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Weak saturation stability

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ABSTRACT

The paper studies wsat(G, H) which is the minimum number of edges in a weakly H-saturated subgraph of G. We prove that wsat(K_n, H) is 'stable' – remains the same after independent removal of every edge of K_n with constant probability – for all pattern graphs H such that there exists a 'local' set of edges percolating in K_n. This is true, for example, for cliques and complete bipartite graphs. We also find a threshold probability for the weak $K_{1,t}$-saturation stability.

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1. Introduction

Let G and H be graphs (below, we refer to them as *host* and *pattern* graphs respectively). Let $F \subset G$ be a spanning subgraph of G. Let us call a sequence of graphs $F = F_0 \subset \ldots \subset F_m = G$ an *H-bootstrap percolation process*, if F_i is obtained from F_{i-1} by adding an edge that belongs to a copy of H in F_i. F is *weakly (G, H)-saturated*, if G can be obtained from F in an H-bootstrap percolation process (i.e., there exists an ordering $e_1, \ldots, e_m$ of the edges of $G \setminus F$ such that, for every $i \in [m] := \{1, \ldots, m\}$, $F \sqcup \{e_1, \ldots, e_i\}$ has a copy of H that contains e_i). The smallest number of edges in a weakly (G, H)-saturated graph is called *the weak saturation number* and is denoted by wsat(G, H). This notion was first introduced by Bollobás in 1968 [6].

In this paper, we study the phenomena of *stability* of the weak saturation number. It was observed by Korándi and Sudakov [12] that wsat$(G = K_n, K_s)$ remains the same after a deletion of each edge of K_n independently with a constant positive probability (as usual, K_n denotes a complete graph on n vertices). We show that this stability property holds for a wider class of pattern graphs H, and conjecture that it actually holds for all H. We also find a threshold probability for the stability of the weak $(K_n, K_{1,t})$-saturation number. Before stating the results, let us recall known values of

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the weak saturation number when $G = K_n$. We denote by $K_{s,t}$ a complete bipartite graph with parts of size s and t.

The exact value of $\mathrm{wsat}(K_n, K_s)$ was achieved by Lovász [14]: if $n \geq s \geq 2$ then

$$\mathrm{wsat}(K_n, K_s) = \binom{n}{2} - \binom{n-s+2}{2}.$$

The value of $\mathrm{wsat}(K_n, K_{s,t})$ for an arbitrary choice of parameters is still unknown. The most general result was obtained by Kalai [11] in 1985 and Kronenberg, Martins and Morrison [13] in 2020. They proved that

$$\mathrm{wsat}(K_n, K_{t,t}) = (t-1)(n+1-t/2),$$

$$\mathrm{wsat}(K_n, K_{t,t+1}) = (t-1)(n+1-t/2)+1$$

if $t \geq 2$ and $n \geq 3t-3$. In [13], general bounds for arbitrary choice of parameters s, t were also obtained:

$$\mathrm{wsat}(K_n, K_{s,t}) \leq (s-1)(n-s) + \binom{t}{2} \tag{1}$$

if $t > s \geq 2$ and $n \geq 2(s+t)-3$ and

$$\mathrm{wsat}(K_n, K_{s,t}) \geq (s-1)(n-t+1) + \binom{t}{2} \tag{2}$$

if $t > s \geq 2$ and $n \geq 3t-3$.

Notice that, for $s = 1$, the exact value of the weak saturation number is straightforward:

$$\mathrm{wsat}(K_n, K_{1,t}) = \binom{t}{2}.$$

Indeed, let us call the *central* vertex of $K_{1,t}$ (or, simply, *quote*) the vertex that is lying in the part of size one (i.e. has degree t). Now consider a $K_{1,t}$-bootstrap percolation process in K_n. The *sequence of quotes* $v_1, \ldots, v_m$ is defined in the following way: v_i is the central vertex of one of the copies of $K_{1,t}$ used at the step $j \geq i$ such that j is the first step when this quote is used, i.e. $v_1, \ldots, v_m$ are distinct vertices where each v_i appears in the sequence at the first time it is used as a quote. Note that, for every $i \in [t-1]$, the number of neighbours of v_i in $V(F) \setminus \{v_1, \ldots, v_{i-1}\}$ is at least $t-i$. Then, $\mathrm{wsat}(K_n, K_{1,t}) \geq \binom{t}{2}$. Eventually, $\mathrm{wsat}(K_n, K_{1,t}) \leq \binom{t}{2}$ since we can use $F = K_t$ to restore all edges of K_n (first, we restore all edges with one endpoint in the clique, then all the rest).

There are also many results about weak saturation numbers for other specific pairs of host and pattern graphs (e.g., for both G and H being complete bipartite [16], for multipartite graphs [1], for disjoint copies of graphs [8], for hypercubes and grids [3,4,15]). The weak saturation number for hypergraphs has also been studied [1,3,7,8,16,19,20].

In 2017, Korándi and Sudakov [12] proved that, if $s \geq 3$, then $\mathrm{wsat}(K_n, K_s)$ is *stable*, i.e., for constant $p \in (0, 1)$,

$$\mathrm{wsat}(G(n,p), K_s) = \mathrm{wsat}(K_n, K_s)$$

with high probability. Here, we denote by $G(n,p)$ the binomial random graph on the vertex set $[n]$, where every pair of distinct $i, j \in [n]$ is adjacent with probability p independently of the others. Hereinafter, we say that some property holds *with high probability*, or *whp*, if its probability tends to 1 as $n \to \infty$.

In this paper, we prove a transference theorem that can be used to derive such stability results. It immediately implies the result of Korándi and Sudakov as well as stability results for all complete bipartite pattern graphs (despite the fact that the exact value of $\mathrm{wsat}(K_n, K_{s,t})$ is not known for almost all pairs s and t). Below, we denote by $\delta(H)$ the minimum degree of graph H. Without loss of generality, we set $V(K_n) = [n]$.

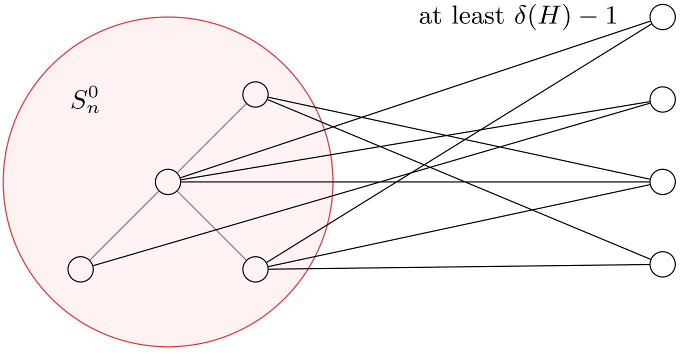

Fig. 1. structure of F_n^0.

Theorem 1. *Let H be a graph without isolated vertices, and let $p \in (0, 1)$, $C \geq \delta(H) - 1$ be constants. For every $n \in \mathbb{N}$, let F_n^0 be a weakly (K_n, H)-saturated graph containing a set of vertices $S_n^0 \subset [n]$ with $|S_n^0| \leq C$, such that every vertex from $[n] \setminus S_n^0$ is adjacent to at least $\delta(H) - 1$ vertices of S_n^0 (see Fig. 1). Then whp there exists a subgraph $F_n \subset G(n, p)$ which is weakly $(G(n, p), H)$-saturated, and F_n has $\min\{|E(G(n, p)), |E(F_n^0)|\}$ edges.*

This theorem implies

Corollary 1. *Let $p \in (0, 1)$ be constant. For an arbitrary graph H without isolated vertices, whp*

$$\mathrm{wsat}(G(n, p), H) = \mathrm{wsat}(K_n, H),$$

if, for every $n \in \mathbb{N}$, there exists a minimum (having $\mathrm{wsat}(K_n, H)$ edges) weakly (K_n, H)-saturated graph with the property described in Theorem 1.

Proof. Indeed, assume that the condition of Theorem 1 is satisfied. Then, it immediately implies that whp $\mathrm{wsat}(G(n, p), H) \leq \mathrm{wsat}(K_n, H)$. Assume that, with a probability bounded away from 0, $\mathrm{wsat}(G(n, p), H)$ is strictly less than $\mathrm{wsat}(K_n, H)$. Since whp every pair of vertices of $G(n, p)$ has at least $|V(H)|$ pairwise adjacent common neighbors [18], whp $G(n, p)$ is weakly (K_n, H)-saturated — a contradiction. □

This corollary immediately implies stability for several pattern graphs. Notice that the graph obtained by removing a copy of K_{n-s+2} from K_n is weakly (K_n, K_s)-saturated, has the structure described in Theorem 1 and has $\mathrm{wsat}(K_n, K_s)$ edges. Therefore, the result of Korándi and Sudakov can be immediately deduced from Corollary 1.

The constructions of Kronenberg, Martins and Morrison [13] of weakly $(K_n, K_{t,t})$-saturated and weakly $(K_n, K_{t,t+1})$-saturated graphs with $\mathrm{wsat}(K_n, K_{t,t})$ and $\mathrm{wsat}(K_n, K_{t,t+1})$ edges respectively also have the structure described in Theorem 1. Therefore, by Corollary 1, for every $p \in (0, 1)$, whp

$$\mathrm{wsat}(G(n, p), K_{t,t}) = \mathrm{wsat}(K_n, K_{t,t}), \quad \mathrm{wsat}(G(n, p), K_{t,t+1}) = \mathrm{wsat}(K_n, K_{t,t+1}).$$

The bounds (1), (2) imply that Corollary 1 can be also applied to pattern graphs $K_{s,t}$ for all possible values of $s \leq t$. To see this it is sufficient to show that, for every $n \in \mathbb{N}$, there exists a minimum weakly $(K_n, K_{s,t})$-saturated graph with the property described in Theorem 1. Note that due to (1) and (2)$\mathrm{wsat}(K_n, K_{s,t}) = (s-1)n+O(1)$, and s is the minimum degree of $K_{s,t}$. Moreover, it is clear that there exists a constant $C = C(s, t)$ such that, for all n large enough, $\mathrm{wsat}(K_n, K_{s,t}) = (s-1)n + C$. Indeed, otherwise there exist $C_1 < C_2$ and two infinite sequences $\{n_i^1\}_{i\in\mathbb{N}}$ and $\{n_i^2\}_{i\in\mathbb{N}}$ such that

$$\mathrm{wsat}(K_{n_i^1}, K_{s,t}) = (s-1)n_i^1 + C_1 \text{ and } \mathrm{wsat}(K_{n_i^2}, K_{s,t}) = (s-1)n_i^2 + C_2.$$

Let us choose sufficiently large i and j such that $n_1 := n_i^1 < n_j^2 =: n_2$, and let F_1 be weakly $(K_{n_1}, K_{s,t})$-saturated. Then, the graph F_2 on $[n_2]$ obtained from F_1 by adding $s-1$ edges from each of the vertices from $[n_2] \setminus [n_1]$ to $[n_1]$ is $(K_{n_2}, K_{s,t})$-weakly saturated and has $(s-1)n_i^2 + C_1 < (s-1)n_i^2 + C_2$ edges — a contradiction. From this, the existence of a weakly $(K_n, K_{s,t})$-saturated graph with the desired property is straightforward. Indeed, let n_0 be so large that $\mathrm{wsat}(K_{n_0}, K_{s,t}) = (s-1)n_0 + C$, and let $F^0_{n_0}$ be a weakly $(K_{n_0}, K_{s,t})$-saturated graph with $(s-1)n_0 + C$ edges. For every $n > n_0$, set $S^0_n = [n_0]$ and define F^0_n as the union of $F^0_{n_0}$ with a graph consisting of $s-1$ edges going from each of the vertices from $[n] \setminus S^0_n$ to S^0_n. The graph F^0_n is weakly $(K_n, K_{s,t})$-saturated and has the desired number of edges. Therefore, by Corollary 1, we get that, for every constant $p \in (0, 1)$ and all $1 \le s \le t$, whp

$$\mathrm{wsat}(G(n, p), K_{s,t}) = \mathrm{wsat}(K_n, K_{s,t}).$$

This stability result demonstrates the efficiency of Theorem 1: it can be used to prove stability even when the exact value of $\mathrm{wsat}(K_n, K_{s,t})$ is unknown.

However, there are graphs H such that minimum weakly (K_n, H)-saturated graphs do not satisfy the condition of Theorem 1. For such graphs, the same stability result cannot be proven using Corollary 1.

For example, consider H being the, so called, t-barbell graph consisting of two vertex-disjoint t-cliques together with a single edge that has an endpoint in each clique. It is clear that $\mathrm{wsat}(K_n, H) \le \binom{t}{2}\frac{n}{t} \sim \frac{tn}{2}$ for n divisible by t since the disjoint union of n/t t-cliques is weakly (K_n, H)-saturated. Nevertheless, any subgraph of K_n with the property described in Theorem 1 has at least $(t-2)n + O(1)$ vertices. Therefore, it is not the minimum possible one, and Corollary 1 is not applicable.

Nevertheless, some extra work (it is very technical, so we omit the details) is required to show that the union of disjoint cliques has the minimum possible number of edges and that, since whp $G(n, p)$ contains a K_t-factor (see [10]), we get stability for t-barbell graph as well. In some sense, the situation covered by Theorem 1 is less pleasant. Indeed, if a graph on $[n]$ has a bounded maximum degree, then whp $G(n, p)$ contains its isomorphic copy as a spanning subgraph (see [2]). So, for such weakly saturated graphs, stability is straightforward. In Theorem 1, we consider an opposite scenario — F^0_n has vertices with degrees $n - O(1)$ that are not likely to be in $G(n, p)$. Having that in mind, we conjecture that, for any constant $p \in (0, 1)$ and every graph H, whp $\mathrm{wsat}(G(n, p), H) = \mathrm{wsat}(K_n, H)$.

Let us now switch to the case $p = o(1)$.

Korándi and Sudakov [12] claim that their result can be easily extended to the range $n^{-\varepsilon(s)} \le p \le 1$. It can be seen that the same is true for Theorem 1 with ε depending only on H. However, for smaller p Theorem 1 or Corollary 1 may not hold.

Korándi and Sudakov [12] pose the following question: what is the exact probability range where whp $\mathrm{wsat}(G(n, p), K_s) = \mathrm{wsat}(K_n, K_s)$? In 2020, Bidgoli et al. [5] proved the existence of threshold probability for this stability property. Moreover, they obtained bounds on the threshold:

- there exists c such that, if $p < cn^{-\frac{2}{s+1}}(\ln n)^{\frac{2}{(s-2)(s+1)}}$, then whp $\mathrm{wsat}(G(n, p), K_s) \ne \mathrm{wsat}(K_n, K_s)$,
- if $p > n^{-\frac{1}{2s-3}}(\ln n)^2$, then whp $\mathrm{wsat}(G(n, p), K_s) = \mathrm{wsat}(K_n, K_s)$.

In this paper, we estimate the threshold probability for the *weak $K_{1,t}$-saturation stability property* $\mathrm{wsat}(G(n, p), K_{1,t}) = \mathrm{wsat}(K_n, K_{1,t})$.

Theorem 2. *Let $t \ge 3$. Denote $p(n, t) = n^{-\frac{1}{t-1}}[\ln n]^{-\frac{t-2}{t-1}}$.*

- *There exists $c > 0$ such that, if $\frac{1}{n^2} \ll p < cp(n, t)$, then whp* $\mathrm{wsat}(G(n, p), K_{1,t}) \ne \mathrm{wsat}(K_n, K_{1,t})$.
- *There exists $C > 0$ such that, if $p > Cp(n, t)$, then whp* $\mathrm{wsat}(G(n, p), K_{1,t}) = \mathrm{wsat}(K_n, K_{1,t})$.

Note that Theorem 2 does not cover the case $t = 2$ as well as $p = O(1/n^2)$. But these cases are much easier. Below we consider them separately.

First, if $p < \frac{Q}{n^2}$ for some constant $Q > 0$, then whp $G(n, p)$ consists of isolated vertices and isolated edges (it simply follows from Markov's inequality applying to the number of P_3 in $G(n, p)$). Therefore, whp there are no copies of $K_{1,t-1}$ for $t \ge 3$ in $G(n, p)$, and there are no weakly $(G(n,p), K_{1,t})$-saturated subgraphs other than the entire graph. So, whp there is stability only if the number of edges of the graph is exactly $\binom{t}{2}$. The latter property holds with probability $\binom{\binom{n}{2}}{\binom{t}{2}} p^{\binom{t}{2}}(1-p)^{\binom{n}{2}-\binom{t}{2}}$. It tends to 0 when $p \ll \frac{1}{n^2}$ and is bounded away both from 0 and from 1 when $\frac{q}{n^2} < p < \frac{Q}{n^2}$ for some $0 < q < Q$.

The case $t = 2$ is also trivial. Clearly, for a graph G on $[n]$,

$$\mathrm{wsat}(G, K_{1,t}) = \mathrm{wsat}(K_n, K_{1,t}) = \binom{t}{2} = 1$$

if and only if G has exactly one non-trivial (having at least one edge) connected component. Using the standard first and second moment methods (see, e.g., [9, Chapter 1]), it can be proven that, for every $\varepsilon > 0$,

- if $p > (1+\varepsilon)\frac{\ln n}{2n}$, then whp $G(n,p)$ contains a unique non-trivial connected component;
- if $\frac{1}{n^2} \ll p < (1-\varepsilon)\frac{\ln n}{2n}$, then whp $G(n,p)$ contains at least two non-trivial connected components;
- if $\frac{q}{n^2} < p < \frac{Q}{n^2}$ for some $0 < q < Q$, then whp all edges in $G(n,p)$ are disjoint, and, arguing as above, we get that stability happens only if the graph contains $\binom{t}{2} = 1$ edges;
- if $p \ll \frac{1}{n^2}$, then whp $G(n,p)$ is empty.

Therefore,

1. if $p > (1+\varepsilon)\frac{\ln n}{2n}$, then whp $\mathrm{wsat}(G(n,p), K_{1,t}) = \mathrm{wsat}(K_n, K_{1,t})$;
2. if $\frac{1}{n^2} \ll p < (1-\varepsilon)\frac{\ln n}{2n}$, then whp $\mathrm{wsat}(G(n,p), K_{1,t}) \neq \mathrm{wsat}(K_n, K_{1,t})$;
3. if $\frac{q}{n^2} < p < \frac{Q}{n^2}$ for some $0 < q < Q$, then
$$\mathsf{P}\Big[\mathrm{wsat}(G(n,p), K_{1,t}) = \mathrm{wsat}(K_n, K_{1,t})\Big] =$$
$$\mathsf{P}(G(n,p) \text{ contains exactly 1 edge}) + o(1) = \binom{n}{2} p(1-p)^{\binom{n}{2}-1} + o(1)$$
is bounded away both from 0 and 1,
4. if $p \ll \frac{1}{n^2}$, then whp $\mathrm{wsat}(G(n,p), K_{1,t}) = 0 \neq \mathrm{wsat}(K_n, K_{1,t})$.

The structure of the paper is the following. In Section 2, we prove Theorem 1. In Section 3, we prove Theorem 2.

2. Proof of Theorem 1

We denote $d = \delta(H) - 1$, $r = |V(H)|$. First, for convenience, let us state an equivalent modification of Theorem 1 which at first sight seems to be weaker. However, there is equivalence, and it is easier to prove this version. In Section 2.1, we prove the equivalence of the two statements, and then prove the modified version.

Lemma 1. *Let H be an arbitrary fixed graph without isolated vertices, and let $p \in (0,1)$, $C \geq d$ be constants. For every $n \in \mathbb{N}$, let F_n^1 be a weakly (K_n, H)-saturated graph containing a set of vertices S_n^1 of size at most C such that every vertex of $[n] \setminus S_n^1$ is adjacent to **exactly** d vertices of S_n^1, and there are no edges between vertices of $[n] \setminus S_n^1$. Then whp there exists a subgraph $F_n \subset G(n,p)$ which is weakly $(G(n,p), H)$-saturated, and F_n has the same number of edges as F_n^1.*

2.1. Proof of equivalence

Clearly, Theorem 1 implies Lemma 1. Let us prove that the opposite implication is also true.

We first notice that $\mathrm{wsat}(K_n, H) \leq \binom{r}{2} + d(n-r)$. Indeed, we can construct a weakly (G_n, H)-saturated subgraph with at most so many edges in the following way. Let F_0 be a weakly (K_r, H)-saturated subgraph on $[r]$, the first r vertices of K_n. The desired graph F_1 has the same edges as F_0

on $[r]$, and from every other vertex there are exactly d edges to F_0. The existence of a bootstrap percolation process that starts on F_1 and finishes on K_n is straightforward: first restore all edges of K_r, then restore all edges going to $[r]$ and finally restore all the remaining edges.

We assume that Lemma 1 holds. The constructed graph F_1 satisfies the conditions of this lemma. Then, whp there exists a weakly $(G(n,p),H)$-saturated subgraph F'_n such that

$$|E(F'_n)|=|E(F_1)|\leq \binom{r}{2}+d(n-r)=dn+\left(\binom{r}{2}-dr\right).$$

Let F^0_n be a graph that satisfies the condition of Theorem 1.

If $|E(F^0_n)|\geq |E(F'_n)|$, then a subgraph of $G(n,p)$ obtained by adding $\min\left\{|E(F^0_n)|,|E(G(n,p))|\right\}-|E(F'_n)|$ edges to F'_n is weakly $(G(n,p),H)$-saturated and F_n has $\min\left\{|E(F^0_n)|,|E(G(n,p))|\right\}$ edges.

If $|E(F^0_n)|<|E(F'_n)|$, then there are at most a vertices with degree more than d in F^0_n outside S^0_n, where a does not depend on n. We can add them (let us call the set of these vertices A) to S^0_n, and then $|S^0_n\cup A|$ will be bounded from above by $C+a$. So, F^0_n satisfies the condition of Lemma 1 with $C:=C+a$. Therefore, whp there exists a weakly $(G(n,p),H)$-saturated subgraph with $|E(F^0_n)|$ edges. □

Below, we prove Lemma 1. Let us outline the proof. In Section 2.2, we describe sufficient properties of a spanning subgraph G of K_n that allow to find a weakly (G,H)-saturated subgraph with the same number of edges as in a weakly (K_n,H)-saturated subgraph. In Section 2.3, we prove that these properties are indeed sufficient for the described transference property. In Section 2.4, we prove that whp $G(n,p)$ has the described properties and, thus, finish the proof of Lemma 1.

Let us now switch to the proof of Lemma 1. Assume that the requirements of the lemma hold. We let n to be even in order to avoid overloading with floor and ceiling functions notations. This does not affect the proof anyhow.

Let us denote the vertices of H as $w_1,\ldots,w_r$ where w_1 is a vertex with degree $d+1$ and $w_2,\ldots,w_{d+2}$ are its neighbours. Let H' be obtained from H by deleting the vertices w_1,w_2 and let H'' be obtained from H by deleting the edge $\{w_1,w_2\}$ (but preserving the vertices w_1,w_2).

In what follows, for a graph F and its vertex v, we denote by $N_F(v)$ the set of all neighbours of v in F.

2.2. Sufficient properties

We want to find in $G(n,p)$ a subgraph having similar structure to a weakly saturated subgraph in K_n. However, it cannot be done immediately since whp all vertices in $G(n,p)$ have degrees $np(1+o(1))$ which is far away from $n-O(1)$. Nevertheless, we can find a clique K in $G(n,p)$ of size $\Theta(\ln n)$, and first reconstruct the edges of the clique. For that, we fix a weakly saturated spanning subgraph with the minimum possible number of edges and the desired structure in K. In other words, we choose a subset $S\subset K$ playing the role of $S^0_{|K|}$. After reconstructing the edges of K we might hope that it is sufficient to use d edges of $G(n,p)$ incident to every vertex outside K to reconstruct all the other edges of $G(n,p)$. The properties that allow to do this are described below.

We start from distinguishing several subsets of $[n]$ that we use to describe the properties. Everywhere below, by $G|_V$ we denote the induced subgraph of G on the vertex set V (where $V\subset V(G)$).

Let $c>0$. Let G be a graph on the vertex set $[n]$. Let

- $G_1=G|_{[n/2]}$, $G_2=G|_{[n]\setminus[n/2]}$;
- $K\subset V_1:=V(G_1)$ be a set of size $k\geq c\ln n$, where $c>0$ is constant,
  S be a subset of K of size $|S^1_k|$,
  D be a subset of S of size d (clearly, the requirements in Lemma 1 imply that $|S^1_k|\geq d$);
- Z be the set of all common neighbours of D in $V(G_2)$;
- $Z=Z_1\sqcup Z_2\sqcup Z_3$ be an (almost) equal partition of Z.
- R be an arbitrary set of r vertices from $K\setminus S$ and $T\subset Z$ be the set of all common neighbors from Z of vertices from R.

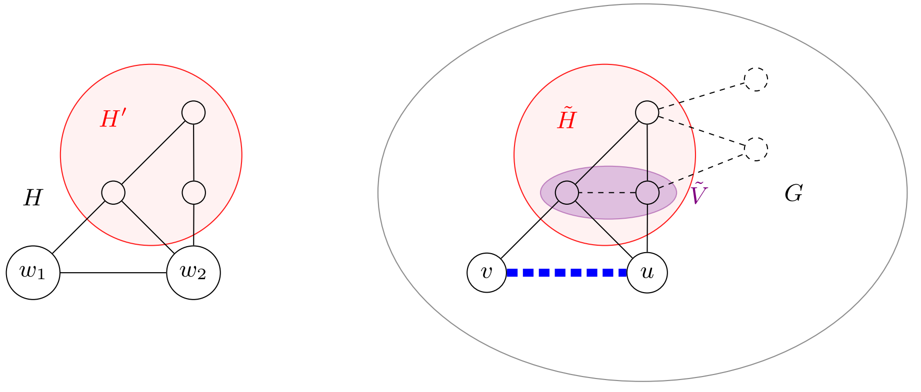

Fig. 2. H-completable tuples $(v, u, \tilde{H})$ and $(v, u, \tilde{H}, \tilde{V})$. Black edges are edges mapped to edges of H, dashed edges are other edges of G, the blue dashed edge can be immediately reconstructed in a bootstrap percollation process.

For vertices v, u of G and a subgraph $\tilde{H} \cong H'$ of G, we call the tuple $(v, u, \tilde{H})$ *H-completable in G* (see Fig. 2), if there exists an embedding f (we call it *$(v, u, \tilde{H})$-embedding*) from H to $G|_{V(\tilde{H})\sqcup\{v,u\}}$ such that $f(w_1) = v, f(w_2) = u$ and f maps H' to $\tilde{H}$, i.e. the graph with the set of vertices $V(\tilde{H})$ and the set of edges $\{\{f(x), f(y)\}, \{x, y\} \in E(H')\}$ equals $\tilde{H}$ (see Fig. 2). In plain words, it means that we are able to immediately reconstruct the edge $\{u, v\}$.

For a vertex v of G and a subgraph $\tilde{H} \cong H'$ of G, we call the pair $(v, \tilde{H})$ *H-completable in G*, if there exists an embedding f (we call it *$(v, \tilde{H})$-embedding*) from $H|_{V(H)\setminus\{w_2\}}$ to $G|_{V(\tilde{H})\sqcup\{v\}}$ such that $f(w_1) = v$ and f maps H' to $\tilde{H}$. Intuitively, it means that nothing prevents us from adding an edge from v to some other vertex w (such that $(v, w, \tilde{H})$ is H-completable in G). Thus, for such a tuple, the possibility of completing $G|_{V(\tilde{H})\cup\{v\}}$ to a copy of H depends only on the existence of a vertex w with the same neighbourhood in $V(\tilde{H}) \cup \{v\}$ as w_2 has in H. The difference from the previous definition is that if $(v, u, \tilde{H})$ is H-completable, we can immediately draw the edge $\{v, u\}$ and thus complete a copy of H, and if $(v, \tilde{H})$ is H-completable, we can *possibly* draw some edges from v depending on the existence of a suitable w. There may be several such edges, or there may be none.

Let $(v, u, \tilde{H})$ be H-completable in G and let $\tilde{V} \subset V(\tilde{H})$ have exactly d vertices. We call the tuple $(v, u, \tilde{H}, \tilde{V})$ *H-completable in G*, if there exists a $(v, u, \tilde{H})$-embedding f such that f maps $N_H(w_1)\setminus\{w_2\}$ onto $\tilde{V}$. In Fig. 2, $\tilde{V}$ is shown in violet colour. We need the parameter $\tilde{V}$ to distinguish the set of v's neighbours in some suitable $\tilde{H}$.

Similarly, for an H-completable $(v, \tilde{H})$ and $\tilde{V} \subset V(\tilde{H})$ of size d, we call the tuple $(v, \tilde{H}, \tilde{V})$ *H-completable in G*, if there exists an $(v, \tilde{H})$-embedding f such that f maps $N_H(w_1) \setminus \{w_2\}$ onto $\tilde{V}$.

Let us now describe the desired properties. We define two properties of G. The first one is used to reconstruct all the edges but those between $S \setminus D$ and $[n] \setminus (K \sqcup Z)$. All the other edges are reconstructed using the second property.

For a vertex v of G, a subgraph $\tilde{H} \cong H'$ of G and a set of vertices $\tilde{V} \subset V(\tilde{H})$ of size d, denote by $U_v(\tilde{H}, \tilde{V})$ the set of all neighbours u of v such that $u \in Z_2$ and the tuple $(v, u, \tilde{H}, \tilde{V})$ is H-completable.

Let us say that the tuple $(G; K, S, D, Z_1, Z_2, Z_3, T)$ satisfies the *first H-saturation property* or, simply, *the first property*, if (note that, among the members of the tuple, G is a graph, and all the others are sets of vertices; recall also that $Z_1 \sqcup Z_2 \sqcup Z_3$ is a partition of Z)

1. K induces a clique in G;
2. for any adjacent (in G) pair $v_1, v_2 \in Z \setminus T$, there exists a copy H_{v_1,v_2} of H' in $G|_T$ such that (v_1, v_2, H_{v_1,v_2}) is H-completable in G;
3. for every $v \notin K \sqcup Z$, there exists $H_v \cong H'$ inside Z_1 and $V_v \subset V(H_v)$ such that (v, H_v, V_v) is H-completable in G;

4. for any pair $v_1, v_2 \notin K \sqcup Z$, there exists a copy of H' in $G|_{U_{v_1}(H_{v_1},V_{v_1})\cap U_{v_2}(H_{v_2},V_{v_2})}$;
5. for any $v \notin K \sqcup Z$, $u \in (K \setminus S) \sqcup Z_2 \sqcup Z_3$, there exists a copy of H' in $G|_{U_v(H_v,V_v)\cap N_G(u)}$;
6. for any $v \notin K \sqcup Z$, $u \in Z_1$, there exists a copy of H' in $G|_{Z_3\cap N_G(v)\cap N_G(u)}$;
7. for any $u \notin S$, there exists a copy of H' in $G|_{Z\cap N_G(u)}$.

Finally, let $V(G_2) = V^1 \sqcup V^2 \sqcup V^3$ be an (almost) equal partition. Let us say that the tuple $(G; V^1, V^2, V^3, S, D)$ satisfies the *second H-saturation property* or, simply, *the second property*, if

1. there exists $H_D \subset G|_{D\cup V^1}$ such that $H_D \cong H'$ and, for every $v \in S \setminus D$, $u \in [V(G_1) \setminus S] \sqcup V^3$, there is a copy of H' inside $G|_{\hat{U}_v\cap N_G(u)}$, where $\hat{U}_v$ is the set of all neighbours z of v in V^2 such that (v, z, H_D, D) is H-completable.
2. for every $v \in S \setminus D$, $u \in V^1 \sqcup V^2$, there is a copy of H' inside $G|_{V^3\cap N_G(u)\cap N_G(v)}$.

2.3. Proof for a non-random graph

Assume that tuples $(G; K, S, D, Z_1, Z_2, Z_3, T)$ and $(G; V^1, V^2, V^3, S, D)$ satisfy the first and the second property respectively.

Clearly, in G, there are at least $\binom{k}{2} + d(n-k) > |E(F_n^1)|$ (for n large enough, since F_n^1 has $dn + O(1)$ edges due to the requirements in Lemma 1, and $k \geq c \ln n$) edges. Let us prove that there exists a weakly (G, H)-saturated graph with $|E(F_n^1)|$ edges for sufficiently large n. Clearly, it is sufficient to prove the existence of a weakly (G, H)-saturated graph with at most $|E(F_n^1)|$ edges.

Without loss of generality, assume that, for every n, $|V(S_n^1)| = C$ (we can add $C - |V(S_n^1)|$ vertices of $[n] \setminus S_n^1$ to S_n^1, and then we will preserve conditions of Lemma 1). We can also assume that, for every n, F_n^1 has the minimum number of edges among all graphs that satisfy conditions of Lemma 1 and have $|V(S_n^1)| = C$.

Let us now construct a spanning subgraph $F \subset G$ with *at most* $|E(F_n^1)|$ edges. After that, we will prove that this graph is weakly (G, H)-saturated.

Let us first define those edges of F that are induced by K. Let φ be a bijection from K to $V(K_k)$ such that $S \subset K$ is mapped onto S_k^1. Then, we construct a spanning graph on K (and we let $F|_K$ to be exactly this graph) isomorphic to F_k^1 such that φ is an isomorphism of $F|_K$ and F_k^1 (i.e., an edge $\{u, v\}$ belongs to $F|_K$ iff $\{\varphi(u), \varphi(v)\}$ belongs to F_k^1).

Second, for every vertex $v \in Z$, keep the edges of G going from v to D (there are d of them). Moreover, for every vertex outside $K \sqcup Z$, keep *specific* d edges of G going from v to Z (so many edges exist due to List 3 of the first property). The choice of edges between $[n] \setminus (K \sqcup Z)$ and Z will be explained later.

Clearly,

$$|E(F)| = |E(F_n^1)| + \left|E\left(F_k^1|_{S_k^1}\right)\right| - \left|E\left(F_n^1|_{S_n^1}\right)\right|. \tag{3}$$

Let us prove that $\left|E\left(F_k^1|_{S_k^1}\right)\right| = \left|E\left(F_n^1|_{S_n^1}\right)\right|$ for n large enough. It is enough to prove that there exists $N \in \mathbb{N}$ such that, for every $n_1, n_2 \geq N$, $\left|E\left(F_{n_1}^1|_{S_{n_1}^1}\right)\right| = \left|E\left(F_{n_2}^1|_{S_{n_2}^1}\right)\right|$. Assume the contrary: for every $N \in \mathbb{N}$, there exist $n_1 > n_2 \geq N$ such that $\left|E\left(F_{n_1}^1|_{S_{n_1}^1}\right)\right| > \left|E\left(F_{n_2}^1|_{S_{n_2}^1}\right)\right|$. Let N be large enough. Since $|S_{n_1}^1| = |S_{n_2}^1| = C$, we get that $|E(F_{n_1}^1)| > |E(F_{n_2}^1)| + d(n_1 - n_2)$. Since $F_{n_2}^1$ is weakly (K_{n_2}, H)-saturated, we get that a graph on $[n_1]$ obtained from $F_{n_2}^1$ by adding d edges from each vertex of $[n_1] \setminus [n_2]$ to $F_{n_2}^1$ is weakly (K_{n_1}, H)-saturated. This contradicts with the minimality of the number of edges in $F_{n_1}^1$.

From (3), we get that $|E(F)| = |E(F_n^1)|$ for large n.

Now let us show that F is weakly (G, H)-saturated and, on the way, specify the edges from $[n] \setminus (K \sqcup Z)$ to Z.

We first sequentially add the following bunch of edges to F: edges inside K, edges from K to Z, edges inside T, edges between Z and T, edges inside Z.

(1) Here, we restore the edges of G that are inside K. This is straightforward since K is a clique (by List 1 of the first property), $F|_K \cong F^1_k$ and there exists a bootstrap percolation process that starts on F^1_k and finishes on K_k. Let $F_1 = F \cup G|_K$.

(2) Let us restore the edges of G between K and Z. The edges between Z and D are already in F_1. Consider $u \in K \setminus D$, $v \in Z$. Let K' be a set of $r-d-2$ vertices of $K \setminus [D \sqcup \{u\}]$. Then, for any graph $\tilde{H} \cong H'$ on the vertex set $K' \sqcup D$ such that $N_H(w_1)$ is mapped onto D (such a mapping exists since K induces a clique in G), the tuple $(v, u, \tilde{H})$ is H-completable in $(V(F_1), E(F_1) \sqcup \{u, v\})$ since v is adjacent to every vertex from D in G. So, we can restore $\{u, v\}$. Let F_2 be obtained from F_1 by adding all the edges of G between K and Z.

(3) Let us switch to the edges that are entirely in T. Consider $u, v \in T$. Recall that R is an arbitrary set of r vertices in $K \setminus S$. The edges inside R and between $\{u, v\}$ and R are already in F_2 (since $T \subset Z$, $R \subset K$, and the edges inside K and between K and Z are already restored). Let $\tilde{H} \cong H'$ be inside R (recall that R is a clique). Then $(v, u, \tilde{H})$ is H-completable in $(V(F_2), E(F_2) \sqcup \{u, v\})$ and, therefore, we are able to restore $\{u, v\}$. We get $F_3 = F_2 \cup G|_T$.

(4) Let us restore the edges between $Z \setminus T$ and T. Consider $v \in Z \setminus T$, $u \in T$. Since u is adjacent to all vertices from $R \cup D$, v is adjacent to all vertices in D, $|R| = r$ and $|D| = d$, we get that there exists $\tilde{H} \cong H'$ in $F_3|_{R \cup D}$ such that $(v, u, \tilde{H})$ is H-completable in $(V(F_3), E(F_3) \sqcup \{u, v\})$. Let F_4 be obtained from F_3 by adding all the edges of G between $Z \setminus T$ and T.

(5) Let us restore the remaining edges inside Z. Consider adjacent (in G) $v_1, v_2 \in Z \setminus T$. By List 2 of the first property, there is $H_{v_1,v_2} \cong H'$ inside $G|_T = F_4|_T$ such that (v_1, v_2, H_{v_1,v_2}) is H-completable. Since all the edges from $G|_T$ and between $\{v_1, v_2\}$ and T are already in F_4, we get that (v_1, v_2, H_{v_1,v_2}) is H-completable in $(V(F_4), E(F_4) \sqcup \{v_1, v_2\})$ and we can restore $\{v_1, v_2\}$. We get $F_5 = F_4 \cup G|_Z$.

Next we restore the edges that are entirely outside $K \sqcup Z$.

(6) Let $v \notin K \sqcup Z$. Notice that *we have to specify d edges going from v to Z in F*. Let us do that. By List 3 of the first property, there exists a copy $H_v \subset G|_{Z_1}$ of H' such that (v, H_v) is H-completable. Let $V_v \subset V(H_v)$ be the neighbours of v in H_v. We *specify* the d edges drawn from v in F as edges from v to V_v.
Let us now restore the edges between v and $U_v(H_v, V_v)$. The edges inside $H_v \subset Z$ and between v and V_v are already in F_5, so by the definition of $U_v(H_v, V_v)$, we can restore all the edges between v and $U_v(H_v, V_v)$. Let F_6 be obtained from F_5 by adding edges between every $v \notin K \sqcup Z$ and $U_v(H_v, V_v)$.

(7) Here, we consider the edges that have both vertices outside $K \sqcup Z$. Consider $v_1, v_2 \notin K \sqcup Z$. The edges from v_1 and v_2 to $U_{v_1}(H_{v_1}, V_{v_1}) \cap U_{v_2}(H_{v_2}, V_{v_2})$ and the edges inside $U_{v_1}(H_{v_1}, V_{v_1}) \cap U_{v_2}(H_{v_2}, V_{v_2})$ are already in F_6. By List 4 of the first property, there is a copy of H' inside $G|_{U_{v_1}(H_{v_1}, V_{v_1}) \cap U_{v_2}(H_{v_2}, V_{v_2})} = F_6|_{U_{v_1}(H_{v_1}, V_{v_1}) \cap U_{v_2}(H_{v_2}, V_{v_2})}$, so the edge between v_1 and v_2 can be restored. Let $F_7 = F_6 \cup G|_{[n] \setminus [K \sqcup Z]}$.

It remains to restore only the edges between $K \sqcup Z$ and $[n] \setminus (K \sqcup Z)$.

(8) Let us restore all the edges between $(K \sqcup Z) \setminus S$ and $[n] \setminus (K \sqcup Z)$. Let $v \notin K \sqcup Z$.
First, let $u \in (K \setminus S) \sqcup Z_2 \sqcup Z_3$. The edges inside $U_v(H_v, V_v) \cap N_G(u) \subset Z$, the edges from u to $N_G(u) \cap Z_2$ and the edges from v to $U_v(H_v, V_v)$ are already in F_7. By List 5 of the first property, there is a copy of H' inside $U_v(H_v, V_v) \cap N_G(u)$. So, we can restore the edge between u and v.
Second, let $u \in Z_1$. The edges from v to Z_3 are just restored. The edges inside Z_3 and the edges between u and Z_3 are already in F_7. By List 6 of the first property, there is a copy of H' inside $Z_3 \cap N_G(u) \cap N_G(v)$, so we can restore $\{u, v\}$.
The graph F_8 is obtained from F_7 by adding all the edges of G between $(K \sqcup Z) \setminus S$ and $[n] \setminus (K \sqcup Z)$.
It remains to restore only the edges between S and $[n] \setminus (K \sqcup Z)$.

(9) Here, we restore the edges between D and $[n] \setminus (K \sqcup Z)$. Let $v \in D$, $u \in [n] \setminus (K \sqcup Z)$. Then, the edges between u and Z, the edges between v and Z and the edges inside Z are in F_8, and $Z \subset N_G(v)$. By List 7 of the first property, there is a copy of H' inside $Z \cap N_G(u)$. So, we can restore $\{u, v\}$. Let F_9 be obtained from F_8 by adding all the edges of G between D and $[n] \setminus (K \sqcup Z)$.

(10) It remains to restore the edges between $S \setminus D$ and $[n] \setminus (K \sqcup Z)$. Consider $v \in S \setminus D$. By the definition of $\hat{U}_v$ (given in List 1 of the second property), the edges from v to $\hat{U}_v$ can be restored immediately, as the edges inside H_D, the edges between $\hat{U}_v$ and H_D and the edges from v to D

are in F_9. For $u \in (V(G_1) \setminus S) \sqcup V^3$, by List 1 of the second property, there is a copy of H' inside $\hat{U}_v \cap N_G(u)$. The edges inside $\hat{U}_v \cap N_G(u) \subset V^2$ and the edges between $\{u, v\}$ and $\hat{U}_v \cap N_G(u) \subset V^2$ are already restored, so $\{u, v\}$ can be restored. Finally, consider $u \in V^1 \sqcup V^2$. The edges between S and V^3 have just been restored. By List 2 of the second property, there is a copy of H' inside $V^3 \cap N_G(u) \cap N_G(v)$, so we can restore $\{u, v\}$ as well.

2.4. Random graph has the properties

Let us first recall some results on the distribution of small subgraphs in the binomial random graph.

Given a graph Y, it is well known that the number of subgraphs isomorphic to Y in $G(n, p)$ is well-concentrated around its expectation. In particular, Janson's inequality implies that (see, e.g., [9, Theorem 2.14]) the probability that $G(n, p)$ does not contain an isomorphic copy of K_ℓ (ℓ is a positive integer constant) is at most $e^{-\Omega(n^2)}$. By the union bound, we get

Claim 1. *Let $\varepsilon > 0$. Whp, for any subset $A \subset [n]$ such that $|A| \geq \varepsilon n$, there exists a copy of K_ℓ in $G(n, p)|_A$.*

Since K_ℓ contains as a subgraph any graph on ℓ vertices, we get that the statement of Claim 1 is also true for any graph Y.

Below, we use a notion of (X, Y)-extension introduced by Spencer in [17]. Let $x \in \mathbb{N}$ and $X = \{\omega_1, \ldots, \omega_x\}$ be a set of x vertices called *roots*. Let Y be a graph on $\{\omega_1, \ldots, \omega_y\}$, $y > x$. Then a graph $\tilde{Y}$ on $\{\tilde{\omega}_1, \ldots, \tilde{\omega}_y\}$ is called *(X, Y)-extension of* $\tilde{X} = \{\tilde{\omega}_1, \ldots, \tilde{\omega}_x\}$, if, for distinct $i \in [y]$, $j \in [y] \setminus [x]$, the presence of the edge $\{\omega_i, \omega_j\}$ in Y implies the presence of the edge $\{\tilde{\omega}_i, \tilde{\omega}_j\}$ in $\tilde{Y}$.

In [17], it is proven (by a straightforward application of another Janson's inequality, [9, Theorem 2.18 (i)]) that $[x]$ does not have an (X, Y)-extension with probability at most $e^{-\Omega(n)}$. By the union bound, this observation implies the following.

Claim 2. *Let $\varepsilon > 0$, $\tilde{n} \in (\varepsilon n, n]$ be a sequence of positive integers. Then whp*

- *for any pair $u, v \notin [\tilde{n}]$ of adjacent in $G(n, p)$ vertices, there exists a copy H_{uv} of H' in $G(n, p)|_{[\tilde{n}]}$ such that (v, u, H_{uv}) is H-completable in $G(n, p)$;*
- *for any $v \in [n] \setminus [\tilde{n}]$, there exists a copy H_v of H' in $G(n, p)|_{[\tilde{n}]}$ such that (v, H_v) is H-completable in $G(n, p)$.*

Let $b \in \mathbb{N}$. The number of common neighbours of $[b]$ in $G(n, p)$ has binomial distribution with parameters $n - b$ and p^b. By the Chernoff bound, this number is smaller than $\frac{1}{2}p^b(n - b)$ with probability at most $e^{-\Omega(n)}$. By the union bound, we get the following.

Claim 3. *Let $\varepsilon > 0$, $\tilde{n} \in (\varepsilon n, n]$ be a sequence of positive integers, $b \in \mathbb{N}$. Then whp, any subset of $[n] \setminus [\tilde{n}]$ of size at most b has at least $\frac{\varepsilon}{2}p^b n$ common neighbours in $G(n, p)|_{[\tilde{n}]}$.*

Now, let us prove that there exist $c > 0$ and sets $K, S, D, Z_1, Z_2, Z_3, T, V^1, V^2, V^3$ such that

- the tuple $(G(n, p), c, K, S, D, Z_1, Z_2, Z_3, T)$ whp satisfies the first H-saturation property,
- the tuple $(G(n, p), V^1, V^2, V^3, S, D)$ whp satisfies the second H-saturation property.

Let us start with the first H-saturation property. We will define the parameters and prove that whp each condition (out of 7 from the definition of the first property) holds for these parameters at the same time.

1. Since $G_1 \overset{d}{=} G(n/2, p)$, whp, in G_1, there is a clique of size at least $c \ln n$ for some positive constant c (see [9], Theorem 7.1). Let K be this clique of size $k \geq c \ln n$. So, List 1 of the first property holds whp.

Let

- S be a set of $|S_k^0|$ vertices of K,
- D be a set of d vertices of S,
- Z be the set of all common neighbours of D from $V(G_2)$.

Notice that the appearances of the edges (in $G(n,p)$) between D and $V(G_2)$ do not depend on the choice of K, S and D. Then, $|Z|$ has binomial distribution with parameters $n/2$ and p^d. Therefore, whp $\frac{1}{4}p^d n < |Z| < \frac{3}{4}p^d n$ (say, by Chebyshev's inequality).

2. Recall that R is an arbitrary set of r vertices in $K \setminus S$. Then T is the set of all common neighbours of $D \sqcup R$ in $V(G_2)$.
The appearances of the edges between $V(G_2)$ and $D \sqcup R$ in $G(n,p)$ do not depend on the choice of D and R. Then, $|T|$ has binomial distribution with parameters $n/2$ and p^{r+d}. Therefore, whp $\frac{1}{4}p^{r+d}n < |T| < \frac{3}{4}p^{r+d}n$.
Notice that appearances (in $G(n,p)$) of the edges between T and $Z \setminus T$ and the edges inside T are independent of the choice of Z and T, so, conditioned on Z and T, they have independent Bernoulli distributions. Then, by Claim 2, whp for every adjacent in $G(n,p)$ pair $v_1, v_2 \in Z \setminus T$, there exists a copy $H_{v_1,v_2} \cong H'$ inside $G(n,p)|_T$ such that (v_1, v_2, H_{v_1,v_2}) is H-completable.
3. Let $Z = Z_1 \sqcup Z_2 \sqcup Z_3$ be an almost equal partition of Z (the sizes of the parts differ by at most one). Then whp $|Z_1| = \Omega(n)$ and appearances (in $G(n,p)$) of the edges inside Z_1 and between Z_1 and $[n] \setminus (K \sqcup Z)$ do not depend on the choice of K, Z and Z_1. Therefore, by Claim 2, whp for every $v \notin K \sqcup Z$, there exists $H_v \cong H'$ in $G(n,p)|_{Z_1}$ such that (v, H_v) is H-completable. It implies that there is some copy $\hat{H} \cong H|_{V(H)\setminus\{v\}}$ such that w_1 is mapped onto v and H' is mapped onto H_v. Clearly, it implies the existence of $V_v \subset V(H_v) \setminus \{v\}$ (by the definition) such that (v, H_v, V_v) is H-completable.
4. Recall that, for a vertex $v \notin K \sqcup Z$, we denote by $U_v(H_v, V_v)$ the set of all neighbours u of v in Z_2 such that the tuple (v, u, H_v, V_v) is H-completable.
Set $\tilde{U}_v := U_v(H_v, V_v)$. Fix $v_1, v_2 \notin K \sqcup Z$. Notice that if $u \in Z_2$ is a common neighbour of $V(H_{v_1}) \cup V(H_{v_2}) \cup \{v_1, v_2\}$ then u lies in $\tilde{U}_{v_1} \cap \tilde{U}_{v_2}$. Notice that $V(H_{v_1}) \cup V(H_{v_2}) \subset Z_1$ and $|V(H_{v_1}) \cup V(H_{v_2})| \leq 2r$. Appearances (in the random graph) of the edges between vertices of $[n] \setminus (K \sqcup Z)$ and vertices of Z_2 are independent of the choice of $[n] \setminus (K \sqcup Z)$ and Z_2. The appearances of the edges between Z_1 and Z_2 are independent of the choice of Z_1 and Z_2 as well, and, moreover, $|Z_2| = \Omega(n)$. So, applying Claim 3, we get that whp $|\tilde{U}_{v_1} \cap \tilde{U}_{v_2}| = \Omega(n)$. By Claim 1, whp there is a copy of H' in every $\tilde{U}_{v_1} \cap \tilde{U}_{v_2}$.
5. Fix $v \notin K \sqcup Z$, $u \in (K \setminus S) \sqcup Z_2 \sqcup Z_3$. Notice that the appearances of the edges between $([n] \setminus K \sqcup Z) \sqcup (K \setminus S) \sqcup Z_2 \sqcup Z_3 = [n] \setminus S \setminus Z_1$ and Z_2 are independent of the choice of S, K, Z, Z_1, Z_2, Z_3 and so are the edges between Z_1 and Z_2. By the Chernoff bound, with probability $1 - e^{-\Omega(n)}$, vertices from $V(H_v) \cup \{u, v\}$ have $\Omega(n)$ common neighbors in Z_2. Since there are at most n^2 choices of $v \notin K \sqcup Z$ and $u \in (K \setminus S) \sqcup Z_2 \sqcup Z_3$, by the union bound, we get that whp there are $\Omega(n)$ common neighbours in Z_2 for each element of $\{V(H_v) \cup \{u, v\} \mid v \notin K \sqcup Z, u \in (K \setminus S) \sqcup Z_2 \sqcup Z_3\}$. Notice that, if $u' \in Z_2$ is common neighbour of $V(H_v) \cup \{u, v\}$, then $u' \in \tilde{U}_v \cap N_{G(n,p)}(u)$. So, whp, for every $v \notin K \sqcup Z$, $u \in (K \setminus S) \sqcup Z_2 \sqcup Z_3$, $|\tilde{U}_v \cap N_{G(n,p)}(u)| = \Omega(n)$. By Claim 1, whp there is a copy of H' in every $\tilde{U}_v \cap N_{G(n,p)}(u)$.
6. Notice that the appearances of the edges between $([n] \setminus K \setminus Z) \sqcup Z_1$ and Z_3 do not depend on the choice of these sets, and whp $|Z_3| = \Omega(n)$, therefore whp any $v \notin K \sqcup Z, u \in Z_1$ has $\Omega(n)$ common neighbours in Z_3 by Claim 3. Indeed, such pairs $\{v, u\}$ are subsets of $[n] \setminus Z_3$ of size 2. By Claim 3 whp all subsets of $[n] \setminus Z_3$ of size 2 have at least $\frac{\varepsilon p^2}{2} n = \Omega(n)$ common neighbours in Z_3, where ε is such that $|Z_3| \geq \varepsilon n$. So, by Claim 1, whp there is a copy of H' in $Z_3 \cap N_{G(n,p)}(u) \cap N_{G(n,p)}(v)$ for every $u \in Z_1$, $v \notin K \sqcup Z$.
7. Notice that, if $u \notin S$, then the appearances of the edges between u and Z do not depend on the choice of Z and u. If $u \notin S$ then, by the Chernoff bound, with probability $1 - e^{-\Omega(n)}$, the vertex u has $\Omega(n)$ neighbours in Z. Since there are at most n choices of $u \notin S$, by the union bound, we get that whp, every vertex $u \notin S$ has at least $\Omega(n)$ neighbours in Z. So, we can apply Claim 1 and get that whp there is a copy of H' in $N_{G(n,p)}(u) \cap Z$ for every $u \notin S$.

Now let us prove the second property. Let $V(G_2) = V^1 \sqcup V^2 \sqcup V^3$ be an equal partition.

1. Let us find a copy of H' such that $N_H(w_1) \setminus \{w_2\}$ is mapped onto D and other vertices of this copy lie in V^1. Notice that the appearances of the edges between D and V^1 and inside V^1 have independent Bernoulli distributions. Recall that D induces a clique in $G(n, p)$ and $|V^1|= \Omega(n)$. So, Claim 3 implies the existence (whp) of $\Omega(n)$ common neighbours of D in V^1. By Claim 1, there exists an r-clique in the set of all common neighbors of D. This immediately implies the existence of the desired H_D.
   Notice that all common neighbours of $V(H_D) \cup \{v\}$ in V^2 lie in $\hat{U}_v$ for every $v \in S \setminus D$ as $D \subset N_{G(n,p)}(v)$. The appearances of the edges between $D \sqcup V^1 \sqcup (V(G_1) \setminus S) \sqcup V^3$ and V^2 do not depend on the choice of these sets. So, by Claim 3, whp, for every $u \in V(G_1) \setminus S \sqcup V^3$, there are $\Omega(n)$ common neighbours of $V(H_D) \sqcup \{v, u\}$ in V^2 (and so $|\hat{U}_v \cap N_{G(n,p)}(u)|= \Omega(n)$). By Claim 1, whp, for any $v \in S \setminus D$, $u \in (V(G_1) \setminus S) \sqcup V^3$, there is a copy of H' in $\hat{U}_v \cap N_{G(n,p)}(u)$.
2. The appearances of the edges between $(S \setminus D) \sqcup V^1 \sqcup V^2$ and V^3 do not depend on the choice of these sets, so, by Claim 3, for any $v \in (S \setminus D)$, $u \in V^1 \sqcup V^2$, whp there are $\Omega(n)$ common neighbours of $\{u, v\}$ in V^3. By Claim 1, whp, for any $v \in (S \setminus D)$, $u \in V^1 \sqcup V^2$, there is a copy of H' inside $V^3 \cap N_{G(n,p)}(u) \cap N_{G(n,p)}(v)$.

3. Proof of Theorem 2

Recall that $t \geq 3$ is assumed.

Let us first notice that if $G = G_1 \sqcup \ldots \sqcup G_m$ consists of m connected components $G_1, \ldots, G_m$, then

$$\mathrm{wsat}(G, K_{1,t}) = \sum_{i=1}^{m} \mathrm{wsat}(G_i, K_{1,t}).$$

Therefore, $\mathrm{wsat}(G, K_{1,t})$ is at least the number of non-empty components in G.

Let $\frac{1}{n^2} \ll p \leq \frac{\ln n}{2n}$. Consider $w_n > 0$ such that

- $w_n \to \infty$ as $n \to \infty$,
- $w_n \leq \frac{\ln n}{2}$ for n large enough,
- $\frac{w_n}{n^2} \leq p$ for n large enough.

Then, for n large enough, $\frac{w_n}{n} \leq pn \leq \frac{\ln n}{2}$. Let X be the number of isolated edges in $G(n, p)$. Since

$$\mathsf{E}X = \binom{n}{2} p(1-p)^{2(n-2)} \sim \frac{1}{2} \exp\left[\ln n + \ln(np) - 2pn\right] \geq \frac{w_n}{2}(1 + o(1))$$

and

$$\mathrm{Var}X = \mathsf{E}X + \binom{n}{2}\binom{n-2}{2} p^2(1-p)^{4+4(n-4)} - (\mathsf{E}X)^2 = \mathsf{E}X + O\left((\mathsf{E}X)^2 \frac{\ln n}{n}\right),$$

by Chebyshev's inequality, we get that

$$\mathsf{P}\left(X < \frac{w_n}{3}\right) \leq \frac{\mathrm{Var}X}{(\mathsf{E}X - w_n/3)^2} \to 0, \quad n \to \infty.$$

Therefore, whp

$$\mathrm{wsat}(G(n, p), K_{1,t}) \geq w_n > \binom{t}{2} = \mathrm{wsat}(K_n, K_{1,t}).$$

Now, let $p > \frac{\ln n}{2n}$. For such p, for any $\varepsilon > 0$, whp there exists a connected component G_n in $G(n, p)$ of size at least $(1 - \varepsilon)n$ [9, Theorem 5.4]. Clearly, $\mathrm{wsat}(G(n, p), K_{1,t}) \geq \mathrm{wsat}(G_n, K_{1,t})$.

Below, for a connected graph G, we give a necessary and sufficient condition, in terms of the existence of a subgraph from a certain class, for the stability property $\mathrm{wsat}(G, K_{1,t}) = \binom{t}{2}$.

Let $t \leq y < x \leq n$ be integers and G be a graph on $[n]$. Let $F' \subset F \subset G$, $V(F) = \{v_1, \ldots, v_x\}$, $V(F') = \{v_1, \ldots, v_y\}$. Let us call F a *saturating structure of length x in G with the core F'*, if every v_i, $y+1 \leq i \leq x$, sends exactly $t-1$ edges to the previous vertices $v_1, \ldots, v_{i-1}$ in F, i.e. $|N_F(v_i) \cap \{v_1, \ldots, v_{i-1}\}| = t-1$. We call y *the size* of the core. The vector $\mathbf{v} = (v_1, \ldots, v_x)$ is called a *saturating ordering of F*.

Claim 4. *Let G be connected.*

(1) If G contains a saturating structure of length n with a core $F' \cong K_t$*, then* $\mathrm{wsat}(G, K_{1,t}) = \binom{t}{2}$.
(2) If $\mathrm{wsat}(G, K_{1,t}) = \binom{t}{2}$ *and G has at least* μ *vertices with degrees at least* $t-1$*, then G contains a saturating structure of length* μ *with a core of size at most* $\binom{t+1}{2}$.

Now, due to Claim 4 and the fact that whp $G(n,p)$ has at least $n/2$ vertices with degrees at least $t-1$ (this can be proven by a straigtforward application of Chebyshev's inequality to the number of vertices with such degrees), Theorem 2 immediately follows from

Claim 5.

1. *There exists* $c > 0$ *such that, if* $p < cp(n,t)$*, then whp there is no saturating structure of length* $\lfloor \ln n \rfloor$ *and with a core of size at most* $\binom{t+1}{2}$ *in* $G(n,p)$.
2. *There exists* $C > 0$ *such that, if* $p > Cp(n,t)$*, then whp there exists a saturating structure of length n with a core isomorphic to* K_t *in* $G(n,p)$.

It remains to prove Claims 4 and 5. We give the proof of Claim 4 in Section 3.1 and the proof of Claim 5 in Sections 3.2 and 3.3. The most involved part is the proof of the second part of Claim 5 given in Section 3.3. At first we prove the existence of a saturating structure of size $x = \lfloor \ln n \rfloor$ whp using the second moment method. After that we extend this structure to size $y = \frac{t^2}{p}$ whp. Finally, we extend this structure to a structure of size n whp, which finishes the proof.

3.1. Proof of Claim 4

First, assume that G contains a saturating structure F of length n with a core $F' \cong K_t$. It is clear that F is both weakly $(G, K_{1,t})$-saturated and weakly $(K_n, K_{1,t})$-saturated. In particular, it implies that G is weakly $(K_n, K_{1,t})$-saturated and, therefore, $\mathrm{wsat}(G, K_{1,t}) \geq \mathrm{wsat}(K_n, K_{1,t})$. Since F is weakly $(G, K_{1,t})$-saturated, it remains to prove that $\mathrm{wsat}(F, K_{1,t}) \leq \binom{t}{2}$. Let $(v_1, \ldots, v_n)$ be a saturating ordering of F. Then $F' = F|_{\{v_1,\ldots,v_t\}} \cong K_t$ has exactly $\binom{t}{2}$ edges. Let us show that F' is weakly $(F, K_{1,t})$-saturated. Since each vertex of $v_1, \ldots, v_t$ has degree $t-1$ in F', we can restore all the edges in F adjacent to one of these vertex. In particular, we restore all the edges going from v_{t+1} to $v_1, \ldots, v_t$. Proceeding in this way by induction, we restore all the edges of F.

Now, let $\mathrm{wsat}(G, K_{1,t}) = \binom{t}{2}$ and G have at least μ vertices with degrees at least $t-1$. Let F' be a weakly $(G, K_{1,t})$-saturated graph with $\binom{t}{2}$ edges and y non-isolated vertices. Let us order these vertices of F' in a way $v_1, \ldots, v_y$ such that v_i plays the role of the ith central vertex of $K_{1,t}$ in a $K_{1,t}$-bootstrap percolation process that starts on F' and finishes on G. Clearly, for every $i \in [t-1]$, the vertex v_i sends at least $t-i$ edges to $F'|_{\{v_{i+1},\ldots,v_y\}}$. Since the total number of these edges is $\binom{t}{2}$, F' cannot contain any other edge. The bound $y \leq t + \binom{t}{2} = \binom{t+1}{2}$ follows.

Consider a $K_{1,t}$-bootstrap pecolation process that starts on F' and finishes on G. Let $e_1, \ldots, e_m$ be the appearing sequentially in this process edges that contain at least one vertex outside $\{v_1, \ldots, v_y\}$. Let $w_1, \ldots, w_{\mu-y}$ be vertices of G outside $\{v_1, \ldots, v_y\}$ with degrees at least $t-1$ ordered in the following way.

- Let $i \in [m]$ be such that e_i contains w_1 and a vertex from $\{v_1, \ldots, v_y\}$, there are exactly $t-2$ edges among $e_1, \ldots, e_{i-1}$ that contain w_1, and all of them have the second end in $\{v_1, \ldots, v_y\}$.
- For $j \in \{2, \ldots, \mu-y\}$, let $i_j \in [m]$ be such that e_{i_j} contains w_j and a vertex from $\{v_1, \ldots, v_y, w_1, \ldots, w_{j-1}\}$, there are exactly $t-2$ edges among $e_1, \ldots, e_{i_j-1}$ that contain w_j, and all of them have the second end in $\{v_1, \ldots, v_y, w_1, \ldots, w_{j-1}\}$.

Such an ordering exists due to the definition of the $K_{1,t}$-bootstrap pecolation process. Then, the desired saturating structure of length μ is obtained from $F'|_{\{v_1,\ldots,v_y\}}$ by adding w_i, $i \in [\mu-y]$, with the $t-1$ edges going to the previous vertices $v_1, \ldots, v_y, w_1, \ldots, w_{i-1}$.

3.2. *Proof of Claim 5.1*

Let $x = \lfloor \ln n \rfloor$, $y = \binom{t+1}{2}$, $c < e^{-(y+1)/(t-1)}$. Let $p < cp(n,t)$.

Let X be the number of subgraphs F in $G(n,p)$ on x vertices such that there exist $i \in \{t-1, t, \ldots, y\}$ and $v_1, \ldots, v_i \in V(F)$ satisfying the following property:

> for every $v \in V(F) \setminus \{v_1, \ldots, v_i\}$, there exists a set N_v of its $t-1$ neighbours in F such that, for every $u \in N_v$, $v \notin N_u$.

Clearly, an ordered saturating structure of size x with a core of size at most y is a subgraph with the above property. Therefore, it is sufficient to prove that $\mathsf{P}(X \geq 1) \to 0$ as $n \to \infty$.

Let us bound $\mathsf{E}X$ from above. By the linearity of expectation, we get

$$\mathsf{E}X \leq \binom{n}{x} \sum_{i=t-1}^{y} \binom{x}{i} \binom{x}{t-1}^{x-i} p^{(x-i)(t-1)}.$$

Indeed, $\binom{n}{x}$ is the number of ways to choose the set of vertices of F. Then, for every $i = t-1, \ldots, y$ we choose the vertices $v_1, \ldots, v_i$ out of the set $V(F)$ in $\binom{x}{i}$ ways. After that we choose N_v for every $v \in V(F) \setminus \{v_1, \ldots, v_i\}$ (there are $x-i$ vertices and at most $\binom{x}{t-1}$ choices for one vertex). Since edges from v to N_v should be distinct for different $v \in V(F) \setminus \{v_1, \ldots, v_i\}$, the probability to draw them is $p^{(x-i)(t-1)}$. So,

$$\mathsf{E}X \leq \frac{n^x x^y}{x!} \sum_{i=t-1}^{y} \left(\binom{x}{t-1} p^{t-1}\right)^{x-i} \leq \frac{y n^x x^y}{x!} (xp)^{(t-1)(x-y)} = e^{x \ln n - x \ln x + x + (x-y)(t-1)\ln(xp) + O(\ln \ln n)}.$$

Since

$$\ln p < \ln(cp(n,t)) = -\frac{1}{t-1} \ln n - \frac{t-2}{t-1} \ln \ln n - \ln \frac{1}{c},$$

we get that

$$\mathsf{E}X \leq \exp\left[y \ln n - y \ln \ln n + x - (x-y)(t-1) \ln \frac{1}{c} + O(\ln \ln n)\right] =$$
$$\exp\left[\ln n \left(y + 1 - (t-1) \ln \frac{1}{c}\right) + O(\ln \ln n)\right] \to 0, \quad n \to \infty.$$

Markov's inequality implies $\mathsf{P}(X \geq 1) \to 0$.

3.3. *Proof of Claim 5.2*

Set $p = Cp(n,t)$ where C is a large positive constant (for example, any C bigger than $2^{\frac{2t-1}{t-1}} t(t-1)$ is sufficient). Since 'containing a saturating structure of length n with a core isomorphic to K_t' is an increasing property, it is sufficient to prove that it holds whp for this value of p.

The structure of the proof is the following: at first we prove the existence of a saturating structure of size $x = \lfloor \ln n \rfloor$ whp, then we extend this structure to size $y = \frac{t^2}{p}$ whp, and finally, we extend it to the desired size n whp.

3.3.1. *Saturating structure of size $x = \lfloor \ln n \rfloor$*

Let X be the number of saturating structures F of size x in $G(n,p)$ with a core isomorphic to K_t and a saturating ordering $(v_1, \ldots, v_x)$ such that $v_1 < v_2 < \cdots < v_x$. Let us call such an ordering a *canonical* saturating ordering. Let S be the set of all such structures in K_n. Then $X = \sum_{A \in S} I_A$ where I_A indicates that A belongs to $G(n,p)$. We have

$$\mathsf{E}X \sim \frac{n^x}{x!} p^{\binom{t-1}{2}} \prod_{i=t}^{x} \left[\binom{i-1}{t-1} p^{t-1}\right]. \qquad (4)$$

Notice that

$$\prod_{i=t}^{x}\binom{i-1}{t-1}=\frac{1}{[(t-1)!]^{x-t}}\frac{t!\cdot\ldots\cdot(x-1)!}{1!\cdot\ldots\cdot(x-t)!}=$$
$$\frac{1}{[(t-1)!]^{x-t}}\frac{(x-t+1)!\cdot\ldots\cdot(x-1)!}{1!\cdot\ldots\cdot(t-1)!}\geq\frac{[(x-t+1)!]^{t-1}}{[(t-1)!]^{x-t}}\sim$$
$$\sim\frac{(2\pi x)^{(t-1)/2}(x-t+1)^{(x-t+1)(t-1)}}{e^{(x-t+1)(t-1)}[(t-1)!]^{x-t}}>(x-t+1)^{(x-t+1)(t-1)}.$$

So,

$$\mathsf{E}X\geq(1+o(1))\frac{n^x}{\sqrt{2\pi x}(x/e)^x}(x-t+1)^{(t-1)(x-t+1)}p^{\binom{t-1}{2}+(t-1)(x-t+1)}=$$
$$\exp\left\{\frac{t}{2}\ln n+x[1+(t-2)\ln x+(t-1)\ln C-(t-2)\ln\ln n]+o(x)\right\}=$$
$$\exp\left\{\ln n\left[\frac{t}{2}+(1+(t-1)\ln C)\right]+o(\ln n)\right\}.$$

As $C>e^{-\frac{1}{t-1}}$, we get $\mathsf{E}X\to\infty$.

Now let us estimate $\mathrm{Var}X$. Since, for disjoint $A,B\in S$, I_A,I_B are independent, we get

$$\mathsf{E}\left[\sum_{\substack{A,B\in S,\\ V(A)\cap V(B)=\varnothing}}I_AI_B\right]=\sum_{\substack{A,B\in S,\\ V(A)\cap V(B)=\varnothing}}\mathsf{E}I_A\mathsf{E}I_B\leq\mathsf{E}\left[\sum_{A\in S}I_A\right]\mathsf{E}\left[\sum_{B\in S}I_B\right]=(\mathsf{E}X)^2$$

and

$$\mathrm{Var}X=\mathsf{E}X^2-(\mathsf{E}X)^2=$$

$$\mathsf{E}\left[\sum_{\substack{A,B\in S,\\ V(A)\cap V(B)=\varnothing}}I_AI_B\right]+\mathsf{E}\left[\sum_{\substack{A,B\in S,\\ V(A)\cap V(B)\neq\varnothing}}I_AI_B\right]-(\mathsf{E}X)^2\leq\sum_{\substack{A,B\in S,\\ V(A)\cap V(B)\neq\varnothing}}\mathsf{E}I_AI_B. \quad (5)$$

Let $A,B\in S$ have a non-empty intersection $W=V(A)\cap V(B)$ and let $w_1<\cdots<w_d$ be the vertices of W.

Since the number of edges in W is maximum when each w_i sends all $\min\{t-1,i-1\}$ edges to the previous $w_1,\ldots,w_{i-1}$, we get that A and B have at most

$$M:=\left[\binom{t-1}{2}+(d-t+1)(t-1)\right]I(d\geq t)+\binom{d}{2}I(d<t).$$

common edges. Notice that $M=(d-t/2)(t-1)$ when $d\geq t$. Denote by $\mathrm{cnt}(d,m_1,m_2)$ the number of pairs $A,B\in S$ such that $|V(A)\cap V(B)|=d$, $A\cap B$ has m_1 edges inside the core of B and m_2 edges outside the core of B. In this case A and B have m_1+m_2 common edges, so the probability to draw edges of $A\cup B$ is $p^{2z-m_1-m_2}$, where $z=\binom{t-1}{2}+(x-t+1)(t-1)=(x-t/2)(t-1)$ is the number of edges in each of A and B. So,

$$\sum_{\substack{A,B\in S,\\ A\cap B\neq\emptyset}}\mathsf{E}I_AI_B=\sum_{\substack{A,B\in S,\\ A\cap B\neq\emptyset}}P(A\subset G(n,p),B\subset G(n,p))=\sum_{d}\sum_{m_1}\sum_{m_2}\mathrm{cnt}(d,m_1,m_2)p^{2z-m_1-m_2}. \quad (6)$$

Now let us bound $cnt(d,m_1,m_2)$ from above. Notice that $\mathsf{E}X$ is the number of choices of an A of size x multiplied by the probability to draw z edges. So, we can choose A in $\frac{\mathsf{E}X}{p^z}$ ways. After A is chosen, we select d common vertices of A and B in $\binom{x}{d}$ ways. Then, we choose other vertices of B (it can be done in at most $\frac{n^{x-d}}{(x-d)!}$ ways). After that we have to choose edges in B. Notice that, for a fixed j and for the ith vertex of B that has j outcoming edges (i.e. edges between the ith vertex and vertices that are labelled by numbers less than i) that should be entirely in A, the number of choices of these common edges is at most $\binom{t-1}{j}$, and the number of choices of all the other outcoming edges from the ith vertex is at most $\binom{i-1}{t-1-j}$. Then,

$$\mathrm{cnt}(d, m_1, m_2) \leq \frac{\mathsf{E}X}{p^z}\binom{x}{d}\frac{n^{x-d}}{(x-d)!}\max_{j_{t+1},\ldots,j_x \in J(d,m_2)}\prod_{i=t+1}^{x}\binom{i-1}{t-1-j_i}\binom{t-1}{j_i}, \tag{7}$$

where $J(d, m_2)$ is the set of all tuples $(j_{t+1}, \ldots, j_x) \in \{0, 1, \ldots, t-1\}^{t-x}$ such that $j_{t+1}+\cdots+j_x = m_2$ and the number of non-zero j_i is at most d. Clearly, for a $(j_{t+1}, \ldots, j_x) \in J(d, m_2)$, we have $\prod_{i=t+1}^{x}\binom{t-1}{j_i} \leq 2^{(t-1)d}$. Moreover,

$$\prod_{i=t+1}^{x}\binom{i-1}{t-1-j_i} = \prod_{i=t+1}^{x}\frac{(t-1)!(i-t)!}{(t-1-j_i)!(i-t+j_i)!}\binom{i-1}{t-1}$$
$$\leq \prod_{i=t+1}^{x}\left(\frac{t-1}{i-t+1}\right)^{j_i}\binom{i-1}{t-1} = (t-1)^{m_2}\prod_{i=t+1}^{x}\frac{\binom{i-1}{t-1}}{(i-t+1)^{j_i}}.$$

The function $g(j_{t+1}, \ldots, j_x) = \prod_{i=t+1}^{x}\frac{1}{(i-t+1)}^{j_i}$ defined on the intersection of $\{0, 1, \ldots, t-1\}^{x-t}$ with the hyperplane $j_{t+1}+\cdots+j_x = m_2$ achieves its maximum when

$$j_i = \begin{cases} t-1, & t+1 \leq i \leq t+\left\lfloor\frac{m_2}{t-1}\right\rfloor, \\ m_2 \mod t-1, & i = t+\left\lfloor\frac{m_2}{t-1}\right\rfloor+1, \\ 0, & i > t+\left\lfloor\frac{m_2}{t-1}\right\rfloor+1, \end{cases}$$

since $-\ln g = \sum_{i=t+1}^{x}\alpha_i j_i$ is linear and the coefficients α_i increase as i grows.

Therefore, $\prod_{i=t+1}^{x}\frac{1}{(i-t+1)^{j_i}} = O\left[\left(\frac{1}{\lceil\frac{m_2}{t-1}\rceil!}\right)^{t-1}\right]$. Combining this with (4) and (7), we get

$$f(d, m_1, m_2) := \frac{\mathrm{cnt}(d, m_1, m_2)p^{2z-m_1-m_2}}{(EX)^2} =$$
$$= O\left(\binom{x}{d}x^d n^{-d}2^{(t-1)d}\frac{(t-1)^{m_2}}{(1/\lceil\frac{m_2}{t-1}\rceil!)^{t-1}}p^{-m_1-m_2}\right) =$$
$$= O\left(\left(\frac{xe2^{(t-1)}}{d}\right)^d x^d n^{-d}p^{-m_1}\left(\frac{(t-1)^2e}{m_2p}\right)^{m_2}\right).$$

Since $\left(\frac{(t-1)^2e}{m_2p}\right)^{m_2}$ increases (as a function of m_2) on $(0, M]$ (the maximum is achieved at $\frac{(t-1)^2}{p} \gg M$), we get that

$$p^{-m_1}\left(\frac{(t-1)^2e}{m_2p}\right)^{m_2} \leq p^{-m_1}\left(\frac{(t-1)^2e}{(M-m_1)p}\right)^{M-m_1} = p^{-M}\left(\frac{(t-1)^2e}{M-m_1}\right)^{M-m_1}. \tag{8}$$

This expression achieves its maximum when $m_1 = M-(t-1)^2$. Since $m_1 \leq \binom{t-1}{2}$ by its definition, and M may be either large or small depending on the value of d, below, we distinguish several scenarios: $d \geq 2t-2$, $t \leq d < 2t-2$ and $d < t$.

1. If $d\geq 2t-2$, then $M\geq(3t/2-2)(t-1)=(t-1)^2+\binom{t-1}{2}$. Therefore, $M-(t-1)^2\geq\binom{t-1}{2}$. It means that the bound to the right in (8) increases with m_1, and its maximum value is achieved at $m_1=\binom{t-1}{2}$. Then,

$$f(d,m_1,m_2)=O\left(\left(\frac{x^2e2^{t-1}}{dn}\right)^d\left(\frac{(t-1)e}{(d-t+1)}\right)^{(t-1)(d-t+1)}p^{-M}\right)=$$
$$=O\left(e^{d[2\ln x+1-\ln d-\ln n+(t-1)(\ln 2-\ln(d-t+1)+\ln(t-1)+1-\ln p)]}\times\right.$$
$$\left.\times\ e^{(t/2)(t-1)\ln p+o(x)}\right).$$

Notice that $\ln(d-t+1)\geq\ln\frac{d}{t}$ (since $d\geq t$). So,

$$f(d,m_1,m_2)=O\left(e^{d\gamma(d)+(t/2)(t-1)\ln p+o(x)}\right),$$

where

$$\gamma(d)=2\ln x+1-t\ln d+(t-2)\ln\ln n+(t-1)(\ln t-\ln C+\ln(t-1)+1+\ln 2).$$

Notice that $[d\gamma(d)]'=\gamma(d)-t$ and γ decreases. Therefore, $d_0=\gamma^{-1}(t)$ is a point of global maximum of $d\gamma(d)$. Clearly,

$$d_0=\left(\frac{2(t-1)t}{C}\right)^{\frac{t-1}{t}}\left(x^2[\ln n]^{t-2}\right)^{\frac{1}{t}}=\left(\frac{2(t-1)t}{C}\right)^{\frac{t-1}{t}}x(1+o(1)).$$

Therefore, $d_0\gamma(d_0)=\left(\frac{2(t-1)t}{C}\right)^{\frac{t-1}{t}}xt(1+o(1))$. As $C>2^{\frac{2t-1}{t-1}}t(t-1)$, we get that, for $\varepsilon>0$ small enough, $f(d,m_1,m_2)\leq n^{-\varepsilon}$.

2. Let $t\leq d<2t-2$. Then $\left(\frac{(t-1)^2e}{(M-m_1)}\right)^{M-m_1}\leq e^{(t-1)^2}$. Therefore,

$$f(d,m_1,m_2)=O\left(\left(\frac{x^2}{n}\right)^d p^{-M}\right)\ =O\left(\left(\frac{x^2}{np^{t-1}}\right)^d p^{\frac{t(t-1)}{2}}\right)$$
$$=O\left(\frac{[\ln n]^{td}}{(n[\ln n]^{t-2})^{t/2}}\right)=O\left(\frac{1}{n}\right).$$

3. Finally, let us switch to the case $d<t$. Since $M-(t-1)^2<0$, we get that $\left(\frac{(t-1)^2e}{M-m_1}\right)^{M-m_1}$ is maximal when $m_1=0$. Therefore,

$$f(d,m_1,m_2)=O\left(\left(\frac{x^2}{n}\right)^d p^{-M}\right)=O\left(\left(\frac{x^2}{np^{\frac{d-1}{2}}}\right)^d\right)$$
$$=O\left(\left(\frac{x^2}{n^{1-\frac{d-1}{2(t-1)}}[\ln n]^{\frac{(t-2)(d-1)}{2(t-1)}}}\right)^d\right)=O\left(\frac{\ln^2 n}{n}\right).$$

Combining the above bounds with (5) and (6), we get, by Chebyshev's inequality, that, for n large enough,

$$\mathsf{P}(X=0)\leq\frac{\mathrm{Var}X}{(\mathsf{E}X)^2}\leq\sum_d\sum_{m_1}\sum_{m_2}f(d,m_1,m_2)\leq xz^2n^{-\varepsilon}=o(1).$$

Therefore, whp there exists a saturating structure of size x.

3.3.2. Saturating structure of size $y = \frac{t^2}{p}$

First, we divide the random graph into two parts:

$$G_1 = G(n,p)|_{[\lfloor n/2 \rfloor]}, \quad G_2 = G(n,p)|_{[n]\setminus[\lfloor n/2 \rfloor]}.$$

Let F_0 be a saturating structure of size $x = \lfloor \ln(\lfloor n/2 \rfloor) \rfloor$ in G_1 (if exists). Let A_0 be the event that F_0 exists.

Let us enlarge the saturating structure by induction. Suppose that $\hat{C}$ is a constant greater than 1 (it will be defined later). Take $\delta > 0$ such that $(1-\delta)\hat{C} > 1$. Let, for every $i \in \{0, 1, \ldots, \ell\}$,

$$C_i^- = ((1-\delta)\hat{C})^{\frac{(t-1)^i - 1}{t-2}},$$

Let $U_1' \subset V(G_2)$ be a set of vertices connected to at least $t-1$ vertices of F_0 and let U_1 comprise arbitrary $\min(|U_1'|, \lceil C_1^- x \rceil)$ vertices of U_1'. For $i = 1, 2, \ldots$, let $U_{i+1}' \subset V(G_2) \setminus (U_1 \sqcup \ldots \sqcup U_i)$ be the set of vertices connected to at least $t-1$ vertices of U_i, and let U_{i+1} be composed of arbitrary $\min(|U_{i+1}'|, \lceil C_{i+1}^- x \rceil)$ vertices of U_{i+1}'. If, for some ℓ, we get $|V(F_0) \sqcup U_1 \sqcup \ldots \sqcup U_\ell| \geq y$, then we immediately get a saturating structure of size at least y. We will define the desired ℓ later. But we assume for a while that some value of ℓ is fixed.

Set $X_i = |U_i'|$ for $i \in [\ell]$. Notice that the appearances of the edges between G_2 and F_0 do not depend on the choice of F_0 and have independent Bernoulli distributions. Therefore,

$$\mathsf{E}(X_1 \mid A_0) = \left\lceil \frac{n}{2} \right\rceil P, \quad \mathrm{Var}(X_1 \mid A_0) = \left\lceil \frac{n}{2} \right\rceil P(1-P),$$

where

$$P = \sum_{k=t-1}^{x} \binom{x}{k} p^k (1-p)^{x-k}.$$

We get

$$\mathsf{E}(X_1 \mid A_0) \sim \frac{n}{2} \binom{x}{t-1} p^{t-1} (1-p)^{x-t+1} \sim \frac{n}{2} \frac{x^{t-1}}{(t-1)!} p^{t-1} = \frac{C^{t-1}}{2(t-1)!} x$$

As $C > (2(t-1)!)^{\frac{1}{t-1}}$, we get that $\hat{C} := \frac{C^{t-1}}{2(t-1)!} > 1$. Also,

$$\frac{\mathrm{Var}(X_1 \mid A_0)}{(\mathsf{E}(X_1 \mid A_0))^2} \sim \frac{1}{\mathsf{E}(X_1 \mid A_0)} \sim \frac{1}{\hat{C}x}.$$

By Chebyshev's inequality, for every $\delta > 0$,

$$\mathsf{P}(|X_1 - \hat{C}x| > \delta \hat{C}x \mid A_0) \leq (1 + o(1)) \frac{1}{\delta^2 \hat{C}x}. \tag{9}$$

Notice that $C_1^- = (1-\delta)\hat{C}$. Therefore (recall how U_i is constructed),

$$\mathsf{P}(|U_1| < \lceil C_1^- x \rceil \mid A_0) = \mathsf{P}(|U_1| < C_1^- x \mid A_0) \leq (1 + o(1)) \frac{1}{\delta^2 \hat{C}x}.$$

Let, for every $i \in \{1, \ldots, \ell\}$,

$$A_i = A_{i-1} \cap \{|U_i| = \lceil C_i^- x \rceil\}.$$

Now let us define ℓ. Let $z = \frac{y}{x} \sim \frac{t^2}{p \ln n}$. Let

$$\ell = \left\lceil \log_{t-1} \left[\left(\frac{\ln z}{\ln((1-\delta)\hat{C})} \right) (t-2) + 1 \right] \right\rceil.$$

Due to our choice of ℓ,

$$\frac{\ln z}{\ln((1-\delta)\hat{C})} \leq \frac{(t-1)^{\ell}-1}{t-2},$$

which implies $C_{\ell}^{-} \geq z$ and $C_{\ell-1}^{-} < z$ and $C_i^{-} = o(z)$ for $i < \ell - 1$.

Then $C_{\ell}^{-}x \geq y$, $C_{\ell-1}^{-}x \leq y$ and $C_i^{-}x = o(y)$, $i \in [\ell-2]$.

Let us prove that

$$\mathsf{P}(C_i^{-}x \leq X_i | A_{i-1}) \geq 1 - (1+o(1))\frac{1-\delta}{\delta^2 C_i^{-} x}, \quad (10)$$

uniformly over all $i \in [\ell - 1]$ and that

$$\mathsf{P}(y \leq X_{\ell} | A_{\ell-1}) \geq 1 - (1+o(1))\frac{1-\delta}{\delta^2 y}, \quad (11)$$

As seen above, it is true for $i = 1$.

Now suppose that $i \in [\ell - 1]$ and (10) holds. Notice that the appearances of the edges between $V(G_2) \setminus \bigcup_{j=1}^{i} U_j$ and U_i still have independent Bernoulli distributions.

Let $a := |U_i| = \lceil C_i^{-}x \rceil$ on A_i.

Since, on A_i, almost all vertices of $V(G_2)$ may be included in U_{i+1} (i.e., $|U_1 \sqcup \ldots \sqcup U_i| = (1+o(1))\ln n(C_1^{-} + \cdots + C_i^{-}) = o(n)$ as $i \leq \ell - 1$) we get

$$\mathsf{E}(X_{i+1}|A_i) \sim \frac{n}{2}\sum_{k=t-1}^{a}\binom{a}{k}p^k(1-p)^{a-k} \geq (1+o(1))\frac{n}{2}\frac{a^{t-1}}{(t-1)!}p^{t-1}e^{-ap};$$

$$\frac{\mathsf{Var}(X_{i+1}|A_i)}{(\mathsf{E}(X_{i+1}|A_i))^2} \leq \frac{1}{\mathsf{E}(X_{i+1}|A_i)}.$$

Notice that in this case

$$\mathsf{E}(X_{i+1}|A_i) \geq (1+o(1))\frac{n}{2}\frac{(C_i^{-}x)^{t-1}}{(t-1)!}p^{t-1}e^{-ap} = (1+o(1))\frac{C_{i+1}^{-}}{1-\delta}xe^{-ap}.$$

Notice that for $i < \ell - 1$, it equals $(1+o(1))\frac{C_{i+1}^{-}}{1-\delta}x$.

Therefore, if $i < \ell - 1$, by Chebyshev's inequality,

$$\mathsf{P}\left(X_{i+1} < C_{i+1}^{-}x \middle| A_i\right) \leq \mathsf{P}\left(|X_{i+1} - \mathsf{E}(X_{i+1}|A_i)| > \delta\mathsf{E}(X_{i+1}|A_i) \middle| A_i\right)$$
$$\leq \frac{\mathsf{Var}(X_{i+1}|A_i)}{(\delta\mathsf{E}(X_{i+1}|A_i))^2} \leq \frac{(1+o(1))(1-\delta)}{\delta^2 C_{i+1}^{-}x} \to 0, \quad n \to \infty.$$

This finishes the proof of (10).

Now let us look at the case of $i = \ell - 1$. Let $\overline{C}$ be such that $(1-\delta)\overline{C}^{t-1}\hat{C}x = y$. Notice that $C_{\ell-1}^{-} \geq \overline{C}$, since $(1-\delta)(C_{\ell-1}^{-})^{t-1}\hat{C}x = C_{\ell}^{-}x \geq y$. Let W be a subset of $\lceil \overline{C}x \rceil$ vertices of $U_{\ell-1}$ (we suppose now that $A_{\ell-1}$ holds). Therefore, U_{ℓ}' contains at least all the vertices of $V(G_2) \setminus (U_1 \sqcup \ldots U_{\ell-1})$ that are connected to at least $t-1$ vertices of W. Also, $|W| = \lceil \overline{C}x \rceil = o(\frac{1}{p})$, as $\overline{C}^{t-1}x^{t-1}p^{t-1} = O(\frac{1}{np}) \to 0$. Therefore,

$$\mathsf{E}(X_{\ell}|A_{\ell-1}) \geq \frac{n}{2}\sum_{k=t-1}^{|W|}\binom{|W|}{k}p^k(1-p)^{|W|-k} \geq (1+o(1))\overline{C}^{t-1}\hat{C}x = (1+o(1))\frac{y}{1-\delta};$$

$$\frac{\mathsf{Var}(X_{\ell}|A_{\ell-1})}{(\mathsf{E}(X_{\ell}|A_{\ell-1}))^2} \leq \frac{1}{\mathsf{E}(X_{\ell}|A_{\ell-1})}.$$

So, by Chebyshev's inequality

$$\mathsf{P}\left(X_\ell < y \middle| A_{\ell-1}\right) \le \mathsf{P}\left(|X_\ell - \mathsf{E}(X_\ell|A_i)| > \delta\mathsf{E}(X_\ell|A_i) \middle| A_i\right)$$
$$\le \frac{\mathrm{Var}(X_\ell|A_i)}{(\delta\mathsf{E}(X_\ell|A_i))^2} \le \frac{(1+o(1))(1-\delta)}{\delta^2 y} \to 0, \quad n\to\infty,$$

which proves (11).

Notice that (9), (10) and (11) imply

$$\mathsf{P}(\neg A_\ell) \le o(1) + \sum_{i=1}^{\ell} \mathsf{P}(C_i^- x \le X_i | A_{i-1})$$
$$\le o(1) + \sum_{i=1}^{\ell-1} \frac{(1+o(1))(1-\delta)}{\delta^2 C_i^- x} + \frac{(1+o(1))(1-\delta)}{\delta^2 y} \to 0, \quad n\to\infty.$$

Here, we used the fact that $(1-\delta)\hat{C} > 1$.

So, whp there exists a saturating structure of size y.

3.3.3. *Saturating structure of size n*

In this section, we prove that, for every proper $S \subset [n]$ of size at least $y = \frac{t^2}{p}$, in $G(n,p)$ there exists a vertex outside S such that it is connected to at least $t-1$ vertices of S. Clearly, this observation finishes the proof of Claim 5.2.

For a fixed set S of size $z \ge y$, the probability that some fixed vertex outside S has no more than $t-2$ neighbours is

$$\sum_{k=0}^{t-2}\binom{z}{k}p^k(1-p)^{z-k} \le (1-p)^z + \sum_{k=1}^{t-2}\left(\frac{zep}{(1-p)k}\right)^k e^{-pz}$$
$$\le (t-1)\exp\left[(t-2)\ln\left(\frac{zep}{(1-p)(t-2)}\right) - zp\right]$$

since the function $(t-2)\ln\left(\frac{zep}{(1-p)(t-2)}\right)$ increases in $x \in (0,t)$.

By the union bound, the probability that there exists a set $S \subset [n]$ of size $z \in [y, n/\ln n]$ such that every vertex outside S has less than $t-1$ neighbours in S is at most

$$\sum_{z=y}^{\lfloor n/\ln n\rfloor}\binom{n}{z}\left((t-1)\exp\left[(t-2)\ln\left(\frac{zep}{(1-p)(t-2)}\right) - zp\right]\right)^{n-z} \le$$
$$\le \sum_{z=y}^{\lfloor n/\ln n\rfloor} \exp\left[z(\ln n + 1 - \ln z) - \Omega(n)\right] = n\exp[-\Omega(n)] \to 0, \quad n\to\infty,$$

since $(t-1)\exp\left[(t-2)\ln\left(\frac{zep}{(1-p)(t-2)}\right) - zp\right]$ is bounded away from 1. Indeed, $(t-2)\ln(zp) - zp$ decreases as zp increases, so its maximum is achieved at $zp = t^2$. Thus the expression under the exponent is at most $(t-2)[\ln(t^2/(t-2)) - t^2/(t-2)] - o(1) < -t^2/2 - o(1)$.

Finally, the probability that there exists a proper $S \subset [n]$ of size $z > n/\ln n$ such that every vertex outside S has less than $t-1$ neighbours in S is at most

$$\sum_{z=\lceil n/\ln n\rceil}^{n-1}\binom{n}{n-z}\left((t-1)\exp\left[(t-2)\ln\left(\frac{zep}{(1-p)(t-2)}\right) - zp\right]\right)^{n-z} \le$$
$$\le \sum_{z=\lceil n/\ln n\rceil}^{n-1} \exp\left[(n-z)\left(\ln n - zp(1+o(1))\right)\right] \to 0, \quad n\to\infty.$$

Claim 5.2 and Theorem 2 follows.

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