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Structured Observer Synthesis via Static Output Feedback

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Abstract— **This paper considers the problem of structured observer synthesis for linear-time-invariant (LTI) systems. The importance of the problem is demonstrated through two motivational examples: utilizing high-fidelity black-box models as part of observers, and anti-windup gain design. The problem is connected with its control counterpart: static output feedback, and the process is illustrated through the design of an anti-windup gain for a multi-variable PID controller. A numerical example shows how the resulting closed-loop system converges faster than a recent method in the literature, with the added benefit of solving the problem with a non-iterative method, which decreases the computational effort.**

Keywords— **Structured observer, anti-windup, static output feedback, multi-variable control.**

I. Introduction

State-feedback controllers rely on knowledge of the system states, however, these are rarely measured, at least not completely. An observer aims at estimating these unmeasured states by combining knowledge about the system dynamics with the available measurements [1].

In addition, one may think of other inaccessible system variables that might be of importance depending on the designer needs. For instance, those cases where some inputs are not measured, e.g. unknown disturbances, or variation in system parameters, such as the drift caused by deterioration. For these, the unknown input observer (UIO) offers an estimate of said inputs [2]. Other requirements may be motivated by the need of having accessible certain unmeasurable outputs, hence, an output observer may be required [3].

All these classical approaches assume that the predicted system state will be corrected *fully*, i.e. that the error between the predicted outputs and the respective measurements will be used to correct all the states simultaneously. Therefore, if there are restrictions in *how* the corrections are performed, alternative synthesis procedures become necessary. These restrictions might be, for example, that only a subset of the predicted states are to be corrected, or that different subsets of measurement are used to correct each state.

These restrictions are motivated in Section II via two practical examples: an UIO using black-box models, and design of anti-windup gains.

As we show in Section III-B, the problem of the structured observer design for linear-time-invariant (LTI) systems can be posed as its counterpart in control design: the static output feedback (SOF) synthesis, which allows taking advantage of recent results from [4].

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A. Motivating examples

The UIO using black-box models aims at utilizing the best available representation of a system as part of the observer. However, this type of model may have its dynamics inaccessible to us, and hence, the corrections introduced by an observer will not be able to alter *directly* the state transition equation. Instead, this will be achieved *indirectly* through some auxiliary input, i.e. the correction is restricted in the way it is introduced to the state. This restriction is imposed by the architecture of the observer.

In the case of multi-variable anti-windup design, most modern literature [5, 6, 7] poses the design problem as a controller synthesis with a sector non-linearity. A perhaps simpler approach is the observer-based synthesis [8, 9], which regards the unsaturated closed-loop system as an imperfect estimator of the *true* system, i.e. the system with saturated inputs. The restriction in this case is imposed by the problem, as only the controller state can be directly corrected.

B. Notation

Throughout the paper, the standard mathematical notation is used, with lowercase array variables, e.g. x, uppercase matrix variables, e.g. A. The symbols $\mathbb{R}$ and $\mathbb{S}$ are the sets of real and symmetric numbers respectively. The dimensions of the sets are specified as exponents, and any restrictions as subscripts, e.g. $\mathbb{S}^{n_x}_{++}$ ($\mathbb{S}^{n_x}_{+}$) is the set of positive definite (positive semi-definite) symmetric matrices of order n_x.

The sum of a square matrix with its transpose is represented by $\{A\}^{\mathrm{S}} = A + A^T$. The symbol $\star$ in the upper-diagonal blocks is used to indicate their equivalence to the transpose of the corresponding lower-diagonal blocks in a symmetric matrix.

$$\begin{bmatrix} A & \star \\ B & C \end{bmatrix} \equiv \begin{bmatrix} A & B^T \\ B & C \end{bmatrix}. \tag{1}$$

The definiteness and semi-definiteness of a matrix are represented by the mathematical symbols $\prec$ and $\preceq$. The symbol I represents the identity matrix of appropriate dimensions. Finally, $\mathbb{E}\left[\cdot\right]$ performs the expected value operation.

C. Aim and contribution of this paper

In this paper we consider the structured observer synthesis, firstly by acknowledging its relevance in realistic engineering applications, and secondly by framing this problem as the estimator equivalent to static output feedback (SOF) synthesis, i.e. the design of a gain constrained in both its row and column spaces.

This paper extends the non-iterative result of [4] to observer design, and establishes a common framework to the synthesis of structured observers. The process is illustrated by applying the technique to an anti-windup strategy, showing how when a PID is tuned, the backstepping gain can be readily determined as a function of the system dynamics, and the controller gains.

II. Problem statement and motivating examples

This section contextualizes the problem addressed in this paper by first discussing two motivating examples. The problem is then formalized in a common framework.

A. Unknown input observer design using black-box models

High-fidelity models representing complex engineering and/or social systems present an accurate means to better understand these systems, and to obtain unmeasured variables that can be used for monitoring or control purposes.

It is therefore desirable to use these models within observer frameworks, so the accuracy of the estimations is significantly improved. However, these models are often built as black-box software applications, which makes it difficult to use the standard observer design paradigm.

Observers are usually designed making the assumption that the state transition equation is available, and can be corrected, e.g. the standard, or extended, Kalman filter modifies the state of the model during the correction step [10]. This is somewhat incompatible with the nature of black-box models, where the internal state transition cannot be directly modified, as only system inputs and outputs are available. Hence, to modify the dynamics when a black-box model is to be used as part of the observer, the designer is left only with the possibility of changing the inputs to the system.

If all inputs are known, the designer might choose to introduce small variations to the existing inputs, and if some of the system inputs are unknown, the designer might choose to use these unknown inputs, or a subset of them, adapting the model to better resemble the real system.

All these different alternatives can be regarded as unknown input observer (UIO) problems, with structure constraints, i.e. with restrictions on how the system state is corrected. Fig. 1 shows an example of an UIO using a black-box model, where the signal $d(t)$ needs to be estimated in order to adapt the black-box model.

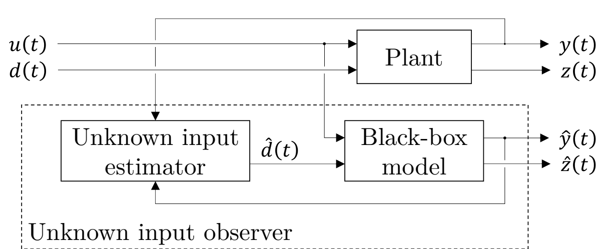

Fig. 1. Unknown-input observer using a black-box model.

In the UIO problem, the dynamics of the unknown input can be regarded as a random walk using a Gaussian variable [11], effectively defining the random nature of the changes in this input from the system's perspective. A linear version of this system can be represented in state space as in (2) using augmented state formulation, where the time dependency of the variables is omitted for simplicity of notation.

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \dot{d} \end{bmatrix} = \begin{bmatrix} A & G \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} w_x \\ w_d \end{bmatrix}, \\ y \quad = \begin{bmatrix} C & H \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix} + D\,u + v\,, \end{cases} \tag{2}$$

where $x \in \mathbb{R}^{n_x}$ is the system state, $u \in \mathbb{R}^{n_u}$ is the measured input, $d \in \mathbb{R}^{n_d}$ is the unknown input, $y \in \mathbb{R}^{n_y}$ is the measurement, the matrices A, B, C, D, G, and H are of appropriate dimensions, and the stochastic variables $w_x \in \mathbb{R}^{n_x}$, $w_d \in \mathbb{R}^{n_d}$ and $v \in \mathbb{R}^{n_y}$ are the sources of uncertainty in the state, the unknown input, and the measurement respectively.

The observer from Fig. 1 can then be formulated as per (3). where it can be seen that the correction enters the augmented state only through the unknown input d, as the dynamic equation is not available in the said architecture.

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{d}} \end{bmatrix} = \begin{bmatrix} A & G \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{d} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ I \end{bmatrix} K\,\tilde{y}\,, \tag{3}$$

where $\tilde{y} = (y - \hat{y})$ is the measurement error, and K is the observer gain of appropriate dimensions.

B. Observer-based anti-windup design

A more general setting would be that where the corrections are introduced to the state through a desired structure, i.e. the correction does not modify the state transition equation freely, but through a (non-invertible) matrix.

As an illustrative example, we will consider here the design of a backstepping gain for the anti-windup of a proportional-integral-derivative (PID) controller, following the observer-based interpretation of anti-windup techniques [9]. The procedure followed here is applicable to both single- and multi-variable controllers.

The relevance of the PID, being the most used control paradigm in industry [12], is well recognized. The availability of a tool to systematically design anti-windup gains as part of the tuning process of a PID is therefore pertinent and timely, considering the latest developments on static output feedback (SOF) synthesis.

The main challenge when designing anti-windup backstepping gains arises when trying to find a feasible solution to the bilinear matrix inequality (BMI) reached during the synthesis process. This BMI is non-linearizable due to the restriction in how the corrections enter the closed-loop system, since, like with the UIO from Section II-A, only part of the state can be modified, i.e. the corrections in an anti-windup scheme can be made to the controller integrator, its input, or its output, however, none of these corrections will reach the system when the controller output is saturated.

Fig. 2 shows a common PID scheme, where the backstepping correction is introduced through the error signal [5].

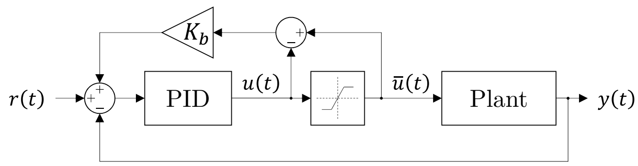

Fig. 2. Backstepping anti-windup scheme for a PID regulator.

The closed-loop LTI system from Fig 2 can be formulated in state-space form as in (4), where the correction enters the state through the input of the controller. This, like in (3), illustrates a case with restrictions to the state correction.

$$\begin{bmatrix} \dot{x}_{(t)} \\ \dot{x}_{c_{(t)}} \end{bmatrix} = A_{cl} \begin{bmatrix} x_{(t)} \\ x_{c_{(t)}} \end{bmatrix} + \begin{bmatrix} 0 \\ G_c \end{bmatrix} K_b \, \tilde{u}_{(t)} \,, \quad (4)$$

where $x \in \mathbb{R}^{n_x}$ and $x_c \in \mathbb{R}^{n_c}$ are the system and controller states respectively, $\tilde{u} = (\bar{u} - u)$ is the difference between the system input and the controller output, A_{cl} is the closed-loop transition matrix, G_c is the controller state correction matrix, and K_b is the backstepping gain.

This type of setup can be easily generalized to any type of restriction in the correction of the observer, as it will be clear in Section II-C, and it is chosen to illustrate the process of designing the structured observer, as it is a well-understood control paradigm by the broader engineering community.

C. The structured observer design problem

Let us consider the following LTI continuous-time system:

$$\begin{cases} \dot{x}_{(t)} = A\, x_{(t)} + B\, u_{(t)} + G\, \omega_{(t)} \;, \\ y_{(t)} = C\, x_{(t)} + D\, u_{(t)} + H\, \omega_{(t)} \;, \end{cases} \quad (5)$$

where $x_{(t)} \in \mathbb{R}^{n_x}$, $u_{(t)} \in \mathbb{R}^{n_u}$, and $y_{(t)} \in \mathbb{R}^{n_y}$ represent respectively the state, input, and measurement of the system, $\omega^T_{(t)} = [\, w^T_{(t)} \; v^T_{(t)} \,]$, with $w_{(t)} \in \mathbb{R}^{n_w}$ being the system uncertainty, and $v_{(t)} \in \mathbb{R}^{n_v}$ being the measurement noise. Matrices A, B, C, D, G, and H are of appropriate dimensions.

The uncertainty $w_{(t)}$ and the measurement noise $v_{(t)}$ are regarded as Gaussian variables with zero mean and covariances given by (6).

$$\mathbb{E}\left[\begin{bmatrix} w_{(t)} \\ v_{(t)} \end{bmatrix} \begin{bmatrix} w_{(\tau)} \\ v_{(\tau)} \end{bmatrix}^T \right] = \begin{bmatrix} Q & \star \\ S & R \end{bmatrix} \delta_{(t-\tau)} \;, \quad (6)$$

where $Q \in \mathbb{S}^{n_w}$ and $R \in \mathbb{S}^{n_v}$ are the covariance matrices of the uncertainty and the measurement noise respectively, $S \in \mathbb{R}^{n_v \times n_w}$ is the cross-covariance, and $\delta_{(\cdot)}$ is the Dirac delta function. Finally, the covariance of $\omega_{(t)}$ is defined in (7), where Q_ω is defined by blocks in (6).

$$\mathbb{E}\left[\omega_{(t)}\, \omega^T_{(\tau)} \right] = Q_\omega \, \delta_{(t-\tau)} \;. \quad (7)$$

The structured observer design problem consists in designing a linear observer gain $K \in \mathbb{R}^{n_q \times n_y}$, such that the estimated state $\hat{x}_{(t)}$ in (8) converges to a value that minimizes the covariance of the estimation error defined by $\tilde{x}_{(t)} = x_{(t)} - \hat{x}_{(t)}$, i.e. K minimizes $P_{(t)} = \mathbb{E}\left[\tilde{x}_{(t)} \tilde{x}^T_{(t)} \right]$.

$$\begin{cases} \dot{\hat{x}}_{(t)} = A\, \hat{x}_{(t)} + B\, u_{(t)} + J\, K \left(y_{(t)} - \hat{y}_{(t)} \right) \;, \\ \hat{y}_{(t)} = C\; \hat{x}_{(t)} \;, \end{cases} \quad (8)$$

where $J \in \mathbb{R}^{n_x \times n_q}$ is the correction matrix. The dynamics of the error between the system and the observer are:

$$\dot{\tilde{x}}_{(t)} = A\, \tilde{x}_{(t)} + G\, \omega_{(t)} - J\, K \left(C\, \tilde{x}_{(t)} + H\, \omega_{(t)} \right) \,. \quad (9)$$

In contrast to the traditional Luenberger [1] or Kalman filter [10] observers, the uncertainty matrix G from (5) can now be included in the estimator by setting $J = G$ in (8), in order to take advantage of our knowledge about how the uncertainty enters the system. Furthermore, in an even more general case, the correction matrix in the observer (8) might even differ from the one in the system (5), i.e. $J \neq G$, or be non-trivial, i.e. $J \neq I$.

From this definition, it follows that the UIO observer problem presented in II-A is a particular case of the structured observer from (8) using an augmented system representation, and a correction matrix $J = \left[\, 0 \;\, I \,\right]^T$. Similarly, the PID anti-windup design from II-B can also be considered a structured observer with $J = \left[\, 0 \;\, G^T_c \,\right]^T$.

The challenge arises precisely when considering a non-invertible correction matrix, which leads to a non-linearizable bilinear matrix inequality. This will be solved by means of SOF methods such as the one in [4], for which first we will need to solve a traditional observer design, e.g. the continuous-time Kalman filter [13].

III. Structured observer synthesis

This section presents our proposed method to solve the structured observer synthesis, and its application to PID anti-windup design. First, some useful lemmas are enunciated. Second, it is shown how the structured observer design can be reformulated as a SOF problem, with the resulting BMI being dilated following the approach in [4]. Finally, the process is illustrated through the anti-windup design for a PID controller.

A. Useful lemmas

Lemma 1: Continuous-time Kalman filter. The traditional Kalman filter in continuous time follows (10).

$$\dot{\hat{x}}_{(t)} = A\; \hat{x}_{(t)} + B\; u_{(t)} + L \left(y_{(t)} - \hat{y}_{(t)} \right) , \quad (10)$$

where the observer state is fully corrected via multiplying the measurement error by the gain $L \in \mathbb{R}^{n_x \times n_y}$. It is well-known [13] that solving the Riccati inequality in (11) provides the optimal observer gain L in the sense of minimum error covariance when the observer is constructed as in (10).

$$(A - LC)\, P_{(t)} + P_{(t)}\, (A - LC)^T + [I \;\; -L] \begin{bmatrix} Q_s & \star \\ S_s & R_s \end{bmatrix} \begin{bmatrix} I \\ -L^T \end{bmatrix} \preceq 0, \quad (11)$$

where $P_{(t)} \in \mathbb{S}^{n_x}_{++}$ is the covariance of the state estimation error, the matrices A, C, G, and H are defined in (5), and the system covariance matrices are defined by (12).

$$Q_s = G\, Q_\omega\, G^T \,, \qquad S_s = H\, Q_\omega\, G^T \,, \qquad R_s = H\, Q_\omega\, H^T \,. \quad (12)$$

The optimum is achieved with $P_{(t)} = P_o$ and $L = L_o$ such that the equality holds.

Lemma 2: Structured observer. The pair (P, K) minimising the trace of $P_{(t)}$ subject to (13) is the optimum solution to the structured observer problem.

$$
\begin{aligned}
&(A - JKC)\,P_{(t)} + P_{(t)}\,(A - JKC)^T \\
&\quad + [I \;\; -JK] \begin{bmatrix} Q_s & \star \\ S_s & R_s \end{bmatrix} \begin{bmatrix} I \\ -K^T J^T \end{bmatrix} \preceq 0,
\end{aligned}
\tag{13}
$$

where $K \in \mathbb{R}^{n_w \times n_y}$ is the structured gain, and $P_{(t)} \in \mathbb{S}^{n_x}_{++}$. The proof follows from setting $L = JK$ in (11).

For clarity of notation, the time-dependency of all signals and matrices will be omitted from the notation hereafter.

Lemma 3: Linear quadratic static output feedback. For the closed-loop system given by:

$$
\dot{x} = (A + BK_{sof}C)\,x\,,
\tag{14}
$$

where $K_{sof} \in \mathbb{R}^{n_u \times n_y}$ is the static output feedback gain, the tuple (P, X, Y), which minimizes the trace of P subject to (15), provides a K_{sof} that is locally optimal, among the class of SOF gains, with respect to the linear quadratic (LQ) criterion [4].

$$
\begin{bmatrix} \left\{(A + BK_{lqr})^T P\right\}^{\mathrm{S}} + Q_{lqr} & \star \\ B^T P + S_{lqr}^T - XK_{lqr} + YC & R_c - \{X\}^{\mathrm{S}} \end{bmatrix} \preceq 0,
\tag{15}
$$

where $K_{lqr} \in \mathbb{R}^{n_u \times n_x}$ is the optimal state-feedback gain with respect to the LQ criterion, Q_{lqr} and S_{lqr} are defined by (16) and (17) respectively, and $R_c \in \mathbb{S}^{n_u}_{+}$ is the actuation penalty.

$$
Q_{lqr} = Q_c + \{S_c K_{lqr}\}^{\mathrm{S}} + K_{lqr}^T R_c K_{lqr}\,,
\tag{16}
$$

$$
S_{lqr} = S_c + K_{lqr}^T R_c\,,
\tag{17}
$$

where $Q_c \in \mathbb{S}^{n_x}_{+}$ is the state penalty, and $S_c \in \mathbb{R}^{n_x \times n_u}$ is the cross-penalty. The proof can be found in [4], where it is discussed how this approach also minimizes the conservatism of the solution. The SOF gain is given by $K_{sof} = X^{-1}Y$.

B. From structured observer to SOF formulation

Recognizing that the optimum correction in an observer is given when the correction matrix is $J = I$, the full observer gain L is first derived by solving (11). This gain is then used to reformulate (13) in terms of the error between a structured gain K with its correction matrix J, and the optimal full observer gain L.

$$
\begin{aligned}
&\{(A - LC)P + (L - JK)\,(CP + S_L)\}^{\mathrm{S}} \\
&\quad + Q_L + (L - JK)\,R_s\,(L - JK)^T \preceq 0,
\end{aligned}
\tag{18}
$$

where the new matrices S_L and Q_L are defined by (19) and (20) respectively.

$$
S_L = S_s - R_s L^T\,,
\tag{19}
$$

$$
Q_L = Q_s - \{L\,S_s\}^{\mathrm{S}} + L\,R_s\,L^T\,.
\tag{20}
$$

This formulation enables the dilation of the BMI from (18) into the following form:

$$
\begin{aligned}
&\begin{bmatrix} \{(A - LC)P\}^{\mathrm{S}} + Q_L & \star \\ CP + S_L & R_s \end{bmatrix} \\
&\quad + \left\{ \begin{bmatrix} (L - JK) \\ -I \end{bmatrix} [F \quad X] \right\}^{\mathrm{S}} \preceq 0,
\end{aligned}
\tag{21}
$$

where $F \in \mathbb{R}^{n_y \times n_x}$ and $X \in \mathbb{R}^{n_y \times n_y}$ are two new slack matrices.

The slack matrix F can be safely set to 0 in (21), thanks to $(A - LC)$ being Hurwitz by design of L, which leaves the resulting dilated inequality as linear after applying the change of variable $Y = KX$. This leads to (22), which can be solved by any readily available LMI solver. For more details on this dilation the reader is referred to [4].

$$
\begin{bmatrix} \{(A - LC)P\}^{\mathrm{S}} + Q_L & \star \\ CP + S_L + X^T L^T - Y^T J^T & R_s - \{X\}^{\mathrm{S}} \end{bmatrix} \preceq 0.
\tag{22}
$$

The gain $K = YX^{-1}$ resulting from minimizing the trace of P subject to (22) can be used directly as the structured observer correction gain, or can be used as a starting point for iteration algorithms, such as the one proposed in [14], which may numerically refine the gain further.

This development takes us to formulate the following result. The proof follows from the appropriate changes of variables in *Lemma 3*.

Proposition 1: The triplet (P, X, Y) that minimizes the trace of P subject to (22) provides a locally optimal solution to the structured observer problem by letting $K = YX^{-1}$.

Remark 1: As discussed in [4], this solution is locally optimal due to the latent conservatism of the method. Nevertheless, this conservatism is minimized as part of the solution.

Remark 2: This result can also be understood from the perspective of the known duality between optimal control and estimation [15], as it pivots on a result from controller synthesis.

C. PID anti-windup design

The second motivating example given in the introduction is the design of a backstepping anti-windup gain that corrects the error fed back to a PID controller, to avoid windup.

Anti-windup, and more generally AWBT (anti-windup bump-less transfer) techniques have been studied extensively by the control community for decades, with early connections to static output feedback noticed in [16] and [17], where even an example of a decentralized anti-windup gain is given.

A multi-variable PID controller, like the one in Fig. 2, before introducing any anti-windup corrections, can be formulated in state-space as in (23), adapted from [18].

$$
\begin{cases}
\dot{x}_c = \begin{bmatrix} -T^{-1} & 0 \\ 0 & 0 \end{bmatrix} x_c + \begin{bmatrix} T^{-1} \\ I \end{bmatrix} e\,, \\
u = \begin{bmatrix} -T^{-1}K_d & K_i \end{bmatrix} x_c + \left(K_p - T^{-1}K_d\right) e\,,
\end{cases}
\tag{23}
$$

where $x_c \in \mathbb{R}^{2n_e}$ is the controller state, which contains twice as many states as errors being fed to the controller, as derivative filters and integrators, $u \in \mathbb{R}^{n_u}$ is the control command given by the PID controller, $e \in \mathbb{R}^{n_e}$ is the error signal, $T \in \mathbb{S}^{n_e}$ is a diagonal matrix containing the time constants of the derivative filters (to provide causality), and K_p, K_i, and K_d are the multi-variable PID gains of appropriate dimensions. These gains are assumed tuned to control the plant in (5) as part of an independent process.

A more generic formulation of the controller when adding anti-windup compensation is provided in (24).

$$\begin{cases} \dot{x}_c = A_c\, x_c + B_c\, e + G_c\, K_b\, (\bar{u} - u)\ , \\ u = C_c\, x_c + D_c\, e + H_c\, K_b\, (\bar{u} - u)\ , \end{cases} \tag{24}$$

where A_c, B_c, C_c, and D_c are the controller matrices, defined in (23) for the PID case, while G_c and H_c are the state and output correction matrices of appropriate dimensions.

In order to illustrate the example from Fig. 2, the matrices G_c and H_c are defined by (25), as the correction enters through the error signal. However, other definitions are possible depending on the choice of correction.

$$G_c = B_c\,, \qquad H_c = D_c\,. \tag{25}$$

Let us now define a *non-linear* closed-loop system, where the controller output is exactly the plant input at all times, and the *non-linearity* is represented in the controller by additive uncertainty as per (26).

$$\begin{cases} \dot{\bar{x}} = A\bar{x} + B\bar{u}\,, & \bar{y} = C\bar{x} + D\bar{u}\,, \\ \dot{\bar{x}}_c = A_c\bar{x}_c - B_c\bar{y} + w\,, & \bar{u} = C_c\bar{x}_c - D_c\bar{y} + v\,. \end{cases} \tag{26}$$

where $w \in \mathbb{R}^{n_c}$, and $v \in \mathbb{R}^{n_u}$ are the uncertainties in the controller state and its output, and the reference signal from Fig. 2 has been set to 0 for simplicity of notation.

If now we define the plant represented by the matrices (A, B, C, D), in connection to the controller with anti-windup from (24), as our estimator with corrections, then, the dynamics that define the error between our estimator and the *non-linear* closed-loop system can be defined by (27), after performing the necessary algebraic operations.

$$\begin{cases} \dot{\tilde{x}}_{cl} = A_{cl}\, \tilde{x}_{cl} + G_{cl}\, \omega - B_{cl}\, K_b\, \tilde{u}\,, \\ \tilde{u} = C_{cl}\, \tilde{x}_{cl} + H_{cl}\, \omega - D_{cl}\, K_b\, \tilde{u}\,, \end{cases} \tag{27}$$

where $x_{cl}^T = \left[\, x^T\, x_c^T \,\right]$ is the closed-loop state, containing both the system and the controller states, and $\omega^T = [\, w^T\, v^T \,]$. The resulting closed-loop matrices from this type of interconnection, or the process to obtain them, can be seen in the literature, see for example [7] and references therein.

This formulation follows ideas from [9], by regarding the difference between the saturated and the unsaturated closed-loop systems as caused by the uncertainty affecting the controller state and the command signal. The error between these two systems is the result in (27).

Due to the presence of the feedthrough matrix D_{cl}, a change of variable is needed in order to obtain the original SOF formulation. By using the transformation from [19]: $\varGamma_b = K_b(I + D_{cl}\, K_b)^{-1}$, the error is redefined in (28).

$$\dot{\tilde{x}}_{cl} = A_{cl}\, \tilde{x}_{cl} + G_{cl}\, \omega - B_{cl}\, \varGamma_b\, \left(C_{cl}\, \tilde{x} + H_{cl}\, \omega\right)\,. \tag{28}$$

The synthesis of the gain $\varGamma_b$ can be seen as a case of structured observer design, by letting $A = A_{cl}$, $G = G_{cl}$, $J = B_{cl}$, $C = C_{cl}$, and $H = H_{cl}$ in (9). Finally, once the gain $\varGamma_b$ has been calculated, the backstepping gain can be recovered by reverting the transformation as in (29).

$$K_b = (I - \varGamma_b\, D_{cl})^{-1}\, \varGamma_b\,. \tag{29}$$

Remark 3: The presence of feedthrough is not desirable in practice. In order to avoid it in the anti-windup scheme, a first order transfer function can be used to filter the error $\tilde{u}$ before multiplying it by the backstepping gain. This additional time constant can be easily incorporated to the design by augmenting the state.

Remark 4: In general, all observer cases where the number of correction signals is lower than the number of states, can be represented by a non-invertible matrix J.

Remark 5: when solving the problem with feedthrough, the non-singularity of $(I - \varGamma_b\, D_{cl})$ can be imposed through the LMI in (30).

$$\begin{bmatrix} -I & \star \\ D_{cl} + Y^T & -D_{cl}D_{cl}^T - \{X\}^S \end{bmatrix} \prec 0\,. \tag{30}$$

The proof follows from applying, to the inequality $(I - \varGamma_b\, D_{cl})\, (I - \varGamma_b\, D_{cl})^T \succ 0$, a similar dilation to the one used in III-B.

IV. Numerical example

This section shows the results of synthesizing a backstepping gain by the proposed method, in comparison to the method from [7], where a PI controller connected to an example plant from [20] is presented. Computation of anti-windup gains for this system is claimed to be challenging using other existing approaches.

Two anti-windup strategies are analyzed and computed to solve the problem in [7]: Algorithm 1 is used to obtain a gain that corrects the controller state x_c directly, while Algorithm 2 is used to synthesize a gain to modify both the controller state x_c and the control signal u before saturation.

These two examples can be understood in the framework of Section III-C by setting the matrices B_{cl} and D_{cl} to the values in (31) and (32) respectively for each case, where the matrices Δ_u and Δ_y are as defined in [7].

$$B_{cl}^{Case1} = \begin{bmatrix} 0 \\ I \end{bmatrix}\,,\ D_{cl}^{Case1} = \begin{bmatrix} 0 \end{bmatrix}\,. \tag{31}$$

$$B_{cl}^{Case2} = \begin{bmatrix} 0 & B\Delta_u \\ I & B_c\Delta_y D \end{bmatrix}\,,\ D_{cl}^{Case2} = \begin{bmatrix} 0 & \Delta_u \end{bmatrix}\,. \tag{32}$$

The gains obtained for two different sets of covariance matrices are in Tab. I, together with the ones obtained with the method from [7]. The sign of the latter has been inverted, since the error signal in [7] is calculated as $(u - \bar{u})$.

The simulation results are shown in Fig. 3. The left two plots show the system output, and the saturated control signal, when the anti-windup correction only modifies the controller state. The right two plots show the same signals, but with the anti-windup correction applied to both the controller state, and the control signal.

TABLE I

ANTI-WINDUP GAINS FOR THE EXAMPLE FROM [7]

Scenario	Priuli et al. [7]	Section III-C ($Q=1, R=1$)	Section III-C ($Q=1, R=0.1$)
Case 1	1.6491	1.2959	5.9674
Case 2	$\begin{bmatrix} 1.0020 \\ -0.4307 \end{bmatrix}$	$\begin{bmatrix} 0.6311 \\ -0.8795 \end{bmatrix}$	$\begin{bmatrix} 8.3672 \\ 1.0148 \end{bmatrix}$

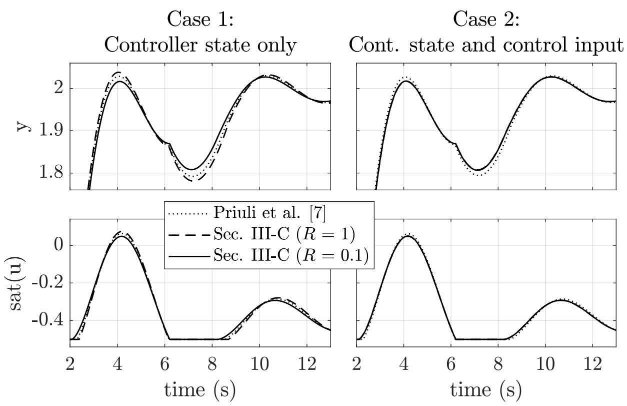

Fig. 3. Simulation comparison with the gains synthesized in [7].

In Case 1, all responses are similar, with the first tuning slightly slower than [7], and the second tuning slightly faster. In Case 2, both tunings (overlapping in the figure) are slightly faster than [7].

The most noticeable difference is in the computing times. Our solution takes advantage of the recent results for SOF synthesis from [4], which does not need to perform iterations, resulting in faster computing times, as reported in Tab. II.

TABLE II

COMPUTING TIMES FOR THE GAINS IN TAB. I

Scenario	Priuli et al. [7]	Section III-C ($Q=1, R=1$)	Section III-C ($Q=1, R=0.1$)
Case 1	35.49 s	0.14 s	0.11 s
Case 2	51.24 s	0.19 s	0.12 s

The times in Tab. II were obtained on an Intel Core i7-7500U CPU @ 2.70GHz $\times$ 2 with 8 GB of RAM, using YALMIP for MATLAB, with MOSEK as LMI solver.

V. CONCLUSIONS

This paper presents the structured observer synthesis problem for LTI systems, motivated by two examples of practical significance: the use of black-box models in observer designs, and the synthesis of anti-windup gains.

The problem is formalized in an unified framework, and connected with its control design counterpart: the static output feedback (SOF) synthesis.

As an application example, the design of an anti-windup gain is illustrated, and posed as a structured observer problem. Finally, the method is applied to an example from the literature, recognized to be challenging to solve [7].

The results show that our method achieves slightly faster responses in the example system, with the addition of providing the capability to tune the gain. Furthermore, if the method from [4] is used, the gain can be computed in a fast and non-iterative manner, resulting in a computation performed 200 to 300 times faster.

The structured observer synthesis has been illustrated here for linear continuous-time invariant systems. Extension to discrete-time is trivial, via formulation of the equivalent ARE and BMI conditions. Similarly, extension to LPV systems is possible by means of the *Lyapunov shaping paradigm* [21].

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