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Discrete Integrable Principal Chiral Field Model and Its Involutive Reduction

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ABSTRACT

We discuss an integrable discretization of the principal chiral field models equations and its involutive reduction. We present a Darboux transformation and general construction of soliton solutions for these discrete equations.

1 | Introduction

Given a differential equation of mathematical physics that is integrable, one can search for a difference (discrete) equation such that, first, in a continuum limit goes to the differential one, second, is integrable as well. This article can be viewed as the first step in a larger program of integrable discretizations of chiral field model equations (or σ -models equations). Therefore, before we summarize the results of this article we would like to glance at the problem from a broad perspective and to present our motivations first. Namely, by the nonautonomous (or nonisospectral) extension of principal $GL(N)$ chiral field model equations, we understand the equation

$$(r\Phi_{,v}\Phi^{-1})_{,u} + (r\Phi_{,u}\Phi^{-1})_{,v} = 0, \quad r_{,uv} = 0, \quad (1)$$

where Φ is a function of two independent u and v variables (in the hyperbolic case u and v are real variables while in the elliptic case u and v are treated as complex variables such that one variable is complex conjugation of the second) that takes values in invertible square matrices of given size N with complex entries, while r is a scalar (in general complex valued) function of the independent variables.

Dedicated to the memory of David Kaup.

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The chiral models (or nonlinear σ models) appeared in theoretical physics in the 1960s in two classical articles of particle physics [1, 2]. The history of these models in mathematics is even longer for in the beginning of the previous century [3], Bianchi constructed the Bäcklund transformation for the system

$$\mathbf{N}_{,uv} = f\mathbf{N}, \quad \mathbf{N} \cdot \mathbf{N} = U(u) + V(v), \quad (2)$$

where $\mathbf{N}$ is a vector valued function of variables u and v , dot denotes (pseudo)-scalar product, and $U(u)$ and $V(v)$ are prescribed functions of indicated variable. Nowadays, Bianchi system (2) is referred to as $O(p, q)$ σ -model ((p, q) is the signature of the dot product). In the case $p = 3, q = 0$, the Bianchi system is equivalent to an involutive reduction of $U(2)$ version of model (1) and (in the case $p = 2, q = 1$) to hyperbolic Ernst equation of general relativity. Indeed, it turns out the following two constraints are compatible with Equations (1)

1. involutive constraint

$$\Phi^2 = \mathbb{I} \quad (3)$$

which gives after the substitution $\Phi = \mathbb{I} - 2P$ idempotent constraint on P , that is, $P^2 = P$.

2. (pseudo)-unitary constraint

$$\Phi^\dagger \eta \Phi \eta = \mathbb{I} \quad \eta := \text{diag} \left(\underbrace{1, \dots, 1}_i, \underbrace{-1, \dots, -1}_j \right). \quad (4)$$

In the case $N = 2$, the constraints give

$$\Phi = \frac{1}{\sqrt{r}} \begin{bmatrix} n_1 & n_2 + in_3 \\ \epsilon(n_2 - in_3) & -n_1 \end{bmatrix}, \quad n_1^2 + \epsilon(n_2^2 + n_3^2) = r, \quad \epsilon = \pm 1$$

and Equation (1) becomes Equation (2) for $\mathbf{N} := [n_1, n_2, n_3]$, where $\mathbf{N} \cdot \mathbf{N} := n_1^2 + \epsilon(n_2^2 + n_3^2) = r$. After the stereographical projection $n_2 + in_3 = \sqrt{r} \frac{2\xi}{1+\epsilon\bar{\xi}\xi}$, $n_1 = \sqrt{r} \frac{1-\epsilon\bar{\xi}\xi}{1+\epsilon\bar{\xi}\xi}$, we come to the equation

$$(\bar{\xi}\xi + \epsilon) \left(\xi_{,uv} + \frac{r_{,v}}{2r} \xi_{,u} + \frac{r_{,u}}{2r} \xi_{,v} \right) = 2\bar{\xi}\xi_{,u} \xi_{,v} \quad (5)$$

$$r_{,uv} = 0$$

which is (in the elliptic case and $\epsilon = -1$) the Ernst equation, that is, the equation the problem of finding vacuum solutions of Einstein equations outside axisymmetric and stationary distribution of matter can be reduced to [4]. A bit more tedious calculation shows that in the case $N = 3$ and $U(1, 2)$ and involutive reduction leads to the Ernst Einstein–Maxwell equations [5]

$$\begin{aligned} (\bar{\xi}\xi - \bar{\chi}\chi - 1) \left(\xi_{,uv} + \frac{r_{,v}}{2r} \xi_{,u} + \frac{r_{,u}}{2r} \xi_{,v} \right) &= \xi_{,u} (\bar{\xi}\xi_{,v} - \bar{\chi}\chi_{,v}) + \xi_{,v} (\bar{\xi}\xi_{,u} - \bar{\chi}\chi_{,u}) \\ (\bar{\xi}\xi - \bar{\chi}\chi - 1) \left(\chi_{,uv} + \frac{r_{,v}}{2r} \chi_{,u} + \frac{r_{,u}}{2r} \chi_{,v} \right) &= \chi_{,u} (\bar{\xi}\xi_{,v} - \bar{\chi}\chi_{,v}) + \chi_{,v} (\bar{\xi}\xi_{,u} - \bar{\chi}\chi_{,u}) \end{aligned} \quad (6)$$

$$r_{,uv} = 0.$$

Electromagnetic field outside the distribution of matter manifests in this description as an additional complex variable χ . The essential fact from the point of view of this article is that the differential equations presented above are integrable. There is a large number of publications on the topic. We confine ourselves to listing the most important ones

- In 1905 [3], Bianchi published Bäcklund transformation for the system (2).
- In 1972, Geroch found an infinite-dimensional group (the Geroch group K) equation (5) is covariant under [6]. Later on, Kinnersley extended Geroch's result to electrovacuum equations (6) finding a covariance group (the Kinnersley group K') for Equation (6) [7].
- In 1978, the Ernst equations became a part of “soliton” theory [8–13].

- In 1978, Zakharov and Mikhailov established the general scheme for the integration of chiral field equations and their reductions [14–16].

There are partial results in discretization of system (1) and its reductions, we are going to recall them now.

- To the best of our knowledge, an integrable discretization of autonomous (or isospectral) ($r = \text{const}$) version of Bianchi system (2) was given by Orfanidis [17] and reads

$$T_1 T_2 \vec{n} + \vec{n} = F(T_1 \vec{n} + T_2 \vec{n}) \quad \vec{n} \cdot \vec{n} = 1, \quad (7)$$

where, this time, the dependent variables are $\vec{n} : \mathbb{Z}^2 \mapsto T\mathbb{E}^n$ (where $T\mathbb{E}^n$ denotes tangent space of (pseudo)-Euclidean space), $F : \mathbb{Z}^2 \mapsto \mathbb{R}$, is a scalar function that can be expressed in terms of $\vec{n}$ as follows $F = \frac{2\vec{n} \cdot (T_1 \vec{n} + T_2 \vec{n})}{(T_1 \vec{n} + T_2 \vec{n}) \cdot (T_1 \vec{n} + T_2 \vec{n})}$. Equation (7) can be written in the case $n = 3$ as

$$\Delta_1(\Delta_2(\Phi)\Phi) + \Delta_2(\Delta_1(\Phi)\Phi) = 0, \quad \Phi^\dagger \Phi = \mathbb{I}, \quad \Phi^2 = \mathbb{I}, \quad (8)$$

where Φ is 2×2 matrix such that

$$\Phi := \begin{bmatrix} n_0 & n_1 + in_2 \\ n_1 - in_2 & -n_0 \end{bmatrix}, \quad \vec{n} := (n_0, n_1, n_2).$$

- As for discretization of the nonisospectral (nonautonomous) case, only two results are known so far. The first one is the system introduced by Schief [18]

$$\begin{aligned} T_1 T_2 \vec{N} + \vec{N} &= F(T_1 \vec{N} + T_2 \vec{N}), \\ \Delta_1 \Delta_2 \left[(T_1 T_2 \vec{N} + \vec{N}) \cdot (T_1 \vec{N} + T_2 \vec{N}) \right] &= 0 \end{aligned} \quad (9)$$

as a permutability theorem of Calapso equation. It has been demonstrated that Schief's system can be regarded as discretization of Bianchi system (2) [19–21] but not the only one. The second result is that the system (9) is a potential version of the system [20]

$$\begin{aligned} \frac{T_1 T_2 \vec{\mathcal{N}}}{T_1 T_2 \sqrt{F}} + \frac{\vec{\mathcal{N}}}{\sqrt{F}} &= T_1 \sqrt{F} T_1 \vec{\mathcal{N}} + T_2 \sqrt{F} T_2 \vec{\mathcal{N}}, \\ \Delta_1 \Delta_2 (\vec{\mathcal{N}} \cdot \vec{\mathcal{N}}) &= 0. \end{aligned} \quad (10)$$

The solution spaces of (9) and (10) are related by

$$\vec{\mathcal{N}} = \sqrt{F}(T_1 \vec{N} + T_2 \vec{N}).$$

- Autonomous discrete principal $GL(n)$ σ -models has been introduced by Cherednik [22] and read

$$\Delta_1(\Delta_2(\Phi)\Phi^{-1}) + \Delta_2(\Delta_1(\Phi)\Phi^{-1}) = 0. \quad (11)$$

A Lax pair for the equation and discussion on some of its integrable features can be found in [23, 24].

In the present article, we add to the list the system

$$\Delta_1(\Delta_2(\Phi)\Phi) + \Delta_2(\Delta_1(\Phi)\Phi) = 0, \quad \Phi^2 = \mathbb{I}, \quad (12)$$

which is a discrete integrable version of the autonomous ($r = 1$), hyperbolic version of Equation (1), subjected to involutive constraint. We also discuss Equation (8) (i.e., system (12) subjected to unitary constraint) in the case of 2×2 matrices, and show that system matrices (12) of larger size than 2, in contrast to the continuous case, do not admit unitary reduction.

2 | The Classical Principal Chiral Model and Involutive Reduction

We study one possible integrable discretization of the classical principal chiral field model:

$$(\Phi_{,u} \Phi^{-1})_{,v} + (\Phi_{,v} \Phi^{-1})_{,u} = 0, \quad \Phi \in GL(N, \mathbb{C}), \quad (13)$$

where Φ is $GL(N, \mathbb{C})$ -valued function of two independent real variables u and v .

The integrability of the principal chiral field model was established in [14], where Equation (13) was represented as the compatibility condition of two linear problems:

$$\Psi_{,u} = \frac{\mathcal{A}}{\lambda - 1} \Psi, \quad \Psi_{,v} = \frac{\mathcal{B}}{\lambda + 1} \Psi. \quad (14)$$

Here, λ is a spectral parameter.

The compatibility conditions of (14) read

$$\mathcal{A}_{,v} - \mathcal{B}_{,u} = 0, \quad \mathcal{A}_{,v} + \mathcal{B}_{,u} + \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A} = 0,$$

and guarantee the existence of a common fundamental solution $\Psi(u, v; \lambda)$. From the second equation, we infer that there exists potential Φ such that

$$\mathcal{A} = -\Phi_{,u} \Phi^{-1}, \quad \mathcal{B} = \Phi_{,v} \Phi^{-1}.$$

Substituting these expressions into the first equation yields Equation (13). The potential Φ can be identified with $\Psi(u, v; 0)$. Equation (13) can naturally be reduced to any complex or real classical Lie group [25], assuming that the variables $\mathcal{A}, \mathcal{B}$ are elements of the corresponding Lie algebra. In [14], it was shown that the principal chiral field model (13) admits a remarkable (see the introduction) integrable reduction:

$$\Phi^2 = \mathbb{I}. \quad (15)$$

We refer to this as the involutive reduction or projective (idempotent) reduction, since condition (15) implies the representation $\Phi = \mathbb{I} - 2P$, where P is a projector ($P^2 = P$). In terms of P , Equation (13) takes the elegant form [14]:

$$[P_{uv}, P] = 0.$$

In this article, we study an integrable discretization

$$\Delta_1((\Delta_2 \Phi) \Phi^{-1}) + \Delta_2((\Delta_1 \Phi) \Phi^{-1}) = 0 \quad (16)$$

of the principle chiral model (13), where Φ is $GL(N, \mathbb{C})$ -valued function of two discrete variables $(m_1, m_2) \in \mathbb{Z}^2$ and Δ_i denotes forward difference operator in variable m_i , that is, $\Delta_1 f(m_1, m_2) := f(m_1 + 1, m_2) - f(m_1, m_2)$, $\Delta_2 f(m_1, m_2) := f(m_1, m_2 + 1) - f(m_1, m_2)$, for any function $f : \mathbb{Z}^2 \mapsto \mathbb{C}$. In Section 3, we construct explicitly two poles Darboux-Bäcklund transformations for Equation (13). In Section 4, we show that the involutive reduction $\Phi^2 = \mathbb{I}$ is compatible with the discrete system (13) (a constraint is compatible with an equation iff its imposition on initial conditions implies that due to the equation the constraint holds at every point of the domain), leading to an additional symmetry of the Lax representation and the corresponding fundamental solution. In Section 4.1, we construct a two-pole Darboux matrix and the corresponding soliton solution of the reduced system

$$\Delta_1((\Delta_2 \Phi) \Phi) + \Delta_2((\Delta_1 \Phi) \Phi) = 0, \quad \Phi^2 = \mathbb{I}, \quad (17)$$

where $\Phi \in GL(N, \mathbb{C})$. This solution is parameterized by two points on the complex Grassmannian $G_{k,N}$ and two complex parameters λ_0 and μ_0 (see Corollary 2). In the case $N = 2$, solutions for the system (17) are presented in Section 4.2. This particular case ($N = 2$) admits the unitary reduction $\Phi^{-1} = \Phi^\dagger = \Phi$ and solutions for the system (17) subjected unitary constraint $\Phi^\dagger \Phi = \mathbb{I}$ are presented in Section 5. In this case, the solution obtained is parameterized by one point on the CP^1 and one complex parameter λ_0 .

3 | The $GL(n)$ Principal σ -Models in the Discrete Case

We start our considerations from the pair of linear equations (the so-called Lax pair) on a function $\Psi(m_1, m_2; \lambda)$ [23] (two independent variables are integers and are omitted to make formulas shorter)

$$T_1 \Psi(\lambda) = \left(\mathbb{I} + \frac{1}{\lambda - 1} \mathbb{A} \right) \Psi(\lambda) \quad T_2 \Psi(\lambda) = \left(\mathbb{I} + \frac{1}{\lambda + 1} \mathbb{B} \right) \Psi(\lambda), \quad (18)$$

where λ is a complex valued parameter referred to as spectral parameter and in general can depend on m_1 and m_2 , however, in the present article, we confine ourselves to the isospectral case, that is, to the case when λ does not depend on the independent variables. We assume that the square matrices $\mathbb{A}$ and $\mathbb{B}$ do not depend on λ and do depend on m_1 and m_2 , $\mathbb{I}$ denotes the unit matrix and we recall that T_j ($j = 1, 2$) denotes forward shift operators: $T_1 \Psi(m_1, m_2) := \Psi(m_1 + 1, m_2)$, $T_2 \Psi(m_1, m_2) := \Psi(m_1, m_2 + 1)$. The function Ψ is a fundamental matrix solution of the system (18) so it takes values in square matrices. Shortly, we have

$$T_1 \Psi(\lambda) = U(\lambda) \Psi(\lambda), \quad T_2 \Psi(\lambda) = V(\lambda) \Psi(\lambda),$$

where U and V are characterized as having simple poles at $\lambda = 1$, and $\lambda = -1$, respectively, and

$$\lim_{\lambda \rightarrow \infty} U(\lambda) = \mathbb{I}, \quad \lim_{\lambda \rightarrow \infty} V(\lambda) = \mathbb{I}. \quad (19)$$

We want the compatibility conditions of system (18)

$$(T_2 \mathbb{A} - \mathbb{A} - T_1 \mathbb{B} + \mathbb{B})\lambda + (T_2 \mathbb{A} - \mathbb{I})(\mathbb{B} + \mathbb{I}) - (T_1 \mathbb{B} + \mathbb{I})(\mathbb{A} - \mathbb{I}) = 0 \quad (20)$$

to be valid for all $\lambda \in \bar{\mathbb{C}} \setminus \{1, -1\}$, so we arrive at

$$\Delta_2 \mathbb{A} = \Delta_1 \mathbb{B} \quad (21)$$

$$[T_2(\mathbb{A} - \mathbb{I})](\mathbb{B} + \mathbb{I}) = [T_1(\mathbb{B} + \mathbb{I})](\mathbb{A} - \mathbb{I}). \quad (22)$$

Due to Equation (22), we can introduce¹ “potential” Φ

$$\mathbb{A} - \mathbb{I} = -(T_1 \Phi) \Phi^{-1} \quad \mathbb{B} + \mathbb{I} = (T_2 \Phi) \Phi^{-1}. \quad (23)$$

Thus,

$$\mathbb{A} = -(\Delta_1 \Phi) \Phi^{-1}, \quad \mathbb{B} = (\Delta_2 \Phi) \Phi^{-1}. \quad (24)$$

The potential Φ can be defined as

$$\Phi = \Psi(0). \quad (25)$$

Substitution (24) to Equation (21) gives

$$T_1 T_2 \Phi (T_1 \Phi^{-1} + T_2 \Phi^{-1}) = (T_1 \Phi + T_2 \Phi) \Phi^{-1} \quad (26)$$

which is nothing but Equation (16). To prove the integrability of (26), we will construct its Darboux–Bäcklund transformation.

3.1 | Construction of a Darboux Matrix

We consider the Darboux transformation

$$\tilde{\Psi}(\lambda) = \mathbb{D}(\lambda) \Psi(\lambda) \quad (27)$$

which leads to the Darboux–Bäcklund transformation

$$\tilde{U}(\lambda) = (T_1 \mathbb{D}(\lambda))U(\lambda)\mathbb{D}(\lambda)^{-1}, \quad \tilde{V}(\lambda) = (T_2 \mathbb{D}(\lambda))V(\lambda)\mathbb{D}(\lambda)^{-1}, \quad (28)$$

where $\mathbb{D}$ is usually referred to as Darboux matrix and depends on m_1, m_2 , and λ .

The normalization $\mathbb{N}$ of the Darboux matrix is defined by

$$\mathbb{N} := \lim_{\lambda \rightarrow \infty} \mathbb{D}(\lambda), \quad (29)$$

we tacitly assume that the limit exists. First, we are going to check whether condition (19) is preserved by the transformation. Taking into account (19) and (29), we have

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \tilde{U}(\lambda) &= (T_1 \mathbb{N})(\lim_{\lambda \rightarrow \infty} U(\lambda))\mathbb{N}^{-1} = (T_1 \mathbb{N})\mathbb{N}^{-1}, \\ \lim_{\lambda \rightarrow \infty} \tilde{V}(\lambda) &= (T_2 \mathbb{N})(\lim_{\lambda \rightarrow \infty} V(\lambda))\mathbb{N}^{-1} = (T_2 \mathbb{N})\mathbb{N}^{-1}. \end{aligned} \quad (30)$$

These limits are equal to $\mathbb{I}$ if and only if $T_1 \mathbb{N} = T_2 \mathbb{N} = \mathbb{N}$.

Corollary 1. *If $\mathbb{N} = \text{const}$, then the Darboux transformation (28) preserves the constraints (19).*

We assume throughout the article, $\mathbb{N} = \mathbb{I}$.

There is another, more fundamental, constraint to be preserved by the transformation (28), namely, the dependence of U, V on λ (the divisors of poles of U and V). We adapt the classical results on N -pole Darboux matrix for continuous $GL(n)$ principal σ -model to the discrete $GL(n)$ principal σ -model (16), that is, we confine ourselves to the following form of $\mathbb{D}$ and its inverse $\mathbb{D}^{-1}$:

$$\mathbb{D} = \mathbb{I} + \sum_{i=1}^N \frac{\lambda_i P_i}{\lambda - \lambda_i}, \quad \mathbb{D}^{-1} = \mathbb{I} + \sum_{j=1}^N \frac{\mu_j F_j}{\lambda - \mu_j}, \quad \forall i, j : \lambda_i \neq \mu_j, \quad \lambda_i, \mu_i \neq 1, \quad (31)$$

where the rank of all matrices P_j, F_j is the same and equals k , that is,

$$P_j = |p_{1j}\rangle\langle p_{2j}|, \quad F_j = |f_{1j}\rangle\langle f_{2j}|, \quad (32)$$

where “bras” are $k \times n$ matrices of rank k while “kets” are $n \times k$ matrices of rank k .

The conditions $\mathbb{D}\mathbb{D}^{-1} = \mathbb{D}^{-1}\mathbb{D} = \mathbb{I}$ impose the following constraints:

$$\begin{aligned} \langle p_{2i} | \mathbb{D}^{-1}(\lambda_i) &= 0, & \mathbb{D}(\mu_i) | f_{1i} \rangle &= 0, \\ \mathbb{D}^{-1}(\lambda_i) | p_{1i} \rangle &= 0, & \langle f_{2i} | \mathbb{D}(\mu_i) &= 0, \end{aligned} \quad (33)$$

which allow us to express $|p_{1i}\rangle$ and $\langle f_{2i}|$ in terms of $|f_{1i}\rangle$ and $\langle p_{2i}|$.

To ensure that the transformation (28) preserves divisors of poles of U and V , we demand that the residua at λ_i, μ_i ($i = 1, \dots, N$) of the right-hand sides of (28) vanish:

$$\begin{aligned} T_1(P_i)U(\lambda_i)\mathbb{D}^{-1}(\lambda_i) &= 0, \\ T_2(P_i)V(\lambda_i)\mathbb{D}^{-1}(\lambda_i) &= 0, \\ T_1(\mathbb{D}(\mu_i))U(\mu_i)F_i &= 0, \\ T_2(\mathbb{D}(\mu_i))V(\mu_i)F_i &= 0. \end{aligned} \quad (34)$$

The following theorem takes place in full analogy with the continuous case [14, 26, 27]:

Theorem 1. *If $\langle p_{2i} | = \langle p_{0i} | \Psi^{-1}(\lambda_i)$ and $|f_{1i}\rangle = \Psi(\mu_i) |f_{0i}\rangle$, where $\Psi(\lambda)$ is a solution of the linear problem (18) and $\langle p_{0i} |$ and $|f_{0i}\rangle$ are, respectively, constant $k \times n$ matrices of rank k and $n \times k$ matrices of rank k and $|p_{1i}\rangle$ and $\langle f_{2i}|$ are given via formulas (33), then Equations (34) are satisfied.*

Proof. The proof is straightforward. It is enough to substitute $\langle p_{2i}| = \langle p_{0i}| \Psi^{-1}(\lambda_i)$ and $|f_{1i}\rangle = \Psi(\mu_i) |f_{0i}\rangle$ to (34) and to take into account $T_1\Psi(\lambda) = U(\lambda)\Psi(\lambda)$, $T_2\Psi(\lambda) = V(\lambda)\Psi(\lambda)$ evaluated at $\lambda = \lambda_i$ or $\lambda = \mu_i$. The resulting expression vanishes by virtue of (33), for example, for the first equation of (34) we have

$$\begin{aligned} T_1(P_i)U(\lambda_i)\mathbb{D}^{-1}(\lambda_i) &= (T_1|p_{1i}\rangle)(T_1\langle p_{2i}|)U(\lambda_i)\mathbb{D}^{-1}(\lambda_i) = \\ &= (T_1|p_{1i}\rangle)\langle p_{0i}|(T_1\Psi^{-1}(\lambda_i))U(\lambda_i)\mathbb{D}^{-1}(\lambda_i) = (T_1|p_{1i}\rangle)\langle p_{0i}|\Psi^{-1}(\lambda_i)\mathbb{D}^{-1}(\lambda_i) = \\ &= (T_1|p_{1i}\rangle)\langle p_{2i}|\mathbb{D}^{-1}(\lambda_i) = 0. \end{aligned}$$

□

Remark 1. The factors $|p_{1i}\rangle$, $\langle p_{2i}|$, $|f_{1i}\rangle$, and $\langle f_{2i}|$ in (32) are not uniquely defined. Indeed, a transformation

$$|p_{1i}\rangle \mapsto |p_{1i}\rangle W_i^{-1}, \quad \langle p_{2i}| \mapsto W_i \langle p_{2i}|, \quad |f_{1i}\rangle \mapsto |f_{1i}\rangle \hat{W}_i^{-1}, \quad \langle f_{2i}| \mapsto \hat{W}_i \langle f_{2i}|,$$

where $W_i, \hat{W}_i$ are any invertible $k \times k$ matrices, does not change the matrices P_i, F_i . Now, it follows from (33) and Theorem 1 that the dressing Darboux matrix (31) is parameterized by $2N$ points $\langle p_{0i}|, |f_{0i}\rangle$ on the Grassmanian $G_{k,N}$ and by the set of $2N$ complex parameters $\lambda_i, \mu_i, i = 1, \dots, N$.

Remark 2. In (32), we assumed that all poles of the Darboux matrices in the spectral parameter λ are simple, that the number of poles in $\mathbb{D}$ and $\mathbb{D}^{-1}$ coincide, and that the ranks of matrices P_i, F_i are the same and do not depend on i . A more general Darboux dressing matrix can be easily constructed without the above assumptions and in the same way but it would make the paper less readable.

3.2 | Darboux Matrix With Two Poles

Since in the case of reduction $\Phi^2 = \mathbb{I}$ considered in the next section, it is sufficient to consider the Darboux matrix with two poles; we discuss the case $N = 2$ in detail. In the case $N = 2$ Darboux transformation, the Darboux matrix and its inverse take, respectively, forms

$$\tilde{\Psi}(\lambda; \lambda_1, \lambda_2) = \mathbb{D}(\lambda; \lambda_1, \lambda_2)\Psi(\lambda), \quad (35)$$

$$\mathbb{D}(\lambda) := \mathbb{I} + \frac{\lambda_1 \mathbb{P}}{\lambda - \lambda_1} + \frac{\lambda_2 \mathbb{Q}}{\lambda - \lambda_2}, \quad (36)$$

$$\mathbb{D}^{-1}(\lambda) := \mathbb{I} + \frac{\mu_1 \mathbb{F}}{\lambda - \mu_1} + \frac{\mu_2 \mathbb{G}}{\lambda - \mu_2}. \quad (37)$$

Taking into account $\mathbb{D}\mathbb{D}^{-1} = \mathbb{D}^{-1}\mathbb{D} = \mathbb{I}$, we arrive at the equations

$$\begin{aligned} \mathbb{P} \left(\mathbb{I} + \frac{\mu_1}{\lambda_1 - \mu_1} \mathbb{F} + \frac{\mu_2}{\lambda_1 - \mu_2} \mathbb{G} \right) &= 0 \\ \mathbb{Q} \left(\mathbb{I} + \frac{\mu_1}{\lambda_2 - \mu_1} \mathbb{F} + \frac{\mu_2}{\lambda_2 - \mu_2} \mathbb{G} \right) &= 0 \\ \left(\mathbb{I} + \frac{\lambda_1}{\mu_1 - \lambda_1} \mathbb{P} + \frac{\lambda_2}{\mu_1 - \lambda_2} \mathbb{Q} \right) \mathbb{F} &= 0 \\ \left(\mathbb{I} + \frac{\lambda_1}{\mu_2 - \lambda_1} \mathbb{P} + \frac{\lambda_2}{\mu_2 - \lambda_2} \mathbb{Q} \right) \mathbb{G} &= 0. \end{aligned} \quad (38)$$

We assume that all matrices $\mathbb{P}, \mathbb{Q}, \mathbb{F}, \mathbb{G}$ are of the same rank $k > 0$ so can be written as

$$\mathbb{P} = |p_1\rangle\langle p_2|, \quad \mathbb{Q} = |q_1\rangle\langle q_2|, \quad \mathbb{F} = |f_1\rangle\langle f_2|, \quad \mathbb{G} = |g_1\rangle\langle g_2|,$$

where “bras” are $k \times n$ matrices of rank k while “kets” are $n \times k$ matrices of rank k . Equations (38) take form

$$\begin{aligned} \langle p_2 | + \frac{\mu_1}{\lambda_1 - \mu_1} \langle p_2 | f_1 \rangle \langle f_2 | + \frac{\mu_2}{\lambda_1 - \mu_2} \langle p_2 | g_1 \rangle \langle g_2 | &= 0 \\ \langle q_2 | + \frac{\mu_1}{\lambda_2 - \mu_1} \langle q_2 | f_1 \rangle \langle f_2 | + \frac{\mu_2}{\lambda_2 - \mu_2} \langle q_2 | g_1 \rangle \langle g_2 | &= 0 \\ |f_1\rangle + \frac{\lambda_1}{\mu_1 - \lambda_1} |p_1\rangle \langle p_2 | f_1 \rangle + \frac{\lambda_2}{\mu_1 - \lambda_2} |q_1\rangle \langle q_2 | f_1 \rangle &= 0 \\ |g_1\rangle + \frac{\lambda_1}{\mu_2 - \lambda_1} |p_1\rangle \langle p_2 | g_1 \rangle + \frac{\lambda_2}{\mu_2 - \lambda_2} |q_1\rangle \langle q_2 | g_1 \rangle &= 0, \end{aligned} \quad (39)$$

where $\langle p_2 | f_1 \rangle$, $\langle p_2 | g_1 \rangle$, $\langle q_2 | f_1 \rangle$, and $\langle q_2 | g_1 \rangle$ are matrices $k \times k$. The system (39) can be solved with respect to $|p_1\rangle$, $|q_1\rangle$, $\langle f_2 |$, and $\langle g_2 |$, therefore $|p_1\rangle$, $|q_1\rangle$, $\langle f_2 |$, and $\langle g_2 |$ are given in terms of $\langle p_2 |$, $\langle q_2 |$, $|f_1\rangle$, and $|g_1\rangle$. The latter set, according to Theorem 1, evolves in a simple way and can be integrated, namely, $\langle p_2 | = \langle p_{20} | \psi(\lambda_1)$, $\langle q_2 | = \langle q_{20} | \psi(\lambda_2)$, $|f_1\rangle = |f_{10}\rangle \psi(\mu_1)$, and $|g_1\rangle = |g_{10}\rangle \psi(\mu_2)$ where $\langle p_{20} |$, $\langle q_{20} |$, $|f_{10}\rangle$, and $|g_{10}\rangle$ are constant bras and kets. Upon the observation that in Equation (39) $\langle p_2 |$ and $\langle q_2 |$ are combinations of $|f_2\rangle$, $|g_2\rangle$ only, while $|f_1\rangle$ and $|g_1\rangle$ are combinations of $\langle p_1 |$ and $\langle q_1 |$ only, one can obtain formulas for $|p_1\rangle$, $|q_1\rangle$, $\langle f_1 |$, $\langle g_1 |$ by inverting 2×2 block matrices. We will need here only formulas for $|p_1\rangle$, $|q_1\rangle$ which are

$$\begin{aligned} |p_1\rangle &= \frac{1}{\lambda_1} \left((\mu_1 - \lambda_2) |f_1\rangle \langle q_2 | f_1 \rangle^{-1} - (\mu_2 - \lambda_2) |g_1\rangle \langle q_2 | g_1 \rangle^{-1} \right) \left(\frac{\mu_2 - \lambda_2}{\mu_2 - \lambda_1} \langle p_2 | g_1 \rangle \langle q_2 | g_1 \rangle^{-1} - \frac{\mu_1 - \lambda_2}{\mu_1 - \lambda_1} \langle p_2 | f_1 \rangle \langle q_2 | f_1 \rangle^{-1} \right)^{-1}, \\ |q_1\rangle &= \frac{1}{\lambda_2} \left((\mu_1 - \lambda_1) |f_1\rangle \langle p_2 | f_1 \rangle^{-1} - (\mu_2 - \lambda_1) |g_1\rangle \langle p_2 | g_1 \rangle^{-1} \right) \left(\frac{\mu_2 - \lambda_1}{\mu_2 - \lambda_2} \langle q_2 | g_1 \rangle \langle p_2 | g_1 \rangle^{-1} - \frac{\mu_1 - \lambda_1}{\mu_1 - \lambda_2} \langle q_2 | f_1 \rangle \langle p_2 | f_1 \rangle^{-1} \right)^{-1}, \end{aligned} \quad (40)$$

formulas for $\langle f_2 |$ and $\langle g_2 |$ will not be used and we omit them.

4 | Involution Reduction $\Phi^2 = \mathbb{I}$

As far as the reduction $\Phi^2 = \mathbb{I}$ is concerned, we are guided by the results obtained in the continuous case (see [14]). The first observation is that the constraint $\Phi^2 = \mathbb{I}$ is a valid reduction of (16), for it is preserved under propagation of Φ by means of Equation (16). Indeed, if at some point of the lattice $\Phi^2 = \mathbb{I}$ holds and at neighboring points we have $T_1 \Phi^2 = \mathbb{I}$ and $T_2 \Phi^2 = \mathbb{I}$, then due to (16), we have

$$\begin{aligned} T_1 T_2 \Phi^2 &= (T_1 \Phi + T_2 \Phi) \Phi^{-1} (T_1 \Phi^{-1} + T_2 \Phi^{-1})^{-1} (T_1 \Phi + T_2 \Phi) \Phi^{-1} (T_1 \Phi^{-1} + T_2 \Phi^{-1})^{-1} = \\ &= (T_1 \Phi + T_2 \Phi) \Phi (T_1 \Phi + T_2 \Phi)^{-1} (T_1 \Phi + T_2 \Phi) \Phi (T_1 \Phi + T_2 \Phi)^{-1} = \mathbb{I}. \end{aligned} \quad (41)$$

The second observation is that if the constraint

$$\Phi^2 = \mathbb{I} \quad (42)$$

holds, then the matrices

$$U(\lambda) := \frac{\lambda \mathbb{I} - (T_1 \Phi) \Phi^{-1}}{\lambda - 1}, \quad V(\lambda) := \frac{\lambda \mathbb{I} + (T_2 \Phi) \Phi^{-1}}{\lambda + 1}$$

of the Lax pair

$$T_1 \Psi(\lambda) = U(\lambda) \Psi(\lambda), \quad T_2 \Psi(\lambda) = V(\lambda) \Psi(\lambda) \quad (43)$$

has the property

$$U(\lambda^{-1}) = (T_1 \Phi) U(\lambda) \Phi^{-1}, \quad V(\lambda^{-1}) = (T_2 \Phi) V(\lambda) \Phi^{-1}. \quad (44)$$

Due to the property (44), matrix $\Phi \Psi(\lambda^{-1})$ is a fundamental solution of the Lax pair (43) thus we have

$$\Psi(\lambda^{-1}) = \Phi \Psi(\lambda) \mathbb{S} \quad (45)$$

valid for every $\lambda \in \mathbb{C}$ and where $\mathbb{S}$ is an invertible constant matrix. In general, matrix $\mathbb{S}$ can depend on λ , but we shall see that assumption that $\mathbb{S}$ does not depends on λ leads to nontrivial Darboux transformation. We will usually choose such a fundamental solution of $\Psi(\lambda)$ such that $\Psi(0) = \Phi$. In the latter case, evaluating (45) at $\lambda = 0$ and $\lambda = \infty$ we get

$$\Psi(\infty) = \mathbb{S}, \quad \Psi(\infty)\mathbb{S} = \mathbb{I}, \quad (46)$$

that is,

$$\mathbb{S}^2 = \mathbb{I}.$$

After the above preliminary considerations, we are ready to construct Darboux transformation that preserves the constraint $\Phi^2 = \mathbb{I}$.

4.1 | The Darboux–Bäcklund Transformation Preserving Constraint $\Phi^2 = \mathbb{I}$

Now we demand the constraints (45), to be preserved under the Darboux–Bäcklund transformation

$$\tilde{\Psi}(\lambda) = \mathbb{D}(\lambda)\Psi(\lambda), \quad (47)$$

that is,

$$\tilde{\Psi}(\lambda^{-1}) = \tilde{\Phi}\tilde{\Psi}(\lambda)\tilde{\mathbb{S}}. \quad (48)$$

Due to chosen normalization of the Darboux matrix $D(\infty) = \mathbb{I}$ we get from (47) $\tilde{\Psi}(\infty) = \Psi(\infty)$ and due to (46), $\tilde{\mathbb{S}} = \mathbb{S}$ holds. We immediately (by evaluating (48) at $\lambda = \infty$) get that $\tilde{\Psi}(0) = \tilde{\Phi}$ and we arrive at the following constraint on the Darboux matrix $\mathbb{D}$:

$$\mathbb{D}(\lambda^{-1})\Phi = \mathbb{D}(0)\Phi\mathbb{D}(\lambda). \quad (49)$$

Conversely, if $D(\lambda)$ obey the constraint (49) then matrix $\tilde{\Psi}(\lambda)$ given by (47) satisfies (48) where $\tilde{\Phi} = \tilde{\Psi}(0)$ and as a consequence

$$(\mathbb{D}(0)\Phi)^2 = \mathbb{I}, \quad (50)$$

that is,

$$\tilde{\Phi}^2 = \mathbb{I}. \quad (51)$$

From (49), it follows that the set of poles of $\mathbb{D}(\lambda)$ coincides with the set of poles of $\mathbb{D}(1/\lambda)$. In the case of the two-pole Darboux matrix (36), it means that either $\lambda_1\lambda_2 = 1$ or $\lambda_1^2 = \lambda_2^2 = 1$. We confine ourselves to the first (generic) case. The same considerations applies to the inverse of $\mathbb{D}$, see (37). We also confine ourselves to the case $\mu_1\mu_2 = 1$. Therefore, we assume function $\mathbb{D}(\lambda)$ has two poles disposed symmetrically with respect to inversions in the unit sphere

$$\lambda_1 = \lambda_0, \quad \lambda_2 = \lambda_0^{-1}, \quad \lambda_0 \neq 0, 1, \quad (52)$$

that is,

$$\mathbb{D}(\lambda) := \mathbb{I} + \frac{\lambda_0\mathbb{P}}{\lambda - \lambda_0} + \frac{\mathbb{Q}}{\lambda\lambda_0 - 1}, \quad (53)$$

and its inverse is of the same form ($\mu_1 = \mu_0, \mu_2 = 1/\mu_0, \mu_0 \neq 0$), that is,

$$\mathbb{D}^{-1}(\lambda) := \mathbb{I} + \frac{\mu_0\mathbb{F}}{\lambda - \mu_0} + \frac{\mathbb{G}}{\lambda\mu_0 - 1}, \quad (54)$$

that is,

$$\mu_1 = \mu_0, \quad \mu_2 = \mu_0^{-1}, \quad \mu_0 \neq 0, 1. \quad (55)$$

There are two simple conditions sufficient to satisfy (49):

Theorem 2. If

$$\langle q_2| = \langle p_2|\Phi, \quad |g_1\rangle = \Phi|f_1\rangle, \quad (56)$$

then constraint (49) is satisfied.

Proof: Indeed, condition (49) $\mathbb{D}(\lambda^{-1})\Phi\mathbb{D}(\lambda)^{-1} = \mathbb{D}(0)\Phi$ after substitution of assumed forms of $\mathbb{D}$ (53) and $\mathbb{D}^{-1}$ (54) reads

$$\left(\mathbb{I} + \frac{\lambda\lambda_0\mathbb{P}}{1-\lambda\lambda_0} + \frac{\lambda\mathbb{Q}}{\lambda_0-\lambda}\right)\Phi\left(\mathbb{I} + \frac{\mu_0\mathbb{F}}{\lambda-\mu_0} + \frac{\mathbb{G}}{\lambda\mu_0-1}\right) = (\mathbb{I} - \mathbb{P} - \mathbb{Q})\Phi. \quad (57)$$

It is clear that residual at ∞ of left- and right-hand sides of (57) are the same. The remaining residue of left-hand side at $\lambda_0, \lambda_0^{-1}, \mu_0, \mu_0^{-1}$ vanishes iff, respectively,

$$\begin{aligned} |q_1\rangle\langle p_2| \left(\mathbb{I} + \frac{\mu_0}{\lambda_0-\mu_0}\mathbb{F} + \frac{1}{\lambda_0\mu_0-1}\mathbb{G}\right) &= 0, \\ |p_1\rangle\langle q_2| \left(\mathbb{I} + \frac{\lambda_0\mu_0}{1-\lambda_0\mu_0}\mathbb{F} + \frac{\lambda_0}{\mu_0-\lambda_0}\mathbb{G}\right) &= 0, \\ \left(\mathbb{I} + \frac{\lambda_0\mathbb{P}}{\mu_0-\lambda_0} + \frac{1}{\lambda_0\mu_0-1}\mathbb{Q}\right)|f_1\rangle\langle g_2| &= 0 \\ \left(\mathbb{I} + \frac{\lambda_0\mu_0\mathbb{P}}{1-\lambda_0\mu_0} + \frac{\mu_0}{\lambda_0-\mu_0}\mathbb{Q}\right)|g_1\rangle\langle f_2| &= 0 \end{aligned} \quad (58)$$

hold, where we have used conditions (56). The point is that Equations (58) are satisfied because we already made sure that Equations (38) hold and Equations (38) in virtue of (52) and (55) take form

$$\begin{aligned} |p_1\rangle\langle p_2| \left(\mathbb{I} + \frac{\mu_0}{\lambda_0-\mu_0}\mathbb{F} + \frac{1}{\lambda_0\mu_0-1}\mathbb{G}\right) &= 0, \\ |q_1\rangle\langle q_2| \left(\mathbb{I} + \frac{\lambda_0\mu_0}{1-\lambda_0\mu_0}\mathbb{F} + \frac{\lambda_0}{\mu_0-\lambda_0}\mathbb{G}\right) &= 0, \\ \left(\mathbb{I} + \frac{\lambda_0\mathbb{P}}{\mu_0-\lambda_0} + \frac{1}{\lambda_0\mu_0-1}\mathbb{Q}\right)|f_1\rangle\langle f_2| &= 0 \\ \left(\mathbb{I} + \frac{\lambda_0\mu_0\mathbb{P}}{1-\lambda_0\mu_0} + \frac{\mu_0}{\lambda_0-\mu_0}\mathbb{Q}\right)|g_1\rangle\langle g_2| &= 0 \end{aligned} \quad (59)$$

and are clearly equivalent to (58). $\square$

To assure that constraint $\Phi^2 = \mathbb{I}$ is preserved under Darboux transformation we used relations $\langle q_2| = \langle p_2|\Phi$ and $|g_1\rangle = \Phi|f_1\rangle$, but the quantities involved in these relations, by virtue of Theorem 1 obey the following evolution:

$$\begin{aligned} \langle p_2| &= \langle p_{20}|\Psi^{-1}(\lambda_0), \quad |f_1\rangle = \Psi(\mu_0)|f_{10}\rangle \\ \langle q_2| &= \langle q_{20}|\Psi^{-1}(\lambda_0^{-1}), \quad |g_1\rangle = \Psi(\mu_0^{-1})|g_{10}\rangle, \end{aligned} \quad (60)$$

where $\Psi(\lambda)$ is the solution of the Lax pair, and $\langle p_{20}|, \langle q_{20}|, |f_{10}\rangle, |g_{10}\rangle$ are constant vectors (initial data), so the question arises whether these relations are compatible with the evolution? The answer is positive.

Due to the reduction (45), we have

$$\Psi(\lambda_0^{-1}) = \Phi\Psi(\lambda_0)\mathbb{S}, \quad \Psi(\mu_0^{-1}) = \Phi\Psi(\mu_0)\mathbb{S}, \quad (61)$$

which means that

$$\langle q_2| = \langle q_{20}|\mathbb{S}\Psi^{-1}(\lambda_0)\Phi, \quad |g_1\rangle = \Phi\Psi(\mu_0)\mathbb{S}|g_{10}\rangle, \quad (62)$$

and, as a conclusion we get that to satisfy relations in Lemma 2 it is enough to set $\langle q_{20}| = \langle p_{20}|\mathbb{S}$ and $|g_{10}\rangle = \mathbb{S}|f_{10}\rangle$.

We can now formulate the main conclusion of the article.

Corollary 2. Having a solution Φ of system (17) and a corresponding solution $\Psi(\lambda)$ of Lax pair (18), by virtue of Theorem 1 and considerations above we have

$$\langle p_2 | = \langle p_{20} | \Psi^{-1}(\lambda_0), \quad |f_1\rangle = \Psi(\mu_0) |f_{10}\rangle, \quad (63)$$

where $\langle p_{20} |, |f_{10}\rangle$, are constant vectors (initial data). Equations (40) under the reduction considered in this section with

$$\lambda_1 = \lambda_0, \quad \lambda_2 = \lambda_0^{-1}, \quad \mu_1 = \mu_0, \quad \mu_2 = \mu_0^{-1}, \quad \lambda_0, \mu_0 \neq 0, 1 \quad (64)$$

and after introducing quantities

$$x := \frac{\langle p_2 | f_1 \rangle}{\lambda_0 - \mu_0}, \quad y := \frac{\langle p_2 | \Phi | f_1 \rangle}{1 - \lambda_0 \mu_0} \quad (65)$$

take form

$$|p_1\rangle = \lambda_0^{-1} (|f_1\rangle y^{-1} + \mu_0^{-1} \Phi |f_1\rangle x^{-1}) (xy^{-1} - yx^{-1})^{-1} \quad (66)$$

$$|q_1\rangle = -(|f_1\rangle x^{-1} + \mu_0^{-1} \Phi |f_1\rangle y^{-1}) (xy^{-1} - yx^{-1})^{-1}. \quad (67)$$

Then the Darboux matrix that respects the constraint $\Phi^2 = \mathbb{I}$ is

$$\mathbb{D}(\lambda) = \mathbb{I} + \frac{\lambda_0 |p_1\rangle \langle p_2|}{\lambda - \lambda_0} + \frac{|q_1\rangle \langle p_2| \Phi}{\lambda \lambda_0 - 1}$$

and family of functions $\tilde{\Phi}$ given by

$$\tilde{\Phi} = \Phi - |p_1\rangle \langle p_2| \Phi - |q_1\rangle \langle p_2| \quad (68)$$

is family of solutions of system (17).

4.2 | Solutions

We apply the procedure described in Corollary (2) to background solution Φ of system (17)

$$\Phi = \begin{bmatrix} 0 & a^{m_1} b^{m_2} \\ \frac{1}{a^{m_1}} & \frac{1}{b^{m_2}} \\ 0 & 0 \end{bmatrix},$$

where a and b are complex constants. For this solution, the Lax pair (18) takes form

$$T_1 \Psi(\lambda) = \begin{bmatrix} \frac{\lambda - a}{\lambda - 1} & 0 \\ 0 & \frac{a\lambda - 1}{a(\lambda - 1)} \end{bmatrix} \Psi(\lambda), \quad T_2 \Psi(\lambda) = \begin{bmatrix} \frac{b + \lambda}{\lambda + 1} & 0 \\ 0 & \frac{b\lambda + 1}{b(\lambda + 1)} \end{bmatrix} \Psi(\lambda)$$

from which we get the following fundamental solution:

$$\Psi(\lambda) = \begin{bmatrix} 0 & \left(\frac{\lambda - a}{\lambda - 1}\right)^{m_1} \left(\frac{b + \lambda}{\lambda + 1}\right)^{m_2} \\ \left(\frac{1 - a\lambda}{a - a\lambda}\right)^{m_1} \left(\frac{b\lambda + 1}{b\lambda + b}\right)^{m_2} & 0 \end{bmatrix}.$$

Taking

$$\langle p_{20} | = [c_2, c_1], \quad |f_{10}\rangle = \begin{bmatrix} c_4 \\ c_3 \end{bmatrix},$$

where c_i are arbitrary complex constants we obtain from procedure described in Corollary (2) solution $\tilde{\Phi}$ of system (17)

$$\tilde{\Phi} = \frac{1}{D} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & -M_{11} \end{bmatrix},$$

where

$$D = \lambda_0 \mu_0 (\gamma + \delta - \alpha - \beta)(\alpha + \beta + \gamma + \delta)$$

$$M_{12} = a^{m_1} b^{m_2} (\mu_0 \gamma + \lambda_0 \delta - \alpha - \lambda_0 \mu_0 \beta)(\alpha + \lambda_0 \mu_0 \beta + \mu_0 \gamma + \lambda_0 \delta)$$

$$M_{21} = a^{-m_1} b^{-m_2} (\lambda_0 \gamma + \mu_0 \delta - \lambda_0 \mu_0 \alpha - \beta)(\lambda_0 \mu_0 \alpha + \beta + \lambda_0 \gamma + \mu_0 \delta)$$

$$M_{11} = \lambda_0 (\mu_0^2 - 1)(\alpha \gamma - \beta \delta) + \mu_0 (\lambda_0^2 - 1)(\alpha \delta - \beta \gamma),$$

and where

$$\alpha := c_1 c_4 (\lambda_0 - \mu_0)(a \lambda_0 - 1)^{m_1} (a \mu_0 - 1)^{m_1} (b \lambda_0 + 1)^{m_2} (b \mu_0 + 1)^{m_2}$$

$$\beta := c_2 c_3 (\lambda_0 - \mu_0)(\lambda_0 - a)^{m_1} (\mu_0 - a)^{m_1} (b + \lambda_0)^{m_2} (b + \mu_0)^{m_2}$$

$$\gamma := c_1 c_3 (\lambda_0 \mu_0 - 1)(a \lambda_0 - 1)^{m_1} (\mu_0 - a)^{m_1} (b \lambda_0 + 1)^{m_2} (b + \mu_0)^{m_2}$$

$$\delta := c_2 c_4 (\lambda_0 \mu_0 - 1)(\lambda_0 - a)^{m_1} (a \mu_0 - 1)^{m_1} (b + \lambda_0)^{m_2} (b \mu_0 + 1)^{m_2}.$$

In this particular case ($N = 2$), we are able to find the large class of solutions that satisfy also unitary constraint.

5 | Unitary Reduction $\Phi^\dagger \Phi = \mathbb{I}$ in the Case $N = 2$

In general, unitary constraint $\Phi^\dagger \Phi = \mathbb{I}$ is not compatible with Equation (16). However, in the particular case $N = 2$, set of matrices

$$S = \left\{ \begin{bmatrix} a & b + ic \\ b - ic & -a \end{bmatrix} \mid a, b, c \in \mathbb{R}, a^2 + b^2 + c^2 = 1 \right\}$$

is not only unitary, Hermitian, and involutive at the same time $\Phi^{-1} = \Phi^\dagger = \Phi$, but it follows that anticommutator of two elements of S gives

$$\begin{bmatrix} a_1 & b_1 + ic_1 \\ b_1 - ic_1 & -a_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 + ic_2 \\ b_2 - ic_2 & -a_2 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 + ic_2 \\ b_2 - ic_2 & -a_2 \end{bmatrix} \begin{bmatrix} a_1 & b_1 + ic_1 \\ b_1 - ic_1 & -a_1 \end{bmatrix} = 2(a_1 a_2 + b_1 b_2 + c_1 c_2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and therefore if at some point of the lattice $\Phi \in S$ and at neighboring points $T_1 \Phi \in S$ and $T_2 \Phi \in S$ then due to (16) we have $(T_1 T_2 \Phi)^2 = \mathbb{I}$ and moreover $(T_1 T_2 \Phi)^\dagger (T_1 T_2 \Phi) = \mathbb{I}$. As we mentioned in the introduction, this particular case is equivalent to Orfanidis system (7) and was discussed in article [33]. The results of Section 4 can be used to obtain solutions in this case as follows.

Applying the procedure from Section 4.2 to seed solution (clearly, $\Phi^\dagger \Phi = \mathbb{I}$ and $\Phi^2 = \mathbb{I}$)

$$\Phi = \begin{bmatrix} 0 & e^{iam_1} e^{ibm_2} \\ e^{-iam_1} e^{-ibm_2} & 0 \end{bmatrix},$$

where a and b this time are real parameters, it is enough to set

$$\begin{bmatrix} c_4 \\ c_3 \end{bmatrix} = \begin{bmatrix} \bar{c}_2 \\ \bar{c}_1 \end{bmatrix}, \quad \mu_0 = \bar{\lambda}_0$$

to guarantee that $\tilde{\Phi}^\dagger \tilde{\Phi} = \mathbb{I}$. Therefore,

$$\tilde{\Phi} = \frac{1}{|\lambda_0|^2((s + \bar{s})^2 + (r_1 + r_2)^2)} \times \begin{bmatrix} ir_1[\Lambda s - \bar{\Lambda} \bar{s}] + ir_2[\bar{\Lambda} s - \Lambda \bar{s}] & e^{im_1 a} e^{im_2 b} \left((s + |\lambda_0|^2 \bar{s})^2 + (\bar{\lambda}_0 r_1 + \lambda_0 r_2)^2 \right) \\ e^{-im_1 a} e^{-im_2 b} \left((|\lambda_0|^2 s + \bar{s})^2 + (\lambda_0 r_1 + \bar{\lambda}_0 r_2)^2 \right) & -ir_1[\Lambda s - \bar{\Lambda} \bar{s}] - ir_2[\bar{\Lambda} s - \Lambda \bar{s}] \end{bmatrix}, \quad (69)$$

where

$$\begin{aligned} \Lambda &= \lambda_0(\bar{\lambda}_0^2 - 1) \\ r_1 &= (|\lambda_0|^2 - 1)|c_1|^2 |e^{ia} \lambda_0 - 1|^{2m_1} |e^{ib} \lambda_0 + 1|^{2m_2} \\ r_2 &= (|\lambda_0|^2 - 1)|c_2|^2 |e^{ia} \bar{\lambda}_0 - 1|^{2m_1} |e^{ib} \bar{\lambda}_0 + 1|^{2m_2} \\ s &= i(\bar{\lambda}_0 - \lambda_0)c_1 \bar{c}_2 (e^{ia} \lambda_0 - 1)^{m_1} (\bar{\lambda}_0 - e^{-ia})^{m_1} (e^{ib} \lambda_0 + 1)^{m_2} (\bar{\lambda}_0 + e^{-ib})^{m_2} \end{aligned}$$

is a family of solution of the system

$$\Delta_1((\Delta_2 \Phi) \Phi) + \Delta_2((\Delta_1 \Phi) \Phi) = 0, \quad \Phi^2 = \mathbb{I}, \quad \Phi^\dagger \Phi = \mathbb{I}, \quad (70)$$

that is, for the discrete chiral field that takes values on two-dimensional sphere S^2 .

6 | Conclusions

We have constructed the Darboux transformation for the difference equation

$$\Phi(m_1 + 1, m_2 + 1)(\Phi^{-1}(m_1 + 1, m_2) + \Phi^{-1}(m_1, m_2 + 1)) = (\Phi(m_1 + 1, m_2) + \Phi(m_1, m_2 + 1))\Phi^{-1}(m_1, m_2) \quad (71)$$

and then have reduced the transformation so that it preserves the involutive constraints

$$(\Phi(m_1, m_2))^2 = 1. \quad (72)$$

Unlike the continuous case, system (71–72) in general is not compatible with the unitary constraint (4). An open problem remains to find an integrable discretization of the chiral field models (1) and (13) that admits reductions to classical Lie algebras (or Lie groups) and further involutive reductions. Moreover, to the best of our knowledge, integrable discretization of elliptic chiral field model (i.e., when in (1) $u = x + iy$ and $v = x - iy$, where x, y are real independent variables) has not been discussed in the literature, so far. In particular, integrable discretization of the original (i.e., elliptic) Ernst equation is not known, yet. However, in the autonomous case, one can replace the discrete operator in (7) with discrete Schrödinger operator [28–30] or equivalently to apply the so-called sublattice approach [31, 32] to first equation of (7) to get its elliptic version.

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Data Availability Statement

No data sets were created or analyzed in this research.

Endnotes

¹ Due to (21), one can introduce “potential” Γ via $\mathbb{A} = \Delta_1 \Gamma + \mathbb{I}$, $\mathbb{B} = \Delta_2 \Gamma - \mathbb{I}$ and, plugging it in (22), get $T_2(\Delta_1 \Gamma) \Delta_2 \Gamma = T_1(\Delta_2 \Gamma) \Delta_1 \Gamma$. Alternatively, one can put $\mathbb{A} = 2\Delta_1 H$, $\mathbb{B} = \Delta_2 H$ and get $\Delta_1 \Delta_2 H = T_2(\Delta_1 H) \Delta_2 H - T_1(\Delta_2 H) \Delta_1 H$.

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