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The attack-and-defense conflict with the gun-and-butter dilemma

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Abstract

We analyze a general equilibrium model of attack and defense with production. One attacker and one defender allocate fixed endowments between producing butter and guns. We characterize the unique interior and unique corner equilibrium, and find that (i) the defenders may spend more resources on conflict than the attacker even without loss aversion or other preferential bias, (ii) the attackers may expend all their resources only in conflict, and (iii) the interior and the corner equilibria cannot coexist. These results may help explain Ukraine's sustained high defense effort and Russia's militarized economy, or the excessive conflict expenditure by Hamas etc.

KEYWORDS
attack and defense, conflict, contest, gun and butter, production

JEL CLASSIFICATION
C72, D74, D23, Q34

1 | INTRODUCTION

The fundamental tension between resource allocation for production versus conflict has long been a central theme in economic thought. Hirshleifer (2000) famously articulated this dichotomy, stating that "There are two main ways of making a living: by production or by conflict. Consequently, two distinct technologies must be distinguished: the familiar technology of production and exchange on one hand, and the technology of conflict and struggle on the other." In many such conflicts, people with limited resources allocate it between consumption (butter) and conflict (gun) (Samuelson, 1948). These choices carry profound real-world consequences for consumption and other productive activities. For instance, in 2022, Middle Eastern and North African countries allocated 4.6% of their GDP to defense, exceeding the global average of 2.3%, while spending only 3.8% on education, below the global average of 4.5%. Similarly, the net opportunity cost of conflict with Pakistan for India is estimated to be 2.5% of its GNP (Navaz & Guruswamy, 2014). Whereas the cost of the first intifada is estimated to be about $2000 per capita per year for Israel

Abbreviations: CSF, contest success function; FARC, Fuerzas Armadas Revolucionarias de Colombia.
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(Horiuchi & Mayerson, 2015). Such opportunity cost inherent in conflict is also observed in sub-Saharan Africa, where the incidence of civil war has a negative effect on the size of the manufacturing sector (Caruso, 2010).

In the absence of strict property rights, which is often the case for disputed territories or in many developing countries, the means of conflict becomes important. The expenditure on conflict is usually sunk and unproductive. However, even when one is not interested or capable of appropriating others' property, they have to engage in conflict in order to defend their own. This specific form of conflict is "attack and defense" in which one type of agent attacks, and the other type defends. As Mill (1848, p. 979) pointed out "[...] the energies now spent by mankind in injuring one another, or in protecting themselves against injury [...]," both attacking others and defending oneself entail opportunity cost of not producing consumption goods. Such situations include guerrilla war, siege, terrorism, malware, bank fraud etc. Hence, the issues of production versus conflict, and attack versus defense are very important as well as related.

Both also are popular separate research topics in economics, political science, and conflict studies. Many recent or ongoing conflicts (e.g., Maoist insurgency in India, conflict in Colombia, Russia-Ukraine war, and Israel-Hamas conflict etc.) present poignant and high-stakes illustrations of the attack-and-defense dynamic coupled with the gun-and-butter dilemma. Ukraine, as the defender, faces the challenge of allocating resources to military defense ("guns") while simultaneously attempting to maintain essential public services, reconstruct damaged infrastructure, and sustain its economy ("butter"). Russia, as the attacker, also grapples with similar trade-offs: it diverts significant portions of its national budget and productive capacity toward military objectives. Russia is projected to bear the greatest absolute cost of the war, estimated at \$1.69 trillion over 6 years, while Ukraine faces the most severe relative loss, estimated at 193% of its GDP, or \$386 billion over 6 years (Gorodnichenko & Vasudevan, 2025).

Similarly, the October 7, 2023, attack by Hamas involved coordinated rocket launches and ground incursions into Israeli territory, targeting both civilian and military sites. Israel's Iron Dome missile defense system intercepted many incoming rockets, exemplifying the defender's strategy of deploying resources to protect populated areas and critical infrastructure. In response, Israel conducted aerial campaigns and ground operations within the Gaza Strip to neutralize Hamas' capabilities (Council on Foreign Relations, 2025).

Finally, the literature on insurgency and counter-insurgency (Kilcullen, 2006) deals with situation in which one party (usually a state) has to allocate scarce resources between productive investment and counter-insurgency efforts, whereas another party does the same with insurgency effort. These real-world scenarios directly mirror the tension where both conflict versus production (gun-and-butter) and attack-and-defense dynamics are present. However, despite the extensive research on both areas separately, a notable scarcity exists in analyses that combine these two fields.[1] As a result, a microfoundation of such real-life scenarios is not available, neither is it possible to explain or predict outcomes in the field. In this study we aim to fill in this gap and contribute theoretically to both these areas of literature.

To do so, we consider a two-player contest with attack and defense in a general equilibrium structure. One attacker and one defender start with their own fixed endowments and can allocate it either to produce guns or to produce butter. The guns determine the winner of the conflict through a ratio form (Tullock, 1980) contest success function. If the Attacker wins then she appropriates the butter of the defender, otherwise they consume their own butter. We fully characterize the equilibria of this game, and show that unique interior solution exists, and unique corner solution exists. In both the equilibria, defenders may spend more on conflict than the attacker even without assuming loss aversion (unlike in the literature). In the corner solution, attackers decide only to attack and not to produce any butter. Comparative statics results show further non-intuitive results.

The insights from our model have significant implications for policymakers and security agencies that the existing models do not serve. A unique contribution of this study is its capacity to explain counter-intuitive outcomes, such as a defender allocating more resources to conflict than an attacker, without recourse to behavioral economics or complex network theory; or explanation of conditions under which an attacker produces no butter at all. This parsimonious explanation enhances the model's predictive power for complex and often puzzling conflict dynamics observed in real-world scenarios, including the Ukraine conflict. For example, the patterns of defense expenditure in Ukraine can be explained through the first result, whereas the budget and offensive expenditure of Hamas is explained through the latter result.

The rest of the paper is organized as follows. Section 2 covers the existing literature, Section 3 presents the model and outlines the results, and Section 4 concludes.

2 | LITERATURE REVIEW

Both the areas of gun-and-butter and attack-and-defense attracted adequate attention. Seminal studies such as Hirshleifer (1995), Skaperdas (1992) and the ones following these studies (e.g., Grossman & Kim, 1995; Neary, 1997; Noh, 2002) focus on the tension between consumable production (butter) versus resources in conflict (gun) under budget constraints.[2]

Other notable contributions in the game of production and conflict include Hausken (2005), Hafer (2006), and Kolmar (2008). Hausken (2005) compares the production and conflict model with rent seeking model and find the effects of group size in both. Hafer (2006) show theoretically the emergence of two specific type of agents (haves and have-nots) in a steady state equilibrium. Kolmar (2008) uses a model of sequential attack and defense in Tullock contest to find conditions for endogenously arising property rights. He finds that even when perfectly secure property rights emerge, the incentive to produce remains inefficient. Kornienko (2020) analyzes an n-player all-pay auction in which the reward is the residual of resources of all players after their bid. She finds condition under which the equilibrium payoff becomes identical to the one in single-object independent private value auctions. In these studies, however, the players are either not defined as attacker and defender, or they are defined in terms of their budget.

There is also a long literature on attack and defense. Bester and Konrad (2004) find the asymmetry between an attacker and a defender to be responsible for delay in contests. Clark and Konrad (2007) consider a multi-battle contest in which the attacker needs to win at most one battlefield, whereas the defender will have to win all the battlefields. Similar structures were used in Arce et al. (2012) and Kovenock and Roberson (2018). Bose and Konrad (2020) study multiple attackers attacking multiple defenders. When defense effort is observable, defenders compete between themselves to avoid becoming a weak target. If it is not, this competition disappears. Dziubiński and Goyal (2013) use a network model where a defender forms costly links among a number of nodes and invests on defending a subset of them, whereas an attacker tries to eliminate the nodes. They find that the defendable network can be either sparse or dense depending on whether the cost is small or large. Goyal and Vigier (2014) investigate contagion in a network with attack and defense. They find conditions under which some specific type of defense network emerges.

Deck and Sheremeta (2012) use a sequential series of all-pay auctions between two players on many battlefields. To win, the attacker needs to win one battlefield, but the defender will have to win all the battlefields. They show theoretically and experimentally that when the valuation of the winning is not high enough, the defender stops fighting, whereas it tries to fight all the battlefields in case the valuation is high. Chowdhury et al. (2018), instead, consider theoretically and experimentally a standard contest in which the defender owns the prize to begin with (attacker starts with nothing), and loses it to the attacker if s/he loses the contest. They find that defenders expend more effort than their attacker counterpart explain such behavior in terms of loss aversion. De Dreu et al. (2019), in a series of experiments, find similar results. They show two further psychological pathways through which the lack of attack can be explained. De Dreu et al. (2019) find that people with higher level of social preferences (such as empathy) and people who make cautious decisions attack less. However. Those qualities do not explain the defender behavior. See Kovenock et al. (2019) for further experimental evidence.

Attack and defense is also analyzed in group settings. Chowdhury and Topolyan (2016a, b) analyze group contests in which the attacker group has a best-shot impact function whereas the defender group has a weakest link impact function. Aloni and Sela (2012) consider pairwise individual contests between two groups. The attackers need to win one such contest, whereas the defenders will need to win all. Biologists and psychologists also consider the attack and defense structure, but mostly with a prisoner's dilemma structure. See, for example, De Dreu and Gross (2019) for a survey on various behavioral aspects of attack-and-defense. None of these studies, however, consider production while analyzing attack and defense.

None of these studies combine attack and defense with gun and butter. The only study that combines these two is the closest to ours is by Yektaş et al. (2009). Here the whole butter of the defender and a fraction of the butter of the attacker constitute the prize. They focus on the effect of that fraction (when it is 1versus when it is <1), and the difference in production function on the behavior of the players. They make several assumptions to achieve unique Nash equilibrium and find only interior equilibrium (attacker never expends the whole resource in producing guns). They also focus on the choice of being an attacker or a defender that we do not do, since many of the attackers and defenders in real life are exogenously and historically determined.

3 | MODEL

Consider a setting with 2 risk-neutral players: identified as the Attacker (A) and the Defender (D). Each Player $i(=A,D)$ has their own given endowment $E_i > 0$ that they can allocate either to create gun (i.e., conflict effort: x_i) or to produce butter (i.e., consumption good: y_i). The conversion from endowment to consumption goods is assumed to be a direct one-to-one, whereas the constant marginal cost of conflict is $c_i > 0$ for Player i (and hence, $y_i \leq E_i - c_i x_i$). Any endowment that is not allocated to produce either guns or butter is wasted. The utility or payoff of each player depends directly on the amount of butter they consume. The conflict effort in itself does not provide any utility to the players, the final payoff depends only on their own consumption.

The players can be asymmetric in terms of their level of endowment (E_i) as well as in terms of their conflict cost (c_i). Nonetheless, the main difference between the players arises from their nature in the conflict: Player A attacks and Player D defends. If the attacker wins, then s/he appropriates the defender's butter (along with their own). However, if the defender wins or there is no conflict, then each player consumes only the butter they produce on their own. One can view the attacker and the defender as a terrorist organization and a defending government, or an attacking country and a country defending its land from the attacker etc.

The winner of the conflict is determined by the amount of guns produced by each player with a ratio form (Tullock, 1980) contest success function.[3] In particular, when at least one of the players decides to produce guns, then the probability that the defender wins is given by $p_D = x_D/(x_A + x_D)$ if $(x_A, x_D) \neq (0,0)$. However, if no player decides to produce guns, then by default the players consume their own butter; or in other words, $p_D = 1, if$ $(x_A, x_D) = (0,0)$. The probability that the attacker wins is given always by $p_A = 1 - p_D$, that is, no stalemate is possible.

Hence, the payoff functions for the Attacker and the Defender, respectively, are:

$$u_A = \begin{cases} (E_D - c_D x_D)\dfrac{x_A}{x_A + x_D} + (E_A - c_A x_A) & if\ \ (x_A, x_D) \neq (0,0) \\ E_A & if\ \ (x_A, x_D) = (0,0) \end{cases}, \tag{1}$$

$$u_D = \begin{cases} (E_D - c_D x_D)\dfrac{x_D}{x_A + x_D} & if\ \ (x_A, x_D) \neq (0,0) \\ E_D & if\ \ (x_A, x_D) = (0,0) \end{cases}. \tag{2}$$

Equation (1) states that the attacker always consumes their own butter $(E_A - c_A x_A)$, and also consumes the butter of the defender $(E_D - c_D x_D)$ when they win with the probability p_A. Equation (2), on the other hand, states that the defender can consume only their own butter $(E_D - c_D x_D)$, only when they win with the probability p_D.

This attack-and-defense game is defined as $\Gamma(A, D; \Omega)$, where $\Omega = \{\mathrm{E_A}, E_D, c_a, c_D\}$ is the set of parameters. The objective function for Player $i(=A,D)$ in this attack-and-defense game Γ is:

$$\max_{x_i} u_i \text{ subject to the budget constraint } x_i \in [0, E_i/c_i], i = A, D. \tag{3}$$

We focus on equilibria in pure strategies. Whether equilibrium in mixed strategies exists and if so, its characterization, is a difficult problem that is beyond the scope of this paper.[4]

We show later in Table 1 that $(x_A, x_D) = (0,0)$ can never be an equilibrium. For the time being, provided that $(x_A, x_D) \neq (0,0)$, the players' marginal utilities are,

$$\frac{\partial u_A}{\partial x_A} = \frac{x_D(E_D - c_D x_D)}{(x_A + x_D)^2} - c_A, \tag{4}$$

$$\frac{\partial u_D}{\partial x_D} = \frac{x_A E_D - x_D c_D(2x_A + x_D)}{(x_A + x_D)^2}. \tag{5}$$

It is easy to note that $\frac{\partial u_i}{\partial x_i}$ is decreasing in x_i $(i = A, D)$, so each Player i's payoff function u_i is strictly concave in their own decision variable x_i. Consequently, Player i's best response is unique for any $x_j \neq 0, j \neq i$. Note that Player A's best

TABLE 1 Possible types of equilibria in $\Gamma(A, D; \Omega)$.

x_A^*	x_D^*	Note	Explanation
0	0	Not possible	A can increase payoff by exerting small $x_A^* = \varepsilon > 0$
0	E_D/c_D	Not possible	D can increase payoff by exerting $x_D^* = 0$
0	$(0, E_D/c_D)$	Not possible	If $x_A^* = 0$, then D wants to decrease x_D^* to 0
E_A/c_A	0	Not possible	A can increase payoff by exerting small $x_A^* = \varepsilon > 0$
E_A/c_A	E_D/c_D	Not possible	A can increase payoff by exerting $x_A^* \in (0, E_A)$
$(0, E_A/c_A)$	0	Not possible	A can increase payoff by lowering x_A^* even further to a smaller yet positive level
$(0, E_A/c_A)$	E_D/c_D	Not possible	A can increase payoff by exerting $x_A^* = 0$
$(0, E_A/c_A)$	$(0, E_D/c_D)$	Possible	Standard interior solution
E_A/c_A	$(0, E_D/c_D)$	Possible	A's low budget may make it feasible

response is undefined when $x_D = 0$, because the attacker always wants to deviate to an infinitesimally small yet positive effort. Player A's best response is zero when $x_D \geq \frac{E_D}{c_A + c_D}$ (this follows from the fact that $\frac{\partial u_A}{\partial x_A}$ is decreasing in x_A and $\frac{\partial u_A}{\partial x_A} \leq 0$ when $x_A = 0$ and $x_D \geq \frac{E_D}{c_A + c_D}$). Using a similar reasoning, we conclude that Player D's best response is zero when $x_A = 0$ and strictly positive when $x_A > 0$.

In what follows in our characterization of equilibria, we distinguish between interior equilibria, that is, those with $x_i \in (0, E_i/c_i)$ for each $i = A, D$; and corner equilibria, such that for some i, $x_i = 0$ or $x_i = E_i/c_i$. From Equations (3–5), 9 such possible cases can arise in which both the players either exert no effort, or the highest effort possible, or something in between. However, it is easy to show that many of such cases cannot be an equilibrium. Table 1 summarizes all the cases.

Hence, in the continuation, we study only the equilibria possible to attain. First, we investigate interior equilibria and find a unique equilibrium that is characterized in Proposition 1.

Proposition 1. *There exists a unique interior equilibrium characterized by* $x_A^* = \frac{E_D}{4c_D}\left(\frac{c_A + 4c_D}{\sqrt{c_A(c_A + 4c_D)}} - 1\right)$ *and* $x_D^* = \frac{E_D}{2c_D}\left(1 - \frac{\sqrt{c_A}}{\sqrt{c_A + 4c_D}}\right)$, *if and only if* $\frac{E_D}{4c_D}\left(\sqrt{1 + \frac{4c_D}{c_A}} - 1\right) < \frac{E_A}{c_A}$.

Proof. See Appendix A. ∎

From Equation (A3) in Appendix A, it is easy to show that as long as interior equilibrium exists, the defender's best response x_D is increasing in x_A, indicating strategic complementarity (Amir, 2005). However, the defender's best response x_D is, in general, non-monotone in x_A. Figure 1 depicts best responses and (interior) equilibrium when $E_A = 1.5$, $E_D = 1$, $c_A = c_D = 1$. Note that $(0, 0)$ is not an equilibrium since best responses are discontinuous at zero.

Note that the interior equilibrium exists only if the attacker has either a sufficiently large endowment relative to the defender, or a sufficiently small cost of conflict—when the endowment is not large enough. This matches with the field observations in which a fringe terrorist group, a small group of bandits (or a lone wolf), having a very small endowment compared to the defending government, often expend all their resources in conflict. However, a relatively larger terrorist group, such as the Maoists in India (Mahadevan, 2012) spend their resources both to arrange their own economic system, as well as to engage into conflict with the government. Similarly, the Fuerzas Armadas Revolucionarias de Colombia, while engaging in armed conflict, also maintained significant economic operations including drug production and taxation, representing a blend of "gun" and "butter" activities within their operational model in Colombia (Rubiano, 2021).

Note also that the endowment of the attacker, E_A does not enter into the equilibrium conflict allocation of either the attacker or the defender. This is because it is a fixed amount that is not contested, whereas due to the conflict over the residual of E_D and x_D, E_D enters the calculation. As expected, the larger the defender's endowment that is redistributed in conflict, the higher both the attacker's and the defender's efforts in conflict. We provide some further results derived from this proposition in Corollary 1.

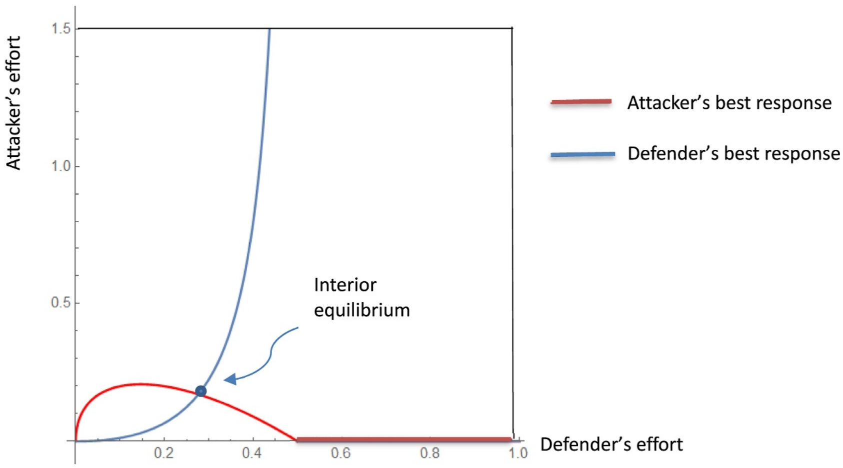

FIGURE 1 Best responses and the interior equilibrium.

Corollary 1. *Consider the unique interior equilibrium described by Proposition 1.*

(i) The defender exerts more effort than the attacker $(x_D > x_A)$ *if and only if* $c_D < (3/4)c_A$.
(ii) Each player's effort $(x_i, i = A, D)$ *is decreasing in their own marginal cost of effort* (c_i)*, as well as their rival's marginal cost of effort* $(c_j, j \neq i)$.
(iii) The probability of each player's success $(p_i, i = A, D)$ *is decreasing in their own marginal cost of effort* (c_i) *and increasing in the rival's marginal cost of effort* $(c_j, j \neq i)$.
(iv) The payoff of the attacker is increasing in their own endowment but increasing in defender's endowment only if $2c_D > c_A$. *The payoff of the defender is increasing in their own endowment, but independent of the attacker's endowment.*

Proof. See Appendix A. ■

The first result in the corollary is of great interest. This shows that it is possible for the defender to exert even more effort than the attacker if the defender has enough cost advantage. In this case the defender can produce an adequate amount of butter for consumption, but at the same time can also produce enough guns to protect their butter. This adds to more intuition and arguably provides a more generic support of those observations from the field where defenders indeed exert more conflict effort. When the defender has sufficiently high volume of endowment compared to the attacker (e.g., the US Govt. vs. Al-Qaeda) then it can easily spend more on guns than the attacker. However, we show that even otherwise a cost advantage can give the same result. Earlier literature in Attack and Defense (without production) has derived such result either due to network externalities (Clark & Konrad, 2007) or due to the loss aversion of the defender (Chowdhury et al., 2018). We show here that while optimizing between consumption and conflict, defender may be more conflictual than the attacker even without loss aversion or externalities. When the attacker has more endowment than the defender, then the "weaker" defender wins with higher probability.

The second result shows that an increase in either own conflict cost or opponent's conflict cost reduces own conflict effort. While the effect of own conflict cost is intuitive, the effect of opponent's cost is counterintuitive, especially in the context of standard contest models (e.g., Baik, 2004). This is because unlike standard contests with no production, an increase in the opponent's cost of conflict provides higher incentives for the opponent to produce more butter. This prompts the player concerned also to exert less conflict effort and produce more butter themselves. The third result shows that own as well as opponent marginal cost has a monotonic (although) opposite effect on winning. Unfortunately, though, since own costly effort and own probability of winning get opposite effect from marginal costs, the overall effect on payoff is ambiguous.

The third result is quite uncomplicated. However, the fourth result is one of the most interesting ones. It shows that an increase in own endowment always increases own payoff—a result similar to standard contests. However, due to the production and consumption tradeoff in the game and increase in the defender's endowment increases attacker's payoff only when they have a relative cost advantage. Moreover, due to the attack and defense aspect of the game, the endowment of the attacker does not affect the payoff of the defender.

We now aim to investigate the possibility of a corner equilibrium. We find that a unique corner solution exists in which the attacker spends all the endowment for conflict and the defender allocates only a part of its endowment for conflict. We also find that the interior and the corner equilibria do not coexist. This is summarized in the next proposition.

Proposition 2. *There exists a unique corner equilibrium characterized by* $x_A^* = \frac{E_A}{c_A}$ *and* $x_D^* = -\frac{E_A}{c_A} + \sqrt{\frac{E_A^2}{c_A^2} + \frac{E_A E_D}{c_A c_D}}$, *if and only if* $\frac{E_D}{4c_D}\left(\sqrt{1 + \frac{4c_D}{c_A}} - 1\right) \geq \frac{E_A}{c_A}$ *and* $E_A \leq \frac{E_D\sqrt{c_A}}{2c_D}\left(\frac{c_A + 2c_D}{\sqrt{c_A + 4c_D}} - \sqrt{c_A}\right)$.

Proof. See Appendix A. ■

This result shows that when the attacker is constrained with a very low endowment, then they decide not to produce butter at all and allocate the whole endowment in conflict. As discussed earlier, it reflects the situations such as group of bandits or a mercenary who live off conflict, or a hacker who earns a living by hacking—in contrast to an organization that engages in productive work as well as hacking. The defender, however, expends only a part of the endowment for conflict. Interestingly, when the endowment of the defender increases, they increase producing gun (and butter), but it cannot affect the decision of the attacker who is already constrained. However, unlike the interior equilibrium, if the endowment of the attacker increases up to the relevant range, it triggers the defender to spend more resources to produce gun and defend their butter.

Figure 2 depicts best responses and the corner equilibrium when $E_A = 1, E_D = 8, c_A = c_D = 1$. As before, note that $(0, 0)$ is not an equilibrium since best responses are discontinuous at zero.

This result—that as long as the attacker has comparative advantage in conflict and the potential award is high, he specializes in producing guns—is straightforward, intuitive, and connects to the existing result in the literature without production. For example, with a similar result Konrad and Skaperdas (2012) explain how professional bandits may emerge in society. It is interesting, however, to further investigate which player exerts higher conflict effort, and how the cost structure affects their decisions. These are included in the following corollary.

Corollary 2. *Consider the unique corner equilibrium described by Proposition 2.*

(i) *The defender exerts more effort than the attacker if and only if* $(E_D/c_D) > 3(E_A/c_A)$.
(ii) *Each player's effort is decreasing in their marginal cost of effort. Moreover, the defender's effort is decreasing in the attacker's marginal cost.*
(iii) *The probability of each player's success is decreasing in their own marginal cost of effort and increasing in the opponent's marginal cost of effort.*

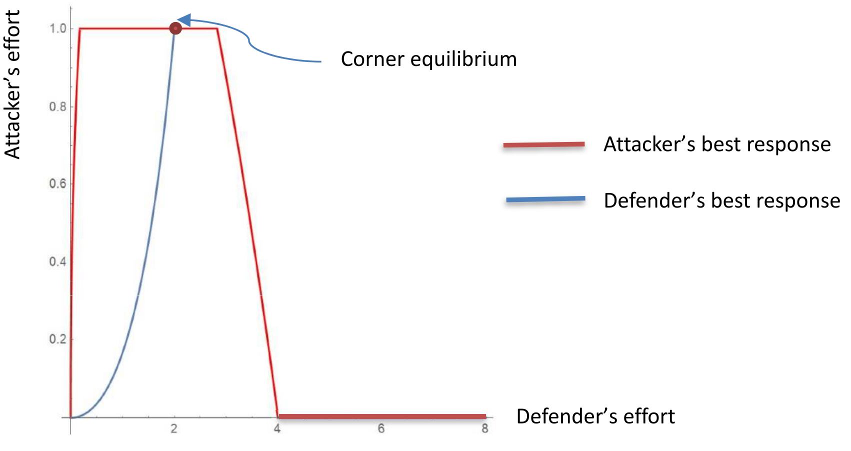

FIGURE 2 Best responses and the corner equilibrium.

(iv) *Each player's payoff is decreasing in own marginal cost of effort and increasing in the opponent's marginal cost of effort.*

Proof. See Appendix A. ■

Note that when the attacker exerts their whole budget in conflict, then the defender exerts more conflict effort than the attacker only if the defender's budget (normalized by cost) is very large (at least three times the normalized budget of the attacker). Once again, this result shows that it is possible for the defender to be more conflictive even without loss aversion or network effects.

It is interesting that even when the corner equilibrium arises and the attacker's effort is restricted by her low endowment, it is possible that the attacker exerts more effort than the defender, as the following example displays. This also shows that the result in part (i) of Corollary 2 is non-trivial.

Example 1. Suppose $E_A = 1$, $E_D = 9.5$, $c_A = 1$, and $c_D = 4$, It is easy to check that the condition for the corner equilibrium given in Proposition 2 is satisfied. At the same time, calculations show that $x_A^* = 1$ while $x_D^* = 0.83$.

We now summarize the results in the following theorem.

Theorem. *In the attacker-defender game* $\Gamma(A, D; \Omega)$, *there exists a unique interior equilibrium characterized by* $x_A^* = \frac{E_D}{4c_D}\left(\frac{c_A + 4c_D}{\sqrt{c_A(c_A + 4c_D)}} - 1\right)$ *and* $x_D^* = \frac{E_D}{2c_D}\left(1 - \frac{\sqrt{c_A}}{\sqrt{c_A + 4c_D}}\right)$, *if and only if* $\frac{E_D}{4c_D}\left(\sqrt{1 + \frac{4c_D}{c_A}} - 1\right) < \frac{E_A}{c_A}$; *and a unique corner equilibrium characterized by* $x_A^* = \frac{E_A}{c_A}$ *and* $x_D^* = -\frac{E_A}{c_A} + \sqrt{\frac{E_A^2}{c_A^2} + \frac{E_A E_D}{c_A c_D}}$, *if and only if* $\frac{E_D}{4c_D}\left(\sqrt{1 + \frac{4c_D}{c_A}} - 1\right) \geq \frac{E_A}{c_A}$ *and* $E_A \leq \frac{E_D\sqrt{c_A}}{2c_D}\left(\frac{c_A + 2c_D}{\sqrt{c_A + 4c_D}} - \sqrt{c_A}\right)$.

Proof. Comes directly from Table 1, Proposition 1, and Proposition 2. ■

It can be shown easily that the conditions required for the existence of the interior equilibrium, and the corner equilibrium cannot simultaneously hold. This is because the concavity of the payoff functions ensures the existence and the uniqueness of a pure strategy Nash equilibrium, regardless of it occurs interiorly or at the boundary. As a result, interior and corner equilibria do not coexist. At the same time, it is possible to find a range of parameters for which neither interior nor corner equilibria exist, as the following example shows.

Example 2. Suppose $E_A = 1$ and $E_D = 8$. Figure 1 depicts the regions in (c_A, c_D) plane where a unique interior and a unique corner equilibrium exists. Note that the two regions do not overlap and for some parameter values, no equilibrium exists (green regions in Figure 3).

4 | DISCUSSION

We analyze a two-player attack and defense model with production. Both the attacker and the defender can allocate their endowments between production and conflict. The main difference of this model with the existing literature comes from the outcome of the conflict. In the status quo, or in case the defender wins, both players consume their own production of butter. However, the attacker appropriates all the butter produced in the economy if they win. We show that there are two types of equilibria. If the attacker is highly budget constrained, then they spend the whole endowment in conflict. In another case, both the attacker and the defender allocate a part of the endowment to produce butter. We find that depending on the cost of conflict, the defender may be more conflict prone than the attacker even without loss aversion or network externalities.

These results contribute both to the gun and butter and the attack and defense literature. Production with conflict (gun and butter) literature rarely consider the asymmetry in the objectives of an attacker and a defender. It also abstracts away from the difference in outcome of such a conflict. This paper fills in that gap and results from our model

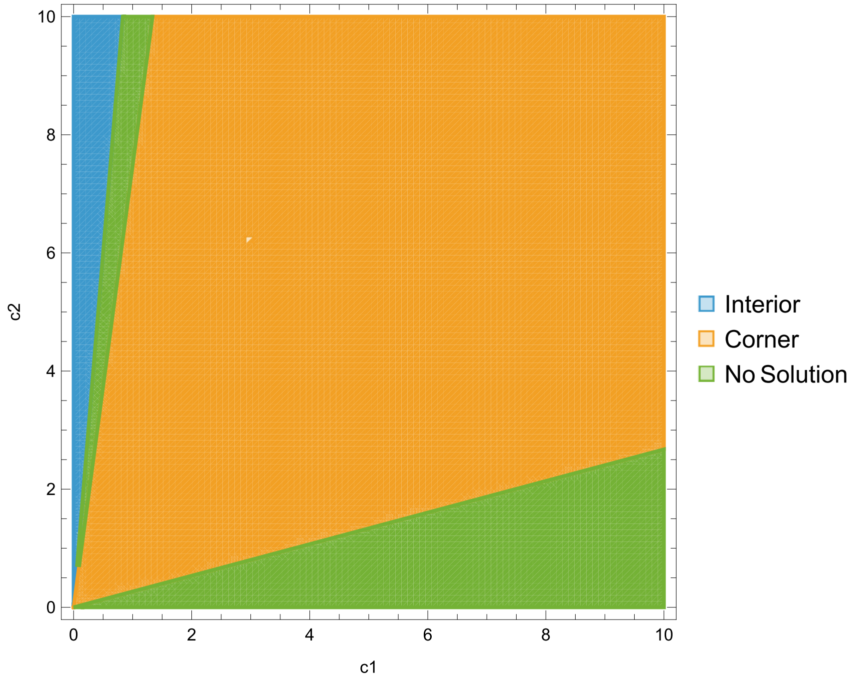

FIGURE 3 Existence of interior and corner equilibria.

reflect real life conflict situations that cannot be explained through the lens of the existing literature. Below we provide some examples.

The finding in Corollary 1(i), that the defender may exert more effort than the attacker due to a cost advantage, offers a unique perspective on the ongoing war in Ukraine. Despite Russia's initially overwhelming military might (a larger *endowment* in traditional terms), Ukraine's fierce resistance and the significant allocation of its resources to defense, often surpassing initial expectations, could be interpreted through this lens. In 2024, Ukraine's military spending reached an astonishing \$64.7 billion, representing 34.0% of its GDP (SIPRI, 2025). This figure marks a dramatic increase from its pre-war average of 2.7% between 2017 and 2020 (Global Data, 2025). In contrast, Russia's military spending in 2024, while higher in absolute terms at \$149 billion, constituted only 7.1% of its GDP (SIPRI, 2025). Despite Russia's significantly larger economy, Ukraine's substantially greater proportionate effort aligns with the model's prediction that a defender can exert more effort. One could also view that the Ukraine's efficient utilization of resources (perhaps due to strong international support), effectively translates into a "cost advantage" in terms of converting resources into its capability relative to the attacker's objectives. This contrasts with standard models that might predict a purely endowment-driven outcome.

Similarly, in asymmetrical conflicts such as the US government versus Al-Qaeda, while the US government clearly possesses a vastly larger endowment, its "cost advantage" in sophisticated defense technologies and intelligence infrastructure enables it to expend more resources on defense (e.g., intelligence gathering, counter-terrorism operations) compared to the attacker's total effort, while still maintaining a big economy. These are new results stemmed from our model that could not be derived from existing disconnected strands of literature on attack and defense, and on gun and butter.

The prediction that low-endowment attackers specialize in conflict (Proposition 2 and Corollary 2) also fits the situation in the Israel conflict and Hamas' revealed allocation choices. Open-source estimates put Hamas' operational/military budget around \$600 million per year are prioritized for rockets, training, and the subterranean network rather than civilian consumption (Monroe, 2023; Mourenza, 2024). In 2014, Hamas reportedly spent about \$90 million to build about 36 cross-border tunnels, and some analyses assess the broader tunnel network's cost at over \$1 billion (Spencer, 2024). Meanwhile, large tranches of civilian outlays for Gaza have come from external donors rather than from Hamas' own endowment (Monroe, 2023). This is consistent with a strategy in which the group channels scarce internal resources toward attack capacity while outsourcing "butter" to aid flows. By contrast, Israel, as a state, must always maintain substantial "butter" (public services) alongside "guns." Even during the war, its defense burden rose to about 8.8% of GDP in 2024 (Liang et al., 2025). This underscores the structural constraint that a defender cannot fully specialize in conflict without paying large opportunity costs on butter. Our results provides a microfoundation for such observation for the first time in literature. Similar examples of corner solution can be drawn for non-state actors such as the ISIS or Boko Haram, and cybercriminal group such as Conti or DarkSide. For the non-state actors, their endowment is predominantly acquired through direct conflict, territorial control, and illicit activities, rather than a conventional productive economy. For the cybercriminals, their endowment is exclusively derived from their attack activities, with no parallel butter production. Their production is the act of cyberattack itself.

The theoretical model also connects to the literature of psychology/neuroscience and economics experiments in attack and defense. As pointed out in Chowdhury (2019), economics literature focuses on theoretical benchmark to understand behavioral mechanisms, whereas neuroscientists use tools such as the fMRI for the same cause, but not necessarily with microfoundation. Moreover, whereas economics experiments on attack and defense do not consider production, psychology and neuroscience experiments do so without a rigorous theory. This paper can be considered as a bridge between these two distinct approaches and a source to provide a theoretical structure.

It is possible to extend this study both in theory and in applications. We mention some of such obvious future avenues. A natural extension would be to include more than two players. Group dynamics would be another important extension. Considering network externalities would make the structure more realistic. It will also be useful to include social preference aspects such as spite or retaliation. These aspects can also be investigated in a laboratory setting. This model also helps extending the empirical conflict literature. As mentioned earlier, the Ukraine conflict serves as a real-time empirical laboratory for validating and refining theoretical models of conflict economics. The extensive data emerging from this conflict will allow for a more refined understanding of the gun-and-butter dilemma and attacker-defender dynamics than previously attainable.

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CONFLICT OF INTEREST STATEMENT

The authors declare no conflicts of interest.

DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

ENDNOTES

[1] Neuroscience experiments investigate the neurocognitive and hormonal foundations of attack and defense, but do not consider production (De Dreu et al., 2019).

[2] See also Durham et al. (1998) for an experimental result.

[3] Another popular CSF that is implemented in the attack-and-defense is the All-pay auction (see e.g., Aloni & Sela, 2012; Chowdhury & Topolyan, 2016b). We choose the ratio form (Tullock, 1980) since it captures the stochastic nature of a conflict outcome and allows us to obtain closed form solution in interpretable pure strategies.

[4] As the prize value depends on the defender's choice, the results of Nti (1999) and Ewerhart (2015) do not apply.

[5] Note: $c_A + 4c_D = 2c_D + (c_A + c_A + 4c_D)/2$ and the geometric mean of two distinct (positive) numbers (in our case, $\sqrt{c_A(c_A + 4c_D)}$ is the geometric mean of c_A and $c_A + 4c_D$) is always strictly less than their arithmetic mean.

REFERENCES

Aloni, E. & Sela, A. (2012) The assortative matching scheme in a survival battle. *Economics Letters*, 117(1), 272–275. Available from: https://doi.org/10.1016/j.econlet.2012.05.038

Amir, R. (2005) Supermodularity and complementarity in economics: an elementary survey. *Southern Economic Journal*, 71(3), 636–660. Available from: https://doi.org/10.1002/j.2325-8012.2005.tb00664.x

Arce, D.G., Kovenock, D. & Roberson, B. (2012) Weakest-link attacker-defender games with multiple attack technologies. *Naval Research Logistics*, 59(6), 457–469. Available from: https://doi.org/10.1002/nav.21500

Baik, K.H. (2004) Two-player asymmetric contests with ratio-form contest success functions. *Economic Inquiry*, 42(4), 679–689. Available from: https://doi.org/10.1093/ei/cbh089

Bester, H. & Konrad, K.A. (2004) Delay in contests. *European Economic Review*, 48(5), 1169–1178. Available from: https://doi.org/10.1016/s0014-2921(03)00048-5

Bose, G. & Konrad, K.A. (2020) Devil take the hindmost: deflecting attacks to other defender. *Reliability Engineering & System Safety*, 204, 107205. Available from: https://doi.org/10.1016/j.ress.2020.107205

Caruso, R. (2010) Butter, guns, and ice cream: theory and evidence from Sub-Saharan Africa. *Defence and Peace Economics*, 21(3), 269–283. Available from: https://doi.org/10.1080/10242690903568975

Chowdhury, S.M. (2019) The attack and defense mechanisms–perspectives from behavioral economics and game theory. *Behavioral and Brain Sciences*, 42, e121. Available from: https://doi.org/10.1017/s0140525x19000815

Chowdhury, S.M. & Topolyan, I. (2016a) Best-shot versus weakest-link in political lobbying: an application of group all-pay auction. *Social Choice and Welfare*, 47(4), 959–971. Available from: https://doi.org/10.1007/s00355-016-1000-1

Chowdhury, S.M. & Topolyan, I. (2016b) The attack-and-defense group contests: best shot versus weakest link. *Economic Inquiry*, 54(1), 548–557. Available from: https://doi.org/10.1111/ecin.12246

Chowdhury, S.M., Jeon, J.Y. & Ramalingam, A. (2018) Property rights and loss aversion in contests. *Economic Inquiry*, 56(3), 1492–1511. Available from: https://doi.org/10.1111/ecin.12505

Clark, D.J. & Konrad, K.A. (2007) Asymmetric conflict: weakest link against best shot. *Journal of Conflict Resolution*, 51(3), 457–469.

Council on Foreign Relations. (2025) Israeli-Palestinian conflict, by the Center for Preventive Action. Available from: https://www.cfr.org/global-conflict-tracker/conflict/israeli-palestinian-conflict [Accessed 16th August 2025].

De Dreu, C.K. & Gross, J. (2019) Revisiting the form and function of conflict: neurobiological, psychological, and cultural mechanisms for attack and defense within and between groups. *Behavioral and Brain Sciences*, 42, e116. Available from: https://doi.org/10.1017/s0140525x18002170

De Dreu, C.K., Giacomantonio, M., Giffin, M.R. & Vecchiato, G. (2019) Psychological constraints on aggressive predation in economic contests. *Journal of Experimental Psychology: General*, 148(10), 1767–1781. Available from: https://doi.org/10.1037/xge0000531

Deck, C. & Sheremeta, R.M. (2012) Fight or flight? Defending against sequential attacks in the game of siege. *Journal of Conflict Resolution*, 56(6), 1069–1088.

Durham, Y., Hirshleifer, J. & Smith, V.L. (1998) Do the rich get richer and the poor poorer? Experimental tests of a model of power. *The American Economic Review*, 88(4), 970–983.

Dziubiński, M. & Goyal, S. (2013) Network design and defence. *Games and Economic Behavior*, 79, 30–43. Available from: https://doi.org/10.1016/j.geb.2012.12.007

Ewerhart, C. (2015) Mixed equilibria in Tullock contests. *Economic Theory*, 60(1), 50–71. Available from: https://doi.org/10.1007/s00199-014-0835-x

Global Data. (2025) Ukraine defense budget as a percentage of GDP and per capita (2017–2026). Available from: https://www.globaldata.com/data-insights/macroeconomic/ukraine-defense-budget-as-a-percentage-of-gdp-and-per-capita/

Gorodnichenko, Y. & Vasudevan, V. (2025) The (projected) cost of Russian aggression. CEPR Vox EU Column. Available from: https://cepr.org/voxeu/columns/projected-cost-russian-aggression

Goyal, S. & Vigier, A. (2014) Attack, defence, and contagion in networks. *The Review of Economic Studies*, 81(4), 1518–1542. Available from: https://doi.org/10.1093/restud/rdu013

Grossman, H.I. & Kim, M. (1995) Swords or plowshares? A theory of the security of claims to property. *Journal of Political Economy*, 103(6), 1275–1288. Available from: https://doi.org/10.1086/601453

Hafer, C. (2006) On the origins of property rights: conflict and production in the state of nature. *The Review of Economic Studies*, 73(1), 119–143. Available from: https://doi.org/10.1111/j.1467-937x.2006.00371.x

Hausken, K. (2005) Production and conflict models versus rent-seeking models. *Public Choice*, 123(1–2), 59–93. Available from: https://doi.org/10.1007/s11127-005-1717-3

Hirshleifer, J. (1995) Theorizing about conflict. *Handbook of Defense Economics*, 1, 165–189.

Hirshleifer, J. (2000) The macrotechnology of conflict. *Journal of Conflict Resolution*, 44(6), 773–792. Available from: https://doi.org/10.1177/0022002700044006004

Horiuchi, Y. & Mayerson, A. (2015) The opportunity cost of conflict: statistically comparing Israel and synthetic Israel. *Political Science Research and Methods*, 3(3), 609–618. Available from: https://doi.org/10.1017/psrm.2014.47

Kilcullen, D. (2006) Counter-insurgency redux. *Survival*, 48(4), 111–130. Available from: https://doi.org/10.1080/00396330601062790

Kolmar, M. (2008) Perfectly secure property rights and production inefficiencies in Tullock contests. *Southern Economic Journal*, 75(2), 441–456. Available from: https://doi.org/10.1002/j.2325-8012.2008.tb00913.x

Konrad, K. & Skaperdas, S. (2012) The market for protection and the origin of the state. *Economic Theory*, 50(2), 417–443. Available from: https://doi.org/10.1007/s00199-010-0570-x

Kornienko, T. (2020) *Expropriation auctions: predation contests and wallet games*. Mimeo. Working paper at the University of Edinburgh. Available from https://www.research.ed.ac.uk/en/publications/expropriation-auctions-predation-contests-and-wallet-games

Kovenock, D. & Roberson, B. (2018) The optimal defense of networks of targets. *Economic Inquiry*, 56(4), 2195–2211. Available from: https://doi.org/10.1111/ecin.12565

Kovenock, D., Roberson, B. & Sheremeta, R.M. (2019) The attack and defense of weakest-link networks. *Public Choice*, 179(3-4), 175–194. Available from: https://doi.org/10.1007/s11127-018-0618-1

Liang, X., Tian, N., Da Silva, D.L., Carazzato, L., Karim, Z. & Ricard, J.G. (2025) *Trends in world military expenditure, 2024*. SIPRI Fact Sheet. Available from: https://www.sipri.org/sites/default/files/2025-04/2504_fs_milex_2024.pdf

Mahadevan, P. (2012) The Maoist insurgency in India: between crime and revolution. *Small Wars and Insurgencies*, 23(2), 203–220. Available from: https://doi.org/10.1080/09592318.2012.642204

Mill, J.S. (1848) *Principles of political economy*. Oxford, UK: Oxford University Press.

Monroe, B. (2023) *Unraveling a complex web: a primer on Hamas funding sources, Iranian support, global connections, and compliance concerns, considerations*. Association of Certified Financial Crime Specialists. Available from: https://www.acfcs.org/unraveling-a-complex-web-a-primer-on-hamas-funding-sources-iranian-support-global-connections-and-compliance-concerns-considerations

Mourenza, A. (2024) *Hamas Ltd: the financial muscle of the Palestinian Islamist militia*. El Pais English. Available from: https://english.elpais.com/international/2024-01-09/hamas-ltd-the-financial-muscle-of-the-palestinian-islamist-militia.html

Navaz, S. & Guruswamy, M. (2014) *India and Pakistan: The opportunity cost of conflict*. Atlantic Council, South Asia Center. Available from: https://www.atlanticcouncil.org/in-depth-research-reports/report/india-and-pakistan-the-opportunity-cost-of-conflict/

Neary, H.M. (1997) Equilibrium structure in an economic model of conflict. *Economic Inquiry*, 35(3), 480–494. Available from: https://doi.org/10.1111/j.1465-7295.1997.tb02026.x

Noh, S.J. (2002) Production, appropriation, and income transfer. *Economic Inquiry*, 40(2), 279–287. Available from: https://doi.org/10.1093/ei/40.2.279

Nti, K.O. (1999) Rent-seeking with asymmetric valuations. *Public Choice*, 98(3-4), 415–430. Available from: https://doi.org/10.1023/a:1018391504741

Rubiano, A.M.P. (2021) *How to feed 10,000 rebel fighters for 50 years*. Gastro Obscura. Available from: https://www.atlasobscura.com/articles/what-farc-ate-colombia

Samuelson, P. (1948). *Military expenditure and capitalism, in economics: an introductory analysis*, vol. 1. New York: McGraw-Hill, pp. 61–76.

SIPRI. (2025) Unprecedented rise in global military expenditure as European and Middle East spending surges. Available from: https://www.sipri.org/media/press-release/2025/unprecedented-rise-global-military-expenditure-european-and-middle-east-spending-surges

Skaperdas, S. (1992) Cooperation, conflict, and power in the absence of property rights. *The American Economic Review*, 720–739.

Spencer, J. (2024) *Israel's new approach to tunnels: a paradigm shift in underground warfare*. Modern Ware Institute, West Point University. Available from: https://mwi.westpoint.edu/israels-new-approach-to-tunnels-a-paradigm-shift-in-underground-warfare/

Tullock, G. (1980) Efficient rent seeking. In: Buchanan, J.M., Tollison, R.D. & Tullock, G. (Eds.) *Toward a theory of the rent-seeking society*. College Station, TX: Texas A&M University Press, pp. 97–112.

Yektaş, H., Hoffmann, M., Hentschel, F. & Hodler, R. (2019) Wars of conquest and independence. *Journal of Institutional and Theoretical Economics*, 175(4), 617–640. Available from: https://doi.org/10.1628/jite-2019-0037

SUPPORTING INFORMATION

Additional supporting information can be found online in the Supporting Information section at the end of this article.

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APPENDIX A

Proof of Proposition 1. The first-order necessary conditions for an interior equilibrium are:

$$\frac{\partial u_A}{\partial x_A} = \frac{x_D(E_D - c_D x_D)}{(x_A + x_D)^2} - c_A = 0, \tag{A1}$$

$$\frac{\partial u_D}{\partial x_D} = \frac{x_A E_D - x_D c_D(2x_A + x_D)}{(x_A + x_D)^2} = 0. \tag{A2}$$

Since the payoff functions are strictly concave, the above first-order necessary conditions are also sufficient. When $x_A > 0$, Equation (A2) implies that

$$x_A = \frac{c_D x_D^2}{E_D - 2c_D x_D}. \tag{A3}$$

Equation (A1) further shows that an interior equilibrium exists only if $x_D < \frac{E_D}{c_A + c_D}$. Under that condition, $\frac{x_D(E_D - c_D x_D)}{(x_A + x_D)^2} = c_A$. This, combined with Equation (A3), implies:

$$c_A\left(\frac{c_D x_D^2}{E_D - 2c_D x_D} + x_D\right)^2 = x_D(E_D - c_D x_D). \tag{A4}$$

Equation (A4) has four solutions, $x_D = 0$, $x_D = \frac{E_D}{c_D}$, and $x_D = \frac{E_D}{2c_D}\left(1 \pm \frac{\sqrt{c_A}}{\sqrt{c_A + 4c_D}}\right)$. The first two are not consistent with an interior equilibrium (the second one does not satisfy $x_D < \frac{E_D}{c_A + c_D}$). Note that $x_D = \frac{E_D}{2c_D}\left(1 + \frac{\sqrt{c_A}}{\sqrt{c_A + 4c_D}}\right)$ implies $E_D - 2c_D x_D < 0$, and hence $x_A < 0$, which is impossible. Thus, the only candidate for an interior equilibrium is $x_D = \frac{E_D}{2c_D}\left(1 - \frac{\sqrt{c_A}}{\sqrt{c_A + 4c_D}}\right)$, which is an equilibrium only if $\frac{E_D}{2c_D}\left(1 - \frac{\sqrt{c_A}}{\sqrt{c_A + 4c_D}}\right) < \frac{E_D}{c_A + c_D}$. It can be verified that this inequality is always satisfied.

Then $x_A = \frac{E_D}{4c_D}\left(\frac{c_A + 4c_D}{\sqrt{c_A(c_A + 4c_D)}} - 1\right)$. One can verify that $x_A > 0$ for all $c_A, c_D > 0$.[5] We thus have a unique candidate for an interior equilibrium:

$$x_A^* = \frac{E_D}{4c_D}\left(\frac{c_A + 4c_D}{\sqrt{c_A(c_A + 4c_D)}} - 1\right), \tag{A5}$$

$$x_D^* = \frac{E_D}{2c_D}\left(1 - \frac{\sqrt{c_A}}{\sqrt{c_A + 4c_D}}\right). \tag{A6}$$

To ensure that it is indeed an interior equilibrium, we need to verify that $x_i^* < E_i/c_i$. Earlier we found that $x_D^* < E_D/(c_A + c_D)$, hence the defender's budget constraint is never binding in Equation (A6). Turning to the attacker's budget constraint, we need to ensure that

$$\frac{E_D}{4c_D}\left(\frac{c_A + 4c_D}{\sqrt{c_A(c_A + 4c_D)}} - 1\right) < \frac{E_A}{c_A}. \tag{A7}$$

Equivalently,

$$\frac{E_D}{4c_D}\left(\sqrt{1 + \frac{4c_D}{c_A}} - 1\right) < \frac{E_A}{c_A}. \tag{A8}$$

Hence, Equations (A5–A8) fully characterize the interior equilibrium. ■

Proof of Corollary 1.

(i) To investigate who exerts a higher effort—the attacker or the defender, note from Equations (A5) and (A6) that

$$x_A^* - x_D^* = \frac{E_D}{4c_D\sqrt{c_A(c_A + 4c_D)}}\left(3c_A + 4c_D - 3\sqrt{c_A(c_A + 4c_D)}\right).$$

This difference is positive if and only if $c_D > (3/4)c_A$, which tells us that the attacker exerts more effort than the defender if and only if her cost is sufficiently lower than that of the defender's. ■

(ii) $\frac{\partial x_A^*}{\partial c_A} = -\frac{E_D}{2c_A^{3/2}(c_A + 4c_D)^{1/2}} < 0$ for all $c_A, c_D > 0$.

$\frac{\partial x_D^*}{\partial c_D} = \frac{E_D}{2c_D^2}\left(-1 + \sqrt{\frac{c_A}{c_A + 4c_D}} \cdot \frac{c_A + 6c_D}{c_A + 4c_D}\right) < 0$ for all $c_A, c_D > 0$.

As for the cross effects:

$\frac{\partial x_A^*}{\partial c_D} = -\frac{E_D}{4c_D^2} \cdot \frac{(c_A + 2c_D)}{\sqrt{c_A(c_A + 4c_D)}} < 0$ for all $c_A, c_D > 0$.

$\frac{\partial x_D^*}{\partial c_A} = -\frac{E_D}{\sqrt{c_A}(c_A + 4c_D)^{3/2}} < 0$ for all $c_A, c_D > 0$. ■

(iii) Plugging Equations (A5) and (A6) into $p_A = x_A/(x_A + x_D)$ and simplifying yields:

$$p_A = \frac{\sqrt{c_A + 4c_D}}{\sqrt{c_A + 4c_D} + 2\sqrt{c_A}}.$$

This implies $\frac{\partial p_A}{\partial c_A} = -\frac{4c_D}{(c_A+4c_D)\left(\sqrt{c_A+4c_D}+2\sqrt{c_A}\right)^2\sqrt{c_A}} < 0$ and $\frac{\partial p_A}{\partial c_D} = \frac{4\sqrt{c_A}}{\left(\sqrt{c_A+4c_D}+2\sqrt{c_A}\right)^2\sqrt{c_A+4c_D}} > 0.$

Since $p_D = 1 - p_A$, we conclude that $\frac{\partial p_D}{\partial c_A} > 0$ and $\frac{\partial p_D}{\partial c_D} > 0.$ ∎

(iv) $u_A = (E_D - c_D x_D)\frac{x_A}{x_A+x_D} + (E_A - c_A x_A) = \left[E_D - c_D\frac{E_D}{2c_D}\left(1 - \frac{\sqrt{c_A}}{\sqrt{c_A+4c_D}}\right)\right]\frac{\sqrt{c_A+4c_D}}{\sqrt{c_A+4c_D}+2\sqrt{c_A}}$

$+ \left[E_A - \frac{c_A E_D}{4c_D}\left(\frac{c_A+4c_D}{\sqrt{c_A(c_A+4c_D)}} - 1\right)\right]$

Or, $u_A = \frac{E_D}{2}\left[\frac{\sqrt{c_A+4c_D}+\sqrt{c_A}}{\sqrt{c_A+4c_D}+2\sqrt{c_A}}\right] + \frac{1}{4c_D}\left[4c_D E_A - c_A E_D\left(\frac{(c_A+4c_D)-\sqrt{c_A(c_A+4c_D)}}{\sqrt{c_A(c_A+4c_D)}}\right)\right]$

Let $\sqrt{c_A + 4c_D} = m$ and $\sqrt{c_A} = n$

then $u_A = \frac{E_D}{2}\left[1 - \frac{n}{m+2n}\right] + \left[E_A - E_D\frac{n}{(m+n)}\right]$

Hence, $\frac{\partial u_A}{\partial E_A} = 1 > 0$ (an increase in attacker resources always increases attacker's payoff).

And $\frac{\partial u_A}{\partial E_D} = \frac{1}{2} - \frac{n}{2(m+2n)} - \frac{n}{(m+n)} = \frac{c_A + 4C_D - 3C_A}{2(m+2n)(m+n)}$

Hence, $\frac{\partial u_A}{\partial E_D} > 0$ if $2c_D > c_A$. So, an increase in defender resources increases the payoff of attacker—only if the attacker cost is relatively low enough.

Similarly, $u_D = (E_D - c_D x_D)\frac{x_D}{x_A+x_D}$

Or, $u_D = \left(E_D - c_D\frac{E_D}{2c_D}\left(1 - \frac{\sqrt{c_A}}{\sqrt{c_A+4c_D}}\right)\right)\frac{2\sqrt{c_A}}{\sqrt{c_A+4c_D}+2\sqrt{c_A}}$

Or, $u_D = \left(\frac{\sqrt{c_A+4c_D}+\sqrt{c_A}}{2\sqrt{c_A+4c_D}}\right)\frac{2E_D\sqrt{c_A}}{\sqrt{c_A+4c_D}+2\sqrt{c_A}}$

As $\sqrt{c_A + 4c_D} = m$ and $\sqrt{c_A} = n$ then $u_D = \frac{n(m+n)}{m(m+2n)}E_D$

It is straightforward that $\frac{\partial u_D}{\partial E_D} > 0$ and $\frac{\partial u_D}{\partial E_A} = 0.$ ∎

Proof of Proposition 2. From Equations (A1) and (A2), it is easy to see that the equilibria such that $(x_A = 0, x_D > 0)$ or $(x_A > 0, x_D = 0)$ do not exist, provided that $c_A, c_D > 0$. Thus, the only candidate is $x_A = x_D = 0$. In this case, the attack succeeds with probability 1/2. But then Player A improves the payoff by exerting a very small yet positive effort level. Thus, $x_A = x_D = 0$ is not an equilibrium either.

The only remaining candidates for corner equilibria are profiles with $x_A = E_A/c_A$ or $x_D = E_D/c_D$. Earlier we established that when $x_D = E_D/c_D$, the attacker's best response is $x_A^* = 0$. But this cannot be sustained in equilibrium since the defender is better-off by reducing the defense effort.

Suppose that $x_A = E_A/c_A$, then we can solve for the defender's best response from Equation (A2) and obtain

$$x_D^* = \min\left\{-\frac{E_A}{c_A} + \sqrt{\frac{E_A^2}{c_A^2} + \frac{E_A E_D}{c_A c_D}}, \frac{E_D}{c_D}\right\}.$$

The above formula reflects the fact that the defender's budget constraint may or may not be binding. It is straightforward to check that $-\frac{E_A}{c_A} + \sqrt{\frac{E_A^2}{c_A^2} + \frac{E_A E_D}{c_A c_D}} < \frac{E_D}{c_D}$. So, in fact, the defender's best response is always in the interior of the strategy space. Thus,

$$x_D^* = -\frac{E_A}{c_A} + \sqrt{\frac{E_A^2}{c_A^2} + \frac{E_A E_D}{c_A c_D}}. \tag{A9}$$

It remains to check whether $x_A^* = E_A/c_A$ is the attacker's best response against x_D^*. This is the case if and only if $\frac{\partial u_A}{\partial x_A}\left(x_A^*, x_D^*\right) \geq 0$. Substituting x_A^* and x_D^* into Equation (4), yields

$$E_A \leq \frac{E_D\sqrt{c_A}}{2c_D}\left(\frac{c_A + 2c_D}{\sqrt{c_A + 4c_D}} - \sqrt{c_A}\right). \tag{A10}$$

Equation (A10), in conjunction with $\frac{E_D}{4c_D}\left(\sqrt{1 + \frac{4c_D}{c_A}} - 1\right) \geq \frac{E_A}{c_A}$, the latter ruling out an interior equilibrium, represents the necessary and sufficient condition for the existence of a corner equilibrium. ■

Proof of Corollary 2.

(i) Note from Proposition 2, and after some manipulation that: $x_D^* - x_A^* = \frac{E_A}{c_A}\left(\frac{E_D}{c_D} - 3\frac{E_A}{c_A}\right)$. Hence, $x_D^* > x_A^*$ if and only if $(E_D/c_D) > 3(E_A/c_A)$. ■

(ii) It is easy to see from Proposition 2 that in the corner equilibrium, provided that it exists, $\frac{\partial x_A^*}{\partial c_A} = -\frac{E_A}{c_A^2} < 0$ and $\frac{\partial x_D^*}{\partial c_D} = -\frac{E_A E_D}{c_A c_D^2} \cdot \frac{1}{2\left(\frac{E_A^2}{c_A^2} + \frac{E_A E_D}{c_A c_D}\right)^{0.5}} < 0$.

Furthermore, numerical methods show that $\frac{\partial x_D^*}{\partial c_A} < 0$. ■

(iii) Note that the probability of the attack's success is given by $p_A = \frac{E_A}{\sqrt{E_A^2 + \frac{c_A E_A E_D}{c_D}}}$.
Using numerical methods, one can show that $\frac{\partial p_A}{\partial c_A} < 0$ and $\frac{\partial p_A}{\partial c_D} > 0$.
Using the identity $p_D = 1 - p_A$, we conclude that $\frac{\partial p_D}{\partial c_A} > 0$ and $\frac{\partial p_D}{\partial c_D} > 0$. ■

(iv) $x_A^* = \frac{E_A}{c_A}$ and $x_D^* = -\frac{E_A}{c_A} + \sqrt{\frac{E_A^2}{c_A^2} + \frac{E_A E_D}{c_A c_D}}$

Let $\frac{E_A}{c_A} = m$ and $\frac{E_D}{c_D} = n$, then $p_A = \frac{\sqrt{m}}{\sqrt{m+n}}$ and $p_D = \frac{\sqrt{m+n} - \sqrt{m}}{\sqrt{m+n}}$

$$u_A = (E_D - c_D x_D)\frac{x_A}{x_A + x_D} + (E_A - c_A x_A) = c_D\sqrt{m}\left(\sqrt{m+n} - \sqrt{m}\right)$$

$$\begin{aligned}\frac{\partial u_A}{\partial c_A} &= c_D\left[\frac{1}{2\sqrt{m}}\left(-\frac{E_A}{c_A^2}\right)\left(\sqrt{m+n} - \sqrt{m}\right) + \sqrt{m}\left(\frac{1}{2\sqrt{m+n}}\left(-\frac{E_A}{c_A^2}\right) - \frac{1}{2\sqrt{m}}\left(-\frac{E_A}{c_A^2}\right)\right)\right] \\ &= -\frac{E_A}{c_A^2}\frac{c_D}{2}\left[\frac{\sqrt{m+n} - \sqrt{m}}{\sqrt{m+n}} + \frac{\sqrt{m}}{\sqrt{m+n}} - 1\right]\end{aligned}$$

$$\frac{\partial u_A}{\partial c_A} = -\frac{E_A c_D}{c_A^2}\left[\frac{\left(\frac{m}{2} + \frac{m+n}{2}\right) - \sqrt{m(m+n)}}{\sqrt{m(m+n)}}\right] < 0$$

The above inequality holds because the geometric mean of two distinct (positive) numbers (in our case, $\sqrt{m(m+n)}$ is the geometric mean of m and $m+n$) is always strictly less than their arithmetic mean.

$$\frac{\partial u_A}{\partial c_D} = \sqrt{m}\left[\sqrt{m+n} - \sqrt{m} + \frac{c_D}{2\sqrt{m+n}}\left(-\frac{E_D}{c_D^2}\right)\right] = \sqrt{m}\left[\sqrt{m+n} - \sqrt{m} - \frac{n}{2\sqrt{m+n}}\right] = \frac{\left(\frac{m}{2} + \frac{m+n}{2}\right) - \sqrt{m(m+n)}}{\sqrt{m+n}}$$

> 0.

The previous inequality holds for the same reason, arithmetic mean being greater than the geometric mean.

$$u_D = (E_D - c_D x_D)\frac{x_D}{x_A + x_D} = c_D\left(\sqrt{m+n} - \sqrt{m}\right)^2$$

$$\frac{\partial u_D}{\partial c_A} = 2c_D\left(\sqrt{m+n} - \sqrt{m}\right)\left(-\frac{E_A}{c_A^2}\right)\left[\frac{1}{2\sqrt{m+n}} - \frac{1}{2\sqrt{m}}\right] > 0$$

$$\frac{\partial u_D}{\partial c_D} = \left(\sqrt{m+n} - \sqrt{m}\right)^2 + c_D\left(\sqrt{m+n} - \sqrt{m}\right)\frac{1}{2\sqrt{m+n}}\left(-\frac{E_D}{c_D^2}\right) = \left(\sqrt{m+n} - \sqrt{m}\right)\left[\sqrt{m+n} - \sqrt{m} - \frac{n}{\sqrt{m+n}}\right]$$

$$\frac{\partial u_D}{\partial c_D} = \left(\sqrt{m+n} - \sqrt{m}\right)\frac{m - \sqrt{m(m+n)}}{\sqrt{m+n}} < 0. \blacksquare$$