Article:

Blackwell, P.G. orcid.org/0000-0002-3141-4914 (2025) A tractable symmetric multivariate logistic distribution with bounded score function. Bulletin of the Australian Mathematical Society. pp. 1-7. ISSN: 0004-9727

https://doi.org/10.1017/s0004972725100439

A TRACTABLE SYMMETRIC MULTIVARIATE LOGISTIC DISTRIBUTION WITH BOUNDED SCORE FUNCTION

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(Submitted June 5, 2025)

Abstract

Existing multivariate versions of the logistic probability distribution generally lack some of the useful properties of the univariate logistic distribution, such as its bounded score function or the tractability of its density function, or lack the rotational symmetry necessary for many applications. This paper clarifies some of the properties of such distributions, and proposes a multivariate distribution closely related to the univariate logistic that has a tractable density, including the necessary normalising constant, bounded score function, and elliptical symmetry. Some properties of its marginal distributions are explored, particularly in the bivariate case.

Mathematics subject classification: primary 60E05
Keywords and phrases: multivariate logistic distribution, bivariate distribution, elliptical symmetry

1. Introduction

The logistic distribution in one dimension has simple forms for its cumulative distribution function and its probability density function, and has the desirable property, for some modelling purposes, that its score function $f'(x)/f(x)$

The author gratefully acknowledges the support of the Leverhulme Trust through Research Fellowship RF-2020-241.

is bounded. While the Laplace distribution has similar advantages, it lacks the smoothness of the logistic at the origin, where its PDF is not differentiable. The motivation of the present work is to find a multivariate distribution with the advantages of the logistic distribution that also has rotational symmetry.

The motivation comes from applied problems where tractability is important and where rotational symmetry is necessary for a meaningful model, for example any application in 2-dimensional 'geographical' space where the choice of co-ordinate system is arbitrary. For a particular case where the tail behaviour of the distribution is also crucial, see Section 5.

2. Existing multivariate logistic distributions

There are several existing multivariate distributions linked in some way to the univariate logistic distribution. One class of such distributions has marginal distributions that are logistic, but lacks rotational symmetry, including the bivariate case in [4], and more general forms in [8] and [1], plus the special case of independence.

Circular symmetry of a multivariate distribution, as well as rotational symmetry of a whole family of distributions, can be achieved by considering elliptically symmetric distributions [9]. Typically, such distributions are written in the form

$$f_n(\mathbf{x}) = |\Sigma|^{1/2} g_n((\mathbf{x}-\mu)^T \Sigma^{-1}(\mathbf{x}-\mu))$$

where Σ is a positive definite $n \times n$ matrix, $\mathbf{x}, \mu \in \mathbb{R}^n$, and $g_n(\cdot)$ is known as the kernel or density generator; often g_n depends on the dimension n only through a normalising constant c_n [6]. There are then two existing ways in which a link has been made with the logistic distribution.

The first, giving what is widely referred to as *the* elliptically symmetric logistic distribution, involves taking

$$g_n(u) = \frac{c_n \exp(-u)}{(1+\exp(-u))^2}$$

i.e. using a density generator based on the standard logistic distribution [5, 17]. However, this approach does not have the property that the score function is bounded—even when $n = 1$, as this distribution does not reduce to the usual logistic in that case. See Section 5 for details.

The second approach is to require that $\mathbf{x}$ is elliptically symmetric with marginal distributions given by the univariate logistic distribution. Volodin [15] discusses this case—see references therein for its origins—and gives a general construction for elliptically symmetric distributions with marginals $f(\cdot)$, taking very different forms in odd and even dimensions. For even dimension $n = 2m$ the density involves m-fold differentiation with respect to $r = \sqrt{(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)})$ of $\int_0^\infty f(\sqrt{z^2+r})\mathrm{d}z$ [15, equation (2)]. In the logistic case, Volodin [15] shows that the integral can be rewritten as an infinite series. For odd dimension $n = 2m+1$, the density is somewhat simpler, but still involves m-fold differentiation of $f(\cdot)$. Though this distribution has logistic marginals and elliptical symmetry, it lacks the required tractability, particularly when $n = 2$.

3. The proposed distribution

Staying with the elliptically symmetric framework, instead of taking the density generator to be the univariate logistic, we can simply take the square root first, to give

$$g_n(u) = \frac{c_n \exp(-\sqrt{u})}{(1+\exp(-\sqrt{u}))^2}. \tag{3.1}$$

Then we have

$$f_n(\mathbf{x}|\mu,\Sigma) = |\Sigma|^{-1/2} c_n \frac{\exp(-r(\mathbf{x}))}{(1+\exp(-r(\mathbf{x})))^2}, \tag{3.2}$$

where

$$r(\mathbf{x}) = \sqrt{(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)}. \tag{3.3}$$

When $n = 1$, this reduces to the usual univariate logistic distribution, with $c_1 = 1$. In two dimensions, it is straightforward to get the normalising constant c_2. Since the Jacobian for transforming to polar co-ordinates (r,θ) is just r, we need

$$\begin{aligned} c_2^{-1} &= \int_{\theta=-\pi}^{\pi}\int_{r=0}^{\infty} r f_n(r,\theta)\,\mathrm{d}r\,\mathrm{d}\theta \\ &= 2\pi \int_{r=0}^{\infty} r f_n(r)\,\mathrm{d}r \\ &= 2\pi \int_{r=0}^{\infty} (r\exp(-r))/(1+\exp(-r))^2\,\mathrm{d}r \\ &= 2\pi\log(2). \end{aligned}$$

Explicitly, we have

$$f_2(\mathbf{x}|\mu, \Sigma) = \frac{|\Sigma|^{-1/2}}{2\pi \log(2)} \frac{\exp(-r(\mathbf{x}))}{(1 + \exp(-r(\mathbf{x})))^2}.$$

Normalising constants for $n \geq 3$ are given in Section 4.

Clearly this proposed distribution is both simple in form and elliptically symmetric. It occurs as a particular case of the very general family defined by Wang and Yin [16]; the next section makes that link explicitly, and builds on results of [16] to obtain the normalising constant c_n in general. In later sections, I will show that its score function is bounded (Section 5) and explore its marginal distributions (Section 6).

4. Normalising constants

Wang and Yin [16] define a very general family of multivariate densities that includes the proposed density of equation (3.2) as a special case. They consider density generators of the form

$$g_n(u) = \frac{c_n u^{N-1} \exp(-a u^{s_1})}{(1 + \exp(-b u^{s_2})^{2r}}$$

(their parameter r is not to be confused with the function $r(\cdot)$ used here for (scaled) distance from the origin) which includes the proposed distribution when $N = 1, s_1 = s_2 = 1/2, f = 1, a = b = 1, r = 1$. Wang and Yin [16] do not give much information about the particular distribution proposed in equation (3.1); this case is not included in any of the eight sub-families that they explore in more detail. However, their Theorem 2.1 relates the normalising constants c_n to the generalized Hurwitz-Lerch zeta function Φ^*, as defined there [16, Section 2.2], in cases where $s_1 = s_2$. For the case proposed here, the expression for c_n simplifies in one of two ways. For $n \geq 4$, the sufficient conditions to express Φ^* in summation form are met. For $n = 2, 3$ an integral expression for Φ^* is given, but direct calculation shows that the summation form still holds even though Wang and Yin's sufficient conditions are not met. For all n, the necessary value of Φ^* simplifies to $\eta(n-1)$, where

$$\eta(t) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i^t}$$

is the Dirichlet eta function or alternating zeta function, and

$$c_n = \frac{\Gamma(n/2)}{2\pi^{n/2}\Gamma(n)\eta(n-1)}. \tag{4.1}$$

5. Score functions

The linear score function,

$$\nabla \log(f(\mathbf{x})) = \left(\frac{\partial \log f}{\partial x_1}, \ldots, \frac{\partial \log f}{\partial x_1}\right)^T,$$

is of particular interest because of its rôle in constructing models in movement ecology that are tailored to specific stationary distributions, which can be thought of as analogous to the construction of Markov Chain Monte Carlo algorithms for specific target distributions [10–12]. For example, the Langevin algorithm for a target $\pi(\mathbf{x})$ has a drift term proportional to $-\nabla \log(\pi(\mathbf{x}))$ [see 10], and the Bouncy Particle Sampler [3] involves 'bounce' events with a rate that depends on $\nabla \log(\pi(\mathbf{x}))$. Boundedness of the score function is relevant to both the realism and the practical implementation of such models.

For a general elliptically symmetric distribution with density generator $g(\cdot)$, and defining

$$\begin{aligned} q(\cdot) : q(\mathbf{x}) &= (\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu), \\ h_n(\cdot) : h_n(u) &= \frac{\mathrm{d} \log(g_n(u))}{\mathrm{d}u}, \end{aligned}$$

we have

$$\nabla \log(f_n(\mathbf{x})) = 2 h_n(q(\mathbf{x})) \Sigma^{-1} (\mathbf{x} - \mu). \tag{5.1}$$

For the existing elliptically symmetric multivariate logistic, with $g_n(\cdot)$ proportional to the univariate logistic,

$$h_n(u) = -\frac{1 - \exp(-u)}{1 + \exp(-u)}$$

and so $h_n(u) \to -1$ as $u \to \infty$ and $h_n(u) \to 1$ as $u \to -\infty$. Substituting into equation (5.1), clearly the score is not bounded.

On the other hand, if we take $g_n(\cdot)$ as in equation (3.1), the proposed 'square root' version, we have

$$h_n(u) = -\frac{1}{2\sqrt{u}} \cdot \frac{1 - \exp(-\sqrt{u})}{1 + \exp(-\sqrt{u})}.$$

To see that the score is bounded in this case, it is simplest to consider the case $\Sigma = I, \mu = \mathbf{0}$. Then

$$\begin{aligned}
\nabla \log(f_n(\mathbf{x})) &= \frac{-\mathbf{x}}{\sqrt{\mathbf{x}^T\mathbf{x}}} \cdot \frac{1 - \exp(-\sqrt{\mathbf{x}^T\mathbf{x}})}{1 + \exp(-\sqrt{\mathbf{x}^T\mathbf{x}})}, \\
|\nabla \log(f_n(\mathbf{x}))| &= \frac{1 - \exp(-\sqrt{\mathbf{x}^T\mathbf{x}})}{1 + \exp(-\sqrt{\mathbf{x}^T\mathbf{x}})}.
\end{aligned}$$

So the score is bounded in this case, and by linear transformation for general Σ and μ too.

6. Marginal distributions

The marginal distributions for the distribution proposed in Section 3 can be evaluated by considering first the standard circular case $\Sigma = I_n, \mu = \mathbf{0}$. There, the marginal (for any x_i) is not exactly a logistic distribution, except when $n = 1$; empirically, it is close to one, although not with scale 1. Before making this relationship more precise, it is useful to assess the scale by evaluating the marginal at 0, which we can do exactly.

For any standard elliptically symmetric distribution where the density generator is of the form

$$g_n(u) = c_n g_0(u),$$

we have

$$f_{x_n}(x|\mu = \mathbf{0}, \Sigma = I_n) = \int_{\mathbf{x}_{(n)}} g_n((\mathbf{x}_{(n)}, x_n)^T(\mathbf{x}_{(n)}, x_n))\mathrm{d}\mathbf{x}_{(n)}$$

where with a slight abuse of notation we write $f_{x_n}(\cdot)$ for the univariate marginal for x_n derived from $f_n(\cdot)$, and $\mathbf{x}_{(n)}$ for $(x_1, \ldots, x_{n-1})$. Hence

$$\begin{aligned}
f_{x_n}(0|\mu = \mathbf{0}, \Sigma = I_n) &= \int_{\mathbf{x}_{(n)}} g_n(\mathbf{x}_{(n)}^T\mathbf{x}_{(n)})\mathrm{d}\mathbf{x}_{(n)} \\
&= \int_{\mathbf{x}_{(n)}} c_n g_0(\mathbf{x}_{(n)}^T\mathbf{x}_{(n)})\mathrm{d}\mathbf{x}_{(n)} \\
&= \frac{c_n}{c_{n-1}} \int_{\mathbf{x}_{(n)}} c_{n-1} g_0(\mathbf{x}_{(n)}^T\mathbf{x}_{(n)})\mathrm{d}\mathbf{x}_{(n)} \\
&= \frac{c_n}{c_{n-1}}.
\end{aligned}$$

A univariate logistic with scale σ has a density at zero of $1/4\sigma$; re-arranging means that the marginal density at zero matches that of a univariate logistic with scale

$$\tau_n = \frac{c_{n-1}}{4c_n}.$$

In particular, for $g_n(\cdot)$ given by equation (3.1) we have $c_1 = 1, c_2 = (2\pi \log(2))^{-1}$, $c_3 = (3/2)\pi^{-3}$, and so $\tau_2 = \pi \log(2)/2 \approx 1.088793, \tau_3 = \pi^2/12 \log(2) \approx 1.186569$.

In both these cases, calculation of the marginal distribution by numerical integration (using the standard R routine [14]) shows that the main parts of the marginal densities are well approximated by logistic distributions with scale parameters τ_2 and τ_3 respectively. However, in the extreme tails, there is some divergence in shape, with densities decaying more quickly than $\exp(-\tau_n x)$ but less quickly than $\exp(-x)$.

For $n = 2$ and $x_2 \in [-6.5, 6.5]$, containing more than the central 99% of mass, the the marginal density $f_{x_2}(x|\mu = \mathbf{0}, \Sigma = I_2)$ and the univariate logistic density with scale parameter τ_2 agree to within a factor of 1.01. A more precise expression for the density in the extreme tails is suggested by comparison with the standard bivariate Laplace distribution [7], which has joint density

$$\pi^{-1} K_0 \left(\sqrt{2(x_1^2 + x_2^2)} \right),$$

where $K_0(\cdot)$ is the modified Bessel function of the second kind, and marginal density

$$2^{-1/2} \exp(-x\sqrt{2}).$$

The asymptotic form

$$K_0(z) \sim \sqrt{\pi/(2z)} \exp(-z)$$

[e.g. 13] suggests

$$\sqrt{x/\pi} \exp(-x)$$

for the marginal of the proposed logistic with $n = 2$, and numerical experiments show that it is indeed extremely close for sufficiently large x.

7. Implementation

Calculation of the proposed density is straightforward once the normalising constants are known. In R [14], the Dirichlet eta function is available in the `pracma` package [2], and so calculation of c_n from equation (4.1) is trivial. Some improvement in speed can be achieved by taking advantage of the existing function for the univariate logistic, $f_1(\cdot)$, writing

$$f_n(\mathbf{x}|\mu, \Sigma) = |\Sigma|^{-1/2} c_n f_1(r(\mathbf{x})|0, 1)$$

where $r(\mathbf{x})$ is as defined in equation (3.3).

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