Proceedings Paper:

Govindan, S., Laraki, R. and Pahl, L. orcid.org/0000-0002-0268-371X (2020) On Sustainable Equilibria. In: Biró, P. and Hartline, J., (eds.) EC '20: Proceedings of the 21st ACM Conference on Economics and Computation. EC '20: The 21st ACM Conference on Economics and Computation, 13-17 Jul 2020, Virtual Event (Hungary). ACM, pp. 767-768. ISBN: 9781450379755.

https://doi.org/10.1145/3391403.3399514

On Sustainable Equilibria

SRIHARI GOVINDAN, Department of Economics, University of Rochester.
RIDA LARAKI, CNRS (Lamsade, Dauphine-PSL) and Computer Science Department (University of Liverpool)
LUCAS PAHL, Department of Economics, University of New South Wales, Sydney.

Following the ideas laid out in Myerson [5], Hofbauer [3] defined an equilibrium of a game as sustainable if it can be made the unique equilibrium of a game obtained by deleting a subset of the strategies that are inferior replies to it, and then adding others. Hofbauer also formalized Myerson's conjecture about the relationship between the sustainability of an equilibrium and its index: for a generic class of games, an equilibrium is sustainable iff its index is +1. Von Schemde and von Stengel [8] proved this conjecture for bimatrix games. This paper shows that the conjecture is true for all finite games. More precisely, we prove that an isolated Nash equilibrium of a given game has index +1 if and only if it can be made the unique equilibrium of a game obtained by adding finitely many inferior reply strategies.[1]

CCS Concepts: • **Theory of computation → Algorithmic game theory and mechanism design**.

Additional Key Words and Phrases: Nash equilibria, +1 index, Hopf extension theorem, Delaunay triangulation

ACM Reference Format:
Srihari Govindan, Rida Laraki, and Lucas Pahl. 2020. On Sustainable Equilibria. In *Proceedings of the 21st ACM Conference on Economics and Computation (EC '20), July 13–17, 2020, Virtual Event, Hungary*. ACM, New York, NY, USA, 2 pages. https://doi.org/10.1145/3391403.3399514

Myerson [5] proposes a refinement of Nash equilibria of finite games, which he calls *sustainable equilibria*. He reasons that the two pure-strategy equilibria in the Battle-of-Sexes game are sustainable while the mixed equilibrium is not and concludes with a conjecture that the index of an equilibrium is a determinant of its sustainability.[2]

Hofbauer [3] distills the ideas in Myerson's paper to provide a definition of sustainable equilibria. Hofbauer posits that a minimum requirement of sustainability should be that if an equilibrium of a game is sustainable, it should remain sustainable in the game obtained by restricting players' strategies to the set of best replies to the equilibrium.[3] If one also accepts that an equilibrium that is unique is sustainable, then one is lead to the following definition. Say that a game-equilibrium pair is *equivalent* to another such pair if the restrictions of the two games to the set of best replies to their respective equilibria are the same game (modulo a relabelling of the players and their strategies) and the two equilibria coincide (under the same identification). An equilibrium of a game is *sustainable* if it has an equivalent pair where the equilibrium is unique. Call an equilibrium *regular* if locally the equilibrium is a differentiable function of the game payoffs.

[1]Link to the paper: https://arxiv.org/abs/2005.14094
[2]Interestingly, Myerson speculates that one could perhaps develop a theory of index of equilibria based on fixed-point theory, seemingly unaware, as Hofbauer [3] observes, of an existent theory in the literature (see Gül, Pearce and Stacchetti [2] and Ritzberger [6]). For an account of index theory written primarily for economists, see McLennan [4].
[3]For regular games, this is equivalent to restricting players' strategies to the support of the equilibrium

EC '20, July 13–17, 2020, Virtual Event, Hungary

ACM ISBN 978-1-4503-7975-5/20/07.
https://doi.org/10.1145/3391403.3399514

Following Myerson, Hofbauer conjectured that a regular equilibrium is sustainable iff its index is +1. Von Stengel and von Schemde [8] proved this conjecture for bimatrix games. In this paper, we show that the conjecture holds for all N-person games.

For example, the battle of the sexes G below has three regular Nash equilibria: two are pure (t, l) and (b, r) (with index +1), and one is mixed (with index -1).

$$G = \begin{array}{c|c|c|} & l & r \\ \hline t & (3,2) & (0,0) \\ \hline b & (0,0) & (2,3) \\ \hline \end{array}$$

By adding one strategy to each player (x for player 1, and y for player 2, see the game $\bar{G}$ below), b and r are now strictly dominated. Removing them yield a game where x and y are strictly dominated, making the regular +1 equilibrium $\sigma = (t, l)$ of G, a quasi-strict[4] and the unique equilibrium of $\bar{G}$.

$$\bar{G} = \begin{array}{c|c|c|c|} & l & r & y \\ \hline t & (3,2) & (0,0) & (0,1) \\ \hline b & (0,0) & (2,3) & (-2,4) \\ \hline x & (1,0) & (4,-2) & (-1,-1) \\ \hline \end{array}$$

This construction extends to all games: we show that a regular (and more generally an isolated) Nash equilibrium of a game has index +1 if and only if it can be made the unique equilibrium of a game obtained by adding finitely many inferior reply strategies.

In a bimatrix game, a player's payoff function is linear in his opponent's strategy. Von Schemde and von Stengel [8] exploit those features and use tools from the theory of polytopes to establish the construction. In general games, their technique is inapplicable.

Our proof, similar in spirit to that in Govindan and Wilson [1], works as follows. Suppose that an equilibrium σ of G is sustainable, and that (G, σ) is equivalent to $(\bar{G}, \sigma)$ where σ is the unique equilibrium of $\bar{G}$. Let G^* be the game obtained from G by deleting strategies that are inferior replies to σ. It follows from a property of the index that the index of σ in G can be computed as the index of σ in G^*. The game G^* is also obtained from $\bar{G}$ by deleting inferior replies there. Therefore, the index of σ in G^* can be computed as the index of σ in $\bar{G}$. As σ is the unique equilibrium of $\bar{G}$, its index is +1. Going the other way, if we have a +1 index equilibrium σ of G, and since the sum of indices over all equilibria is +1, then the sum of the indices of the other equilibria is zero. Now, we can take a map f whose fixed points are the Nash equilibria of G and alter it outside a neighbourhood of σ so that the new map $\bar{f}$ has no fixed points other than σ.[5] By a careful addition of strategies and specification of payoffs for these strategies, we obtain $(\bar{G}, \sigma)$ equivalent to (G, σ) where any equilibrium of $\bar{G}$ must translate into a fixed point of $\bar{f}$, making σ the unique equilibrium in $\bar{G}$.

REFERENCES

[1] S. Govindan and R. Wilson. 2005. Essential Equilibria. *Proceedings of the National Academy of Sciences* 102 (2005).

[2] F. Gül, D. Pearce, and E. Stacchetti. 1993. A Bound on the Proportion of Pure Strategy Equilibria in Generic Games. *Mathematics of Operations Research* 18 (1993).

[3] J. Hofbauer. 2010. Some Thoughts on Sustainable/Learnable Equilibria. *Mimeo* (2010).

[4] A. McLennan. 2018. *Advanced Fixed Point Theory for Economics.* Springer-Verlag.

[5] R.B. Myerson. 1996. *Understanding Strategic Interaction: Essays in Honor of Reinhard Selten.* Name of chapter: Sustainable Equilibria in Culturally Familiar Games.

[6] K. Ritzberger. 1994. The Theory of Normal Form Games from the Differentiable Viewpoint. *International Journal of Game Theory* 23 (1994).

[7] E.H. Spanier. 2016. *Algebraic Topology.* Springer-Verlag.

[8] A. von Schemde and B. von Stengel. 2008. Strategic Characterization of the Index of an Equilibrium. *SAGT* 1 (2008).

[4] An equilibrium is quasi-strict if all unused strategies are inferior replies.

[5] The possibility of this construction follows from a deep result in algebraic topology, the Hopf Extension Theorem, see [7].