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Brief paper

Two-layer nonlinear control of DC–DC buck converters with meshed network topology[☆]

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ABSTRACT

In this paper, we analyse a buck converter network containing arbitrary, up to mild regularity assumptions, loads. Our analysis begins with the primary controller where we propose a novel decentralised Lyapunov function for the interconnection between currents and a bounded integrator. We leverage on this result to study the network as a cascaded interconnection between voltages and bounded currents. We, in addition, propose a distributed optimal secondary control framework to steer voltages close to their nominal operating values. We employ the properties of the Laplacian kernel to show recursive feasibility and input-to-state stability of the closed loop. We demonstrate our results in a meshed topology network containing 6 power converters, each converter feeding an individual constant power load with values changing arbitrarily within a pre-specified range.

1. Introduction

The presence of DC networks has become an important feature in modern complex systems such as electric vehicles and aircraft (Elsayed et al., 2015). In addition, a DC Microgrid (MG) can grow from simple photovoltaic and storage implementations to highly complex and constantly evolving systems with meshed topology with an increasing tendency to decentralisation (Planas et al., 2015). Traditionally, control problems in DC MG, as mentioned in Tucci, Riverso et al. (2018), involve voltage stabilisation, current or load sharing, and voltage balancing. The growing size of DC systems calls for non-centralised control techniques for voltage regulation which tackle plug-and-play methods (Sadabadi & Shafiee, 2020), $\mathcal{L}_2$ gain-based loop shaping methods (Sadabadi, 2021b), and optimisation-based controller methods using control barrier functions (Kosaraju et al., 2022). Approaches that include in its formulation current regulation and sharing include (Sadabadi, 2021a; Trip et al., 2019). On the other hand, De Persis et al. (2018) propose a power-sharing controller which employs nonlinear consensus to obtain effective load sharing while keeping voltages bounded to a compact set. Despite the rapid developments of control techniques and deeper system theoretic understanding gained in the past decades, two main problems have not yet been fully understood: safe operation during transients for input currents combined with voltage regulation, and the role of current sharing in a meshed network topology. Among distributed control techniques, receding horizon controllers offer a methodology including constraints in its formulation (Maestre & Negenborn, 2014). For the MG case, receding horizon techniques have been used predominantly in their distributed optimisation form as mentioned in the excellent review of Hu et al. (2021). Robust control methods, however, have not yet, to the best of the authors' knowledge, established a foothold in a MG setting. The source of limitation is the ubiquitous assumption on the size of the interaction strength (Baldivieso-Monasterios, 2018). In a MG setting, the interactions represent currents flowing through the network whose magnitude is comparable to that of local states. Therefore, robust control methods for distributed receding horizon control require adjustments to how they handle interactions among individual elements of the network.

In this paper, we aim to rigorously analyse the system theoretic properties of a DC network with a meshed topology where each converter feeds generic nonlinear loads with mild continuity assumptions. We address two important issues: current constraint satisfaction, and voltage regulation with an implicit notion of load sharing. We use a novel Lyapunov-based analysis for the decentralised primary controller where we prove asymptotic stability within a compact set. We, then, study the interconnection

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properties between voltages and currents under the scope of the cascaded systems approach; our results show that the kernel of the network Laplacian matrix defines an attractive set which yields an ISS-type result. This analysis is, however, not enough to guarantee convergence to a particular equilibrium point. To regulate voltages, we employ a distributed voltage regulation based on concepts of robust distributed model predictive controllers. The latter implies that information sharing occurs only once each sampling period as opposed to distributed approaches. On this vein, we extend, to a nonlinear setting, the approach of Baldivieso Monasterios and Trodden (2018) which proposes an MPC technique capable of handling exogenous information. The resulting distributed controller is, to the best of the authors' knowledge, the first attempt to use a non-iterative distributed predictive controller. We analyse the nominal equilibrium behaviour of the proposed control law, and show that, assuming a bounded load deviation, it remains in a neighbourhood of the "real" equilibrium, *i.e.*, the one considering uncertain loads. Similarly, we show that our distributed control law at steady state lies in the equilibrium manifold of the system.

Our contributions are:

(i) We propose a Lyapunov function for the Bounded Integral Controller (BIC)-based primary controller in closed loop form with each node current as opposed to Konstantopoulos and Baldivieso-Monasterios (2020) where only local stability results are obtained based on linearisation. With this Lyapunov function, we can conclude the asymptotic stability of all the current dynamics.

(ii) We show that under the cascaded system interpretation, a buck converter network satisfies a weaker ISS property. The state remains close to the kernel of the Laplacian and not around an equilibrium point.

(iii) We propose a novel non-iterative distributed receding horizon voltage controller to steer the voltages towards a given equilibrium point which does not rely on the ubiquitous assumption of weak coupling between components, as seen for example in Riverso et al. (2018) and Trodden and Maestre (2017). In the proposed controller, each node employs information about the voltage values of its neighbours to compute its own control law and exploit the influence of neighbouring nodes. The proposed approach extends the results of Baldivieso Monasterios and Trodden (2018) to a nonlinear setting to guarantee recursive feasibility.

Notation: A MG is a connected undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where the set of nodes $\mathcal{V}$ represents inverters and local loads; the set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ defining the MG topology is characterised by the incidence matrix $\mathcal{B} \in \mathbb{R}^{|\mathcal{E}| \times |\mathcal{V}|}$. The 2-norm is denoted $|x| = \|x\|_2$; the distance of a point $x \in \mathbb{R}^n$ to a set $\mathcal{A} \subset \mathbb{R}^n$ is $|x|_{\mathcal{A}} = \inf\{|x - y| : y \in \mathcal{A}\}$. A set $\mathcal{A} \subset \mathbb{R}^n$ is a *C-set* if it is convex and compact; *PC-set* is a C-set with the origin in its nonempty interior. A class $\mathcal{K}$-function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, strictly increasing with $\alpha(0) = 0$; $\alpha(\cdot)$ is $\mathcal{K}_\infty$ if in addition $\lim_{r\to\infty} \alpha(r) = \infty$. A function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is $\mathcal{KL}$ if $\beta(\cdot, t)$ is $\mathcal{K}$ for all $t \in \mathbb{R}^+$; $\beta(r, \cdot)$ is continuous, strictly decreasing, and $\lim_{t\to\infty} \beta(r, t) = 0$ for all $r \in \mathbb{R}^+$. A set $\mathcal{P}$ is *positively invariant* for $\dot{x} = f(x)$ if for any $x \in \mathcal{P}$, the state trajectory $x(t) \in \mathcal{P}$ for all $t \geq 0$. A set $\mathcal{R} \subset \mathbb{X}$ is *control invariant* for $\dot{x} = f(x, u)$ and constraint sets $(\mathbb{X}, \mathbb{U})$ if for any $x \in \mathcal{R}$, there exists a control law $\mu : \mathcal{R} \to \mathbb{U}$, such that the closed loop system $\dot{x} = f(x, \mu(x))$ satisfies $x(t) \in \mathcal{R}$ for all $t \geq 0$ with $x(0) = x_0$. A $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ is a *control Lyapunov function* for a system $\dot{x} = f(x, u)$ with $u \in \mathbb{U}$ if it is positive definite $V(x) \geq 0$ for all $x \in \mathbb{R}^n$, radially unbounded $\lim_{|x|\to\infty} V(x) = \infty$, and there exists $\alpha_V : \mathbb{R} \to \mathbb{R}$ nondecreasing and radially unbounded such that $\inf_{u\in\mathbb{U}} \frac{\partial V}{\partial x} f(x, u) + \alpha_V(|x|) < 0$. A set-valued map $\Phi : U \to 2^X$ is: (*i*) *upper semi-continuous* (u.s.c) at $x_0 \in U$ if for an open neighbourhood $V_U \subset U$ of x_0, for all $x \in V_U$, $\Phi(x) \subset V_X$ for an open neighbourhood $V_X \subset X$; (*ii*) *lower semi-continuous* (l.s.c) at $x_0 \in U$ if for any $y_0 \in \Phi(x_0)$ and a neighbourhood $V_X \subset X$, there exists $V_U \subset U$ such that for all $x \in V_U$, $\Phi(x) \cap V_X \neq \emptyset$.; and *continuous* if it is u.s.c. and l.s.c.

2. Preliminaries

2.1. Modelling of DC-DC buck converters

The system in consideration consists of a network of M converters connected through M_E lines characterised by an undirected graph $(\mathcal{V}, \mathcal{E})$. The dynamics of the network for each $i \in \mathcal{V} = \{1, \dots, M\}$ and $e = (i, j) \in \mathcal{E}$ is

$$L_i \frac{di_i}{dt} = -r_i i_i - v_i + V_i^{\mathrm{IN}} \tilde{m}_i, \tag{1a}$$

$$C_i \frac{dv_i}{dt} = -f_{L,i}(v_i, P_{L,i}) + i_i - \mathcal{B}_i^\top i_{\mathcal{E}}, \tag{1b}$$

$$L_e \frac{di_e}{dt} = -r_e i_e + \mathcal{B}_e v, \tag{1c}$$

where the states are $(\{i_i, v_i\}_{i\in\mathcal{V}}, \{i_e\}_{e\in\mathcal{E}})$ converter currents and voltages together with line currents. In addition, (L_i, r_i) are the converter inductance and parasitic resistance; C_i denotes the converter capacitance; $f_{L,i}(\cdot)$ represents the load current according to power reference $P_{L,i}$; V_i^{IN} is the input voltage; $\tilde{m}_i$ is the converter duty ratio; and (L_e, r_e) are line eth inductance and resistance. The network affects the ith converter via $\mathcal{B}_i^\top i_{\mathcal{E}}$ with $\mathcal{B}_i^\top$ corresponding to the ith column of the incidence matrix $\mathcal{B}$ and $i_{\mathcal{E}}$ the current running through the network lines, and, conversely, line $e = (i, j) \in \mathcal{E}$ depends on voltage differences characterised by $\mathcal{B}_e$, the eth row of $\mathcal{B}$.

2.2. Current controller structure

The decentralised primary current controller, which has a similar structure to Konstantopoulos and Baldivieso-Monasterios (2020), ensures current limitation $|i_i - \frac{1}{2} I_i^{\max}| \leq \frac{1}{2} I_i^{\max}$. The nonlinear PI controller that allows tracking any reference $|i_i^{\mathrm{ref}}| \leq I_i^{\max}$ is

$$V_i^{\mathrm{IN}} \bar{m}_i = v_i - k_{P,i} i_i + \frac{k_{P,i} + r_i}{2} I_i^{\max} + M_i \sigma_i, \tag{2a}$$

$$\frac{d\sigma_i}{dt} = \frac{k_{I,i}}{M_i} (i_i^{\mathrm{ref}} - i_i)(1 - \sigma_i^2), \tag{2b}$$

where $k_{P,i} > 0$, $k_{I,i} > 0$, $M_i = \frac{1}{2}(r_i + k_{P,i}) I_i^{\max}$ are controller gains. The closed loop equilibrium points of (1) with (2) are given by the solution to[1]

$$i_i^{\mathrm{eq}} = \mathrm{sat}(i_i^{\mathrm{ref}}, 0, I_i^{\max}), \tag{3a}$$

$$\sigma_i^{\mathrm{eq}} = \mathrm{sat}\left(\frac{2 i_i^{\mathrm{ref}} - I_i^{\max}}{I_i^{\max}}, -1, 1 \right), \tag{3b}$$

$$i^{\mathrm{eq}} = f_L(v^{\mathrm{eq}}, P_L) + \mathcal{B}^\top r_E^{-1} \mathcal{B} v^{\mathrm{eq}}, \tag{3c}$$

$$i_{\mathcal{E}}^{\mathrm{eq}} = r_E^{-1} \mathcal{B} v^{\mathrm{eq}}, \tag{3d}$$

where $f_L(v, P_L) = (f_{L,1}(v_1, P_{L,1}), \dots, f_{L,M}(v_M, P_{L,M}))$ and $r_E = diag(r_1, \dots, r_{M_E})$. The controller (2) decouples the equilibrium point between local (currents and integrator) and networked (voltages and line currents) equilibria. The latter depends on the current drawn by each load which are required to satisfy the following assumption:

[1] The saturation function is defined as $\mathrm{sat}(x, y, z) = x$ when $z \leq x \leq y$, $\mathrm{sat}(x, y, z) = y$ when $x > y$, and $\mathrm{sat}(x, y, z) = z$ when $x < z$.

Assumption 1. For each $i \in \mathcal{V}$, the current drawn by each load is $f_{L,i}(v_i, P_{L,i}) = g_{L,i}(v_i)d_{L,i}$ with $g_{L,i}: \mathbb{R} \to \mathbb{R}^{n}_{L,i}{}^{*}$ being $\mathcal{C}^1$ in an open neighbourhood $A \subset \mathbb{R}$ of a voltage $v \neq 0$, and $d_{L,i} \in \mathbb{R}^{n_{L,i}}$ defining the load characteristics.

Remark 1. Assumption 1 is not restrictive; the current drawn by a ZIP load can be written as $f_{L,i} = \frac{1}{R_{L,i}} v_i + I_{L,i} + \frac{P_{L,i}}{v_i}$ for an impedance $R_{L,i}$, constant current $I_{L,i}$, and constant power $P_{L,i}$. Therefore, $g_{L,i}(v_i) = (v_i, 1, v_i^{-1})$, $d_i = (R_{L,i}, I_{L,i}, P_{L,i})$.

2.3. Constraints and control objectives

The overall state for each node is $x_i = (v_i, i_i, \sigma_i)$ with input $u_i = i_i^{\text{ref}}$. The system is subject to constraints on the inputs and states such that for all $i \in \mathcal{V}$, $x_i \in \mathbb{X}_i$, $u_i \in \mathbb{U}_i$ which satisfy the following assumption:

Assumption 2 (*Network Constraints*)**.** For each $i \in \mathcal{V}$,

(i) the input constraint set $\mathbb{U}_i \subset \mathbb{R}$ is a PC-set, while the state constraint $\mathbb{X}_i$ is a C-set.
(ii) the load characteristics $d_{L,i} \in \mathbb{R}^{n_{L,i}}$ are constrained to a PC-set $\mathbb{D}_i \subseteq \mathbb{R}^{n_{L,i}}$.

In addition, we impose the following assumption on the closed-loop dynamics on the time-scale separation between lines and nodes

Assumption 3 (*Time-scale Separation*)**.** The network parameters satisfy

$$\min_{i \in \mathcal{V}} \left\{ \frac{L_i}{r_i + k_{P,i}}, \frac{4C_i P_{L,i}}{(I_i^{\max})^2}, \frac{r_i + k_{P,i}}{k_{I,i}} \right\} \gg \max_{e \in \mathcal{E}} \left\{ \frac{L_e}{r_e} \right\}$$

where $P_{L,i}$ is the load power at node $i \in \mathcal{V}$.

Assumption 3 essentially requires the time constants for each power line to be sufficiently small. A consequence of Assumption 3 is that line currents satisfy the algebraic relation (3d), and the analysis focuses only on nodes. This Assumption takes into account the case of low voltage microgrids, *i.e.*, $L_e = 0$. The aim is to solve the following optimal control problem: from a state $x = (x_1, \ldots, x_M)$, determine the control policy, *i.e.*, reference currents, that minimises the criteria

$$J(x, u, v^*) = \int_0^\infty (\mathbb{1}_{|\mathcal{V}|} v^* - Hx)^\top (\mathbb{1}_{|\mathcal{V}|} v^* - Hx) + \gamma(u) dt.$$

where node voltages are outputs defined by the linear map $H: \mathbb{R}^{3M} \to \mathbb{R}^M$ such that $v = (v_1, \ldots, v_M) = Hx$. This criterion encourages these to operate near a common point v^*, and each source to feed its associated local load while minimising its operating costs $\gamma(\cdot)$.

3. Primary controller and interconnection analysis

In this section, we aim to analyse the properties of the interconnection between the decentralised current controller (2) with the rest of the network. First, in Section 3.1, we prove the invariance properties of the bounded integral control, then using a Lyapunov function, we infer the stability properties. Later in Section 3.2, the dynamics of each node are decomposed into two constituting parts: the *driving system* given by currents and integrators; and the *driven system* composed of node voltages. We derive stability properties, not of an equilibrium point, but of a neighbourhood of the kernel of the network Laplacian.

3.1. Current controller properties

The next result is an adaptation to our setting of Konstantopoulos and Baldivieso-Monasterios (2020, Proposition 3) and is followed by our first novel result,

Lemma 1 (*Bounded Integral Control*)**.** *For all $i \in \mathcal{V}$, the set $\mathbb{Z}_i = [0, I_i^{\max}] \times [-1, 1]$ is positively invariant for (1a) and (2) with $I_i^{\max} = \frac{2M_i}{(r_i + k_{P,i})}$ for all $0 \le |i_i^{\text{ref}}| \le I_i^{\max}$.*

Proposition 1 (*Bounded Integrator Lyapunov Function*)**.** *For each $i \in \mathcal{V}$ and any $u_i \in (0, I_i^{\max})$, the $\mathcal{C}^1$ function $W_i: \text{int}(\mathbb{Z}_i) \to \mathbb{R}$ defined as*

$$\begin{aligned} W_i(i_i, \sigma_i) =& \frac{1}{2} L_i (i_i - u_i)^2 + \frac{M_i^2}{k_{I,i}} \frac{u_i}{I_i^{\max}} \ln \left| \frac{\frac{u_i}{I_i^{\max}}}{1 + \sigma_i} \right| \\ &+ \frac{M_i^2}{k_{I,i}} \left(1 - \frac{u_i}{I_i^{\max}}\right) \ln \left| \frac{1 - \frac{u_i}{I_i^{\max}}}{1 - \sigma_i} \right| + \ln 2 \end{aligned} \tag{4}$$

is a Lyapunov function for the driving subsystem in $\text{int}(\mathbb{Z}_i)$.

Proof. For $W_i(\cdot)$ to be a Lyapunov function, it needs to be positive definite and has a negative time derivative along the trajectories of (i_i, σ_i). For the latter, the time derivative is $\dot{W}_i = \frac{\partial W}{\partial \tilde{i}_i} f(i, \sigma_i) + \frac{\partial W}{\partial \sigma_i} g(i, \sigma_i, u_i)$ which results in $\dot{W}_i < -\gamma_i (r + k_{P,i})(i_i - u_i)^2$ with $\gamma_i \in (0, 1)$. For the former, the positive definiteness condition for W_i follows from an analysis of the Hessian of W_i at its extrema. The Hessian of W_i satisfies $\nabla^2 W_i = \text{diag}\big(L_i, \big(1 - \frac{u_i}{I_i^{\max}}\big)\frac{1}{(1+\sigma)^2} + \frac{u_i}{I_i^{\max}} \frac{1}{(1-\sigma)^2}\big) > 0$ since $u_i \in (0, I_i^{\max})$. Then $W_i(i_i^{\text{eq}}, \sigma_i^{\text{eq}}) \le W_i(i_i, \sigma_i)$ for all $(i_i, \sigma_i) \in \text{int}(\mathbb{Z}_i)$ and $W_i(i_i^{\text{eq}}, \sigma_i^{\text{eq}}) = 0$. As a result, the candidate function is a Lyapunov function. □

An immediate consequence of the above result is

Corollary 1 (*Asymptotic Stability Driving Subsystem*)**.** *Suppose Assumption 2 holds. For all $i \in \mathcal{V}$ and any $u_i \in (0, I_i^{\max})$, $|(i_i(t), \sigma_i(t))|_{(i_i^{\text{eq}}, \sigma_i^{\text{eq}})} \to 0$ as $t \to \infty$.*

Proof. Let $\Omega_{i,(c,u_i)} = \{(i_i, \sigma_i): W_i(i_i, \sigma_i) \le c\}$ be a c-level set for W_i. The normal vector, ∇W_i, of each level set satisfies $\nabla W_i f_i(i_i, \sigma_i, u_i) \le 0$ with $f_i(\cdot, \cdot, \cdot)$ defining the system dynamics. Next, by construction, we note that $W_i(i_i, \sigma_i) = 0$ only at the equilibrium $(\tilde{i}_i^{\text{eq}}, \sigma_i^{\text{eq}}) \in \Omega_{i,(c,u_i)}$. As a result, the asymptotic stability of the equilibrium point follows by applying the invariance principle. □

The case for which $u_i \in \{0, I_i^{\max}\}$ corresponds to a Lyapunov function $W_i(i, \sigma_i) = \frac{1}{2} L_i i_i^2 + \frac{M_i^2}{k_{I,i}} \ln \left| \frac{2}{1 \pm \sigma_i} \right|$. The domain of definition of $W(\cdot, \cdot)$ is a consequence of the multi-stability properties of (2); those equilibrium points corresponding to $(I_i^{\max}, 1)$ and $(0, -1)$ for $u_i \in (0, I_i^{\max})$ are not stable which ensures that the basin of attraction remains inside $\mathbb{Z}_i$.

3.2. Cascaded structure

We now exploit interconnection structure between node voltages and currents with their associated integrators. This state of each node can be written as: $x_i = (v_i, z_i)$ with $z_i = (\tilde{i}_i, \sigma_i)$ as the driving states with dynamics $\dot{v}_i = f_i(v_i, 0) + h(v_i, z_i)$ and $\dot{z}_i = g_i(z_i, u_i)$ where $h_i(v_i z_i) = f_i(v_i, z_i) - f_i(v_i, 0)$. The stability analysis of cascaded systems has been thoroughly explored, see for example Sepulchre et al. (1997), and two main methods exist to infer the asymptotic stability of the cascaded system:

(i) Asymptotic stability for the driving system and zero dynamics of the driven system, plus a linear growth restriction on the interaction between these two components.
(ii) Input-to-state Stable (ISS) for the *driven* subsystem and asymptotic stability for the *driving* subsystem.

In our setting, the driving subsystem satisfies both conditions; we will now check conditions for the driven subsystem. An initial guess of a Lyapunov function is the energy stored in the capacitors $S = \sum_{i\in\mathcal{V}} \frac{1}{2}C_i v_i^2$. To account for a nonzero equilibrium point, we employ a Bergman Function $\mathcal{S} = \sum_{i\in\mathcal{V}} \frac{1}{2}C_i(v_i^2 - v_i^{\mathrm{eq}2}) - C_i v_i^{\mathrm{eq}}(v_i - v_i^{\mathrm{eq}})$. The time derivative of $\mathcal{S}$ yields: $\dot{\mathcal{S}} = (v - v^{\mathrm{eq}})^\top(-\mathcal{L}v - g_L(v)d_L + i)$, where $d_L = (d_{L,1}, \dots, d_{L,|\mathcal{V}|}) \in \prod_{i\in\mathcal{V}} \mathbb{D}_i$ is the collection of load characteristics and $\mathcal{L} = \mathcal{B}^\top r_E^{-1}\mathcal{B} \in \mathcal{R}^{|\mathcal{V}|\times|\mathcal{V}|}$ the network Laplacian. First, we check the voltage zero dynamics; then $\dot{\mathcal{S}}\big|_{i=0} = (v - v^{\mathrm{eq}})^\top(-\mathcal{L}v - g_L(v)d_L)$ which results in a stable system only if the load is resistive *i.e.*, $g_L(v)d_L = \mathrm{diag}(R_{L,i}^{-1})v$. This, however, is not true for the general case; for example, when the network contains constant power loads $g_L(v)d_L = \mathrm{diag}(v_i^{-1})d_L$, the impedance is locally negative, *i.e.*, $\frac{\partial g_L}{\partial v}d_L = -\mathrm{diag}(v_i^{-2})d_L$. As a result, the origin becomes a singularity point; and there are multiple unstable and stable equilibria.

However, all is not lost; the voltage dynamics have a Laplacian structure that can be exploited. The network dynamics are dominated by the Laplacian of the connectivity graph and $\ker\mathcal{L} = \mathrm{span}\,\mathbb{1}_{|\mathcal{V}|}$ determines the synchronisation manifold. In the absence of loads, the kernel is stable (Bullo, 2020); in fact the state converges to a weighted average of the initial state, *i.e.*, $v \to \frac{\sum_{i\in\mathcal{V}} C_i v_i(0)}{\sum_{i\in\mathcal{V}} C_i}\mathbb{1}_{|\mathcal{V}|} \in \ker\mathcal{L}$. The next ancillary result provides estimates for $v(\cdot)$ subject to bounded perturbations.

Lemma 2 (*Network Dynamics*)**.** *The set $\mathcal{R}_\eta = \ker\mathcal{L} \oplus \eta\mathbb{B}_{|\mathcal{V}|}$ is attractive with $\eta > 0$ for Laplacian dynamics $C\dot{v} = -\mathcal{L}v + u$ with $|u(t)| \le B_u$ for $B_u > 0$ and all $t \in \mathbb{R}$*

Proof. Given the Laplacian matrix pencil $\lambda C + \mathcal{L}$ with eigenvalues $\{\lambda_0, \dots, \lambda_{|\mathcal{V}|-1}\}$ and the state transformation $y_i = \langle \xi_i, Cv\rangle$ with $\xi_i \in \mathbb{R}^{|\mathcal{V}|}$ the eigenvector associated with λ_i for $i \in \{0, \dots, |\mathcal{V}| - 1\}$. The dynamics of each y_i satisfy $\dot{y}_i = -\lambda_i y_i + \tilde{u}_i$ with $\tilde{u}_i = \langle \xi_i, u\rangle$. The solution for each $i \in \{0, \dots, |\mathcal{V}| - 1\}$ is $y_i(t) = e^{-\lambda_i t}y_i(0) + \int_0^t e^{-\lambda_i(t-\tau)}\tilde{u}_i d\tau$ yielding for $v = \sum y_i\xi_i$:

$$v = \left(\frac{\langle \mathbb{1}_{|\mathcal{V}|}, Cv(0)\rangle}{\mathbb{1}_{|\mathcal{V}|}^\top C\mathbb{1}_{|\mathcal{V}|}} + \int_0^t \frac{\langle \mathbb{1}_{|\mathcal{V}|}, u(\tau)\rangle}{\mathbb{1}_{|\mathcal{V}|}^\top C\mathbb{1}_{|\mathcal{V}|}}d\tau\right)\mathbb{1}_{|\mathcal{V}|} + \sum_{i=2}^{n}\int_0^t e^{-\lambda_i(t-\tau)}\langle \xi_i, u(\tau)\rangle d\tau\,\xi_i.$$

Given $\alpha\mathbb{1}_n \in \ker\mathcal{L}$, the distance from any state to this set is $d(v(t), \ker\mathcal{L}) = \min\{|v(t) - \alpha\mathbb{1}_n| : \alpha \in \mathbb{R}\}$. The explicit minimum occurs at $\alpha^*(v) = \bar{v} \implies d(v, \ker\mathcal{L}) = |v - \bar{v}\mathbb{1}_n|$. The desired bound is $d(v(t), \ker\mathcal{L}) \le B_u\sum_{i=2}^{n-1}\frac{1-e^{-\lambda_i t}}{\lambda_i}$ and in steady state $d(v_{ss}, \ker\mathcal{L}) \le B_u\sum_{i=2}^{n-1}\frac{1}{\lambda_i}$. Setting $\eta = B_u\sum_{i=2}^{n-1}\frac{1}{\lambda_i}$ implies the set $\mathcal{R} = \ker\mathcal{L} \oplus \eta\mathbb{B}_{|\mathcal{V}|}$ is attractive for $\dot{v} = \mathcal{L}v + u$. □

The above lemma concludes an ISS type behaviour with respect to the kernel, this however might not imply the solutions are stable in the ordinary sense. For example, a constant input produces a ramp output that remains close to the kernel.

4. Distributed voltage regulation

In this section, we exploit the kernel stability properties to obtain closed-loop stability in the classic sense and optimal performance with respect to an optimisation cost. First, we define the distributed optimisation problem, then we proceed to analyse both recursive feasibility and stability of the overall network. Section 4.2 defines the optimal control problem in terms of exogenous information; we have two steps to prove recursive feasibility. First, we assume neighbouring measurements do not change and prove stability under unchanging information. The second step allows for changes in exogenous information and exploit the regularity of the set-valued control law defining the OCP. This last step represents a generalisation to nonlinear systems to the result presented in Baldivieso Monasterios and Trodden (2018). We use the robust stability of each local controller to conclude recursive feasibility, and, in Section 4.2.2, use the kernel properties to show the overall stability of the network.

4.1. Optimal control problem

The system objective is to minimise the infinite horizon cost for the network. A tractable solution to the problem is to use finite horizon approximations and a separable cost $J(x, u, v^*) = \sum_{i\in\mathcal{V}} J_i(x_i, u_i, v^*)$. The motivation behind this is twofold: a separable cost allows us to use robust-based methods without introducing additional coordination, and ease of the computational burden. Each node is subject to parametric, $d_{L,i} \in \mathbb{D}_i$, and "coupling", $w_i = \sum_{j\in\mathcal{N}_j}\mathcal{L}_{ij}v_j$ arising from the interconnection with neighbouring nodes, uncertainty.[2] The dynamics of each node are $C_i\dot{v}_i = -\mathcal{L}_{ii}v_i - g_{L,i}(v_i)\bar{d}_{L,i} + i_i + w_i + w_{L,i}(v_i)$ where $w_{L,i}(v_i) = g_{L,i}(v)(\bar{d}_{L,i} - d_{L,i})$ and $\bar{d}_{L,i}$ is a nominal load. Robust methods can be used to handle parametric uncertainty; these methods, however, are not suitable for coupling uncertainty because DC networks do not satisfy weak coupling assumptions that permeate robust distributed methods (Baldivieso-Monasterios, 2018) as seen in the next example.

Example 1. Consider a network $(\{1, 2\}, (1, 2))$ with voltage constraints $\mathbb{V}_i = [0, \bar{V}]$ and line admittance $Y_{12} = 10$[S]. The effect of 1 on 2 lies inside $\mathbb{W}_1 = C_1^{-1}Y_{12}\mathbb{V}_2$. The weak coupling assumption for tube MPC methods requires $\mathbb{W}_i \subset \mathbb{V}_i$ for $i = 1, 2$ which is not the case for this simple example. This implies that there are no robust invariant sets capable to account for the interconnection disturbance.

The above example illustrates one limitation of robust distributed approaches for electrical networks. However, we can exploit the Laplacian structure, by virtue of Lemma 2, to bound the effect of neighbours on each converter.The distributed optimisation problem $\mathbb{P}_i(x_i, \bar{d}_{L,i}, w_i)$ for each node $i \in \mathcal{V}$ consists in minimising

$$J_i(x_i, u_i, v^*) = \int_0^T \ell_i(x_i, u_i, v^*)dt + \psi_i(x_i(T)), \tag{5}$$

where $\ell_i(x_i, u_i, v^*) = (v^* - H_i x_i)^2 + \gamma_i(u_i)$, $x_i = (v_i, \tilde{i}_i, \sigma_i)$, $v_i = H_i x_i$, and $u_i = i_i^{\mathrm{ref}}$, subject to

$$\dot{x}_i = F_i(x_i, u_i, \bar{d}_{L,i}) + E_i w_i, \tag{6a}$$
$$x_i \in \mathbb{X}_i, \quad u_i \in \mathbb{U}_i, \tag{6b}$$
$$x_i(0) = x_i, \quad x_i(T) \in \mathbb{X}_{f,i}. \tag{6c}$$

where $E_i \in \mathbb{R}^{3\times 1}$ determines how the coupling w_i affects the local dynamics. The constraint set is a set-valued map $\mathcal{U}_i^N(x_i, \bar{d}_{L,i}, w_i) \subset \mathbb{U}_i$. The terminal ingredients, $\psi_i : \mathbb{X}_{f,i} \to \mathbb{R}^+$ and $\mathbb{X}_{f,i} \subseteq \mathbb{X}_i$, together with the stage cost $\ell_i(\cdot,\cdot,\cdot)$ satisfy the following assumptions

[2] For a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the set of neighbours of node i is $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$.

Assumption 4. For each $i \in \mathcal{V}$, the terminal conditions for $\mathbb{P}_i(x_i, \bar{d}_{L,i}, w_i)$ satisfy

(i) For all $x \in \mathbb{X}_{f,i}$, the terminal cost satisfies

$$\min_{u\in\mathbb{U}, x_i(t)\in\mathbb{X}_{f,i}} \left\{\psi_i(x_i(t)) + \int_0^t \ell_i(x_i, u_i)dt\right\} \leq \psi_i(x_i) \tag{7}$$

(ii) The set $\mathbb{X}_{f,i} \subset \mathbb{X}$ is a *control invariant* for $\dot{x} = F_i(x_i, u_i, d_{L,i})$ with input constraints $\mathbb{U}_i \subset \mathbb{R}$.

Assumption 5 (*Positive Definite Stage Cost*)**.** $\ell_i : \mathbb{X}_i \times \mathbb{U}_i \times \mathbb{R} \to \mathbb{R}$ is for each $i \in \mathcal{V}$ is continuous and positive definite.

In the context of $\mathbb{P}_i(x_i, \bar{d}_{L,i}, w_i)$, each local controller needs only access to a measure of w_i which is kept constant across the prediction horizon $[t, t+T]$. The solution at time $t \geq 0$ is a piece-wise constant optimal control input $u_i : [t, t+T] \to \bar{\mathbb{U}}_i$. This input is applied to the system only during the interval $[t, t+\delta]$ with $\delta > 0$ acting as a sampling rate. At $t+\delta$, each subsystem measures its states and collects neighbouring information to form w_i, then (5) is solved with the updated information.

4.2. Properties of the OCP

In this section, the analysis focuses on local recursive feasibility and robustness to load changes. Then, we focus on stability properties for the overall network.

4.2.1. Recursive feasibility

We invoke the following assumption to make precise the concepts used in the OCP formulation.

Assumption 6 (*Information Available to the Controller*)**.** For each $i \in \mathcal{V}$,

1. The state $x_i(\delta k)$, interconnection information $w_i(\delta k)$, and nominal load $\bar{d}_{L,i} \in \mathbb{D}_i$ are known exactly at time $t = \delta k \geq 0$ with $k \in \mathbb{N}$.
2. There exists $\gamma_{d,i} > 0$ such that $d_{L,i} \in \mathbb{D}_i$ satisfies $|\bar{d}_{L,i} - d_{L,i}| < \gamma_{d,i}$.

If the load variations do not satisfy the second part of Assumption 6, then a robust approach can be used to regulate the voltages to a neighbourhood of the equilibrium point. In fact, a tube-based approach can be included in our approach by a suitable constraint tightening. At a time δk, the OCP is solved by discretising the continuous problem $\mathbb{P}_i(x_i, \bar{d}_{L,i}, w_i)$ such that the solution is a sequence of N optimal current references $\mathbf{u}_i^0(x_i, \bar{d}_{L,i}, w_i) = \{u_i^0(0), \ldots, u_i^0(N-1)\}$. The interconnection w_i is kept constant across the horizon, $\mathbf{w}_i = w_i \mathbb{1}_{N+1}$; at the next sampling time, this sequence, following Assumption 6, is allowed to change. To account for this, the analysis will focus on two scenarios: prove recursive feasibility for an unchanging interconnection, *i.e.*, $w_i(k+1) = w_i(k)$. Then proving that recursive feasibility holds for the general case by leveraging on the structure of the OCP.

The first hurdle to overcome is to find suitable terminal ingredients satisfying Assumption 4. These conditions depend on the equilibrium of the local dynamics $F_i(x_i, \bar{d}_{L,i}, u_i) + E_i w_i = 0$. This equation admits a solution, since F_i is $\mathcal{C}^1$, by the implicit function theorem $(x_i^{\rm eq}, u_i^{\rm eq}) = \xi_i^{ss}(\bar{d}_{L,i}, w_i)$. The terminal set is computed using the approach used in Baldivieso-Monasterios and Konstantopoulos (2020); given polytopic set $\mathbb{X}_{f,i}(x_i^{\rm eq}) \subseteq \mathbb{X}_i$, it is possible to find a control action $u_i = u_i^{\rm eq} + \kappa_{f,i}(x_i - x_i^{\rm eq})$ such that $x(t) \in \mathbb{X}_{f,i}(x_i^{\rm eq})$ for all $t \geq 0$. This control action is built based on the one-step reachability properties of the discretised system and is the solution of $\min\{|u_i - u_i^{\rm eq}|^2 : x_i(\delta(k+1)) \in \lambda\mathbb{X}_{f,i}(x_i^{\rm eq}),\ u_i - u_i^{\rm eq} \in \mathbb{U}_i\}$, where $\lambda \in [0, 1]$ is a design parameter to adjust the "aggressiveness" of the controller with minimisers $\mathcal{U}_{f,i}(x_i, \bar{d}_{L,i}, w_i) \subset \mathbb{U}_i$. The following proposition summarises the properties of the terminal ingredients for the OCP.

Proposition 2 (*Shifted Terminal Ingredients*)**.** *Suppose Assumptions 1 and 5 hold. (i) For a fixed $\bar{d}_{L,i} \in \mathbb{D}_i$, if there exists a control action $u_i \in \mathbb{U}_i$ such that*

$$|u_i| \geq \max_{y\in\mathbb{X}_{f,i}(x_i^{\rm eq})} \min_{z\in\lambda\mathbb{X}_{f,i}(x_i^{\rm eq})} |F_i(y, \bar{d}_{L,i}, u) - F_i(0, 0, u) - z|,$$

then $\mathcal{U}_i(x_i, \bar{d}_{L,i}, w_i) \neq \emptyset$ for all $x_i \in \mathbb{X}_{f,i}(x_i^{\rm eq})$. Furthermore, (ii) the function $\Psi_{f,i}(x_i) = \inf\{r \in \mathbb{R}^+ : x_i \in r\mathbb{X}_{f,i}\}$ is a control Lyapunov function. Lastly, (iii) the set $\mathbb{X}_{f,i}$ is control invariant for $\dot{x}_i = F_i(x_i, u_i, \bar{d}_i) + E_i w_i$.

Proof. The proof for (*i*) is based on a modification to our setting of Baldivieso-Monasterios and Konstantopoulos (2020, Theorem 3). The second assertion follows by construction, when $x_i = x_i^{\rm eq}$, then $\Psi_{f,i}(x_i^{\rm eq}) = 0$. Furthermore, the difference $\Psi_{f,i}(x_i^+) - \Psi_{f,i}(x_i) < 0$ if $u_i \in \mathcal{U}_i(x_i, \bar{d}_{f,i}, w_i)$. In particular, it is possible to construct the function $\alpha_{V,i}(x_i) = \int_t^{t+\delta} \ell(x_i, \kappa_i(x_i, \bar{d}_i, w_i))dt$ with $\kappa_i(\cdot,\cdot,\cdot)$ a selection of the set-valued map $\mathcal{U}_i(\cdot)$. Therefore, $\Psi_{f,i}(\cdot)$ satisfies an integral version of the definition of a CLF. The control invariance stated in (*iii*) follows from Baldivieso-Monasterios and Konstantopoulos (2020, Corollary 1). □

Proposition 2 shows that our terminal conditions satisfy Assumption 4 which allows us to define the feasible regions for $T > \delta > 0$ and w_i as:

$$\begin{aligned}\mathcal{X}_i^{\delta(k+1)}(\bar{d}_{L,i}, w_i) &= \{x_i \in \mathbb{X}_i : \exists u_i \in \mathbb{U}_i, x_i(\delta) \in \mathcal{X}_i^{\delta k}\}\\ \mathcal{X}_i^0(\bar{d}_{L,i}, w_i) &= \mathbb{X}_{f,i}(x_i^{\rm eq}).\end{aligned} \tag{8}$$

The existence of a feasible set $\mathcal{X}_i^{N\delta}$ is linked with the existence of a sequence of control actions $\mathbf{u}_i \in \mathbb{U}_i^N$ which is related to the OCP solution. The next result states feasibility under unchanging interconnection information.

Proposition 3 (*Recursive Feasibility Under Unchanging w_i*)**.** *Suppose Assumptions 1–6 hold. For each $i \in \mathcal{V}$, if $\mathbf{w}_i^+ = \mathbf{w}_i(t+\delta) = \mathbf{w}_i(t)$, then (i) $x_i \in \mathcal{X}_i^T(\bar{d}_i, \mathbf{w}_i)$ implies that $x(t+\delta) \in \mathcal{X}_i^T(\bar{d}_{L,i}, \mathbf{w}_i^+)$. (ii) the set $\mathcal{X}_i^T(\bar{d}_{L,i}, \mathbf{w}_i)$ is control invariant for $\dot{x}_i = F_i(x_i, u_i, \bar{d}_{L,i}) + E_i w_i$ and $\mathbb{U}_i$.*

The proof of these results follows the line of argument of Baldivieso Monasterios and Trodden (2018, Proposition 2) albeit modified to account for the nonlinear nature of the system. Standard MPC results, see for example (Grüne & Pannek, 2016, Chapter 5), *i.e.*, feasibility implies stability, lead to the following Corollary

Corollary 2 (*Local Lyapunov Function*)**.** *Suppose Assumptions 1, 2–5 hold. For each node $i \in \mathcal{V}$ and a fixed $w_i \in \mathbb{W}_i$ and $\bar{d}_{L,i} \in \mathbb{D}_i$, if $x_i^{\rm eq} \in \mathbb{X}_i$ is an equilibrium of node i, then there exist $\mathcal{K}$ functions α_{ih} with $h \in \{1, 2, 3\}$ such that the value function $V_{N,i}^0(\cdot,\cdot,\cdot)$ satisfies*

$$\begin{aligned}\alpha_{i1}(|x_i|_{x_i^{\rm eq}}) \leq V_{N,i}^0(x_i, \bar{d}_{L,i}, w_i) &\leq \alpha_{i2}(|x_i|_{x_i^{\rm eq}})\\ V_{N,i}^0(x_i^+, \bar{d}_{L,i}, w_i) - V_{N,i}^0(x_i, \bar{d}_{L,i}, w_i) &\leq -\alpha_{i3}(|x_i|_{x_i^{\rm eq}})\end{aligned} \tag{9}$$

The above results allow us to conclude that for a fixed $w_i \in \mathbb{W}_i$ and $\bar{d}_{L,i} \in \mathcal{D}_i$, the closed-loop system $\dot{x}_i = F_i(x_i, \kappa_{N,i}(x_i, \bar{d}_{L,i}, w_i), \bar{d}_{L,i}) + E_i w_i$ is asymptotically stable with respect to $\{x_i^{\rm eq}(\bar{d}_{L,i}, w_i)\}$. This next result is a generalisation to a nonlinear setting of Baldivieso Monasterios and Trodden (2018, Corollary 1); the nonlinearity of our setting makes it impossible to apply the previous result. In the following lemma, we study the regularity of the value function using the set-valued minimisers and local convexity properties.

Lemma 3 (*$\mathcal{K}$-continuity of the Value Function*). *Suppose Assumptions 1, 2, and 5 hold, fix $\bar{d}_{L,i} \in \mathbb{D}_i$. The value function $V^0_{N,i}(\cdot)$ for each $i \in \mathcal{V}$ satisfies $|V^0_{N,i}(z) - V^0_{N,i}(\hat{z})| \le \sigma_V(|z-\hat{z}|)$ over $\mathcal{Z}^N_i$ and σ_V is a $\mathcal{K}$-function and any $z, \hat{z}$ in its domain.*

Proof. Continuity of the value function implies the result by Limon et al. (2009, Lemma 1) and the Heine–Cantor theorem. To prove continuity, we claim that for any neighbourhood $\mathcal{T}$ of the minimisers $\mathcal{S}_i(x_i, \bar{d}_{L,i}, w_i)$, there exists a neighbourhood Z of (x_i, w_i) such that $\mathcal{T} \cap \mathcal{U}^N_i(x_i, \bar{d}_{L,i}, w_i) \neq \emptyset$. To prove this statement, we consider two cases an optimal point $u^0_i \in \mathcal{U}^N_i(x_i, \bar{d}_{L,i}, w_i)$ lies either in the interior of $\mathcal{U}^N_i(x_i, \bar{d}_{L,i}, w_i)$ or at the boundary $\mathcal{U}^N_i(x_i, \bar{d}_{L,i}, w_i)$. Each neighbourhood of the optimal point u^0_i satisfies $V_U \cap \mathcal{U}^N_i(x_i, \bar{d}_{L,i}, w_i) \neq \emptyset$. Given the graph of $\mathcal{U}^N_i(\cdot,\cdot,\cdot)$ is a closed set, there exists a sequence $\{(x^k_i, w^k_i, u^k_i)\}$ converging to (x_i, w_i, u^0_i), *i.e.*, $(x^k_i, w^k_i, u^k_i) \in Z \times V_U$ for $k > K$ with $K > 0$. Since $\mathbf{U}^N_i(\cdot,\cdot,\cdot)$ is closed, then $\mathrm{cl}(Z \times V_U)$ is compact. On the other hand, there exists $Z^k \times V_U \subset \mathrm{cl}Z \times V_U$ for each element of the sequence that forms a covering of $\mathrm{cl}(Z \times V_U)$, and there exists a finite subcover by compactness such that for any element of $(\tilde{x}_i, \tilde{w}_i) \in Z = \bigcap_{h=1}^{H} Z^{k_h}$, $V_U \cap \mathcal{U}^N_i(\tilde{x}_i, \bar{d}_{L,i}, \tilde{w}_i) \neq \emptyset$. This statement together with continuity of the cost function $J_i(\cdot,\cdot,\cdot)$ and closedness of the graph of $\mathcal{U}^N_i(\cdot,\cdot,\cdot)$ fulfils the hypothesis of Bonnans and Shapiro (2000, Proposition 4.4) which asserts the continuity of the value function of perturbed optimisation problems. □

In the next step in order to prove recursive feasibility, we need to investigate the effect of a changing disturbance, *i.e.*, $w_i(k+1) \neq w_i(k)$. To this aim, we invoke the following assumption:

Assumption 7 (*Bounded Interconnection*). For each $i \in \mathcal{V}$, the interconnection effect at a time $t+\delta$ satisfies $w^+_i = w_i(t+\delta) = w_i(t) + \Delta w_i$ where $\Delta w_i \in \Delta\mathbb{W}_i$. The set $\Delta\mathbb{W}_i$ is chosen such that $\lambda_i = \max\{|w - \tilde{w}| : w, \tilde{w} \in \mathbb{W}_i,\ w - \tilde{w} \in \Delta\mathbb{W}_i\}$ satisfies $\lambda_i \le \sigma_V^{-1} \circ \alpha_{i3} \circ \alpha_{i2}^{-1}(\beta_i)$ where $\beta_i > 0$.

The following result, the proof of which follows from a modification to the nonlinear case of Baldivieso Monasterios and Trodden (2018, Theorem 1), asserts recursive feasibility under changing interconnection effect.

Theorem 1 (*Recursive Feasibility*). *Suppose Assumptions 1, and 2–7 hold. If at time $t > 0$ and for a fixed $\bar{d}_{L,i} \in \mathbb{D}_i$ the state satisfies $x_i \in \mathcal{X}^N_i(w_i, \bar{d}_{L,i})$, then at time $t+\delta$, the state satisfies $x_i(t+\delta) \in \mathcal{X}^N_i(w^+_i, \bar{d}_{L,i})$.*

4.2.2. Closed-loop stability

In the previous section, we have shown that each node is recursively feasible. In this section, we analyse the stability and construct a Lyapunov function for the complete network. In the analysis, we emphasise on bounding interactions and guaranteeing local steady state convergence. Local equilibria, characterised by $F_i(x_i, u_i, \bar{d}_{L,i}) = 0$ and $u_i \in [0, I^{\max}_i]$, satisfy $(i_i, \sigma_i) \in \mathbb{Z}_i$. This allows us to analyse the voltage equilibrium pairs using

$$\begin{aligned}&\mathbb{P}^{ss}_i(\bar{d}_{L,i}, w_i) \colon \min\{|v_i - v^*|^2 : u_i \in [0, I^{\max}_i],\\ &v_i \in [V^{\min}_i, V^{\max}_i], \mathcal{L}_{ii} v_i + g_i(v_i)\bar{d}_{L,i} = u_i + w_i\}\end{aligned} \tag{10}$$

The following property sheds some light on the properties of the steady state optimisation for each node

Proposition 4. *If the optimal current $u^{ss}_i \in (0, I^{\max}_i)$, then the optimal steady state voltage satisfies $v^{ss}_i = v^*$.*

Proof. The KKT system for $\mathbb{P}^{ss}_i(\bar{d}_i)$ is given by

$$\begin{aligned}(v_i - v^*) + \lambda \nabla_v h(v_i, u_i, w_i \bar{d}_{L,i}) + \mu_v^\top \nabla_v r(v_i, u_i) &= 0\\ \lambda \nabla_u h(v_i, u_i, w_i \bar{d}_{L,i}) + \mu_u^\top \nabla_u r(v_i, u_i) &= 0\\ 0 \le \mu \perp r(v_i, u_i) \ge 0,\ h(v_i, u_i, w_i, \bar{d}_i) &= 0\end{aligned}$$

where $(\lambda, \mu_v, \mu_v) \in \mathbb{R}^5$ are the dual variables; the inequality and equality constraints are $h(v_i, u_i, w_i, \bar{d}_{L,i})$ and $r(v_i, u_i)$ respectively; and $a \perp b = a^\top b$ with $a, b \in \mathbb{R}^n$. Since by assumption u^{ss}_i is an interior point, then $\mu_u = 0$. The reduced KKT system can be expressed as $(v_i - v^*) + \mu_{v1} - \mu_{v2} = 0$, $\min(\mu_{v1}, -v_i + V^{\max}_i) = 0$, and $\min(\mu_{v2}, v_i - V^{\min}_i) = 0$ where we obtain two cases: $\mu_{v2} = 0$ or $v_i = V^{\min}_i$. The latter condition implies $\mu_{v1} = 0$ and $\mu_{v2} < 0$. The first case yields $\mu_{v1} = v^* - v_i$ and two further cases: $\mu_{v1} = 0$ or $v_i = V^{\max}_i$. Therefore $\mu_{v1} = 0$ implying $v^{ss}_i = v^*$. □

On the other hand, the network steady state pairs (x^{eq}, u^{eq}) lie in $\mathcal{H} = \{(v, u) \in \mathbb{R}^{|\mathcal{V}|} : -\mathcal{L}v + u + i_s - g_L(v)d_L = 0, d_L \in \mathbb{D},\ u \in \mathbb{U}\}$ which is a level set of a locally surjective map $\Phi_{d_L} : \mathbb{R}^{|\mathcal{V}|} \times \mathbb{R}^{|\mathcal{V}|} \to \mathbb{R}^{|\mathcal{V}|}$. The implicit function theorem guarantees $x^{eq} = \xi(u)$ such that $\Phi_{d_L}(\xi(u), u) = 0$ for all $u \in \mathbb{U}$. A natural step is to investigate the deviation of $u^{ss} = (u^{ss}_1(w_1, \bar{d}_{L,1}), \ldots, u^{ss}_{\mathcal{V}}(w_{\mathcal{V}}, \bar{d}_{L,|\mathcal{V}|}))$ and u^{eq}. The desired difference is $|u^{ss} - u^{eq}| \le (|\mathcal{L}^D| + G)|v^{eq} - v^* \mathbb{1}_{|\mathcal{V}|}| + |g_L(v^{eq})| \sum_{i \in \mathcal{V}} \gamma_{L,i}$ with $\mathcal{L}^D_{ii} = \mathcal{L}_{ii}$, $G > 0$ the Lipschitz constant for the load, and $\gamma_{L,i} >$ from Assumption 6. The resulting closed-loop network voltage dynamics are

$$\begin{aligned}C\dot{v} = &-(\mathcal{L}^D + G)(v - v^* \mathbb{1}_{|\mathcal{V}|}) + u + \tilde{s}(v - v^* \mathbb{1}_{|\mathcal{V}|})\\ &+ \tilde{r}(i - u) - g_L(v)(d_L - \bar{d}_L).\end{aligned} \tag{11}$$

where $\tilde{r}(\cdot)$ measures the error between current and its reference, $\lim_{v \to v^* \mathbb{1}_{|\mathcal{V}|}} \frac{|\tilde{s}(v - v^* \mathbb{1}_{|\mathcal{V}|})|}{|v - v^* \mathbb{1}_{|\mathcal{V}|}|} = 0$ and G is the derivative of $g_L(\cdot)\bar{d}_L$ evaluated at the reference voltage. We note that when using $u = u^{ss}$, the resulting equilibrium manifold is $\mathcal{L}^D(v - \mathbb{1}_{|\mathcal{V}|}) + (g_L(v) - g_L(v^* \mathbb{1}_{\mathcal{V}}))\bar{d}_L = g_L(v)(d_L - \bar{d}_L)$, the solution of which is clearly $v^{eq} = v^* \mathbb{1}_{\mathcal{V}}$ when $\bar{d}_L - d_L = 0$, *i.e.*, when the controller has perfect knowledge of the load. A perturbation analysis yields $v^{eq} = v^* + |d_L - \bar{d}_L| \nu_1(v^*, \bar{d}_L) + |d_L - \bar{d}_L|^2 \nu_2(v^*, \bar{d}_L) + \cdots$ where ν_1, ν_2, etc. are functions of both operating conditions and nominal load which results in a bound on $|v^{eq} - v^* \mathbb{1}_{\mathcal{V}}| \le \varepsilon |\sum_{k=1}^{\infty} \nu_k \varepsilon^{k-1}|$ with $\varepsilon = \sum_{i \in \mathcal{V}} \gamma_{L,i}$. The following result establishes the properties of the closed loop system around a neighbourhood of the nominal operating point.

Proposition 5 (*Closed Loop Control Invariance Near the Equilibrium*). *Suppose Assumptions 1–6 hold. There exists a set $\mathbb{S} \subset \mathbb{R}^{|\mathcal{V}|}$ that is control invariant for the voltage dynamics $\dot{z} = Az + u + \tilde{s}(z) + \tilde{r}(\tilde{i} - u) - g_L(z + v^* \mathbb{1}_{|\mathcal{V}|})(d - \bar{d})$ and constraint sets $(\mathbb{V}, \mathbb{U})$.*

Proof. The proof follows from the OCP formulation and the boundedness of the voltage deviations, currents, and loads w.r.t $v^* \mathbb{1}_{|\mathcal{V}|}$, $\kappa_N(v, \bar{d})$ and d_L respectively. □

The importance of the above proposition is that it ensures the existence of a control action that counteracts both the potential instability introduced by the loads and yields an equilibrium pair $(v^{eq}, \tilde{u}^{eq} + \kappa_N(v, \bar{d})) \in \mathcal{H}$. The final part of the puzzle is the analysis of the interconnection disturbance, *i.e.*, $\Delta w_i = w_i(\delta + t) - w_i(t)$ which can be bounded for all $i \in \mathcal{V}$ as $|\Delta w_i| \le (|(\mathcal{L} - \mathcal{L}^D)_i||v^+ - v|)$. The difference between solutions of $\mathbb{P}^{ss}_i(w_i(t+\delta), \bar{d}_{L,i})$ and $\mathbb{P}^{ss}_i(w_i(t), \bar{d}_{L,i})$ behaves similarly, *i.e.*, $|u^{ss}_i(w_i(t+\delta), \bar{d}_{L,i}) - u^{ss}_i(w_i, \bar{d}_{L,i})| \le |\mathcal{L}^C_i||v^+ - v|$. Moreover, from (11), $|v^+ - v| \le (G + |\mathcal{L}^D|)\int_t^{t+\delta} |v - v^* \mathbb{1}_{\mathcal{V}}| dt + \delta|u| + |g_L(v)| \sum_{i \in \mathcal{V}} \gamma_{L,i} + \alpha_V(|v - v^* \mathbb{1}_{\mathcal{V}}|) + \delta(|u| + \Gamma \sum_{i \in \mathcal{V}} \gamma_{L,i})$ holds[3]

[3] In the last inequality, we have used the following well known property of $\mathcal{K}$-functions: the integral of a class $\mathcal{K}$-function is also class $\mathcal{K}$.

with $|g_L(v)| \leq \Gamma$ for all v. We are now in position to state the main result of this paper:

Theorem 2 (*Closed-loop Stability*). *Suppose Assumptions 1, 2–6 hold. If in addition, $\sigma_V(r) = L_V r$ for $r > 0$ and $L_V > 0$, and there exists $\eta_i > 0$ for all $i \in \mathcal{V}$ such that $\alpha(r) = \sum_{i\in\mathcal{V}} \eta_i \alpha_{3i}(r) - L_V(\sum_{i\in\mathcal{V}} \eta_i |(\mathcal{L} - \mathcal{L}^D)_i|)\alpha_V(r)$ is a $\mathcal{K}$-function, then the network of buck converters (1) is input to state stable (ISS) in closed loop with the control law (2a) and current $i_i^{\text{ref}} = \kappa_{N,i}(x_i, w_i, \bar{d}_{L,i})$.*

Proof. Following Corollary 2, the value function for each $i \in \mathcal{N}$ is a Lyapunov function with respect to an equilibrium point $x_i^{\text{eq}}(\bar{d}_{L,i}, w_i) \in \mathbb{X}_i$ depending on nominal loads and interactions. Furthermore, Proposition 1 provides us with a Lyapunov function for the driving subsystem for each $i \in \mathcal{V}$. Our strategy of proving asymptotic stability of the closed-loop system hinges on showing $\Psi(x) = \sum_{i\in\mathcal{V}} \eta_i(V_{N,i}^0(x_i, \bar{d}_{L,i}, w_i) + W_i(x_i))$ is a Lyapunov function. Here, $V_{N,i}^0(\cdot, \cdot)$ is the OCP value function and $W_i(\cdot)$ is defined in (4); $\eta_i > 0$ is a suitable weight as in Siljak (2007). The variation $\Delta\Psi(x) = \Psi(x(t+\delta)) - \Psi(x(t))$, following Corollary 2 and Proposition 1, is $\Delta\Psi(x) \leq -\alpha(|v - v^*\mathbb{1}_{\mathcal{V}}|) - \tilde{\alpha}(|i - \kappa_N(x, \bar{d}_{L,i})|) - \hat{\alpha}(|\sigma - \frac{1}{I^{\max}}\kappa_N(x, \bar{d}_{L,i})|) + L_V \sum_{i\in\mathcal{V}} \eta_i |(\mathcal{L} - \mathcal{L}^D)_i| \delta(|u| + \Gamma \sum_{i\in\mathcal{V}} \gamma_{L,i})$. Furthermore $\tilde{\alpha} = \sum_{i\in\rangle} \eta_i \tilde{\alpha}_{4i}$ where $\tilde{\alpha}_{4i} r = \frac{1}{3}\alpha_{3i}(r) + \int_t^{t+\delta} |r| dt$ and $\hat{\alpha} = \sum_{i\in\mathcal{V}} \frac{\eta_i}{3}\alpha_{3i}$ are $\mathcal{K}$-functions. Following that $|u| \to u^{\text{eq}}$ as the system approaches its equilibrium, and the uncertainty of the load is bounded by Assumption 6, the ISS of the buck converter network follows. □

5. Simulations

In this section, we explore the behaviour of a network of $|\mathcal{V}| = 6$ interconnected, without ignoring line dynamics, buck converters where each source feeds a constant power load, *i.e.*, each $g_{L,i}(v_i) = v_i^{-1}$. The input is $V_{in} = 800$ V, the operating voltage is set to $v^* = 560$ V. The rated power for each converter is given as $P_{C,i}^{\max} = \{43, 39, 46, 39, 50, 42\}$ kW; the voltage constraint sets are $\mathbb{V}_i = V_{\text{in}}[0.3, 1]$ yielding: $I_i^{\max} = \frac{P_{C,i}^{\max}}{0.3V_{\text{in}}}$. The primary controller is given by (3a) and the translation $\tilde{i} = i_i - i_s$ with $i_s = \frac{1}{2}I_i^{\max}$. The cost used or each $i \in \mathcal{V}$ is $\ell_i(x_i, u_i) = q_i|v_i - v^*|^2 + n_i|u_i - u_i^{\text{ss}}(w_i, \bar{d}_i)|$. The terminal set is $\mathbb{X}_i^f = v^* + [-\Delta V_i, \Delta V_i] \times [0, I_i^{\max}] \times [-1, 1]$ with $\Delta V_i = 10$ V. Finally, the sampling time used in this simulation is $\delta = 2$ ms, and the simulation horizon is 14 s. The simulation shows the behaviour of the network to *a priori* unknown load changes switching to values that exceed their converter power rate, albeit the condition $\sum_{i\in\mathcal{V}} P_{L,i} \leq \sum_{i\in\mathcal{V}} P_{C,i}^{\max}$ always holds. When power converter i cannot feed its own load, the rest of the network aids this converter while accounting for losses in the network. The load steps in the following way: $P_{L,1} = 0.95P_{C,i}^{\max}$ and at $t = 0.6$ s it switches to $P_{L,1} = 0.735P_{C,1}^{\max}$; $P_{L,2} = 1.14P_{C,2}^{\max}$ and at $t = 1.24$ s it switches to $P_{L,2} = 0.73P_{C,2}^{\max}$; $P_{L,4} = 0.5P_{C,4}^{\max}$ and at $t = 0.93$ s it switches to $P_{L,4} = 1.03P_{C,4}^{\max}$; and $P_{L,6} = 0.66P_{C,6}^{\max}$ and at $t = 0.3$ s it switches to $P_{L,6} = 1.05P_{C,6}^{\max}$. As seen in Fig. 1, the voltage of each node reacts to a change in load without leaving the terminal set, we note the non-minimal phase type behaviour exhibited by all the voltages when reacting to a load step. From Fig. 1, the voltages clearly converge to neighbourhoods of the equilibrium points. As the converters converge towards a neighbourhood of the equilibrium point because of the uncertainty introduced in the load. We see the effect of this deviation from the nominal point in Fig. 2 where we portray the distance of the centralised equilibrium v^{eq} computed for the uncertain load to the operating point $v^*\mathbb{1}_{|\mathcal{V}|}$ and the voltage v.

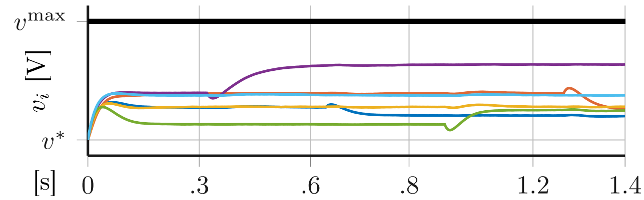

Fig. 1. Voltage responses of the network to arbitrary and uncertain load variations. All voltages remain within a compact neighbourhood of the operating voltage v^*.

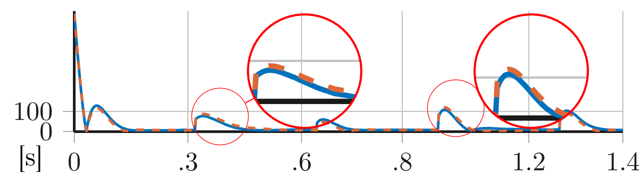

Fig. 2. Deviation from voltage centralised equilibrium, (——) represents $|v^{\text{eq}} - v^*\mathbb{1}_{|\mathcal{V}|}|$ and (- - -) represents $|v^{\text{eq}} - v|$.

6. Conclusions

In this paper, we have proposed a decentralised primary controller, together with a distributed secondary controller for a DC network of Buck power converters connected in meshed topology. For the primary controller, we have proposed a Lyapunov function candidate defined over a compact set which is considered a safety region. Using this Lyapunov function, we have proven the asymptotic stability of an equilibrium point using the invariance principle. We have analysed the interconnection between voltages and currents to conclude the attractiveness of a neighbourhood of the kernel of the network Laplacian matrix. We leverage this result to prove the ISS behaviour of the closed-loop system. We have also proven the recursive feasibility of the controller. Future lines of research may involve coupled constraint satisfaction which would account for line saturation, an economic network operation that would relax the need for terminal ingredients in the MPC framework, and different converter topologies.

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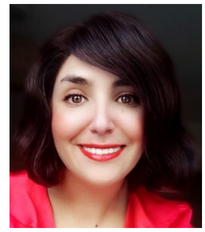

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