



This is a repository copy of *Quantum-corrected three-dimensional AdS space-time*.

White Rose Research Online URL for this paper:

<https://eprints.whiterose.ac.uk/id/eprint/232648/>

Version: Preprint

Preprint:

Thompson, J.C. and Winstanley, E. orcid.org/0000-0001-8964-8142 (Submitted: 2025)
Quantum-corrected three-dimensional AdS space-time. [Preprint - arXiv] (Submitted)

<https://doi.org/10.48550/arxiv.2508.14697>

© 2025 The Author(s). This preprint is made available under a Creative Commons Attribution 4.0 International License. (<https://creativecommons.org/licenses/by/4.0/>)

Reuse

This article is distributed under the terms of the Creative Commons Attribution (CC BY) licence. This licence allows you to distribute, remix, tweak, and build upon the work, even commercially, as long as you credit the authors for the original work. More information and the full terms of the licence here:

<https://creativecommons.org/licenses/>



Takedown

If you consider content in White Rose Research Online to be in breach of UK law, please notify us by emailing eprints@whiterose.ac.uk including the URL of the record and the reason for the withdrawal request.



eprints@whiterose.ac.uk
<https://eprints.whiterose.ac.uk/>

Quantum-corrected three-dimensional AdS space-time

Jacob C. Thompson  and Elizabeth Winstanley 

School of Mathematical and Physical Sciences, University of Sheffield, Hicks Building, Hounsfield Road, Sheffield S3 7RH, United Kingdom.

E-mail: JThompson16@sheffield.ac.uk, E.Winstanley@sheffield.ac.uk

Abstract. We study the backreaction of a quantum scalar field on anti-de Sitter (AdS) space-time. The renormalized expectation value of the stress-energy tensor operator (RSET) for a quantum scalar field acts as a source term on the right-hand-side of the Einstein equations for the quantum-corrected metric. We find the RSET for a nonrotating thermal state on global AdS in three dimensions and compare this with results from relativistic kinetic theory. We then solve the linearized quantum-corrected Einstein equations. We interpret these quantum-corrected metrics as asymptotically-AdS solitons.

1 Introduction

In general relativity, all energy and matter gravitates, and thus results in the curvature of space-time, governed by the classical Einstein equations:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}, \quad (1)$$

where $G_{\mu\nu}$ is the Einstein tensor for the metric $g_{\mu\nu}$, the energy and matter distribution is described by the stress-energy tensor $T_{\mu\nu}$, and Λ is the cosmological constant (we use units in which $8\pi G = c = \hbar = k_B = 1$). Let us consider a simple type of matter field, namely a massless, conformally-coupled scalar field. In four space-time dimensions, the only asymptotically flat solutions of the Einstein equations (1) with $T_{\mu\nu}$ for such a scalar field are black holes [1]. However, in asymptotically anti-de Sitter (AdS) space-time with $\Lambda < 0$ such a scalar field can generate soliton-like solutions [2].

A natural question is whether solitons can also be generated when the classical scalar field is replaced by a massless, conformally-coupled, *quantum* scalar field. In this situation the classical stress-energy tensor $T_{\mu\nu}$ in (1) is replaced by the renormalized expectation value of the stress-energy tensor operator (RSET) $\langle \hat{T}_{\mu\nu} \rangle$. The RSET is computed on a *fixed* AdS background, and a suitable quantum state has to be chosen for the scalar field. On pure AdS space-time, the RSET for the global vacuum state results simply in a renormalization of the cosmological constant Λ [3], so we consider instead a global thermal state. In four dimensions, it is found [4] that the thermal RSET $\langle \hat{T}_{\mu\nu}[g^0] \rangle$ (where g^0 is the background AdS metric) generates *quantum-corrected* (QC) solitons.

In this note, we consider the simpler situation of three-dimensional AdS, inspired by the work of [5] on quantum-corrections to the three-dimensional, asymptotically AdS, BTZ black hole. While in [5] the background space-time possesses a singularity, here we work on the regular global AdS space-time. In our previous work [4], we solved the nonlinear Einstein equations. However, it is expected that the RSET (and therefore quantum corrections to the AdS metric) will be small, of order the Planck length ℓ_P . We therefore follow [5] and take a perturbative approach, writing the QC metric as $g^{\text{QC}} = g^0 + \ell_P \delta g + \mathcal{O}(\ell_P^2)$, and solve the $\mathcal{O}(\ell_P)$ linearized quantum-corrected Einstein equations (LQCEE)

$$\delta G_{\mu\nu} + \Lambda \delta g_{\mu\nu} = \langle \hat{T}_{\mu\nu}[g^0] \rangle, \quad (2)$$

where $\delta G_{\mu\nu}$ is the linearized perturbation of the Einstein tensor. In the following we briefly review the properties of the RSET source $\langle \hat{T}_{\mu\nu}[g^0] \rangle$ [6] before discussing soliton-like solutions of the LQCEE (2). Since this is a brief note, we consider only static solitons, and plan to return to rotating solitons in a future work [7].

2 RSET on AdS₃

Our background is three-dimensional global AdS space-time with metric $g_{\mu\nu}^0$ given by the line element

$$ds_0^2 = a^{-2} (1 - r^2)^{-2} \left[- (1 + r^2)^2 dt^2 + 4 dr^2 + 4r^2 d\theta^2 \right], \quad (3)$$

where a is the inverse AdS length scale, related to the negative cosmological constant Λ by $a = \sqrt{-\Lambda}$ and the coordinates have ranges $-\infty < t < \infty$, $0 \leq r < 1$ and $0 \leq \theta < 2\pi$. We consider a massless, conformally-coupled, quantum scalar field in a thermal state on the background metric (3). The quantum field theory (QFT) expectation value $\langle \hat{T}_{\mu\nu}[g^0] \rangle$ is computed in [6] for a rigidly-rotating thermal state at inverse temperature β (using a different coordinate system to that employed here). Setting the angular speed of rotation Ω to zero, and changing to the coordinate system in (3), the QFT-RSET has nonvanishing components ${}^{\text{QFT}}T_t^t$, ${}^{\text{QFT}}T_r^r$ and ${}^{\text{QFT}}T_\theta^\theta$ given by

$${}^{\text{QFT}}T_r^r = \frac{a^3}{16\sqrt{2}\pi} \frac{(1 - r^2)^3}{(1 + r^2)^3} \sum_{j=1}^{\infty} \left[\frac{1}{4\sqrt{2}} [3 + \cosh(j\beta)] \operatorname{cosech}^3\left(\frac{j\beta}{2}\right) - \mathcal{A}_j^{-\frac{3}{2}} \sinh^2\left(\frac{j\beta}{2}\right) \right], \quad (4a)$$

$${}^{\text{QFT}}T_\theta^\theta = {}^{\text{QFT}}T_r^r - \frac{3a^3 r^2}{2\sqrt{2}\pi} \frac{(1 - r^2)^3}{(1 + r^2)^5} \sum_{j=1}^{\infty} \mathcal{A}_j^{-\frac{5}{2}} \sinh^2\left(\frac{j\beta}{2}\right), \quad (4b)$$

with ${}^{\text{QFT}}T_t^t + {}^{\text{QFT}}T_r^r + {}^{\text{QFT}}T_\theta^\theta = 0$ and $\mathcal{A}_j = \cosh(j\beta) + (1 + r^2)^{-2} (1 - 6r^2 + r^4)$. Since the expressions (4) are somewhat unwieldy, we also take an alternative approach, namely relativistic kinetic theory (RKT). In RKT, the quantum scalar field is modelled as a classical gas of massless bosonic particles with a Bose-Einstein distribution function at inverse temperature β . The resulting classical RKT-SET takes the much simpler form (here $\zeta(3)$ is the Riemann-zeta function) [6]

$${}^{\text{RKT}}T_\mu^\nu = \frac{a^3 \zeta(3)}{2\pi\beta^3} \frac{(1 - r^2)^3}{(1 + r^2)^3} \operatorname{Diag}\{-2, 1, 1\}. \quad (5)$$

In Fig. 1 we show the energy density $E = -T_t^t$ for low temperatures (left-hand-plot) and high temperatures (middle plot), with the RKT results given by dashed lines and the QFT results by solid lines. At high temperatures (small β), the RKT-SET (5) is a very good approximation to the QFT-RSET (4), but it ceases to be a good approximation at low temperatures (large β) [6]. For all fixed temperatures, the energy density computed using RKT is larger than than in QFT at the same value of the radial coordinate r . In both QFT and RKT, the energy density is positive throughout the space-time and has a soliton-like profile: for fixed inverse temperature β , the energy density has a maximum at the origin $r = 0$ and decreases to zero as $r \rightarrow 1$ and the AdS boundary is approached [6].

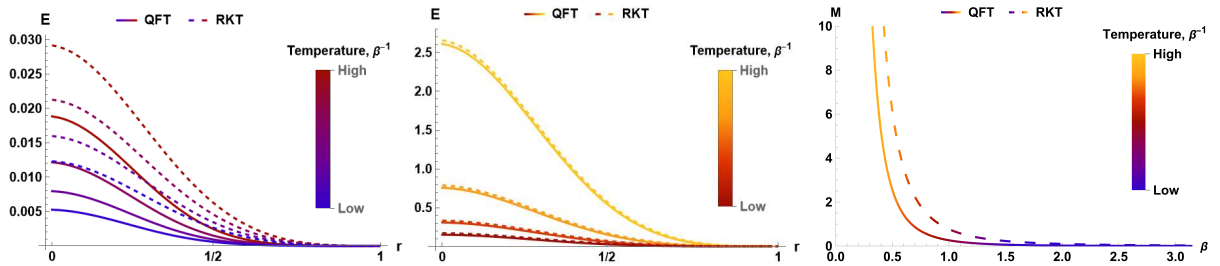


Figure 1: Energy density $E = -T_t^t$ for $\beta \in \{3\pi/4, 5\pi/6, 11\pi/12, \pi\}$ (left) and $\beta \in \{\pi/6, \pi/4, \pi/3, 5\pi/12\}$ (middle). QC soliton mass M (10) (right) as a function of inverse temperature β . Dotted lines denote results from RKT, while solid lines are those from QFT. We set the AdS inverse length scale $a = 1$.

3 QC AdS₃ solitons

We now use the QFT-RSET (4) and the RKT-SET (5) as source terms in the LQCEE (2) and seek soliton-like solutions of these equations. We assume that the QC metric is static and spherically-symmetric, and takes the form

$$ds_{\text{QC}}^2 = a^{-2} \left[-A(r) dt^2 + B(r) dr^2 + C(r) d\theta^2 \right]. \quad (6)$$

We write each of the metric functions $A(r)$, $B(r)$ and $C(r)$ as their expressions for the background metric (3) plus small $\mathcal{O}(a\ell_P)$ perturbations $\delta A(r)$, $\delta B(r)$ and $\delta C(r)$. By a suitable choice of gauge (corresponding to a perturbation of the radial coordinate r) we may take the perturbation of the metric function $B(r)$ to be

$$\delta B(r) = \frac{(1-r^2)^2}{r^2(1+r^2)^2} \delta A(r). \quad (7)$$

With this choice, the LQCEE (2) reduce to two ordinary differential equations, a second order equation for $\delta A(r)$ and a first order equation for $\delta A(r) + \delta C(r)$. Employing, respectively, the method of variation of parameters and an integrating factor, these can be solved to give explicit expressions for the metric perturbations in terms of integrals over the RSET components (for details, see [7]):

$$\delta A(r) = \mathcal{C}_1 + \frac{\mathcal{C}_2 r^2}{(1-r^2)^2} - \int \frac{1+r^2}{ar(1-r^2)} T_{\theta\theta} dr + \frac{r^2}{(1-r^2)^2} \int \frac{(1-r^4)}{ar^3} T_{\theta\theta} dr, \quad (8a)$$

$$\delta A(r) + \delta C(r) = \frac{r^2}{(1-r^2)^2} \left[\mathcal{C}_3 + 2 \int \frac{1-r^4}{ar} T_{rr} dr \right], \quad (8b)$$

where \mathcal{C}_1 , \mathcal{C}_2 and \mathcal{C}_3 are arbitrary constants, which are fixed by the requirement that the metric perturbations are regular at the origin $r = 0$ and that the QC space-time approaches pure AdS as $r \rightarrow 1$. Substituting the RKT-SET (5) and QFT-RSET (4) into the integrals in (8), explicit algebraic expressions for the metric perturbations can be obtained [7], which are quite complicated in the QFT case.

While the metric ansatz (6) is useful for explicitly finding the metric perturbations, in order to aid our soliton interpretation of the QC metric, we change to a new radial coordinate R , given by $R^2 = C(r) \in [0, \infty)$ and consider the alternative metric (following the QC BTZ ansatz in [5]):

$$ds_{\text{QC}}^2 = a^{-2} [-N(R)f(R) dt^2 + N(R)^{-1} dR^2 + R^2 d\theta^2], \quad (9)$$

where $N(R) = R^2 + 1 + a\ell_P m(R) + \mathcal{O}(a^2\ell_P^2)$ and $f(R) = 1 + \mathcal{O}(a\ell_P)$. It is the metric perturbation $m(R)$ which is of particular interest for our purposes. This is given as a linear combination of the metric perturbations δA and δC and satisfies the differential equation $m'(R) = 2RE/a^2$, where $E = -T_t^t$ is the local scalar field energy density [7]. In the BTZ black hole metric, $m(R)$ is a constant, corresponding to the mass of the black hole [8]. In order to have a regular origin at $R = 0$ (which corresponds to $r = 0$), it must be the case that $m(0) = 0$ (which is not satisfied for the BTZ black hole, hence there is a singularity at the origin). The condition $m(0) = 0$ is satisfied by an appropriate choice of the integration constants in (8) [7]. Using both the RKT-SET and QFT-RSET in the LQCEE, since $E > 0$, we find that $m(R)$ is a monotonically increasing function of R and tends to a finite limit as $R \rightarrow \infty$ (which corresponds to the AdS boundary at $r = 1$). By analogy with the BTZ metric [8], we interpret this limit as the mass M of the QC solitons, which takes the simple expressions

$$M = \lim_{R \rightarrow \infty} m(R) = \begin{cases} \frac{2\zeta(3)}{\pi\beta^3}, & \text{RKT,} \\ \frac{1}{32\pi} \sum_{j=1}^{\infty} [3 + \cosh(j\beta) - \sinh(j\beta)] \operatorname{cosech}^3\left(\frac{j\beta}{2}\right), & \text{QFT,} \end{cases} \quad (10)$$

when evaluated using the RKT-SET and QFT-RSET, respectively. The soliton mass M is shown in the right-hand plot in Fig. 1 as a function of the inverse temperature β . It can be seen that the mass computed using the RKT-SET is larger than that derived from the QFT-RSET, as expected since the local energy density E is greater in RKT than QFT. Furthermore, the mass increases rapidly as the temperature of the thermal state increases.

4 Conclusions

In this note we have studied the backreaction of a massless, conformally-coupled, quantum scalar field in a static thermal state on three-dimensional global AdS space-time. Using a perturbative approach, we have solved the linearized, quantum-corrected, Einstein equations (LQCEE) to find the metric perturbations of pure AdS, with the scalar field RSET acting as a source. We have computed the RSET in two complementary approaches: RKT and QFT. We find globally regular, static soliton solutions of the LQCEE, whose mass increases rapidly as the temperature of the thermal state increases. We are currently working on the corresponding rotating quantum-corrected solitons sourced by a rigidly-rotating thermal state on three-dimensional AdS, and plan to report our results in the near future [7].

Acknowledgements

We thank Siva Namasivayam for helpful discussions. The work of J.T. is supported by an EPSRC studentship. The work of E.W. is supported by STFC grant number ST/X000621/1.

References

- [1] Xanthopoulos B C and Zannias T 1991 *J. Math. Phys.* **32** 1875–1880
- [2] Radu E and Winstanley E 2005 *Phys. Rev. D* **72** 024017
- [3] Kent C and Winstanley E 2015 *Phys. Rev. D* **91** 044044
- [4] Thompson J C and Winstanley E 2024 *Phys. Rev. D* **110** 125003
- [5] Casals M, Fabbri A, Martínez C and Zanelli J 2019 *Phys. Rev. D* **99** 104023
- [6] Thompson J C and Winstanley E 2025 *Phys. Lett. B* **868** 139780
- [7] Thompson J C and Winstanley E, work in progress.
- [8] Banados M, Teitelboim C and Zanelli J 1992 *Phys. Rev. Lett.* **69** 1849–1851