KLINGEN EISENSTEIN SERIES CONGRUENCES AND MODULARITY

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Abstract. We construct a mod ℓ congruence between a Klingen Eisenstein series (associated to a classical newform ϕ of weight k) and a Siegel cusp form f with irreducible Galois representation. We use this congruence to show non-vanishing of the Bloch-Kato Selmer group $H^1_f(\mathbf{Q}, \mathrm{ad}^0\rho_\phi(2-k) \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell)$ under certain assumptions and provide an example. We then prove an $R = dvr$ theorem for the Fontaine-Laffaille universal deformation ring of $\overline{\rho}_f$ under some assumptions, in particular, that the residual Selmer group $H^1_f(\mathbf{Q}, \mathrm{ad}^0\overline{\rho}_\phi(k-2))$ is cyclic. For this we prove a result about extensions of Fontaine-Laffaille modules. We end by formulating conditions for when $H^1_f(\mathbf{Q}, \mathrm{ad}^0\overline{\rho}_\phi(k-2))$ is non-cyclic and the Eisenstein ideal is non-principal.

1. Introduction

The construction of Eisenstein congruences has a long and consequential history. Interesting in their own right, their significance is amplified by the existence of Galois representations attached to the congruent forms, as the ones attached to Eisenstein series are always reducible while the ones attached to cusp forms are often irreducible. Using various generalizations of the result known as Ribet's Lemma, they lead to the construction of non-zero elements in Selmer groups. This direction was first explored by Ribet himself in the context of the group GL_2 in [Rib76] and later used by many other authors in a variety of different settings e.g. [Wil90], [Bro07], [SU14].

In a different direction, such congruences can play a crucial role in proving modularity of deformations of reducible residual Galois representations $\overline{\rho}$, see e.g. [SW97], [BK13], [BK20], [BK23], [Cal06], [WWE20], and [Wak23]. In [Cal06] Calegari introduced a method of proving modularity assuming $\overline{\rho}$ is unique up to isomorphism, which relies on proving the principality of the ideal of reducibility of the universal deformation ring R of $\overline{\rho}$. This method was developed further by Berger and Klosin [BK11, BK13, BK20] and Wake and Wang-Erickson [WWE20] and successfully applied in many contexts (see also [Ake23, Hua23]). It relies heavily on the ideas of Bellaiche and Chenevier [BC09] and their study of Generalized Matrix Algebras (GMAs).

In this paper we pursue both of these directions in the case of Klingen Eisenstein series of level one on the group Sp_4. More precisely, let $k > 4$ be an even integer and ϕ a classical weight k Hecke eigenform of level 1 (i.e., on the group $\mathrm{GL}_{2/\mathbf{Q}}$). Write $E^{2,1}_\phi$ for the (appropriately normalized) Klingen Eisenstein series on Sp_4 induced from ϕ. It is a Siegel modular form of weight k and full level. Congruences between Klingen Eisenstein series and cusp forms have been studied previously by Katsurada and Mizumoto [KM12, Miz86] and Urban (unpublished). Katsurada and Mizumoto obtain congruences as an application of the doubling method. In this paper, we produce congruences via a much shorter argument using results of Yamauchi [Yam21]. The trade-off is that while our proof is much shorter, we obtain congruences only modulo a prime ℓ whereas Katsurada and Mizumoto obtain congruences modulo powers of ℓ. However, the hypotheses required for our result are different and less restrictive than those needed in [KM12]. We show that under certain conditions $E^{2,1}_\phi$ is congruent to some cusp form f of the same weight and level with irreducible Galois representation (Theorem 3.5). This is the first main result of the paper. These congruences are

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governed by the numerator of the (algebraic part) of the symmetric square L-function $L_{\mathrm{alg}}(2k-2, \mathrm{Sym}^2\phi)$ of ϕ. We also exhibit a concrete example when the assumptions of Theorem 3.5 are satisfied (see Example 3.6).

We then proceed to show that these congruences give rise (under some assumptions) to non-trivial elements in the Selmer group $H_{2-k} := H^1_f(\mathbf{Q}, \mathrm{ad}\,\rho_\phi(2-k)\otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell)$. Here ρ_ϕ is the Galois representation attached to ϕ by Deligne and we use the Fontaine-Laffaille condition at ℓ. Assuming the Vandiver Conjecture for ℓ we also deduce the non-triviality of the Selmer group $H^1_f(\mathbf{Q}, \mathrm{ad}^0\,\rho_\phi(2-k)\otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell)$ (Corollary 5.9 and Remark 5.8). This is our second main result and gives evidence for new cases of the Bloch-Kato conjecture. This conjecture was studied for other twists of $\mathrm{ad}\,\rho_\phi$ by [DFG04] and [Klo09]. In [Urb01] Urban assumed the existence of Klingen Eisenstein congruences to prove a result towards the main conjecture of Iwasawa theory for the adjoint L-function.

To properly analyze these Selmer groups we require some results on extensions of Fontaine-Laffaille modules whose proofs appear to be absent in the literature. In Section 4 we carefully study certain aspects of Fontaine-Laffaille theory, in particular, prove the Hom-tensor adjunction formula and give a precise definition of Selmer groups with coefficient rings of finite length.

Given the congruence $E^{2,1}_\phi \equiv f \pmod{\ell}$ we also study deformations of a non-semi-simple Galois representation $\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_4(\overline{\mathbf{F}}_\ell)$ whose semi-simplification arises from the Klingen Eisenstein series. Such a representation is reducible with two 2-dimensional Jordan-Holder blocks and more precisely one has

$$\overline{\rho} = \begin{bmatrix} \overline{\rho}_\phi & * \\ & \overline{\rho}_\phi(k-2) \end{bmatrix}.$$

Conjecturally such representations should arise as mod ℓ reductions of Galois representations attached to Siegel cusp forms which are congruent to $E^{2,1}_\phi$ mod ℓ. We assume that $\dim H_{2-k}[\ell] = 1$, where $[\ell]$ indicates ℓ-torsion. This can be seen as a refinement of the uniqueness assumption of [SW97] similar to the one in [BK13] and as in [BK13, Cal06] we prove the principality of the reducibility ideal of the universal deformation. However, this principality cannot be achieved through the method of [BK13] because the representation in question fails to satisfy the strong self-duality property required for the method of [loc.cit.]. Instead we improve on a recent result of Akers [Ake23] which replaces the self-duality condition with a one-dimensionality assumption on the Selmer group $H_{k-2} := H^1_f(\mathbf{Q}, \mathrm{ad}\,\overline{\rho}_\phi(k-2))$ of the 'opposite' Tate twist of $\mathrm{ad}\,\rho_\phi$. With these assumptions in place we are able to show that the universal deformation ring R is a discrete valuation ring and prove a modularity result guaranteeing that the unique deformation of $\overline{\rho}$ indeed arises from a Siegel cusp form congruent to $E^{2,1}_\phi$ (Theorem 6.20). This is the third main result of the paper.

We then proceed to formulate conditions for non-cyclicity of the Selmer group H_{k-2}. While many results in the literature give bounds on the orders of Selmer groups (in particular, Corollary 5.9 gives such a lower bound on H_{2-k}), the structure of these groups is notoriously mysterious. In this paper we prove that if the (local) Klingen Eisenstein ideal $J_{\mathfrak{m}}$ is not principal then H_{k-2} is not cyclic (Corollary 7.3). We further refine this result by providing a criterion for non-principality in terms of the depth of congruences between cusp forms and $E^{2,1}_\phi$ (Corollary 7.5). An intriguing feature of these results is that H_{k-2} is non-critical, i.e. this Selmer group is not controlled by a critical L-value in the sense of Deligne.

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2. Background and notation

2.1. **Galois representation.** Given a field F we denote by G_F its absolute Galois group. Fix a rational prime $\ell > 2$. If M is a topological $\mathbf{Z}_\ell[G_F]$-module (i.e. the G_F-action is continuous) we set $H^i(F, M)$ to be the ith cohomology group of continuous cocycles.

For such M we set

$$M^\vee = \mathrm{Hom}_{\mathrm{cont}}(M, \mathbf{Q}_\ell/\mathbf{Z}_\ell)$$

to be the Pontryagin dual of M and we will write $M(n) = M \otimes \epsilon^n$ for the n-th Tate twist where ϵ denotes the ℓ-adic cyclotomic character.

For each prime p, we fix an embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. This is equivalent to choosing a prime $\overline{p}$ of $\overline{\mathbf{Q}}$ lying over p and fixes an isomorphism $D_p \cong G_{\mathbf{Q}_p}$, where D_p is the decomposition group of $\overline{p}$. We will denote by $I_p \subset D_p$ the corresponding inertia group. We also fix an isomorphism $\overline{\mathbf{Q}}_\ell \cong \mathbf{C}$.

Let E denote a finite extension of $\mathbf{Q}_\ell$ with valuation ring $\mathcal{O}$, uniformizer λ, and residue field $\mathbf{F}$.

For a continuous homomorphism $\rho : G_F \to \mathrm{GL}_n(\mathcal{O})$ we write $\overline{\rho} : G_F \to \mathrm{GL}_n(\mathbf{F})$ for the mod λ reduction of ρ. We note that given a continuous representation $\rho : G_F \to \mathrm{GL}_n(E)$ one can always find a G_F-stable lattice inside its space and thus get a continuous homomorphism valued in $\mathrm{GL}_n(\mathcal{O})$. While the isomorphism class of this homomorphism may depend on the choice of this lattice, the semi-simplification of its reduction does not, so $\overline{\rho}^{\mathrm{ss}}$ is always well-defined.

2.2. **Siegel modular forms.** Let n be a positive integer. Let Mat_n denote the affine group scheme over $\mathbf{Z}$ of $n \times n$ matrices. For $\gamma \in \mathrm{Mat}_n$, we write $|\gamma|$ for the determinant of γ. Given a matrix $\gamma \in \mathrm{Mat}_{2n}$, we will often write it as

$$\gamma = \begin{bmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{bmatrix}$$

where the blocks are in Mat_n. We will denote by GL_n the affine group scheme over $\mathbf{Z}$ of invertible $n \times n$ matrices. Recall that the degree n symplectic group is defined by

$$\mathrm{GSp}_{2n} = \left\{ g \in \mathrm{GL}_{2n} : {}^t g J_n g = \mu_n(g) J_n, \mu_n(g) \in \mathrm{GL}_1 \right\}$$

where $J_n = \begin{bmatrix} 0_n & -1_n \\ 1_n & 0_n \end{bmatrix}$ where 1_n is the n by n identity matrix, and $\mu_n : \mathrm{GL}_{2n} \to \mathrm{GL}_1$ is the homomorphism defined via the equation given in the definition. Write $\mathrm{GSp}^+_{2n}(\mathbf{R})$ for the subgroup of $\mathrm{GSp}_{2n}(\mathbf{R})$ consisting of elements g with $\mu_n(g) > 0$. We set $\mathrm{Sp}_{2n} = \ker(\mu_n)$ and denote $\mathrm{Sp}_{2n}(\mathbf{Z})$ by Γ_n to ease notation. Note that $\mathrm{Sp}_2 = \mathrm{SL}_2$, the subgroup scheme of GL_2 of matrices of determinant one.

The Siegel upper half-space is given by

$$\mathfrak{h}_n = \{ z = x + iy \in \mathrm{Mat}_n(\mathbf{C}) : x, y \in \mathrm{Mat}_n(\mathbf{R}),\, {}^t z = z, y > 0 \}$$

where we write $y > 0$ to indicate that y is positive definite. The group $\mathrm{GSp}^+_{2n}(\mathbf{R})$ acts on $\mathfrak{h}_n$ via $\gamma z = (a_\gamma z + b_\gamma)(c_\gamma z + d_\gamma)^{-1}$.

Let $f : \mathfrak{h}_n \to \mathbf{C}$ be a function. Set

$$(f|_\kappa \gamma)(z) = \mu_n(\gamma)^{nk/2} j(\gamma, z)^{-k} f(\gamma z)$$

for $\gamma \in \mathrm{GSp}^+_{2n}(\mathbf{R})$ and $z \in \mathfrak{h}_n$ where $j(\gamma, z) = \det(c_\gamma z + d_\gamma)$. A Siegel modular form of weight k and level Γ_n is a holomorphic function $f : \mathfrak{h}_n \to \mathbf{C}$ satisfying

$$(f|_k \gamma)(z) = f(z)$$

for all $\gamma \in \Gamma_n$. If $n = 1$, we also require the standard growth condition at the cusp. We denote the $\mathbf{C}$-vector space of Siegel modular forms of weight k and level Γ_n as $M_k(\Gamma_n)$. Any $f \in M_k(\Gamma_n)$ has a Fourier expansion of the form

$$f(z) = \sum_{T \in \Lambda_n} a(T; f) e(\mathrm{Tr}(Tz))$$

where Λ_n is defined to be the set of n by n half-integral (diagonal entries are in $\mathbf{Z}$, off diagonal are allowed to lie in $\frac{1}{2}\mathbf{Z}$) positive semi-definite symmetric matrices and $e(w) := e^{2\pi i w}$. Given a ring $A \subset \mathbf{C}$, we write $f \in M_k(\Gamma_n; A)$ if $a(T; f) \in A$ for all $T \in \Lambda_n$.

We define the Siegel operator $\Phi : M_k(\Gamma_n) \to M_k(\Gamma_{n-1})$ by

$$\Phi(f)(z) = \lim_{t \to \infty} f\left(\begin{bmatrix} z & 0 \\ 0 & it \end{bmatrix} \right).$$

We say $f \in M_k(\Gamma_n)$ is a cuspform if $\Phi(f) = 0$. Set $S_k(\Gamma_n) = \ker(\Phi)$.

We will now introduce certain Eisenstein series, which will play a prominent role in this paper. For $n \geq 1$ and $0 \leq r \leq n$ define the parabolic subgroup

$$P_{n,r} = \left\{ \begin{bmatrix} a_1 & 0 & b_1 & * \\ * & u & * & * \\ c_1 & 0 & d_1 & * \\ 0 & 0 & 0 & {}^t u^{-1} \end{bmatrix} \in \Gamma_n : \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in \Gamma_r, u \in \mathrm{GL}_{n-r}(\mathbf{Z}) \right\}.$$

We define projections

$$\begin{aligned} \star : \mathfrak{h}_n &\to \mathfrak{h}_r \\ z = \begin{bmatrix} z^\star & * \\ * & * \end{bmatrix} &\mapsto z^\star \end{aligned}$$

and

$$\begin{aligned} \star : P_{n,r} &\to \Gamma_r \\ \gamma &\mapsto \gamma^\star = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}. \end{aligned}$$

These allow us to define the Eisenstein series of interest. Let $\phi \in S_k(\Gamma_1)$. The Klingen Eisenstein series attached to ϕ is the series

$$E_\phi^{2,1}(z) = \sum_{\gamma \in P_{2,1} \backslash \Gamma_2} \phi((\gamma z)^\star) j(\gamma, z)^{-k}$$

where $z \in \mathfrak{h}_2$. The Eisenstein series converges for $k > 4$, see [Kli90] Theorem 1 page 67 for example. Note that [Kli90] Proposition 5 page 68 gives $\Phi(E_\phi^{2,1}) = \phi$.

Given two Siegel modular forms $f_1, f_2 \in M_k(\Gamma_n)$ with at least one a cusp form, we define the Petersson product of f_1 and f_2 by

$$\langle f_1, f_2 \rangle = \int_{\Gamma_n \backslash \mathfrak{h}_n} f_1(z) \overline{f_2(z)} (\det y)^k d\mu z,$$

where $z = x + iy$ with $x = (x_{\alpha,\beta})$, $y = (y_{\alpha,\beta}) \in \mathrm{Mat}_n(\mathbf{R})$,

$$d\mu z = (\det y)^{-(n+1)} \prod_{\alpha \leq \beta} dx_{\alpha,\beta} \prod_{\alpha \leq \beta} dy_{\alpha,\beta}$$

with $dx_{\alpha,\beta}$ and $dy_{\alpha,\beta}$ the usual Lebesgue measure on $\mathbf{R}$.

Given $\gamma \in \mathrm{GSp}_{2n}^+(\mathbf{Q})$, we write $T(\gamma)$ to denote the double coset $\Gamma_n \gamma \Gamma_n$. We define the usual action of $T(\gamma)$ on Siegel modular forms by setting

$$T(\gamma) f = \sum_i f|_k \gamma_i$$

where the γ_i are given by the finite decomposition $\Gamma_n \gamma \Gamma_n = \coprod_i \Gamma_n \gamma_i$ and $f \in M_k(\Gamma_n)$. Let p be prime and define Hecke operators

$$T^{(n)}(p) = T(\mathrm{diag}(1_n, p1_n))$$

and for $1 \leq i \leq n$ set

$$T_i^{(n)}(p^2) = T(\mathrm{diag}(1_{n-i}, p1_i, p^2 1_{n-i}, p1_i)).$$

The spaces $M_k(\Gamma_n)$ and $S_k(\Gamma_n)$ are both stable under the action of $T^{(n)}(p)$ and $T_i^{(n)}(p^2)$ for $1 \leq i \leq n$ and all p. We say a nonzero $f \in M_k(\Gamma_n)$ is an eigenform if it is an eigenvector of $T^{(n)}(p)$ and $T_i^{(n)}(p^2)$ for all p and all $1 \leq i \leq n$. As we will be focused on the case $n = 2$, we specialize to that case.

We let $\mathbf{T}'$ denote the $\mathbf{Z}$-subalgebra of $\mathrm{End}_{\mathbf{C}}(S_k(\Gamma_2))$ generated by the Hecke operators $T^{(2)}(p)$ and $T_1^{(2)}(p^2)$ for all primes p.

Recall that $E/\mathbf{Q}_\ell$ denotes a finite extension with valuation ring $\mathcal{O}$ and uniformizer λ. Given eigenforms $f_1, f_2 \in M_k(\Gamma_n; \mathcal{O})$, we write $f_1 \equiv_{\mathrm{ev}} f_2 \pmod{\lambda}$ if $\lambda_{f_1}(T) \equiv \lambda_{f_2}(T) \pmod{\lambda}$ for all $T \in \mathbf{T}'$ where $Tf_i = \lambda_{f_i}(T) f_i$.

Let $f \in S_k(\Gamma_n)$ be an eigenform. Associated to f is a cuspidal automorphic representation π_f of $\mathrm{PGSp}_{2n}(\mathbf{A})$, where $\mathbf{A}$ denotes the ring of adeles of $\mathbf{Q}$. We can decompose π_f into local components $\pi_f = \bigotimes' \pi_{f,p}$, where the tensor product is the restricted tensor product and runs over all primes p with $\pi_{f,p}$ an Iwahori spherical representation of $\mathrm{PGSp}_{2n}(\mathbf{Q}_\ell)$. The representation $\pi_{f,p}$ is given as $\pi(\chi_0, \chi_1, \dots, \chi_n)$ for χ_i unramified characters of $\mathbf{Q}_\ell^\times$. One can see Section 3.2 of [AS01] for the definition of this spherical representation. Let $\alpha_0(p; f) = \chi_0(p), \dots, \alpha_n(p; f) = \chi_n(p)$ denote the p-Satake parameters of f. Note these are normalized so that

$$\alpha_0(p; f)^2 \alpha_1(p; f) \cdots \alpha_n(p; f) = 1.$$

We drop f and/or p in the notation for the Satake parameters when they are clear from context. Set $\widetilde{\alpha_0} = p^{\frac{2nk - n(n+1)}{4}} \alpha_0$ and

$$L_p(X, f; \mathrm{spin}) = (1 - \widetilde{\alpha_0} X) \prod_{j=1}^{n} \prod_{1 \le i_1 \le \cdots \le i_j \le n} (1 - \widetilde{\alpha_0} \alpha_{i_1} \cdots \alpha_{i_j} X).$$

The spinor L-function associated to f is given by

$$L(s, f; \mathrm{spin}) = \prod_p L_p(p^{-s}, f; \mathrm{spin})^{-1}.$$

The product converges for $\Re(s) > k+1$, has meromorphic continuation to the entire complex plane, and satisfies a functional equation. We will be interested in the cases of $n = 1$ and $n = 2$. In the case $n = 1$, we set

$$L(s, \phi) := L(s, \phi; \mathrm{spin}).$$

Note that in this case

$$L_p(p^{-s}, \phi; \mathrm{spin}) = (1 - \lambda_\phi(p) p^{-s} + p^{k-1-2s})$$

and we write $\lambda_\phi(p)$ is the eigenvalue of $T(p) := T^{(1)}(p)$ corresponding to the eigenform $\phi \in S_k(\Gamma_1)$. In the case that $n = 2$, one has

$$\begin{aligned} L_p(p^{-s}, f, \mathrm{spin}) =& (1 - \lambda_f(p) p^{-s} + (\lambda_f(p)^2 - \lambda_f(p; T_1^{(2)}(p^2)) - p^{2k-4}) p^{-2s} \\ & - \lambda_f(p) p^{2k-3-3s} + p^{4k-6-4s}) \end{aligned}$$

where we write $\lambda_f(p)$ is the eigenvalue of $T^{(2)}(p)$ corresponding to f and $\lambda_f(p; T_1^{(2)}(p^2))$ for the eigenvalue $T_1^{(2)}(p^2)$ corresponding to f.

In the $n = 1$ case we will also be interested in the symmetric square L-function attached to ϕ:

$$L(s, \mathrm{Sym}^2 \phi) = \prod_p L_p(p^{-s}, \mathrm{Sym}^2 \phi)^{-1}$$

where

$$L_p(X, \mathrm{Sym}^2 \phi) = (1 - \widetilde{\alpha_0}^2 X)(1 - \widetilde{\alpha_0}^2 \alpha_1 X)(1 - \widetilde{\alpha_0}^2 \alpha_1^2 X).$$

The symmetric square L-function converges in the right half-plane $\Re(s) > k$, satisfies a functional equation, and has analytic continuation to the entire complex plane.

The following result of Laumon and Weissauer attaches an ℓ-adic Galois representation to any eigenform $f \in S_k(\Gamma_2)$.

Theorem 2.1 ([Wei05] Theorem 1). *Let $f \in S_k(\Gamma_2)$ be an eigenform. For a sufficiently large finite extension $F/\mathbf{Q}_\ell$ one has $L_q(X, f, \mathrm{spin}) \in F[X]$ for all primes $q \neq \ell$ and there is a four dimensional semisimple continuous representation*

$$\rho_{f,\varpi} : G_{\mathbf{Q}} \to \mathrm{GL}_4(F)$$

which is unramified outside of ℓ so that for $q \neq \ell$ one has

$$L_q(X, f; \mathrm{spin}) = \det(1 - \rho_{f,\varpi}(\mathrm{Frob}_q)X)$$

where ϖ is the uniformizer of F. The eigenvalues of $\rho_{f,\varpi}(\mathrm{Frob}_q)$ are algebraic integers for $q \neq \ell$.

We will write ρ_f for $\rho_{f,\varpi}$ when ϖ is clear from context.

3. Congruence

We keep the notation of section 2. Throughout this section we fix an even weight $k > 4$ and an odd prime ℓ and make the following assumption.

Assumption 3.1. Given an even weight $k > 4$ and prime ℓ assume that $E/\mathbf{Q}_\ell$ is sufficiently large to contain the fields F from Theorem 2.1 for all forms $f \in S_k(\Gamma_2)$. We also assume that for every eigenform $\phi \in S_k(\Gamma_1)$ the field E contains all the Hecke eigenvalues of ϕ as well as the value $L_{\mathrm{alg}}(2k-2, \mathrm{Sym}^2 \phi)$ (see (3.1) for the definition). In addition we suppose that E contains a primitive cube root of unity.

Recall that we denote the valuation ring of E by $\mathcal{O}$. Let $\phi \in S_k(\Gamma_1)$ be an eigenform and consider the Klingen Eisenstein series $E_\phi^{2,1}$. In this section we show under certain conditions that $E_\phi^{2,1}$ is eigenvalue-congruent to a cuspidal Siegel modular form with irreducible Galois representation.

Write

$$E_\phi^{2,1}(z) = \sum_{T \geq 0} a(T; E_\phi^{2,1}) e(\mathrm{Tr}(Tz)).$$

For T that are singular, i.e., $|T| = 0$, one has T is unimodularly equivalent to $\begin{bmatrix} n & 0 \\ 0 & 0 \end{bmatrix}$ for some $n \in \mathbf{Z}_{\geq 0}$. For such T, one has $a(T; E_\phi^{2,1}) = a(n; \phi)$ where $\phi(z) = \sum_{n>0} a(n; \phi) e(nz)$.

We use the following result to prove our congruence.

Corollary 3.2 ([Yam21] Corollary 2.3). *Assume $\ell \geq 7$. Let g be a Hecke eigenform in $M_k(\Gamma_2; \mathcal{O})$ with Fourier expansion $g(z) = \sum_{T \geq 0} a(T; g) e(\mathrm{Tr}(Tz))$. Assume that $\lambda \mid a(T; g)$ for all T with $|T| = 0$ and that there exists at least one $T > 0$ with $a(T; g) \in \mathcal{O}^\times$. Then there exists a Hecke eigenform $f \in S_k(\Gamma_2; \mathcal{O})$ so that $g \equiv_{\mathrm{ev}} f \not\equiv_{\mathrm{ev}} 0 \pmod{\lambda}$.*

For $T = \begin{bmatrix} m & r/2 \\ r/2 & n \end{bmatrix}$, we say T is primitive if $\gcd(m, n, r) = 1$. We set $|2T| = \Delta(T)\mathfrak{f}^2$ for a positive integer $\mathfrak{f}$ and where $-\Delta(T)$ is the discriminant of the quadratic field $\mathbf{Q}(\sqrt{-|2T|})$. We set $\chi_T = \left(\frac{-\Delta(T)}{\cdot}\right)$, the quadratic character associated to the field $\mathbf{Q}(\sqrt{-|2T|})$.

Define

$$\begin{aligned} \vartheta_T(z) &= \sum_{a,b \in \mathbf{Z}^2} e(z(ma^2 + rab + nb^2)) \\ &= \sum_{n \geq 0} b(n; \vartheta_T) e(nz). \end{aligned}$$

Given $v \in \mathbf{Z}_{\geq 1}$, set

$$\vartheta_T^{(v)}(z) = \sum_{n \geq 0} b(v^2 n; \vartheta_T) e(nz).$$

One can check that $\vartheta_T^{(v)} \in M_1(\Gamma(4|T|))$ where $\Gamma(N) = \ker(\mathrm{SL}_2(\mathbf{Z}) \to \mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z}))$ and $M_k(\Gamma(N))$ denotes the modular forms of weight k and level $\Gamma(N)$. Set

$$D(s, \phi, \vartheta_T^{(v)}) = \sum_{n \geq 1} a(n;\phi) b(v^2 n; \vartheta_T) n^{-s}.$$

We have that $D(s, \phi, \vartheta_T^{(v)})$ converges in a right half-plane with meromorphic continuation to the entire complex plane ([Shi76]). Set

$$L_{\mathrm{alg}}(2k-2, \mathrm{Sym}^2 \phi) := \frac{L(2k-2, \mathrm{Sym}^2 \phi)}{\pi^{3k-3}\langle \phi, \phi \rangle}, \tag{3.1}$$

$$L_{\mathrm{alg}}(k-1, \chi_T) = \frac{\Delta(T)^{k-3/2} L(k-1, \chi_T)}{\pi^{k-1}},$$

and

$$D_{\mathrm{alg}}(k-1, \phi, \vartheta_T^{(v)}) = \frac{D(k-1, \phi, \vartheta_T^{(v)})}{\pi^{k-1}\langle \phi, \phi \rangle}.$$

We have each of these terms is algebraic, see ([Shi76], [Stu80], [Zag77]). Moreover, we have via [Zag77] Equation (22) that if $\ell > k-1$, then $L_{\mathrm{alg}}(k-1, \chi_T)$ is ℓ-integral.

Theorem 3.3. [Miz84] *Let $\phi \in S_k(\Gamma_1)$ be a normalized eigenform with a Fourier expansion as above. Let $T > 0$ be primitive. We have*

$$a(T; E_\phi^{2,1}) = (-1)^{k/2} \frac{(k-1)!}{(2k-2)!} 2^{k-1} \frac{L_{\mathrm{alg}}(k-1, \chi_T)}{L_{\mathrm{alg}}(2k-2, \mathrm{Sym}^2 \phi)} \cdot \sum_{\substack{m | \mathfrak{f} \\ m > 0}} M_T(\mathfrak{f} m^{-1}) \sum_{\substack{t | m \\ t > 0}} \mu(t) D_{\mathrm{alg}}(k-1, \phi, \vartheta_T^{(m/t)})$$

where

$$M_T(a) = \sum_{\substack{d | a \\ d > 0}} \mu(d) \chi_T(d) d^{k-2} \sigma_{2k-3}(ad^{-1})$$

and

$$\sigma_s(d) = \sum_{\substack{g | d \\ g > 0}} g^s.$$

Note that while this theorem is only stated for Fourier coefficients indexed by primitive T, we have that Fourier coefficients indexed by non-primitive T are an integral linear combination of Fourier coefficients indexed by primitive T by [Miz84] Equation 1.3, so we only need to consider the primitive T to guarantee the hypotheses of Corollary 3.2 are satisfied.

We have the following congruence result.

Lemma 3.4. *Assume $\ell > 4k-7$. Let $f \in S_k(\Gamma_2; \mathcal{O})$ be an eigenform. If there exists a normalized eigenform $\phi \in S_k(\Gamma_1; \mathcal{O})$ so that $f \equiv_{\mathrm{ev}} E_\phi^{2,1} \pmod{\lambda}$ and that $\overline{\rho}_\phi$ is irreducible, then ρ_f is irreducible.*

Proof. We know via [Wei05] that if ρ_f is reducible, then the automorphic representation associated to f is either CAP or a weak endoscopic lift. Moreover, by [PS09] Corollary 4.5 since $f \in S_k(\Gamma_2)$ and $k > 2$, the automorphic representation attached to f can be CAP only with respect to the Siegel parabolic, i.e., f is a classical Saito-Kurokawa lift. Suppose that f is a Saito-Kurokawa lift of $\psi \in S_{2k-2}(\Gamma_1)$. Then we have $\overline{\rho}_f^{\mathrm{ss}} = \overline{\rho}_\psi \oplus \overline{\epsilon}^{k-1} \oplus \overline{\epsilon}^{k-2}$ where $\overline{\epsilon}$ is the mod ℓ reduction of the ℓ-adic cyclotomic character. Using the fact that $f \equiv_{\mathrm{ev}} E_\phi^{2,1} \pmod{\lambda}$ and that the eigenvalues of $E_\phi^{2,1}$ are given by $\lambda(p; E_\phi^{2,1}) = a(p;\phi) + p^{k-2} a(p;\phi)$, the Brauer-Nesbitt and Chebotarev Theorems give that

$\overline{\rho}_f^{\rm ss} = \overline{\rho}_\phi \oplus \overline{\rho}_\phi(k-2)$, where recall that we write $\overline{\rho}_\phi(k-2)$ for $\overline{\rho}_\phi \otimes \overline{\epsilon}^{k-2}$. This is a contradiction if $\overline{\rho}_\phi$ is irreducible. Thus, f cannot be a Saito-Kurokawa lift. It remains to show that the automorphic representation associated to f is not a weak endoscopic lift. The possible decompositions of ρ_f are given in [SU06] Theorem 3.2.1 under the assumption that $\ell > 4k-7$. Of these, the only case remaining to check is Case B(v), which states if $\rho_f = \sigma \oplus \sigma'$ with σ and σ' both 2-dimensional, then $\det(\sigma) = \det(\sigma')$. In our case, this would require $\det(\rho_\phi) = \det(\rho_\phi(k-2))$, i.e., $\overline{\epsilon}^{k-1} = \overline{\epsilon}^{2k-3}$, which is impossible by our assumption that $\ell > 4k-7$. Thus, ρ_f is irreducible. $\square$

Theorem 3.5. *Assume that $\ell > 4k-7$. Let $\phi \in S_k(\Gamma_1; \mathcal{O})$ be a normalized eigenform. Suppose that $\lambda \mid L_{\rm alg}(2k-2, \mathrm{Sym}^2 \phi)$. Furthermore, assume there exists $T_0 > 0$ so that*

$$\mathrm{val}_\lambda \left(L_{\rm alg}(2k-2, \mathrm{Sym}^2 \phi) a(T_0, E_\phi^{2,1}) \right) \leq 0.$$

Then there exists an eigenform $f \in S_k(\Gamma_2; \mathcal{O})$ so that

$$E_\phi^{2,1} \equiv_{\rm ev} f \pmod{\lambda}.$$

If in addition $\overline{\rho}_\phi$ is irreducible, then ρ_f is irreducible.

Proof. Set $H_\phi^{2,1}(z) = L_{\rm alg}(2k-2, \mathrm{Sym}^2 \phi) E_\phi^{2,1}(z)$. For $T \geq 0$, define

$$c(T) = \mathrm{val}_\lambda(a(T; H_\phi^{2,1})).$$

Let $c = \min_{T \geq 0} c(T)$. Since $H_\phi^{2,1} \in M_k(\Gamma_2)$, the Fourier coefficients $a(T; H_\phi^{2,1})$ have bounded denominators so c is well-defined ([Shi75]). Moreover, our assumption that there is a $T_0 > 0$ with $\mathrm{val}_\lambda(a(T_0; H_\phi^{2,1})) = \mathrm{val}_\lambda \left(L_{\rm alg}(2k-2, \mathrm{Sym}^2 \phi) a(T_0, E_\phi^{2,1}) \right) \leq 0$ gives that $c \leq 0$. Set

$$G_\phi^{2,1}(z) = \lambda^{-c} H_\phi^{2,1}(z).$$

We have $a(T; G_\phi^{2,1}) \in \mathcal{O}$ for all $T \geq 0$ since $c(T) - c \geq 0$ for all $T \geq 0$. Observe that for T with $|T| = 0$, we have $a(T; G_\phi^{2,1}) = \lambda^{-c} L_{\rm alg}(2k-2, \mathrm{Sym}^2 \phi) a(n; \phi)$ for some $n \in \mathbf{Z}_{\geq 0}$. Since $a(n; \phi) \in \mathcal{O}$ by assumption and $-c \geq 0$, this gives $\lambda \mid a(T; G_\phi^{2,1})$ for all T with $|T| = 0$, i.e., all the Fourier coefficients indexed by singular T vanish modulo λ. Moreover, since $c = c(\widetilde{T})$ for some $\widetilde{T}$, we have $a(\widetilde{T}; G_\phi^{2,1}) \in \mathcal{O}^\times$ for some $\widetilde{T}$. Since $c \leq 0$ and $\lambda \mid a(T; G_\phi^{2,1})$ for all singular T, we have $\widetilde{T} > 0$. Thus, Corollary 3.2 and the fact that $G_\phi^{2,1}$ and $E_\phi^{2,1}$ have the same eigenvalues gives a non-trivial eigenform $f \in S_k(\Gamma_2; \mathcal{O})$ so that

$$E_\phi^{2,1} \equiv_{\rm ev} f \pmod{\lambda}.$$

One now applies Lemma 3.4 to obtain that ρ_f is irreducible. $\square$

Example 3.6. Consider the space $M_{26}(\Gamma_2)$. This space has dimension seven and is spanned by $E^{2,0}$ (Siegel Eisenstein series), $E_\phi^{2,1}$ (Klingen Eisenstein series), three Saito-Kurokawa lifts, and two non-lift forms Υ_1 and Υ_2 where here $\phi \in S_{26}(\Gamma_1)$ is the unique newform given by

$$\phi(z) = e(z) - 48e(2z) - 195804e(3z) + \cdots.$$

We have via [Dum01] that

$$L_{\rm alg}(50, \mathrm{Sym}^2 \phi) = \frac{2^{41} \cdot 163 \cdot 187273}{3^{26} \cdot 5^{10} \cdot 7^7 \cdot 11^4 \cdot 13^2 \cdot 17^2 \cdot 19 \cdot 23^2 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 657931}$$

We consider $\ell \in \{163, 187273\}$. We provide a different argument for each prime in producing an example.

The Klingen Eisenstein series associated to ϕ is given in the beta version of LMFDB. By considering the Fourier coefficients indexed by $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ one can see that the Klingen Eisenstein series given there, say E_ϕ^{LMFDB}, is given by

$$E_\phi^{2,1}(z) = -\frac{E_\phi^{\mathrm{LMFDB}}(z)}{2^6 \cdot 3^3 \cdot 11 \cdot 19 \cdot 163 \cdot 187273}.$$

We have from LMFDB that

$$a\left(\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}; E_\phi^{2,1}\right) = \frac{2^2 \cdot 5 \cdot 43}{11 \cdot 19 \cdot 163 \cdot 187273}$$

Consider $G_\phi^{2,1}(z) = L_{\mathrm{alg}}(50, \mathrm{Sym}^2 \phi) E_\phi^{2,1}(z)$. We have for ℓ as above that $\ell \mid a(T; G_\phi^{2,1})$ for all T with $|T| = 0$ and $a\left(\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}; G_\phi^{2,1}\right) \not\equiv 0 \pmod{\ell}$. Thus by Theorem 3.5 there exists a non-trivial Hecke eigenform $f \in S_k(\Gamma_2; \mathbf{Z}_\ell)$ with $E_\phi^{2,1} \equiv_{\mathrm{ev}} f \pmod{\ell}$.

Consider first the prime $\ell = 163$ and suppose that $\overline{\rho}_{\phi,163}^{\mathrm{ss}} = \psi_1 \oplus \psi_2$ for some characters ψ_1, ψ_2. Since $\overline{\rho}_\phi$ is unramified for all $p \neq \ell$ we see that ψ_1 and ψ_2 are each an integer power of $\overline{\epsilon}$ (see the proof of Lemma 5.3). As $163 \nmid a(163; \phi)$ we know ϕ is ordinary at 163 and we get $\overline{\rho}_{\phi,163}^{\mathrm{ss}} = \overline{\epsilon}^{25} \oplus 1$. By [Rib76] Proposition 2.1 we can find a lattice such that

$$\overline{\rho}_{\phi,163} = \begin{bmatrix} 1 & * \\ 0 & \overline{\epsilon}^{25} \end{bmatrix} \not\cong 1 \oplus \overline{\epsilon}^{25}.$$

One can use ordinarity of ϕ to show that $*$ gives an unramified 163-extension of $\mathbf{Q}(\zeta_{163})$ (see e.g. the proof of Theorem 4.28 in [BK23].) By Herbrand's Theorem this implies that $163 \mid B_{26}$. However, one can check this is not true, so we must have that $\overline{\rho}_{\phi,163}$ is irreducible and so $E_\phi^{2,1}$ must be congruent (modulo 163) to a cuspform f that is not a Saito-Kurokawa lift, i.e. ρ_f is irreducible by Theorem 3.5. One uses LMFDB to check that $f = \Upsilon_2$.

Now consider the case that $\ell = 187273$. In this case it is less practical to calculate $a(187273; \phi)$, so we directly eliminate the possibility that $E_\phi^{2,1}$ is congruent to a Saito-Kurokawa lift modulo 187273. The space to consider is $S_{50}(\Gamma_1)$. This space has one Galois conjugacy class of newforms consisting of three newforms, call them ψ_1, ψ_2, and ψ_3. Each newform has a field of definition K_{ψ_i} generated by a root α_i of

$$c(x) = x^3 + 24225168x^2 - 566746931810304x - 13634883228742736412672.$$

One has that $\lambda(2, E_\phi^{2,1}) = -805306416$ and that $\lambda(2, \psi_i) = 2^{49} + 2^{48} + \alpha_i$. One uses SAGE to check that $\lambda(2, E_\phi^{2,1}) \not\equiv \lambda(2, \psi_i) \pmod{187273}$, so $E_\phi^{2,1}$ must be congruent to a cusp form that is not a Saito-Kurokawa lift. One uses LMFDB to see that $E_\phi^{2,1} \equiv_{\mathrm{ev}} \Upsilon_1 \pmod{187273}$.

4. Extensions of Fontaine-Laffaille modules

In this section we gather various facts (in particular Proposition 4.9 and Proposition 4.24) about extensions of Fontaine-Laffaille modules, which we use in this article but which to the best of our knowledge have not been published elsewhere.

4.1. **Definitions.** We keep our assumption that ℓ is an odd prime. We fix integers a, b such that $0 \leq b - a \leq \ell - 2$. In this section let E be an arbitrary finite extension of $\mathbf{Q}_\ell$ with ring of integers $\mathcal{O}$, uniformizer λ and residue field $\mathbf{F}$. Write $\mathrm{LCA}_\mathcal{O}$ (respectively $\mathrm{LCN}_\mathcal{O}$) for the category of local complete Artinian (respectively Noetherian) $\mathcal{O}$-algebras with residue field $\mathbf{F}$. For a category $\mathcal{C}$ we will write $X \in \mathcal{C}$ to mean that X is an object of $\mathcal{C}$.

Definition 4.1 ([Kal19] Definition 2.3/[Boo19] Definition 4.1)**.**

(1) A Fontaine-Laffaille module is a finitely generated $\mathbf{Z}_\ell$-module M together with a decreasing filtration by $\mathbf{Z}_\ell$-module direct summands M^i for $i \in \mathbf{Z}$ such that there exists $k \leq l$ with $M^i = M$ for $i \leq k$ and $M^{i+1} = 0$ for $i \geq l$, and a collection of $\mathbf{Z}_\ell$-linear maps $\phi_M^i : M^i \to M$ such that $\phi_M^i|_{M^{i+1}} = \ell \phi_M^{i+1}$ for all i and $M = \sum_i \phi_M^i(M^i)$. The category of all Fontaine-Laffaille modules is denoted $MF^f_{\mathbf{Z}_\ell}$. Morphisms in this category are $\mathbf{Z}_\ell$-linear maps $f : M \to N$ satisfying $f(M^i) \subset N^i$ and $f \circ \phi_M^i = \phi_N^i \circ f|_{M^i}$ for all i. We will write $MF^f_{\mathrm{tor},\mathbf{Z}_\ell}$ for the full subcategory whose objects are of finite length as a $\mathbf{Z}_\ell$-modules.

(2) For a fixed interval $[k,l]$ we denote the full subcategory of $MF^f_{?,\mathbf{Z}_\ell}$ whose objects M have a filtration satisfying $M^k = M$ and $M^{l+1} = 0$ by $MF^{f,[k,l]}_{?,\mathbf{Z}_\ell}$ for $? \in \{\emptyset, \mathrm{tor}\}$.

(3) For any $A \in \mathrm{LCA}_{\mathcal{O}}$, a Fontaine-Laffaille module over A consists of an object $M \in MF^{f,[a,b]}_{\mathrm{tor},\mathbf{Z}_\ell}$ together with a map $\theta : A \to \mathrm{End}_{MF^{f,[a,b]}_{\mathrm{tor},\mathbf{Z}_\ell}}(M)$ that makes M into a free finitely generated module over A in such a way that M^i is an A-direct summand of M for each i. A morphism between two such objects is required to additionally preserve the A-structure. We will denote this category of Fontaine-Laffaille modules over A as $MF^{f,[a,b]}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A$.

(4) For $M \in MF^{f,[a,b]}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A$ any integer i for which $M^i/M^{i+1} \neq 0$ is called a Fontaine-Laffaille weight for M. The set of Fontaine-Laffaille weights for M will be denoted by $\mathrm{FL}(M)$.

Remark 4.2. We impose the stronger restriction on the length of the filtration as in [BK90] Section 4 and [CHT08] Section 2.4.1, compared to that in Section 1.1.2 of [DFG04] or [Kal19] Definition 2.3 (which allow the length to be $\ell - 1$).

Remark 4.3. One can also replace $\mathbf{Z}_\ell$ in $MF^{f,[a,b]}_{?,\mathbf{Z}_\ell}$ for $? = \emptyset, \mathrm{tor}$ by the integer ring $\mathcal{O}_K$ in a finite unramified extension K of $\mathbf{Q}_\ell$ and require that the maps ϕ_M^i are Frobenius-semilinear. The modules $M \in MF^{f,[a,b]}_{\mathrm{tor},\mathcal{O}_K} \otimes_{\mathbf{Z}_\ell} A$ need to be free modules over $\mathcal{O}_K \otimes_{\mathbf{Z}_\ell} A$. Via the Fontaine-Laffaille functor described below this gives rise to crystalline G_K-representations, in particular the equivalence between $MF^{f,[a,b]}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A$ and $\mathrm{Rep}^{\mathrm{cris},[-b,-a]}_{\mathrm{free},A}(G_{\mathbf{Q}_\ell})$ holds with $\mathbf{Z}_\ell$ replaced by $\mathcal{O}_K$ and $G_{\mathbf{Q}_\ell}$ replaced by G_K (see Theorem 4.16 and [Kal19] Theorem 2.10). As we will have no use for the general case, to simplify the exposition, we restrict to $G_{\mathbf{Q}_\ell}$-representations. We refer the reader to [Boo19] Section 4.2 and [Kal19] Definition 2.3 and 2.8 for the general case.

Definition 4.4. We introduce the following examples of Fontaine-Laffaille modules:

(i) If $0 \in [a,b]$ we write $\mathbf{1} \in MF^{f,[a,b]}_{\mathbf{Z}_\ell}$ for the Fontaine-Laffaille module defined by $\mathbf{1}^i = \mathbf{Z}_\ell$ for $i \leq 0$ and $\mathbf{1}^i = 0$ for $i > 0$. We set $\phi^i : \mathbf{1}^i \to \mathbf{1}$ to be given by $x \mapsto \ell^{-i}x$ for $i \leq 0$.

(ii) For $n \in [a,b] \cap \mathbf{Z}$ define $M_n \in MF^{f,[a,b]}_{\mathrm{tor},\mathbf{Z}_\ell}$ to be the 1-dimensional $\mathbf{F}_\ell$-vector space with filtration $M_n^i = M_n = \mathbf{F}_\ell$ for $i \leq n$, $M_n^{n+1} = 0$ and $\phi^i : M_n^i \to M_n$ the 0-map for all $i \neq n$, and the identity map for $i = n$.

(iii) For any $A \in \mathrm{LCA}_{\mathcal{O}}$ we define $M_{n,A} \in MF^{f,[a,b]}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A$ to be the free rank one A-module equipped with the filtration $M^i_{n,A} = A$ for $i \leq n$, $M^{n+1}_{n,A} = 0$ and $\phi^i : M^i_{n,A} \to M_{n,A}$ given by $x \mapsto \ell^{n-i}x$ for $i \leq n$. We put $\mathbf{1}_A = M_{0,A}$.

Definition 4.5 ([Boo19] Definition 4.9)**.** For $M \in MF^{f,[a,b]}_{\mathrm{tor},\mathbf{Z}_\ell}$ and $s \in \mathbf{Z}$ define $M(s)$ to be the same underlying $\mathbf{Z}_\ell$-module, but change the filtration to $M(s)^i = M^{i-s}$ for any $i \in \mathbf{Z}$. This means that $M(s) \in MF^{f,[a+s,b+s]}_{\mathrm{tor},\mathbf{Z}_\ell}$.

4.2. **Extensions.** To ease notation in the rest of this section we put $\mathcal{C}^I_A = MF^{f,I}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A$ for $A \in \mathrm{LCA}_{\mathcal{O}}$. Here $I = [a,b]$.

Definition 4.6 (Definition/Lemma)**.** Given $M, N \in \mathcal{C}_A^I$ define a filtration on the A-module $\mathrm{Hom}_A(M,N)$ by

$$\mathrm{Hom}_A(M,N)^i = \{f \in \mathrm{Hom}_A(M,N) \mid f(M^j) \subset N^{j+i} \text{ for all } j \in \mathbf{Z}\}$$

and $\mathbf{Z}_\ell$-linear maps $\phi^i : \mathrm{Hom}_A(M,N)^i \to \mathrm{Hom}_A(M,N)$ by

$$\phi^i(f)(\phi_M^j(m)) = \phi_N^{i+j}(f(m))$$

(note that $M = \sum \phi_M^j(M^j)$) for $f \in \mathrm{Hom}_A(M,N)^i$ and all $m \in M^j$ and $j \in \mathbf{Z}$. We claim this defines a Fontaine-Laffaille structure and we get

$$\mathrm{Hom}_A(M,N) \in MF_{\mathrm{tor},\mathbf{Z}_\ell}^{f,[a-b,b-a]} \otimes_{\mathbf{Z}_\ell} A.$$

Proof. First note that there exists a canonical A-module homomorphism $\psi : M^\vee \otimes_A N \to \mathrm{Hom}_A(M,N)$, where $M^\vee = \mathrm{Hom}_A(M,A)$. Definition 4.19 in [Boo19] defines a Fontaine-Laffaille structure on $M^\vee$ (and Lemma 4.20 and 4.21 prove that this structure is well-defined and so we get an object in $MF_{\mathrm{tor},\mathbf{Z}_\ell}^{f,[-b,-a]} \otimes_{\mathbf{Z}_\ell} A$). Definition 4.17 in [Boo19] then gives us the Fontaine-Laffaille structure on $M^\vee \otimes_A N$.

We claim that transferring this structure on $M^\vee \otimes_A N$ via ψ to $\mathrm{Hom}_A(M,N)$ matches our definition.

Recall from [Boo19] that

$$(M^\vee)^i = \{f \in \mathrm{Hom}_A(M,A) | f(M^k) \subset \mathbf{1}_A^{i+k} \text{ for all } k \in \mathbf{Z}\}$$

and

$$(M^\vee \otimes N)^n = \sum_{i+j=n} (M^\vee)^i \otimes_A N^j.$$

We will first show that $\psi((M^\vee \otimes N)^n) \subset \mathrm{Hom}_A(M,N)^n$. Let $f_i \otimes n_j \in (M^\vee)^i \otimes_A N^j$. Then $\psi(f_i \otimes n_j) : m \in M^k \mapsto f_i(m)n_j \in N^j$. In fact, the image lies in N^{n+k}. This is clear for $j \geq n+k$. If $j < n+k$ (and hence $0 < i+k$) it follows since $f_i(m) \in \mathbf{1}_A^{i+k} = 0$. To show the reverse inclusion $\psi((M^\vee \otimes N)^n) \supset \mathrm{Hom}_A(M,N)^n$ consider $f \in \mathrm{Hom}_A(M,N)^n$ and let j be maximal among integers l such that $f(M) \subset N^l$. To satisfy $f(M^k) \subset N^{k+n}$ for all integers k we need $f(M^k) = 0$ for $k+n > j$ by maximality of j. This means that we need f to factor through M/M^{1-i} for $i := n-j$. By [Boo19] Lemma 4.20 we have $(M^\vee)^i = \mathrm{Hom}_A(M/M^{1-i}, A)$ so we get

$$(M^\vee)^i \otimes N^j = \mathrm{Hom}_A(M/M^{1-i}, A) \otimes N^j \overset{\psi}{\cong} \mathrm{Hom}_A(M/M^{1-i}, N^j).$$

We conclude that $f \in \psi^{-1}((M^\vee)^i \otimes N^j) \subset \psi^{-1}((M^\vee \otimes N)^n)$.

Now we check the $\mathbf{Z}_\ell$-linear maps: Recall from [Boo19] that for $f \in M^\vee$ we have

$$\phi_{M^\vee}^i(f)(\phi_M^j(m)) = \phi^{i+j}(f(m))$$

for all $m \in M^j$ and $j \in \mathbf{Z}$. We also have

$$\phi_{M^\vee \otimes_A N}^n = \sum_{i+j=n} \phi_{M^\vee}^i \otimes \phi_N^j.$$

We claim that

$$\phi_{\mathrm{Hom}_A(M,N)}^n \circ \psi = \psi \circ \phi_{M^\vee \otimes_A N}^n : (M^\vee \otimes N)^n \to \mathrm{Hom}_A(M,N).$$

For this one calculates that both sides map $f \otimes n \in (M^\vee)^i \otimes N^{n-i}$ to the homomorphism, for which

$$\phi_M^k(m) \mapsto \begin{cases} 0 & \text{if } i+k \geq 0 \\ \phi_N^{n+k}(f(m)x) & \text{if } i+k \leq 0 \end{cases}$$

for any $m \in M^k$ (for $\psi \circ \phi^n_{M^\vee \otimes_A N}$ this uses $\phi_N^{n+k}|_{N^{n-i}} = \ell^{k+i}\phi_N^{n-i}$ for $i+k \leq 0$). This claim, combined with the results in [Boo19] shows that the definition of $\phi^n_{\mathrm{Hom}_A(M,N)}$ is well-defined and satisfies the requirements for $\mathrm{Hom}_A(M,N)$ to be a Fontaine-Laffaille module in $MF^{f,[a-b,b-a]}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A$. $\square$

For $M, N \in \mathcal{C}_A^I$ consider the map $\phi - 1 : \mathrm{Hom}_A(M,N)^0 \to \mathrm{Hom}_A(M,N)$ which takes f to the homomorphism that sends $m = \sum_j \phi^j_M(m_j)$ to

$$\sum_j \phi^j_N(f(m_j)) - f(m) = \sum_j \left(\phi^j_N(f(m_j)) - f(\phi^j_M(m_j))\right).$$

Note that $\ker(\phi - 1) = \mathrm{Hom}_{\mathcal{C}_A^I}(M,N)$.

Proposition 4.7 ([CHT08] Lemma 2.4.2, [Kal19] Proposition 2.17)**.** *Given $M, N \in \mathcal{C}_A^I$ we have an exact sequence of A-modules (note that $\mathrm{Hom}_{Fil,A}(M,N)$ in [Kal19] equals $\mathrm{Hom}_A(M,N)^0$)*

$$0 \to \mathrm{Hom}_{\mathcal{C}_A^I}(M,N) \to \mathrm{Hom}_A(M,N)^0 \overset{\phi-1}{\to} \mathrm{Hom}_A(M,N) \to \mathrm{Ext}^1_{\mathcal{C}_A^I}(M,N) \to 0.$$

Given $M, N \in \mathcal{C}_A^I$ we write $\mathrm{FL}(M) > \mathrm{FL}(N)$ if there is an integer j such that all elements of $\mathrm{FL}(M)$ are greater than or equal to j, and all elements of $\mathrm{FL}(N)$ are strictly less than j.

Proposition 4.8. *The extension group $\mathrm{Ext}^1_{\mathcal{C}_A^I}(M,N)$ is a finitely generated A-module. Furthermore one has*

(i) If $\mathrm{FL}(M) > \mathrm{FL}(N)$ then $\mathrm{Ext}^1_{\mathcal{C}_A^I}(M,N) \cong \mathrm{Hom}_A(M,N)$, in particular it is a free A-module and $\mathrm{rk}_A(\mathrm{Ext}^1_{\mathcal{C}_A^I}(M,N)) = \mathrm{rk}_A(M)\mathrm{rk}_A(N)$.
(ii) If $\mathrm{FL}(M) < \mathrm{FL}(N)$ then $\mathrm{Ext}^1_{\mathcal{C}_A^I}(M,N) = 0$.

Proof. This follows from Proposition 4.7. In particular, $\mathrm{Ext}^1_{\mathcal{C}_A^I}(M,N)$ is a quotient of the finitely generated A-module $\mathrm{Hom}_A(M,N)$. The calculation on [Kal19] p. 238 ("two notable cases") is carried out for $MF^{f,[0,\ell-1]}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A$, but applies verbatim to $\mathcal{C}_A^I$. If $\mathrm{FL}(M) > \mathrm{FL}(N)$ then this calculation shows that $\mathrm{Hom}_A(M,N)^0 = 0$, while if $\mathrm{FL}(M) < \mathrm{FL}(N)$ then one gets $\mathrm{Hom}_A(M,N)^0 = \mathrm{Hom}_A(M,N)$. $\square$

Proposition 4.9 (Hom-tensor adjunction)**.** *Let $M, N \in \mathcal{C}_A^I$. Assume that $\mathrm{Hom}_A(M,N)$ equipped with the filtration as in Definition 4.6 is an object in $\mathcal{C}_A^I$ and that $0 \in I$. Then there exists a canonical isomorphism of A-modules:*

$$\mathrm{Ext}^1_{\mathcal{C}_A^I}(M,N) \cong \mathrm{Ext}^1_{\mathcal{C}_A^I}(\mathbf{1}_A, \mathrm{Hom}_A(M,N)).$$

Proof. The statement follows from the existence of the following commutative diagram with exact columns:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathrm{Hom}_{\mathcal{C}_A^I}(M,N) \arrow[d] & \mathrm{Hom}_{\mathcal{C}_A^I}(\mathbf{1}_A, \mathrm{Hom}_A(M.N)) \arrow[d] \\
\mathrm{Hom}_A(M,N)^0 \arrow[r, "\psi'"] \arrow[d, "\phi-1"'] & \mathrm{Hom}_A(\mathbf{1}_A, \mathrm{Hom}_A(M,N))^0 \arrow[d, "\phi-1"] \\
\mathrm{Hom}_A(M,N) \arrow[r, "\psi"] \arrow[d, "\alpha"'] & \mathrm{Hom}_A(A, \mathrm{Hom}_A(M,N)) \arrow[d] \\
\mathrm{Ext}^1_{\mathcal{C}_A^I}(M,N) \arrow[r, "\tilde{\psi}"] \arrow[d] & \mathrm{Ext}^1_{\mathcal{C}_A^I}(\mathbf{1}_A, \mathrm{Hom}_A(M,N)) \arrow[d] \\
0 & 0
\end{tikzcd} \tag{4.1}$$

The exactness of both columns follows from Proposition 4.7. The second horizontal arrow is the usual isomorphism ψ of A-modules given by $f \mapsto (a \mapsto af)$ (recall that the underlying module of the object $\mathbf{1}_A$ is A) with the inverse map sending g to $g(1)$, where 1 is the multiplicative identity of A.

The first horizontal arrow is the restriction ψ' of ψ to $\mathrm{Hom}_A(M,N)^0$ (note that $\mathrm{Hom}_A(M,N)^0$ is a subgroup of $\mathrm{Hom}_A(M,N)$ even though $\phi - 1$ is not necessarily injective). We need to check that ψ' lands in $\mathrm{Hom}_A(\mathbf{1}_A, \mathrm{Hom}_A(M,N))^0$. By its definition we need to check if $f(\mathbf{1}_A^j) \subset \mathrm{Hom}_A(M,N)^j$. If $j > 0$ there is nothing to check as then $\mathbf{1}_A^j = 0$, so assume that $j \leq 0$. Then $\mathbf{1}_A^j = A$ and $\mathrm{Hom}_A(M,N)^j \supset \mathrm{Hom}_A(M,N)^0$. So, it is enough to show that if $f \in \mathrm{Hom}_A(M,N)^0$ then $\psi'(f)(A) \subset \mathrm{Hom}_A(M,N)^0$. Let $a \in A$. Then $\psi'(f)(a) = af$ which clearly lies in $\mathrm{Hom}_A(M,N)^0$ as $\mathrm{Hom}_A(M,N)^0$ is an A-module.

Now let $g \in \mathrm{Hom}_A(\mathbf{1}_A, \mathrm{Hom}_A(M,N))^0$. We need to show that $\psi^{-1}(g)$ lands in $\mathrm{Hom}_A(M,N)^0$. Again we need to consider $\psi^{-1}(g)(\mathbf{1}_A^j)$. If $j > 0$, then $g = 0$, hence we are done. Assume that $j \leq 0$. Then $\mathbf{1}_A^j = A$ and $\psi^{-1}(g) = g(1)$. As $1 \in \mathbf{1}_A^0$ and $g \in \mathrm{Hom}_A(\mathbf{1}_A, \mathrm{Hom}_A(M,N))^0$ we must have that $g(1) \in \mathrm{Hom}_A(M,N)^0$. So, we are done again.

This shows that ψ' is a bijection, hence an isomorphism. Hence by the second Four Lemma $\tilde{\psi}$ is injective, and since it is clearly surjective, it is an isomorphism.

□

Remark 4.10. The proof of [Kal19] Proposition 2.17 defines (see also [DFG04] p. 711) the homomorphism $\alpha : \mathrm{Hom}_A(M,N) \to \mathrm{Ext}^1_{\mathcal{C}_A^I}(M,N)$ as follows: Given $f \in \mathrm{Hom}_A(M,N)$ one defines an extension $N \oplus_f M$ as follows: The module is given by $N \oplus M$ with filtration $(N \oplus M)^i = N^i \oplus M^i$ and $\phi^i : (N \oplus M)^i \to N \oplus M$ defined by

$$(n^i, m^i) \mapsto (\phi_N^i(n^i) + f(\phi_M^i(m^i)), \phi_M^i(m^i)).$$

One can show that any extension in $\mathrm{Ext}^1_{\mathcal{C}_A^I}(M,N)$ is of the form $N \oplus_f M$ for some $f \in \mathrm{Hom}_A(M,N)$. We note that $\tilde{\psi}$ defined in the proof of the Proposition 4.9 maps this extension to the extension

$$\mathrm{Hom}_A(M,N) \oplus_{a \mapsto af} \mathbf{1}_A.$$

The commutative diagram (4.1) shows that this is well-defined and gives an isomorphism

$$\tilde{\psi} : \mathrm{Ext}^1_{\mathcal{C}^I_A}(M,N) \to \mathrm{Ext}^1_{\mathcal{C}^I_A}(\mathbf{1}_A, \mathrm{Hom}_A(M,N)).$$

Let us now give an alternative description of the map $\tilde{\psi}$. Starting with an extension

$$0 \to N \to E \to M \to 0$$

in $\mathcal{C}^I_A$ (note that as an A-module it is isomorphic to $N \oplus M$ as N and M are free and by [Kal19] we know there exists $f \in \mathrm{Hom}_A(M,N)$ such that it is explicitly given by $N \oplus_f M$), it gives rise to an exact sequence as A-modules (where we use freeness to conclude right exactness)

$$0 \to \mathrm{Hom}_A(M,N) \to \mathrm{Hom}_A(M,E) \to \mathrm{Hom}_A(M,M) \to 0.$$

This is an exact sequence in $\mathcal{C}^I_A$ since the maps preserve the extra structure (exactness of sequences in $\mathcal{C}^I_A$ is equivalent to exactness as A-modules). Now take the pullback of this extension with respect to $A \hookrightarrow \mathrm{Hom}_A(M,M)$. More specifically we have the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_A(M,N) \arrow[r] \arrow[d, "\mathrm{id}"] & \tilde{\psi}(E) \arrow[r] \arrow[d, hook] & A \arrow[r] \arrow[d, hook, "a\mapsto(m\mapsto am)"] & 0 \\ 0 \arrow[r] & \mathrm{Hom}_A(M,N) \arrow[r] & \mathrm{Hom}_A(M,E) \arrow[r] & \mathrm{Hom}_A(M,M) \arrow[r] & 0 \end{tikzcd} \tag{4.2}$$

This realizes $\tilde{\psi}(E)$ as a submodule of $\mathrm{Hom}_A(M,E)$ of in the following way:

$$\tilde{\psi}(N \oplus_f M) = \{h \in \mathrm{Hom}_A(M, N \oplus M) | \exists g \in \mathrm{Hom}_A(M,N), a \in A \text{ such that } h(m) = (g(m), am)\}.$$

If one equips $\mathrm{Hom}(M, N \oplus_f M)$ with the $\mathcal{C}^I_A$-structure as in Definition 4.6 one can check that $\tilde{\psi}(E)$ corresponds to

$$\mathrm{Hom}_A(M,N) \oplus_{a \mapsto af} \mathbf{1}.$$

4.3. **Fontaine-Laffaille Galois representations.** Fix an interval $I = [a,b]$ with $a, b \in \mathbf{Z}$ and $b - a \leq \ell - 2$. In this subsection we introduce certain categories of $G_{\mathbf{Q}_\ell}$-representations and define a covariant version V_I of the functor in [FL82] from the categories of Fontaine-Laffaille modules defined in section 4.1 to these categories of Galois representations.

Let A_{cris} and B_{cris} denote the usual Fontaine's ℓ-adic period rings (see Definition 7.3 and 7.7 in [FO22] and [Fon82]). We recall that a $\mathbf{Q}_\ell[G_{\mathbf{Q}_\ell}]$-module V is called crystalline if $\dim_{\mathbf{Q}_\ell} V = \dim_{\mathbf{Q}_\ell} H^0(\mathbf{Q}_\ell, V \otimes_{\mathbf{Q}_\ell} B_{\mathrm{cris}})$. Our convention is that the Hodge-Tate weight of the cyclotomic character is $+1$.

Definition 4.11. Let $A \in \mathrm{LCA}_{\mathcal{O}}$. We introduce the following categories:

(i) $\mathrm{Rep}^f_{\mathbf{Z}_\ell}(G_{\mathbf{Q}_\ell})$, the category of $\mathbf{Z}_\ell[G_{\mathbf{Q}_\ell}]$-modules that are finitely generated as $\mathbf{Z}_\ell$-modules.

(ii) $\mathrm{Rep}^f_{\mathrm{tor},\mathbf{Z}_\ell}(G_{\mathbf{Q}_\ell})$, the full subcategory of $\mathrm{Rep}^f_{\mathbf{Z}_\ell}(G_{\mathbf{Q}_\ell})$ whose objects are required to be of finite length as $\mathbf{Z}_\ell[G_{\mathbf{Q}_\ell}]$-modules.

(iii) $\mathrm{Rep}^{\mathrm{cris},I}_{\mathbf{Z}_\ell}(G_{\mathbf{Q}_\ell})$, the full subcategory of $\mathrm{Rep}^f_{\mathbf{Z}_\ell}(G_{\mathbf{Q}_\ell})$ whose objects are isomorphic to T/T', where T and T' are $G_{\mathbf{Q}_\ell}$-stable finitely generated submodules of a crystalline $\mathbf{Q}_\ell$-representation with Hodge-Tate weights in I.

(iv) $\mathrm{Rep}^{\mathrm{cris},I}_{\mathrm{tor},\mathbf{Z}_\ell}(G_{\mathbf{Q}_\ell})$, the full subcategory of $\mathrm{Rep}^f_{\mathrm{tor},\mathbf{Z}_\ell}(G_{\mathbf{Q}_\ell})$ whose objects are isomorphic to T/T', where T and T' are $G_{\mathbf{Q}_\ell}$-stable lattices in a crystalline $\mathbf{Q}_\ell$-representation with Hodge-Tate weights in I. We refer to the objects in $\mathrm{Rep}^{\mathrm{cris},I}_{\mathbf{Z}_\ell}(G_{\mathbf{Q}_\ell})$ as *torsion crystalline representations (with weights in I).*

(v) $\mathrm{Rep}^{\mathrm{cris},I}_{\mathrm{free},A}(G_{\mathbf{Q}_\ell})$, the category of free finite rank A-modules M with an A-linear $G_{\mathbf{Q}_\ell}$-action, for which there exists a crystalline representation of $G_{\mathbf{Q}_\ell}$ defined over E with Hodge-Tate weights in I containing $G_{\mathbf{Q}_\ell}$-stable $\mathcal{O}$-lattices $T' \subset T$, and an $\mathcal{O}$-algebra map $A \to \mathrm{End}_{\mathcal{O}}(T/T')$ such that M is isomorphic as an $A[G_{\mathbf{Q}_\ell}]$-module to T/T'. We will call objects of this category *Fontaine-Laffaille A-representations (with weights in I).*

Remark 4.12. Definition 4.11(v) matches Definition 2.1 in [Kal19] .

Definition 4.13 ([BK90] p. 363, [Boo19] Definition 4.7+4.9)**.** Similar to [Boo19] we define the following two functors.

(i) A covariant functor $T_{\mathrm{cris}} : MF^{f,[2-\ell,0]}_{\mathbf{Z}_\ell} \to \mathrm{Rep}^f_{\mathbf{Z}_\ell}(G_{\mathbf{Q}_\ell})$ defined via

$$T_{\mathrm{cris}}(M) := \ker\left(1-\phi^0_{A_{\mathrm{cris}}\otimes_{\mathbf{Z}_\ell} M} : \mathrm{Fil}^0(A_{\mathrm{cris}} \otimes_{\mathbf{Z}_\ell} M) \to A_{\mathrm{cris}} \otimes_{\mathbf{Z}_\ell} M\right).$$

(ii) A covariant functor $V_I : MF^{f,[a,b]}_{\mathbf{Z}_\ell} \to \mathrm{Rep}^f_{\mathbf{Z}_\ell}(G_{\mathbf{Q}_\ell})$, defined via

$$V_I(M) = T_{\mathrm{cris}}(M(-b))(-b). \tag{4.3}$$

Recall that $M(-b)$ was defined in Definition 4.5, while $(-b)$ on the outside denotes the Tate twist as defined in Section 2.1.

Remark 4.14. We note that for $? \in \{\emptyset, \mathrm{tor}\}$ the category $MF^{f,[a,b]}_{?,\mathbf{Z}_\ell}$ is a full subcategory of $MF^{f,[a,a+\ell-2]}_{?,\mathbf{Z}_\ell}$, since they are both full subcategories of $MF^{f}_{?,\mathbf{Z}_\ell}$ (cf. Definition 4.1), so in particular (4.3) makes sense.

Remark 4.15. Note that V_I extends T_{cris} to general I (in particular $V_{[2-\ell,0]} = T_{\mathrm{cris}}$). Also observe that for $M \in MF^{f,[a,b]}_{\mathrm{tor},\mathbf{Z}_\ell}$ we have $M(-b) \in MF^{f,[2-\ell,0]}_{\mathrm{tor},\mathbf{Z}_\ell}$ since $M(-b)^1 = M^{b+1} = 0$ and $M(-b)^{2-\ell} = M^{2-\ell+b} = M$ as $b+2-\ell \le a$. In particular, the definition of V_I makes sense.

Compared to [Boo19] we work with the more restrictive interval $[2-\ell, 0]$ for T_{cris} and correct a sign error in the Galois twist in [Boo19] Definition 4.9.

Theorem 4.16 ([BK90] Theorem 4.3, [Niz93] Section 2, [DFG04] Section 1.1.2, [Hat19] Section 2.2, [Boo19] Fact 4.10, [Kal19] Theorem 2.10)**.** *We have:*

(i) The covariant functor $V_{[a,b]} : MF^{f,[a,b]}_{\mathbf{Z}_\ell} \to \mathrm{Rep}^f_{\mathbf{Z}_\ell}(G_{\mathbf{Q}_\ell})$ is well-defined, exact and fully faithful.

(ii) For $M \in MF^{f,[a,b]}_{\mathbf{Z}_\ell}$ one has $V_{[a,b]}(M) = \varprojlim_n V_{[a,b]}(M/\ell^n)$.

(iii) The essential image of $V_{[a,b]}$ is closed under formation of sub-objects, quotients and finite direct sums. It is given by the subcategory $\mathrm{Rep}^{\mathrm{cris},[-b,-a]}_{\mathbf{Z}_\ell}(G_{\mathbf{Q}_\ell})$. For $M \in MF^{f,[a,b]}_{\mathrm{tor},\mathbf{Z}_\ell}$ the lengths of M and $V_I(M)$ as $\mathbf{Z}_\ell$-modules agree; in particular the essential image of $MF^{f,[a,b]}_{\mathrm{tor},\mathbf{Z}_\ell}$ under $V_{[a,b]}$ is $\mathrm{Rep}^{\mathrm{cris},[-b,-a]}_{\mathrm{tor},\mathbf{Z}_\ell}(G_{\mathbf{Q}_\ell})$.

(iv) For $A \in \mathrm{LCA}_{\mathcal{O}}$, the functor $V_{[a,b]}$ induces a functor from $MF^{f,[a,b]}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A$ to the category of free finite rank A-modules with an A-linear $G_{\mathbf{Q}_\ell}$-action, which we will also denote by $V_{[a,b]}$. Its essential image is given by $\mathrm{Rep}^{\mathrm{cris},[-b,-a]}_{\mathrm{free},A}(G_{\mathbf{Q}_\ell})$. In fact, $V_{[a,b]}$ gives an equivalence of categories between $MF^{f,[a,b]}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A$ and $\mathrm{Rep}^{\mathrm{cris},[-b,-a]}_{\mathrm{free},A}(G_{\mathbf{Q}_\ell})$.

Remark 4.17.

(1) Note that for $M \in MF^{f,[a,b]}_{\mathrm{tor},\mathbf{Z}_\ell}$ we have $V_{[a+s,b+s]}(M(s)) = V_{[a,b]}(M)(-s)$.
(2) For $I = [a,b] = [0,\ell-2]$ the functor V_I agrees with that of the functor $\mathbb{V}$ in [DFG04] p. 670 by [Bre98] Proposition 3.2.1.7.

(3) For $M \in MF^{f,[a,b]}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A$ the Hodge-Tate weights of $V_I(M)$ (in the sense of Definition 4.11(3)) equal the negatives of the Fontaine-Laffaille weights of M, defined in Definition 4.1(3), due to our convention that the Hodge-Tate weight of the cyclotomic character is $+1$.

Example 4.18. For $n \in [2-\ell, 0]$ and $A \in \mathrm{LCA}_{\mathcal{O}}$ we have that $T_{\mathrm{cris}}(M_{n,A}) = A(-n)$, where $(-n)$ denotes the Tate twist as defined in Section 2.1. In particular, for any $n \in \mathbf{Z} \cap [a,b]$ we get $V_{[a,b]}(M_n) = \mathbf{F}_\ell(-n) = \mathbf{F}_\ell(-n+1-\ell)$, but $M_n \in MF^{f,[a,b]}_{\mathrm{tor},\mathbf{Z}_\ell}$, whereas $M_n(\ell-1) = M_{n+\ell-1} \in MF^{f,[a+\ell-1,b+\ell-1]}_{\mathrm{tor},\mathbf{Z}_\ell}$.

As an immediate consequence of the equivalence of categories in Theorem 4.16(iv) we obtain the following corollary.

Corollary 4.19. *For any $M, N \in MF^{f,I}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A$ there is an isomorphism of A-modules*

$$\mathrm{Ext}^1_{MF^{f,I}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A}(M,N) \cong \mathrm{Ext}^1_{\mathrm{Rep}^{\mathrm{cris},-I}_A(G_{\mathbf{Q}_\ell})}(V_I(M), V_I(N)). \tag{4.4}$$

4.4. **Local Selmer groups.** Let $I = [a,b]$ be an interval as in the previous section (so $0 \leq b-a \leq \ell-2$) but we now also require that $0 \in I$ (so that $\mathbf{1} \in MF^{f,I}_{\mathbf{Z}_\ell}$, see Definition 4.4).

For an extension between two objects M, N in $\mathrm{Rep}_A(G_{\mathbf{Q}_\ell})$

$$0 \to M \to E \to N \to 0$$

we define the n-th Tate twist of the extension to be the extension

$$0 \to M(n) \to E(n) \to N(n) \to 0.$$

For a subgroup G of $\mathrm{Ext}^1_{\mathrm{Rep}_A(G_{\mathbf{Q}_\ell})}(M,N)$ we define $G(n)$ to consist of extensions which are the n-th Tate twists of the elements of G.

Given an extension $\mathcal{E} \in \mathrm{Ext}^1_{MF^{f,I}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A}(M_3, M_1)$ represented by an exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

we will write $V_I(\mathcal{E})$ for the extension in $\mathrm{Ext}^1_{\mathrm{Rep}^{\mathrm{cris},-I}_{\mathrm{free},A}(G_{\mathbf{Q}_\ell})}(V_I(M_3), V_I(M_1))$ represented by

$$0 \to V_I(M_1) \to V_I(M_2) \to V_I(M_3) \to 0.$$

This uses the exactness of the functor V_I (cf. Theorem 4.16(i)).

Since we defined $V_I(M) = T_{\mathrm{cris}}(M(-b))(-b)$ (see Equation (4.3)) we conclude the following lemma:

Lemma 4.20. *For $A \in \mathrm{LCA}_{\mathcal{O}}$ and $M \in MF^{f,I}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A$ we have*

$$V_I(\mathrm{Ext}^1_{MF^{f,I}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A}(\mathbf{1}_A, M)) = \mathrm{Ext}^1_{\mathrm{Rep}^{\mathrm{cris},-I}_{\mathrm{free},A}(G_{\mathbf{Q}_\ell})}(T_{\mathrm{cris}}(M_{-b,A})(-b), T_{\mathrm{cris}}(M(-b))(-b)) \cong$$
$$\cong \mathrm{Ext}^1_{\mathrm{Rep}^{\mathrm{cris},[0,\ell-2]}_{\mathrm{free},A}(G_{\mathbf{Q}_\ell})}(A(b), T_{\mathrm{cris}}(M(-b)))(-b).$$

Note that the latter is naturally isomorphic to $\mathrm{Ext}^1_{\mathrm{Rep}^{\mathrm{cris},[0,\ell-2]}_{\mathrm{free},A}(G_{\mathbf{Q}_\ell})}(A(b), T_{\mathrm{cris}}(M(-b)))$ and they give rise to the same subgroup of $H^1(\mathbf{Q}_\ell, V_I(M))$, see Definition 4.21.

Definition 4.21. For $M \in MF^{f,I}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A$ let $H^1_{f,I}(\mathbf{Q}_\ell, V_I(M)) = V_I(\mathrm{Ext}^1_{MF^{f,I}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A}(\mathbf{1}_A, M)) \subset H^1(\mathbf{Q}_\ell, V_I(M))$.

Remark 4.22. Lemma 4.20 shows that all the extensions in $H^1_{f,I}(\mathbf{Q}_\ell, V_I(M))$ arise from objects in $\mathrm{Ext}^1_{MF^{f,[2-\ell,0]}_{\mathrm{tor},\mathbf{Z}_\ell}}$, in particular, extensions between representations of Hodge-Tate weights in the interval $[0, \ell-2]$.

Remark 4.23. (1) This is a more precise version of the definition made in [BK13] Section 5.2.1. In [BK13] we worked (implicitly) with $I = [0, p-2]$, but the results in [BK13] Section 5 (in particular Corollary 5.4 and Proposition 5.8 restated below) carry over to $H^1_{f,I}$ defined here for general I.

(2) T.B. and K.K. would like to clarify how certain definitions and results in some of our papers fit in with this more precise description of the groups $H^1_{f,I}$: In [BK19] the relevant interval I is $I = [1-k, k-1]$ for Section 5, and the bound on the prime p should be strengthened to $p - 1 > 2k - 2$. The examples in Section 6 of [loc. cit] satisfy this stronger condition. Similarly in [BK20] the relevant interval is $I = [3-2k, 2k-3]$ and p should satisfy $p - 1 > 4k - 6$ (instead of $p > 2k - 2$). This stronger bound should be used for Theorem 10.2, but does not affect the paramodular conjecture application for $k = 2$ since we had excluded $p = 3$ already in Proposition 2.10. In [BK13] Section 6 the suitable interval I is such that $\mathrm{Hom}_{\mathcal{O}}(\tilde{\rho}_2, \tilde{\rho}_1)$ has Hodge-Tate weights in I. The Selmer groups which we refer to are $H^1_\Sigma(F, \mathrm{Hom}_{\mathbf{F}}(\rho_i, \rho_j))$ for $i, j \in \{1, 2\}$, for which the local condition is $H^1_{f,I}(F_v, \mathrm{Hom}_{\mathbf{F}}(\tilde{\rho}_i, \tilde{\rho}_j))$ for $v \mid p$. In [loc.cit.] Section 9 the relevant interval is $I = [-1, 1]$, with corresponding bound $p - 1 > 2$ (which is already assumed). In [loc.cit.] Section 10 the relevant interval is $I = [1-k, k-1]$, and the correct bound is $p - 1 > 2k - 2$. In [BK15] Sections 7 and 8 the same comment applies as for [BK13] Section 9.

In J.B.'s paper [Bro07] the argument in sections 8 and 9 to show the splitting at ℓ of $\begin{pmatrix} \overline{\epsilon}^{k-2} & * \\ 0 & \overline{\epsilon}^{k-1} \end{pmatrix}$ by relating it to $H^1_f(\mathbf{Q}_\ell, \mathbf{F}(-1)) = 0$ requires an interval I containing -1 and $2k-3$, so would need $p - 1 > 2k - 2$. However, one could instead not twist and invoke Proposition 4.8 to deduce the splitting directly from Fontaine-Laffaille theory.

(3) Similar comments apply to other results in the literature, e.g. in [DFG04] Corollary 2.3 the expression $H^1_f(\mathbf{Q}_\ell, \mathrm{ad}^0_\kappa \overline{L})$ is only indirectly defined by $H^1_f(\mathbf{Q}_\ell, \mathrm{ad}_\kappa \overline{L}) = H^1_f(\mathbf{Q}_\ell, \mathrm{ad}^0_\kappa \overline{L}) \oplus H^1_f(\mathbf{Q}_\ell, \kappa)$. To define the Selmer group for the trace zero endomorphisms and prove this identity requires ad^0_κ to lie in the essential image of the Fontaine-Laffaille functor, and therefore $I = [1-k, k-1]$ should be specified, rather than $I = [0, \ell - 2]$ as in [DFG04] Section 1.1.2.

If $M, N \in \mathrm{Rep}^{\mathrm{cris},I}_{\mathrm{free},A}(G_{\mathbf{Q}_\ell})$, then $M \oplus N \in \mathrm{Rep}^{\mathrm{cris},I}_{\mathrm{free},A}(G_{\mathbf{Q}_\ell})$ and it is clear that

$$H^1_{f,I}(\mathbf{Q}_\ell, M \oplus N) = H^1_{f,I}(\mathbf{Q}_\ell, M) \oplus H^1_{f,I}(\mathbf{Q}_\ell, N) \tag{4.5}$$

because the extension groups as well as the functor V_I commute with direct sums.

Proposition 4.24. *For any $n \in [2-\ell, \ell-2]$ such that $0, -n \in I$ the group $H^1_{f,I}(\mathbf{Q}_\ell, V_I(M_{-n}))$ is independent of I. In fact we have*

$$H^1_{f,I}(\mathbf{Q}_\ell, \mathbf{F}(n)) = \begin{cases} 0 & n < 0 \\ H^1_{\mathrm{un}}(\mathbf{Q}_\ell, \mathbf{F}) & n = 0 \\ H^1_{\mathrm{fl}}(\mathbf{Q}_\ell, \mu_\ell) & n = 1 \\ H^1(\mathbf{Q}_\ell, \mathbf{F}(n)) & n > 1, \end{cases}$$

where

$$H^1_{\mathrm{un}}(\mathbf{Q}_\ell, \mathbf{F}) := \ker(H^1(\mathbf{Q}_\ell, \mathbf{F}) \to H^1(I_\ell, \mathbf{F})) \cong \mathrm{Hom}(G_{\mathbf{Q}_\ell}/I_\ell, \mathbf{F})$$

and $H^1_{\mathrm{fl}}(\mathbf{Q}_\ell, \mu_\ell)$ denotes the peu ramifiée classes, namely those classes corresponding to $\mathbf{Z}_\ell^\times/(\mathbf{Z}_\ell^\times)^\ell \subset \mathbf{Q}_\ell^\times/(\mathbf{Q}_\ell^\times)^\ell \cong H^1(\mathbf{Q}_\ell, \mathbf{F}(1))$. For $n \geq 0$ we note that $\dim_{\mathbf{F}} H^1_{f,I}(\mathbf{Q}_\ell, \mathbf{F}(n)) = 1$.

Remark 4.25.
(1) Proposition 4.24 justifies writing $H^1_\Sigma(\mathbf{Q}_\ell, V_I(M_n))$ as we did in [BK19], without specifying the interval I, as long as I contains $-n$. Under the conditions of Proposition 4.27 (see comment after Proposition 5.1), once we have fixed a suitable interval I we will also drop the subscript I in this paper.
(2) Note that the definition of $H^1_{f,I}(\mathbf{Q}_\ell, V_I(M_n))$ depends on $n \in \mathbf{Z}$, even though the coefficients $V_I(M_n) = \mathbf{F}(n)$ only depend on $n \mod \ell - 1$.
(3) [Niz93] Section 9.3 states a version of this result for the local crystalline cohomology of unramified extensions of $\mathbf{Q}_\ell$ and with $\mathbf{Z}_\ell/\ell^m(n)$ coefficients for $m \in \mathbf{Z}_{>0}$.

Proof. We first note that $H^1(\mathbf{Q}_\ell, \mathbf{F}(n))$ is 1-dimensional for $n \neq 0, 1$, which follows from local Tate duality and the Euler characteristic formula, see e.g. [Was97] Theorem 1 and Proposition 3.

For $n = 0$ we refer the reader to [CHT08] Corollary 2.4.4 for identifying $H^1_{f,I}(\mathbf{Q}_\ell, \mathbf{F}(n))$ with $H^1_{\mathrm{un}}(\mathbf{Q}_\ell, \mathbf{F})$. That $H^1_{\mathrm{un}}(\mathbf{Q}_\ell, \mathbf{F})$ is 1-dimensional follows since $\#H^1(G_{\mathbf{Q}_\ell}/I_\ell, \mathbf{F}) = \#H^0(\mathbf{Q}_\ell, \mathbf{F})$.

Since $0 \in I$ we know $b \geq 0$. If $I = [a, b]$ and $n < 0$ (in particular $-n \notin [2-\ell, 0]$) then

$$H^1_{f,I}(\mathbf{Q}_\ell, V_I(M_{-n,\mathbf{F}})) = V_I(\mathrm{Ext}^1_{MF^{f,I}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} \mathbf{F}}(M_{0,\mathbf{F}}, M_{-n,\mathbf{F}})).$$

By Proposition 4.8(ii) $\mathrm{Ext}^1_{MF^{f,I}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} \mathbf{F}}(M_{0,\mathbf{F}}, M_{-n,\mathbf{F}}) = 0$ since the Fontaine-Laffaille weights satisfy the inequality $-n > 0$.

On the other hand, if $n > 0$ then

$$H^1_{f,I}(\mathbf{Q}_\ell, V_I(M_{-n})) = V_I(\mathrm{Ext}^1_{MF^{f,I}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} \mathbf{F}}(M_{0,\mathbf{F}}, M_{-n,\mathbf{F}}))$$

is 1-dimensional by Proposition 4.8(i) since $-n < 0$. For $n > 1$ this equals $H^1(\mathbf{Q}_\ell, \mathbf{F}(n))$ by our observation at the start of the proof.

For $n = 1$ we have $H^1(\mathbf{Q}_\ell, \mathbf{F}(1)) \cong \mathbf{Q}_\ell^\times/(\mathbf{Q}_\ell^\times)^\ell$ is 2-dimensional, and one can identify the Fontaine-Laffaille extensions with the peu ramifiée classes (see e.g. [Bre01] Lemma 8.1.3). ☐

Remark 4.26. Note that $[2-\ell, 0]$ contains both 0 and $2-\ell$ (and is the only interval of this length that contains both). Then since $\mathbf{F}(-1) = \mathbf{F}(\ell-2) = V_{[2-\ell,0]}(M_{2-\ell})$ we get

$$H^1_{f,[2-\ell,0]}(\mathbf{Q}_\ell, \mathbf{F}(-1)) = H^1_{f,[2-\ell,0]}(\mathbf{Q}_\ell, \mathbf{F}(\ell-2)) = H^1_{f,[2-\ell,0]}(\mathbf{Q}_\ell, V_{[2-\ell,0]}(M_{2-\ell})) \neq 0,$$

corresponding to the crystalline non-split extension $\begin{pmatrix} \overline{\epsilon}^{\ell-2} & * \\ 0 & 1 \end{pmatrix}$. Note that $1 \notin [2-\ell, 0]$, so this does not contradict the result in Proposition 4.24 that $H^1_{f,I}(\mathbf{Q}_\ell, \mathbf{F}(-1)) = 0$ for I as in Proposition 4.24.

However for all other intervals $I \subset [2-\ell, \ell-2]$ of length $\ell - 2$ we have $1 \in I$ and so then

$$H^1_{f,I}(\mathbf{Q}_\ell, \mathbf{F}(-1)) = V_I(\mathrm{Ext}^1_{MF^{f,[a,b]}_{\mathrm{tor},\mathbf{Z}_\ell}}(M_0, M_1)) = T_{\mathrm{cris}}(\mathrm{Ext}^1_{MF^{f,[2-\ell,0]}_{\mathrm{tor},\mathbf{Z}_\ell}}(M_{-b}, M_{1-b}))(-b) = 0$$

as per Proposition 4.24.

This demonstrates that $H^1_{f,I}(\mathbf{Q}_\ell, \mathbf{F}(n))$ is only independent of I for I containing $-n$.

Following [BK90] we define for a $\mathbf{Q}_\ell[G_{\mathbf{Q}_\ell}]$-module V

$$H^1_f(\mathbf{Q}_\ell, V) = \ker\left(H^1(\mathbf{Q}_\ell, V) \to H^1(\mathbf{Q}_\ell, V \otimes_{\mathbf{Q}_\ell} B_{\mathrm{cris}})\right),$$

where B_{cris} is Fontaine's ring of ℓ-adic periods, see [Fon82].

Let V be a finite-dimensional E-vector space and $T \subset V$ a $G_{\mathbf{Q}_\ell}$-stable $\mathcal{O}$-lattice, i.e., T is a free $\mathcal{O}$-submodule of V that spans V as a vector space over E. We set $W = V/T$ and $W[\lambda^m] = \{w \in W : \lambda^m w = 0\} \cong T/\lambda^m T$ for any $m \in \mathbf{Z}_{>0}$. Note that $W[\lambda^m]$ lies in $\mathrm{Rep}^{\mathrm{cris},-I}_{\mathcal{O}/\lambda^m}(G_{\mathbf{Q}_\ell})$ if V is crystalline with Hodge-Tate weights in $-I$. We let $H^1_f(\mathbf{Q}_\ell, W)$ be the image of $H^1_f(\mathbf{Q}_\ell, V)$ under the natural map $H^1(\mathbf{Q}_\ell, V) \to H^1(\mathbf{Q}_\ell, W)$.

Proposition 4.27 ([DFG04] Proposition 2.2)**.** *Assume V is a crystalline $E[G_{\mathbf{Q}_\ell}]$-module as above with Hodge-Tate weights in $-I = [-b, -a]$ (and $0 \in I$). For $T \subset V$ and $W = V/T$ as above we then have $H^1_f(\mathbf{Q}_\ell, W) = \varinjlim_m H^1_{f,I}(\mathbf{Q}_\ell, W[\lambda^m])$.*

Proof. We note that the proof of [DFG04] Proposition 2.2 carries over from $[0, \ell-2]$ to general I (in particular one has Proposition 4.7) and apply the argument with (in their notation) V_1 the trivial $G_{\mathbf{Q}_\ell}$-representation and $V_2 = V$. □

Corollary 4.28 ([DFG04] (33), [BK13] Corollary 5.4)**.** *For every $m \in \mathbf{Z}_{>0}$ we have an exact sequence of $\mathcal{O}$-modules*

$$0 \to H^0(\mathbf{Q}_\ell, W)/\lambda^m \to H^1_{f,I}(\mathbf{Q}_\ell, W[\lambda^m]) \to H^1_f(\mathbf{Q}_\ell, W)[\lambda^m] \to 0.$$

Corollary 4.29. *For $n \in \mathbf{Z}$ with $0, n \in I \subset [2-\ell, \ell-2]$ and $n \neq 0$ we have*

$$H^1_{f,I}(\mathbf{Q}_\ell, V_I(M_{-n,\mathbf{F}})) = H^1_f(\mathbf{Q}_\ell, E/\mathcal{O}(n))[\lambda].$$

Proof. Note that $H^0(\mathbf{Q}_\ell, E/\mathcal{O}(n)[\lambda]) = 0$ since $n \not\equiv 0 \mod \ell-1$. This implies $H^0(\mathbf{Q}_\ell, E/\mathcal{O}(n)) = 0$, hence we are done by Corollary 4.28. □

5. Selmer Groups

5.1. **Definitions.** For M a topological $\mathbf{Z}_\ell[G_{\mathbf{Q}}]$-module we set

$$H^1_{\mathrm{un}}(\mathbf{Q}_p, M) := \ker\left(H^1(\mathbf{Q}_p, M) \to H^1(I_p, M)\right)$$

for every prime p.

Let $E/\mathbf{Q}_\ell$ be a finite extension with valuation ring $\mathcal{O}$, uniformizer λ, and residue field $\mathbf{F} = \mathcal{O}/\lambda$. Let V be a finite dimensional E-vector space on which one has a continuous E-linear $G_{\mathbf{Q}}$ action. For finite primes p with $p \neq \ell$, we set

$$H^1_f(\mathbf{Q}_p, V) = H^1_{\mathrm{un}}(\mathbf{Q}_p, V).$$

For $p = \ell$, we recall from Section 4 that

$$H^1_f(\mathbf{Q}_\ell, V) = \ker\left(H^1(\mathbf{Q}_\ell, V) \to H^1(\mathbf{Q}_\ell, V \otimes_{\mathbf{Q}_\ell} B_{\mathrm{cris}})\right).$$

Let $T \subset V$ be a $G_{\mathbf{Q}}$-stable $\mathcal{O}$-lattice, i.e., T is a free $\mathcal{O}$-submodule of V that spans V as a vector space over E. We set $W = V/T$ and $W[\lambda^n] = \{w \in W : \lambda^n w = 0\} \cong T/\lambda^n T$. For every p we let $H^1_f(\mathbf{Q}_p, W)$ be the image of $H^1_f(\mathbf{Q}_p, V)$ under the natural map $H^1(\mathbf{Q}_p, V) \to H^1(\mathbf{Q}_p, W)$. We have $H^1_f(\mathbf{Q}_p, W) = H^1_{\mathrm{un}}(\mathbf{Q}_p, W)$ for all $p \neq \ell$, as long as V is unramified at p, which for us will always be the case.

We define the global Selmer group of W as

$$H^1_f(\mathbf{Q}, W) = \ker\left\{H^1(\mathbf{Q}, W) \to \bigoplus_p \frac{H^1(\mathbf{Q}_p, W)}{H^1_f(\mathbf{Q}_p, W)}\right\}.$$

We note that as $H^1_f(\mathbf{Q}_\ell, W)$ commutes with direct sums and so clearly does $H^1_{\mathrm{un}}(\mathbf{Q}_\ell, W)$, we get that $H^1_f(\mathbf{Q}, W)$ does as well.

Let $I = [a, b]$ with $a, b \in \mathbf{Z}$ and $b - a \leq \ell - 2$ and assume that $0 \in I$. If V is crystalline with Hodge-Tate weights in $-I$ we define

$$H^1_{f,I}(\mathbf{Q}, W[\lambda^n]) = \ker\left\{H^1(\mathbf{Q}, W[\lambda^n]) \to \bigoplus_{p \neq \ell} \frac{H^1(\mathbf{Q}_p, W[\lambda^n])}{H^1_{\mathrm{un}}(\mathbf{Q}_p, W[\lambda^n])} \oplus \frac{H^1(\mathbf{Q}_\ell, W[\lambda^n])}{H^1_{f,I}(\mathbf{Q}_\ell, W[\lambda^n])}\right\}.$$

As noted in (4.5) $H^1_f(\mathbf{Q}_\ell, W[\lambda^n])$ also commutes with direct sums and so we get that $H^1_{f,I}(\mathbf{Q}, W[\lambda^n])$ does as well.

We record a slight strengthening of [BK13] Proposition 5.8:

Proposition 5.1. *Assume that the interval $I = [a, b]$ contains 0 and V is a $E[G_{\mathbf{Q}}$-module which is finite-dimensional as an E-vector space and crystalline as a $G_{\mathbf{Q}_\ell}$-module with Hodge-Tate weights in $-I$. If $H^0(\mathbf{Q}, W[\lambda]) = 0$ then we have*

$$H^1_f(\mathbf{Q}, W)[\lambda^n] \cong H^1_{f,I}(\mathbf{Q}, W[\lambda^n]).$$

Proof. [BK13] Proposition 5.8 proves the claim under the assumption $H^0(\mathbf{Q}, W) = 0$.

Suppose we have $\alpha \in H^0(\mathbf{Q}, W)$. We know every element of W is annihilated by some power of λ, so if $\alpha \neq 0$ there is an integer m so that $\lambda^m \alpha = 0$ but $\lambda^n \alpha \neq 0$ for all $0 < n < m$. However, this gives $\lambda^{m-1}\alpha \in H^0(\mathbf{Q}, W[\lambda]) = 0$, so it must be that $\alpha = 0$. Thus, $H^0(\mathbf{Q}, W) = 0$ as desired. $\square$

After a suitable interval I has been fixed we will therefore also drop the subscript I and write $H^1_f(\mathbf{Q}, W[\lambda^n])$.

Let G be a group, R a commutative ring with identity, and M_i finitely generated free R-modules with R-linear action given by $\rho_i : G \to \mathrm{GL}_R(M_i)$ for $i = 1, 2$. The action of G on $\mathrm{Hom}_R(\rho_2, \rho_1)$ is given by

$$(g \cdot \varphi)(v) = \rho_1(v)\varphi(\rho_2(g^{-1})v).$$

In particular, if $\rho_1 = \rho_2 = \rho$, we define the adjoint representation of ρ to be the $R[G]$-module $\mathrm{ad}\,\rho = \mathrm{Hom}_R(\rho, \rho)$. We write $\mathrm{ad}^0\,\rho$ for the $R[G]$-submodule of $\mathrm{ad}\,\rho$ consisting of endomorphisms of trace zero.

If ρ is of rank n and $2n \in R^\times$ then we have an isomorphism of $R[G]$-modules

$$\mathrm{ad}\,\rho \cong \mathrm{ad}^0\,\rho \oplus R. \tag{5.1}$$

5.2. **Non-vanishing of a Selmer group.** In this section we explain how the congruence of a Siegel cuspform to the Klingen Eisenstein series in Section 3 leads to a non-zero element of $H^1_f(\mathbf{Q}, \mathrm{ad}^0(\rho_{\phi,\lambda})(2-k) \otimes E/\mathcal{O})$.

From now on, we fix the weight $k > 4$ even and the prime ℓ satisfying $\ell > 4k - 5$ and impose Assumption 3.1 on the field $E/\mathbf{Q}_\ell$. Let $\phi \in S_k(\Gamma_1)$ be a normalized eigenform. Let ρ_ϕ be the λ-adic Galois representation associated to ϕ and assume $\overline{\rho}_\phi$ is irreducible. Let $f \in S_k(\Gamma_2)$ be an eigenform with irreducible Galois representation ρ_f so that f is eigenvalue congruent to $E^{2,1}_\phi$ modulo λ.

The following result shows we can choose a lattice so that the residual Galois representation gives rise to a non-split extension.

Lemma 5.2. *There exists a $G_{\mathbf{Q}}$-stable lattice in the space of ρ_f such that with respect to this lattice*

$$\overline{\rho}_f = \begin{bmatrix} \overline{\rho}_\phi & * \\ & \overline{\rho}_\phi(k-2) \end{bmatrix} \not\cong \overline{\rho}_\phi \oplus \overline{\rho}_\phi(k-2).$$

Proof. Using the compactness of $G_{\mathbf{Q}}$ one can show that there exists a $G_{\mathbf{Q}}$-stable lattice Λ' in the space of ρ_f. In other words, there exists a $\mathrm{GL}_4(E)$-conjugate $\rho_{f,\Lambda'}$ of ρ_f which has image in $\mathrm{GL}_4(\mathcal{O})$. As

$$\mathrm{tr}\,\rho_f(\mathrm{Frob}_p) \equiv \mathrm{tr}\,\rho_\phi(\mathrm{Frob}_p) + \mathrm{tr}\,\rho_\phi(k-2)(\mathrm{Frob}_p) \pmod{\lambda}$$

for all primes $p \neq \ell$ one uses Brauer-Nesbitt Theorem together with the Chebotarev Density Theorem to conclude that $\overline{\rho}^{\mathrm{ss}}_{f,\Lambda'} = \overline{\rho}_\phi \oplus \overline{\rho}_\phi(k-2)$. Now the existence of the desired lattice which gives the non-split extension follows from Theorem 4.1 in [BK20]. $\square$

From now on, whenever we write ρ_f, we assume we have made a choice of lattice as in Lemma 5.2, so we consider ρ_f as a map from $G_{\mathbf{Q}}$ to $\mathrm{GL}_4(\mathcal{O})$.

We now choose the interval $I = [3 - 2k, 2k - 3]$ so that it contains all the Hodge-Tate weights of ρ_f, ρ_ϕ, $\rho_\phi(k-2)$, $\mathrm{ad}\,\rho_\phi(2-k)$, and $\mathrm{ad}\,\rho_\phi(k-2)$. Note that $-I = I$. We assume that $\ell - 2 \geq 4k - 6$. When we write H^1_f from now on this refers to $H^1_{f,I}$ as defined in Section 5.1.

Let ρ be any of the representations above and write V for the representation space of ρ. We choose a $G_{\mathbf{Q}}$-stable lattice $T \subset V$ and recall that the isomorphism class of the semi-simplification of the $\mathbf{F}[G_{\mathbf{Q}}]$-representation $T/\lambda T$ is independent of the choice of T. It is well-known that if $T/\lambda T$ is irreducible then the $\mathcal{O}$-length of $H^1_f(\mathbf{Q}, W)$ is independent of T, where as before $W = V/T$. By Proposition 5.1 we then conclude that also the $\mathcal{O}$-length of $H^1_f(\mathbf{Q}, W[\lambda^n])$ is independent of the choice of T as long as $H^0(\mathbf{Q}, W) = 0$.

Lemma 5.3. *Under our assumptions (in particular, $\overline{\rho}_\phi$ irreducible and $\ell > 4k-5$) the mod λ reduction of $\operatorname{ad}^0 \rho_\phi$ is irreducible.*

Proof. Assume the three-dimensional representation $\operatorname{ad}^0 \overline{\rho}_\phi$ is reducible. Then it either has a one-dimensional $G_{\mathbf{Q}}$-stable subspace or quotient. Since $\operatorname{ad} \rho_\phi$ and $\mathbf{1}$ are self-dual, so is $\operatorname{ad}^0 \overline{\rho}_\phi$. Hence we can assume without loss of generality that $\operatorname{ad}^0 \overline{\rho}_\phi$ has a $G_{\mathbf{Q}}$-stable line. Write ψ for the character by which $G_{\mathbf{Q}}$ acts on the line.

As $\overline{\rho}_\phi$ is unramified away from ℓ, the splitting field of ψ must be a subfield of $\mathbf{Q}(\zeta_{\ell^\infty})$. As the order of ψ is prime to ℓ, this splitting field must be a subfield of $\mathbf{Q}(\mu_\ell)$, so $\psi = \overline{\epsilon}^a$ for some integer $a \in I$.

This would require $H^0(\mathbf{Q}, \operatorname{ad}^0 \overline{\rho}_\phi(-a)) \neq 0$. Note that

$$H^0(\mathbf{Q}, \operatorname{ad} \overline{\rho}_\phi(-a)) = \operatorname{Hom}_{G_{\mathbf{Q}}}(\overline{\rho}_\phi(a), \overline{\rho}_\phi).$$

If $a \equiv 0 \pmod{(\ell-1)}$, then this space is one-dimensional by Schur's Lemma since $\overline{\rho}_\phi$ is irreducible. So, $H^0(\mathbf{Q}, \operatorname{ad}^0 \overline{\rho}_\phi) = 0$, contradiction.

If $a \not\equiv 0 \pmod{(\ell-1)}$, then

$$H^0(\mathbf{Q}, \operatorname{ad} \overline{\rho}_\phi(-a)) = H^0(\mathbf{Q}, \operatorname{ad}^0 \overline{\rho}_\phi(-a)) \neq 0.$$

This means that $\overline{\rho}_\phi$ is isomorphic to $\overline{\rho}_\phi(a)$. Considering the determinant, $\overline{\epsilon}^a$ must be the trivial character or the quadratic character $\overline{\epsilon}^{(\ell-1)/2}$. Both are ruled out since $a \in I = [3-2k, 2k-3]$ by our assumption that $\ell > 4k-5$. $\square$

Remark 5.4. From Lemma 5.3 we conclude that when $\rho \in \{\rho_\phi, \rho_\phi(k-2), \operatorname{ad}^0 \rho_\phi(2-k), \operatorname{ad}^0 \rho_\phi(k-2)\}$, the $\mathcal{O}$-length of $H^1_f(\mathbf{Q}, W)$ and $H^1_f(\mathbf{Q}, W[\lambda^n])$ are independent of the choice of T. As we will ever only be interested in the order of these groups, the choice of T is immaterial and we will simply assume that such a choice was made. So, for example we will use the notation $H^1_f(\mathbf{Q}, \operatorname{ad}^0 \rho_{\phi,\lambda}(k-2) \otimes E/\mathcal{O})$, thus assuming that when we write $\operatorname{ad}^0 \rho_{\phi,\lambda}(k-2)$, we have made a choice of a lattice for this representation. Likewise any one-dimensional representation ρ is irreducible, so the $\mathcal{O}$-length of $H^1_f(\mathbf{Q}, \rho \otimes E/\mathcal{O})$ is independent of the choice of T.

For the representation $\operatorname{ad} \rho(m)$, $m \in \{k-2, 2-k\}$ (which is reducible) we choose a lattice which is a direct sum of a lattice inside $\operatorname{ad}^0 \rho(m)$ and a lattice inside $E(m)$. So, from now on whenever we write $\operatorname{ad} \rho(m)$ we mean such a lattice. Since the formation of Selmer groups commutes with direct sums we then get

(5.2)
$$H^1_f(\mathbf{Q}, \operatorname{ad} \rho_\phi(m) \otimes E/\mathcal{O}) = H^1_f(\mathbf{Q}, \operatorname{ad}^0 \rho_\phi(m) \otimes E/\mathcal{O}) \oplus H^1_f(\mathbf{Q}, E/\mathcal{O}(m)) \quad \text{for } m \in \{k-2, 2-k\}.$$

Note that the $\mathcal{O}$-length (and in particular, the non-triviality) of $H^1_f(\mathbf{Q}, \operatorname{ad} \rho(m) \otimes E/\mathcal{O})$ is independent of the choice of a lattice inside $\operatorname{ad} \rho_\phi(m)$ as long as it is the direct sum of lattices in $\operatorname{ad}^0 \rho_\phi(m)$ and $E(m)$.

Theorem 5.5. *With the set-up as above we have $H^1_f(\mathbf{Q}, \operatorname{ad} \rho_\phi(2-k) \otimes E/\mathcal{O}) \neq 0$.*

Proof. We have via Lemma 5.2 that there is a lattice $T_f \subset V_f$ so that the residual representation $\overline{\rho}_f : G_{\mathbf{Q}} \to \mathrm{GL}_4(\mathbf{F})$ has the form

$$\overline{\rho}_f = \begin{bmatrix} \overline{\rho}_\phi & \psi \\ 0 & \overline{\rho}_\phi(k-2) \end{bmatrix} \tag{5.3}$$

and is not semisimple. The fact that ψ as in (5.3) gives a non-trivial class $[\psi]$ in $H^1(\mathbf{Q}, \mathrm{Hom}_{\mathbf{F}}(\overline{\rho}_2, \overline{\rho}_1)) = H^1(\mathbf{Q}, \mathrm{ad}\,\rho_\phi(2-k)\otimes E/\mathcal{O}[\lambda])$ is clear. We need to show that $[\psi]$ lies in $H^1_f(\mathbf{Q}, \mathrm{ad}\,\rho_\phi(2-k)\otimes E/\mathcal{O}[\lambda])$ and that the latter group injects into $H^1_f(\mathbf{Q}, \mathrm{ad}\,\rho_\phi(2-k)\otimes E/\mathcal{O})$.

We first show that $[\psi]$ satisfies the conditions to be in $H^1_f(\mathbf{Q}, \mathrm{ad}\,\rho_\phi(2-k)\otimes E/\mathcal{O}[\lambda])$. We have that ρ_f is unramified at all primes $p \neq \ell$, so the local conditions are satisfied for all primes $p \neq \ell$.

Since f has level one and weight k, $\rho_f|_{D_\ell}$ is crystalline with Hodge-Tate weights in $[0, 2k-3] \subset I = -I$. Hence $\overline{\rho}_f$ (considered as a $G_{\mathbf{Q}_\ell}$-module) belongs to $\mathrm{Rep}^{\mathrm{cris},I}_{\mathrm{free},\mathbf{F}}(G_{\mathbf{Q}_\ell})$ and gives rise to an element of

$$\mathrm{Ext}^1_{\mathrm{Rep}^{\mathrm{cris},I}_{\mathrm{free},\mathbf{F}}(G_{\mathbf{Q}_\ell})}(\overline{\rho}_\phi(k-2), \overline{\rho}_\phi) \subset \mathrm{Ext}^1_{\mathbf{F}[G_{\mathbf{Q}_\ell}]}(\rho_\phi(k-2)\otimes E/\mathcal{O}[\lambda], \rho_\phi \otimes E/\mathcal{O}[\lambda]).$$

By our choice of I we can use (4.4) and Proposition 4.9 to get a non-zero element in

$$\mathrm{Ext}^1_{\mathrm{Rep}^{\mathrm{cris},I}_{\mathrm{free},\mathbf{F}}(G_{\mathbf{Q}_\ell})}(\mathbf{F}, \mathrm{ad}\,\rho_\phi(2-k)\otimes E/\mathcal{O}[\lambda]) \subset \mathrm{Ext}^1_{\mathbf{F}[G_{\mathbf{Q}_\ell}]}(\mathbf{F}, \mathrm{ad}\,\rho_\phi(2-k)\otimes E/\mathcal{O}[\lambda]).$$

As this extension maps to $[\psi|_{G_{\mathbf{Q}_\ell}}]$ in $H^1(\mathbf{Q}_\ell, \mathrm{ad}\,\rho_\phi(2-k)\otimes E/\mathcal{O}[\lambda])$ under the canonical isomorphism $\mathrm{Ext}^1_{\mathbf{F}[G_{\mathbf{Q}_\ell}]}(\mathbf{F}, \mathrm{ad}\,\rho_\phi(2-k)\otimes E/\mathcal{O}[\lambda]) \cong H^1(\mathbf{Q}_\ell, \mathrm{ad}\,\rho_\phi(2-k)\otimes E/\mathcal{O}[\lambda])$, we conclude that

$$[\psi|_{G_{\mathbf{Q}_\ell}}] \in H^1_f(\mathbf{Q}_\ell, \mathrm{ad}\,\rho_\phi(2-k)\otimes E/\mathcal{O}[\lambda]) \subset H^1(\mathbf{Q}_\ell, \mathrm{ad}\,\rho_\phi(2-k)\otimes E/\mathcal{O}[\lambda]).$$

Therefore we have established that

$$[\psi] \in H^1_f(\mathbf{Q}, \mathrm{ad}\,\rho_\phi(2-k)\otimes E/\mathcal{O}[\lambda]).$$

By Proposition 5.1 this group is isomorphic to $H^1_f(\mathbf{Q}, \mathrm{ad}\,\rho_\phi(2-k)\otimes E/\mathcal{O})[\lambda]$ if $H^0(\mathbf{Q}, \mathrm{ad}\,\rho_\phi(2-k)\otimes E/\mathcal{O}[\lambda]) = 0$. The latter holds since

$$\mathrm{ad}\,\rho_\phi(2-k)\otimes E/\mathcal{O}[\lambda]^{G_{\mathbf{Q}}} = \mathrm{Hom}_{G_{\mathbf{Q}}}(\overline{\rho}_\phi(k-2), \overline{\rho}_\phi) = 0 \tag{5.4}$$

as $\overline{\rho}_\phi$ and $\overline{\rho}_\phi(k-2)$ are absolutely irreducible (by assumption) and non-isomorphic since $k-2 \not\equiv 0, \frac{\ell-1}{2} \pmod{\ell-1}$ as $\ell > 4k-5$ and $k \neq 2$ (cf. the proof of Lemma 5.3). □

Lemma 5.6. *Let n be an even integer satisfying $3-2k < n \leq 0$. Assuming $\ell \nmid \#\mathrm{Cl}^{\overline{\epsilon}^n}_{\mathbf{Q}(\zeta_\ell)^+}$, one has $H^1_f(\mathbf{Q}, \mathbf{F}(n)) = 0$ and, if additionally $n \neq 0$, $H^1_f(\mathbf{Q}, E/\mathcal{O}(n)) = 0$.*

Proof. We see from Proposition 4.24 that any cohomology class in $H^1_f(\mathbf{Q}, \mathbf{F}(n))$ must vanish when restricted to I_ℓ. As all classes in $H^1_f(\mathbf{Q}, \mathbf{F}(n))$ are unramified away from ℓ, we therefore get that they are unramified everywhere. Using inflation-restriction sequence where $H = \mathrm{Gal}(\mathbf{Q}(\zeta_\ell)^+/\mathbf{Q})$ we see that

$$H^1(\mathbf{Q}, \mathbf{F}(n)) \cong H^1(\mathbf{Q}(\zeta_\ell)^+, \mathbf{F}(n))^H = \mathrm{Hom}_H(G_{\mathbf{Q}(\zeta_\ell)^+}, \mathbf{F}(n)).$$

Note that everywhere unramified classes map to homomorphisms that kill all the inertia groups. Hence the image of $H^1_f(\mathbf{Q}, \mathbf{F}(n))$ lands inside $\mathrm{Hom}\left(\mathrm{Cl}^{\overline{\epsilon}^n}_{\mathbf{Q}(\zeta_\ell)^+}, \mathbf{F}\right) = 0$.

Note that a torsion $\mathcal{O}$-module M is zero if and only if $M[\lambda] = 0$. Therefore the vanishing of $H^1_f(\mathbf{Q}, E/\mathcal{O}(n))$ follows from Proposition 5.1, which tells us that $H^1_f(\mathbf{Q}, E/\mathcal{O}(n))[\lambda] = H^1_f(\mathbf{Q}, \mathbf{F}(n))$ if $H^0(\mathbf{Q}, E/\mathcal{O}(n)) = 0$. We know that $H^0(\mathbf{Q}_\ell, E/\mathcal{O}(n)[\lambda]) = H^0(\mathbf{Q}, \mathbf{F}(n)) = 0$ for $n \neq 0$ since $n \not\equiv 0 \pmod{\ell-1}$ under our assumption $\ell > 4k-5$. □

Corollary 5.7. *Let $\phi \in S_k(\Gamma_1)$ be as in Theorem 3.5 and assume the hypotheses of Theorem 3.5 are satisfied. Assuming $\ell \nmid \# \mathrm{Cl}_{\mathbf{Q}(\zeta_\ell)^+}^{\overline{\epsilon}^{2-k}}$ one has $H^1_f(\mathbf{Q}, \mathrm{ad}^0\, \rho_\phi(2-k) \otimes E/\mathcal{O}) \neq 0$.*

Proof. This follows from Theorem 5.5, Lemma 5.6 and isomorphism (5.2). □

Remark 5.8. If we assume Vandiver's conjecture for the prime ℓ, this gives that $\ell \nmid \# \mathrm{Cl}_{\mathbf{Q}(\zeta_\ell)^+}^{\overline{\epsilon}^{2-k}}$.

Thus, we obtain the following corollary.

Corollary 5.9. *Let $\phi \in S_k(\Gamma_1)$ be as in Theorem 3.5 and assume the hypotheses of Theorem 3.5 are satisfied. Furthermore, assume Vandiver's conjecture is true for the prime ℓ. Then we have*

$$\ell \mid \# H^1_f(\mathbf{Q}, \mathrm{ad}^0\, \overline{\rho}_\phi(2-k)).$$

6. Modularity

Recall that we fixed the weight $k > 4$ even and prime $\ell > 4k-5$ and imposed Assumption 3.1 on the field $E/\mathbf{Q}_\ell$. We also fixed the Fontaine-Laffaille interval $I = [3-2k, 2k-3]$. Let $\phi \in S_k(\Gamma_1)$ be a newform such that $\overline{\rho}_\phi$ is irreducible.

The goal of this section is to prove a modularity theorem under the following assumption:

Assumption 6.1. For k, and ϕ as above we assume that

(i) there exists $f \in S_k(\Gamma_2)$ such that $f \equiv_{\mathrm{ev}} E^{2,1}_\phi \pmod{\lambda}$, and
(ii) $\# H^1_f(\mathbf{Q}, \mathrm{ad}^0 \rho_\phi(2-k) \otimes_{\mathcal{O}} E/\mathcal{O}) = \# \mathcal{O}/\lambda$ (recall that the left hand side is independent of the choice of lattice, see Remark 5.4), and
(iii) $H^1_f(\mathbf{Q}, \mathrm{ad}^0\, \overline{\rho}_\phi) = 0$.

We impose Assumption 6.1 and fix f as in Assumption 6.1 in what follows.

Remark 6.2. Assumption 6.1 (i) is satisfied under the assumptions of Theorem 3.5, and so is one inequality in Assumption 6.1 (ii) under the assumptions of Corollary 5.9.

As before we denote the ℓ-adic Galois representation attached to f by $\rho_f : G_{\mathbf{Q}} \to \mathrm{GL}_4(E)$ (see Theorem 2.1). In particular, such ρ_f is unramified away from ℓ. Lemma 3.4 gives that ρ_f is irreducible.

We will write $G_{\{\ell\}}$ for the Galois group of the maximal Galois extension of $\mathbf{Q}$ unramified away from ℓ. Clearly the representation ρ_f factors through $G_{\{\ell\}}$.

We will use Mazur's deformation theory which is well-known, and we refer the reader to standard references such as [CSS97, Ram93] for the definitions and basic properties.

Definition 6.3. For $B \in \mathrm{LCN}_{\mathcal{O}}$ we say that a representation $\rho : G_{\mathbf{Q}_\ell} \to \mathrm{GL}_n(B)$ is *Fontaine-Laffaille* (with Hodge-Tate weights in $-I$) if $\rho \otimes_B A$ lies in $\mathrm{Rep}^{\mathrm{cris},-I}_{\mathrm{free},A}(G_{\mathbf{Q}_\ell})$ (see Definition 4.11(v)) for every Artinian quotient A of B. By Theorem 4.16(iv) this is equivalent to requiring $\rho \otimes_B A$ to lie in the essential image of the Fontaine-Laffaille functor.

Remark 6.4. We know that any choice of $\mathcal{O}$-lattice ρ_L in ρ_ϕ or ρ_f is Fontaine-Laffaille in this sense, since their restrictions to $G_{\mathbf{Q}_\ell}$ lie in $\mathrm{Rep}^{\mathrm{cris},-I}_{\mathbf{Z}_\ell}(G_{\mathbf{Q}_\ell})$ and therefore in the essential image of the Fontaine-Laffaille functor by Theorem 4.16(iii). Since they are also free $\mathcal{O}$-modules this implies by Theorem 4.16 (iii) and (iv) that $\rho_L \otimes B$ lies in $\mathrm{Rep}^{\mathrm{cris},-I}_{\mathrm{free},A}(G_{\mathbf{Q}_\ell})$ for every Artinian quotient B of $\mathcal{O}$.

For any local complete Noetherian $\mathcal{O}$-algebra A with residue field $\mathbf{F}$ by a deformation of a residual Galois representation $\tau : G_{\{\ell\}} \to \mathrm{GL}_n(\mathbf{F})$ we will mean a strict equivalence class of lifts $\tilde{\tau} : G_{\{\ell\}} \to \mathrm{GL}_n(A)$ of τ that are Fontaine-Laffaille at ℓ. This deformation condition is introduced in [BK13] Section 5.3 and [CHT08] p.35.

As is customary, we will denote a strict equivalence class of deformations by any of its members. If τ has scalar centralizer then this deformation problem is representable by a local complete Noetherian $\mathcal{O}$-algebra which we will denote by R_τ [Ram02]. In particular, the identity map in $\mathrm{Hom}_{\mathcal{O}-\mathrm{alg}}(R_\tau, R_\tau)$ furnishes what is called the universal deformation $\tau^{\mathrm{univ}} : G_{\{\ell\}} \to \mathrm{GL}_n(R_\tau)$.

Lemma 6.5. *One has $R_{\overline{\rho}_\phi} \cong R_{\overline{\rho}_\phi(k-2)} \cong \mathcal{O}$. Furthermore, ρ_ϕ (resp. $\rho_\phi(k-2)$) is the unique deformation of $\overline{\rho}_\phi$ (resp. $\overline{\rho}_\phi(k-2)$) to $\mathrm{GL}_2(\mathcal{O})$.*

Proof. Using the fact that our deformation condition is the property of being Fontaine-Laffaille one has (see e.g., Section 2.4.1 [CHT08])

$$\# \mathrm{Hom}_{\mathbf{F}}(\mathfrak{m}/(\mathfrak{m}^2, \lambda), \mathbf{F}) = \# \mathrm{Hom}_{\mathcal{O}-\mathrm{alg}}(R_{\overline{\rho}_\phi}, \mathbf{F}[X]/X^2) = \# H^1_f(\mathbf{Q}, \mathrm{ad}\, \overline{\rho}_\phi), \tag{6.1}$$

where $\mathfrak{m}$ is the maximal ideal of $R_{\overline{\rho}_\phi}$. We have $H^1_f(\mathbf{Q}, \mathrm{ad}\, \overline{\rho}_\phi) = H^1_f(\mathbf{Q}, \mathrm{ad}^0\, \overline{\rho}_\phi) \oplus H^1_f(\mathbf{Q}, \mathbf{F}) = 0$ as $H^1_f(\mathbf{Q}, \mathbf{F}) = 0$ by Lemma 5.6 and we have imposed Assumption 6.1(iii).

There is a short exact sequence of $\mathbf{F}$-vector spaces

$$0 \to (\mathfrak{m}^2, \lambda)/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}/(\mathfrak{m}^2, \lambda) \to 0.$$

Note that $\lambda^2 \in \mathfrak{m}^2$ and $(\mathfrak{m}^2, \lambda)/\mathfrak{m}^2 \cong \lambda R_{\overline{\rho}_\phi}/(\mathfrak{m}^2 \cap \lambda R_{\overline{\rho}_\phi})$. The multiplication by λ induces a surjection $R_{\overline{\rho}_\phi} \to \lambda R_{\overline{\rho}_\phi}/\lambda \mathfrak{m} R_{\overline{\rho}_\phi}$ which descends to a surjection $\mathbf{F} \cong R_{\overline{\rho}_\phi}/\mathfrak{m} \to \lambda R_{\overline{\rho}_\phi}/\lambda \mathfrak{m} R_{\overline{\rho}_\phi}$ followed by a surjection $\lambda R_{\overline{\rho}_\phi}/\lambda \mathfrak{m} R_{\overline{\rho}_\phi} \to \lambda R_{\overline{\rho}_\phi}/(\mathfrak{m}^2 \cap \lambda R_{\overline{\rho}_\phi})$. Hence $\dim_{\mathbf{F}}(\mathfrak{m}, \lambda)/\mathfrak{m}^2 \leq 1$. As in our case $\mathfrak{m}/(\mathfrak{m}^2, \lambda) = 0$, we conclude that $\mathfrak{m}/\mathfrak{m}^2$ is one-dimensional and thus $\mathfrak{m}$ is a cyclic $\mathcal{O}$-module by a complete version of Nakayama's Lemma (cf. [Eis95], Exercise 7.2 or [Mat89] Theorem 8.4). Let us briefly explain why this version applies here. As $R_{\overline{\rho}_\phi} \otimes_{\mathcal{O}} \mathbf{F} \neq 0$, we see that $\lambda \in \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of $R_{\overline{\rho}_\phi}$. Hence

$$\bigcap_n \lambda^n R_{\overline{\rho}_\phi} \subset \bigcap_n \mathfrak{m}^n. \tag{6.2}$$

The latter intersection is zero, since $R_{\overline{\rho}_\phi}$ is complete, so separated with respect to $\mathfrak{m}$. Hence (6.2) implies that $R_{\overline{\rho}_\phi}$ is separated with respect to $\lambda R_{\overline{\rho}_\phi}$ allowing for the application of the complete version of Nakayama's Lemma.

As ρ_ϕ is a deformation to $\mathcal{O}$, we conclude that $R_{\overline{\rho}_\phi} = \mathcal{O}$. [Alternatively, we could have argued as in the proof of Corollary 6.12 using the triviality of $H^1_f(\mathbf{Q}, \mathrm{ad}\, \overline{\rho}_\phi)$ and the second equality in (6.1) in lieu of Lemma 6.9.]

This implies that if $\rho : G_{\{\ell\}} \to \mathrm{GL}_2(\mathcal{O})$ is any deformation of $\overline{\rho}_\phi$, one has $\rho \cong \rho_\phi$. Similarly, if $\rho : G_{\{\ell\}} \to \mathrm{GL}_2(\mathcal{O})$ is a deformation of $\overline{\rho}_\phi(k-2)$ then $\rho(2-k)$ is a deformation of $\overline{\rho}_\phi$. Note that our choice of $I = [3-2k, 2k-3]$ means that this twisting stays inside our category of Fontaine-Laffaille representations. Hence we get that $\rho(2-k) \cong \rho_\phi$, and so we are done. $\square$

Remark 6.6. Note that the determinant of our deformations is automatically fixed as $H^1_f(\mathbf{Q}, \mathrm{ad}\, \overline{\rho}_\phi) = H^1_f(\mathbf{Q}, \mathrm{ad}^0\, \overline{\rho}_\phi)$ under our assumptions. This means that all deformations ρ of $\overline{\rho}_\phi$ (respectively $\overline{\rho}_\phi(k-2)$) satisfy $\det \rho = \epsilon^{k-1}$ (respectively $\det \rho = \epsilon^{2k-3}$).

Remark 6.7. Regarding Assumption 6.1(iii) we note that if one additionally assumes that $\overline{\rho}_\phi$ is absolutely irreducible when restricted to $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}(\sqrt{(-1)^{(\ell-1)/2}\ell})$ then [DFG04] Theorem 3.7 (see also [Hid00] Theorem 5.20) relates $H^1_f(\mathbf{Q}, \mathrm{ad}^0 \rho_\phi \otimes E/\mathcal{O})$ (via an $R_{\overline{\rho}_\phi} = \mathbf{T}$ theorem) to a congruence ideal $\eta^\emptyset_\phi$. One can use Proposition 5.1 this implies that $H^1_f(\mathbf{Q}, \mathrm{ad}^0\, \overline{\rho}_\phi) = H^1_f(\mathbf{Q}, \mathrm{ad}^0 \rho_\phi \otimes E/\mathcal{O})[\lambda] = 0$ if $\eta^\emptyset_\phi$ is coprime to ℓ.

Lemma 6.8. *Let G be a group and F be a field. For $i \in \{1, 2\}$, let $n_i \in \mathbf{Z}_+$ and $\rho_i : G \to \mathrm{GL}_{n_i}(F)$ be an irreducible representation with $\rho_1 \not\cong \rho_2$. Let $\rho : G \to \mathrm{GL}_{n_1+n_2}(F)$ be a representation such* *that*

$$\rho = \begin{bmatrix} \rho_1 & a \\ & \rho_2 \end{bmatrix} \not\cong \rho_1 \oplus \rho_2.$$

Then ρ has scalar centralizer.

Proof. Note that $\tilde{a} : g \to \rho_2(g)^{-1}a(g)$ defines a cocycle from G to $\mathrm{Hom}(\rho_2, \rho_1)$. The fact that ρ is non-semi-simple implies that $\tilde{a}$ is not a coboundary. For $i, j \in \{1, 2\}$ let $A_{i,j} \in \mathrm{Mat}_{n_1,n_2}(F)$ be such that $\begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}$ centralizes ρ. Using the fact that ρ_1, ρ_2 are irreducible and non-isomorphic one gets that $A_{2,1} = 0$. Then Schur's Lemma gives us that $A_{1,1} = \alpha I_{n_1}$ and $A_{2,2} = \delta I_{n_2}$ for scalars α, δ. Finally one gets

$$\alpha a + A_{1,2}\rho_2 = \rho_1 A_{1,2} + a\delta. \tag{6.3}$$

If $\alpha \neq \delta$, then one gets

$$\tilde{a} = \frac{1}{\alpha - \delta}(\rho_1 A_{1,2}\rho_2^{-1} - A_{1,2}),$$

contradicting the fact that it is not a coboundary. Thus $\alpha = \delta$ and (6.3) now gives that $A_{1,2} = 0$. $\square$

Fix a lattice in the space of ρ_f as in Lemma 5.2, i.e. such that $\overline{\rho}_f = \begin{bmatrix} \overline{\rho}_\phi & * \\ & \overline{\rho}_\phi(k-2) \end{bmatrix} : G_{\{\ell\}} \to \mathrm{GL}_4(\mathbf{F})$ is non-semisimple. For simplicity, we will write R for the universal deformation ring $R_{\overline{\rho}_f}$ of $\overline{\rho}_f$ and $\rho^{\mathrm{univ}} : G_{\{\ell\}} \to \mathrm{GL}_4(R)$ for the universal deformation. Note that the deformation problem is representable because $\overline{\rho}_f$ is non-semisimple with irreducible, mutually non-isomorphic Jordan-Holder factors, hence by Lemma 6.8 the centralizer of $\overline{\rho}_f$ consists of only scalar matrices. We say that a deformation $\tilde{\rho}$ is *upper-triangular* if $\tilde{\rho}$ is strictly equivalent to a deformation of $\overline{\rho}_f$ of the form

$$\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$$

with the stars representing 2×2 blocks.

Lemma 6.9. *There do not exist any non-trivial deformations of $\overline{\rho}_f$ into $\mathrm{GL}_4(\mathbf{F}[X]/X^2)$ that are* upper-triangular*.*

Proof. We use Proposition 7.2 in [BK13] noting that Assumption 6.1(i) in [loc.cit.] is satisfied because we impose the current Assumption 6.1(ii). On the other hand, Assumption 6.1(ii) in [loc.cit.] is satisfied because of Lemma 6.5. $\square$

Definition 6.10. The smallest ideal I of R such that $\mathrm{tr}\,\rho^{\mathrm{univ}}$ is the sum of two pseudocharacters mod I will be called the *reducibility ideal* of R. We will denote this ideal by I_{re}.

Proposition 6.11. *Let $I \subset R$ be an ideal such that R/I is an Artin ring. Then $I \supset I_{\mathrm{re}}$ if and only if ρ^{univ} (mod I) is* upper-triangular*.*

Proof. This is proved as Corollary 7.8 in [BK13]. $\square$

Corollary 6.12. *The structure map $\mathcal{O} \to R/I_{\mathrm{re}}$ is surjective and descends to an isomorphism $\mathcal{O}/\lambda^s \to R/I_{\mathrm{re}}$ for some $s \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$. In fact, one has*

$$R/I_{\mathrm{re}} \cong \mathcal{O}/\lambda.$$

Proof. By Theorem 7.16 in [Eis95] we know that any local complete Noetherian $\mathcal{O}$-algebra with residue field $\mathbf{F}$ is a quotient of $\mathcal{O}[[X_1, \dots, X_n]]$ for some positive integer n. Hence $S := R/(I_{\mathrm{re}} + \lambda R) \cong \mathbf{F}[[X_1, \dots, X_n]]/J$ for some ideal J. Suppose first that J is not maximal. Then by Lemma 6.13 we know that S admits a surjection φ to $\mathbf{F}[T]/T^2$. This means that there exists a non-trivial (because the image of φ is not contained in $\mathbf{F}$) deformation of ρ to $\mathbf{F}[T]/T^2$ which is upper-triangular (by Proposition 6.11), which contradicts Lemma 6.9. Thus, indeed, $S = \mathbf{F}$.

Hence, $R/I_{\rm re}$ is generated by one element over $\mathcal{O}$ by the complete version of Nakayama's Lemma (one can argue as in the proof of Lemma 6.5 to justify that this version applies). This proves that the structure map $\mathcal{O} \to R/I_{\rm re}$ is surjective and so $R/I_{\rm re} \cong \mathcal{O}/\lambda^s$ for some $s \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$.

The composition of $\rho^{\rm univ}$ with the map $R \to R/I_{\rm re}$ gives rise to a deformation $\rho_{\rm re} : G_{\{\ell\}} \to \mathrm{GL}_4(R/I_{\rm re}) = \mathrm{GL}_4(\mathcal{O}/\lambda^s)$. By Proposition 6.11, this deformation is upper triangular, i.e., one has

$$\rho_{\rm re} = \begin{bmatrix} *_1 & *_2 \\ & *_3 \end{bmatrix}.$$

As the property of being Fontaine-Laffaille is preserved by subobjects and quotients, we see that $*_1$ and $*_3$ are Fontaine-Laffaille representations with values in $\mathrm{GL}_2(R/I_{\rm re}) = \mathrm{GL}_2(\mathcal{O}/\lambda^s)$. Thus by Lemma 6.5 we can conclude that

$$*_1 = \rho_\phi, \quad *_3 = \rho_\phi(k-2) \pmod{\lambda^s}.$$

Thus by (5.4) and Proposition 5.1 $*_2$ gives rise to a class in $H^1_f(\mathbf{Q}, \mathrm{ad}^0\rho_\phi(2-k) \otimes_{\mathcal{O}} E/\mathcal{O})$ as $\rho_{\rm re}$ is Fontaine-Laffaille. As ρ is non-semi-simple, we conclude that $*_2$ is not annihilated by λ^{s-1}, i.e., the class of $*_2$ gives rise to a subgroup of $H^1_f(\mathbf{Q}, \mathrm{ad}^0\rho_\phi(2-k) \otimes_{\mathcal{O}} E/\mathcal{O})$ isomorphic to $\mathcal{O}/\lambda^s$. Thus $s \leq 1$ as $\# H^1_f(\mathbf{Q}, \mathrm{ad}^0\rho_\phi(2-k) \otimes_{\mathcal{O}} E/\mathcal{O}) \leq \#\mathcal{O}/\lambda$ by Assumption 6.1(ii). Finally, $s > 0$ as $\overline{\rho}_f$ itself is reducible. This concludes the proof. □

Lemma 6.13. *If J is an ideal of $\mathbf{F}[[X_1, \dots, X_n]]$ that is strictly contained in the maximal ideal, then $\mathbf{F}[[X_1, \dots, X_n]]/J$ admits an $\mathbf{F}$-algebra surjection to $\mathbf{F}[T]/T^2$.*

Proof. For a positive integer k let I_k be the ideal of $\mathbf{F}[[X_1, \dots, X_k]]$ generated by all the monomials of degree at least 2. Set $S_k := \mathbf{F}[[X_1, \dots, X_k]]/I_k$ and write $\phi_k : \mathbf{F}[[X_1, \dots, X_k]] \to S_k$ for the canonical $\mathbf{F}$-algebra surjection. If $\phi_n(J) = 0$, then composing ϕ_n with the map $S_n \to \mathbf{F}[[T]]/T^2$ sending X_1 to T and X_i for $i > 1$ to zero gives the desired surjection.

Now suppose $\phi_n(J) \neq 0$. Without loss of generality (renumbering the variables if necessary) we may assume then that J contains an element of the form $u := X_n + f(X_1, \dots, X_{n-1}) + g(X_1, \dots, X_n)$, where f is homogeneous of degree one and all the terms in g have degree at least 2. Note that we can assume without loss of generality that some power of X_n appears in g. (Indeed, if g contains no X_n then we replace u by $u + u^2 \in J$.) By Theorem 7.16(a) in [Eis95] there is a unique $\mathbf{F}$-algebra map from $\mathbf{F}[[X_1, \dots, X_n]]$ to itself sending X_n to $-f-g$ and X_i to itself for $i < n$. In other words, for any power series $h(X_1, \dots, X_n)$, the element $h(X_1, \dots, X_{n-1}, -f-g)$ also lives in $\mathbf{F}[[X_1, \dots, X_n]]$ and we denote it by $h'(X_1, \dots, X_n)$. Clearly $h - h' \in J$.

Thus for any power series h where the smallest total degree of any term containing X_n is s we have

$$h \equiv h' \pmod{J}$$

for some power series h' with the smallest total degree of any term containing X_n equal to $s' > s$. By the same process we get an h'' such that $h' \equiv h'' \mod J$ and the smallest total degree of any term X_n in h'' is strictly greater than s'. This way we can construct a sequence of power series h_s where for every s we have the smallest total degree of any term containing X_n being greater than or equal to s and such that $h - h_s \in J$ for every s. We note that h_s is a Cauchy sequence with respect to the $(X_1, \dots, X_n)$-adic topology (indeed, for $t, u > s$ we see that $h_t - h_u$ lies in $(X_1, \dots, X_n)^s$). Set $h_0 = \lim_{s\to\infty} h_s$. As J is a closed ideal, we get that $h_0 - h \in J$. For every s we have

$$h_0 \equiv h_s \equiv w_s \mod X_n^s,$$

for some $w_s \in \mathbf{F}[[X_1, \dots, X_{n-1}]]$. Note that the w_s also form a Cauchy sequence since h_s does. Set $w := \lim_{s\to\infty} w_s \in \mathbf{F}[[X_1, \dots, X_{n-1}]]$. Thus $h_0 \equiv w$ modulo $\bigcap_s (X_n^s) \subset \bigcap_s (X_1, \dots, X_n)^s = 0$, so $h_0 \in \mathbf{F}[[X_1, \dots, X_{n-1}]]$.

Hence the natural $\mathbf{F}$-algebra map $\psi_{n-1} : \mathbf{F}[[X_1, \dots, X_{n-1}]] \to \mathbf{F}[[X_1, \dots, X_n]]/J$ given by $h_0 \mapsto h_0+J$ is surjective. Thus we get an $\mathbf{F}$-algebra isomorphism $\mathbf{F}[[X_1, \dots, X_n]]/J \to \mathbf{F}[[X_1, \dots, X_{n-1}]]/J_{n-1}$, where $J_{n-1} = \ker \psi_{n-1}$.

If $\phi_{n-1}(J_{n-1}) \neq 0$, continue this way obtaining a sequence of ideals $J_{n-2}, J_{n-3}, \dots$. If at any stage $(1 \leq r \leq n-2)$ we get $\phi_{n-r}(J_{n-r}) = 0$, then we are done. Otherwise we can eliminate all but one variable and get $\mathbf{F}[[X_1, \dots, X_n]]/J \cong \mathbf{F}[[X_1]]/J_1$ and now we must have $\phi_1(J_1) = 0$ as otherwise J_1 and hence J is maximal. $\square$

The following Proposition does not use Assumption 6.1(ii).

Proposition 6.14. *Assume that* $\dim H^1_f(\mathbf{Q}, \mathrm{ad}\,\overline{\rho}_\phi(k-2)) \leq 1$. *Then the ideal* I_{re} *is a principal ideal.*

Proof. Since ρ^{univ} is a trace representation in the sense of Section 1.3.3 of [BC09] Lemma 1.3.7 in [loc.cit.] tells us that we can conjugate ρ^{univ} by a matrix $P \in \mathrm{GL}_2(R)$ (here we use that every finite type projective R-module is free since R is local) to get ρ^{univ} adapted to a data of GMA idempotents for $R[G_{\{\ell\}}]/\ker \rho^{\mathrm{univ}}$. By [BC09] Lemma 1.3.8 we then get an isomorphism of R-modules

$$R[G_{\{\ell\}}]/\ker \rho^{\mathrm{univ}} \cong \begin{bmatrix} \mathrm{Mat}_2(R) & \mathrm{Mat}_2(B) \\ \mathrm{Mat}_2(C) & \mathrm{Mat}_2(R) \end{bmatrix}$$

for ideals $B, C \subset R$. By [BC09] Proposition 1.5.1 we further know that $I_{\mathrm{re}} = BC$.

[BC09] Theorem 1.5.5 proves that there are injections

$$\mathrm{Hom}_R(B, \mathbf{F}) \hookrightarrow H^1(G_{\{\ell\}}, \mathrm{ad}\,\overline{\rho}_\phi(2-k))$$

and

$$\mathrm{Hom}_R(C, \mathbf{F}) \hookrightarrow H^1(G_{\{\ell\}}, \mathrm{ad}\,\overline{\rho}_\phi(k-2)).$$

Arguing as in [Ake23] Proposition 4.2 (see also [WWE19] Theorem 4.3.5 and Remark 4.3.6) one sees that the images are contained in the Selmer groups $H^1_f(\mathbf{Q}, \mathrm{ad}\,\overline{\rho}_\phi(2-k))$ and $H^1_f(\mathbf{Q}, \mathrm{ad}\,\overline{\rho}_\phi(k-2))$, respectively. From Assumption 6.1 (ii) and Proposition 5.1 we see that $H^1(\mathbf{Q}, \mathrm{ad}\,\overline{\rho}_\phi(2-k)) \cong \mathbf{F}$. Together with the assumption $\dim H^1_f(\mathbf{Q}, \mathrm{ad}\,\overline{\rho}_\phi(k-2)) \leq 1$ we deduce by Nakayama's Lemma that both B and C, and therefore also I_{re} are principal ideals of R. Note that Nakayama's Lemma applies since B and C are ideals in R, which is Noetherian, hence they are finitely generated over R. $\square$

Remark 6.15. Note that there is a natural anti-involution on $R[G_{\{\ell\}}]$ given by $g \mapsto \epsilon^{2k-3}(g)g^{-1}$, however it swaps $\overline{\rho}_\phi$ with $\overline{\rho}_\phi(k-2)$, so the results of section 2 of [BK13] guaranteeing the principality of the reducibility ideal do not apply in this case.

[Ake23] Proposition 3.10 proves the principality of the reducibility ideal of the reduced Fontaine-Laffaille deformation ring R^{red} for any residual representations with two Jordan-Hölder factors. Our argument (whilst relying on [Ake23] Proposition 4.2) is slightly more general as it allows us to treat the case of non-reduced deformation rings.

Remark 6.16. By (5.2) we have

$$H^1_f(\mathbf{Q}, \mathrm{ad}\,\overline{\rho}_\phi(k-2)) = H^1_f(\mathbf{Q}, \mathrm{ad}^0\,\overline{\rho}_\phi(k-2)) \oplus H^1_f(\mathbf{Q}, \mathbf{F}(k-2)).$$

However, as opposed to the case of the $(2-k)$-twist of the trivial representation (cf. proof of Lemma 5.6), there is no simple relation between $H^1_f(\mathbf{Q}, \mathbf{F}(k-2))$ and part of a class group except for the case $k = 2$ by Proposition 4.24. By the same Proposition for $2 < k \leq \ell$ the group $H^1_f(\mathbf{Q}, \mathbf{F}(k-2))$ requires no ramification condition at ℓ, so equals $H^1(G_{\{\ell\}}, \mathbf{F}(k-2))$.

We have the following results about $H^1(G_{\{\ell\}}, \mathbf{F}(n))$ for $n > 0$:

Proposition 6.17 ([BK19] Proposition 6.5). *Suppose $n \in \mathbf{Z}_{>0}$ and $n \not\equiv 1 \mod \ell - 1$. Assume that $\ell \nmid \# \mathrm{Cl}_{\mathbf{Q}(\zeta_\ell)}^{\overline{\epsilon}^n}$. Then $\dim H^1(G_{\{\ell\}}, \mathbf{F}(n)) \leq 1$.*

Proposition 6.18. *Let $n > 0$ be an even integer. Assume $\ell \nmid B_n$ (the n-th Bernoulli number) and $n \not\equiv 0 \mod \ell - 1$. Then $H^1(G_{\{\ell\}}, \mathbf{F}(n)) = 0$.*

Proof. Since n is even and $H^0(G_{\{\ell\}}, \mathbf{F}(n)) = 0$ as $n \not\equiv 0 \mod \ell - 1$ we know $\dim_{\mathbf{F}} H^1(G_{\{\ell\}}, \mathbf{F}(n)) = \dim_{\mathbf{F}} H^2(G_{\{\ell\}}, \mathbf{F}(n))$ by [NSW08] Corollary 8.7.5 (Euler Poincare characteristic). [Ass95] Proposition 1.3 (condition (ii, β)) proves that $H^2(G_{\{\ell\}}, \mathbf{F}(n)) = 0$ if $n \not\equiv 1 \mod \ell - 1$ (which is automatically satisfied for even n) and $\ell \nmid \# \mathrm{Cl}_{\mathbf{Q}(\zeta_\ell)}^{\overline{\epsilon}^{1-n}}$. By Herbrand's Theorem (see e.g. [Was82] Theorem 6.17) the latter follows from our assumption that $\ell \nmid B_n$ (here we use again $n \not\equiv 0 \mod \ell - 1$). □

Remark 6.19. Note that the assumption $\ell \nmid B_n$ is stronger than $\ell \nmid \# \mathrm{Cl}_{\mathbf{Q}(\zeta_\ell)}^{\overline{\epsilon}^n}$ in [BK19] Proposition 6.5. As noted in the proof of Proposition 6.18 $\ell \nmid B_n$ implies $\ell \nmid \# \mathrm{Cl}_{\mathbf{Q}(\zeta_\ell)}^{\overline{\epsilon}^{\ell - n}}$ by Herbrand's Theorem. By the "reflection theorem" [Was82] Theorem 10.9 this means that also $\ell \nmid \mathrm{Cl}_{\mathbf{Q}(\zeta_\ell)}^{\overline{\epsilon}^n}$.

This allows us to prove the following modularity theorem.

Theorem 6.20. *Recall that we impose Assumptions 3.1 and 6.1. Furthermore, assume that $\dim H^1_f(\mathbf{Q}, \mathrm{ad}\overline{\rho}_\phi(k-2)) \leq 1$. Then the structure map $\iota : \mathcal{O} \to R$ is an isomorphism. In particular, if $\tau : G_{\mathbf{Q}} \to \mathrm{GL}_4(E)$ is any continuous irreducible homomorphism unramified outside ℓ, crystalline at ℓ with Hodge-Tate weights in $[3 - 2k, 2k - 3]$ and such that*

$$\overline{\tau}^{\mathrm{ss}} = \overline{\rho}_\phi \oplus \overline{\rho}_\phi(k-2),$$

then $\tau \cong \rho^{\mathrm{univ}} \cong \rho_f$, i.e., in particular τ is modular.

Proof. It follows from Corollary 6.12 that I_{re} is a maximal ideal of R. As the deformation ρ_f induces a surjective map $j : R \to \mathcal{O}$, we get the following commutative diagram of $\mathcal{O}$-algebra maps

$$\begin{tikzcd} \mathcal{O} \arrow[r, "\iota"'] \arrow[d] \arrow[rr, bend left, "\mathrm{id}"] & R \arrow[r, "j"'] \arrow[d] & \mathcal{O} \arrow[d] \\ \mathcal{O}/\lambda \arrow[r, "\overline{\iota}"] \arrow[rr, bend right, "\mathrm{id}"'] & R/I_{\mathrm{re}} \arrow[r, "\overline{j}"] & \mathcal{O}/\lambda \end{tikzcd} \tag{6.4}$$

As $\overline{\iota}$ is an isomorphism, we get that so is $\overline{j}$. So, using the fact that I_{re} is principal (Proposition 6.14), we can now apply Theorem 6.9 in [BK11] to the right square to conclude that j is an isomorphism.

Now, let τ be as in the statement of the Theorem. Then τ factors through a representation of $G_{\{\ell\}}$. Using that τ is irreducible, Theorem 4.1 in [BK20] allows us to find a lattice in the space of τ such that with respect to that lattice one has

$$\overline{\tau} = \begin{bmatrix} \overline{\rho}_\phi & * \\ & \overline{\rho}_\phi(k-2) \end{bmatrix}$$

that is non-semi-simple. Using Remark 6.4 we see that this lattice is Fontaine-Laffaille, so the star gives rise to a non-zero element in $H^1_f(\mathbf{Q}, \mathrm{ad}^0\rho_\phi(2-k) \otimes_{\mathcal{O}} E/\mathcal{O})$. As the latter group has order $\#\mathcal{O}/\lambda$ by Assumption 6.1(ii), we conclude that $\overline{\tau} \cong \rho$. In particular, τ is a deformation of ρ. Hence τ gives rise to an $\mathcal{O}$-algebra map $R \to \mathcal{O}$, which must equal j by the first part of the theorem. □

Remark 6.21. We note that unlike in many minimal deformation problems, the definition of R does not require that the deformations have a fixed determinant, but rather this is a consequence of Theorem 6.20.

Remark 6.22. We return to Example 3.6 and note that Assumption 6.1 (i) holds, as discussed earlier. Since $\ell = 163$ or 187273 do not divide $(2k-1)(2k-3)k!$ for $k = 26$ and $\overline{\rho}_\phi$ is irreducible, [DFG04] Lemma 2.5 proves that $\overline{\rho}_\phi$ stays irreducible when restricted to $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}(\sqrt{(-1)^{(\ell-1)/2}\ell})$. Via Remark 6.7 we can therefore check that $H^1_f(\mathbf{Q}, \mathrm{ad}^0\,\overline{\rho}_\phi) = 0$ as ϕ is the only cuspform of weight 26 and level 1, so in particular, ϕ is not congruent mod ℓ to other forms. Since in addition $L_{\mathrm{alg}}(50, \mathrm{Sym}^2\phi)$ has ℓ-valuation 1 for both $\ell = 163$ and 187273 the Bloch-Kato conjecture for $\# H^1_f(\mathbf{Q}, \mathrm{ad}^0\rho_\phi(2-k) \otimes E/\mathcal{O}) = \#\mathcal{O}/\lambda$ (see [Dum09] Conjecture (5.2) and (5)) would imply that Assumption (ii) holds.

We do not know how to check $\dim H^1_f(\mathbf{Q}, \mathrm{ad}\overline{\rho}_\phi(k-2)) \leq 1$, as the corresponding divisible Selmer group is not critical (in the sense of Deligne). Note that $\dim H^1_f(\mathbf{Q}, \mathrm{ad}\overline{\rho}_\phi(k-2)) = \dim H^1_f(\mathbf{Q}, \mathrm{ad}^0\overline{\rho}_\phi(k-2))$ by Proposition 6.18 since neither prime ℓ divides B_{24}.

7. (Non-)principality of Eisenstein ideals

In this section we will formulate conditions under which the Eisenstein ideal of the local Hecke algebra acting on $S_k(\Gamma_2)$ is non-principal and

$$\dim_{\mathbf{F}} H^1_f(\mathbf{Q}, \mathrm{ad}^0\,\overline{\rho}_\phi(k-2)) > 1.$$

In particular, in that case $R \not\cong \mathcal{O}$.

Recall in section 2 we defined $\mathbf{T}'$ to be the $\mathbf{Z}$-subalgebra of $\mathrm{End}_{\mathbf{C}}(S_k(\Gamma_2))$ generated by the Hecke operators $T^{(2)}(p)$ and $T^{(2)}_1(p^2)$ for all primes p. Let $\mathbf{T}$ denote the $\mathcal{O}$-subalgebra of $\mathbf{T}' \otimes_{\mathbf{Z}} \mathcal{O}$ generated by the operators $T^{(2)}(p)$ and $T^{(2)}_1(p^2)$ for all primes $p \nmid \ell$. Since strong multiplicity one holds in the level one case, we can choose an orthogonal basis $\mathcal{N}'$ of $S_k(\Gamma_2)$ consisting of eigenforms for all the operators in $\mathbf{T}$.

Each $g \in \mathcal{N}'$ gives rise to an element ψ_g of the finite set $\mathrm{Hom}_{\mathcal{O}-\mathrm{alg}}(\mathbf{T}, \mathcal{O})$ where $\psi_g(T) = \lambda_g(T)$, with $\lambda_g(T)$ the eigenvalue of the operator T corresponding to g. Thus we get a map $\Psi : \mathcal{N}' \to \mathrm{Hom}_{\mathcal{O}-\mathrm{alg}}(\mathbf{T}, \mathcal{O})$ given by $g \mapsto \lambda_g$, which by strong multiplicity one (which holds for level Γ_2) is an injection.

Lemma 7.1. *The natural $\mathcal{O}$-algebra map*

$$\mathbf{T} \to \prod_{g \in \mathcal{N}'} \mathcal{O} \quad \text{given by} \quad T \mapsto (\psi_g(T))_g \tag{7.1}$$

is injective and has finite cokernel, i.e. $\mathbf{T}$ can be viewed as a lattice in $\prod_{g\in\mathcal{N}'} \mathcal{O}$.

Proof. Any $t \in \mathbf{T}$ in the kernel of this map kills every $g \in \mathcal{N}'$. As the elements of $\mathcal{N}'$ form a basis of $S_k(\Gamma_2)$, the operator t is the zero endomorphism. This proves injectivity.

We will now show that the map has finite cokernel. Note that the (set) map $\Psi \otimes \overline{\mathbf{Q}}_\ell : \mathcal{N}' \to \mathrm{Hom}_{\overline{\mathbf{Q}}_\ell-\mathrm{alg}}(\mathbf{T} \otimes \overline{\mathbf{Q}}_\ell, \overline{\mathbf{Q}}_\ell) \hookrightarrow \mathrm{Hom}_{\overline{\mathbf{Q}}_\ell}(\mathbf{T} \otimes \overline{\mathbf{Q}}_\ell, \overline{\mathbf{Q}}_\ell)$ given by $g \mapsto \lambda_g \otimes \overline{\mathbf{Q}}_\ell$ is injective (because Ψ is injective). Suppose there is a linear relation $\sum_{g\in\mathcal{N}'} c_g \lambda_g = 0$. Consider the form $g_0 = \sum_{g\in\mathcal{N}'} c_g g \in S_k(\Gamma_2)$. It is clear that g_0 is an eigenform for all the operators in $\mathbf{T} \otimes \overline{\mathbf{Q}}_\ell$ with all eigenvalues zero. By strong multiplicity one we conclude that $g_0 = 0$. As the elements of $\mathcal{N}'$ (being an orthogonal basis of $S_k(\Gamma_2)$) form a linearly independent set, we get that $c_g = 0$ for all $g \in \mathcal{N}'$. Thus the set $\{\lambda_g \mid g \in \mathcal{N}'\}$ is a linearly independent subset of $\mathrm{Hom}_{\overline{\mathbf{Q}}_\ell}(\mathbf{T} \otimes \overline{\mathbf{Q}}_\ell, \overline{\mathbf{Q}}_\ell)$. Hence

$$\dim_{\overline{\mathbf{Q}}_\ell} \mathbf{T} \otimes \overline{\mathbf{Q}}_\ell = \dim_{\overline{\mathbf{Q}}_\ell} \mathrm{Hom}_{\overline{\mathbf{Q}}_\ell}(\mathbf{T} \otimes \overline{\mathbf{Q}}_\ell, \overline{\mathbf{Q}}_\ell) \geq \#\mathcal{N}'. \tag{7.2}$$

Tensoring the map (7.1) with $\overline{\mathbf{Q}}_\ell$ we get a corresponding map $\mathbf{T} \otimes \overline{\mathbf{Q}}_\ell \to \prod_{g\in\mathcal{N}'} \overline{\mathbf{Q}}_\ell$, which is injective because (7.1) is. Thus it must also be surjective by (7.2). Hence the map (7.1) has finite cokernel. $\square$

We now identify $\mathbf{T}$ with the image of the map (7.1). We note that $\mathbf{T}$ is a semi-local, complete, reduced $\mathcal{O}$-algebra and one has the following decomposition

$$\mathbf{T} = \prod_{\mathfrak{m} \in \mathrm{MaxSpec}\mathbf{T}} \mathbf{T}_{\mathfrak{m}},$$

where $\mathbf{T}_{\mathfrak{m}}$ is the localization of $\mathbf{T}$ at the maximal ideal $\mathfrak{m}$. Let $\mathcal{N}$ be the subset of $\mathcal{N}'$ consisting of all the $g \in \mathcal{N}'$ which satisfy

$$\psi_g(T) \equiv \lambda_{E_\phi^{1,2}}(T) \pmod{\lambda} \quad \text{for all } T \in \mathbf{T}.$$

Then there is a maximal ideal $\mathfrak{m} \in \mathrm{MaxSpec}\mathbf{T}$ such that the map $\mathbf{T} \to \prod_{g\in\mathcal{N}'} \mathcal{O} \to \prod_{g\in\mathcal{N}} \mathcal{O}$ factors through $\mathbf{T}_{\mathfrak{m}}$. We fix this $\mathfrak{m}$ from now on.

Set $J \subset \mathbf{T}$ to be the Eisenstein ideal, i.e., J is the ideal of $\mathbf{T}$ generated by the set

$$\{T^{(2)}(p) - (\mathrm{tr}\,\rho_\phi(\mathrm{Frob}_p) + \mathrm{tr}\,\rho_\phi(k-2)(\mathrm{Frob}_p)) \mid p \neq \ell\}.$$

Write $J_{\mathfrak{m}}$ to be the image of J under the canonical map $\mathbf{T} \to \mathbf{T}_{\mathfrak{m}}$.

Recall that we fixed in Section 5.2 the weight $k > 4$ even and prime $\ell > 4k-5$ and imposed Assumption 3.1 on the field $E/\mathbf{Q}_\ell$. We also fixed the Fontaine-Laffaille interval $I = [3-2k, 2k-3]$. Let $\phi \in S_k(\Gamma_1)$ be a newform such that $\overline{\rho}_\phi$ is irreducible.

For the rest of this section we also impose Assumption 6.1 and fix the corresponding $f \in S_k(\Gamma_2)$. Then $f \in \mathcal{N}$, i.e., $\mathbf{T}_{\mathfrak{m}}/J_{\mathfrak{m}} \neq 0$. Let $R = R_{\overline{\rho}_f}$ be the universal deformation ring defined in Section 6.

Theorem 7.2. *Recall that we impose Assumptions 3.1 and 6.1. Then there exists a surjective $\mathcal{O}$-algebra map $\varphi : R \to \mathbf{T}_{\mathfrak{m}}$ such that $\varphi(I_{\mathrm{re}}) = J_{\mathfrak{m}}$ and $J_{\mathfrak{m}}$ is a maximal ideal of $\mathbf{T}_{\mathfrak{m}}$. If, in addition $\dim_{\mathbf{F}} H^1_f(\mathbf{Q}, \mathrm{ad}\overline{\rho}_\phi(k-2)) \leq 1$, then all of the following are true:*

- *the map φ is an isomorphism;*
- *the Hecke ring $\mathbf{T}_{\mathfrak{m}}$ is isomorphic to $\mathcal{O}$;*
- *the Eisenstein ideal $J_{\mathfrak{m}}$ is principal.*

Proof. Let $g \in \mathcal{N}$. Then by Lemma 5.2 there exists a $G_{\mathbf{Q}}$-stable lattice with respect to which one has $\overline{\rho}_g = \begin{bmatrix} \overline{\rho}_\phi & * \\ & \overline{\rho}_\phi(k-2) \end{bmatrix}$ and is not semi-simple. Hence the $*$ gives rise to an element in $H^1_f(\mathbf{Q}, W[\lambda])$, where $W = \mathrm{ad}^0\rho_\phi(2-k) \otimes_{\mathcal{O}} E/\mathcal{O}$.

By (5.4) and Proposition 5.1 we get $H^1_f(\mathbf{Q}, W[\lambda]) = H^1_f(\mathbf{Q}, W)[\lambda]$. The latter group is cyclic by Assumption 6.1 (ii), so we must have that $\overline{\rho}_g \cong \overline{\rho}_f$, and so after adjusting the basis if necessary we get that ρ_g is a deformation of $\overline{\rho}_f$.

This implies that for every $g \in \mathcal{N}$ we get an $\mathcal{O}$-algebra (hence continuous) map $\varphi_g : R \to \mathcal{O}$ with the property that $\mathrm{tr}\,\rho^{\mathrm{univ}}(\mathrm{Frob}_p) \mapsto \lambda_g(T^{(2)}(p))$. This property completely determines φ_g because R is topologically generated by the set $\{\mathrm{tr}\,\rho^{\mathrm{univ}}(\mathrm{Frob}_p) \mid p \neq \ell\}$ by Proposition 7.13 in [BK13]. Putting these maps together we get an $\mathcal{O}$-algebra map $\varphi : R \to \prod_{g\in\mathcal{N}} \mathcal{O}$ whose image is an $\mathcal{O}$-subalgebra of $\prod_{g\in\mathcal{N}} \mathcal{O}$ generated by $\{T^{(2)}(p) \mid p \neq \ell\}$. We now claim that this subalgebra equals $\mathbf{T}_{\mathfrak{m}}$. Indeed, one clearly has $\varphi(R) \subset \mathbf{T}_{\mathfrak{m}}$. To see the opposite inclusion consider the characteristic polynomial $f_p(X) \in R[X]$ of $\rho^{\mathrm{univ}}(\mathrm{Frob}_p)$ for $p \neq \ell$. Combining Theorem 2.1 with the definition of $L_p(X, f; \mathrm{spin})$ we see that the coefficient at X^2 is mapped by φ to $T^{(2)}(p)^2 - T_1^{(2)}(p^2) - p^{2k-4} \in \prod_{g\in\mathcal{N}} \mathcal{O}$. As $T^{(2)}(p)$ and p^{2k-4} both belong to $\varphi(R)$, so therefore must $T_1^{(2)}(p^2)$. Hence $\varphi(R)$ contains all the Hecke operators away from ℓ, i.e., $\varphi(R) = \mathbf{T}_{\mathfrak{m}}$. We denote the resulting $\mathcal{O}$-algebra epimorphism $R \to \mathbf{T}_{\mathfrak{m}}$ again by φ. We claim that $\varphi(I_{\mathrm{re}}) \subset J_{\mathfrak{m}}$.

Indeed, as $\varphi(\mathrm{tr}\,\rho^{\mathrm{univ}}(\mathrm{Frob}_p)) = T^{(2)}(p)$ and

$$T^{(2)}(p) - (\mathrm{tr}\,\rho_\phi(\mathrm{Frob}_p) + \mathrm{tr}\,\rho_\phi(k-2)(\mathrm{Frob}_p)) \in J_{\mathfrak{m}}$$

for all primes $p \neq \ell$, one has

$$\operatorname{tr}\rho^{\mathrm{univ}}(\mathrm{Frob}_p) - (\operatorname{tr}\rho_\phi(\mathrm{Frob}_p) + \operatorname{tr}\rho_\phi(k-2)(\mathrm{Frob}_p)) \in \varphi^{-1}(J_{\mathfrak{m}}).$$

By the Chebotarev Density Theorem, this implies that

$$\operatorname{tr}\rho^{\mathrm{univ}} \equiv \operatorname{tr}\rho_\phi + \operatorname{tr}\rho_\phi(k-2) \pmod{\varphi^{-1}(J_{\mathfrak{m}})},$$

so $I_{\mathrm{re}} \subset \varphi^{-1}(J_{\mathfrak{m}})$. As φ is a surjection, this implies that $\varphi(I_{\mathrm{re}}) \subset J_{\mathfrak{m}}$. Hence φ gives rise to a sequence of $\mathcal{O}$-algebra surjections $R/I_{\mathrm{re}} \to \mathbf{T}_{\mathfrak{m}}/\varphi(I_{\mathrm{re}}) \to \mathbf{T}_{\mathfrak{m}}/J_{\mathfrak{m}}$. As $R/I_{\mathrm{re}} = \mathbf{F}$ by Corollary 6.12 we conclude that all these surjections are isomorphisms (note that $\mathbf{T}_{\mathfrak{m}}/J_{\mathfrak{m}} \neq 0$), hence $\varphi(I_{\mathrm{re}}) = J_{\mathfrak{m}}$ and $J_{\mathfrak{m}}$ is maximal. This proves the first claim.

Now assume in addition that $\dim H^1_f(\mathbf{Q}, \mathrm{ad}\overline{\rho}_\phi(k-2)) \leq 1$. Then Theorem 6.20 gives us that $R = \mathcal{O}$, so we get that φ is an isomorphism, and so $R \cong \mathbf{T}_{\mathfrak{m}} \cong \mathcal{O}$. In particular, $J_{\mathfrak{m}}$ is a principal ideal. □

Corollary 7.3. *If $J_{\mathfrak{m}}$ is not principal, then*

$$\dim_{\mathbf{F}} H^1_f(\mathbf{Q}, \mathrm{ad}\,\overline{\rho}_\phi(k-2)) > 1.$$

If in addition $\ell \nmid B_{k-2}$ then

$$\dim_{\mathbf{F}} H^1_f(\mathbf{Q}, \mathrm{ad}^0\,\overline{\rho}_\phi(k-2)) > 1.$$

Proof. The first inequality is just a restatement of one of the claims of Theorem 7.2. The second follows from the first one and Proposition 6.18. □

The latter can potentially be done by counting the depth of Eisenstein congruences as the following result shows.

Proposition 7.4. *For each $g \in \mathcal{N}$ write m_g for the largest positive integer m such that $g \equiv E^{\phi}_{2,1}$ mod λ^m. If*

$$\mathrm{val}_\ell(\#\mathbf{T}_{\mathfrak{m}}/J_{\mathfrak{m}}) < [\mathbf{F} : \mathbf{F}_\ell] \cdot \sum_{g \in \mathcal{N}} m_g \tag{7.3}$$

then $J_{\mathfrak{m}}$ is not principal.

Proof. Set $A = \prod_{g \in \mathcal{N}} A_g$, where $A_g = \mathcal{O}$ for all $g \in \mathcal{N}$. Let $\phi_g : A \to A_g$ be the canonical projection. Since by Lemma 7.1 $\mathbf{T}$ is a full rank $\mathcal{O}$-submodule of $\prod_{g \in \mathcal{N}'} \mathcal{O}$ we conclude that the local complete $\mathcal{O}$-subalgebra $\mathbf{T}_{\mathfrak{m}} \subset A$ is of full rank as an $\mathcal{O}$-submodule and $J_{\mathfrak{m}} \subset \mathbf{T}_{\mathfrak{m}}$ is an ideal of finite index. Set $T_g = \phi_g(\mathbf{T}_{\mathfrak{m}}) = A_g = \mathcal{O}$ and $J_g = \phi_g(J_{\mathfrak{m}}) = \lambda^{m_g}\mathcal{O}$. Hence we are in the setup of section 2 of [BKK14]. Assume $J_{\mathfrak{m}}$ is principal. Then Proposition 2.3 in [BKK14] gives us that

$$\#\mathbf{T}_{\mathfrak{m}}/J_{\mathfrak{m}} = \prod_{g \in \mathcal{N}} \#T_g/J_g. \tag{7.4}$$

Note that one has

$$\mathrm{val}_\ell\left(\prod_{g \in \mathcal{N}} \#T_g/J_g\right) = [\mathbf{F} : \mathbf{F}_\ell] \cdot \sum_{g \in \mathcal{N}} m_g. \tag{7.5}$$

This equality together with (7.4) contradicts the inequality (7.3). □

Corollary 7.5. *Let m_g be defined as in Proposition 7.4. If*

$$\sum_{g \in \mathcal{N}} m_g > 1 \tag{7.6}$$

then $J_{\mathfrak{m}}$ is not principal and

$$\dim_{\mathbf{F}} H^1_f(\mathbf{Q}, \mathrm{ad}\,\overline{\rho}_\phi(k-2)) > 1.$$

If in addition $\ell \nmid B_{k-2}$ *then*

$$\dim_{\mathbf{F}} H^1_f(\mathbf{Q}, \mathrm{ad}^0\,\overline{\rho}_\phi(k-2)) > 1.$$

Proof. Note that from the proof of Theorem 7.2 we get that $\mathbf{T}_{\mathfrak{m}}/J_{\mathfrak{m}} = \mathbf{F}$, even without assuming $\dim_{\mathbf{F}} H^1_f(\mathbf{Q}, \mathrm{ad}\,\overline{\rho}_\phi(k-2)) \leq 1$. Assume that $J_{\mathfrak{m}}$ is principal. Then from (7.4) and (7.5) we conclude that $\sum_{g\in\mathcal{N}} m_g = 1$, which contradicts assumption (7.6). Hence $J_{\mathfrak{m}}$ is not principal.

The Selmer group inequalities now follow from Corollary 7.3. □

Remark 7.6. Corollary 7.3 directly ties the cyclicity of the Selmer group $H^1_f(\mathbf{Q}, \mathrm{ad}\,\overline{\rho}_\phi(k-2))$ with the principality of the Eisenstein ideal $J_{\mathfrak{m}}$. We note that Assumption 6.1(ii) implies the equality $\mathbf{T}_{\mathfrak{m}}/J_{\mathfrak{m}} = \mathbf{F}$. Contrary to what one might think, the existence of several forms $g \equiv E^{\phi}_{2,1} \bmod \lambda$ does not preclude this equality. For example, if there are exactly two linearly independent eigenforms $g_1, g_2 \in \mathcal{N}$ with $m_{g_1} = m_{g_2} = 1$ such that $g_1 \not\equiv g_2 \mod \lambda^2$ then $\mathbf{T}_{\mathfrak{m}} \cong \mathcal{O} \times_{\mathbf{F}} \mathcal{O} = \{(a,b) \in \mathcal{O}\times\mathcal{O} \mid a \equiv b \mod \lambda\}$ and in this case $J_{\mathfrak{m}}$ is the maximal ideal, i.e. $\mathbf{T}_{\mathfrak{m}}/J_{\mathfrak{m}} = \mathbf{F}$, so Corollary 7.5 applies and $\dim_{\mathbf{F}} H^1_f(\mathbf{Q}, \mathrm{ad}\,\overline{\rho}_\phi(k-2)) > 1$.

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