



The minimal primes of localizations of rings

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ABSTRACT

The set of minimal primes of a ring is a very important set as far the spectrum of a ring is concerned as every prime contains a minimal prime. So, knowing the minimal primes is the first (important and difficult) step in describing the spectrum. In the algebraic geometry, the minimal primes of the algebra of regular functions on an algebraic variety determine/respond to the irreducible components of the variety. The aim of the paper is to obtain several descriptions of the set of minimal prime ideals of localizations of rings under several natural assumptions. In particular, the following cases are considered: a localization of a semiprime ring with finite set of minimal primes; a localization of a prime rich ring where the localization respects the ideal structure of primes and primeness of certain minimal primes; a localization of a ring at a left denominator set generated by normal elements, and others. As an application, for a semiprime ring with finitely many minimal primes, a description of the minimal primes of its largest left/right quotient ring is obtained. For a semiprime ring R with finitely many minimal primes $\min(R)$, criteria are given for the map

$$\rho_{R,\min} : \min(R) \rightarrow \min(Z(R)), \quad \mathfrak{p} \mapsto \mathfrak{p} \cap Z(R)$$

being a well-defined map or surjective where $Z(R)$ is the centre of R .

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1. Introduction

Notation. In this paper, module means a left module, and the following notation is fixed:

- R is a ring with 1, $R^\times$ is its group of units, $Z(R)$ is the centre of R , $\mathfrak{n}_R$ is the prime radical and $\text{rad}(R)$ is the radical of R , and $\min(R)$ is the set of minimal primes of R ;
- $\text{Spec}(R)$ is the prime spectrum and $\text{Spec}_c(R)$ is the completely prime spectrum of R ;
- $\text{Den}_l(R, \mathfrak{a})$ is the set of left denominator sets S of R with $\text{ass}_l(S) = \mathfrak{a}$ where $\mathfrak{a}$ is an ideal of R and $\text{ass}_l(S) := \{r \in R \mid sr = 0 \text{ for some } s \in S\}$;
- For $S \in \text{Den}_l(R, \mathfrak{a})$, $\min(R, S) := \{\mathfrak{p} \in \min(R) \mid \mathfrak{p} \cap S = \emptyset\}$ and $\min(R, S, \text{id}) := \{\mathfrak{p} \in \min(R) \mid S^{-1}\mathfrak{p}S^{-1}R \neq S^{-1}R\}$;
- $\max.\text{Den}_l(R)$ is the set of maximal left denominator sets of R (it is a *non-empty* set, [2]);
- $\mathcal{C}_R$ is the set of regular elements of the ring R (i.e. $\mathcal{C}_R$ is the set of non-zero-divisors of the ring R);
- $Q_{l,cl}(R) := \mathcal{C}_R^{-1}R$ is the *left quotient ring* (the *classical left ring of fractions*) of the ring R (if it exists);
- $S_{l,\mathfrak{a}}(R)$ is the *largest element* of the partially ordered set (poset) $(\text{Den}_l(R, \mathfrak{a}), \subseteq)$ and $Q_{l,\mathfrak{a}}(R) := S_{l,\mathfrak{a}}(R)^{-1}R$ is the *largest left quotient ring associated to* $\mathfrak{a}$, $S_{l,\mathfrak{a}}(R)$ exists, [2, Theorem 2.1.(2)];
- In particular, $S_l(R) = S_{l,0}(R)$ is the largest element of the poset $(\text{Den}_l(R, 0), \subseteq)$ and $Q_l(R) := S_0^{-1}R$ is the largest left quotient ring of R ;
- For a commutative ring R and $\mathfrak{p} \in \text{Spec}(R)$, $R_{\mathfrak{p}} := (R \setminus \mathfrak{p})^{-1}R$ is the localization of R at the prime ideal $\mathfrak{p}$.

Semiprime ideals and minimal primes. Prime and semiprime rings are large and important classes of rings both in commutative and non-commutative algebra. They also have geometric flavour. The algebra of regular functions on an affine algebraic variety is a semiprime algebra and the corresponding ideal of definition of the variety is a semiprime ideal. If, in addition, the variety is irreducible then the algebra of regular functions on it is a prime algebra and the defining ideal is a prime ideal. In the algebraic geometry, the minimal primes of the algebra of regular functions correspond to the irreducible components of the algebraic variety and generators of minimal primes are the equations of the corresponding irreducible component of the variety. In the commutative situation, localizations respect many operations and concepts. This is not the case in the noncommutative setting (in general, a localization of an ideal is not an ideal, a localization of a prime ideal is not a prime ideal, there is little connection between prime ideals of a ring and its localization, etc).

Let us describe the main results and the structure of the paper. At the beginning of Section 2, we recall some results on localizations that are used in the paper. Suppose that $\mathfrak{a}$ is an ideal of a ring R . In general, the left ideal $S^{-1}\mathfrak{a}$ of the ring $S^{-1}R$ is *not* an ideal. Proposition 2.4 is a criterion for the left ideal $S^{-1}\mathfrak{a}$ being an ideal of $S^{-1}R$. Corollary 2.7 presents criteria for a localization of a prime ideal being a prime ideal (under certain conditions).

The set $\min(S^{-1}R)$ where R is a prime rich ring and S respects the ideal structure of primes. A ring is called a **prime rich ring** if each ideal contains a finite product of prime ideals that contain it. Proposition 1.1 is a criterion for a ring being prime rich where $\min(\mathfrak{a})$ is the set of minimal primes over an ideal $\mathfrak{a}$ (its proof is given in Section 2).

Proposition 1.1. *For a ring R the following conditions are equivalent:*

1. *The ring R is a prime rich ring.*
2. *Every ideal $\mathfrak{a}$ of R contains a finite product of its minimal primes.*
3. *For every ideal $\mathfrak{a}$ of R , $|\min(\mathfrak{a})| < \infty$ and the prime radical $\mathfrak{n}_{R/\mathfrak{a}}$ of the factor ring $R/\mathfrak{a}$ is a nilpotent ideal.*

Every one-sided Noetherian ring R is prime rich. A left denominator set S of a ring R is called **an ideal preserving left denominator set** (or a left denominator set that **respects the ideal structure** of R) if the map

$$S^{-1} : \mathcal{I}(R) \rightarrow \mathcal{I}(S^{-1}R), \quad I \mapsto S^{-1}I$$

is a well-defined map, i.e. for each ideal I of R , the left ideal $S^{-1}I$ of the ring $S^{-1}R$ is an ideal (where $\mathcal{I}(R)$ is the set of all ideals of R). We say that a left denominator set S of a ring R **respects the ideal structure of primes** of R if for each prime ideal $\mathfrak{p} \in \text{Spec}(R)$ the left ideal $S^{-1}\mathfrak{p}$ of the ring $S^{-1}R$ is an ideal. All the left denominator sets of a left Noetherian ring are ideal preserving and as a result also respect the ideal structure of primes.

For a prime rich ring R and its left denominator set $S \in \text{Den}_l(R, \mathfrak{a})$ that respects the ideal structure of primes, Proposition 1.2 describes the set of minimal primes of the ring $S^{-1}R$ via the minimal primes of the ring R .

Proposition 1.2. *Suppose that R is a prime rich ring, a left denominator set $S \in \text{Den}(R, \mathfrak{a})$ respects the ideal structure of primes and $S^{-1}\mathfrak{p} \in \text{Spec}(S^{-1}R)$ for all $\mathfrak{p} \in \min(R, S)$. Then*

1. $1 \leq |\min(R, S)| \leq |\min(R)| < \infty$, the set

$$\min(S^{-1}R) = \min\{S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \min(R, S)\}$$

is a finite set and $|\min(S^{-1}R)| \leq |\min(R, S)| < \infty$.

2. $\min(S^{-1}R) = \{S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \min(R, S)\}$ iff the ideals $\{S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \min(R, S)\}$ are incomparable.
3. If in addition, for each prime ideal $\mathfrak{p} \in \min(R, S)$, the ideal $\sigma_S^{-1}(S^{-1}\mathfrak{p})$ is a finitely generated right R -module (e.g. R is a right Noetherian ring) then

$$\min(S^{-1}R) = \{S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \min(R, S)\}$$

where $\sigma_S : R \rightarrow S^{-1}R, r \mapsto \frac{r}{1}$.

Corollary 1.3. *Suppose that R is a right Noetherian ring, $S \in \text{Den}(R, 0)$ and $S^{-1}R$ is a left Noetherian ring (e.g. R is a Noetherian ring). Then*

$$\min(S^{-1}R) = \{S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \min(R, S)\}.$$

In particular, $\min(R, S) \neq \emptyset$.

A proof of Proposition 1.2 and Corollary 1.3 are given in Section 2.

Description of the set of minimal primes of a localization of a semiprime ring at a regular denominator set. The aim of Section 3 is to prove Theorem 1.4.(2) that provides an explicit description of the set of minimal primes of localizations of semiprime rings provided that they have only finitely many minimal primes (statement 1 is a known result). Theorem 1.4.(2) is one of the key facts in understanding the structure of embeddings of localizations of semiprime rings into semisimple or semiprimary rings, see [5].

Theorem 1.4. *Let R be a semiprime ring and $S \in \text{Den}_l(R, 0)$. Then*

1. The ring $S^{-1}R$ is a semiprime ring.
2. If, in addition, $|\min(R)| < \infty$ then $\min(S^{-1}R) = \{S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \min(R)\}$, i.e. the map $\min(R) \rightarrow \min(S^{-1}R), \mathfrak{p} \mapsto S^{-1}\mathfrak{p}$ is a bijection and $|\min(S^{-1}R)| = |\min(R)|$.

3. If, in addition, $|\min(R)| < \infty$ then $\pi_{\mathfrak{p}}(S) \in \text{Den}_l(R/\mathfrak{p}, 0)$ and

$$\pi_{\mathfrak{p}}(S)^{-1}(R/\mathfrak{p}) \simeq S^{-1}R/S^{-1}\mathfrak{p} \simeq S^{-1}(R/\mathfrak{p})$$

where $\pi_{\mathfrak{p}} : R \rightarrow R/\mathfrak{p}$, $r \mapsto r + \mathfrak{p}$.

In general, for a ring R , its *classical left quotient ring* of R ,

$$Q_{l,cl}(R) := \mathcal{C}_R^{-1}R,$$

does not exist (e.g. the algebras of polynomial integro-differential operators, [1]) but for each ring R there is the largest left Ore set $S_l(R)$ of regular elements of R and the ring

$$Q_l(R) := S_l(R)^{-1}R$$

is called the *largest left quotient ring* of R , [1,2]. As a corollary, for a semiprime ring R with finitely many minimal primes, Corollary 3.3 describes the set of minimal primes of the largest left quotient ring $Q_l(R)$ of R .

Corollary 3.4 is a generalization of Theorem 1.4 to the case when the denominator set S contains zero divisors of R . To be more precisely, $S \in \text{Den}_l(R, \mathfrak{a})$ where $\mathfrak{a} \in \text{Spec}(R)$.

Theorem 3.9 is a generalization of Theorem 1.4.(2,3) for left denominator sets that contain zero divisors.

Proposition 1.5, which is a corollary of Theorem 3.9, describes the set of minimal primes, $\min(S^{-1}R)$, of a localization of a semiprime ring R under conditions that hold for all commutative semiprime rings (namely, for all $\mathfrak{p} \in \min(R, S)$, $\mathfrak{p}$ is a completely prime ideal of R and $S^{-1}\mathfrak{p}$ is an ideal of $S^{-1}R$).

Proposition 1.5. *Let R be a semiprime ring with $|\min(R)| < \infty$ and $S \in \text{Den}(R, \mathfrak{a})$. Suppose that for all $\mathfrak{p} \in \min(R, S)$, $\mathfrak{p}$ is a completely prime ideal of R and $S^{-1}\mathfrak{p}$ is an ideal of $S^{-1}R$. Then*

1. *The ring $S^{-1}R$ is a semiprime ring with $\min(S^{-1}R) = \{S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \min(R, S)\} \subseteq \text{Spec}_c(S^{-1}R)$.*
2. *In statement 1, all ideals in the set $\min(S^{-1}R)$ are distinct, i.e. $|\min(S^{-1}R)| = |\min(R, S)|$.*

Proofs of Theorem 1.4 and Proposition 1.5 are given in Section 3.

Description of minimal primes of localizations of rings at multiplicative sets generated by normal elements. Inversion of elements in a ring is an important and difficult operation that ‘simplifies’ the situation as a rule. Ore’s method of localization is an example of a theory of *one-sided fractions* where by definition only the elements of a *denominator set* can be inverted and the result is a ring of one-sided fractions which always exists, i.e. it is not equal to zero, [13,9,12]. In [1, Theorem 4.15], it is proven that the elements of an *arbitrary* (left and right) *Ore set* can be inverted and the result is also a ring of one-sided fractions (which always exists) but in general Ore set is not a denominator set. This was the starting point in [3] and [4] where the most general theory of one-sided fractions was presented.

Theorem 1.6. ([3, Theorem 1.6.(1,2)]) *Let R be a ring and $S \in \text{Ore}(R)$.*

1. *$\mathfrak{a} := \{r \in R \mid srt = 0 \text{ for some elements } s, t \in S\}$ is an ideal of R such that $\mathfrak{a} \neq R$.*
2. *Let $\pi : R \rightarrow \overline{R} := R/\mathfrak{a}$, $r \mapsto \overline{r} = r + \mathfrak{a}$. Then $\overline{S} := \pi(S) \in \text{Den}(\overline{R}, 0)$, $\text{ass}_R(S) = \mathfrak{a}$, S is a localizable set, and $R\langle S^{-1} \rangle \simeq \overline{S}^{-1}\overline{R}$, an R -isomorphism (the ring $R\langle S^{-1} \rangle$ is defined in (7)).*

Let R be a ring and S be a multiplicative subset of R which is generated by normal elements of R . Recall that an element $x \in R$ is called a *normal element* of R if $Rx = xR$. Clearly, $S \in \text{Ore}(R)$ is an Ore set of R . By Theorem 1.6, S is a localizable set of R such that $\text{ass}_R(S) = \mathfrak{a}$ and $R\langle S^{-1} \rangle \simeq \overline{S}^{-1}\overline{R}$, see Section 4 for details. We keep the notation of Theorem 1.6. Let

$$\tilde{\pi} : \overline{R} \rightarrow \tilde{R} := \overline{R}/\mathfrak{n}_{\overline{R}}, \quad \bar{r} \mapsto \tilde{r} = \tilde{\bar{r}} = \bar{r} + \mathfrak{n}_{\overline{R}}$$

where $\mathfrak{n}_{\overline{R}}$ is the prime radical of the ring $\overline{R}$. There are bijections

$$\min(\mathfrak{a}) \xrightarrow{\pi} \min(\overline{R}) \xrightarrow{\tilde{\pi}} \min(\tilde{R}), \quad \mathfrak{p} \mapsto \bar{\mathfrak{p}} \mapsto \tilde{\mathfrak{p}}.$$

Let $\min(\mathfrak{a}, S) := \{\mathfrak{p} \in \min(\mathfrak{a}) \mid \mathfrak{p} \cap S = \emptyset\}$.

Theorem 1.7 describes the set of minimal primes of the ring $R\langle S^{-1} \rangle$. In general, the ring $R\langle S^{-1} \rangle$ is not a semiprime ring but precisely the same result as Theorem 1.4 holds for it but without the ‘semiprimeness’ assumption, see Theorem 1.7.(4a). The key ideal of the proof of this result is to reduce it to the semiprime case by considering the semiprime ring $R\langle S^{-1} \rangle/\mathfrak{n}_{R\langle S^{-1} \rangle}$.

Theorem 1.7. *Let R be a ring and S be a multiplicative subset of R which is generated by normal elements and such that $|\min(\mathfrak{a})| < \infty$ where $\mathfrak{a}$ is defined in Theorem 1.6.(1), e.g. R is a one-sided Noetherian ring. For each $\mathfrak{p} \in \min(\mathfrak{a})$, let $\pi_{\mathfrak{p}} : R \rightarrow R/\mathfrak{p}$, $r \mapsto r + \mathfrak{p}$. Then*

1. *The map $\min(\mathfrak{a}) \rightarrow \text{Spec}(R\langle S^{-1} \rangle) = \text{Spec}(\overline{S}^{-1}\overline{R})$, $\mathfrak{p} \mapsto \overline{S}^{-1}\bar{\mathfrak{p}}$ is an injection and $\mathfrak{n}_{\overline{S}^{-1}\overline{R}} \subseteq \overline{S}^{-1}\mathfrak{n}_{\overline{R}}$.*
2. *$\tilde{\tilde{S}} \in \text{Den}(\tilde{R}, 0)$, $\overline{S}^{-1}\overline{R}/\overline{S}^{-1}\mathfrak{n}_{\overline{R}} \simeq \tilde{\tilde{S}}^{-1}\tilde{R}$, and the map $\min(\mathfrak{a}) \rightarrow \min(\tilde{\tilde{S}}^{-1}\tilde{R})$, $\mathfrak{p} \mapsto \tilde{\tilde{S}}^{-1}\tilde{\mathfrak{p}}$ is a bijection.*
3. *For each $\mathfrak{p} \in \min(\mathfrak{a})$, $\pi_{\mathfrak{p}}(S) \in \text{Den}(R/\mathfrak{p}, 0)$ and $\pi_{\mathfrak{p}}(S)^{-1}(R/\mathfrak{p}) \simeq \tilde{\tilde{S}}^{-1}\tilde{R}/\tilde{\tilde{S}}^{-1}\tilde{\mathfrak{p}} \simeq \overline{S}^{-1}\overline{R}/\overline{S}^{-1}\bar{\mathfrak{p}}$ and there is a commutative diagram:*

$$\begin{array}{ccccccc} \pi_{\mathfrak{p}} : & R & \xrightarrow{\pi} & \overline{R} & \xrightarrow{\tilde{\pi}} & \tilde{R} & \xrightarrow{\tilde{\pi}_{\mathfrak{p}}} & R/\mathfrak{p} \\ & & & \downarrow & & \downarrow & & \downarrow \\ & & & \overline{S}^{-1}\overline{R} & \rightarrow & \tilde{\tilde{S}}^{-1}\tilde{R} & \rightarrow & \pi_{\mathfrak{p}}(S)^{-1}(R/\mathfrak{p}) \\ & & & \downarrow & \nearrow \simeq & \downarrow & \nearrow \simeq & \\ & \overline{S}^{-1}\overline{R}/\mathfrak{n}_{\overline{S}^{-1}\overline{R}} & \rightarrow & \overline{S}^{-1}\overline{R}/\overline{S}^{-1}\mathfrak{n}_{\overline{R}} & \rightarrow & \tilde{\tilde{S}}^{-1}\tilde{R}/\tilde{\tilde{S}}^{-1}\tilde{\mathfrak{p}} & & \\ & & & \downarrow & \nearrow \simeq & & & \\ & & & \overline{S}^{-1}\overline{R}/\overline{S}^{-1}\mathfrak{p} & & & & \end{array}$$

4. *Suppose that the prime radical $\mathfrak{n}_{\overline{R}}$ is a nilpotent ideal (e.g. the ring $\overline{R}$ is a one-sided Noetherian). Then*
 - (a) *The map $\min(\mathfrak{a}) \rightarrow \min(R\langle S^{-1} \rangle) = \min(\overline{S}^{-1}\overline{R})$, $\mathfrak{p} \mapsto \overline{S}^{-1}\bar{\mathfrak{p}}$ is a bijection, i.e. $\min(R\langle S^{-1} \rangle) = \{\overline{S}^{-1}\bar{\mathfrak{p}} \mid \mathfrak{p} \in \min(\mathfrak{a})\}$.*
 - (b) *$\tilde{\tilde{S}} \in \text{Den}(\tilde{R}, 0)$, $\overline{S}^{-1}\mathfrak{n}_{\overline{R}} = \mathfrak{n}_{\overline{S}^{-1}\overline{R}}$ and $\overline{S}^{-1}\overline{R}/\overline{S}^{-1}\mathfrak{n}_{\overline{R}} \simeq \overline{S}^{-1}\overline{R}/\mathfrak{n}_{\overline{S}^{-1}\overline{R}} \simeq \tilde{\tilde{S}}^{-1}\tilde{R}$.*

The proof of Theorem 1.7 is given in Section 4.

Lemma 4.2 is a generalization of Theorem 1.7 to a more general situation.

Minimal primes of a semiprime ring and of its centre. For a ring R , the **restriction map**

$$\rho_R : \text{Spec}(R) \rightarrow \text{Spec}(Z(R)), P \mapsto P \cap Z(R) \quad (1)$$

is a well-defined map. In general, the map

$$\rho_{R,\min} : \min(R) \rightarrow \min(Z(R)), \quad \mathfrak{p} \mapsto \mathfrak{p} \cap Z(R) \quad (2)$$

is not a well-defined map, see Lemma 5.2. For each natural number $n \geq 1$, Lemma 5.2 provides an example of a finitely generated algebra A_n over a field K such that A_n is a domain, the centre $Z(A_n) = P_n = K[z_1, \dots, z_n]$ is polynomial algebra in n variables and

$$\rho_{A_n}(\min(A_n)) = \{\{0\}, \mathfrak{p}_I := (x_i)_{i \in I} \mid \emptyset \neq I \subseteq \{1, \dots, n\}\}.$$

The set $\rho_{A_n}(\min(A_n))$ contains prime ideals of all possible heights.

Lemma 5.2 provides an example of a countably generated algebra A_∞ over a (countable) field K such that the set $\min(A_\infty)$ is uncountable and every minimal prime of A_∞ is a completely prime ideal. Notice that if the field K is a countable field then the algebra A_∞ contains a countably many elements.

For a semiprime ring R with $|\min(R)| < \infty$, Proposition 1.8 is a criterion for the map $\rho_{R,\min}$ being a well-defined map.

Proposition 1.8. *Let R be a semiprime ring with $|\min(R)| < \infty$. Then the following statements are equivalent:*

1. $\mathcal{C}_{Z(R)} \subseteq \mathcal{C}_R$.
2. $\mathcal{C}_{Z(R)} \cap \mathfrak{p} = \emptyset$ for all $\mathfrak{p} \in \min(R)$.
3. The map $\rho_{R,\min}$ is a well-defined map.

Proof. Proposition 1.8 is equivalent to Proposition 5.4. $\square$

Proposition 1.9 is a criterion for the map $\rho_{R,\min}$ being a surjection.

Proposition 1.9. *Let R be a semiprime ring with $|\min(R)| < \infty$ and $|\min(Z(R))| < \infty$. Then the following statements are equivalent:*

1. $\mathcal{C}_{Z(R)} \subseteq \mathcal{C}_R$.
2. $\mathcal{C}_{Z(R)} \cap \mathfrak{p} = \emptyset$ for all $\mathfrak{p} \in \min(R)$.
3. The map $\rho_{R,\min}$ is a well-defined map.
4. The map $\rho_{R,\min}$ is a surjection.

Proofs of Proposition 1.8 and Proposition 1.9 are given in Section 5.

2. Descriptions of minimal primes of localizations of semiprime rings

At the beginning of the section, we recall some results on localizations that are used in the paper. Suppose that $\mathfrak{a}$ is an ideal of a ring R . In general, the left ideal $S^{-1}\mathfrak{a}$ of the ring $S^{-1}R$ is *not* an ideal. Proposition 2.4 is a criterion for the left ideal $S^{-1}\mathfrak{a}$ being an ideal of $S^{-1}R$. Corollary 2.7 presents criteria for a localization of a prime ideal being a prime ideal (under certain conditions). For a ring R and its left denominator set $S \in \text{Den}_l(R)$, Proposition 1.2 describes the set of minimal primes of the ring $S^{-1}R$ via the minimal primes of the ring R provided some general conditions hold.

The largest regular left Ore set and the largest left quotient ring of a ring. Let R be a ring. A **multiplicatively closed subset** S of R or a **multiplicative subset** of R (i.e., a multiplicative sub-semigroup of $(R, \cdot)$ such that $1 \in S$ and $0 \notin S$) is said to be a **left Ore set** if it satisfies the **left Ore condition**: for each $r \in R$ and $s \in S$,

$$Sr \cap Rs \neq \emptyset.$$

Let $\text{Ore}_l(R)$ be the set of all left Ore sets of R . For $S \in \text{Ore}_l(R)$, $\text{ass}_l(S) := \{r \in R \mid sr = 0 \text{ for some } s \in S\}$ is an ideal of the ring R .

A left Ore set S is called a **left denominator set** of the ring R if $rs = 0$ for some elements $r \in R$ and $s \in S$ implies $tr = 0$ for some element $t \in S$, i.e. $r \in \text{ass}(S)$. Let $\text{Den}_l(R)$ be the set of all left denominator sets of R . For $S \in \text{Den}_l(R)$, the ring

$$S^{-1}R = \{s^{-1}r \mid s \in S, r \in R\}$$

is called the **left localization** of the ring R at S (the **left quotient ring** of R at S). In Ore's method of localization one can localize precisely at left denominator sets. In an obvious way, the **right Ore condition**, **right Ore** and **right denominator sets** of R are defined and they are denoted by $\text{Ore}_r(R)$ and $\text{Den}_r(R)$, respectively. For $S \in \text{Den}_r(R)$, the ring

$$RS^{-1} = \{rs^{-1} \mid s \in S, r \in R\}$$

is called the **right localization** of the ring R at S (the **right quotient ring** of R at S). For $S \in \text{Ore}_r(R)$, $\text{ass}_r(S) := \{r \in R \mid rs = 0 \text{ for some } s \in S\}$ is an ideal of the ring R .

A left and right Ore or denominator set is called an **Ore set** or a **denominator set** and these sets are denoted by $\text{Ore}(R)$ and $\text{Den}(R)$, respectively. If $S \in \text{Den}(R)$ then

$$S^{-1}R \simeq RS^{-1}.$$

For an ideal $\mathfrak{a}$ of R , let $\text{Den}_*(R, \mathfrak{a}) := \{S \in \text{Den}_*(R) \mid \text{ass}_*(S) = \mathfrak{a}\}$ where $*$ $\in \{l, r, \emptyset\}$.

In general, the set $\mathcal{C}_R$ of regular elements of a ring R is neither left nor right Ore set of the ring R and as a result neither left nor right classical quotient ring ($Q_{l,cl}(R) := \mathcal{C}_R^{-1}R$ and $Q_{r,cl}(R) := R\mathcal{C}_R^{-1}$) exists. There exists the largest (w.r.t. $\subseteq$) regular left Ore set $S_l(R)$, [2]. This means that the set $S_l(R)$ is an Ore set of the ring R that consists of regular elements (i.e. $S_l(R) \subseteq \mathcal{C}_R$) and contains all the left Ore sets in R that consist of regular elements. Also, there exists the largest regular right (respectively, left and right) Ore set $S_r(R)$ (respectively, $S_r(R)$, $S_{l,r}(R)$) of the ring R . In general, the sets $\mathcal{C}_R$, $S_l(R)$, $S_r(R)$ and $S_{l,r}(R)$ are distinct, for example, when $R = \mathbb{I}_1 = K\langle x, \partial, f \rangle$ is the ring of polynomial integro-differential operators over a field K of characteristic zero, [1]. In [1], these four sets are explicitly described for $R = \mathbb{I}_1$.

Definition 2.1. ([1,2].) The ring

$$Q_l(R) := S_l(R)^{-1}R$$

(respectively, $Q_r(R) := RS_r(R)^{-1}$ and $Q(R) := S_{l,r}(R)^{-1}R \simeq RS_{l,r}(R)^{-1}$) is called the **largest left** (respectively, **right and two-sided**) **quotient ring** of the ring R .

In general, the rings $Q_l(R)$, $Q_r(R)$ and $Q(R)$ are not isomorphic, for example, when $R = \mathbb{I}_1$, [1].

Let R be a ring. We say that two left localizations of R are **equal** and write $S^{-1}R = S'^{-1}R$ if the map $S^{-1}R \rightarrow S'^{-1}R$, $s^{-1}r \mapsto s'^{-1}r$, is a well-defined isomorphism. This isomorphism is an R -isomorphism. Then the map $S'^{-1}R \rightarrow S^{-1}R$, $s'^{-1}r \mapsto s^{-1}r$, is also a ring R -isomorphism. So, two localizations are equal iff there is an R -isomorphism between them. So, the relation of 'equality' is an equivalence relation on the set of left localizations of the ring R . The set of all the equivalence classes is denoted by $\text{Loc}_l(R)$. Clearly, $S^{-1}R = S'^{-1}R$ iff $\text{ass}_l(S) = \text{ass}_l(S')$, $\frac{s}{1} \in (S'^{-1}R)^\times$ for all $s \in S$ and $\frac{s'}{1} \in (S^{-1}R)^\times$ for all $s' \in S'$.

The next theorem gives various properties of the ring $Q_l(R)$. In particular, it describes its group of units.

Theorem 2.2. ([2, Theorem 2.8].)

1. $S_l(Q_l(R)) = Q_l(R)^\times$ and $S_l(Q_l(R)) \cap R = S_l(R)$.
2. $Q_l(R)^\times = \langle S_l(R), S_l(R)^{-1} \rangle$, i.e., the group of units of the ring $Q_l(R)$ is generated by the sets $S_l(R)$ and $S_l(R)^{-1} := \{s^{-1} \mid s \in S_l(R)\}$.
3. $Q_l(R)^\times = \{s^{-1}t \mid s, t \in S_l(R)\}$.
4. $Q_l(Q_l(R)) = Q_l(R)$.

A ring R has **finite left rank** (i.e. **finite left uniform dimension**) if there are no infinite direct sums of nonzero left ideals in R . For a non-empty subset T of a ring R , the sets

$$\text{l.ann}_R(T) := \{r \in R \mid rT = 0\} \text{ and } \text{r.ann}_R(T) := \{r \in R \mid Tr = 0\}$$

are called the **left** and **right annihilators** of T , respectively. They are left and right ideals of R , respectively. If they coincide, we write $\text{ann}_R(T)$ for their common value. A ring R is called a **left (right) Goldie ring** if R has finite left (right) uniform dimension and R satisfies the ascending chain condition (the a.c.c.) on left (right) annihilators.

The next theorem is a semisimplicity criterion for the ring $Q_l(R)$ (statements 2-5 are Goldie's Theorem, [7,8,11]).

Theorem 2.3. ([2, Theorem 2.9].) *The following properties of a ring R are equivalent:*

1. $Q_l(R)$ is a semisimple ring.
2. $Q_{l,cl}(R)$ exists and is a semisimple ring.
3. R is a left order in a semisimple ring.
4. R has finite left rank, satisfies the ascending chain condition on left annihilators and is a semi-prime ring.
5. A left ideal of R is essential iff it contains a regular element.

If one of the equivalent conditions holds then $S_0(R) = C_R$ and $Q_l(R) = Q_{l,cl}(R)$.

Let R be a ring, $S \in \text{Den}_l(R, \mathfrak{a})$,

$$\pi_{\mathfrak{a}} : R \rightarrow R/\mathfrak{a}, \quad r \mapsto \bar{r} := r + \mathfrak{a} \text{ and } \sigma_S : R \rightarrow S^{-1}R, \quad r \mapsto \frac{r}{1}.$$

Let $\mathcal{I}_l(R)$ and $\mathcal{I}(R)$ be the sets of left and two-sided ideals of the ring R , respectively. Then the maps below are well-defined:

$$S^{-1} : \mathcal{I}_l(R) \rightarrow \mathcal{I}_l(S^{-1}R), \quad I \mapsto S^{-1}I \text{ and } \sigma_S^{-1} : \mathcal{I}_l(S^{-1}R) \rightarrow \mathcal{I}_l(R), \quad J \mapsto \sigma_S^{-1}(J). \quad (3)$$

Furthermore, $S^{-1}\sigma_S^{-1}(J) = J$ for all $J \in \mathcal{I}_l(S^{-1}R)$. In general, the map $S^{-1} : \mathcal{I}(R) \rightarrow \mathcal{I}(S^{-1}R)$, $I \mapsto S^{-1}I$ is *not* defined (see [6, Example 10L]) but the map

$$\sigma_S^{-1} : \mathcal{I}(S^{-1}R) \rightarrow \mathcal{I}(R), \quad J \mapsto \sigma_S^{-1}(J) \quad (4)$$

is a well-defined map and $S^{-1}\sigma_S^{-1}(J) = J$ for all $J \in \mathcal{I}(S^{-1}R)$. So, the localized ring $S^{-1}R$ has 'less' ideals than expected since the images of some ideals of the ring R are only *left* ideals but not two-sided ideals of the ring $S^{-1}R$. This phenomenon does not occur if the ring R is a commutative ring or the denominator set S consists of central elements of R . Nonetheless all the ideal of $S^{-1}R$ can be obtained from some ideals of

R . The latter are precisely the ideals of R that stay two-sided ideals under the localization at S . [6, Example 10M] is an example of a prime ring $S^{-1}R$ such that the ideal $\sigma_S^{-1}(0)$ of R is not a prime or even semiprime ideal.

Criterion for the left ideal $S^{-1}\mathfrak{a}$ being an ideal of $S^{-1}R$ where $\mathfrak{a}$ is an ideal of R . Let R be a ring, $S \in \text{Den}_l(R, \mathfrak{a})$, $\overline{R} := R/\mathfrak{a}$ and $\pi_{\mathfrak{a}} : R \rightarrow \overline{R}$, $r \mapsto r + \mathfrak{a}$. Then

$$\overline{S} := \pi_{\mathfrak{a}}(S) \in \text{Den}_l(\overline{R}, 0) \text{ and } S^{-1}R \simeq \overline{S}^{-1}\overline{R}.$$

For a right R -module M , the set

$$\text{tor}_{r,S}(M) := \{m \in M \mid ms = 0 \text{ for some } s \in S\}$$

is the **set of S -torsion elements** of M . In general, $\text{tor}_{r,S}(M)$ is *not* a submodule of M unless S is a right denominator set of R .

Suppose that $\mathfrak{b}$ is an ideal of a ring R . In general, the left ideal $S^{-1}\mathfrak{b}$ of the ring $S^{-1}R$ is *not* an ideal. Proposition 2.4 is a criterion for the left ideal $S^{-1}\mathfrak{b}$ being an ideal of $S^{-1}R$.

Proposition 2.4. *Let R be a ring, $S \in \text{Den}_l(R, \mathfrak{a})$ and $\mathfrak{b}$ be an ideal of R . Then the following statements are equivalent:*

1. *The left ideal $S^{-1}\mathfrak{b}$ of the ring $S^{-1}R$ is an ideal.*
2. *For all elements $s \in S$, $\mathfrak{b}s^{-1} \subseteq S^{-1}\mathfrak{b}$.*
3. *If $rs \in \mathfrak{b}$ for some elements $r \in R$ and $s \in S$ then $s'r \in \mathfrak{b}$ for some $s' \in S$.*
4. *$rs \in \mathfrak{a} + \mathfrak{b}$ for some elements $s \in S$ and $r \in R$ then $s'r \in \mathfrak{a} + \mathfrak{b}$ for some $s' \in S$.*
5. *$\text{tor}_{r,\overline{S}}(\overline{R}/\pi_{\mathfrak{a}}(\mathfrak{b})) \subseteq \text{tor}_{l,\overline{S}}(\overline{R}/\pi_{\mathfrak{a}}(\mathfrak{b}))$.*
6. *For each $s \in S$, the ascending chain of left ideals of the rings $S^{-1}R$, $\mathfrak{b}'_0 \subseteq \mathfrak{b}'_1 \subseteq \dots \subseteq \mathfrak{b}'_i \subseteq \dots$, stabilizes where $\mathfrak{b}'_i := \sum_{j=0}^i S^{-1}\mathfrak{b}s^{-j}$ for $i \geq 0$.*

Proof. $(1 \Leftrightarrow 2 \Rightarrow 6)$ Straightforward.

$(2 \Rightarrow 3)$ Suppose that $rs \in \mathfrak{b}$ for some elements $r \in R$ and $s \in S$. Then

$$\frac{r}{1} \in \mathfrak{b}s^{-1} \subseteq S^{-1}\mathfrak{b},$$

and so $s'r \in \mathfrak{b}$ for some $s' \in S$.

$(3 \Rightarrow 2)$ Let $b \in \mathfrak{b}$ and $s \in S$. Then $bs^{-1} = s_1^{-1}r$ for some elements $s_1 \in S$ and $r \in R$ or, equivalently,

$$ts_1b = trs$$

for some element $t \in S$. It follows from the inclusion $(tr)s = ts_1b \in \mathfrak{b}$ that $t_1tr \in \mathfrak{b}$ for some element $t_1 \in S$. Now, $bs^{-1} = (t_1ts_1)^{-1} \cdot t_1tr \in S^{-1}\mathfrak{b}$.

$(3 \Rightarrow 4)$ $rs \in \mathfrak{a} + \mathfrak{b}$ for some elements $s \in S$ and $r \in R$. There is an element $s_1 \in S$ such that $s_1rs \in \mathfrak{b}$. Now, by statement 3, there is an element $s_2 \in S$ such that $s_2s_1r \in \mathfrak{b} \subseteq \mathfrak{a} + \mathfrak{b}$.

$(4 \Rightarrow 3)$ Suppose that $rs \in \mathfrak{b} \subseteq \mathfrak{a} + \mathfrak{b}$ for some elements $r \in R$ and $s \in S$. By statement 4, there is an element $s_1 \in S$ such that $s_1r \in \mathfrak{a} + \mathfrak{b}$. Hence, $s_2s_1r \in \mathfrak{b}$ for some $s_2 \in S$.

$(4 \Leftrightarrow 5)$ The equivalence is an obvious re-writing of the inclusion $\text{tor}_{r,\overline{S}}(\overline{R}/\pi_{\mathfrak{a}}(\mathfrak{b})) \subseteq \text{tor}_{l,\overline{S}}(\overline{R}/\pi_{\mathfrak{a}}(\mathfrak{b}))$ in terms of the ring R (since $\overline{R}/\pi_{\mathfrak{a}}(\mathfrak{b}) \simeq R/(\mathfrak{a} + \mathfrak{b})$).

$(6 \Rightarrow 2)$ For each $s \in S$, there is a natural number $n \geq 0$ such that $\mathfrak{b}'_n = \mathfrak{b}'_{n+1}$. Hence, $S^{-1}\mathfrak{b} \supseteq \sum_{i=0}^n S^{-1}\mathfrak{b}s^i \subseteq S^{-1}\mathfrak{b}s^{-1}$, and statement 2 follows. $\square$

Corollary 2.5. *Let R be a ring, $S \in \text{Den}_l(R, \mathfrak{a})$ and $\mathfrak{b}$ be an ideal of R . Then the left ideal $S^{-1}\mathfrak{b}$ of the ring $S^{-1}R$ is not an ideal iff there is an element $s \in S$ such that the ascending chain of left ideals of the rings $S^{-1}R$, $\mathfrak{b}'_0 \subset \mathfrak{b}'_1 \subset \cdots \subset \mathfrak{b}'_i \subset \cdots$, is strictly increasing where $\mathfrak{b}'_i = \sum_{j=0}^i S^{-1}\mathfrak{b}s^{-j}$ for $i \geq 0$. In this case, neither the ring R nor its localization $S^{-1}R$ is a left Noetherian ring.*

Proof. The corollary follows from Proposition 2.4. $\square$

Lemma 2.6. *If R is a prime ring and $S \in \text{Den}_l(R, 0)$ then the ring $S^{-1}R$ is a prime ring.*

Proof. Since $S \in \text{Den}_l(S, 0)$, the module ${}_R R$ is an essential left R -submodule of $S^{-1}R$. Suppose that $\mathfrak{a}\mathfrak{b} = 0$ for some ideals $\mathfrak{a}$ and $\mathfrak{b}$ of $S^{-1}R$. Then $\mathfrak{a}'\mathfrak{b}' = 0$ where $\mathfrak{a}' = \mathfrak{a} \cap R$ and $\mathfrak{b}' = \mathfrak{b} \cap R$ are ideals of R . The ring R is a prime ring. Therefore, at least one of the ideals $\mathfrak{a}'$ or $\mathfrak{b}'$ is equal to zero, say $\mathfrak{a}' = 0$. Then $\mathfrak{a} = 0$, and so the ring $S^{-1}R$ is a prime ring. $\square$

Corollary 2.7. *Suppose that R is a ring, $S \in \text{Den}_l(R, 0)$ and $\mathfrak{p} \in \text{Spec}(R)$.*

1. *Suppose that $S^{-1}\mathfrak{p} \cap R = \mathfrak{p}$. Then $S^{-1}\mathfrak{p} \in \text{Spec}(S^{-1}R)$ iff the left ideal $S^{-1}\mathfrak{p}$ of $S^{-1}R$ is an ideal.*
2. *Suppose that $S^{-1}\mathfrak{p} \cap R \in \text{Spec}(R)$. Then $S^{-1}\mathfrak{p} \in \text{Spec}(S^{-1}R)$ iff the left ideal $S^{-1}\mathfrak{p}$ of $S^{-1}R$ is an ideal.*

Proof. 1. ($\Rightarrow$) The implication is obvious.

($\Leftarrow$) Suppose that the left ideal $S^{-1}\mathfrak{p}$ of $S^{-1}R$ is an ideal. By the assumption, $S^{-1}\mathfrak{p} \cap R = \mathfrak{p}$. Hence, $S^{-1}\mathfrak{p} \neq S^{-1}R$. Since $S \in \text{Den}_l(S, 0)$, the module ${}_R R$ is an essential left R -submodule of $S^{-1}R$. Suppose that $\mathfrak{a}\mathfrak{b} \subseteq S^{-1}\mathfrak{p}$ for some ideals $\mathfrak{a}$ and $\mathfrak{b}$ of $S^{-1}R$. Then

$$\mathfrak{a}'\mathfrak{b}' \subseteq R \cap S^{-1}\mathfrak{p} = \mathfrak{p}$$

where $\mathfrak{a}' = \mathfrak{a} \cap R$ and $\mathfrak{b}' = \mathfrak{b} \cap R$ are ideals of R . Since $\mathfrak{p} \in \text{Spec}(R)$, at least one of the ideals $\mathfrak{a}'$ or $\mathfrak{b}'$ is contained in $\mathfrak{p}$, say $\mathfrak{a}' \subseteq \mathfrak{p}$. Then $S^{-1}\mathfrak{p} \supseteq S^{-1}\mathfrak{a}' = \mathfrak{a}$, and so $S^{-1}\mathfrak{p} \in \text{Spec}(S^{-1}R)$.

2. By the assumption $\mathfrak{p}' := S^{-1}\mathfrak{p} \cap R \in \text{Spec}(R)$. Then

$$\begin{aligned} S^{-1}\mathfrak{p}' \cap R &= S^{-1}(S^{-1}\mathfrak{p} \cap R) \cap R = S^{-1}\mathfrak{p} \cap S^{-1}R \cap R = S^{-1}\mathfrak{p} \cap R = \mathfrak{p}', \\ S^{-1}\mathfrak{p}' &= S^{-1}(S^{-1}\mathfrak{p} \cap R) = S^{-1}\mathfrak{p} \cap S^{-1}R = S^{-1}\mathfrak{p}. \end{aligned}$$

By statement 1, $S^{-1}\mathfrak{p}' = S^{-1}\mathfrak{p} \in \text{Spec}(S^{-1}R)$ iff the left ideal $S^{-1}\mathfrak{p}' = S^{-1}\mathfrak{p}$ of $S^{-1}R$ is an ideal. $\square$

For a prime ideal $\mathfrak{p}$ and a left denominator set S , Proposition 2.8 provides sufficient conditions for $\sigma_S^{-1}(S^{-1}\mathfrak{p}) = \mathfrak{p}$.

Proposition 2.8. *Suppose that R is a ring, $S \in \text{Den}_l(R, \mathfrak{a})$, $\mathfrak{p} \in \text{Spec}(R)$, $S \cap \mathfrak{p} = \emptyset$ and the ideal $\sigma_S^{-1}(S^{-1}\mathfrak{p})$ of R is a finitely generated right R -module. Then $\sigma_S^{-1}(S^{-1}\mathfrak{p}) = \mathfrak{p}$.*

Proof. Since $\mathfrak{p}$ is an ideal of R , the left ideal of R ,

$$\mathfrak{q} := \sigma_S^{-1}(S^{-1}\mathfrak{p}) = \{r \in R \mid sr \in \mathfrak{p} \text{ for some } s \in S\},$$

is an ideal. Clearly, $\mathfrak{p} \subseteq \mathfrak{q}$ and for each element $q \in \mathfrak{q}$ there is an element $s \in S$ such that $sq = 0$. By the assumption, the right R -module $\mathfrak{q}$ is finitely generated. Let $q_1, \dots, q_m$ be a generating set. Then we can fix an element $s \in S$ such that $sq_i \in \mathfrak{p}$ for all $i = 1, \dots, m$. Hence, $\mathfrak{p} \supseteq s\mathfrak{q}$, and so

$$\mathfrak{p} \supseteq (s)\mathfrak{q},$$

and $\mathfrak{q} \subseteq \mathfrak{p}$ since $s \notin \mathfrak{p}$ (since $\mathfrak{p} \cap S = \emptyset$). Therefore, $\mathfrak{p} = \mathfrak{q}$. $\square$

Proposition 2.9 is a generalization of Lemma 2.6.

Proposition 2.9. *If R is a ring and $S \in \text{Den}_l(R, \mathfrak{q})$ such that $\mathfrak{q} \in \text{Spec}(R)$. Then $S^{-1}R$ is a prime ring.*

Proof. Suppose that $\mathfrak{a}\mathfrak{b} = 0$ for some ideals $\mathfrak{a}$ and $\mathfrak{b}$ of $S^{-1}R$. Then $\mathfrak{a}'\mathfrak{b}' \subseteq \mathfrak{q}$ where $\mathfrak{a}' = \sigma_S^{-1}(\mathfrak{a})$ and $\mathfrak{b}' = \sigma_S^{-1}(\mathfrak{b})$ are ideals of R . Since $\mathfrak{q} \in \text{Spec}(R)$, at least one of the ideals $\mathfrak{a}'$ or $\mathfrak{b}'$ is a subset of $\mathfrak{q}$, say $\mathfrak{a}' \subseteq \mathfrak{q}$. Then $\mathfrak{a} = S^{-1}\sigma_S^{-1}(\mathfrak{a}) \subseteq S^{-1}\mathfrak{q} = 0$, and so the ring $S^{-1}R$ is a prime ring. $\square$

Lemma 2.10 is a criterion for an epimorphic image of a left denominator set being a left denominator set.

Lemma 2.10. *Suppose that R is a ring, $S \in \text{Den}_l(R, \mathfrak{a})$ and $\mathfrak{b}$ be an ideal of R such that $\mathfrak{a} \subseteq \mathfrak{b}$. Let $\pi_{\mathfrak{b}} : R \rightarrow R/\mathfrak{b}$, $r \mapsto \bar{r} = r + \mathfrak{b}$ and $\bar{S} = \pi_{\mathfrak{b}}(S)$. Then $\bar{S} \in \text{Den}_l(R/\mathfrak{b}, 0)$ iff $\bar{S} \subseteq \mathcal{C}_{R/\mathfrak{b}}$.*

Proof. $(\Rightarrow)$ The implication is obvious.

$(\Leftarrow)$ The inclusion $\bar{S} \subseteq \mathcal{C}_{R/\mathfrak{b}}$ implies that $\bar{S} \in \text{Ore}_l(R/\mathfrak{b})$ is a left Ore set of the ring $\mathfrak{b}$ that consists of regular elements of $\bar{R}$, i.e. $\bar{S} \in \text{Den}_l(R/\mathfrak{b}, 0)$. $\square$

Lemma 2.11 is generalization of Lemma 2.10.

Lemma 2.11. *Suppose that R is a ring, $S \in \text{Den}_l(R, \mathfrak{a})$, $\mathfrak{b}$ be an ideal of R such that $S \cap (\mathfrak{a} + \mathfrak{b}) = \emptyset$. Let $\pi_{\mathfrak{a}+\mathfrak{b}} : R \rightarrow \bar{R} := R/(\mathfrak{a} + \mathfrak{b})$, $r \mapsto \bar{r} = r + \mathfrak{a} + \mathfrak{b}$, $\bar{S} = \pi_{\mathfrak{a}+\mathfrak{b}}(S)$, and $\bar{\mathfrak{c}} := \text{ass}_l(\bar{S})$. Let $\pi_{\bar{\mathfrak{c}}} : \bar{R} \rightarrow \bar{R}/\bar{\mathfrak{c}}$, $r \mapsto \tilde{r} = \bar{r} + \bar{\mathfrak{c}}$ and $\tilde{S} = \pi_{\bar{\mathfrak{c}}}(\bar{S})$. Then $\tilde{S} \in \text{Den}_l(\bar{R}/\bar{\mathfrak{c}}, 0)$ iff $\text{tor}_{r,S}(\bar{R}/\bar{\mathfrak{c}}) = 0$ (i.e. if $\bar{r}s \in \bar{\mathfrak{c}}$ for some $\bar{r} \in R$ and $s \in S$ then $\bar{r} \in \bar{\mathfrak{c}}$).*

Proof. The condition that $S \cap (\mathfrak{a} + \mathfrak{b}) = \emptyset$ implies that $\bar{S} \in \text{Ore}_l(\bar{R}, \bar{\mathfrak{c}})$. Then $\tilde{S} \in \text{Ore}_l(\bar{R}/\bar{\mathfrak{c}}, 0)$. Now, the lemma is obvious. $\square$

The set of minimal primes of the ring $S^{-1}R$ where $S \in \text{Den}_l(R)$. For an ideal $\mathfrak{a}$ of a ring R , let $\min(\mathfrak{a})$ be the set of minimal prime ideals over $\mathfrak{a}$. So, the map

$$\min(\mathfrak{a}) \rightarrow \min(R/\mathfrak{a}), \quad \mathfrak{p} \mapsto \mathfrak{p}/\mathfrak{a} \quad (5)$$

is a bijection where $\min(R/\mathfrak{a})$ is the set of minimal prime ideals of the factor ring $R/\mathfrak{a}$. For a ring R , the ideal

$$\mathfrak{n}_R := \bigcap_{\mathfrak{p} \in \text{Spec}(R)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \min(R)} \mathfrak{p}$$

is called the **prime radical** of R . If $\mathfrak{n}_R = 0$, the ring R is called a **semiprime ring**. The ring R is a semiprime ring iff the zero ideal is the only nilpotent ideal of R . Then the map

$$\min(R) \rightarrow \min(R/\mathfrak{n}_R), \quad \mathfrak{p} \mapsto \mathfrak{p}/\mathfrak{n}_R \quad (6)$$

is a bijection. So, in dealing with minimal primes of a ring without loss of generality we may assume that the ring is a *semiprime* ring. For $S \in \text{Den}_l(R)$, let

$$\min(R, S) := \{\mathfrak{p} \in \min(R) \mid \mathfrak{p} \cap S = \emptyset\} \text{ and } \min(R, S, \text{id}) := \{\mathfrak{p} \in \min(R) \mid S^{-1}\mathfrak{p}S^{-1}R \neq S^{-1}R\}.$$

Since $S^{-1}\mathfrak{p} \subseteq S^{-1}\mathfrak{p}S^{-1}R$,

$$\min(R, S) \subseteq \min(R, S, \text{id}).$$

Definition 2.12. A ring is called a **prime rich ring** if each ideal contains a finite product of prime ideals that contain it.

Example 2.13. Every one-sided Noetherian ring R is prime rich (since $|\min(R)| < \infty$ and the prime radical $\mathfrak{n}_R$ is a nilpotent ideal).

Proposition 2.14.

1. Suppose that R is a ring such that $\mathfrak{p}_1 \cdots \mathfrak{p}_n = 0$ for some prime ideals $\mathfrak{p}_i$ of R . Then $\min(R) \subseteq \min\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ and $|\min(R)| \leq n < \infty$.
2. Every ideal of a prime rich ring has only finitely many minimal primes.

Proof. 1. Let $\mathfrak{q} \in \min(R)$. Then the inclusion $\mathfrak{p}_1 \cdots \mathfrak{p}_n = \{0\} \subseteq \mathfrak{q}$ implies the inclusion $\mathfrak{p}_i \subseteq \mathfrak{q}$ for some i , and so $\mathfrak{p}_i = \mathfrak{q}$, by the minimality of $\mathfrak{q}$, and the result follows.

2. Statement 2 follows from statement 1. In more detail, if $\mathfrak{a}$ is an ideal of R then $\mathfrak{q}_1 \cdots \mathfrak{q}_m \subseteq \mathfrak{a}$ for some prime ideals $\mathfrak{q}_i$ of R that contains $\mathfrak{a}$. Let $R \rightarrow \overline{R} = R/\mathfrak{a}$, $r \mapsto \bar{r} = r + \mathfrak{a}$. Then

$$\bar{\mathfrak{q}}_1 \cdots \bar{\mathfrak{q}}_m = 0 \text{ and } \{\bar{\mathfrak{q}}_1, \dots, \bar{\mathfrak{q}}_m\} \subseteq \text{Spec}(\overline{R}).$$

By statement 1, $\min(\overline{\mathfrak{a}}) = \min\{\bar{\mathfrak{q}}_1, \dots, \bar{\mathfrak{q}}_m\}$, and so $\min(\mathfrak{a}) = \min\{\mathfrak{q}_1, \dots, \mathfrak{q}_m\}$ and statement 2 follows. $\square$

Proof of Proposition 1.1. ($1 \Rightarrow 3$) Suppose that the ring R is a prime rich ring. Then each ideal $\mathfrak{a}$ of R contains a product of prime ideals, say $\mathfrak{p}_1 \cdots \mathfrak{p}_n$. By Proposition 2.14.(1),

$$\min(\mathfrak{a}) \subseteq \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}.$$

Recall that the map $\min(\mathfrak{a}) \rightarrow \min(R/\mathfrak{a})$, $\mathfrak{p} \mapsto \mathfrak{p}/\mathfrak{a}$ is a bijection. Now, it follows from the inclusions

$$\mathfrak{a} \supseteq \mathfrak{p}_1 \cdots \mathfrak{p}_n \supseteq \left(\bigcap_{\mathfrak{p} \in \min(\mathfrak{a})} \mathfrak{p} \right)^n$$

that the prime radical $\mathfrak{n}_{R/\mathfrak{a}}$ of the factor ring $R/\mathfrak{a}$ is a nilpotent ideal.

($3 \Rightarrow 2$) Let $\mathfrak{a}$ be an ideal of R . By the assumption, $|\min(\mathfrak{a})| < \infty$ and the prime radical $\mathfrak{n}_{R/\mathfrak{a}}$ of the factor ring $R/\mathfrak{a}$ is a nilpotent ideal. Hence,

$$\mathfrak{a} \supseteq \left(\bigcap_{\mathfrak{p} \in \min(\mathfrak{a})} \mathfrak{p} \right)^n \supseteq \left(\prod_{\mathfrak{p} \in \min(\mathfrak{a})} \mathfrak{p} \right)^n,$$

as required.

($2 \Rightarrow 1$) Straightforward. $\square$

Definition 2.15. A left denominator set S of a ring R is called an **ideal preserving** left denominator set (or a left denominator set that **respects the ideal structure** of R) if the map $S^{-1} : \mathcal{I}(R) \rightarrow \mathcal{I}(S^{-1}R)$, $I \mapsto S^{-1}I$

is a well-defined map, i.e. for each ideal I of R , $S^{-1}I$ is an ideal of the ring $S^{-1}R$. The set of all ideal preserving left denominator sets is denoted by $\text{Den}_{l,ip}(R)$. A ring R is called an **ipl-ring** (an **ideal preserving ring under left localizations**) if $\text{Den}_{l,ip}(R) = \text{Den}_l(R)$, i.e. every left denominator set of R is ideal preserving.

Definition 2.16. We say that a left denominator set S of a ring R **respects the ideal structure of primes** of R if for each prime ideal $\mathfrak{p} \in \text{Spec}(R)$ the left ideal $S^{-1}\mathfrak{p}$ of the ring $S^{-1}R$ is an ideal.

If the left denominator set respects the ideal structure of a ring then it also respects the ideal structure of primes. By Corollary 2.7, if a left denominator set S of a ring R respects the ideal structure of primes then it also respects primeness of those prime ideals of R that satisfy the assumption of Corollary 2.7.

Example 2.17. Every denominator set that belong to the centre of a ring respects the ideal structure. All denominator sets of a commutative ring respect the ideal structure of the ring.

Proposition 2.18. Let R be a ring and $S \in \text{Den}_l(S, \mathfrak{a})$. Suppose that the ring $S^{-1}R$ is a left Noetherian ring (e.g. R is a left Noetherian ring) then S respects the ideal structure of R .

Proof. Let $\mathfrak{b}$ be an ideal of the ring R and $s \in S$. For each natural number for $i \geq 0$, let $\mathfrak{b}_i = \sum_{j=0}^i S^{-1}\mathfrak{b}s^{-j}$. By the assumption, the ring $S^{-1}R$ is a left Noetherian ring. So, the ascending chain of left ideals of the ring $S^{-1}R$,

$$\mathfrak{b} = \mathfrak{b}_0 \subseteq \mathfrak{b}_1 \subseteq \dots$$

stabilizes, say on n 'th step. Then $S^{-1}\mathfrak{b}s^{-n-1} \subseteq \mathfrak{b}_n$, and so

$$S^{-1}\mathfrak{b}s^{-1} \subseteq \mathfrak{b}_n s^n = \sum_{j=0}^n S^{-1}\mathfrak{b}s^j \subseteq S^{-1}\mathfrak{b},$$

i.e. the left ideal $S^{-1}\mathfrak{b}$ of the ring $S^{-1}R$ is an ideal. $\square$

Proof of Proposition 1.2. 1. (i) $|\min(R)| < \infty$: The ring R is prime rich. By Proposition 2.14.(2), $|\min(R)| < \infty$.

(ii) $\min(S^{-1}R) \subseteq \min\{S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \min(R, S)\}$: Let $\mathfrak{q} \in \min(S^{-1}R)$. Then the ideal $\sigma_S^{-1}(\mathfrak{q})$ of the prime rich ring R contains a finite product of prime ideals of R , say $\mathfrak{p}_1 \cdots \mathfrak{p}_n$, that contain $\sigma_S^{-1}(\mathfrak{q})$. The left denominator set $S \in \text{Den}(R, \mathfrak{a})$ respects the ideal structure of primes. So,

$$S^{-1}\mathfrak{p}_1 \cdots S^{-1}\mathfrak{p}_n = S^{-1}\mathfrak{p}_1 \mathfrak{p}_2 \cdots \mathfrak{p}_n \subseteq S^{-1}\sigma_S^{-1}(\mathfrak{q}) = \mathfrak{q}.$$

Hence, $S^{-1}\mathfrak{p}_i \subseteq \mathfrak{q}$ for some i since $\mathfrak{q} \in \text{Spec}(S^{-1}R)$, and so $\mathfrak{p}_i \subseteq \sigma_S^{-1}(S^{-1}\mathfrak{p}_i) \subseteq \sigma_S^{-1}(\mathfrak{q})$. There exists a minimal prime ideal $\mathfrak{p} \in \min(R)$ such that $\mathfrak{p} \subseteq \mathfrak{p}_i$. Since $S^{-1}\mathfrak{p} \subseteq S^{-1}\mathfrak{p}_i \subseteq \mathfrak{q}$, we have that $\mathfrak{p} \in \min(R, S)$. Then, by the assumption, $S^{-1}\mathfrak{p} \in \text{Spec}(S^{-1}R)$. Hence,

$$S^{-1}\mathfrak{p} \subseteq S^{-1}\sigma_S^{-1}(\mathfrak{q}) = \mathfrak{q},$$

and so $\mathfrak{q} = S^{-1}\mathfrak{p}$, by the minimality of $\mathfrak{q}$, and the statement (ii) follows.

(iii) $\min(S^{-1}R) = \min\{S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \min(R, S)\}$: The statement (iii) follows from the statement (ii).

(iv) $1 \leq |\min(R, S)| \leq |\min(R)| < \infty$: The set of minimal primes of an arbitrary ring is a non-empty set. Now, the statement (iv) follows from the statements (i) and (iii).

2. Statement 2 follows from statement 1.

3. (i) For each $\mathfrak{p} \in \min(R, S)$, $\sigma_S^{-1}(S^{-1}\mathfrak{p}) = \mathfrak{p}$: The statement (i) follows from Proposition 2.8

(ii) $\min(S^{-1}R) = \{S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \min(R, S)\}$: In view of statement 2, we have to show that the ideals $S^{-1}\mathfrak{p}$ and $S^{-1}\mathfrak{p}'$ are incomparable for all distinct prime ideals $\mathfrak{p}, \mathfrak{p}' \in \min(R, S)$. Suppose that $S^{-1}\mathfrak{p} \subset S^{-1}\mathfrak{p}'$, a strict inclusion. Then, by the statement (i),

$$\mathfrak{p} = \sigma_S^{-1}(S^{-1}\mathfrak{p}) \subseteq \sigma_S^{-1}(S^{-1}\mathfrak{p}') = \mathfrak{p}'.$$

Hence, $\mathfrak{p} = \mathfrak{p}'$ (since $\mathfrak{p}, \mathfrak{p}' \in \min(R)$), a contradiction. $\square$

Proof of Corollary 1.3. (i) *The ring R is a prime rich ring*: The prime radical of an one-sided Noetherian ring is a nilpotent ideal. For each ideal $\mathfrak{a}$ of R , the factor ring $R/\mathfrak{a}$ is also one-sided Noetherian. Hence, there is a natural number n such that

$$\mathfrak{a} \supseteq \left(\bigcap_{\mathfrak{p} \in \min(\mathfrak{a})} \mathfrak{p} \right)^n \supseteq \left(\prod_{\mathfrak{p} \in \min(\mathfrak{a})} \mathfrak{p} \right)^n.$$

(ii) *S respects the ideal structure of the ring R* : Since the ring $S^{-1}R$ is a left Noetherian, the statement (ii) follows from Proposition 2.18.

(iii) *For each $\mathfrak{p} \in \min(R, S)$, $\sigma_S^{-1}(S^{-1}\mathfrak{p}) = \mathfrak{p}$* : Since the ring R is a right Noetherian ring, the statement (iii) follows from Proposition 2.8.

(iv) *$S^{-1}\mathfrak{p} \in \text{Spec}(S^{-1}R)$ for all $\mathfrak{p} \in \min(R, S)$* : The statement (iv), follows from Corollary 2.7.(1) and the statements (ii) and (iii).

Now, the corollary follows from Proposition statements (i), (iii) and (iv), and 1.2.(3). $\square$

3. Description of the set of minimal primes of a localization of a semiprime ring at a regular denominator set

The aim of this section is to prove Theorem 1.4.(2) that provides an explicit description of the set of minimal primes of localizations of semiprime rings provided that they have only finitely many minimal primes. Theorem 1.4.(2) is one of the key facts in understanding the structure of embeddings of localizations of semiprime rings into semisimple or semiprimary rings, see [5]. As a corollary, for a semiprime ring R with finitely many minimal primes, Corollary 3.3 describes the set of minimal primes of the largest left quotient ring $Q_l(R)$ of R . Corollary 3.3 shows that the largest left quotient ring $Q_l(R)$ of a semiprime ring R is also a semiprime ring and provides an explicit description of the set of the minimal primes $\min(Q_l(R))$ provided that $|\min(R)| < \infty$. Corollary 3.4 is a generalization of Theorem 1.4. Theorem 3.9 is a generalization of Theorem 1.4.(2,3) for left denominator sets that contain zero divisors. A proof of Proposition 1.5 is given in this section.

Characterization of the set of minimal primes of a semiprime ring.

Definition 3.1. A finite set Q of ideals of a ring R with zero intersection is called an **irredundant set** of ideals if $\bigcap_{\mathfrak{q} \in Q \setminus \{\mathfrak{q}'\}} \mathfrak{q} \neq 0$ for all $\mathfrak{q}' \in Q$.

Lemma 3.2 is a useful characterization of the set $\min(R)$ of minimal primes of a semiprime ring R provided $|\min(R)| < \infty$.

Lemma 3.2. *Let R be a semiprime ring with $|\min(R)| < \infty$. Then the set $\min(R)$ is the only irredundant set that consist of prime ideals of R .*

Proof. Evidently, the set $\min(R)$ is an irredundant set that consist of prime ideals of R .

Suppose that Q is an irredundant set that consist of prime ideals of R . We have to show that $Q = \min(R)$. Since the set $\min(R)$ is an irredundant set of prime ideals, it suffices to show that $Q \supseteq \min(R)$. For each $\mathfrak{p} \in \min(R)$,

$$\bigcap_{\mathfrak{q} \in Q} \mathfrak{q} = \{0\} \subseteq \mathfrak{p},$$

and so $\mathfrak{q} \subseteq \mathfrak{p}$ for some $\mathfrak{q} \in Q$. Hence $\mathfrak{p} = \mathfrak{p}$, by the minimality of $\mathfrak{p}$. Therefore, $Q \supseteq \min(R)$. $\square$

Description of the set of minimal primes of a localization of a semiprime ring at a regular denominator set. If T is an ideal of R then non-empty subset T of a ring R , the sets $\text{l.ann}_R(T)$ and $\text{r.ann}_R(T)$ are also ideals. Ideals of that kind are called **annihilator ideals**.

A submodule of a module is called an **essential submodule** if it meets all the nonzero submodules of the module. In a semiprime ring the left annihilator of an ideal is equal to its right annihilator and vice versa.

Proof of Theorem 1.4. Since $S \in \text{Den}_l(S, 0)$, ${}_R R$ is an essential left R -submodule of $S^{-1}R$.

1. Suppose that $\mathfrak{a}$ is a nonzero nilpotent ideal of $S^{-1}R$ then $\mathfrak{a}' = R \cap \mathfrak{a}$ is a nonzero nilpotent ideal of R , a contradiction (since R is a semiprime ring). Therefore, R is a semiprime ring.

2 and 3. The case where $|\min(R)| = 1$, i.e. the ring R is a prime ring, follows from Lemma 2.6.

So, we assume that $|\min(R)| \geq 2$. For each $\mathfrak{p} \in \min(R)$, let $\mathfrak{p}^c := \bigcap_{\mathfrak{q} \in \min(R) \setminus \{\mathfrak{p}\}} \mathfrak{q}$. By Lemma 3.2, $\mathfrak{p}^c \neq 0$.

(i) For all $\mathfrak{p} \in \min(R)$, $\text{ann}(\mathfrak{p}^c) = \mathfrak{p}$: The ring R is a semiprime ring, i.e. $\bigcap_{\mathfrak{p} \in \min(R)} \mathfrak{p} = 0$. Hence, $\mathfrak{p}\mathfrak{p}^c = 0$. Therefore,

$$\mathfrak{p} \subseteq \text{ann}_R(\mathfrak{p}^c).$$

Suppose that $\mathfrak{a}\mathfrak{p}^c = 0$ for some ideal $\mathfrak{a}$ of R . Then $\mathfrak{a}\mathfrak{p}^c \subseteq \mathfrak{p}$, and so

$$\mathfrak{a} \subseteq \mathfrak{p}$$

(since $\mathfrak{p}^c \not\subseteq \mathfrak{p}$), and the statement (i) follows.

(ii) For all $\mathfrak{p} \in \min(R)$, $\text{ann}(\mathfrak{p}) = \mathfrak{p}^c$: By the statement (i), $\text{ann}(\mathfrak{p}) \supseteq \mathfrak{p}^c$. Suppose that $\mathfrak{a}\mathfrak{p} = 0$ for some ideal $\mathfrak{a}$ of R . Then $\mathfrak{a}\mathfrak{p} \subseteq \mathfrak{q}$ for all $\mathfrak{q} \in \min(R) \setminus \{\mathfrak{p}\}$, and so

$$\mathfrak{a} \subseteq \mathfrak{q}$$

(since $\mathfrak{p} \not\subseteq \mathfrak{q}$), i.e. $\mathfrak{a} \subseteq \mathfrak{p}^c$ and the statement (ii) follows.

(iii) For all $\mathfrak{p} \in \min(R)$, $S \cap \mathfrak{p} = \emptyset$: Since $\mathfrak{p}^c \neq 0$, $\mathfrak{p}^c \mathfrak{p} = \mathfrak{p}\mathfrak{p}^c = 0$ (the statement (i)), and $S \subseteq \mathcal{C}_R$, we must have $S \cap \mathfrak{p} = \emptyset$ for all $\mathfrak{p} \in \min(R)$.

Let $\pi_{\mathfrak{p}} : R \rightarrow R/\mathfrak{p}$, $r \mapsto r + \mathfrak{p}$.

(iv) For all $\mathfrak{p} \in \min(R)$, $\pi_{\mathfrak{p}}(S) \subseteq \mathcal{C}_{R/\mathfrak{p}}$ and $\pi_{\mathfrak{p}}(S) \in \text{Den}_l(R/\mathfrak{p}, 0)$: Suppose that $\pi_{\mathfrak{p}}(S) \not\subseteq \mathcal{C}_{R/\mathfrak{p}}$. Then there are elements $s \in S$ and $r \in R \setminus \mathfrak{p}$ such that either $sr \in \mathfrak{p}$ or $rs \in \mathfrak{p}$. Let us to consider the first case as the arguments in the second case are ‘symmetrical’ to the ones in the first case. It follows from the inclusion $sr \in \mathfrak{p}$ and the statement (i) that $s\mathfrak{p}^c = 0$. Hence, $r\mathfrak{p}^c = 0$ since $s \in S \subseteq \mathcal{C}_R$. By the statement (i), $r \in \mathfrak{p}$, a contradiction. The image of a left Ore set under a ring epimorphism is a left Ore set provide that it does not contain 0. Hence, $\pi(S) \in \text{Den}_l(R/\mathfrak{p}, 0)$ since $\pi_{\mathfrak{p}}(S) \subseteq \mathcal{C}_{R/\mathfrak{p}}$.

(v) For each $\mathfrak{p} \in \min(R)$, the left ideal $S^{-1}\mathfrak{p}$ of $S^{-1}R$ is an ideal: By the statement (iv),

$$\text{tor}_{r,S}(R/\mathfrak{p}) = 0.$$

Hence $S^{-1}\mathfrak{p}$ is an ideal of $S^{-1}R$, by Proposition 2.4.(1).

(vi) $\{S^{-1}R \mid \mathfrak{p} \in \min(R)\} \subseteq \text{Spec}(S^{-1}R)$ and $S^{-1}(R/\mathfrak{p}) \simeq S^{-1}R/S^{-1}\mathfrak{p} \simeq \pi_{\mathfrak{p}}(S)^{-1}(R/\mathfrak{p})$: By Lemma 2.6, for each $\mathfrak{p} \in \min(R)$, the ring $S^{-1}(R/\mathfrak{p})$ is a prime ring. By the statements (iv) and (v), $S^{-1}(R/\mathfrak{p}) \simeq S^{-1}R/S^{-1}\mathfrak{p} \simeq \pi_{\mathfrak{p}}(S)^{-1}(R/\mathfrak{p})$, and so

$$S^{-1}\mathfrak{p} \in \text{Spec}(S^{-1}R).$$

(vii) The set $\{S^{-1}R \mid \mathfrak{p} \in \min(R)\}$ is an irredundant set of prime ideals of $S^{-1}R$: Since $S \in \text{Den}_l(R, 0)$, we have that

$$\begin{aligned} 0 &= S^{-1}0 = S^{-1}\left(\bigcap_{\mathfrak{p} \in \min(R)} \mathfrak{p}\right) = \bigcap_{\mathfrak{p} \in \min(R)} S^{-1}\mathfrak{p}, \\ 0 &\neq \bigcap_{\mathfrak{p} \in \min(R) \setminus \{\mathfrak{p}'\}} \mathfrak{p} \subseteq S^{-1}\left(\bigcap_{\mathfrak{p} \in \min(R) \setminus \{\mathfrak{p}'\}} \mathfrak{p}\right) = \bigcap_{\mathfrak{p} \in \min(R) \setminus \{\mathfrak{p}'\}} S^{-1}\mathfrak{p} \text{ for all } \mathfrak{p}' \in \min(R), \end{aligned}$$

and the statement (vii) follows.

(viii) $\min(S^{-1}R) = \{S^{-1}R \mid \mathfrak{p} \in \min(R)\}$: The statement (viii) follows from the statement (vii) and Lemma 3.2. $\square$

Corollary 3.3.(1) shows that the largest left quotient ring $Q_l(R)$ of a semiprime ring R is also a semiprime ring. Corollary 3.3.(2) and provides an explicit description of the set of its minimal primes provided that $|\min(R)| < \infty$.

Corollary 3.3. *Let R be a semiprime ring. Then*

1. *The ring $Q_l(R)$ is a semiprime ring.*
2. *If, in addition, $|\min(R)| < \infty$. Then*
 - (a) $\min(Q_l(R)) = \{S_l(R)^{-1}\mathfrak{p} \mid \mathfrak{p} \in \min(R)\}$.
 - (b) *For all $\mathfrak{p} \in \min(R)$, $\pi_{\mathfrak{p}}(S_l(R)) \in \text{Den}_l(R/\mathfrak{p}, 0)$ and*

$$\pi_{\mathfrak{p}}(S_l(R))^{-1}(R/\mathfrak{p}) \simeq S_l(R)^{-1}R/S_l(R)^{-1}\mathfrak{p} \simeq Q_l(R)/S_l(R)^{-1}\mathfrak{p} \simeq S_l(R)^{-1}(R/\mathfrak{p})$$

where $\pi_{\mathfrak{p}} : R \rightarrow R/\mathfrak{p}$, $r \mapsto r + \mathfrak{p}$.

- (c) *For all $\mathfrak{p} \in \min(R)$, $\pi_{\mathfrak{p}}(S_l(R)) \subseteq S_l(R/\mathfrak{p})$ and $Q_l(R)/S_l(R)^{-1}\mathfrak{p} \subseteq Q_l(R/\mathfrak{p})$.*

Proof. 1 and 2. By the definition, $Q_l(R) = S_l(R)^{-1}R$ and $S_l(R) \in \text{Den}_l(R, 0)$, and statement 1 and 2 follow from Theorem 1.4.(1,2).

3. Statement 3 is a particular case of Theorem 1.4.(3).

4. By the definition, $S_l(R/\mathfrak{p})$ is the largest left denominator set of the ring $R/\mathfrak{p}$ that consists of regular elements of $R/\mathfrak{p}$. By statement 3,

$$\pi_{\mathfrak{p}}(S_l(R)) \in \text{Den}_l(R/\mathfrak{p}, 0),$$

and so $\pi_{\mathfrak{p}}(S_l(R)) \subseteq S_l(R/\mathfrak{p})$ and $Q_l(R)/S_l(R)^{-1}\mathfrak{p} \subseteq Q_l(R/\mathfrak{p})$. $\square$

Corollary 3.4 is a generalization of Theorem 1.4.

Corollary 3.4. *Let R be a ring, $S \in \text{Den}_l(R, \mathfrak{a})$, $\mathfrak{a}$ be a semiprime ideal, $\overline{R} = R/\mathfrak{a}$ and $\overline{S} = \{s + \mathfrak{a} \mid s \in S\}$. Then*

1. The ring $S^{-1}R \simeq \overline{S}^{-1}\overline{R}$ is a semiprime ring.
2. If, in addition, $|\min(\overline{R})| < \infty$ then $\min(S^{-1}R) = \{S^{-1}\mathfrak{p} = \overline{S}^{-1}\mathfrak{p} \mid \mathfrak{p} \in \min(\overline{R})\}$, i.e. the map $\min(\overline{R}) \rightarrow \min(S^{-1}R)$, $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$ is a bijection and $|\min(S^{-1}R)| = |\min(\overline{R})|$.
3. If, in addition, $|\min(\overline{R})| < \infty$ then $\pi_{\mathfrak{p}}(\overline{S}) \in \text{Den}_l(R/\mathfrak{p}, 0)$ and $\pi_{\mathfrak{p}}(\overline{S})^{-1}(\overline{R}/\mathfrak{p}) \simeq \overline{S}^{-1}\overline{R}/\overline{S}^{-1}\mathfrak{p} \simeq \overline{S}^{-1}(\overline{R}/\mathfrak{p})$ where $\pi_{\mathfrak{p}} : \overline{R} \rightarrow \overline{R}/\mathfrak{p}$, $r \mapsto r + \mathfrak{p}$.

Proof. The ideal $\mathfrak{a}$ is a semiprime ideal of R . Hence, the factor ring $\overline{R} = R/\mathfrak{a}$ is a semiprime ring. It follows from the inclusion $S \in \text{Den}_l(S, \mathfrak{a})$ that $\overline{S} \in \text{Den}_l(\overline{R}, 0)$, $S^{-1}R \simeq \overline{S}^{-1}\overline{R}$ is an R -isomorphism. Now, the corollary follow from Theorem 1.4. $\square$

Proposition 3.5. Let R be a semiprime ring with $|\min(R)| < \infty$ and $* \in \{l, r, \emptyset\}$. Then

1. $S_*(R) \subseteq \bigcap_{\mathfrak{p} \in \min(R)} \pi_{\mathfrak{p}}^{-1}(S_*(R/\mathfrak{p}))$.
2. The homomorphism

$$\prod_{\mathfrak{p} \in \min(R)} \pi'_{\mathfrak{p}} : Q_*(R) \rightarrow \prod_{\mathfrak{p} \in \min(R)} Q_*(R/\mathfrak{p})$$

is an R -monomorphism where the homomorphisms $\pi'_{\mathfrak{p}} : Q_*(R) \rightarrow Q_*(R/\mathfrak{p})$ are defined in the commutative diagram below.

Proof. Let us prove the result for $* = l$. The other two cases can be treated in a similar way.

1. By Theorem 1.4.(3), $\pi_{\mathfrak{p}}(S_l(R)) \in \text{Den}_l(R/\mathfrak{p}, 0)$ and so $\pi_{\mathfrak{p}}(S_l(R)) \subseteq S_l(R/\mathfrak{p})$, by the maximality of $S_l(R/\mathfrak{p})$, and there is a commutative diagram of ring homomorphisms:

$$\begin{array}{ccc} R & \xrightarrow{\pi_{\mathfrak{p}}} & R/\mathfrak{p} \\ \sigma \downarrow & & \downarrow \sigma_{\mathfrak{p}} \\ Q_l(R) & \xrightarrow{\pi'_{\mathfrak{p}}} & Q_l(R/\mathfrak{p}) \end{array}$$

where $\sigma(r) = \frac{r}{1}$, $\sigma_{\mathfrak{p}}(\pi_{\mathfrak{p}}(r)) = \frac{\pi_{\mathfrak{p}}(r)}{1}$ and $\pi'_{\mathfrak{p}}(s^{-1}r) = \pi_{\mathfrak{p}}(s)^{-1}\pi_{\mathfrak{p}}(r)$ for all $r \in R$ and $s \in S_l(R)$. Now, statement 1 follows from Theorem 2.2.(1) and the commutative diagram above.

2. By the commutative diagram above, the map

$$\prod_{\mathfrak{p} \in \min(R)} \pi'_{\mathfrak{p}} : Q_*(R) \rightarrow \prod_{\mathfrak{p} \in \min(R)} Q_*(R/\mathfrak{p})$$

is an R -monomorphism. $\square$

For an ideal $\mathfrak{a}$ of a ring R , let $\text{Ore}_l(R, \mathfrak{a}) := \{S \in \text{Ore}_l(R) \mid \text{ass}_l(S) = \mathfrak{a}\}$, $\text{Den}_l(R, \mathfrak{a}) := \{S \in \text{Den}_l(R) \mid \text{ass}(S) = \mathfrak{a}\}$. Let $\text{Ass}_l(R) := \{\text{ass}_l(S) \mid S \in \text{Den}_l(R)\}$.

Definition 3.6. ([2]) Let $S_{l,\mathfrak{a}}(R)$ be the largest element of the poset $(\text{Den}_l(R, \mathfrak{a}), \subseteq)$ ($S_{l,\mathfrak{a}}(R)$ exists, [2, Theorem 2.1.(2)]). The ring

$$Q_{l,\mathfrak{a}}(R) := S_{l,\mathfrak{a}}(R)^{-1}R$$

is called the **largest left quotient ring associated to $\mathfrak{a}$** .

For each denominator set $S \in \text{Den}_l(R, \mathfrak{a})$ where $\mathfrak{a} := \text{ass}_l(S)$, there are natural ring homomorphisms $R \xrightarrow{\pi} R/\mathfrak{a} \rightarrow S^{-1}R$.

Lemma 3.7. ([2, Lemma 3.3.(2)]) *Let $\mathfrak{a} \in \text{Ass}_l(R)$ and $\pi : R \rightarrow R/\mathfrak{a}$, $a \mapsto a + \mathfrak{a}$. Then $\pi^{-1}(S_l(R/\mathfrak{a})) = S_{l,\mathfrak{a}}(R)$, $\pi(S_{l,\mathfrak{a}}(R)) = S_l(R/\mathfrak{a})$ and $Q_{l,\mathfrak{a}}(R) = Q_l(R/\mathfrak{a})$.*

Let R be a semiprime ring with $|\min(R)| < \infty$. For each $\mathfrak{p} \in \min(R)$, the subset of R ,

$$T_l(\mathfrak{p}) := \left(\sigma_{\mathfrak{p}} \pi_{\mathfrak{p}} \right)^{-1} (Q_l(R/\mathfrak{p})^\times) = \pi_{\mathfrak{p}}^{-1}(S_l(R/\mathfrak{p})),$$

is a multiplicative set, see the diagram in the proof of Proposition 3.5.(1) for the definitions of the maps $\pi_{\mathfrak{p}}$ and $\sigma_{\mathfrak{p}}$. The second equality follows from the equality $S_l(R/\mathfrak{p}) = \sigma_{\mathfrak{p}}^{-1}(Q_l(R/\mathfrak{p})^\times)$ (Theorem 2.2.(1)). For each $\mathfrak{p} \in \min(R)$, let

$$\mathfrak{a}_l(\mathfrak{p}) := \text{ass}_l(T_l(\mathfrak{p})) = \{r \in R \mid tr = 0 \text{ for some } t \in T_l(\mathfrak{p})\},$$

$$\mathfrak{a}_r(\mathfrak{p}) := \text{ass}_r(T_l(\mathfrak{p})) = \{r \in R \mid rt = 0 \text{ for some } t \in T_l(\mathfrak{p})\}.$$

By the definition, $\mathfrak{a}_l(\mathfrak{p})$ and $\mathfrak{a}_r(\mathfrak{p})$ are right and left ideals of the ring R , respectively, but not ideals, in general.

Proposition 3.8. *Let R be a semiprime ring with $|\min(R)| < \infty$. Then*

1. $\mathfrak{a}_l(\mathfrak{p}) \subseteq \mathfrak{p}$, $\mathfrak{a}_r(\mathfrak{p}) \subseteq \mathfrak{p}$ and $\bigcap_{\mathfrak{p} \in \min(R)} \mathfrak{a}_l(\mathfrak{p}) = \bigcap_{\mathfrak{p} \in \min(R)} \mathfrak{a}_r(\mathfrak{p}) = 0$.
2. $T_l(\mathfrak{p}) \in \text{Ore}_l(R)$ iff for each pair of elements $(s, p) \in T_l(\mathfrak{p}) \times \mathfrak{p}$ there is a pair of elements $(s', r') \in T_l(\mathfrak{p}) \times R$ such that $s'r - r's \in \mathfrak{a}_l(\mathfrak{p})$.
3. Suppose that $T_l(\mathfrak{p}) \in \text{Den}_l(R)$. Then
 - (a) $T_l(\mathfrak{p})^{-1}\mathfrak{p}$ is an ideal of the ring $T_l(\mathfrak{p})^{-1}R$.
 - (b) $T_l(\mathfrak{p})^{-1}R/T_l(\mathfrak{p})^{-1}\mathfrak{p} \simeq Q_l(R/\mathfrak{p})$.
 - (c) If, in addition, $\mathfrak{a}_l(\mathfrak{p}) = \mathfrak{p}$ then $S_{l,\mathfrak{p}}(R) = T_l(\mathfrak{p})$ and $Q_{l,\mathfrak{p}}(R) \simeq Q_l(R/\mathfrak{p}) \simeq T_l(\mathfrak{p})^{-1}R$.

Proof. 1. Let us prove that statement 1 holds for $\mathfrak{a}_l(\mathfrak{p})$. The second case can be treated in a similar way (using symmetrical arguments). If $r \in \mathfrak{a}_l(\mathfrak{p})$ then $tr = 0$ for some element $t \in T_l(\mathfrak{p})$. By the definition, $\sigma_{\mathfrak{p}}\pi_{\mathfrak{p}}(T_l(\mathfrak{p})) \subseteq Q_l(R/\mathfrak{p})^\times$. It follows that $0 = \sigma_{\mathfrak{p}}\pi_{\mathfrak{p}}(tr) = \sigma_{\mathfrak{p}}\pi_{\mathfrak{p}}(t)\sigma_{\mathfrak{p}}\pi_{\mathfrak{p}}(r)$, and so

$$\sigma_{\mathfrak{p}}\pi_{\mathfrak{p}}(r) = 0,$$

i.e. $\pi_{\mathfrak{p}}(r) = 0$. This means that $r \in \mathfrak{p}$. So, $\mathfrak{a}_l(\mathfrak{p}) \subseteq \mathfrak{p}$. Now, $\bigcap_{\mathfrak{p} \in \min(R)} \mathfrak{a}_l(\mathfrak{p}) \subseteq \bigcap_{\mathfrak{p} \in \min(R)} \mathfrak{p} = 0$, and so $\bigcap_{\mathfrak{p} \in \min(R)} \mathfrak{a}_l(\mathfrak{p}) = 0$.

2. ($\Rightarrow$) Suppose that $T_l(\mathfrak{p}) \in \text{Ore}_l(R)$. Then for each pair of elements $(s, p) \in T_l(\mathfrak{p}) \times \mathfrak{p}$ there is a pair of elements $(s', r') \in T_l(\mathfrak{p}) \times R$ such that $s'r - r's = 0 \in \mathfrak{a}_l(\mathfrak{p})$.

($\Leftarrow$) We have to show that the left Ore condition holds for the multiplicative subset $T_l(\mathfrak{p})$. By the assumption, for each pair of elements $(s, p) \in T_l(\mathfrak{p}) \times \mathfrak{p}$ there is a pair of elements $(s', r') \in T_l(\mathfrak{p}) \times R$ such that $s'r - r's \in \mathfrak{a}_l(\mathfrak{p})$. By the definition of the set $|\mathfrak{a}_l(\mathfrak{p})|$, there is an element $t \in T_l(\mathfrak{p})$ such that

$$t(s'r - r's) = 0.$$

Then $(ts')r = (tr')s$ with $ts' \in T_l(\mathfrak{p})$ and $tr' \in R$, as required.

3. By the assumption, $T_l(\mathfrak{p}) \in \text{Den}_l(R)$, and so $T_l(\mathfrak{p}) \in \text{Den}_l(R, \mathfrak{a}_l(\mathfrak{p}))$. Recall that

$$\sigma_{\mathfrak{p}}\pi_{\mathfrak{p}}(T_l(\mathfrak{p})) \subseteq Q_l(R/\mathfrak{p})^\times.$$

By the universal property of localization, there is a homomorphism

$$\iota_{\mathfrak{p}} : T_l(\mathfrak{p})^{-1}R \rightarrow Q_l(R/\mathfrak{p}), \quad t^{-1}r \mapsto t^{-1}r.$$

Recall that $S_l(R/\mathfrak{p}) = \sigma_{\mathfrak{p}}^{-1}(Q_l(R/\mathfrak{p})^\times)$ (Theorem 2.2.(1)) and $T_l(\mathfrak{p}) = \pi_{\mathfrak{p}}^{-1}(S_l(R/\mathfrak{p}))$ (see the definition of the set $T_l(\mathfrak{p})$). The map $\pi_{\mathfrak{p}} : R \rightarrow R/\mathfrak{p}$, $r \mapsto r + \mathfrak{p}$ is an epimorphism. Hence,

$$\pi_{\mathfrak{p}}(T_l(\mathfrak{p})) = S_l(\mathfrak{p}),$$

and so the homomorphism $\iota_{\mathfrak{p}}$ is an epimorphism. By applying the localization functor $T_l(\mathfrak{p})^{-1}$ (which is an exact functor) to the short exact sequence of left R -modules

$$0 \rightarrow \mathfrak{p} \rightarrow R \rightarrow R/\mathfrak{p} \rightarrow 0,$$

we obtain the short exact sequences of left $T_l(\mathfrak{p})^{-1}R$ -modules

$$0 \rightarrow T_l(\mathfrak{p})^{-1}\mathfrak{p} \rightarrow T_l(\mathfrak{p})^{-1}R \rightarrow T_l(\mathfrak{p})^{-1}(R/\mathfrak{p}) \simeq \left(\pi_{\mathfrak{p}}(T_l(\mathfrak{p}))\right)^{-1}(R/\mathfrak{p}) = S_l(R/\mathfrak{p})^{-1}(R/\mathfrak{p}) = Q_l(R/\mathfrak{p}) \rightarrow 0,$$

i.e. $0 \rightarrow T_l(\mathfrak{p})^{-1}\mathfrak{p} \rightarrow T_l(\mathfrak{p})^{-1}R \xrightarrow{\iota_{\mathfrak{p}}} Q_l(R/\mathfrak{p}) \rightarrow 0$. Hence, $\ker(\iota_{\mathfrak{p}}) = T_l(\mathfrak{p})^{-1}\mathfrak{p}$ is an ideal of the ring $T_l(\mathfrak{p})^{-1}R$ such that

$$T_l(\mathfrak{p})^{-1}R/T_l(\mathfrak{p})^{-1}\mathfrak{p} \simeq Q_l(R/\mathfrak{p}).$$

The proofs of the statements (a) and (b) are complete.

Finally, suppose that $\mathfrak{a}_l(\mathfrak{p}) = \mathfrak{p}$, i.e. $T_l(\mathfrak{p}) \in \text{Den}_l(R, \mathfrak{p})$. Then, by the statement (b),

$$\sigma : R \rightarrow T_l(\mathfrak{p})^{-1}R \simeq Q_l(R/\mathfrak{p}), \quad r \mapsto \frac{r}{1}.$$

By [2, Lemma 3.3.(2)],

$$Q_{l,\mathfrak{a}}(R) = S_{l,\mathfrak{a}}(R)^{-1}R = Q_l(R/\mathfrak{p}) \quad \text{and} \quad S_{l,\mathfrak{a}}(R) = \pi_{\mathfrak{p}}^{-1}(S_l(R/\mathfrak{p})).$$

By the definition of the set $T_l(\mathfrak{p})$, $\pi_{\mathfrak{p}}^{-1}(S_l(R/\mathfrak{p})) = T_l(\mathfrak{p})$. The proof of the statement (c) is complete. $\square$

Description of the set of minimal primes of a localization of a semiprime ring at a not necessarily regular denominator set. An element n of a ring R is called a **normal element** if

$$Rn = nR.$$

Theorem 3.9 is a generalization of Theorem 1.4.(2,3) for left denominator sets that contain zero divisors. The general situation is described by Proposition 1.2 but the goal is to describe a situation that has the ‘same’ answer as in Theorem 1.4.(2,3).

Theorem 3.9. Let R be a semiprime ring with $|\min(R)| < \infty$ and $S \in \text{Den}_l(R, \mathfrak{a})$. Then

$$\min(R, S) := \{\mathfrak{p} \in \min(R) \mid \mathfrak{p} \cap S = \emptyset\} \neq \emptyset$$

and statements 1 and 2 below are equivalent:

1. The ring $S^{-1}R$ is a semiprime ring with $\min(S^{-1}R) = \{S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \min(R, S)\}$.
2. For each $\mathfrak{p} \in \min(R, S)$, $S^{-1}\mathfrak{p}$ is an ideal of $S^{-1}R$ and the factor ring $S^{-1}R/S^{-1}\mathfrak{p}$ is a prime ring.
3. Suppose that the equivalent conditions 1 and 2 hold. Then
 - (a) If the left denominator set S is generated by normal elements of R then all ideals in the set $\min(S^{-1}R)$ are distinct, i.e. $|\min(S^{-1}R)| = |\min(R, S)|$.
 - (b) If for each element $s \in S$ there is an element $t \in S$ such that the element ts is a normal element of R then all ideals in the set $\min(S^{-1}R)$ are distinct, i.e. $|\min(S^{-1}R)| = |\min(R, S)|$.

Remark. In statement 1, not all ideals in the set $\min(S^{-1}R)$ are assumed distinct.

Proof. Suppose that $\min(R, S) = \emptyset$, i.e. $\mathfrak{p} \cap S \neq \emptyset$ for all $\mathfrak{p} \in \min(R)$. Choose an element $s_{\mathfrak{p}} \in \mathfrak{p} \cap S \neq \emptyset$ for each $\mathfrak{p} \in \min(R)$. Then on the one hand

$$0 \neq s := \prod_{\mathfrak{p} \in \min(R)} s_{\mathfrak{p}} \in S$$

where the product is taken in an arbitrary order. On the other hand, $s \in \bigcap_{\mathfrak{p} \in \min(R)} \mathfrak{p} = 0$ (since the ring R is a semiprime ring), a contradiction. Therefore, $\min(R, S) \neq \emptyset$.

(1 $\Rightarrow$ 2) Straightforward.

(2 $\Rightarrow$ 1) By the assumption, for each $\mathfrak{p} \in \min(R, S)$, $S^{-1}\mathfrak{p}$ is an ideal of $S^{-1}R$ and the factor ring $S^{-1}R/S^{-1}\mathfrak{p}$ is a prime ring, i.e. the ideal $S^{-1}\mathfrak{p}$ is a prime ideal of the ring $S^{-1}R$. Since the ring R is a semiprime ring with $|\min(R)| < \infty$, we have that

$$\{0\} = S^{-1}\{0\} = S^{-1} \bigcap_{\mathfrak{p} \in \min(R)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \min(R)} S^{-1}\mathfrak{p} = \bigcap_{\mathfrak{p} \in \min(R, S)} S^{-1}\mathfrak{p} = \bigcap_{\mathfrak{p} \in P} S^{-1}\mathfrak{p}$$

for some subset P of $\min(R, S)$ such that the last intersection is irredundant. By Lemma 3.2, $\min(S^{-1}R) = \{S^{-1}\mathfrak{p} \mid \mathfrak{p} \in P\}$.

3. Suppose that the equivalent conditions 1 and 2 of the theorem hold.

(a) The statement (a) is a particular case of the statement (b).

(b) Suppose that $S^{-1}\mathfrak{p} = S^{-1}\mathfrak{q}$ for some ideals $\mathfrak{p}, \mathfrak{q} \in \min(R, S)$. We have to show that $\mathfrak{p} = \mathfrak{q}$. It suffices to show that

$$\mathfrak{p} \subseteq \mathfrak{q}$$

(since then the inclusions $\mathfrak{p}, \mathfrak{q} \in \min(R)$ imply the equality $\mathfrak{p} = \mathfrak{q}$). For each element $p \in \mathfrak{p}$, $\frac{p}{1} = s^{-1}q$ for some element $q \in \mathfrak{q}$. Hence, $t(sp - q) = 0$ for some element and $t \in S$, and so

$$ts \cdot p = tq \in \mathfrak{q} \text{ and } ts \in S \setminus \mathfrak{q}$$

(since $S \cap \mathfrak{q} = \emptyset$). By the assumption, there is an element $t' \in S$ such that the element $n := t'ts \in S$ is a normal element of R . Now, the inclusion $np \in t'\mathfrak{q} \subseteq \mathfrak{q}$ implies the inclusion of ideals

$$(n)(p) \subseteq \mathfrak{q}$$

(since the element $n \in R$ is a normal element). Therefore, $(p) \subseteq \mathfrak{q}$ since $n \notin \mathfrak{p}$, i.e. $p \in \mathfrak{q}$ for all elements $p \in \mathfrak{p}$. This implies the inclusion $\mathfrak{p} \subseteq \mathfrak{q}$, as required. $\square$

Corollary 3.10 describes the minimal primes of a localization of a semiprime commutative ring.

Corollary 3.10. *Let R be a semiprime commutative ring with $|\min(R)| < \infty$ and $S \in \text{Den}(R, \mathfrak{a})$. Then*

1. *The ring $S^{-1}R$ is a semiprime ring with $\min(S^{-1}R) = \{S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \min(R, S)\}$.*
2. *In statement 1, all ideals in the set $\min(S^{-1}R)$ are distinct, i.e. $|\min(S^{-1}R)| = |\min(R, S)|$.*

Proof. 1. Since the ring R is a commutative ring, for each $\mathfrak{p} \in \min(R, S)$, $S^{-1}\mathfrak{p}$ is an ideal of $S^{-1}R$ and the factor ring

$$S^{-1}R/S^{-1}\mathfrak{p} \simeq S^{-1}(R/\mathfrak{p})$$

is a prime ring since the factor ring $R/\mathfrak{p}$ is a domain (hence so is its localization $S^{-1}(R/\mathfrak{p})$). Therefore, statement 2 of Theorem 3.9 holds, and so statement 1 follows from Theorem 3.9.

2. Statement 2 follows from Theorem 3.9.(3a). $\square$

An ideal $\mathfrak{p}$ of a ring R is called a **completely prime ideal** of R if the factor ring $R/\mathfrak{p}$ is a domain. Every completely prime ideal is a prime ideal but not vice versa, in general. We denote by $\text{Spec}_c(R)$ be the set of all completely prime ideals of R . Notice that $\text{Spec}_c(R) \subseteq \text{Spec}(R)$.

Proof of Proposition 1.5. 1. By the assumption, for each $\mathfrak{p} \in \min(R, S)$, $\mathfrak{p} \in \text{Spec}_c(R)$ and $S^{-1}\mathfrak{p}$ is an ideal of $S^{-1}R$. So, the ring $R/\mathfrak{p}$ is a domain, and so is the factor ring

$$S^{-1}R/S^{-1}\mathfrak{p} \simeq S^{-1}(R/\mathfrak{p}).$$

Therefore, statement 2 of Theorem 3.9 holds, and so statement 1 follows from Theorem 3.9. Clearly, $S^{-1}\mathfrak{p} \in \text{Spec}_c(S^{-1}R)$ (since the factor ring $S^{-1}R/S^{-1}\mathfrak{p} \simeq S^{-1}(R/\mathfrak{p})$ is a domain).

2. Suppose that $S^{-1}\mathfrak{p} = S^{-1}\mathfrak{q}$ for some ideals $\mathfrak{p}, \mathfrak{q} \in \min(R, S)$. We have to show that $\mathfrak{p} = \mathfrak{q}$. It suffices to show that $\mathfrak{p} \subseteq \mathfrak{q}$. For each element $p \in \mathfrak{p}$, $\frac{p}{1} = s^{-1}q$ for some element $q \in \mathfrak{q}$. Hence, $t(sp - q) = 0$ for some element and $t \in S$, and so

$$ts \cdot p = tq \in \mathfrak{q} \quad \text{and} \quad ts \in S \setminus \mathfrak{q}$$

(since $S \cap \mathfrak{q} = \emptyset$). Therefore, $p \in \mathfrak{q}$ for all elements $p \in \mathfrak{p}$, i.e. $\mathfrak{p} \subseteq \mathfrak{q}$ (since the factor ring $R/\mathfrak{q}$ is a domain), as required. $\square$

4. Minimal primes of localizations of rings at multiplicative sets generated by normal elements

Let R be a ring and S be a multiplicative subset of R which is generated by normal elements of R . The goal of the section is to prove Theorem 1.7 that describes the set of minimal primes of the ring $R\langle S^{-1} \rangle$. In general, the ring $R\langle S^{-1} \rangle$ is not a semiprime ring but precisely the same result as Theorem 1.4 holds for it but without the ‘semiprimeness’ assumption, see Theorem 1.7.(4a). Lemma 4.2 is a generalization of Theorem 1.7 to a more general situation.

Description of minimal primes of localizations of rings at multiplicative sets generated by normal elements. Let R be a ring, S be a non-empty subset of R , $R\langle X_S \rangle$ be a ring freely generated by the ring R and

a set $X_S = \{x_s \mid s \in S\}$ of free noncommutative indeterminates (indexed by the elements of the set S). Let I_S be the ideal of $R\langle X_S \rangle$ generated by the set $\{sx_s - 1, x_ss - 1 \mid s \in S\}$. The factor ring

$$R\langle S^{-1} \rangle := R\langle X_S \rangle / I_S. \quad (7)$$

is called the *localization of R at S* . Let $\text{ass}(S) = \text{ass}_R(S)$ be the kernel of the ring homomorphism

$$\sigma_S : R \rightarrow R\langle S^{-1} \rangle, \quad r \mapsto r + I_S. \quad (8)$$

The factor ring $\overline{R} := R/\text{ass}_R(S)$ is a subring of $R\langle S^{-1} \rangle$ and the map $\pi_S : R \rightarrow \overline{R}$, $r \mapsto \overline{r} := r + \text{ass}_R(S)$ is an epimorphism. The ideal $\text{ass}_R(S)$ of R has a sophisticated structure, its description is given in [3, Proposition 2.12] when

$$R\langle S^{-1} \rangle = \{\overline{s}^{-1}\overline{r} \mid s \in S, r \in R\}$$

is a ring of left fractions. There is an example of a domain R and a finite set S such that $\text{ass}(S) = R$, i.e. $R\langle S^{-1} \rangle = \{0\}$ [10, Exercises 9.5].

Definition 4.1. ([3]) A multiplicative set S of a ring R is called a **left localizable set** if

$$R\langle S^{-1} \rangle = \{\overline{s}^{-1}\overline{r} \mid \overline{s} \in \overline{S}, \overline{r} \in \overline{R}\} \neq \{0\}$$

where $\overline{R} = R/\mathfrak{a}$, $\mathfrak{a} = \text{ass}_R(S)$ and $\overline{S} = (S + \mathfrak{a})/\mathfrak{a}$, i.e., every element of the ring $R\langle S^{-1} \rangle$ is a left fraction $\overline{s}^{-1}\overline{r}$ for some elements $\overline{s} \in \overline{S}$ and $\overline{r} \in \overline{R}$. Similarly, a multiplicative set S of a ring R is called a **right localizable set** of R if

$$R\langle S^{-1} \rangle = \{\overline{r}\overline{s}^{-1} \mid \overline{s} \in \overline{S}, \overline{r} \in \overline{R}\} \neq \{0\},$$

i.e., every element of the ring $R\langle S^{-1} \rangle$ is a right fraction $\overline{r}\overline{s}^{-1}$ for some elements $\overline{s} \in \overline{S}$ and $\overline{r} \in \overline{R}$. A right and left localizable set of R is called a **localizable set** of R .

Every left denominator set $S \in \text{Den}_l(R)$ is a left localizable set and $R\langle S^{-1} \rangle \simeq S^{-1}R$.

Let R be a ring and S be a multiplicative subset of R which is generated by normal elements of R . Clearly, $S \in \text{Ore}(R)$ is an Ore set of R . By Theorem 1.6, S is a localizable set of R such that $\text{ass}_R(S) = \mathfrak{a}$ and $R\langle S^{-1} \rangle \simeq \overline{S}^{-1}\overline{R}$. We keep the notation of Theorem 1.6. Let

$$\tilde{\pi} : \overline{R} \rightarrow \tilde{R} := \overline{R}/\mathfrak{n}_{\overline{R}}, \quad \overline{r} \mapsto \tilde{r} = \overline{r} + \mathfrak{n}_{\overline{R}}$$

where $\mathfrak{n}_{\overline{R}}$ is the prime radical of the ring $\overline{R}$. There are bijections

$$\min(\mathfrak{a}) \xrightarrow{\pi} \min(\overline{R}) \xrightarrow{\tilde{\pi}} \min(\tilde{R}), \quad \mathfrak{p} \mapsto \overline{\mathfrak{p}} \mapsto \tilde{\mathfrak{p}}.$$

Let $\min(\mathfrak{a}, S) := \{\mathfrak{p} \in \min(\mathfrak{a}) \mid \mathfrak{p} \cap S = \emptyset\}$.

An element $a \in R$ is called **strongly nilpotent** if any sequence $a = a_0, a_1, a_2, \dots$ such that $a_{i+1} \in a_i R a_i$ is ultimately zero. The prime radical $\mathfrak{n}_R$ of R is precisely the set of strongly nilpotent elements of R , [12, Theorem 0.2.2]. In particular, the prime radical $\mathfrak{n}_R$ is **nil**, i.e. all elements of $\mathfrak{n}_R$ are nilpotent elements.

Proof of Theorem 1.7. 1–3. By Theorem 1.6.(2), $\overline{S} \in \text{Den}(\overline{R}, 0)$ and $R\langle S^{-1} \rangle \simeq \overline{S}^{-1}\overline{R}$. Hence, for each element $\overline{s} \in \overline{S}$, the map

$$\omega_{\bar{s}} : \bar{R} \rightarrow \bar{R}, \quad \bar{r} \mapsto \omega_{\bar{s}}(\bar{r}) \quad \text{where} \quad \bar{s}\bar{r} = \omega_{\bar{s}}(\bar{r})\bar{s},$$

is an automorphism.

(i) For all $\bar{s} \in \bar{S}$, $\omega_{\bar{s}}(\mathfrak{n}_{\bar{R}}) = \mathfrak{n}_{\bar{R}}$: The statement (i) follows from the fact that $\omega_{\bar{s}} \in \text{Aut}(\bar{R})$.

(ii) $\bar{S} \cap \mathfrak{n}_{\bar{S}} = \emptyset$: Each element $\bar{s} \in \bar{S}$ is a regular element of R . In particular, it is not a nilpotent element. Since the prime radical $\mathfrak{n}_{\bar{S}}$ is nil, we have that $\bar{S} \cap \mathfrak{n}_{\bar{S}} = \emptyset$.

(iii) $\tilde{\bar{S}} \in \text{Den}(\tilde{R}, 0)$ and $\omega_{\xi} \in \text{Aut}(\tilde{R})$ for all $\xi \in \tilde{\bar{S}}$: By the statement (ii), the subset $\tilde{\bar{S}}$ of $\tilde{R}$ is a multiplicative set. By the statement (i), the set $\tilde{\bar{S}}$ is generated by regular normal elements. Hence, $\omega_{\xi} \in \text{Aut}(\tilde{R})$ for all $\xi \in \tilde{\bar{S}}$ and $\tilde{\bar{S}} \in \text{Den}(\tilde{R}, 0)$.

(iv) The map $\min(\tilde{R}) \rightarrow \min(\tilde{\bar{S}}^{-1}\tilde{R})$, $\tilde{\mathfrak{p}} \mapsto \tilde{\bar{S}}^{-1}\tilde{\mathfrak{p}}$ is a bijection: By the definition, the ring $\tilde{R}$ is a semiprime ring with

$$|\min(\tilde{R})| = |\min(\mathfrak{a})| < \infty, \quad \text{and} \quad \tilde{\bar{S}} \in \text{Den}(\tilde{R}, 0)$$

(the statement (iii)). Now, the result follows from Theorem 1.4.(2).

(v) For all $\bar{\mathfrak{p}} \in \min(\bar{R})$, $\bar{S} \cap \bar{\mathfrak{p}} = \emptyset$: The result follows from the statement (iv).

(vi) The map $\min(\bar{R}) \rightarrow \{\bar{S}^{-1}\bar{\mathfrak{p}} \mid \bar{\mathfrak{p}} \in \min(\bar{R})\}$, $\bar{\mathfrak{p}} \mapsto \bar{S}^{-1}\bar{\mathfrak{p}}$ is a bijection, $\bar{S}^{-1}\bar{\mathfrak{p}} \neq \bar{S}^{-1}R$ and $\bar{S} \cap \bar{\mathfrak{p}} = \emptyset$: The result follows from the statement (iv).

Notice that $|\min(\bar{R})| = |\min(\mathfrak{a})| < \infty$.

(vii) There is a natural number $n \geq 1$ such that $\omega_{\bar{s}}^n(\bar{\mathfrak{p}}) = \bar{\mathfrak{p}}$ for all elements $\bar{s} \in \bar{S}$ and $\bar{\mathfrak{p}} \in \min(\bar{R})$: Let G be the subgroup of $\text{Aut}(\bar{R})$ which is generated by the automorphisms $\omega_{\bar{s}}$, where $\bar{s} \in \bar{S}$. The group G acts on the finite set $\min(\bar{R})$ in the obvious way. The image of the group G , say G' , in the symmetric group of the finite set $\min(\bar{R})$ is a finite group. So, it suffices to take $n = |G'|$, the order of the group G' .

(viii) For all $\bar{\mathfrak{p}} \in \min(\bar{R})$ and $\bar{s} \in \bar{S}$, $\bar{\mathfrak{p}}\bar{s}^{-1} \subseteq \bar{s}^{-n}\bar{\mathfrak{p}}$, where n is as in the statement (vii), and the left ideal $\bar{S}^{-1}\bar{\mathfrak{p}}$ of the ring $\bar{S}^{-1}\bar{R}$ is an ideal: The second part of the statement (viii) follows from the first (since then $\bar{\mathfrak{p}}\bar{s}^{-1} \subseteq \bar{S}^{-1}\bar{\mathfrak{p}}$ for all element $\bar{s} \in \bar{S}$). Now,

$$\bar{\mathfrak{p}}\bar{s}^{-1} = \bar{s}^{-n} \cdot \bar{s}^n \bar{\mathfrak{p}} \bar{s}^{-n} \cdot \bar{s}^n \bar{s}^{-1} = \bar{s}^{-n} \omega_{\bar{s}}^n(\bar{\mathfrak{p}}) \bar{s}^{n-1} = \bar{s}^{-n} \bar{\mathfrak{p}} \bar{s}^{n-1} \subseteq \bar{s}^{-n} \bar{\mathfrak{p}}.$$

(ix) For all $\bar{\mathfrak{p}} \in \min(\bar{R})$, $\bar{S}^{-1}\bar{\mathfrak{p}} \in \text{Spec}(\bar{S}^{-1}\bar{R})$: By statement (viii), $\bar{S}^{-1}\bar{\mathfrak{p}}$ is an ideal of the ring $\bar{S}^{-1}\bar{R}$. Furthermore, $\bar{S}^{-1}\bar{\mathfrak{p}} \neq \bar{S}^{-1}\bar{R}$, by the statement (vi). The ring $\bar{R}/\bar{\mathfrak{p}}$ is a prime ring. Hence so is its localization

$$\bar{S}^{-1}(\bar{R}/\bar{\mathfrak{p}}) \simeq \bar{S}^{-1}\bar{R}/\bar{S}^{-1}\bar{\mathfrak{p}},$$

by Lemma 2.6. Therefore, $\bar{S}^{-1}\bar{\mathfrak{p}} \in \text{Spec}(\bar{S}^{-1}\bar{R})$.

(x) $\mathfrak{n}_{\bar{S}^{-1}\bar{R}} \subseteq \bar{S}^{-1}\mathfrak{n}_{\bar{R}}$ and $\bar{S}^{-1}\mathfrak{n}_{\bar{R}}$ is an ideal of $\bar{S}^{-1}\bar{R}$: By the statement (ix) and $|\min(\bar{R})| < \infty$,

$$\mathfrak{n}_{\bar{S}^{-1}\bar{R}} = \bigcap_{\mathfrak{q} \in \text{Spec}(\bar{S}^{-1}\bar{R})} \mathfrak{q} \subseteq \bigcap_{\bar{\mathfrak{p}} \in \min(\bar{R})} \bar{S}^{-1}\bar{\mathfrak{p}} = \bar{S}^{-1} \left(\bigcap_{\bar{\mathfrak{p}} \in \min(\bar{R})} \bar{\mathfrak{p}} \right) = \bar{S}^{-1}\mathfrak{n}_{\bar{R}}$$

and the intersection of ideals of $\bar{S}^{-1}\bar{R}$, $\bar{S}^{-1}\mathfrak{n}_{\bar{R}} = \bigcap_{\bar{\mathfrak{p}} \in \min(\bar{R})} \bar{S}^{-1}\bar{\mathfrak{p}}$, is an ideal of $\bar{S}^{-1}\bar{R}$, see the statement (viii).

(xi) $\bar{S}^{-1}\bar{R}/\bar{S}^{-1}\mathfrak{n}_{\bar{R}} \simeq \tilde{\bar{S}}^{-1}\tilde{R}$: By the statement (x), $\bar{S}^{-1}\mathfrak{n}_{\bar{R}}$ is an ideal of the ring $\bar{S}^{-1}\bar{R}$. Now, by the statement (iii), $\tilde{\bar{S}} \in \text{Den}(\tilde{R}, 0)$, and so

$$\bar{S}^{-1}\bar{R}/\bar{S}^{-1}\mathfrak{n}_{\bar{R}} \simeq \bar{S}^{-1}(\bar{R}/\mathfrak{n}_{\bar{R}}) \simeq \tilde{\bar{S}}^{-1}\tilde{R}.$$

(xii) For all $\bar{\mathfrak{p}} \in \min(\bar{R})$, $\bar{R} \cap \bar{S}^{-1}\bar{\mathfrak{p}} = \bar{\mathfrak{p}}$: Clearly, $\bar{\mathfrak{p}} \subseteq \bar{R} \cap \bar{S}^{-1}\bar{\mathfrak{p}}$. Suppose that $(\bar{R} \cap \bar{S}^{-1}\bar{\mathfrak{p}}) \setminus \bar{\mathfrak{p}} \neq \emptyset$. Fix an element $\bar{r} \in (\bar{R} \cap \bar{S}^{-1}\bar{\mathfrak{p}}) \setminus \bar{\mathfrak{p}}$. Then $\bar{s}\bar{r} \in \bar{\mathfrak{p}}$ for some element $\bar{s} \in \bar{S}$. The element $\bar{s}$ is a normal element of the ring $\bar{R}$ and $\bar{s} \notin \bar{\mathfrak{p}}$, by the statement (vi). Therefore, we have the inclusion of ideals

$$(\bar{s})(\bar{r}) \subseteq \bar{\mathfrak{p}}$$

which implies the inclusion $(\bar{r}) \subseteq \bar{\mathfrak{p}}$ (since $\bar{s} \notin \bar{\mathfrak{p}}$ and $\bar{\mathfrak{p}}$ is a prime ideal of the ring $\bar{R}$), and so $\bar{r} \in \bar{\mathfrak{p}}$, a contradiction.

(xiii) The map $\min(\mathfrak{a}) \rightarrow \text{Spec}(R\langle S^{-1} \rangle) = \text{Spec}(\bar{S}^{-1}\bar{R})$, $\mathfrak{p} \mapsto \bar{S}^{-1}\bar{\mathfrak{p}}$ is an injection: The result follows from the statements (ix) and (xii) (if $\bar{S}^{-1}\bar{\mathfrak{p}} = \bar{S}^{-1}\bar{\mathfrak{p}}'$ for some $\mathfrak{p}, \mathfrak{p}' \in \min(\mathfrak{q})$ then, by the statement (xii),

$$\bar{\mathfrak{p}} = \bar{R} \cap \bar{S}^{-1}\bar{\mathfrak{p}} = \bar{R} \cap \bar{S}^{-1}\bar{\mathfrak{p}}' = \bar{\mathfrak{p}}',$$

and so $\mathfrak{p} = \mathfrak{p}'$).

(xiv) For each $\mathfrak{p} \in \min(\mathfrak{a})$, $\pi_{\mathfrak{p}}(S) \in \text{Den}(R/\mathfrak{p}, 0)$: By the statement (v) and (vii),

$$\bar{S} \cap \bar{\mathfrak{p}} = \emptyset \text{ and } \omega_{\bar{\mathfrak{p}}}(\bar{\mathfrak{p}}) = \bar{\mathfrak{p}}.$$

Hence, the subset $\pi_{\mathfrak{p}}(S)$ of $R/\mathfrak{p}$ is a multiplicative set that is generated by regular normal elements, and so $\pi_{\mathfrak{p}}(S) \in \text{Den}(R/\mathfrak{p}, 0)$.

(xv) For each $\mathfrak{p} \in \min(\mathfrak{a})$, $\pi_{\mathfrak{p}}(S)^{-1}(R/\mathfrak{p}) \simeq \bar{S}^{-1}\bar{R}/\bar{S}^{-1}\bar{\mathfrak{p}} \simeq \bar{S}^{-1}\bar{R}/\bar{S}^{-1}\bar{\mathfrak{p}}$: By the statements (iii), (viii) and (xiv),

$$\bar{S}^{-1}\bar{R}/\bar{S}^{-1}\bar{\mathfrak{p}} \simeq \bar{S}^{-1}(\bar{R}/\bar{\mathfrak{p}}) \simeq \bar{S}^{-1}(\tilde{R}/\tilde{\mathfrak{p}}) \simeq \tilde{S}^{-1}\tilde{R}/\tilde{S}^{-1}\tilde{\mathfrak{p}} \simeq \pi_{\mathfrak{p}}(S)^{-1}(R/\mathfrak{p}).$$

The commutativity of the diagram is obvious.

4. (b) (xvi) $\mathfrak{n}_{\bar{S}^{-1}\bar{R}} = \bar{S}^{-1}\mathfrak{n}_{\bar{R}}$: By the assumption, the ideal $\mathfrak{n}_{\bar{R}} = \bigcap_{\bar{\mathfrak{p}} \in \min(\bar{\mathfrak{a}})} \bar{\mathfrak{p}}$ is a nilpotent ideal, i.e. $\mathfrak{n}_{\bar{R}}^m = 0$ for some natural number $m \geq 1$. Since the left ideal $\bar{S}^{-1}\mathfrak{n}_{\bar{R}}$ of the ring $\bar{S}^{-1}\bar{R}$ is an ideal (the statement (x)),

$$(\bar{S}^{-1}\mathfrak{n}_{\bar{R}})^m = \bar{S}^{-1}\mathfrak{n}_{\bar{R}}^m = 0,$$

and so $\bar{S}^{-1}\mathfrak{n}_{\bar{R}} \subseteq \mathfrak{n}_{\bar{S}^{-1}\bar{R}}$. By the statement (x), we have the opposite inclusion, and the statement (xvi) follows.

(xvii) $\bar{S}^{-1}\bar{R}/\bar{S}^{-1}\mathfrak{n}_{\bar{R}} \simeq \bar{S}^{-1}\bar{R}/\mathfrak{n}_{\bar{S}^{-1}\bar{R}}$: The statement (xvii) follows from the statement (xvi).

The proof of the statement (b) is complete (in view of statement 3).

(a) (xviii) The map $\min(\bar{R}) \rightarrow \min(\bar{S}^{-1}\bar{R})$, $\bar{\mathfrak{p}} \mapsto \bar{S}^{-1}\bar{\mathfrak{p}}$ is a bijection: By the statement (xvi),

$$\mathfrak{n}_{\bar{S}^{-1}\bar{R}} = \bar{S}^{-1}\mathfrak{n}_{\bar{R}} = \bar{S}^{-1}\left(\bigcap_{\bar{\mathfrak{p}} \in \min(\bar{R})} \bar{\mathfrak{p}}\right) = \bigcap_{\bar{\mathfrak{p}} \in \min(\bar{R})} \bar{S}^{-1}\bar{\mathfrak{p}}.$$

By the statements (vi) and (ix), the prime ideals $\{\bar{S}^{-1}\bar{\mathfrak{p}} \mid \bar{\mathfrak{p}} \in \min(\bar{R})\}$ of the ring $\bar{S}^{-1}\bar{R}$ are distinct. The second intersection above is irredundant since otherwise

$$\bar{S}^{-1}\bar{\mathfrak{p}}' \subseteq \bigcap_{\bar{\mathfrak{p}} \in \min(\bar{R}) \setminus \{\bar{\mathfrak{p}}'\}} \bar{S}^{-1}\bar{\mathfrak{p}} \subseteq \bar{\mathfrak{p}}.$$

for some $\bar{\mathfrak{p}}' \in \min(\bar{R})$. By the statement (xii),

$$\bar{\mathfrak{p}}' = \bar{R} \cap \bar{S}^{-1} \bar{\mathfrak{p}}' \subseteq \bar{R} \cap \bar{S}^{-1} \bar{\mathfrak{p}} = \bar{\mathfrak{p}},$$

a contradiction (since $\min(\bar{R}) = \{\bar{\mathfrak{p}} \mid \mathfrak{p} \in \min(\mathfrak{a})\}$). Since the second intersection is irredundant, the statement (xviii) follows from Lemma 3.2. $\square$

Lemma 4.2 extends the results of Theorem 1.7 to a more general situation.

Lemma 4.2. *Let R be a ring and $S' \in \text{Den}_l(R, \mathfrak{a})$. Suppose that for each element $s \in S'$ there is an element $t \in S'$ such that the element ts is a normal element of R . Let S be the set of all normal elements in S' . Then*

1. $S \in \text{Den}_l(R, \mathfrak{a})$ and $S^{-1}R \simeq S'^{-1}R$.
2. Theorem 1.7 holds for the denominator set $S' \in \text{Den}_l(R, \mathfrak{a})$.

Proof. 1. As a product of normal elements is a normal element and $S \subseteq S'$, the set S is a multiplicative set of R . The set S consists of normal elements. So, $S \in \text{Ore}(R)$. Let $\pi : R \rightarrow \bar{R} := R/\mathfrak{a}$, $r \mapsto \bar{r} := r + \mathfrak{a}$ and $\bar{S}' := \pi(S')$. Then

$$\bar{S}' := \pi(S') \in \text{Den}_l(\bar{R}, 0) \text{ and } S'^{-1}R \simeq \bar{S}'^{-1}\bar{R}.$$

(i) $\text{ass}_l(S) = \mathfrak{a}$: Since $S \subseteq S'$, $\text{ass}_l(S) \subseteq \text{ass}_l(S')$. The reverse inclusion follows from the fact that for each element $s \in S'$ there is an element $t \in S'$ such that the element ts is a normal element of R . Therefore, $\text{ass}_l(S) = \text{ass}_l(S') = \mathfrak{a}$.

(ii) $S \in \text{Den}_l(R, \mathfrak{a})$: In view of the statement (i), it remains to show that if $rs = 0$ for some elements $s \in S$ and $r \in R$ then $tr = 0$ for some element $t \in S$. Since $S \subseteq S'$ and $S' \in \text{Den}_l(R, \mathfrak{a})$, there is an element $s' \in S'$ such that $s'r = 0$. Choose an element $t' \in S'$ such that the element $t := t's'$ is a normal element of R , i.e. $t \in S$, and so $tr = t's'r = 0$, as required.

(iii) $S^{-1}R \simeq S'^{-1}R$: By the statement (ii) and the inclusion $S \subseteq S'$, there is a monomorphism

$$S^{-1}R \rightarrow S'^{-1}R, \quad s^{-1}r \mapsto s^{-1}r.$$

By the assumption, for each element $s' \in S'$, there is an element $t' \in S'$ such that $t := t's'$ is a normal element of the ring R , i.e. $t \in S$. Then for each $r \in R$,

$$s'^{-1}r = t^{-1}t'r \in S^{-1}R,$$

i.e. the monomorphism is a surjection, and the statement (iii) follows.

2. Statement 2 follows from statement 1 and Theorem 1.7. $\square$

5. Minimal primes of a semiprime ring and of its centre

In this section, proofs of Corollary 1.8 and Proposition 1.9 are given. Recall that Corollary 1.8 is a criterion for the map $\rho_{R, \min} : \min(R) \rightarrow \min(Z(R))$, $\mathfrak{p} \mapsto \mathfrak{p} \cap Z(R)$ being a well-defined map and Proposition 1.9 is a criterion for the map $\rho_{R, \min}$ being a surjection.

Let R be a ring and $Z(R)$ be its centre. For each prime ideal $\mathfrak{q} \in \text{Spec}(Z(R))$, the field

$$k(\mathfrak{q}) := Z(R)_{\mathfrak{q}}/\mathfrak{q}_{\mathfrak{q}} \simeq (Z(R)/\mathfrak{q})_{\mathfrak{q}}$$

is called the **residue field** of the prime ideal $\mathfrak{q}$. By applying the right exact functor $- \otimes_{Z(R)} R$ to the short exact sequence of $Z(R)$ -modules

$$0 \rightarrow \mathfrak{q}_{\mathfrak{q}} \rightarrow Z(R)_{\mathfrak{q}} \rightarrow k(\mathfrak{q}) \rightarrow 0$$

we obtain an exact sequence of right and left R -modules

$$\mathfrak{q}_{\mathfrak{q}} \otimes_{Z(R)} R \rightarrow Z(R)_{\mathfrak{q}} \otimes_{Z(R)} R \simeq R_{\mathfrak{q}} \rightarrow k(\mathfrak{q}) \otimes_{Z(R)} R \rightarrow 0.$$

Therefore,

$$R_{\mathfrak{q}}/R_{\mathfrak{q}}\mathfrak{q} \simeq k(\mathfrak{q}) \otimes_{Z(R)} R \quad (9)$$

is an isomorphism of rings. Let $\mathfrak{q} \in \text{Spec}(Z(R))$ and $\mathfrak{p} \in \text{Spec}(R)$. Then $R/\mathfrak{p}$ is a prime ring. By Lemma 2.6, the ring $\left(R/\mathfrak{p}\right)_{\mathfrak{q}} \simeq R_{\mathfrak{q}}/\mathfrak{p}_{\mathfrak{q}}$ is a prime ring. Therefore, the map

$$\text{Spec}(R) \rightarrow \text{Spec}(R_{\mathfrak{q}}), \quad \mathfrak{p} \mapsto \mathfrak{p}_{\mathfrak{q}} \quad (10)$$

is a well-defined map.

Proposition 5.1. *Let R be a ring and $Z(R)$ be its centre. Then*

1. *Let $\mathfrak{q} \in \text{Spec}(Z(R))$. Then $\mathfrak{q} \in \text{im}(\rho_R)$ iff $R_{\mathfrak{q}} \neq R_{\mathfrak{q}}\mathfrak{q}$.*
2. *For each $\mathfrak{q} \in \text{Spec}(Z(R))$, the maps*

$$\begin{aligned} \{\mathfrak{p} \in \text{Spec}(R) \mid \mathfrak{p} \cap Z(R) = \mathfrak{q}\} &\rightarrow V_{R_{\mathfrak{q}}}(R_{\mathfrak{q}}\mathfrak{q}) \rightarrow \text{Spec}(k(\mathfrak{q}) \otimes_{Z(R)} R), \\ Q &\mapsto Q_{\mathfrak{q}} \mapsto Q_{\mathfrak{q}}/R_{\mathfrak{q}}\mathfrak{q} \simeq k(\mathfrak{q}) \otimes_{Z(R)} Q, \end{aligned}$$

are bijections where $V_{R_{\mathfrak{q}}}(R_{\mathfrak{q}}\mathfrak{q}) := \{P \in \text{Spec}(R_{\mathfrak{q}}) \mid R_{\mathfrak{q}}\mathfrak{q} \subseteq P\}$.

3. *For each $\mathfrak{q} \in \text{im}(\rho_R)$, there is a minimal prime $\mathfrak{p} \in \text{min}(R)$ such that $\mathfrak{p} \cap Z(R) = \mathfrak{q}$.*

Proof. 2. By (10), the first map is a well-defined map. By the definition of the localization at the prime ideal $\mathfrak{q}$ of the centre $Z(R)$ of R , the first map is a bijection. By (9), $R_{\mathfrak{q}}/R_{\mathfrak{q}}\mathfrak{q} \simeq k(\mathfrak{q}) \otimes_{Z(R)} R$, and so the second map is a bijection.

1. Statement 1 follows from statement 2.

3. Since $\mathfrak{q} \in \text{im}(\rho_R)$, there is a prime ideal $\mathfrak{p}' \in \text{Spec}(R)$ such that $\mathfrak{p}' \cap Z(R) = \mathfrak{q}$. The prime ideal $\mathfrak{p}'$ contains a minimal prime ideal, say $\mathfrak{p} \in \text{min}(R)$. Notice that $\mathfrak{p} \cap Z(R) \in \text{Spec}(Z(R))$ and $\mathfrak{p} \cap Z(R) \subseteq \mathfrak{p}' \cap Z(R) = \mathfrak{q}$. Now, by the minimality of $\mathfrak{q}$, $\mathfrak{p} \cap Z(R) = \mathfrak{q}$, and the statement (ii) follows. $\square$

In general, the restriction map $\rho_{R, \text{min}} : \text{min}(R) \rightarrow \text{min}(Z(R))$, $\mathfrak{p} \mapsto \mathfrak{p} \cap Z(R)$ is not well-defined, see Lemma 5.2. Let K be a field, $n \in \{1, 2, \dots\} \cup \{\infty\}$, $[n] := \{i \in \mathbb{N} \mid 1 \leq i \leq n\}$, $F_n = K\langle x_i \mid i \in [n] \rangle$ be a free K -algebra, $P_n = K[z_i \mid i \in [n]]$ be a polynomial K -algebra and

$$A_n = F_n \otimes_K P_n / (x_i z_i \mid i \in [n]).$$

Lemma 5.2.

1. $\text{min}(A_n) = \{\mathfrak{p}_I \mid I \subseteq [n]\}$ where $\mathfrak{p}_I = (x_i, z_j)_{i \in I, j \in CI}$ and $CI = [n] \setminus I$.
2. For every $I \subseteq [n]$, $A_n/\mathfrak{p}_I \simeq F_{CI} \otimes_K P_I$ is a domain where $F_{CI} = K\langle x_j \mid j \in CI \rangle$ and $P_I = K[z_j \mid j \in I]$.

3. If $n = \infty$ then set $\min(A_\infty)$ is an uncountable set.
4. $Z(A_n) = P_n$.
5. For every $I \subseteq [n]$, $\mathfrak{p}_I \cap Z(A_n) = (z_j)_{j \in CI}$.

Proof. 2, 4 and 5. Statements 2, 4 and 5 are obvious.

1. Let $\mathfrak{p}$ be a minimal prime. For all $i \in [n]$,

$$\mathfrak{p} \supseteq \{0\} = (x_i z_i) = (x_i)(z_i).$$

By the minimality of $\mathfrak{p}$, precisely one of the elements x_i or z_i belongs to $\mathfrak{p}$, and statement 1 follows.

3. Statement 3 follows from statement 1 and the fact that the set of all subsets of the set of natural numbers is an uncountable set. $\square$

Lemma 5.3. Let R be a semiprime ring with $|\min(R)| < \infty$. Then the centre $Z(R)$ of R is a semiprime ring with $|\min(Z(R)) \cap \text{im}(\rho_R)| \leq |\min(R)| < \infty$.

Proof. Suppose that the centre $Z(R)$ is not a semiprime ring. Then it contains a nonzero nilpotent ideal, say $\mathfrak{a}$. Then $R\mathfrak{a}$ is a nonzero nilpotent ideal of the semiprime ring R , a contradiction. Therefore, the centre $Z(R)$ is a semiprime ring. By Proposition 5.1.(3), $|\min(Z(R)) \cap \text{im}(\rho_R)| \leq |\min(R)| < \infty$. $\square$

Proposition 5.4. Let R be a semiprime ring with $|\min(R)| < \infty$. Then the following statements are equivalent:

1. $\mathcal{C}_{Z(R)} \not\subseteq \mathcal{C}_R$.
2. $\mathcal{C}_{Z(R)} \cap \mathfrak{p} \neq \emptyset$ for some $\mathfrak{p} \in \min(R)$.
3. The restriction map $\min(R) \rightarrow \min(Z(R))$, $\mathfrak{p} \mapsto \mathfrak{p} \cap Z(R)$ is not a well-defined map.

Proof. (1 $\Rightarrow$ 2) Suppose that $\mathcal{C}_{Z(R)} \not\subseteq \mathcal{C}_R$. Then let us fix an element, say $c \in \mathcal{C}_{Z(R)} \setminus \mathcal{C}_R$, i.e. the central element c is a zero divisor in the ring R . Therefore, $\mathfrak{a} := \text{ann}_R(c)$ is nonzero annihilator ideal. It follows from the equality $(c)\mathfrak{a} = 0$, that the ideal $(c) = Rc$ of R is contained in the annihilator ideal $\text{ann}_R(\mathfrak{a})$. Then

$$c \in \text{ann}_R(\mathfrak{a}) \subseteq \mathfrak{p} \text{ for some } \mathfrak{p} \in \min(R),$$

(since the ring R is a semiprime ring with $|\min(R)| < \infty$), i.e. $\mathcal{C}_{Z(R)} \cap \mathfrak{p} \neq \emptyset$.

(2 $\Rightarrow$ 3) Suppose that $\mathcal{C}_{Z(R)} \cap \mathfrak{p} \neq \emptyset$ for some $\mathfrak{p} \in \min(R)$. Notice that

$$\mathfrak{q} := Z(R) \cap \mathfrak{p} \in \text{Spec}(Z(R)).$$

Then $\mathfrak{q} \notin \min(Z(R))$ since otherwise $\mathfrak{q} \subseteq \mathcal{Z}(R)$ where $\mathcal{Z}(R) := \bigcup_{\mathfrak{l} \in \min(Z(R))} \mathfrak{l}$ is the set of zero divisors of the semiprime commutative ring $Z(R)$ with $|\min(Z(R))| < \infty$ but the non-empty subset $\mathcal{C}_{Z(R)} \cap \mathfrak{p}$ of $\mathfrak{q}$ has zero intersection with $\mathcal{Z}(R)$, a contradiction.

(3 $\Rightarrow$ 2) Suppose that the restriction map $\min(R) \rightarrow \min(Z(R))$, $\mathfrak{p} \mapsto \mathfrak{p} \cap Z(R)$ is not a well-defined map. So, $\mathfrak{q} := \mathfrak{p} \cap Z(R) \notin \min(Z(R))$ for some $\mathfrak{p} \in \min(R)$. Notice that $\mathfrak{q} \in \text{Spec}(Z(R))$. Therefore,

$$\mathfrak{q} \not\subseteq \mathcal{Z}(R) = \bigcup_{\mathfrak{l} \in \min(Z(R))} \mathfrak{l}$$

since otherwise $\mathfrak{q} \subseteq \mathfrak{l}$ for some $\mathfrak{l} \in \min(Z(R))$, i.e. $\mathfrak{q} = \mathfrak{l} \in \min(Z(R))$, a contradiction. Then

$$\mathfrak{p} \cap \mathcal{C}_{Z(R)} = \mathfrak{q} \cap \mathcal{C}_{Z(R)} \neq \emptyset$$

since $\mathcal{C}_{Z(R)} = Z(R) \setminus \mathcal{Z}(R)$.

(2 $\Rightarrow$ 1) Suppose that $\mathcal{C}_{Z(R)} \cap \mathfrak{p} \neq \emptyset$ for some $\mathfrak{p} \in \min(R)$. Fix an element $c \in \mathcal{C}_{Z(R)} \cap \mathfrak{p}$. Recall that $\mathfrak{p}^c = \bigcap_{\mathfrak{q} \in \min(R) \setminus \{\mathfrak{p}\}} \mathfrak{q} \neq 0$ and $\mathfrak{p}\mathfrak{p}^c = 0$. Then $c\mathfrak{p}^c \subseteq \mathfrak{p}\mathfrak{p}^c = 0$, and so $c \in \mathcal{C}_{Z(R)} \setminus \mathcal{C}_R$. $\square$

Proof of Proposition 1.9. The equivalences (1 $\Leftrightarrow$ 2 $\Leftrightarrow$ 3) are equivalent to Corollary 1.8.

(1 $\Rightarrow$ 4) Suppose that $\mathcal{C}_{Z(R)} \subseteq \mathcal{C}_R$. By Lemma 5.3, the centre $Z(R)$ is a semiprime ring. By the assumption, $|\min(Z(R))| < \infty$. Since the centre $Z(R)$ is a commutative semiprime ring with $|\min(Z(R))| < \infty$,

$$\mathcal{C}_{Z(R)} = Z(R) \setminus \bigcup_{\mathfrak{q} \in \min(Z(R))} \mathfrak{q}.$$

Hence, the quotient ring $Q(Z(R)) = \mathcal{C}_{Z(R)}^{-1}Z(R)$ is a semiprime commutative Artinian ring with

$$\min(Q(Z(R))) = \{\mathcal{C}_{Z(R)}^{-1}\mathfrak{q} \mid \mathfrak{q} \in \min(R)\}.$$

Hence, the ring $Q(Z(R))$ is a semisimple Artinian ring with $\text{Spec}(Z(R)) = \min(Z(R))$, and so the quotient ring

$$Q(Z(R)) = \mathcal{C}_{Z(R)}^{-1}Z(R) \simeq \prod_{\mathfrak{q} \in \min(Z(R))} Z(R)_{\mathfrak{q}} = \prod_{\mathfrak{q} \in \min(Z(R))} k(\mathfrak{q})$$

is a finite direct product of fields $Z(R)_{\mathfrak{q}} = k(\mathfrak{q})$. Since $\mathcal{C}_{Z(R)} \subseteq \mathcal{C}_R$,

$$\mathcal{C}_{Z(R)}^{-1}Z(R) \subseteq \mathcal{C}_{Z(R)}^{-1}Z(R) \simeq \prod_{\mathfrak{q} \in \min(Z(R))} Z(R)_{\mathfrak{q}} \otimes_{Z(R)} R \simeq \prod_{\mathfrak{q} \in \min(Z(R))} R_{\mathfrak{q}},$$

and so the map $\rho_{R, \min}$ is a surjection, by Proposition 5.1 (since $R_{\mathfrak{q}} = R_{\mathfrak{q}}/R_{\mathfrak{q}}\mathfrak{q}$).

(4 $\Rightarrow$ 3) The implication is obvious. $\square$

Corollary 5.5. *Let R be a semiprime ring such that $|\min(R)| < \infty$, $|\min(Z(R))| < \infty$ and $\mathcal{C}_{Z(R)} \subseteq \mathcal{C}_R$. Then*

1. $R \subseteq \mathcal{C}_{Z(R)}^{-1}R \simeq \prod_{\mathfrak{q} \in \min(Z(R))} R_{\mathfrak{q}}$.
2. $Z(\mathcal{C}_{Z(R)}^{-1}R) = \mathcal{C}_{Z(R)}^{-1}Z(R) \simeq \prod_{\mathfrak{q} \in \min(Z(R))} Z(R)_{\mathfrak{q}}$ is a finite direct product of fields $Z(R)_{\mathfrak{q}}$.
3. For every $\mathfrak{q} \in \min(Z(R))$, $R_{\mathfrak{q}}$ is a semiprime ring, $\min(R_{\mathfrak{q}}) = \{\mathfrak{p}_{\mathfrak{q}} \mid \mathfrak{p} \in \min(R)\}$ and $|\min(R_{\mathfrak{q}})| \leq |\min(R)| < \infty$.
4. For every $\mathfrak{q} \in \min(Z(R))$, $Z(R_{\mathfrak{q}}) \simeq Z(R)_{\mathfrak{q}}$, i.e. the $Z(R)_{\mathfrak{q}}$ -algebra $R_{\mathfrak{q}}$ is a central algebra.

Proof. 1. Statement 1 was proven in the proof of Proposition 1.9.

2. By the assumption, $\mathcal{C}_{Z(R)} \subseteq \mathcal{C}_R$. Hence, $R \subseteq \mathcal{C}_{Z(R)}^{-1}R$. Let $c^{-1}r \in Z(\mathcal{C}_{Z(R)}^{-1}R)$ for some elements $c \in \mathcal{C}_{Z(R)}$ and $r \in R$. Then $r \in Z(R)$. It follows that $Z(\mathcal{C}_{Z(R)}^{-1}R) = \mathcal{C}_{Z(R)}^{-1}Z(R)$. The isomorphism $\mathcal{C}_{Z(R)}^{-1}Z(R) \simeq \prod_{\mathfrak{q} \in \min(Z(R))} Z(R)_{\mathfrak{q}}$ was proven in the proof of Proposition 1.9.

3. Statement 3 follows Theorem 1.4.(1,2).

4. Statement 4 follows from statements 1 and 3. $\square$

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V.V. Bavula: Project administration, Writing – original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data sharing not applicable – no new data generated.

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