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Homology of matching complexes and representations of symmetric groups

Michael Bate · Brent Everitt · Sam Ford · Eric Ramos [★]

Abstract. We compute the homology of the matching complex $M(\Gamma)$, where Γ is the complete hypergraph on $n \geq 2$ vertices, and analyse the S_n -representations carried by this homology. These results are achieved using standard techniques in combinatorial topology, such as the theory of shellings. We then broaden the scope to the larger class of fibre-closed families of simplicial complexes and consider these through the lens of representation stability. This allows us to prove a number of results of an asymptotic nature, such as an analysis of the growth of Betti numbers and the kinds of irreducible S_n -representations that appear.

Introduction

Let G be a finite group, p a prime dividing the order of G and $\mathcal{S}_p(G)$ the poset of p -subgroups of G ordered by inclusion. Brown's homological Sylow theorem [Bro75] states that the simplicial order complex of $\mathcal{S}_p(G)$ has Euler characteristic congruent to 1 modulo the order of a Sylow p -subgroup of G . This led Quillen [Qui78] to show that $\mathcal{S}_p(G)$ is homotopy equivalent to the poset $\mathcal{A}_p(G)$ of elementary Abelian p -subgroups of G and to formulate a number of fundamental conjectures. These results have established $\mathcal{S}_p(G)$ and $\mathcal{A}_p(G)$ as central objects in the interplay between geometric combinatorics and representation theory.

Even for the symmetric groups these complexes are not well-understood. Bouc [Bou92] thus introduced the matching complex M_n , which is closely related to $\mathcal{S}_2(S_n)$ and $\mathcal{A}_2(S_n)$. In particular, if k is a field of characteristic 0 then the natural S_n -action on M_n gives the reduced homology $\bar{H}_q(M_n, k)$ the structure of a kS_n -module, and Bouc describes its decomposition into irreducibles. On the other hand, the integral homology of M_n displays complicated torsion behavior and is less well-understood. This has led to much work in recent years on matching complexes and their various generalisations.

This paper starts — see Theorem 1 — by computing the homology of one of these generalisations: the matching complex $M(\Gamma)$, where Γ is the complete hypergraph on $n \geq 2$ vertices. (The original matching complex M_n is equal to $M(\Gamma)$ where Γ is the complete *graph* on n vertices.) Our main technical tool here is the theory of shellings.

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We then analyse the resulting S_n -representations carried by the homology and decompose them (Theorem 2 below) as a sum of induced representations from the normalisers of standard Young subgroups S_λ , for partitions $\lambda \vdash n$ having a certain shape. If λ has no repeated parts then this gives a decomposition (6) of the corresponding part of the homology into Specht modules. Such a decomposition in general seems to be out of reach, so in lieu of this we appeal to the theory of representation stability and study the asymptotics of the S_n -representations as n grows large. This, it turns out, we can do for a larger class of simplicial complexes that we term *fibre closed*, and categorical representation theory allows us to expose the overall “shape” of the representations; this we do in Theorem 3.

1. Matching complexes and their homology

We begin by recalling basic terminology and notation for simplicial complexes — standard references for this material are [Mau96, Section 2.3] or [Spa66, Section 3.1]. We then recall the general matching complex $M(\Gamma)$ of a hypergraph Γ ; see [Ber89] for background on hypergraphs, [LP09] for matching theory and [Jon08] for a survey of matching and related complexes. The homology and the representations it carries are given in Section 1.2, along with the shellings that we use to compute it. A nice survey of general simplicial methods in algebra and combinatorics is [Wac07]; see also [Wac03] for representations carried by the homology of matching complexes. For shellings, especially in the non-pure case, we follow [BW96].

1.1. Matching complexes

A (finite) simplicial complex X with vertex set X_0 is a set of distinguished subsets of X_0 with the property that $\{x\} \in X$ for all $x \in X_0$, and if $\sigma \in X$ and $\tau \subseteq \sigma$ then $\tau \in X$. If $\sigma \in X$ with $|\sigma| = q + 1$ then σ is called a q -(dimensional) face of X . The empty set \emptyset is the unique face of dimension -1 . Write X_q for the set of q -faces.

A face σ that is maximal under inclusion is called a facet and a complex where all the facets have the same dimension is said to be pure. The dimension of X is the maximal q for which there exists a q -facet. A q -simplex $\bar{\sigma}$ is the simplicial subcomplex obtained by considering a q -face σ and all of its subsets; in particular, if σ is a facet then $\bar{\sigma}$ is called a maximal subsimplex. A map $f : X \rightarrow Y$ of simplicial complexes is a set map $f : X_0 \rightarrow Y_0$ such that $f(\sigma) \in Y$ for all $\sigma \in X$. A group G acts on X if G acts on the vertices X_0 in such a way that for any $g \in G$ and face σ , the image subset $g\sigma$ is also a face of X .

Now to matching complexes. A hypergraph Γ consists of a set of vertices V and a set of non-empty subsets of V called hyperedges. A hypergraph whose hyperedges all have cardinality ≤ 2 is just a graph. In particular, loops at a vertex of a graph are allowed (these are the hyperedges corresponding to subsets of size 1) but multiple edges between vertices are not (the hyperedges form a set, not a multi-set). A matching of a hypergraph Γ is some set of pairwise disjoint hyperedges. The *matching complex* $M(\Gamma)$ has q -faces the matchings of Γ that contain $q + 1$ hyperedges. Put another way, for $q \geq 0$, the matching complex $M(\Gamma)$ has q -faces those collections of $q + 1$ mutually disjoint non-empty subsets of the vertices V of Γ that happen to be hyperedges; in particular, the vertices of $M(\Gamma)$ are the non-empty subsets of V that are hyperedges.

The fundamental example is when Γ is the complete graph K_n on n vertices — where any two vertices are joined by an edge and no loops are allowed. Then $M(\Gamma) = M(K_n)$ is *the* matching complex M_n . Alternatively, M_n is the simplicial complex whose vertices are the 2-element

subsets of $[n] := \{1, 2, \dots, n\}$ and whose q -faces are the collections of $q + 1$ mutually disjoint 2-element subsets of $[n]$. If instead Γ is the complete bipartite graph $K_{m,n}$ then $M(\Gamma)$ is isomorphic to the *chessboard* complex, whose q -faces can be identified with the placements of $q + 1$ mutually non-attacking rooks on an $m \times n$ chessboard.

For an example where Γ is a genuine hypergraph, rather than just a graph, consider for a fixed $r \geq 2$ the r -uniform complete hypergraph $\Gamma = K_n^r$, whose hyperedges are all the subsets of $[n]$ of size r . The resulting matching complex $M(\Gamma)$ then has q -faces the $q + 1$ mutually disjoint subsets of $[n]$, all of which have size r — see [BLVŽ94]. When $r = 2$ the complex $M(K_n^2)$ is just the matching complex M_n .

Our interest starts with the matching complex $\overline{X}(n) := M(\Gamma)$ where Γ is the complete hypergraph on $n \geq 2$ vertices: the hyperedges of Γ are now *all* the non-empty subsets of $[n]$. The vertices of $\overline{X}(n)$ are the non-empty subsets $x \subseteq [n]$ and a q -face is a partition $x_0|x_1|\dots|x_q$ of some subset $A \subseteq [n]$ into $q + 1$ non-empty blocks. The complex $\overline{X}(n)$ is $(n - 1)$ -dimensional with a unique $(n - 1)$ -face given by the partition of $[n]$ into n blocks of size 1.

The complex $\overline{X}(n)$ has two connected components: one consists of just the vertex $[n]$, which is contained in no other face. On the other hand, if x is any other vertex then there is a path of at most two edges connecting x to the vertex $\{1\}$. For, if $1 \notin x$ then the partition $1|x$ is an edge connecting x to $\{1\}$; if $1 \in x$ then there is an $i \notin x$ and hence edges $1|i$ and $i|x$ connecting x to $\{i\}$ and then $\{i\}$ to $\{1\}$. For this reason, we remove the isolated vertex $[n]$ and write $X(n)$ for the complex whose vertices are the *proper* non-empty subsets $\emptyset \neq x \subsetneq [n]$ and whose q -faces are the partitions of some subset $A \subseteq [n]$ into $q + 1$ proper non-empty blocks.

The facets of $X(n)$ are the partitions $x_0|x_1|\dots|x_q$ of $[n]$ itself. In particular, the number of q -facets equals the number of partitions of the set $[n]$ into $q + 1$ non-empty blocks; this is given by the Stirling number of the second kind:

$$\left\{ \begin{matrix} n \\ q \end{matrix} \right\} = \frac{1}{q!} \sum_{i=0}^q (-1)^i \binom{q}{i} (q - i)^n.$$

Since these numbers are nonzero for multiple values of q , the complex $X(n)$ is not pure.

1.2. Homology

Returning to a general simplicial complex X , we fix a total ordering $x < y$ of the vertices. A q -face is then oriented when it is written as $\sigma = x_0x_1\dots x_q$ with $x_i < x_j$ iff $i < j$. The reduced simplicial homology $\widetilde{H}_*(X, \mathbb{Z})$ is the homology of the chain complex $C_* = C_*(X, \mathbb{Z})$ having q -chains C_q the free \mathbb{Z} -module on the q -faces X_q . The differential $d : C_q \rightarrow C_{q-1}$ is the map induced by $\sigma \mapsto \sum_i (-1)^i d_i \sigma$, where $d_i \sigma = x_0x_1\dots \widehat{x}_i \dots x_q$ ($0 \leq i \leq q$), and $d_0 : C_0 \rightarrow C_{-1} = \mathbb{Z}$ is induced by $x_0 \mapsto 1$.

If a group G acts on X and $\sigma = x_0x_1\dots x_q$ is an oriented face then:

$$g \cdot \sigma = g \cdot x_0x_1\dots x_q = (gx_0)(gx_1)\dots(gx_q) = y_{\pi(0)}y_{\pi(1)}\dots y_{\pi(q)},$$

where $y_0y_1\dots y_q$ is an oriented face and π is in the symmetric group S_q . Define an action of G on the q -chains C_q induced by:

$$g \cdot x_0x_1\dots x_q = (-1)^{\text{sgn}(\pi)} y_0y_1\dots y_q. \quad (1)$$

Then $gd = dg$ and (1) defines an action of G by chain maps on the complex C_* . In particular there is an induced action of G on the homology $\widetilde{H}_q(X, \mathbb{Z})$ for all q .

We compute homology by finding a *shelling* of X . This is a total ordering $\sigma_1, \sigma_2, \dots, \sigma_m$ of the facets such that for all $t > 1$:

$$\overline{\sigma}_t \cap \bigcup_{i=1}^{t-1} \overline{\sigma}_i, \quad (2)$$

is a pure $(\dim \sigma_t - 1)$ -dimensional subcomplex of the simplex $\overline{\sigma}_t$. In this case, define:

$$\mathcal{R}(\sigma_t) = \{x \in \sigma_t : x \in \bigcup_{i=1}^{t-1} \overline{\sigma}_i\}, \quad (3)$$

and call σ a homology facet when $\mathcal{R}(\sigma) = \sigma$.

The homology $\widetilde{H}_q(X, \mathbb{Z})$ is isomorphic to the free \mathbb{Z} -module with a basis in 1-1 correspondence with the homology facets — see [BW96, Corollary 4.4]. If G acts on X by preserving the set of homology facets then G permutes this basis exactly as it permutes the homology facets themselves. This allows the G -module structure of $\widetilde{H}_q(X, \mathbb{Z})$ to be analysed.

Returning to the matching complex $X(n)$ of Section 1.1, from now on we abbreviate a partition $x_0|x_1| \dots |x_q$ of $[n]$ to just $x_0x_1 \dots x_q$. It will also be convenient to isolate those blocks of a partition that are singletons and write $x_0x_1 \dots x_q$ as $x_0 \dots x_kz_1 \dots z_\ell$, where the blocks x_i have size ≥ 2 and the blocks z_j have size 1.

Totally order the facets of $X(n)$ in decreasing order of the number of singleton blocks: i.e. define:

$$x_0 \dots x_kz_1 \dots z_\ell < x'_0 \dots x'_uz'_1 \dots z'_v,$$

if $\ell > v$; if $\ell = v$, then totally order the partitions with ℓ singleton blocks arbitrarily.

Proposition 1. *This order is a shelling of $X(n)$.*

Proof. Let $\sigma_t = x_0 \dots x_kz_1 \dots z_\ell$ be a facet. We claim that the subcomplex (2) is the union of all the simplices $\overline{\tau}$ where:

$$\tau = x_0 \dots \widehat{x}_i \dots x_kz_1 \dots z_\ell, \quad (4)$$

with $0 \leq i \leq k$ and the hat denoting omission. As each such $\overline{\tau}$ is a $(\dim \sigma_t - 1)$ -dimensional sub-simplex of $\overline{\sigma}_t$, the proposition follows. To see the claim, if in (4) the block $x_i = \{x_{i1}, \dots, x_{ij}\}$, then τ is a face of the facet:

$$\sigma_s = x_0 \dots \widehat{x}_i \dots x_kz_1 \dots z_\ell x_{i1} \dots x_{ij},$$

and $\sigma_s < \sigma_t$. Thus $\overline{\tau} \subseteq \overline{\sigma}_s \in \bigcup_{i=1}^{t-1} \overline{\sigma}_i$ and $\overline{\tau}$ is indeed contained in the subcomplex (2). On the other hand, let ω be a face of σ_t of the form:

$$\omega = x_0 \dots x_k\widehat{z}_1 \dots \widehat{z}_jz_{j+1} \dots z_\ell.$$

If σ_s is a facet that contains ω as a face then $\sigma_s = x_0 \dots x_ky_1 \dots y_rz_{j+1} \dots z_\ell$ where $y_1 \dots y_r$ is some partition of the set $\{z_1, \dots, z_j\}$. In particular, $\sigma_s \geq \sigma_t$ and $\overline{\omega} \notin \bigcup_{i=1}^{t-1} \overline{\sigma}_i$. This proves the claim. \square

An immediate consequence of the claim at the beginning of the proof of Proposition 1 is that the homology facets of $X(n)$ are those facets which as partitions have no singleton blocks. We have thus proved:

Theorem 1. *For $n \geq 2$ and $0 \leq q \leq n - 1$, the homology $\widetilde{H}_q(X(n), \mathbb{Z}) \cong \mathbb{Z}^{\beta(n,q)}$ where $\beta(n, q)$ is the number of partitions of $[n]$ into $q + 1$ blocks, with each block a proper non-singleton subset.*

In particular if $q = 0$ or $q > \lfloor \frac{n}{2} \rfloor - 1$, where $\lfloor x \rfloor$ is the largest integer $\leq x$, then the homology $\widetilde{H}_q(X(n), \mathbb{Z})$ vanishes.

For an integer $r \geq 1$, the r -associated Stirling number $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_r$ of the second kind is the number of partitions of $[n]$ into k blocks, each of which has size at least r — see [Com74, page 221]. There are various closed formulas for $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_r$ — see [How80] — and in particular, inclusion-exclusion gives, for $q > 0$:

$$\beta(n, q) = \left\{ \begin{matrix} n \\ q+1 \end{matrix} \right\}_2 = \sum_{i=0}^{q+1} (-1)^i \binom{n}{i} \left\{ \begin{matrix} n-i \\ q-i+1 \end{matrix} \right\}.$$

2. Representations of the symmetric group

We saw in Section 1.2 that we can make the homology groups for $X(n)$ into $\mathbb{Z}S_n$ -modules: by defining a total order on the vertices we can orient each face of $X(n)$, and then by adjusting the natural action with some sign changes we can ensure that orientations are preserved, so that S_n acts on the homology groups. For the fundamentals of symmetric group representations, see [Sag01].

A convenient total order of the vertices of $X(n)$ is to first order subsets of $[n]$ in decreasing order of size, and then lexicographically order the subsets of size k for each $k \leq n$. Thus, any oriented q -face $\sigma = x_0 x_1 \dots x_q \in X(n)$ has $|x_0| \geq \dots \geq |x_q|$. We denote the corresponding orientation-preserving action by $(g, \sigma) \mapsto g(\sigma)$ for $g \in S_n$ and $\sigma \in X(n)$.

Each facet $\sigma \in X(n)$ gives rise to a partition of the number n : define $\lambda(\sigma) := (|x_0|, \dots, |x_q|) \vdash n$. We call the partition $\lambda = \lambda(\sigma)$ the *shape* of σ . For each $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_q)$, and for the rest of this section, let $x_0 x_1 \dots x_q$ with:

$$x_0 = \{1, \dots, \lambda_0\}, x_1 = \{\lambda_0 + 1, \dots, \lambda_0 + \lambda_1\}, \dots, x_q = \left\{ 1 + \sum_{i=0}^{q-1} \lambda_i, \dots, \sum_{i=0}^q \lambda_i \right\}, \quad (5)$$

be the minimal (in the total order above) oriented q -face of shape λ .

Given a partition $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_q) \vdash n$, we can form the standard Young subgroup:

$$S_\lambda \cong S_{\lambda_0} \times \dots \times S_{\lambda_q},$$

where the factor S_{λ_i} is identified as the permutations of the set x_i in the minimal q -face (5). Let $N_\lambda = N_{S_n}(S_\lambda)$ be the normaliser in S_n of S_λ . If we rewrite $\lambda = (\mu_1^{a_1}, \dots, \mu_r^{a_r})$ by collecting together all the parts of λ with the same size, then there is a homomorphism:

$$N_\lambda \rightarrow N_\lambda / S_\lambda \cong S_{a_1} \times \dots \times S_{a_r},$$

with $g \mapsto \bar{g}$ for $g \in N_\lambda$. In particular, if $x_0 x_1 \dots x_q$ is the face (5) then $g \cdot x_0 x_1 \dots x_q = x_{\bar{g}0} x_{\bar{g}1} \dots x_{\bar{g}q}$. (In fact, we can identify N_λ explicitly inside S_n as a direct product of wreath products: for each $1 \leq i \leq r$, we have a factor with base group the direct product of the a_i copies of S_{λ_i} in S_λ and a symmetric group S_{a_i} on top permuting those copies.)

With this notation in hand, we can identify the homology groups as representations of S_n . First identify $\widetilde{H}_q(X(n), \mathbb{Z})$ with the free \mathbb{Z} -module spanned by the homology facets, where a homology facet σ has partition $\lambda(\sigma) \vdash n$ with no parts of size 1. For each such partition it is clear that the free \mathbb{Z} -module spanned by the facets of a fixed shape λ identifies with a $\mathbb{Z}S_n$ -submodule of $\widetilde{H}_q(X(n), \mathbb{Z})$. Denote this submodule by $V(\lambda)$.

Let $V = \mathbb{Z}[v]$ be a 1-dimensional \mathbb{Z} -module and define an N_λ action on V by $g \cdot v = \text{sgn}(\bar{g}) v$. We then have:

Theorem 2. For each partition $\lambda \vdash n$ there is an isomorphism of $\mathbb{Z}S_n$ -modules:

$$V(\lambda) \cong V \uparrow_{N_\lambda}^{S_n},$$

where the right-hand side is the N_λ module V induced up to S_n , and hence

$$\widetilde{H}_q(X(n), \mathbb{Z}) \cong \bigoplus_{\lambda} V \uparrow_{N_\lambda}^{S_n},$$

as $\mathbb{Z}S_n$ -modules, where the sum is over all λ of length $q + 1$ and with no parts of size 1 or n .

Proof. Let $T = \{t_0 = 1, t_1, \dots, t_m\} \in S_n$ be such that the $t_i \cdot x_0 x_1 \dots x_q$, ($0 \leq i \leq m$), are the oriented q -faces of shape λ , where $x_0 x_1 \dots x_q$ is the minimal oriented q -face from (5). In particular, T is a transversal for N_λ in S_n and the induced module $V \uparrow_{N_\lambda}^{S_n} = \mathbb{Z}[t \otimes v]_{t \in T}$. If $h \in S_n$, then for any $t_i \in T$ we have $ht_i = t_j g$ for some $t_j \in T$, $g \in N_\lambda$. The induced S_n -action is then given by $h \cdot (t_i \otimes v) = t_j \otimes g \cdot v = t_j \otimes \text{sgn}(\bar{g})v$. On the other hand $V(\lambda) = \mathbb{Z}[t \cdot x_0 x_1 \dots x_q]_{t \in T}$, and the action (1) is:

$$h \cdot (t_i \cdot x_0 x_1 \dots x_q) = t_j g \cdot x_0 x_1 \dots x_q = t_j \cdot x_{\bar{g}0} x_{\bar{g}1} \dots x_{\bar{g}q} = \text{sgn}(\bar{g}) t_j \cdot x_0 x_1 \dots x_q.$$

The map $t \cdot x_0 x_1 \dots x_q \mapsto t \otimes v$ is then the $\mathbb{Z}S_n$ -isomorphism we seek. \square

Remark 1. Suppose that λ is a partition with no repeated parts. Then $N_\lambda = S_\lambda$ and V is the trivial module. The induced module is thus the permutation module M^λ corresponding to the partition λ .

If we base change to a field of characteristic 0, then the decomposition of this permutation module into simple modules for S_n is well-known:

$$V \uparrow_{N_\lambda}^{S_n} \cong M^\lambda \cong \bigoplus_{\mu \succeq \lambda} \kappa_{\mu\lambda} \text{Sp}(\mu), \quad (6)$$

where $\text{Sp}(\mu)$ is the Specht module corresponding to the partition $\mu \vdash n$ and the sum is over all μ that dominate λ . The coefficient $\kappa_{\mu\lambda}$ is the number of semistandard tableaux of shape μ and content λ (a Kostka number — see for example [Sag01, §2.11]).

If we instead base change to an algebraically closed field of positive characteristic, then there is an analogous decomposition of the permutation module into a direct sum of Young modules, see [Don93, Lemmas (3.4) and (3.5)].

Remark 2. Building on the previous remark, in *all* cases we can identify the induced module $V \uparrow_{N_\lambda}^{S_n}$ as a submodule of the permutation module M^λ , as follows.

The module M^λ has a basis in bijection with tabloids of shape λ . Consider the (standard) tabloid T of shape λ filled left-to-right, top-to-bottom with the numbers $1, \dots, n$. The group S_λ fixes this tabloid, and the group N_λ acts as permutations of the rows. Let:

$$S = \sum_{g \in N_\lambda} \text{sgn}(\bar{g})(g \cdot T).$$

Let V^λ denote the cyclic $\mathbb{Z}S_n$ -submodule of M^λ generated by S — that is, V^λ is \mathbb{Z} -spanned by the elements $g \cdot S$ as g runs over the elements of S_n . We see easily that $V^\lambda \cong V \uparrow_{N_\lambda}^{S_n}$. (If g_1, \dots, g_r is a system of coset representatives for N_λ , then V^λ is the \mathbb{Z} -span of the $g_i \cdot S$, and S_n acts precisely as it does on the induced module.)

3. Representation stability of fibre-closed families

Even over a field k of characteristic 0, a complete decomposition of the kS_n -module $\widetilde{H}_q(X(n), k)$ into irreducibles seems to be out of reach: to analyse the structure of the induced modules appearing becomes very hard away from cases like Remark 1 above. For example, the discrete Hodge-theoretic techniques used in [DW02, FH98] to decompose into irreducibles the homology of the classical matching complexes M_n do not carry over to the complexes $X(n)$.

In lieu of such a decomposition we work in an asymptotic manner and analyse the overall “shape” of the representations in Theorem 2 (or, more precisely, the analogue of Theorem 2 over \mathbb{Q}). We start with two observations: firstly, the Betti numbers $\beta(n, q)$, for any fixed integer $q \geq 0$, of $X(n)$ can be expressed as a $\mathbb{Q}[n]$ -linear combination of exponential terms. Secondly, the Specht modules $\text{Sp}(\lambda)$ appearing in a decomposition of $\widetilde{H}_q(X_n; \mathbb{Q})$ have the property that the length of λ is at most $q + 1$. It turns out that these two observations, along with a few others that are less immediately deduced from Theorem 2, remain true across a more general collection of simplicial complexes than just the $X(n)$ of Sections 1-2.

3.1. Fibre-closed families of simplicial complexes

Let $\{X_n\}_{n \in \mathbb{N}}$ denote a family of simplicial complexes such that the vertices of X_n consist of certain subsets of $[n] = \{1, 2, \dots, n\}$. We call such a family *fibre closed* when for every surjective function $f : [a] \rightarrow [b]$, and every face $\sigma = \{x_0, \dots, x_q\}$ of X_b , the preimage $f^{-1}\sigma = \{f^{-1}x_0, \dots, f^{-1}x_q\}$ is a face of X_a . The condition of being fibre-closed implies that each of the simplicial complexes X_n carries an action by S_n .

The motivating examples for us are the matching complexes $X(n)$ of the complete hypergraphs on n vertices from Sections 1-2. Other examples include the order complex of the lattice of subsets of $[n]$ and the *anti-order* complex (whose faces comprise the anti-chains) of this lattice.

If $m \leq n$ are non-negative integers and $\lambda = (\lambda_1, \dots, \lambda_r) \vdash m$ is a partition of m , write $\lambda[n]$ for the padded partition $\lambda[n] := (n - m, \lambda_1, \dots, \lambda_r)$.

For any $d \geq 0$, we define $\mathbf{X}_d : \bigcup_{n \geq 0} S_n \rightarrow \mathbb{Q}$ (disjoint union) by setting $\mathbf{X}_d(\sigma)$ to be the number of d -cycles in the cycle decomposition of σ . A character polynomial is then defined to be any class function that can be expressed as a polynomial in the variables \mathbf{X}_d . In a similar vein, if $A = 1^{a_1} 2^{a_2} \dots$ and $\nu = 1^{m_1} 2^{m_2} \dots$ denote partitions of two (not necessarily distinct) integers, then we can define a class function:

$$\binom{\mathbf{X}}{\nu} A^{\mathbf{X}-\nu} := \prod_{d \geq 0} \binom{\mathbf{X}_d}{m_d} \left(\sum_{n|d} n a_n \right)^{\mathbf{X}_d - m_d}$$

These particular character functions were introduced in [Tos22, Theorem 1.11], and will be critical in the next theorem.

Here is the promised asymptotic result, where except for the last part, we work over \mathbb{Q} :

Theorem 3. *Let $\{X_n\}_{n \in \mathbb{N}}$ be a fibre-closed family of simplicial complexes, and let $q \geq 0$ be fixed.*

(i). *There exist polynomials $f_1, \dots, f_{2^{q+1}} \in \mathbb{Q}[x]$, such that the q -th Betti number $\beta(n, q)$ of X_n can be expressed as:*

$$\beta(n, q) = f_1(n) + f_2(n)2^n + f_3(n)3^n + \dots + f_{2^{q+1}}(n)(2^{q+1})^n,$$

for all $n \gg 0$.

- (ii). The $\mathbb{Q}S_n$ -module $\widetilde{H}_q(X_n, \mathbb{Q})$ decomposes as a sum of Specht modules $\mathrm{Sp}(\lambda)$ where the partition $\lambda \vdash n$ has length bounded by 2^{q+1} for all $n \gg 0$.
- (iii). For $n \gg 0$, the multiplicity of $\mathrm{Sp}(\lambda[n])$ appearing in $\widetilde{H}_q(X_n; \mathbb{Q})$ grows as a quasi-polynomial in n .
- (iv). The character of $\widetilde{H}_q(X_n, \mathbb{Q})$ can be written:

$$\chi_{\widetilde{H}_q(X_n, \mathbb{Q})} = \sum_{A, \nu} c(\nu, A) \binom{\mathbf{X}}{\nu} A^{\mathbf{X}-\nu},$$

where the constant $c(\nu, A) = 0$ whenever $|A| > 2^{q+1}$.

- (v). There exists an integer $e_{X,q}$, depending only on the family $\{X_n\}_{n \in \mathbb{N}}$ and q , such that the torsion part of the $\mathbb{Z}S_n$ -module $\widetilde{H}_q(X_n, \mathbb{Z})$ has exponent dividing $e_{X,q}$ for all $n \geq 0$.

Remark 3. The expression $\sum_{A, \nu} c(\nu, A) \binom{\mathbf{X}}{\nu} A^{\mathbf{X}-\nu}$ is a genuinely infinite sum, though at any given conjugacy class all but finitely many terms are 0. Note also that if one takes the conjugacy class of the identity of S_n , then:

$$\mathbf{X}_d(id_n) = \begin{cases} n & \text{if } d = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The character formula therefore simplifies to a $\mathbb{Q}[n]$ linear combination of exponential terms, recovering the first part of our theorem.

We defer the proof of Theorem 3 to the next section and explore the ramifications of the theorem in some examples.

If $X_n = X(n) = M(\Gamma)$, the matching complex of the complete hypergraph on $n \geq 2$ vertices, then parts (i) and (ii) of Theorem 3 are the observations made in the preamble to this section. Parts (iii) and (iv) are less obvious, although an alternative argument illustrates them in the case of the trivial representation, where Frobenius reciprocity gives:

$$\mathrm{Hom}_{S_n}(1_n, V \uparrow_{N_\lambda}^{S_n}) \cong \mathrm{Hom}_{N_\lambda}(1_n \downarrow_{N_\lambda}^{S_n}, V),$$

with 1_n the trivial S_n -representation. As $1_n \downarrow_{N_\lambda}^{S_n}$ is trivial and V is irreducible, the right-hand side has dimension 1 iff V is the trivial module, and this in turn happens exactly when λ has no repeated parts. Thus, the multiplicity of the trivial representation in $V \uparrow_{N_\lambda}^{S_n}$ is the number of partitions λ of n with $q+1$ distinct parts, none of which have size 1 or n . This quantity is thus a quasi-polynomial when $n \gg 0$. This fact can also be deduced from classical generating function arguments on collections of partitions.

Finessing this example a little further, we can produce examples where the Betti number growth of the family $\{X_n\}_{n \in \mathbb{N}}$ hits a variety of different types within the description given by Theorem 3(i). For example, let X_n be just the 1-skeleton of the complete hypergraph matching complex $X(n)$. As this graph is connected we have $H_0 \cong \mathbb{Z}$ and so an Euler characteristic calculation gives $1 - \mathrm{rk} H_1 = |X_0| - |X_1|$, recalling that X_q is the set of q -faces. Thus:

$$\mathrm{rk} H_1 = \frac{1}{2}3^n - 2 \cdot 2^n + \frac{7}{2},$$

as $|X_0| = 2^n - 2$ and $|X_1| = \frac{3^n - (2^n + 2^n - 1)}{2}$. If we instead take X_n to be the complete graph on the vertex set of $X(n)$ —i.e. take the same vertices as $X(n)$ but now join any two by an edge—then the rank of H_1 will have a 4^n term, thus meeting the worst-case scenario of Theorem 3(i).

As one final example of the various growth rates, consider our running example of the matching complex of the complete hypergraph. In this case, our theorem on the homologies of these spaces tells us that the rank of H_1 is equal to the number of partitions of $[n]$ into two proper non-singleton blocks. In particular:

$$\text{rk } H_1 = \frac{2^n - 2 - 2n}{2} = 2^{n-1} - (n + 1).$$

This example illustrates the genuine need for the polynomial coefficients when describing the Betti numbers of matching complexes, and fibre-closed families more generally.

Finally, we illustrate that it is even possible for a fibre-closed family to exhibit torsion in its homology groups. We define a fibre-closed family $\{X_n\}_{n \in \mathbb{N}}$ as follows. For $n \leq 7$ let X_n be the empty complex and let X_7 be the matching complex of the complete graph K_7 . Thus the vertices of X_7 consist of all pairs $\{i, j\}$, and a collection of vertices forms a face if and only if none of the pairs share an element. For $n > 7$, define X_n inductively to be the smallest complex containing all possible preimages of faces of X_{n-1} under surjections $f : [n] \rightarrow [n-1]$. By [BLVŽ94, SW07] we have $H_1(X_7) \cong \mathbb{Z}/3\mathbb{Z}$. The final part of Theorem 3 tells us that in any fibre closed family of complexes, and for any fixed $q \geq 0$, the q -th homology group $H_q(X_n)$ has torsion that is bounded independently of n .

3.2. Representation stability of fibre-closed families

The proof of Theorem 3 appeals to categorical representation theory, and in particular the theory of FS^{op} -modules, as found in [PR21, PY17, SS17, Tos22, Tos21].

Write FS for the category whose objects are the sets $[n] = \{1, 2, \dots, n\}$, for $n \geq 0$, and whose morphisms are all possible surjective maps. An FS^{op} module over a ring R is a functor:

$$V : \text{FS}^{\text{op}} \rightarrow {}_R\mathbf{Mod},$$

to the category of (left) R -modules. In other parts of the literature FS^{op} modules are known variously as presheaves, combinatorial sheaves and quiver representations.

We will abbreviate the value of the functor V on the object $[n]$ to just V_n . As the surjective maps from $[n]$ to $[n]$ are precisely the permutations of $[n]$, an FS^{op} -module can also be thought of as a collection of RS_n -modules $\{V_n\}_{n \geq 0}$, such that for any surjection $[a] \rightarrow [b]$, there is an induced map of R -modules $V_b \rightarrow V_a$ that is compatible with the symmetric group actions.

The key idea of this section is that the homology groups, in fixed homological degree, of any fibre-closed family of simplicial complexes carry the structure of an FS^{op} -module. Indeed, given any surjection of finite sets, the action of taking fibres gives one a means for mapping faces of one member of the family to faces of another, thereby inducing a continuous map between the geometric realization of the complexes. For instance, returning to our running example of the matching complex of the complete hypergraph, we can actually determine the precise FS^{op} -module structure. Note that the action of a given surjection is precisely the most natural extension of the action (1). Namely, given a surjection $f : [m] \twoheadrightarrow [n]$ and a facet of X_n realized as a partition of $[n]$ into proper non-singleton blocks, take the preimages of each individual block. The result will once again be a partition of $[m]$ into proper non-singleton blocks. Having decided on an ordering on the blocks, one then adjusts by a sign, if necessary.

The category whose objects are the FS^{op} -modules and whose morphisms are natural transformations of functors is well known to be Abelian. We say that an FS^{op} -module V is finitely

generated if there is a finite collection of elements $\{v_1, \dots, v_r\} \subseteq \bigcup_n V_n$ which no proper FS^{op}-submodule of V contains. Equivalently, V is finitely generated iff each V_n is finitely generated as an R -module and there exists an integer $d \geq 0$ such that for all $n > d$, the module V_n is generated by the images of the module V_{n-1} under the maps induced by surjections $[n] \rightarrow [n-1]$. In this case, we say that V is finitely generated in degrees $\leq d$. More generally, we say that an FS^{op}-module is d -small if it is a subquotient of an FS^{op}-module that is generated in degrees $\leq d$.

We now summarise the results on FS^{op}-modules that we will need. Let R denote a Noetherian ring, and let V be a finitely generated FS^{op}-module over R that is d -small. Then by [SS17, Corollary 8.1.3] all submodules of V are finitely generated. If $R = k$ is a field then there exist polynomials $f_1, \dots, f_d \in \mathbb{Q}[x]$ such that for all $n \gg 0$:

$$\dim V_n = f_1(n) + f_2(n)2^n + \dots + f_d(n)d^n,$$

— see [SS17, Corollary 8.1.4].

If k is a field of characteristic 0, then [PY17, Theorem 4.1] gives that for $n \gg 0$, all Specht modules $\mathrm{Sp}(\lambda)$ appearing as summands of V_n are associated to partitions λ of length $\leq d$. Moreover, if λ is a partition of some integer m , then for $n \gg 0$, the multiplicity in V_n of $\mathrm{Sp}(\lambda[n])$ is a quasi-polynomial in n ; see [Tos21, Theorem 1.17]. Finally, by [Tos21, Theorem 1.11] the character of V_n can be expressed in terms of the character polynomials \mathbf{X}_d as shown above.

Our final ingredient is the following:

Proposition 2. *If $R = \mathbb{Z}$, then there exists an integer $e_V \geq 0$ such that for any n the exponent of torsion appearing in V_n divides e_V .*

Proof. Let V be a finitely generated FS^{op}-module over \mathbb{Z} . We observe that if $v \in V_n$ is any torsion element, then the images of this element under any map induced by surjections are also torsion. It follows that we may define a new FS^{op}-module by setting W_n to be the collection of all torsion elements of V_n for each n . Because W is a submodule of V by construction, it must be finitely generated by the first result quoted above. The integer d_W can then be taken to be the least common multiple of the orders of the (finitely many) generators of W . \square

We are now ready to proceed with the:

Proof (of Theorem 3). Let $\{X_n\}_{n \geq 0}$ be a fibre closed family of simplicial complexes. For any $a > b \geq 0$, and any surjection $f : [a] \rightarrow [b]$, we have a map $(X_b)_0 \rightarrow (X_a)_0$ by sending a subset $x \subseteq [b]$ to its preimage under f . The definition of fibre-closed tells us that this map lifts to a map between the q -faces of X_a and X_b for all q . In particular, the surjection f induces a continuous map between the complexes $f^* : X_b \rightarrow X_a$. This map will descend to a map on homology, which gives us the desired FS^{op}-structure. It only remains to show that these FS^{op}-modules are finitely generated.

Define an FS^{op}-module M over a Noetherian ring R by setting M_n to be the R -linearization of the power set of $[n]$. This FS^{op}-module is finitely generated in degrees ≤ 2 . According to [PR21, Lemma 2.1], the tensor powers $M^{\otimes(q+1)}$ — defined in the obvious point-wise way — are finitely generated in degrees $\leq 2^{q+1}$. It follows that the exterior power $\bigwedge^q M$ is also generated in degrees $\leq 2^{q+1}$.

Next, for $q \geq 0$, define C^q to be the R -module of simplicial q -cochains of X_n — see Section 1.2. For any q the FS^{op}-module C^q is seen to be a submodule of $\bigwedge^{q+1} M$. By the Noetherianity of FS^{op}-modules, we conclude that C^q is 2^{q+1} -small, and therefore the same can be said about the homology groups $\widetilde{H}_q(X_\bullet; R)$. This concludes the proof. \square

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