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Resonant barrier crossing in a modulated classical double well potential

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ABSTRACT

We show, both numerically and analytically, that for the modulated double well potential given by $-\frac{1}{2}\mathrm{m}\omega^2 x^2(1+\epsilon\cos(\Omega t)) + \frac{\mathrm{m}\lambda}{4}x^4$, a particle initially placed with zero momentum at the bottom of one of the wells can oscillate back and forth between the two wells for an infinitesimal value of ϵ when the modulating frequency Ω equals $\sqrt{2}\omega$. The effect is also present, but in a less dramatic fashion where $\Omega = 2\sqrt{2}\omega$. In the $\epsilon - \Omega$ plane, around the point $\Omega = \sqrt{2}\omega$, the boundary separating the region where the tunnelling occurs from the region where it does not, is a fractal curve whose fractal dimension is 1.45.

The issue of tunnelling across a potential barrier has generally been associated with quantum mechanics. The calculational strategy for obtaining the tunnelling time (classical 'dwell-time') has often made use of semi-classical techniques ([1], [2]). The question of 'dwell-time' in a completely classical set-up seems to have been introduced by Mateos and José [3] and Mateos [4] in studying the classical dynamics of a particle inside a rigid box with an internal oscillating square well potential. Scaling properties of the dynamics of a classical particle in a time dependent potential have been extensively investigated by Leonel, and da Silva [5] and by MaClintock and Leonel [6–8]. In this work, we would like to report an apparently overlooked resonant tunnelling in a modulated double well potential. We would like to emphasise that the tunnelling we are referring to occurs within a purely classical context. We will show that for a particular value of the modulation frequency (we call this the resonance frequency), a particle placed at rest in one of the minima of the potential will make a transition to the other well even if the modulation amplitude is a couple of orders of magnitude smaller than the well depth.

Forced nonlinear oscillators have been extensively studied since these are one-dimensional systems which are capable of showing a wide variety of bifurcation and clear demonstration of chaotic behaviour. Our interest is in the double well oscillator shown in Fig. 1.

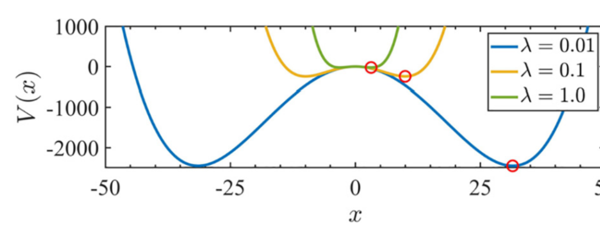

Fig. 1. Double well oscillating potential $V(x,t) = -\mathrm{m}\omega^2 x^2(1+\epsilon\cos(\Omega t))/2 + \mathrm{m}\lambda x^4/4$, is plotted for three values of λ at fixed $\omega = 3, \epsilon = 0.05$ and $\mathrm{m} = 1$. The Minima of the potential at $t = 0$ are, $x_0(t=0) = \pm\omega\sqrt{(1+\epsilon)/\lambda}$, denoted by circles.

The time independent form has attracted more attention in quantum mechanics than in classical mechanics. In particular, quantum dynamics is very important because of the phenomenon of tunnelling in which a particle localised strongly in one of the wells can tunnel back and forth between the two wells. In driven double well quantum oscillators, the regular and irregular dynamics and tunnelling have been studied by Lin and Ballentine [9], Igarashi and Yamada [10], Marthaler and Dykman [11] under different circumstances. Dynamics of moments for the quantum Mathieu oscillator [12] and different facets of parametrically driven classical systems [13] have also been studied. Using a double well

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with a small added stochasticity led to a weak tunnelling in the classical limit [14]. This signal was significantly enhanced if the wells were subjected to an oscillation width of a frequency which was of the order of the frequency (very low) of transferring from one well to another. In this work, we show that in a parametrically modulated well double oscillator with the Hamiltonian,

$$\mathcal{H} = \frac{p^2}{2\mathrm{m}} - \frac{1}{2}\mathrm{m}\omega^2 x^2(1+\epsilon\cos(\Omega t)) + \frac{\mathrm{m}\lambda}{4}x^4, \tag{1}$$

a strongly localised (*i.e.* particle at rest in one of the wells) particle in one well at $t = 0$, can execute oscillations from one well to another for a very tiny modulation strength ϵ_c when the modulating frequency is given by $\Omega = \sqrt{2}\omega + \delta$, where δ is $\mathcal{O}(\epsilon)$. We show the effect very clearly in Fig. 2. In Fig. 2(a), the value of ϵ is

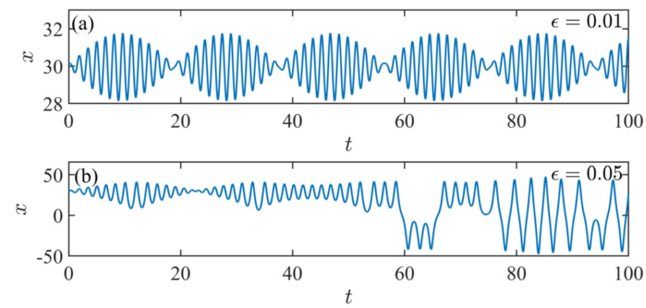

Fig. 2. The particle position (initially placed at $x_0(t=0) = 30\sqrt{1+\epsilon}$) as function of time for $\lambda = 0.01$, $\Omega = 3.9$ and $\omega = 3$. For $\epsilon < \epsilon_c \sim 0.03$, (a), the particle is localised at one well. At very small modulation *i.e.* $\epsilon = 0.05$ (b), the particle tunnels from its initial well.

chosen as 0.01, which is below the threshold required for crossing. We clearly see the particle oscillating in one well. For the choice of $\epsilon = 0.05$ (Fig. 2(b)), it is clear that the particle oscillates between the left and right wells. We see that when the tunnelling begins, small changes in the value of ϵ can lead to large changes in the dynamics. If a particular value of ϵ shows tunnelling, then a small increment can show the absence of tunnelling and vice versa. We show this clearly in Fig. 3 by considering different values of ϵ, which are all quite close to each other. We further find

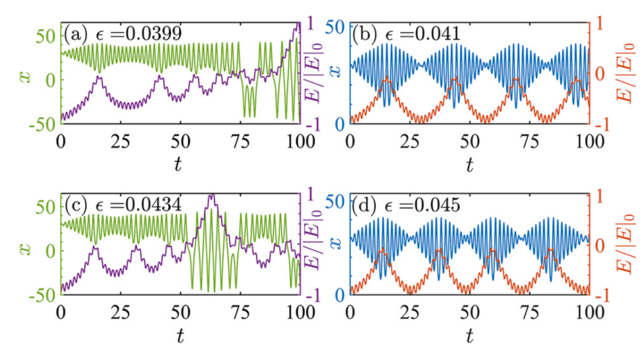

Fig. 3. The particle position (x) and its normalised energy ($E/|E|_0$) as a function of time are shown for various ϵ values, with fixed $\omega = 3$, $\lambda = 0.01$, and $\Omega = 3.9$. Notably, particle tunnelling doesn't exhibit a linear relationship with ϵ values. In panel, (a) for $\epsilon = 0.0399$, a quick particle tunnelling occurs, while in panel (b) with $\epsilon = 0.041$, the particle remains confined to one well side along with its energy. Panel (c) for $\epsilon = 0.0434$, depicts the particle successfully tunnelling to the opposite side, overcoming the energy barrier. Lastly, for $\epsilon = 0.045$ in panel (d), the particle oscillates within its initial well on this time scale, with energy below the barrier height.

that there is a rather unusual boundary in the ϵ vs Ω plane separating regions which exhibit tunnelling from regions which do not. As shown in Fig. 4 (for $\lambda = 0.01$, $\omega = 3$), the tunnelling region originating from $\Omega \simeq \sqrt{2}\omega$ has a structured boundary which can be characterised by the fractal dimension. The resonance near $\Omega \simeq \sqrt{2}\omega$ is stronger than the one near $\Omega \simeq 2\sqrt{2}\omega$ which requires larger values of ϵ for particle tunnelling. Quite surprisingly, this picture is independent of the value of λ. We also find that the

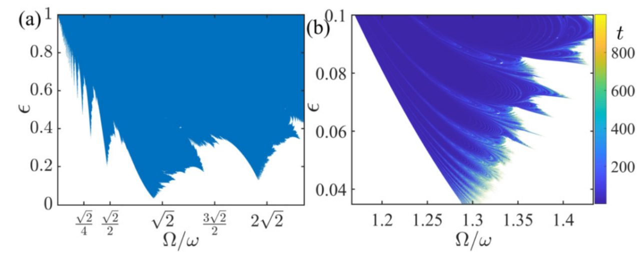

Fig. 4. Solution of the parametric double well oscillator in the $\Omega - \epsilon$ parametric space for $\omega = 3$, where, $\lambda = 0.01$, initial position is at one of well's minima $x_0 = \omega\sqrt{(1+\epsilon)/\lambda}$, and initial velocity $\dot{x}_0 = 0$. Shaded regions correspond to parameters which allow tunnelling, and the white region corresponds to confined oscillation in the initial well. The zoomed region with colourmap near the primary resonance in (b) shows the escaping time t from the initial well. The darker region corresponds to early escape, whereas the lighter region infers a delayed escape from its initial well. The above pictures are similar to the instability zones of the Mathieu equations except for the fact that the modulating frequency is an irrational multiple of the natural frequency and the boundaries are fractal curves [15–17].

boundary curve separating the zone in which the particle can escape to the other well from the zone where the particle is confined in an initial well manifest a fractal nature with a fractal dimension of 1.45 around $\Omega = \sqrt{2}\omega$ and 1.75 for $\Omega = 2\sqrt{2}\omega$.

We now describe the fractal nature of the basin boundary separating the escape zone from the confined one (see Fig. 4(a)). The more spectacular escape is for frequencies $\Omega \simeq \sqrt{2}\omega$ and the boundary, as shown in Fig. 4(b), is a fractal curve of dimension $D = 1.45$. For the frequencies near $\Omega \simeq 2\sqrt{2}\omega$, we have similar fractal behaviour for the separating border, and its fractal dimension is 1.75. We note that the fractal dimensions of the set of points of the attractor for the Poincare maps of chaotic motion in the periodically driven double-well oscillator with dumping can span a range of values around 1.5 as the damping coefficient is varied [18]. However, we are unable to observe any apparent correlation with our specific system.

The dimensions have been computed using the *box counting algorithm*, and the procedure is shown in Fig. 5. For $\Omega \simeq 2\sqrt{2}\omega$,

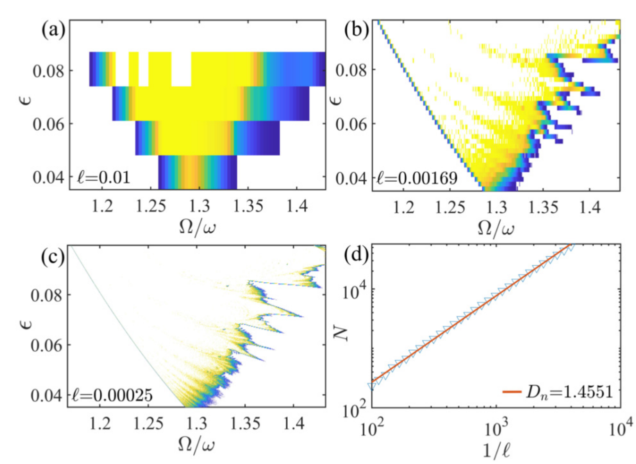

Fig. 5. Box counting for particle tunnelling at differing spatial resolutions of box length ℓ in (a-c). Using the definition of the counting method, the slope of the $\log N$ and $\log 1/\ell$ gives the fractal dimension. The fractal dimension is 1.45 near $\Omega \simeq 3.9$, where $\omega = 3$, $\lambda = 0.01$. The darker colour of the Colourbar indicates in (a-c), a higher number of points in a box, whereas brighter points represent a small number of points counts in a box (see SI Movie 1).

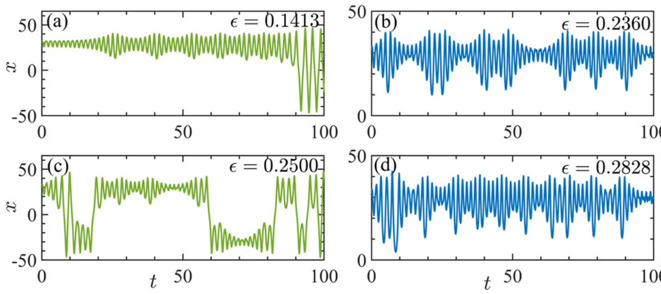

Fig. 6. Position of the particle as a function of time is shown for four different ϵ values around $\Omega = 2\sqrt{2}\omega$, where, $\Omega = 8.1$, $\omega = 3$, $\lambda = 0.01$. Particle tunnelling is not linearly related to the ϵ values. As seen in (a) $\epsilon = 0.1413$ particle can tunnel quickly, however, for a larger value of $\epsilon = 0.236$ in (b) particle is still trapped in one side of the well. Furthermore, with a stronger value of $\epsilon = 0.25$ in (c), the particle tunnels to the other side, however, for $\epsilon = 0.2828$ in (d), the particle is still oscillating in one well in this time scale (see SI Movie 3).

the time series for escape/confinement in the initial well is shown in Fig. 6(a-d). We have also explored how the possibilities of escape from a well change as the initial position of the particle vary. In Fig. 7, we show escape/confined zones for three values of Ω. We vary the initial position from near $x_0 = 0$ to that corresponding to the maximum potential depth ($= 30\sqrt{2}$ where $\epsilon = 1$). The dashed line represents the minimum position of the well, where the rest of the simulation's initial position is considered. It should be noted that Fig. 7(a & c) demonstrate that if the initial position of the particle lies between maxima and minima of the well, escaping possibility is higher for lower values of ϵ. Surprisingly, Fig. 7(b), reveals that the probability of escape is lower when the particle's initial position is intermediate between the highest and lowest points of the barrier, even when $\epsilon > \epsilon_c$. This finding indicates that the probability of escape is associated with the phase relationship between the frequency of the forced oscillation and the frequency of the particle oscillation (see SI Movie 2). It has also been observed that the phenomenon is independent of the value of λ and this is demonstrated in Fig. 8. Our results are independent of the frequency ω as well. Hence the results are independent of the well depth.

To get an analytic handle, we use the coordinate transformation $y = x - \frac{\omega}{\sqrt{\lambda}}$, which centres the coordinate system at the bottom of the unperturbed right well and finds the dynamics to be given by

$$\frac{d^2 y}{dt^2} + 2\omega^2 y + 3\sqrt{\lambda}\omega y^2 + \lambda y^3 = \frac{\epsilon\omega^3}{\sqrt{\lambda}}\cos(\Omega t) + \omega^2\epsilon\cos(\Omega t) y. \tag{2}$$

The interesting thing is that in the above dynamics, there is going to be both forced resonance near $\omega = \frac{\Omega}{\sqrt{2}}$ and a parametric resonance near $\omega = \frac{\Omega}{2\sqrt{2}}$. We note that the quadratic nonlinearity and the cubic one will both contribute at $\mathcal{O}(\lambda)$ and try a solution near $\omega = \frac{\Omega}{\sqrt{2}}$ of the form,

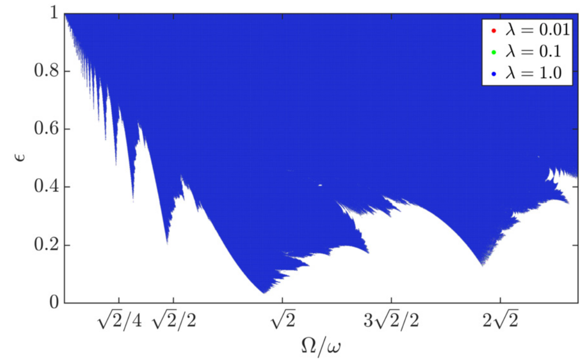

Fig. 8. The boundaries of escape from the initial well and confined in the initial well are independent of the well depth $\omega\sqrt{(1+\epsilon)/\lambda}$. We consider three different $\lambda = 0.01$, 0.1 and 1.0, where all the cases the region boundaries superpose.

$$y = C_1 + C_2 \cos 2(\Omega t + \phi(t)) + A(t)\cos(\Omega t + \phi(t)). \tag{3}$$

In the above C_1 and C_2 are constants of $\mathcal{O}(\sqrt{\lambda})$ and $A(t)$ and $\phi(t)$ are the slowly varying amplitude and phase at $\mathcal{O}(\lambda)$. We set $2\omega^2 = \Omega^2 + \epsilon\Delta$ ($\epsilon << 1$) and by matching terms of the same trigonometric variation obtain

$$2\Omega^2 C_1 = -3\omega\sqrt{\lambda}A^2 - \epsilon\omega^2 A\cos\phi, \tag{4}$$

$$6\Omega^2 C_2 = 3\omega\sqrt{\lambda}A^2 - \epsilon\omega^2 A\cos\phi. \tag{5}$$

Using these values of C_1 and C_2 and the usual Bogoliubov-Krykov technique, we arrive at the slow dynamics given by,

$$2\Omega\dot{A} = -\frac{\epsilon\omega^3}{2\sqrt{\lambda}}\sin\phi\left(1 - \frac{7\lambda}{8\omega^2}A^2\right), \tag{6a}$$

and

$$2\Omega A\dot{\phi} = \epsilon\Delta A - \frac{\epsilon\omega^3}{\sqrt{\lambda}}\cos\phi + \frac{5}{8}\epsilon\omega\sqrt{\lambda}A^2 - 3\lambda A^3. \tag{6b}$$

We have dropped all terms of $\mathcal{O}(\epsilon^2)$ in the above equations. The novel feature in the above set of equations is the phase dependence obtained in Eq. (6a). This is a real effect that can be seen in the movie (see SI), showing the dynamics of escape from the well. From Eq. (6a), we see there are three possible fixed points, (i) $\phi^* = 0$, (ii) $\phi^* = \pi$ and (iii) $A^{*2} = 8\omega^2/7\lambda$. In the case of (i) and (ii), the corresponding A^* has to be found from Eq. (6b). In case of (iii), the phase dynamics will be obtained from Eq. (6b). For the fixed points (i) and (ii), the relevant values of A are found from,

$$3\lambda A^{*3} = \epsilon\left[\Delta A^* + \frac{5}{8}\epsilon\omega\sqrt{\lambda}A^{*2} - \frac{\omega^3}{\sqrt{\lambda}}\cos\phi\right], \tag{7}$$

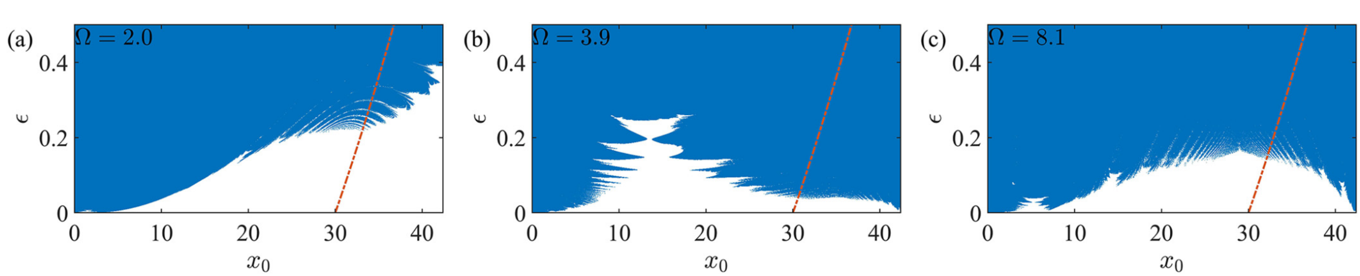

Fig. 7. Dependence on initial conditions for tunnelling at (a) $\Omega = 2.0$ (b) $\Omega = 3.9$ (c) $\Omega = 8.1$ while keeping other parameter fixed at $\omega = 3$ and $\lambda = 0.01$. Blue regions represent the tunnelling region, while white regions correspond to the particle confined in the initial well. The minimum position, $x_0 = \omega\sqrt{(1+\epsilon)/\lambda}$, is indicated by the dotted line.

for $\epsilon << 1$, the largest value is $\mathcal{O}(\epsilon^{1/3})$. This amplitude is not strong enough to provide tunnelling from one well to the other. Thus, the only relevant fixed point is the one for which $A^{*2} = 8\omega^2/7\lambda$. This is a large enough amplitude for the particle to escape from the well. It should be noted that this ability to traverse to the other well is independent of the well depth. This is rather an extraordinary situation where, however deep the well is, a very small modulation is capable of sending a particle from one well to the other. In Fig. 6, we set three values of $\lambda = 1,\ 0.1$, and 0.01 (the potential forms are shown in Fig. 1) keeping other parameters fixed, one observes the same tunnelling phase diagram in $\epsilon - \Omega$ phase space. A similar analysis can be carried out near the parametric resonance case of $\sqrt{2}\omega = \frac{\Omega}{2} + \epsilon\delta(\epsilon << 1)$, but we do not show them here as the results around the resonance is less spectacular than near $\sqrt{2}\omega = \Omega$.

It is interesting to look at the dynamics of the energy E in this context, where $E = \frac{p^2}{2} - \frac{\omega^2 x^2}{2} + \frac{\lambda x^4}{4}$ is the total energy for a particle of unit mass in a double well potential. The rate of charge of energy is

$$\dot{E} = p\dot{p} - \omega^2 x\dot{x} + \lambda x^3\dot{x} = \epsilon\omega^2 xp\cos(\Omega t). \tag{8}$$

To get the approximate dynamics when the particle is near the bottom of the left well, we make the coordinate transformation $x = y - \frac{\omega}{\sqrt{\lambda}}$, which puts the minimum of the left-hand well at $y = 0$, with the dynamics around the minimum governed by the Hamiltonian $H = \frac{p^2}{2} + \omega^2 y^2$ at low energies. In terms of y, the energy variation is

$$\dot{E} = \epsilon\omega^2 \left(y - \frac{\omega}{\sqrt{\lambda}}\right) p\cos(\Omega t). \tag{9}$$

The dynamics at the bottom of the well is approximately $y(t) = A\cos(\sqrt{2}\omega t) + B\sin(\sqrt{2}\omega t)$ and $p = \dot{y} = \sqrt{2}\omega[A\sin(\sqrt{2}\omega t) - B\cos(\sqrt{2}\omega t)]$. On carrying out a time average $\langle yp\cos\Omega t\rangle = 0$, while $\langle p\cos\Omega t\rangle = -B\sqrt{2}\omega\langle\cos\sqrt{2}\omega t\cos\Omega t\rangle = -B\omega/\sqrt{2}$ if $\Omega = \sqrt{2}\omega$. This leads to $\dot{E} = \epsilon\frac{\omega^4 B}{\sqrt{2\lambda}}$ and is the reason for the increase of energy on an average for $\Omega = \sqrt{2}\omega$. As evident in Fig. (3), the particle jumps to the other well while it surpasses the energy barrier height. While the energy is lower than the energy barrier, the particle is confined in its initial well.

Our results have an analogue in the restricted three-body problem under mutual gravitational attraction known as the *Sitnikov* problem. In this three-body dynamics, an amazing setup is achieved where the third body is restricted to move in a line (the z-axis) with the dynamics given by $\ddot{z} + 2Gmz/(r(t)^2 + z^2) = 0$, where G is the gravitational constant and $r(t)$ is the separation distance of the two other bodies. Sitnikov [19] showed for the first time the existence of unbounded oscillations in the restricted three-body problem. The result was first shown for a zero mass situation for the third body and then generalised to finite mass by Alekseev [20] and is analogous to our escape from the bottom of a well with zero initial velocity (see in Fig. 7(b)).

In conclusion, we have shown that the modulated double well potential governed by the Hamiltonian of Eq. (1) has a large number of somewhat unusual characteristics. An extremely small modulation amplitude allows for tunnelling in the most adverse situation namely the particle being rest at the minimum of one of the wells. The tunnelling is found to be independent of the depth of the well in our numerical work and can also be arrived at by deriving an amplitude equation where the amplitude evolution is dependent on the phase. Further, the boundary separating the tunnelling zone from the confined one is a fractal curve with fractal dimensions depending on whether one is near a forced resonance or a parametric resonance.

CRediT authorship contribution statement

Subhadip Biswas: Writing – original draft, Writing – review & editing, Investigation, Data curation, Conceptualization. **Jayanta K. Bhattacharjee:** Writing – original draft, Writing – review & editing, Investigation, Supervision, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

Appendix A. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.physleta.2023.129083.

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