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Version: Accepted Version

Article:
Sharp, B. orcid.org/0000-0002-7238-4993 and Appolloni, L. (Accepted: 2025) A Note on Moving Frames along Sobolev Maps and the Regularity of Weakly Harmonic Maps. Proceedings of the American Mathematical Society. ISSN: 0002-9939 (In Press)

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A Note on Moving Frames along Sobolev Maps and the Regularity of Weakly Harmonic Maps

Luigi Appolloni and Ben Sharp

September 4, 2025

Abstract

The purpose of this note is twofold. First we show that, for weakly differentiable maps between Riemannian manifolds of any dimension, a smallness condition on a Morrey-norm of the gradient is sufficient to guarantee that the pulled-back tangent bundle is trivialised by a finite-energy frame over simply connected regions in the domain. This is achieved via new structure equations for a connection introduced by Rivière in the study of weakly harmonic maps, combined with Coulomb-frame methods and the Hardy-BMO duality of Fefferman-Stein.

We also prove that for weakly harmonic maps from domains of any dimension into closed homogeneous targets, a smallness condition on the BMO seminorm of the map is sufficient to obtain full regularity.

1 Introduction

Since the work of Wente [19], so-called div-curl structures or "Wente terms" have been known to appear crucially in the study of geometric PDE and calculus of variations in the large. Such terms are generally of the form $E \cdot D$ for vector fields in conjugate Lebesgue spaces $E, D \in L^p, L^{p^*}$ for which one is divergence-free whilst the other is curl-free: *a-priori* only in L^1, these terms lie in the slightly smaller Hardy space $\mathcal{H}^1$ which enjoys better properties in relation to weak convergence and under convolution by singular integrals (and hence elliptic regularity theory) see Coiffman-Lions-Mayers-Semmes [4] for generalisations and links to classical harmonic analysis. In his celebrated work on the regularity theory of harmonic maps from surfaces, Frédéric Hélein [9] used a moving frame technique along the map to re-write the equations in such a way that Wente-type terms naturally appear. This was extended by Fabrice Bethuel [3] for weakly harmonic maps $u : B_1^m \to \mathcal{N}$ from higher dimensional domains, under a smallness condition on the $M^{2,m-2}$ Morrey norm of the gradient (which is just the L^2-norm when $m = 2$):

$$\|\mathrm{d}u\|_{M^{2,m-2}(B_1)} := \sup_{x \in B_1^m, r>0} \left(r^{2-m} \int_{B_r(x) \cap B_1^m} |\mathrm{d}u|^2 \, dx \right)^{1/2}.$$

The appearance of Wente terms is sufficient to conclude that a weakly harmonic map whose gradient is small in this Morrey-norm, is in fact smooth. This smallness condition is guaranteed for weakly stationary harmonic maps away from a closed set $S \subset B_1$ satisfying $\mathcal{H}^{m-2}(S) = 0$. Here are throughout, $\mathcal{N}^n \hookrightarrow \mathbb{R}^d$ is an n-dimensional submanifold of $\mathbb{R}^d$ equipped with the induced metric and $B_1^m \subset \mathbb{R}^m$ denotes the open unit ball.

In [13] Tristan Rivière established a deep and remarkable link between the above regularity theory and systems of the form

$$-\Delta u^i = \Omega^i_j \cdot \mathrm{d}u^j \qquad \text{for } u \in W^{1,2}(B^m_1, \mathbb{R}^d) \qquad \text{and } \Omega \in L^2(B^m_1, \mathfrak{so}(d) \otimes \mathrm{T}^*\mathbb{R}^m). \quad (1.1)$$

Rivière found a new gauge in order to re-write the above as a conservation law under an appropriate smallness-condition on Ω, when $m = 2$. In particular he uncovered hidden Wente terms and showed that solutions were continuous. This was extended by Rivière-Struwe [14] to higher-dimensional domains, wherein they recover regularity of such systems under the natural assumption that $\Omega, \mathrm{d}u \in M^{2,m-2}$ and $\|\Omega\|_{M^{2,m-2}}$ is small. This powerful observation concerning systems of the form (1.1) has had deep and fruitful consequences in many nearby regularity problems in geometric PDE both in higher-order and non-local settings, see e.g. [13, 5, 12, 17] and citations thereof.
In particular given a C^2-Riemannian manifold $\mathcal{N} \hookrightarrow \mathbb{R}^d$ and

$$u \in W^{1,2}(B^m_1, \mathcal{N}) := \{v \in W^{1,2}(B^m_1, \mathbb{R}^d) : v(x) \in \mathcal{N} \quad \text{for a.e. } x \in B^m_1\}$$

Rivière introduced the following specific Ω which we henceforth denote with the lower case $\omega = \omega(\mathcal{N}, u) : B^m_1 \to \mathfrak{so}(d) \otimes \mathrm{T}^*\mathbb{R}^m$: let $\mathrm{A} \in \Gamma(\mathrm{T}^*\mathcal{N} \otimes \mathrm{T}^*\mathcal{N} \otimes \mathcal{V}\mathcal{N})$ be the second fundamental form defined by $\mathrm{A}(X, Y) = (D_X Y)^\perp$ which one extends in the obvious way as a section of $\mathrm{T}^*\mathbb{R}^d \otimes \mathrm{T}^*\mathbb{R}^d \otimes \mathrm{T}\mathbb{R}^d$ over $\mathcal{N}$ via $\mathrm{A}(X, Y) := \mathrm{A}(X^\top, Y^\top)$ and can be written in ambient $\mathbb{R}^d$-coordinates $\{z\}$ via $\mathrm{A} = \mathrm{A}^i_{jk} \mathrm{d}z^j \otimes \mathrm{d}z^k \otimes \partial_{z^i}$. For u as above define

$$\omega = \omega^i_j = -(\mathrm{A}^i_{jk}(u) - \mathrm{A}^j_{ik}(u))\mathrm{d}u^k. \quad (1.2)$$

In particular given a vector $v \in \mathbb{R}^d$ we may see that ω may be equivalently defined without coordinates via

$$\omega v = \omega^i_j v^j = -\mathrm{A}(v, \mathrm{d}u) + \langle \mathrm{A}(\mathrm{d}u, \cdot)^\sharp, v\rangle \in \mathbb{R}^d \otimes \mathrm{T}^*\mathbb{R}^m. \quad (1.3)$$

Rivière's interest in such an ω is that weakly harmonic maps u solve (1.1) for this specific ω, ($\Delta u = \omega \cdot \mathrm{d}u = \mathrm{A}(\mathrm{d}u, \mathrm{d}u)$) and thus the regularity theory for harmonic maps may be reduced to the study of systems in the form (1.1). In this article we show that ω as defined in (1.2) is significant even for arbitrary (i.e. not necessarily harmonic) u.

1.1 Moving frames along Sobolev maps

The first half of this article concerns *arbitrary* maps $u \in W^{1,2}(B^m_1, \mathcal{N})$ and the related ω defined in (1.2). We equip $u^*(\mathrm{T}\mathbb{R}^d) = B^m_1 \times \mathbb{R}^d$ with the connection $\nabla^\omega = \mathrm{d} + \omega$. We uncover some fundamental structure equations for this induced connection which, when combined with Coulomb-frame methods, contain Wente terms. Utilising the $\mathcal{H}^1 - BMO$ duality of Fefferman-Stein [6] we have the following:

Theorem 1.1. *Let $\mathcal{N} \hookrightarrow \mathbb{R}^d$ be C^2-Riemannian manifold, $u \in W^{1,2}(B^m_1, \mathcal{N})$ and ω be defined by* (1.2). *There exist $\varepsilon = \varepsilon(m, d) > 0$, $C = C(m, d) < \infty$ so that if*

$$\|\omega\|_{M^{2,m-2}(B_1)} := \sup_{x \in B^m_1, r>0} \left(r^{2-m} \int_{B_r(x) \cap B^m_1} |\omega|^2 \, dx \right)^{1/2} < \varepsilon,$$

then $u^*\mathrm{T}\mathcal{N}$ and $u^*\mathcal{V}\mathcal{N}$ are both trivial in the sense that there exist

$$\{e_i\}^n_{i=1}, \{\nu_j\}^d_{j=n+1} \subset W^{1,2}(B_1, \mathbb{R}^d)$$

so that $\{e_i(x)\}$ *resp.* $\{\nu_j(x)\}$ *form orthonormal bases of* $\mathrm{T}_{u(x)}\mathcal{N}$ *resp.* $\mathcal{V}_{u(x)}\mathcal{N}$ *for a.e.* $x \in B_1$*. Furthermore,* $\{e_i\}$ *resp.* $\{\nu_j\}$ *are finite-energy Coulomb frames in the sense that for all relevant* i, j *we have*

$$\mathrm{d}^*(e_i \cdot \mathrm{d}e_j) = 0, \quad \mathrm{d}^*(\nu_i \cdot \mathrm{d}\nu_j) = 0$$

and

$$\max_{i,j}\{\|\mathrm{d}e_i\|_{M^{2,m-2}(B_1)}, \|\mathrm{d}\nu_j\|_{M^{2,m-2}(B_1)}\} \leq C\|\omega\|_{M^{2,m-2}(B_1)}.$$

Remark 1.2. We may conclude the same result, assuming $\|\mathrm{A}\|_{L^\infty(N)}$ is finite (e.g. if $\mathcal{N}$ is closed), under a smallness condition on $\|\mathrm{d}u\|_{M^{2,m-2}}$ except both ε and C would depend additionally on $\|\mathrm{A}\|_{L^\infty(N)}$. We note that it is not true that a small bound on $\|\mathrm{d}u\|_{M^{2,m-2}}$ implies that the image of u is contained in a small region e.g. when $m = 2$ let $\gamma : [0, \infty) \to \mathcal{N}$ be any Lipschitz curve with $|\gamma'| = 1$ almost everywhere, and consider $\gamma \circ u : B_1 \to \mathcal{N}$ for $u(x) = \varepsilon \log\log(e|x|^{-1})$. However the result above implies that a certain amount of $\|\mathrm{d}u\|_{M^{2,m-2}}$ is required for the image to "wrap around" enough for $u^*\mathrm{T}\mathcal{N}$ to be non-trivial e.g. $u : B_1^3 \to S^2$ defined by $u(x) = \frac{x}{|x|}$ satisfies $r^{-1}\|\mathrm{d}u\|^2_{L^2(B_r(0))} = 8\pi$ for all $r > 0$ and of course the pulled-back tangent bundle is not trivial in this case.

Remark 1.3. We will see below that $\omega = \frac{1}{2}\mathcal{R}^{-1}\mathrm{d}\mathcal{R}$ where $\mathcal{R} : B_1 \to \mathrm{O}(d)$ can be interpreted as $G \circ u$ and $G : \mathcal{N} \to G(n, d) \hookrightarrow \mathrm{O}(d)$ is the Gauss map of $\mathcal{N}$ (see Remark 2.3). Thus the theorem above can be compared directly with Lemma 5.1.4 in [9] and be thought of as an extension of this to higher dimensions - see Theorem 1.5 below.

In light of the above Remarks we pose the following

Question 1.4. Does the above Theorem remain true if instead of imposing that $\|\omega\|_{M^{2,m-2}}$ or $\|\mathrm{d}u\|_{M^{2,m-2}}$ is small, we instead require

$$[\mathcal{R}]_{BMO(B_1)} := \sup_{B_r(x) \subset B_1} \left(r^{-m} \int_{B_r(x)} |\mathcal{R} - \overline{\mathcal{R}}_{r,x}|^2 \right)^{\frac{1}{2}} < \varepsilon, \qquad \overline{\mathcal{R}}_{r,x} := \frac{1}{|B_r(x)|} \int_{B_r(x)} \mathcal{R}$$

or indeed $[u]_{BMO(B_1)} < \varepsilon$. Of course the best one could hope for is that we end up with an equivalent bound on $[e_i]_{BMO(B_1)}$ and $[\nu_j]_{BMO(B_1)}$ in terms of $[\mathcal{R}]_{BMO(B_1)}$.

Theorem 1.1 is a special case of the following

Theorem 1.5. *Suppose that* $\Pi \in W^{1,2}(B_1^m, \mathfrak{gl}(d))$ *is a projection (i.e.* $\Pi^2 = \Pi$ *and* $\mathrm{rank}(\Pi) = n$ *a.e.), equivalently suppose that* $\Pi \in W^{1,2}(B_1, G(n, d))$*. Define* $\omega_\Pi := \Pi\mathrm{d}\Pi - \mathrm{d}\Pi\Pi$*. There exist* $\varepsilon = \varepsilon(m, d) > 0$*,* $C = C(m, d) < \infty$ *so that if* $\|\omega_\Pi\|_{M^{2,m-2}(B_1)} < \varepsilon$ *then there exist*

$$\{e_i\}_{i=1}^n, \{\nu_j\}_{j=n+1}^d \subset W^{1,2}(B_1, \mathbb{R}^d)$$

so that $\{e_i(x)\}$ *resp.* $\{\nu_j(x)\}$ *form orthonormal bases of* $\Pi(x)\mathbb{R}^d$ *resp.* $\Pi^\perp(x)\mathbb{R}^d$ *for a.e.* $x \in B_1$*. Furthermore,* $\{e_i\}$ *resp.* $\{\nu_j\}$ *are finite-energy Coulomb frames in the sense that for all relevant* i, j *we have*

$$\mathrm{d}^*(e_i \cdot \mathrm{d}e_j) = 0, \quad \mathrm{d}^*(\nu_i \cdot \mathrm{d}\nu_j) = 0$$

and

$$\max_{i,j}\{\|\mathrm{d}e_i\|_{M^{2,m-2}(B_1)}, \|\mathrm{d}\nu_j\|_{M^{2,m-2}(B_1)}\} \leq C\|\omega\|_{M^{2,m-2}(B_1)}.$$

In the above $\Pi^\perp(x) := \mathrm{Id} - \Pi$ *is the projection onto the orthogonal complement.*

Remark 1.6. As mentioned in Remark 1.3, when $m = 2$ this result recovers [9, Lemma 5.1.4] with a new proof which works for all m. We also recover (when $m = 2$) $\varepsilon = \sqrt{\frac{4\pi}{3}}$ using the optimal L^2-Wente estimates proved by Ge [7], see Remark 2.6 for the necessary changes in the proof. This constant is equivalent to the $\sqrt{\frac{8\pi}{3}}$ appearing in [9], the discrepancy is due to a differnent choice of metric on $G(n, d)$.

Outline of the Proof of Theorem 1.1 The proof relies on a series of observations concerning the geometry of the connection ∇^ω which we expect to be of interest in their own right.
Let $\mathrm{End}(\mathrm{T}\mathbb{R}^d)$ denote the endomorphism bundle restricted to $\mathcal{N}$ and $\tilde{\mathrm{T}} \in \Gamma(\mathrm{End}(\mathrm{T}\mathbb{R}^d))$ be the projection onto the tangent space of $\mathcal{N}$: $\tilde{\mathrm{T}}(z) = \Pi_{\mathrm{T}_z\mathcal{N}}$. Similarly $\tilde{\mathcal{V}} \in \Gamma(\mathrm{End}(\mathrm{T}\mathbb{R}^d))$ projection onto the normal space $\tilde{\mathcal{V}}(z) = \Pi_{\mathcal{V}_z\mathcal{N}}$. Denote by $\mathrm{T} = \tilde{\mathrm{T}} \circ u \in \Gamma(u^*\mathrm{End}(\mathrm{T}\mathbb{R}^d))$, $\mathcal{V} = \tilde{\mathcal{V}} \circ u \in \Gamma(u^*\mathrm{End}(\mathrm{T}\mathbb{R}^d))$ and $\mathcal{R} = \mathrm{T} - \mathcal{V} \in \Gamma(u^*\mathrm{End}(\mathrm{T}\mathbb{R}^d))$ where we note that $\mathcal{R}$ is of course orthogonal with $\mathcal{R} = \mathcal{R}^{-1} = \mathcal{R}^T$. In fact we have the following identities for ω (see Lemma 2.2):

$$\omega = \frac{1}{2}\mathcal{R}^{-1}\mathrm{d}\mathcal{R} = \mathrm{T}\mathrm{d}\mathrm{T} - \mathrm{d}\mathrm{T}\mathrm{T}.$$

The connection ∇^ω on $u^*(\mathrm{T}\mathbb{R}^d)$ induces a connection on $u^*\mathrm{End}(\mathrm{T}\mathbb{R}^d)$ and we show that T, $\mathcal{V}$ and $\mathcal{R}$ are all parallel sections (see Lemma 2.5), in particular:

$$\nabla^\omega\mathcal{R} = \mathrm{d}\mathcal{R} + [\omega, \mathcal{R}] = 0.$$

Since $\|\omega\|_{M^{2,m-2}}$ is small, we utilise the Coulomb frame constructed by Rivière-Struwe [14] and find a new frame P making the new connection forms divergence free. At which point the gauge-invariant form of the above allows us to write, for $Q = P^{-1}\mathcal{R}P$:

$$\mathrm{d}Q = -[\mathrm{d}^*\xi, Q].$$

Testing this equation with $\mathrm{d}Q$ gives a Wente structure on the right hand side and we may employ $\mathcal{H}^1$-BMO duality to show that $\|\mathrm{d}Q\|^2_{L^2(B_1)} \leq C\varepsilon\|\mathrm{d}Q\|^2_{L^2(B_1)}$ i.e. when ε is sufficently small $Q = P^{-1}\mathcal{R}P$ is a constant. In particular this yields that P itself simultaneously trivialises both the pulled-back tangent and normal bundles. The full details are given in Section 2.

1.2 The regularity of harmonic maps into homogeneous targets

As discussed in the opening weakly harmonic maps are regular in a neighbourhood of a point p if $\|\mathrm{d}u\|_{M^{2,m-2}(B_R(p))}$ is sufficiently small for some radius $R > 0$. By the scale invariance of this norm, this condition may be stated in an equivalent way: as long as all approximate tangent maps, $\hat{u}_{r,q}(x) = u(q + rx)$, for $q \in B_R(p)$ and all $r < R - |p - q|$, are locally $W^{1,2}$-close to a constant, then u is regular near p.
In the case of harmonic maps into homogeneous targets, Hélein [8] utilised Noether's theorem to show that pure Wente structures naturally appear using the presence of a moving frame constructed via the isometries acting on $\mathcal{N}$ (and not via Coulomb gauge methods). In the second part of this article we note that weakly harmonic maps into a homogeneous target are regular in a neighbourhood of a point p if

$$[u]_{BMO(B_R(p))} := \sup_{B_r(x)\subset B_R(p)} \left(r^{-m} \int_{B_r(x)} |u - \overline{u}_{r,x}|^2 \, \mathrm{d}x \right)^{1/2}, \qquad \overline{u}_{r,x} := \frac{1}{|B_r(x)|} \int_{B_r(x)} u$$

is sufficiently small - i.e. when $\mathcal{N}$ is homogeneous we may replace a first order condition on the map u ($\|\mathrm{d}u\|_{M^{2,m-2}}$ small) with a zeroth order one ($[u]_{BMO}$ small) to achieve full regularity. As above, we may re-state this: for weakly harmonic maps into homogeneous targets, if all approximate tangent maps in a neighbourhood of p are locally L^2-close to a constant, then u is regular near p.

Theorem 1.7. *Suppose that $\mathcal{N}^n \hookrightarrow \mathbb{R}^d$ is a closed homogeneous Riemannian manifold isometrically embedded in Euclidean space and equip B_1^m with a smooth metric g. Then there exists $C = C(m, g, \mathcal{N}) < \infty$ so that if $u : (B_1, g) \to \mathcal{N}$ is weakly harmonic then*

$$\|\nabla^2 u\|_{L^1(B_{1/2})} \leq C\|\nabla u\|_{L^2}^2.$$

Furthermore there exists $\varepsilon > 0 = \varepsilon(m, g, \mathcal{N}) > 0$ so that for any weakly harmonic map $u : (B_1, g) \to \mathcal{N}$ satisfying $[u]_{BMO(B_1)} < \varepsilon$ then

$$\|\nabla u\|_{L^\infty(B_{1/2})} \leq C\|u - \overline{u}\|_{L^1(B_1)} \leq C[u]_{BMO(B_1)}.$$

Our starting point for the proof is a result of Hélein [8, Lemmata 1 and 2] which uses the homogeneity of $\mathcal{N}$ and Noether's theorem to uncover Wente terms in the harmonic map PDE, see also [2] for a new interpretation of this via equivariant embeddings. From this we may infer that there exist $K = K(\mathcal{N}) \in \mathbb{N}$, $L = L(\mathcal{N}) < \infty$ so that a harmonic map u as in the Theorem above solves

$$\Delta_g u = X_j \cdot_g \mathrm{d}(Y^j(u)) \tag{1.4}$$

for

$$\{X_j\}_{j=1}^K \subset L^2(B_1, \mathrm{T}^*\mathbb{R}^m), \quad \mathrm{d}_g^*(X_j) = 0 \quad \text{and} \quad |X_j| \leq L|\mathrm{d}u|, \tag{1.5}$$

and

$$\{Y^j\}_{j=1}^K \subset \Gamma(\mathrm{T}\mathcal{N}), \quad |\mathrm{d}(Y^j(u))| \leq L|\mathrm{d}u|, \quad \text{and} \quad [Y^j(u)]_{BMO(B_1)} \leq L[u]_{BMO(B_1)}. \tag{1.6}$$

The proof of Theorem 1.7 now follows from the below, the proof of which is given in Section 3. We note that by translation in $\mathbb{R}^d$ we may assume that (1.4)-(1.6) remain true for $u - \overline{u}$ in place of u, where $\overline{u} = \frac{1}{|B_1|}\int_{B_1} u = 0$:

Theorem 1.8. *Let $1 \leq m, n$ and $B_1^m \subset \mathbb{R}^m$ the unit ball equipped with a smooth Riemannian metric g. Assume that $u \in W^{1,2}(B_1^m, \mathbb{R}^d)$ weakly solves*

$$\Delta_g u = X_j \cdot_g \mathrm{d}Y^j \quad \text{in } B_1^m$$

where $\{X_j\}$, $\{Y^j\}$ satisfy (1.5) and (1.6) and we assume additionally that $\int_{B_1} u = 0$. Then there exists $C = C(K, L, m, g, d)$ so that

$$\|\nabla^2 u\|_{L^1(B_{1/2})} \leq C\|\nabla u\|_{L^2}^2.$$

Furthermore there exits $\varepsilon = \varepsilon(K, L, m, g, d) > 0$ so that $[u]_{BMO(B_1)} < \varepsilon$ implies that

$$\|\nabla u\|_{L^\infty(B_{1/2})} \leq C\|u\|_{L^1(B_1)} \leq C[u]_{BMO(B_1)}.$$

Acknowledgements The authors were supported by the EPSRC grant EP/W026597/1.

2 Proof of Theorems 1.1 and 1.5

Given a C^2-Riemannian manifold $\mathcal{N} \hookrightarrow \mathbb{R}^d$, recall that $\tilde{\mathrm{T}} : \mathcal{N} \to \mathrm{End}(\mathrm{T}\mathbb{R}^d)$ is the projection onto $\mathrm{T}\mathcal{N}$ whilst $\tilde{\mathcal{V}} = \mathrm{Id} - \tilde{\mathrm{T}}$ is the projection onto the normal bundle $\mathcal{V}\mathcal{N}$. Given a map $u : B_1^m \to \mathcal{N}$ we set $\mathrm{T} = \tilde{\mathrm{T}} \circ u$, $\mathcal{V} = \tilde{\mathcal{V}} \circ u$.

Remark 2.1. We note trivially from the definition of T and $\mathcal{V}$ that $\mathrm{T} + \mathcal{V} \equiv \mathrm{Id}$ and so $\mathrm{dT} = -\mathrm{d}\mathcal{V}$, but also $\mathcal{V}\mathrm{dT}\mathcal{V} = 0 = \mathrm{TdTT}$ since:

$$\mathrm{T} = \mathrm{T}^2 \implies \mathrm{dT} = \mathrm{TdT}{+}\mathrm{dTT} \implies \mathrm{TdTT} = 2\mathrm{TdTT} \quad \text{similarly} \quad \mathcal{V}\mathrm{d}\mathcal{V}\mathcal{V} = 0 \implies \mathcal{V}\mathrm{dT}\mathcal{V} = 0.$$

In particular we may decompose dT into the sum of two parts, one which maps tangent vectors to normal, and its transpose:

$$\mathrm{dT} = \mathcal{V}\mathrm{dTT} + \mathrm{TdT}\mathcal{V} = \mathrm{dTT} + \mathrm{TdT}.$$

Lemma 2.2. *Let $\mathcal{N}^n \hookrightarrow \mathbb{R}^d$ be a C^2-Riemannian manifold and $u \in W^{1,2}(B_1, \mathcal{N})$ with ω defined by* (1.2)*. Then we may write ω in the following two equivalent ways:*

1. *Setting $\mathcal{R} = \mathrm{T} - \mathcal{V} \in W^{1,2}(B_1^m, \mathrm{O}(d))$ we have $\omega = \frac{1}{2}\mathcal{R}^{-1}\mathrm{d}\mathcal{R}$.*
2. *$\omega = \mathrm{TdT} - \mathrm{dTT}$.*

Remark 2.3. As may be seen in the work of Uhlenbeck [18, Section 8], or indeed checked directly, $\mathfrak{O} = \{O \in \mathrm{O}(d) : O^2 = \mathrm{Id}\}$ is a disjoint union of totally geodesic submanifolds of $\mathrm{O}(d)$ and that $\tilde{\mathcal{R}} : G(n, d) \to \mathrm{O}(d)$ given by $\tilde{\mathcal{R}}(E) = \Pi_E - \Pi_{E^\perp}$ may be considered a totally geodesic embedding onto $\mathfrak{O}_n = \{O \in \mathfrak{O} : \dim_{+1}(O) = n\}$ where $\dim_{+1}(O)$ is the dimension of the eigenspace corresponding to eigenvalue $+1$. Thus $\mathcal{R}$ above should be thought of as the composition of u with the Gauss map $G : \mathcal{N} \to G(n, d) = \mathfrak{O}_n$.

Remark 2.4. Note that the first point in Lemma 2.2 indicates immediately that the related connection $\nabla^{2\omega} = d + 2\omega$ is flat. We also note later that if we additionally assume that the tension field of u, $\tau(u) := \mathrm{T}\Delta u$ is weakly in L^1 then we may compute (see the Appendix), for any $v \in \mathbb{R}^d$:

$$-\mathrm{d}^*\omega v = -\mathrm{A}(v, \tau(u)) + \langle \mathrm{A}(\tau(u), \cdot)^\sharp, v\rangle - \nabla^{\mathcal{N}}_{\mathrm{d}u}\mathrm{A}(v, \mathrm{d}u) + \langle \nabla^{\mathcal{N}}_{\mathrm{d}u}\mathrm{A}(\mathrm{d}u, \cdot)^\sharp, v\rangle. \tag{2.1}$$

Thus if u is harmonic, and $\mathcal{N}$ is embedded via a parallel second fundamental form ($\nabla^{\mathcal{N}}\mathrm{A} \equiv 0$) then $\mathrm{d}^*\omega = 0$, too. Examples of such $\mathcal{N}$ include round spheres, $\mathrm{O}(d)$, $\mathrm{U}(d)$, and real and complex Grassmannians equipped with their induced metrics e.g. from the embedding $\tilde{\mathcal{R}}$ above.

Proof. For $z_0 \in \mathcal{N}$ fix an orthonormal frame $\{e_i(z)\}_{i=1}^n$ around $z_0 \in \mathcal{N}$ such that $\nabla^{\mathcal{N}}_{e_i} e_j(z_0) = D_{e_i} e_j(z_0) - \mathrm{A}(e_i, e_j) = 0$, where we denote by $\nabla^{\mathcal{N}}$ the Levi-Civita connection on $\mathcal{N}$ and by D the usual Euclidean derivative in $\mathbb{R}^d$. Observe that for all $v \in \mathbb{R}^d$, we have

$$\tilde{\mathrm{T}}(z)v = \sum_{i=1}^n \langle e_i(z), v\rangle e_i(z),$$

thus

$$\mathrm{d}\tilde{\mathrm{T}}(z)\,[e_j]\,v = \sum_{i=1}^d \langle D_{e_j} e_i(z), v\rangle e_i(z) + \langle e_i, v\rangle D_{e_j} e_i(z).$$

Evaluating this at z_0, we obtain

$$\mathrm{d}\tilde{\mathrm{T}}(z_0)\,[e_j]\,v = \sum_{i=1}^{n}\langle \mathrm{A}(e_i,e_j),v\rangle e_i + \langle e_i,v\rangle \mathrm{A}(e_i,e_j).$$

Now, if $X \in T_{z_0}\mathcal{N}$, using Einstein's summation convention, we can write $X = X^j e_j$, and the above formula can be expressed as

$$\mathrm{d}\tilde{\mathrm{T}}(z_0)\,[X]\,v = \sum_{i=1}^{n}\langle \mathrm{A}(e_i,X),v\rangle e_i + \mathrm{A}(v,X) = \langle \mathrm{A}(\cdot,X)^\sharp, v\rangle + \mathrm{A}(v,X). \tag{2.2}$$

Thus

$$\mathrm{dT}(x)v = \langle \mathrm{A}(\cdot,\mathrm{d}u)^\sharp, v\rangle + \mathrm{A}(v,\mathrm{d}u)$$

and clearly

$$\mathrm{dT}(\mathrm{T}v) = \mathrm{A}(v,\mathrm{d}u) \qquad \text{and} \qquad \mathrm{TdT}v = \langle \mathrm{A}(\cdot,\mathrm{d}u)^\sharp, v\rangle$$

which, by comparison with (1.3) gives $\omega v = \mathrm{TdT}v - \mathrm{dTT}v$ for all $v \in \mathbb{R}^d$, proving the second claim.
Now, since T and $\mathcal{V}$ are projections, it is straightforward to verify that $\mathcal{R}^T = \mathcal{R}$ and $\mathcal{R}^2 = \mathrm{Id}$, which implies $\mathcal{R}^{-1} = \mathcal{R}^T = \mathcal{R}$. As a consequence, we have using Remark 2.1

$$\begin{aligned}\frac{1}{2}\mathcal{R}^{-1}\mathrm{d}\mathcal{R} = \frac{1}{2}\mathcal{R}\mathrm{d}\mathcal{R} = \frac{1}{2}\mathcal{R}\left(\mathrm{dT} - \mathrm{d}\mathcal{V}\right) = \mathcal{R}\mathrm{dT} = \left(\mathrm{T} - \mathcal{V}\right)\mathrm{dT} = \mathrm{TdT} - \mathcal{V}\mathrm{dT}\\ = \mathrm{TdT} - \mathrm{dTT}\end{aligned}$$

since $\mathcal{V}\mathrm{dT} = \mathrm{dTT}$.

□

The connection ∇^ω on $u^*\mathrm{T}\mathbb{R}^d$ induces also connection on $u^*(\mathrm{End}(\mathrm{T}\mathbb{R}^d)$ via

$$\nabla^\omega E = \mathrm{d}E + [\omega, E] \qquad \text{for } E \in \Gamma(u^*\mathrm{End}(\mathrm{T}\mathbb{R}^d))$$

from which it is straightforward to check:

Lemma 2.5. $\nabla^\omega \mathrm{T} = 0$, $\nabla^\omega \mathcal{V} = 0$ *and* $\nabla^\omega \mathcal{R} = 0$.

Proof. We will check only the first claim, with the others following trivially from this and Remark 2.1. Using part 2. of Lemma 2.2 and Remark 2.1 we have

$$[\omega, T] = \omega\mathrm{T} - \mathrm{T}\omega = (\mathrm{TdT} - \mathrm{dTT})\mathrm{T} - \mathrm{T}(\mathrm{TdT} - \mathrm{dTT}) = -\mathrm{dTT} - \mathrm{TdT} = -\mathrm{dT}$$

as required. □

Proof of Theorem 1.1 and Theorem 1.5. In the case of Theorem 1.5 we note that Remark 2.1 and Lemma 2.5 holds for Π, $\Pi^\perp$, and ω_Π respectively, and indeed the below proof works by replacing T with Π, $\mathcal{V}$ with $\Pi^\perp$, ω with ω_Π and $\mathcal{R}$ with $\Pi - \Pi^\perp$.
Using the Morrey-space version of the Coulomb frame proven by Rivière-Struwe [14, Lemma 3.1] we know that by choosing $\varepsilon = \varepsilon(m,d) > 0$ sufficiently small there are $P \in H^1(B_1^m, \mathrm{SO}(d))$ and $\xi \in H_0^1(B_1^m, \mathfrak{so}(d) \otimes \wedge^2\mathrm{T}^*\mathbb{R}^m)$ such that

$$P^{-1}\mathrm{d}P + P^{-1}\omega P = \mathrm{d}^*\xi, \quad \text{and} \quad \mathrm{d}\xi = 0 \quad \text{on } B_1^m. \tag{2.3}$$

Moreover, $\mathrm{d}P$ and $D\xi$ belong to $M^{2,m-2}(B_1^m)$ with

$$\|\mathrm{d}P\|_{M^{2,m-2}} + \|D\xi\|_{M^{2,m-2}} \leq C\|\omega\|_{M^{2,m-2}} \leq C\varepsilon. \tag{2.4}$$

In the above $D\xi$ denotes the collection of all first order derivatives of all components of ξ which we distinguish from $\mathrm{d}\xi$ which is simply the exterior derivative of the two-form.
The gauge-invariant version of Lemma 2.5 tells us that

$$\nabla^{\mathrm{d}^*\xi}(P^{-1}\mathrm{T}P) = \mathrm{d}(P^{-1}\mathrm{T}P) + [\mathrm{d}^*\xi, P^{-1}\mathrm{T}P] = 0,$$

similarly $\nabla^{\mathrm{d}^*\xi}(P^{-1}\mathcal{V}P) = 0$ and $\nabla^{\mathrm{d}^*\xi}(P^{-1}\mathcal{R}P) = 0$, as may be checked directly using (2.3).
In particular letting $Q = P^{-1}\mathcal{R}P$ then we have $\mathrm{d}Q = [Q, \mathrm{d}^*\xi]$ giving (where we always sum over repeated indices)

$$\begin{aligned}
|\mathrm{d}Q|^2 &= \mathrm{tr}([Q,\mathrm{d}^*\xi]\cdot \mathrm{d}Q^T) = *(Q^i_k \mathrm{d}^*\xi^k_l \wedge *\mathrm{d}Q^i_l - \mathrm{d}^*\xi^k_l Q^l_i \wedge *\mathrm{d}Q^k_i) \\
&= -*2(\mathrm{d}*\xi^k_l \wedge Q^i_k \mathrm{d}Q^i_l) \\
&= -*2\mathrm{d}(*\xi^k_l \wedge Q^i_k \mathrm{d}Q^i_l) + 2(-1)^{m-2} * (*\xi^k_l \wedge (\mathrm{d}Q^i_k \wedge \mathrm{d}Q^i_l)).
\end{aligned}$$

We may extend ξ by zero to the whole of $\mathbb{R}^m$ without relabelling, and recall that $[\xi]_{BMO(\mathbb{R}^m)} \leq C\|D\xi\|_{M^{2,m-2}(\mathbb{R}^m)}$ in view of the Poincaré inequality. We also extend $Q - \overline{Q}$ to a $W^{1,2}$ function $\tilde{Q}$ on the whole of $\mathbb{R}^m$ so that $\|\mathrm{d}\tilde{Q}\|_{L^2} \leq C\|\mathrm{d}Q\|_{L^2(B_1)}$ and $\mathrm{d}\tilde{Q} = \mathrm{d}Q$ in B_1. By $\mathcal{H}^1$-BMO duality [6] and the fundamental results of Coiffman-Lions-Mayers-Semmes [4] we have, up to a sign which is irrelevnat

$$\begin{aligned}
\|\mathrm{d}Q\|^2_{L^2(B_1)} &= 2\int *\xi^k_l \wedge (\mathrm{d}\tilde{Q}^i_k \wedge \mathrm{d}\tilde{Q}^i_l) \\
&\leq 2[\xi^k_l]_{BMO}\|\mathrm{d}\tilde{Q}^i_k \wedge \mathrm{d}\tilde{Q}^i_l\|_{\mathcal{H}^1} \\
&\leq C\|D\xi\|_{M^{2,m-2}}\|\mathrm{d}Q\|^2_{L^2(B_1)} \\
&\leq C\varepsilon\|\mathrm{d}Q\|^2_{L^2(B_1)}.
\end{aligned}$$

For ε sufficiently small, Q is a constant. Now if $\{E_i\}$ and $\{N_j\}$ is an orthonormal basis $\mathrm{T}_{u(0)}\mathcal{N}$ and $\mathcal{V}_{u(0)}\mathcal{N}$ then we
Claim: $e_i(x) := P(x)P(0)^T E_i$ and $\nu_j(x) := P(x)P(0)^T N_j$ are orthonormal moving frames for $u^*(\mathrm{T}\mathcal{N})$ and $u^*(\mathcal{V}\mathcal{N})$ respectively as required by the Theorem.
Clearly both $\{e_i\}$ and $\{\nu_j\}$ are orthonormal frames satisfying the desired estimate, by (2.4). We will see below that $\mathcal{R}(x)e_i(x) = (\mathrm{T}(x) - \mathcal{V}(x))e_i(x) = e_i(x)$ meaning that $e_i(x) \in \mathrm{T}_{u(x)}\mathcal{N}$ almost everywhere, and similarly $\mathcal{R}(x)\nu_j(x) = -\nu_j(x)$ meaning that $\nu_j(x) \in \mathcal{V}_{u(x)}\mathcal{N}$ almost everywhere. Indeed:

$$\begin{aligned}
\mathcal{R}(x)e_i(x) &= \mathcal{R}(x)P(x)P(0)^T E_i = P(x)(P^T(x)\mathcal{R}(x)P(x))(P(0)^T E_i) \\
&= P(x)(P(0)^T\mathcal{R}(0)P(0))P(0)^T E_i = e_i(x)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{R}(x)\nu_j(x) &= \mathcal{R}(x)P(x)P(0)^T N_j = P(x)(P^T(x)\mathcal{R}(x)P(x))(P(0)^T N_j) \\
&= P(x)(P(0)^T\mathcal{R}(0)P(0))P(0)^T N_j = -\nu_j(x).
\end{aligned}$$

It remains to check that $\mathrm{d}^*(e_i \cdot \mathrm{d}e_j) = 0$ and similarly for $\{\nu_j\}$. We do this for the former leaving the latter to the reader. Assume w.l.o.g. that $P(0) = \mathrm{Id}$ and note that

$$e_i \cdot \mathrm{d}e_j = PE_i \cdot (\mathrm{d}P)E_j = E_i \cdot P^{-1}\mathrm{d}PE_j = E_i \cdot \mathrm{d}^*\xi E_j - E_i \cdot (P^{-1}\omega P)E_j.$$

The result follows from $E_i \cdot (P^{-1}\omega P)E_j = e_i \cdot \omega e_j = 0$ since ωe_j is a normal vector by Remark 2.1. □

Remark 2.6. When $m=2$ one finds $P \in H^1(B_1^m, \mathrm{SO}(d))$ and $\xi \in H_0^1(B_1^m, \mathfrak{so}(d))$ solving

$$P^{-1}\mathrm{d}P + P^{-1}\omega P = *\mathrm{d}\xi \quad \text{on } B, \quad \text{with} \quad \|\mathrm{d}P\|_{L^2} \leq 2\|\omega\|_{L^2} \quad \text{and} \quad \|\mathrm{d}\xi\|_{L^2} \leq \|\omega\|_{L^2}$$

regardless of the size of ε (see e.g. [15, Theorem 8.4]). The optimal L^2 Wente estimate on dics [7] then gives $\|\mathrm{d}Q\|_{L^2}^2 \leq \sqrt{\frac{3}{4\pi}}\|\omega\|_{L^2}\|\mathrm{d}Q\|_{L^2}^2$ which means that $\varepsilon = \sqrt{\frac{4\pi}{3}}$ is sufficient.

3 Proof of Theorem 1.7

Proof of Theorem 1.8. For simplicity we will give the proof only in the case that g is the Euclidean metric, leaving the required changes necessary for the general case to the interested reader.
The first estimate on $\|\nabla^2 u\|_{L^1(B_{1/2})}$ is standard - by suitably extending X_j and Y^j similarly to the below one may use e.g. [9, Theorem 3.2.9].
For the BMO-regularity part we begin by noticing that, from standard L^2-Hodge theory (see for instance [10, Corollary 10.5.1]), there exists $\xi_j \in W^{1,2}(B_1^m, \wedge^2 \mathrm{T}^*\mathbb{R}^m)$ such that $X_j = \mathrm{d}^*\xi_j$, each component of ξ_j has mean zero in B_1 and $\|D\xi_j\|_{L^2(\mathbb{R}^m)} \leq \|X_j\|_{L^2(B_1)}$. We extend ξ_j to $\tilde{\xi}_j \in W^{1,2}(\mathbb{R}^m, \wedge^2\mathrm{T}^*\mathbb{R}^m)$ and for which we still have

$$\|D\tilde{\xi}_j\|_{L^2} \leq C\|X_j\|_{L^2(B_1)} \leq C\|\mathrm{d}u\|_{L^2(B_1)}.$$

We similarly extend each $Y^j - \overline{Y}^j$ to $\tilde{Y}^j$, defined on the whole of $\mathbb{R}^m$ and for which we have $\mathrm{d}\tilde{Y}^j = \mathrm{d}Y^j$ in B_1 and (see [11])

$$[\tilde{Y}^j]_{BMO(\mathbb{R}^m)} \leq C[Y^j]_{BMO(B_1)} \leq C[u]_{BMO(B_1)}.$$

Testing the equation with $\phi \in C_c^\infty(B_1^m, \mathbb{R}^d)$ and applying the divergence theorem, we obtain

$$\begin{aligned}\int X_j \cdot \mathrm{d}Y^j\phi &= \int \mathrm{d}^*\tilde{\xi}_j \cdot \mathrm{d}\tilde{Y}^j\phi = -\int_{B_1} (\mathrm{d}^*\tilde{\xi}_j \cdot \mathrm{d}\phi)\tilde{Y}^j \\ &= \int (\mathrm{d} * \tilde{\xi}_j \wedge \mathrm{d}\phi)\tilde{Y}^j.\end{aligned}$$

Now, [4] gives that

$$\|(\mathrm{d} * \tilde{\xi}_j \wedge \mathrm{d}\phi)\|_{\mathcal{H}^1(\mathbb{R}^m)} \leq C\|D\tilde{\xi}_j\|_{L^2(\mathbb{R}^m)}\|\mathrm{d}\phi\|_{L^2(B_1)} \leq C\|\mathrm{d}u\|_{L^2(B_1)}\|\mathrm{d}\phi\|_{L^2(B_1)}.$$

Combined with the $\mathcal{H}^1 - BMO$ duality we have, for all $\phi \in C_c^\infty(B_1^m, \mathbb{R}^d)$

$$\begin{aligned}\int X_j \cdot \mathrm{d}Y^j\phi &\leq C\|\mathrm{d}u\|_{L^2(B_1)}\|\mathrm{d}\phi\|_{L^2(B_1)}[\tilde{Y}^j]_{BMO(\mathbb{R}^m)} \\ &\leq C\|\mathrm{d}u\|_{L^2(B_1)}[u]_{BMO(B_1)}\|\mathrm{d}\phi\|_{L^2(B_1)},\end{aligned}$$

giving $\|X_j \cdot \mathrm{d}Y^j\|_{H^{-1}(B_1)} \leq C\|\mathrm{d}u\|_{L^2(B_1)}[u]_{BMO(B_1)}$. Now letting $v \in W_0^{1,2}$ be the unique solution to $\Delta v^i = X_j \cdot \mathrm{d}Y^j$ we have

$$\|v\|_{L^2(B_1)}^2 + \|\mathrm{d}v\|_{L^2(B_1)}^2 \leq C\|\mathrm{d}u\|_{L^2(B_1)}^2[u]_{BMO(B_1)}^2. \tag{3.1}$$

We now may essentially follow some of the arguments as in [16] to complete the proof which we outline briefly:

Write $u = h + v$ for h a harmonic function. We first observe that $\|h\|_{L^1(B_1)} \leq C(\|u\|_{L^1(B_1)} + \|v\|_{L^1(B_1)}) \leq C(\|u\|_{L^1(B_1)} + \|\mathrm{d}u\|_{L^2(B_1)}[u]_{BMO(B_1)})$, whence one trivially has that

$$\|\mathrm{d}h\|^2_{L^2(B_{1/2})} \leq C(\|u\|^2_{L^1(B_1)} + \|\mathrm{d}u\|^2_{L^2(B_1)}[u]^2_{BMO(B_1)}).$$

Thus in particular,

$$\|\mathrm{d}u\|^2_{L^2(B_{1/2})} \leq C(\|\mathrm{d}u\|^2_{L^2(B_1)}[u]^2_{BMO(B_1)} + \|u\|^2_{L^1(B_1)})$$

and by repeating the argument above on each ball $B_{2R}(x) \subset B_1$, and by the scaling properties of each quantity we see that

$$\begin{aligned}\|\mathrm{d}u\|^2_{L^2(B_{R/2(x)})} &\leq C(\|\mathrm{d}u\|^2_{L^2(B_R(x))}[u]^2_{BMO(B_1)} + R^{-m-2}\|u\|^2_{L^1(B_1)}) \\ &\leq C\varepsilon^2\|\mathrm{d}u\|^2_{L^2(B_R(x))} + CR^{-m-2}\|u\|^2_{L^1(B_1)}.\end{aligned}$$

Hence exactly as equation (23) in [16] is derived using [16, Lemma A.7] we may obtain that, for ε sufficiently small

$$\|\mathrm{d}u\|_{L^2(B_{7/8})} \leq C\|u\|_{L^1(B_1)}. \tag{3.2}$$

Observe now that we can write $\mathrm{d}u = H + \mathrm{d}v$ where $H = \mathrm{d}h$ is a harmonic one-form and

$$\|\mathrm{d}u\|^2_{L^2(B_1)} = \|H\|^2_{L^2(B_1)} + \|\mathrm{d}v\|^2_{L^2(B_1)}.$$

Since H is harmonic, the quantity $r^{-m}\|H\|^2_{L^2(B_r)}$ is increasing, which implies

$$\begin{aligned}\|H\|^2_{L^2(B_r)} &\leq r^m\|H\|^2_{L^2(B_1)} \\ &\leq r^m\|\mathrm{d}u\|^2_{L^2(B_1)}.\end{aligned} \tag{3.3}$$

for all $0 < r \leq 1$. Using equations (3.1) and (3.3), for any $\delta > 0$, we have

$$\begin{aligned}\|\mathrm{d}u\|^2_{L^2(B_r)} &\leq (\|H\|_{L^2(B_r)} + \|\mathrm{d}v\|_{L^2(B_r)})^2 \\ &\leq (1+\delta)\|H\|^2_{L^2(B_r)} + C_\delta\|\mathrm{d}v\|^2_{L^2(B_1)} \\ &\leq (1+\delta)r^m\|\mathrm{d}u\|^2_{L^2(B_1)} + C_\delta\|\mathrm{d}u\|^2_{L^2}[u]^2_{BMO} \\ &= \left((1+\delta)r^m + C_\delta[u]^2_{BMO}\right)\|\mathrm{d}u\|^2_{L^2(B_1)} \\ &< \left((1+\delta)r^m + C_\delta\varepsilon^2\right)\|\mathrm{d}u\|^2_{L^2(B_1)}\end{aligned} \tag{3.4}$$

where we also used Young's inequality. At this point, given $\alpha \in (0,2)$ to be determined later, choose $\delta, \varepsilon \in (1,2]$ sufficiently small so that

$$\frac{(1+2\delta)}{2^m} + C_\delta\varepsilon^2 = 2^{\alpha-m}.$$

For any $x \in B_{3/4}$, $\varrho = 7/8 - |x| > 1/8$, and any $r \leq \rho$ we may write $2^{-k-1}\varrho \leq r \leq 2^{-k}\varrho$. Deriving (3.4) on $B_\varrho(x)$ and iterating gives (also from (3.2))

$$\|\mathrm{d}u\|^2_{L^2(B_r(x))} \leq \left(2^{-k+1}\right)^{m-\alpha}\|\mathrm{d}u\|^2_{L^2(B_\varrho(x))} \leq 32^{m-\alpha}r^{m-\alpha}\|\mathrm{d}u\|^2_{L^2(B_{7/8})} \leq Cr^{m-\alpha}\|u\|^2_{L^1(B_1)}.$$

The above estimate gives

$$\|\mathrm{d}u\|_{M^{2,m-\alpha}(B_{3/4})} \leq C\|u\|_{L^1(B_1)}.$$

As a consequence, since u solves equation (1.4) and recalling (1.5) and (1.6), we obtain that

$$\|\Delta u\|_{M^{1,m-\alpha}(B_{3/4})} \leq C\|u\|^2_{L^1(B_1)}$$

and thus, fixing α such that α such that $1 < \alpha < 2m/(2m-1)$, the Riesz potential estimates of Adams [1] gives $\mathrm{d}u \in L^{\tilde{p}}(B_{5/8})$ for $\frac{1}{\tilde{p}} = 1 - \frac{1}{\alpha}$. Thus for some $q > 2m$,

$$\|\mathrm{d}u\|_{L^q(B_{5/8})} \leq C\|u\|_{L^1(B_1)}.$$

Going back to (1.5) and (1.6) we obtain $\|\Delta u\|_{L^s(B_{5/8})} \leq C\|u\|^2_{L^1(B_1)}$ for some $s > m$ from which Calderon-Zygmund estimates and Sobolev embedding finishes the proof. □

A Sketch proof of (2.1)

Proof. Given $X, Y \in \Gamma(\mathrm{T}\mathcal{N})$ and assuming that $\nabla^{\mathcal{N}}_X Y = \nabla^{\mathcal{N}}_Y X = 0$ at z_0 we have

$$D^2\tilde{\mathrm{T}}[X,Y]v = D_Y(D_X\tilde{\mathrm{T}})v$$

and starting from (2.2) we have, for any $v \in \mathbb{R}^d$

$$\begin{aligned} D^2\tilde{\mathrm{T}}[X,Y]v &= D_Y(\langle \mathrm{A}(\cdot,X)^\sharp, v\rangle + \mathrm{A}(v,X)) \\ &= \langle D_Y(\mathrm{A}(\cdot,X)^\sharp), v\rangle + D_Y(\mathrm{A}(v,X)) \\ &= \langle \nabla^{\mathcal{N}}_Y \mathrm{A}(\cdot,X)^\sharp, v\rangle - \langle \mathrm{A}(\cdot,X)^\sharp, \mathrm{A}(Y,v)\rangle + \nabla^{\mathcal{N}}_Y \mathrm{A}(v,X) - \langle A(v,X), A(Y,\cdot)^\sharp\rangle \end{aligned}$$

where we have used that if $\eta \in \Gamma(\mathcal{VN})$ then $D_Y\eta = \nabla^{\mathcal{N}}_Y \eta - \langle \eta, \mathrm{A}(Y,\cdot)^\sharp\rangle$.
Thus we have that

$$\begin{aligned} \mathrm{d}^*\omega v &= \Delta \mathrm{T}\mathrm{T}v - \mathrm{T}\Delta\mathrm{T}v \\ &= \mathrm{d}\tilde{\mathrm{T}}[\mathrm{T}\Delta u]\mathrm{T}v - \mathrm{T}\mathrm{d}\tilde{\mathrm{T}}[\mathrm{T}\Delta u]v + D^2\tilde{\mathrm{T}}[\mathrm{d}u,\mathrm{d}u]\mathrm{T}v - \mathrm{T}D^2\tilde{\mathrm{T}}[\mathrm{d}u,\mathrm{d}u]v \\ &= \mathrm{A}(v,\tau(u)) - \langle \mathrm{A}(\tau(u),\cdot)^\sharp, v\rangle + \nabla^{\mathcal{N}}_{\mathrm{d}u}\mathrm{A}(v,\mathrm{d}u) - \langle \nabla^{\mathcal{N}}_{\mathrm{d}u}\mathrm{A}(\mathrm{d}u,\cdot)^\sharp, v\rangle \end{aligned}$$

using the above, and (2.2) again.

□

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L. Appolloni: *l.appolloni@leeds.ac.uk*
B. Sharp: *b.g.sharp@leeds.ac.uk*
School of Mathematics, University of Leeds, Leeds LS2 9JT, UK