy the following completeness relation:

$$\sum_{\sigma\in\{+,-\}}\sum_{n=0}^{\infty}\sum_{l,q}\chi^{(n;l;q)}_{\sigma}(\boldsymbol{\theta_3})\otimes\chi^{(n;l;q)}_{\sigma}(\boldsymbol{\theta'_3})^\dagger = \frac{\delta(\boldsymbol{\theta_3}-\boldsymbol{\theta'_3})}{\sqrt{\tilde{g}}}\mathbb{U}(\boldsymbol{\theta_3},\boldsymbol{\theta'_3}). \tag{C.5}$$

For later convenience, some comments are in order:

• Although the value $n=0$ is allowed in the spectrum of the Dirac operator in (C.3), this value is omitted from the sum in (C.1) as it renders the denominator ill-defined [this also becomes clear in our proof of (C.1) below].

• For $n=0$, for which the allowed values for the rest of the angular momentum numbers are $l=0$ and $q=-1,0$, the spinor spherical harmonics (C.3) coincide with the Killing spinors on S^3, satisfying:

$$\tilde{\nabla}_{\tilde{\mu}}\;\chi^{(0;0;q)}_{\pm} = \pm\frac{i}{2}\tilde{\gamma}_{\tilde{\mu}}\chi^{(0;0;q)}_{\pm}. \tag{C.6}$$

It is clear that this equation is identical with $\tilde{\nabla}^T_{\tilde{\mu}}\chi^{(0;0;q)}_{\pm}=0$ [see eq. (C.2)]. In the main text, the Killing spinors $\chi^{(0;0;q)}_{\pm}$ are denoted as $\tilde{\epsilon}_{\pm,q}$ — see eq. (5.17).

- The commutator of covariant derivatives acting on spinors on S^3 is

$$[\tilde{\nabla}_{\tilde{\mu}},\tilde{\nabla}_{\tilde{\nu}}]=\frac{1}{4}\tilde{R}_{\tilde{\mu}\tilde{\nu}\tilde{\kappa}\tilde{\lambda}}\tilde{\gamma}^{\kappa}\tilde{\gamma}^{\tilde{\lambda}},\tag{C.7}$$

where the Riemann tensor of the unit S^3 is

$$\tilde{R}_{\tilde{\mu}\tilde{\nu}\tilde{\kappa}\tilde{\lambda}}=\tilde{g}_{\tilde{\mu}\tilde{\kappa}}\tilde{g}_{\tilde{\nu}\tilde{\lambda}}-\tilde{g}_{\tilde{\nu}\tilde{\kappa}}\tilde{g}_{\tilde{\mu}\tilde{\lambda}}.\tag{C.8}$$

Also, when acting on spinors on S^3, the squared Dirac operator is related to the Laplace-Beltrami operator as

$$\tilde{g}^{\tilde{\mu}\tilde{\nu}}\tilde{\nabla}_{\tilde{\mu}}\tilde{\nabla}_{\tilde{\nu}}=\tilde{\nabla\!\!\!/}^2+\frac{\tilde{R}}{4},\tag{C.9}$$

where the Ricci scalar is $\tilde{R}=6$. Let us now start proving eq. (C.1).

Proof of (C.1). To prove (C.1), we need to make use of the completeness of the vector-spinor eigenfunctions of the Dirac operator, also known as vector-spinor spherical harmonics, on S^3. There are two kinds of vector-spinor spherical harmonics on S^3 [84]: the transverse-traceless harmonics (3.32) and the longitudinal ones. We denote the latter as $\tilde{\lambda}^{(\mathrm{P};\pm n;l;q)}_{\tilde{\mu}}(\boldsymbol{\theta_3})$ and $\tilde{\lambda}^{(\mathrm{M};\pm n;l;q)}_{\tilde{\mu}}(\boldsymbol{\theta_3})$. To show that $\Delta^{TT}_{\tilde{\mu}\tilde{\nu}'}(\boldsymbol{\theta_3},\boldsymbol{\theta'_3})$ [eq. (3.75)] is given by (C.1), we need to exploit the fact that the TT and longitudinal vector-spinor spherical harmonics form a complete set on S^3. The corresponding completeness relation is

$$\begin{aligned}\frac{\delta(\boldsymbol{\theta_3}-\boldsymbol{\theta'_3})}{\sqrt{\tilde{g}}}\tilde{g}_{\tilde{\mu}\tilde{\nu}'}\mathbb{U}(\boldsymbol{\theta_3},\boldsymbol{\theta'_3})=\;&\Delta^{TT}_{\tilde{\mu}\tilde{\nu}'}(\boldsymbol{\theta_3},\boldsymbol{\theta'_3})+\sum_{\sigma\in\{+,-\}}\sum_{n,l,q}\tilde{\lambda}^{(\mathrm{P};\sigma n;l;q)}_{\tilde{\mu}}(\boldsymbol{\theta_3})\otimes\tilde{\lambda}^{(\mathrm{P};\sigma n;l;q)}_{\tilde{\nu}'}(\boldsymbol{\theta'_3})^{\dagger}\\&+\sum_{\sigma\in\{+,-\}}\sum_{n,l,q}\tilde{\lambda}^{(\mathrm{M};\sigma n;l;q)}_{\tilde{\mu}}(\boldsymbol{\theta_3})\otimes\tilde{\lambda}^{(\mathrm{M};\sigma n;l;q)}_{\tilde{\nu}'}(\boldsymbol{\theta'_3})^{\dagger},\end{aligned}\tag{C.10}$$

where $\Delta^{TT}_{\tilde{\mu}\tilde{\nu}'}(\boldsymbol{\theta_3},\boldsymbol{\theta'_3})$ is the sum over the transverse harmonics [see (3.75)], while the rest of the sums in (C.10) concern the longitudinal harmonics with all the allowed values of the quantum numbers n,l,q (these allowed values are discussed below).

To proceed, we need more information concerning the longitudinal harmonics. The longitudinal vector-spinor harmonics $\tilde{\lambda}^{(\mathrm{P};\pm n;l;q)}_{\tilde{\mu}}(\boldsymbol{\theta_3}),\tilde{\lambda}^{(\mathrm{M};\pm n;l;q)}_{\tilde{\mu}}(\boldsymbol{\theta_3})$ satisfy [84]

$$\tilde{\nabla\!\!\!/}\tilde{\lambda}^{(\mathrm{P};\pm n;l;q)}_{\tilde{\mu}}(\boldsymbol{\theta_3})=+i\sqrt{(n+3/2)^2-2}\;\tilde{\lambda}^{(\mathrm{P};\pm n;l;q)}_{\tilde{\mu}}(\boldsymbol{\theta_3})\tag{C.11}$$

and

$$\tilde{\nabla\!\!\!/}\tilde{\lambda}^{(\mathrm{M};\pm n;l;q)}_{\tilde{\mu}}(\boldsymbol{\theta_3})=-i\sqrt{(n+3/2)^2-2}\;\tilde{\lambda}^{(\mathrm{M};\pm n;l;q)}_{\tilde{\mu}}(\boldsymbol{\theta_3}),\tag{C.12}$$

and they are expressed in terms of the spinor harmonics (C.3) as [84]:

$$\tilde{\lambda}^{(\mathrm{P};\pm n;l;q)}_{\tilde{\mu}}(\boldsymbol{\theta_3})=\frac{c^{(\mathrm{P};\pm n)}}{\sqrt{2}}\left(\tilde{\nabla}_{\tilde{\mu}}+\frac{i}{2}\left\{\mp(n+3/2)+\sqrt{(n+3/2)^2-2}\;\right\}\tilde{\gamma}_{\tilde{\mu}}\right)\chi^{(n;l;q)}_{\pm}(\boldsymbol{\theta_3}),\tag{C.13}$$

$$\tilde{\lambda}^{(\mathrm{M};\pm n;l;q)}_{\tilde{\mu}}(\boldsymbol{\theta_3})=\frac{c^{(\mathrm{M};\pm n)}}{\sqrt{2}}\left(\tilde{\nabla}_{\tilde{\mu}}+\frac{i}{2}\left\{\mp(n+3/2)-\sqrt{(n+3/2)^2-2}\;\right\}\tilde{\gamma}_{\tilde{\mu}}\right)\chi^{(n;l;q)}_{\pm}(\boldsymbol{\theta_3}).\tag{C.14}$$

The normalisation factors $c^{(\mathrm{P};\pm n)}$ and $c^{(\mathrm{M};\pm n)}$ were not introduced in ref. [84]. We introduce them here such that the longitudinal vector-spinor harmonics satisfy

$$\int_{S^3} d\boldsymbol{\theta_3}\sqrt{\tilde{g}}\,\tilde{g}^{\tilde{\mu}\tilde{\nu}}\,\tilde{\lambda}_{\tilde{\mu}}^{(\mathrm{S};\sigma n;l;q)}(\boldsymbol{\theta_3})^\dagger\tilde{\lambda}_{\tilde{\nu}}^{(\mathrm{S}';\sigma' n';l';q')}(\boldsymbol{\theta_3}) = \delta_{\mathrm{SS}'}\delta_{\sigma\sigma'}\delta_{nn'}\delta_{ll'}\delta_{qq'}, \tag{C.15}$$

where S, S$'$ $\in$ {P, M} and $\sigma \in \{+,-\}$. It is straightforward to find that:

$$\left|\frac{c^{(\mathrm{P};\sigma n)}}{\sqrt{2}}\right|^2 = \left(\frac{3}{2}\left((n+3/2)^2-2\right) - \sigma\frac{1}{2}(n+3/2)\sqrt{(n+3/2)^2-2}\right)^{-1}, \tag{C.16}$$

where for $\sigma = -$ we have that all the values of $n \geq 0$ are allowed, while for $\sigma = +$ we have $n \geq 1$ because for $n = 0$ the harmonic $\tilde{\lambda}_{\tilde{\mu}}^{(\mathrm{P};+n;l;q)}$ is identically zero (and, thus, its normalisation factor is not defined) as the spinor harmonic in (C.13) is a Killing spinor (for $n = 0$) and the differential operator acting on it takes the form of the operator in the Killing spinor equation (C.6). Similarly, we find

$$\left|\frac{c^{(\mathrm{M};\sigma n)}}{\sqrt{2}}\right|^2 = \left(\frac{3}{2}\left((n+3/2)^2-2\right) + \sigma\frac{1}{2}(n+3/2)\sqrt{(n+3/2)^2-2}\right)^{-1}, \tag{C.17}$$

where, now, for $\sigma = +$ we have $n \geq 0$, while for $\sigma = -$ we have $n \geq 1$ for the same reason as in the case of $c^{(\mathrm{P};+n)}$ above.

Now that we know the allowed values of the quantum number n, let us re-write (C.10) as

$$\begin{aligned}\Delta^{TT}_{\tilde{\mu}\tilde{\nu}'}(\boldsymbol{\theta_3},\boldsymbol{\theta}'_3) = {}& \frac{\delta(\boldsymbol{\theta_3}-\boldsymbol{\theta}'_3)}{\sqrt{\tilde{g}}}\tilde{g}_{\tilde{\mu}\tilde{\nu}'}\mathbb{U}(\boldsymbol{\theta_3},\boldsymbol{\theta}'_3) - \sum_{n=1}^{\infty}\sum_{l,q}\tilde{\lambda}_{\tilde{\mu}}^{(\mathrm{P};+n;l;q)}(\boldsymbol{\theta_3})\otimes\tilde{\lambda}_{\tilde{\nu}'}^{(\mathrm{P};+n;l;q)}(\boldsymbol{\theta}'_3)^\dagger \\ &-\sum_{n=0}^{\infty}\sum_{l,q}\tilde{\lambda}_{\tilde{\mu}}^{(\mathrm{P};-n;l;q)}(\boldsymbol{\theta_3})\otimes\tilde{\lambda}_{\tilde{\nu}'}^{(\mathrm{P};-n;l;q)}(\boldsymbol{\theta}'_3)^\dagger \\ &-\sum_{n=1}^{\infty}\sum_{l,q}\tilde{\lambda}_{\tilde{\mu}}^{(\mathrm{M};-n;l;q)}(\boldsymbol{\theta_3})\otimes\tilde{\lambda}_{\tilde{\nu}'}^{(\mathrm{M};-n;l;q)}(\boldsymbol{\theta}'_3)^\dagger \\ &-\sum_{n=0}^{\infty}\sum_{l,q}\tilde{\lambda}_{\tilde{\mu}}^{(\mathrm{M};+n;l;q)}(\boldsymbol{\theta_3})\otimes\tilde{\lambda}_{\tilde{\nu}'}^{(\mathrm{M};+n;l;q)}(\boldsymbol{\theta}'_3)^\dagger.\end{aligned} \tag{C.18}$$

Substituting (C.13) and (C.14) into (C.18), and after a long but straightforward calculation, we find that all the $n = 0$ terms cancel among themselves with the help of (C.6), while the remaining terms ($n = 1, 2, \ldots$) can be written in the simpler form:

$$\Delta^{TT}_{\tilde{\mu}\tilde{\nu}'}(\boldsymbol{\theta_3},\boldsymbol{\theta}'_3) = \left(\tilde{g}_{\tilde{\mu}\tilde{\nu}'}\mathbb{U}(\boldsymbol{\theta_3},\boldsymbol{\theta}'_3) - \frac{1}{3}\tilde{\gamma}_{\tilde{\mu}}\mathbb{U}(\boldsymbol{\theta_3},\boldsymbol{\theta}'_3)\tilde{\gamma}_{\tilde{\nu}'}\right)\frac{\delta(\boldsymbol{\theta_3}-\boldsymbol{\theta}'_3)}{\sqrt{\tilde{g}}} + Y_{\tilde{\mu}\tilde{\nu}'}(\boldsymbol{\theta_3},\boldsymbol{\theta}'_3), \tag{C.19}$$

where we have used (C.5), and we have also defined

$$\begin{aligned}&Y_{\tilde{\mu}\tilde{\nu}'}(\boldsymbol{\theta_3},\boldsymbol{\theta}'_3) \\ &= \frac{3}{2}\sum_{\sigma\in\{+,-\}}\sum_{n=1}^{\infty}\sum_{l,q}\frac{1}{-(n+3/2)^2+9/4}\Bigg\{\left(\tilde{\nabla}_{\tilde{\mu}}\chi_{\sigma}^{(n;l;q)}(\boldsymbol{\theta_3}) - \frac{i\sigma(n+3/2)}{3}\tilde{\gamma}_{\tilde{\mu}}\chi_{\sigma}^{(n;l;q)}(\boldsymbol{\theta_3})\right) \\ &\quad\otimes\left(\tilde{\nabla}_{\tilde{\nu}'}\chi_{\sigma}^{(n;l;q)}(\boldsymbol{\theta}'_3)^\dagger + \frac{i\sigma(n+3/2)}{3}\chi_{\sigma}^{(n;l;q)}(\boldsymbol{\theta}'_3)^\dagger\,\tilde{\gamma}_{\tilde{\nu}'}\right)\Bigg\}.\end{aligned} \tag{C.20}$$

Then, using (C.3), it is easy to show that $\Delta^{TT}_{\tilde{\mu}\tilde{\nu}'}(\boldsymbol{\theta_3},\boldsymbol{\theta'_3})$ in (C.19) is equal to the desired expression (C.1). □

Now let us use eq. (C.1) to show that the equal-time anti-commutator (3.91) is local (i.e. vanishes for $\boldsymbol{\theta_3} \neq \boldsymbol{\theta'_3}$) despite that $\Delta^{TT}_{\tilde{\mu}\tilde{\nu}'}$ is non-local due to the appearance of $\left(\tilde{\nabla\!\!\!/}^2 + 9/4\right)^{-1}$ in (C.1). It is clear that the locality of the anti-commutator (3.91) reduces to the locality of the following quantity:

$$\begin{aligned}&\left(\tilde{\nabla\!\!\!/}^2+\frac{1}{4}\right)\Delta^{TT}_{\tilde{\mu}\tilde{\nu}'}(\boldsymbol{\theta_3},\boldsymbol{\theta'_3})\\&=\left(\tilde{\nabla\!\!\!/}^2+\frac{1}{4}\right)\left(\tilde{g}_{\tilde{\mu}\tilde{\nu}'}\mathbb{U}(\boldsymbol{\theta_3},\boldsymbol{\theta'_3})-\frac{1}{3}\tilde{\gamma}_{\tilde{\mu}}\mathbb{U}(\boldsymbol{\theta_3},\boldsymbol{\theta'_3})\tilde{\gamma}_{\tilde{\nu}'}\right)\frac{\delta(\boldsymbol{\theta_3}-\boldsymbol{\theta'_3})}{\sqrt{\tilde{g}}}\\&\quad+\frac{3}{2}\sum_{\sigma\in\{+,-\}}\sum_{n=1}^{\infty}\sum_{l,q}\left(\tilde{\nabla\!\!\!/}^2+\frac{1}{4}\right)\tilde{\nabla}^T_{\tilde{\mu}}\left(\frac{1}{\tilde{\nabla\!\!\!/}^2+9/4}\chi^{(n;l;q)}_{\sigma}(\boldsymbol{\theta_3})\otimes\chi^{(n;l;q)}_{\sigma}(\boldsymbol{\theta'_3})^{\dagger}\right)\overleftarrow{\tilde{\nabla}}^T{}_{\tilde{\nu}'}.\end{aligned}$$

It is straightforward to commutate the two differential operators $\tilde{\nabla\!\!\!/}^2 + 1/4$ and $\tilde{\nabla}^T_{\tilde{\mu}}$ using (C.7), as

$$\left(\tilde{\nabla\!\!\!/}^2+\frac{1}{4}\right)\tilde{\nabla}^T_{\tilde{\mu}}=\left(\tilde{\nabla\!\!\!/}^2+\frac{1}{4}\right)\tilde{\nabla}_{\tilde{\mu}}-\frac{1}{3}\tilde{\gamma}_{\tilde{\mu}}\tilde{\nabla\!\!\!/}\left(\tilde{\nabla\!\!\!/}^2+\frac{1}{4}\right)=\tilde{\nabla}^T_{\tilde{\mu}}\left(\tilde{\nabla\!\!\!/}^2+\frac{9}{4}\right),$$

where we have used that gamma matrices commute with the squared Dirac operator because of (C.9). It is now clear that there is no non-local term in $\left(\tilde{\nabla\!\!\!/}^2+\frac{1}{4}\right)\Delta^{TT}_{\tilde{\mu}\tilde{\nu}'}$, as:

$$\begin{aligned}\left(\tilde{\nabla\!\!\!/}^2+\frac{1}{4}\right)\Delta^{TT}_{\tilde{\mu}\tilde{\nu}'}(\boldsymbol{\theta_3},\boldsymbol{\theta'_3})&=\left(\tilde{\nabla\!\!\!/}^2+\frac{1}{4}\right)\left(\tilde{g}_{\tilde{\mu}\tilde{\nu}'}\mathbb{U}(\boldsymbol{\theta_3},\boldsymbol{\theta'_3})-\frac{1}{3}\tilde{\gamma}_{\tilde{\mu}}\mathbb{U}(\boldsymbol{\theta_3},\boldsymbol{\theta'_3})\tilde{\gamma}_{\tilde{\nu}'}\right)\frac{\delta(\boldsymbol{\theta_3}-\boldsymbol{\theta'_3})}{\sqrt{\tilde{g}}}\\&\quad+\frac{3}{2}\sum_{\sigma\in\{+,-\}}\sum_{n=1}^{\infty}\sum_{l,q}\tilde{\nabla}^T_{\tilde{\mu}}\left(\chi^{(n;l;q)}_{\sigma}(\boldsymbol{\theta_3})\otimes\chi^{(n;l;q)}_{\sigma}(\boldsymbol{\theta'_3})^{\dagger}\right)\overleftarrow{\tilde{\nabla}}^T{}_{\tilde{\nu}'}.\end{aligned}$$

We can now include the value $n = 0$ in the summation as it gives zero contribution because of the Killing spinor equation (C.6). Finally, using the completeness of the spinor spherical harmonics (C.5), we arrive at the local expression

$$\begin{aligned}\left(\tilde{\nabla\!\!\!/}^2+\frac{1}{4}\right)\Delta^{TT}_{\tilde{\mu}\tilde{\nu}'}(\boldsymbol{\theta_3},\boldsymbol{\theta'_3})&=\left(\tilde{\nabla\!\!\!/}^2+\frac{1}{4}\right)\left(\tilde{g}_{\tilde{\mu}\tilde{\nu}'}\mathbb{U}(\boldsymbol{\theta_3},\boldsymbol{\theta'_3})-\frac{1}{3}\tilde{\gamma}_{\tilde{\mu}}\mathbb{U}(\boldsymbol{\theta_3},\boldsymbol{\theta'_3})\tilde{\gamma}_{\tilde{\nu}'}\right)\frac{\delta(\boldsymbol{\theta_3}-\boldsymbol{\theta'_3})}{\sqrt{\tilde{g}}}\\&\quad+\frac{3}{2}\tilde{\nabla}^T_{\tilde{\mu}}\left(\frac{\delta(\boldsymbol{\theta_3}-\boldsymbol{\theta'_3})}{\sqrt{\tilde{g}}}\,\mathbb{U}(\boldsymbol{\theta_3},\boldsymbol{\theta'_3})\right)\overleftarrow{\tilde{\nabla}}^T{}_{\tilde{\nu}'}\,.\end{aligned}\tag{C.21}$$

This shows that the equal-time anti-commutator (3.91) is local.

D Useful expressions concerning the conformal-like symmetry of the graviton

Let $B_{\mu\nu}$ be any (complex or real) symmetric spin-2 tensor field on dS_4. Its conformal-like transformation is defined in (4.31) as

$$T_V B_{\mu\nu} = V^{\rho}\varepsilon_{\rho\sigma\lambda(\mu}\nabla^{\sigma}B^{\lambda}{}_{\nu)}.\tag{D.1}$$

Recall that V^μ is any genuine conformal Killing vector (2.13). One can straightforwardly prove the following:

$$\begin{aligned} g^{\mu\nu}\, T_V B_{\mu\nu} &= 0, \\ \nabla^\alpha\, T_V B_{\alpha\nu} &= \frac{1}{2} V^\rho \varepsilon_{\rho\sigma\lambda\nu} \nabla^\sigma \nabla^\alpha B_\alpha{}^\lambda, \\ \nabla_{(\mu} \nabla^\alpha\, T_V B_{\nu)\alpha} &= \frac{1}{2} V^\rho \varepsilon_{\rho\sigma\lambda(\nu} \nabla^\sigma \nabla_{\mu)} \nabla^\alpha B_\alpha{}^\lambda, \\ \nabla^\nu \nabla^\alpha\, T_V B_{\nu\alpha} &= 0, \\ \Box T_V B_{\mu\nu} &= V^\rho \varepsilon_{\rho\sigma\lambda(\mu} \nabla^\sigma\, \Box B^\lambda{}_{\nu)}. \end{aligned} \tag{D.2}$$

These expressions can be used to prove that T_V is a symmetry of the full linearised Einstein equations (4.3), as well as of the ones in the TT gauge (4.5). Moreover, one can similarly show that both T_V and $\mathcal{T}_V = iT_V$ are symmetries of the non-gauge-fixed complex linearised Einstein equations (4.36), as well as of the complex graviton equations in the TT gauge (4.39).

E Some properties of the field strengths

Let us recall some properties of the field strengths for the complex graviton [eq. (4.59)] and complex gravitino [eq. (3.65)]. Without making use of the equations of motion, it is easy to show that

$$\nabla_{[\kappa} U_{\alpha\beta]\mu\nu} = 0, \quad U_{[\alpha\beta\mu]\nu} = 0, \tag{E.1}$$

and [76]

$$\left(\nabla_{[\kappa} + \frac{i}{2}\gamma_{[\kappa}\right) F_{\alpha\beta]} = 0. \tag{E.2}$$

If the complex graviton satisfies the field equations, then the complex linearised Weyl tensor satisfies

$$g^{\alpha\mu} U_{\alpha\beta\mu\nu} = 0, \quad g^{\beta\nu} U_{\alpha\beta\mu\nu} = 0 \tag{E.3}$$

$$\nabla^\alpha U_{\alpha\beta\mu\nu} = 0. \tag{E.4}$$

The dual, $\widetilde{U}_{\alpha\beta\mu\nu} = \frac{1}{2}\varepsilon_{\alpha\beta}{}^{\kappa\lambda} U_{\kappa\lambda\mu\nu}$, can be also expressed (using the equations of motion) as

$$\widetilde{U}_{\alpha\beta\mu\nu} = \frac{1}{2} U_{\alpha\beta\kappa\lambda}\, \varepsilon^{\kappa\lambda}{}_{\mu\nu}. \tag{E.5}$$

Equation (E.5) can be proved as follows. Let us denote the tensor on the right-hand side of eq. (E.5) as $P_{\alpha\beta\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu}{}^{\kappa\lambda} U_{\alpha\beta\kappa\lambda}$. Contracting $P_{\alpha\beta\mu\nu}$ with $\frac{1}{2}\varepsilon_{\rho\sigma}{}^{\alpha\beta}$, and using eq. (B.6) and the equations of motion, one finds

$$\widetilde{P}_{\rho\sigma\mu\nu} = -U_{\rho\sigma\mu\nu}.$$

Then, contracting this equation with $\frac{1}{2}\varepsilon_{\alpha\beta}{}^{\rho\sigma}$, and using (B.6), we have

$$\frac{1}{2}\varepsilon_{\alpha\beta}{}^{\rho\sigma}\widetilde{P}_{\rho\sigma\mu\nu} = -P_{\alpha\beta\mu\nu} = -\widetilde{U}_{\alpha\beta\mu\nu},$$

thus proving eq. (E.5).

If the complex gravitino satisfies the equations of motion, then its field strength satisfies [76]:

$$\nabla^\alpha F_{\alpha\nu} = \gamma^\alpha F_{\alpha\nu} = 0, \tag{E.6}$$

$$\nabla_{[\kappa} F_{\alpha\beta]} = 0, \tag{E.7}$$

$$\gamma_{[\kappa} F_{\alpha\beta]} = 0, \tag{E.8}$$

$$\nabla\!\!\!/ F_{\mu\nu} = 0, \tag{E.9}$$

$$\gamma_{\mu\nu\alpha\beta} F^{\alpha\beta} = -2F_{\mu\nu}, \quad \text{and thus,} \quad \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta} F^{\alpha\beta} = -i\gamma^5 F_{\mu\nu}. \tag{E.10}$$

E.1 Deriving the SUSY transformation (6.21) of the spin-2 field strength from the initial SUSY transformation (6.20)

To derive eq. (6.21) from eq. (6.20) we have to make use of eq. (6.22). Let us now prove eq. (6.22). We start by considering the following quantity:

$$2\, \varepsilon_{\rho\sigma\alpha\beta}\, \gamma_{[\mu}{}^{\alpha} F^{\beta}{}_{\nu]} = i\gamma^5 \gamma_{\rho\sigma\alpha\beta} \left(\gamma_\mu{}^\alpha F^\beta{}_\nu - \gamma_\nu{}^\alpha F^\beta{}_\mu \right), \tag{E.11}$$

where, on the right-hand side, we have used (2.6) and we have expanded the anti-symmetrisation of the indices μ and ν. Using $\gamma_{\rho\sigma\alpha\beta}\ \gamma_\mu{}^\alpha = g_{\rho\mu}\gamma_{\sigma\beta} - g_{\sigma\mu}\gamma_{\rho\beta} + g_{\beta\mu}\gamma_{\rho\sigma}$, as well as the fact that the spin-3/2 field strength is gamma traceless on-shell, and thus $\gamma_{\rho\beta} F^\beta{}_\nu = -F_{\rho\nu}$, eq. (E.11) gives

$$2\, \varepsilon_{\rho\sigma\alpha\beta}\, \gamma_{[\mu}{}^{\alpha} F^{\beta}{}_{\nu]} = i\gamma^5 \left(2g_{\rho[\mu}F_{\nu]\sigma} - 2g_{\sigma[\mu}F_{\nu]\rho} + 2\gamma_{\rho\sigma}F_{\mu\nu} \right). \tag{E.12}$$

Then, the first two terms on the right-hand side of eq. (E.12) can be re-expressed as

$$2g_{\rho[\mu}F_{\nu]\sigma} - 2g_{\sigma[\mu}F_{\nu]\rho} = \gamma_{\mu\nu}F_{\rho\sigma} - \gamma_{\rho\sigma}F_{\mu\nu}.$$

This can be straightforwardly proved by using $\gamma_{\kappa\lambda} = (\gamma_\kappa\gamma_\lambda - \gamma_\lambda\gamma_\kappa)/2$ on the right-hand side, and then making use of the on-shell property (E.8). Thus, eq. (E.12) gives

$$2\, \varepsilon_{\rho\sigma\gamma\delta}\, \gamma_{[\mu}{}^{\gamma} F^{\delta}{}_{\nu]} = i\gamma^5 \left(\gamma_{\mu\nu}F_{\rho\sigma} + \gamma_{\rho\sigma}F_{\mu\nu} \right). \tag{E.13}$$

Contracting both sides of eq. (E.13) with $\varepsilon^{\alpha\beta\rho\sigma}$ and dividing the result by 2, we have

$$-2\, \gamma_{[\mu}{}^{[\alpha} F^{\beta]}{}_{\nu]} = \frac{i}{2}\gamma^5 \left(\gamma_{\mu\nu}\widetilde{F}^{\alpha\beta} + \frac{1}{2}\varepsilon^{\alpha\beta\rho\sigma}\gamma_{\rho\sigma}F_{\mu\nu} \right).$$

Then, using the on-shell property (E.10), as well as $\varepsilon^{\alpha\beta\rho\sigma}\gamma_{\rho\sigma}/2 = -i\gamma^5\gamma^{\alpha\beta}$, we find

$$-2\, \gamma_{[\mu}{}^{[\alpha} F^{\beta]}{}_{\nu]} = \frac{1}{2} \left(\gamma_{\mu\nu}F^{\alpha\beta} + \gamma^{\alpha\beta}F_{\mu\nu} \right), \tag{E.14}$$

proving eq. (6.22).

Data Availability Statement. This article has no associated data or the data will not be deposited.

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