Minimal Model Renormalization Group Flows: Noninvertible Symmetries and Nonperturbative Description

Federico Ambrosino[1,*] and Stefano Negro[2,†]
[1]Deutsches Elektronen-Synchrotron DESY, Notkestrasse 85, 22607 Hamburg, Germany
[2]Department of Mathematics, University of York, Heslington, York YO10 5DD, United Kingdom

(Received 27 January 2025; revised 16 June 2025; accepted 18 June 2025; published 8 July 2025)

In this Letter we continue the investigation of RG flows between Virasoro minimal models of two-dimensional conformal field theories that are protected by noninvertible symmetries. RG flows leaving unbroken a subcategory of noninvertible symmetries are associated with anomaly matching conditions that we employ systematically to map the space of flows between minimal models beyond the $\mathbb{Z}_2$-symmetric proposed recently in the literature. We introduce a family of nonlinear integral equations that appear to encode the exact finite-size, ground-state energies of these flows, including nonintegrable cases, such as the recently proposed $\mathcal{M}(kq+I,q) \to \mathcal{M}(kq-I,q)$. Our family of NLIEs encompasses and generalizes the integrable flows known in the literature: $\phi_{(1,3)}$, $\phi_{(1,5)}$, $\phi_{(1,2)}$ and $\phi_{(2,1)}$. This work uncovers a new interplay between exact solvability and noninvertible symmetries. Furthermore, our nonperturbative description provides a nontrivial test for all the flows conjectured by anomaly matching conditions, but so far not observed by other means.

DOI: 10.1103/dg1s-5vp6

Introduction—The systematic identification of renormalization group (RG) flows between quantum field theories is a paramount problem in theoretical physics. Global symmetries are central to this quest, providing nonperturbative constraints on the RG flows between the ultraviolet (UV) and infrared (IR) fixed points, and dictating the allowed interactions generated along the flows. By matching their anomalies, we can put strong constraints on the IR theory. Recently, building on the seminal paper [1], a profound effort has been devoted to exploring generalizations to the usual notion of global symmetries, such as *higher-form*, *noninvertible*, or more general *higher-categorical* symmetries, extending the usual *grouplike* structures to the more general algebraic ones of fusion higher categories (for recent reviews see [2,3]). In two-dimensional conformal field theories (CFT), noninvertible symmetries are ubiquitous [4–7]: topological line operators, acting as generators of 0-form symmetries, do not form generically a group but a fusion category. In the case of rational 2d CFTs with diagonal modular invariance, i.e., the Virasoro minimal models $\mathcal{M}(p,q)$, the set of topological line operators coincides with the finitely many Verlinde line defect, forming a *fusion modular category* [4,8]. Hence, the study of RG flows from a minimal model provides a unique arena where we have a complete understanding of the full set of *categorical* symmetries of the UV theory, and it has indeed recently received considerable attention [9–16]. This approach was first undertaken in [4,7] and, more recently, in [17], where the authors predict infinitely many new RG flows between minimal models: $\mathcal{M}(kq+I,q) \to \mathcal{M}(kq-I,q)$, preserving a special A_{q-1} fusion category containing the standard $\mathbb{Z}_2$ symmetry.

Some specific deformations of minimal models are *integrable*, meaning the scattering events are factorized and the two-body S matrix satisfies the Yang-Baxter equation [18–20]. Integrable flows allow for an exact, nonperturbative description through the thermodynamic Bethe ansatz (TBA) equations [21]—equivalently, a nonlinear integral equation (NLIE) [22,23]—encoding their exact, finite-size energy spectrum.

Here, we extend the investigation of RG flows between minimal models predicted by anomaly matching conditions associated with noninvertible symmetries. We also present evidence that the ground-state energy of all these RG flows—not just the integrable ones—admits an explicit NLIE description. We base this statement on the observation that a three-parameter family of NLIE encodes nontrivial features of the RG flows predicted by anomaly-matching conditions. In particular, for multioperator deformations—which is the case for most of the RG flows we looked at—the scaling function obtained from the NLIEs shows clear

*Contact author: federicoambrosino25@gmail.com
†Contact author: stefano.negro@york.ac.uk

signs of multiscale behavior in the UV, with exponents agreeing with the operators predicted by the anomaly matching conditions. Additionally, for the deformations triggered by the $\phi_{(1,3)}$, $\phi_{(1,5)}$, $\phi_{(2,1)}$, and $\phi_{(1,2)}$ operators, the kernels of the NLIEs reduce to the known ones [24–31]. This is substantial evidence supporting the interpretation of the NLIEs as a universal description for RG flows between minimal models. Definitive evidence will come from accurate numerical investigation and by comparison against conformal perturbation theory [21] or Hamiltonian truncation [32]. We will embark on this project in the near future. The NLIEs are a potential nonperturbative description for all the flows $\mathcal{M}(kq+I,q)\xrightarrow{\phi_{(1,2k+1)}}\mathcal{M}(kq-I,q)$ conjectured in [17], providing an explicit proof of their existence. They greatly expand the class of flows that can be studied nonperturbatively, even in the absence of a known integrable structure.

Minimal models RG flows—Consider a UV fixed point described by a Virasoro minimal model $\mathcal{T}_{\mathrm{UV}} = \mathcal{M}(p,q)$. Basic notions about Virasoro minimal models, fixing the conventions used in this Letter, may be found in Appendix A in Supplemental Material [33]. In this Letter, we are interested in studying deformations of the UV theory by one of its relevant ($h_{(r,s)} < 1$) primary fields:

$$\mathcal{H}_{\mathcal{M}(p,q)} + g_{(r,s)} \int \mathrm{d}x \phi_{(r,s)}, \tag{1}$$

where $\mathcal{H}_{\mathcal{M}(p,q)}$ is the Hamiltonian of the minimal model $\mathcal{M}(p,q)$. The IR fixed point at the end of this RG flow may be either *gapped* or *gapless*. The former case is typical for generic deformations, having an IR described by a topological quantum field theory (TQFT). We will not consider these in our analysis, albeit they can be studied with techniques similar to those discussed here [4,7,17,34–37]. We refer the reader to the discussion at the end of this Letter for comments on this matter. In the latter case, when the IR theory is gapless, we assume here it may be described by another minimal model itself:

$$\mathcal{T}_{\mathrm{UV}} = \mathcal{M}(p,q)\xrightarrow{\phi_{(r,s)}}\mathcal{M}(p',q') = \mathcal{T}_{\mathrm{IR}}. \tag{2}$$

An important constraint on $\mathcal{T}_{\mathrm{IR}}$ is given by the c_{eff}-theorem: along RG flows between $\mathcal{PT}$-symmetric nonunitary CFT, the effective central charge

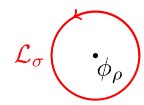
$$c_{\mathrm{eff}}(p,q) = 1 - \frac{6}{pq} \tag{3}$$

is monotonically decreasing [38], and reduces to the usual Zamolodchikov c theorem [39] for the case of unitary CFT. We will assume that $\mathcal{PT}$ symmetry is always preserved along our flows, as tested by now in all the examples considered in the literature [10–12,40,41], where it has been observed that CFT transition happens precisely at the spontaneous $\mathcal{PT}$ breaking locus. Furthermore, $\mathcal{PT}$ symmetry guarantees the reality of the energy spectrum at finite volume (and therefore of conformal dimensions) along the entire flow down to the IR CFT, as is the case for the nonunitary minimal models. Stringent constraints follow from the noninvertible symmetry lines of the minimal models. We will describe a very general strategy we plan to employ also for more general UV fixed points in future work.

Whenever, for any state on the cylinder $|\Phi\rangle$, the line $\mathcal{L}_\sigma$ commutes with the deformation triggering the RG flow,

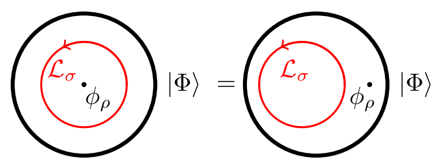
$$[\mathcal{L}_\sigma, \phi_{(r,s)}]|\Phi\rangle = 0, \tag{4}$$

then the line operator $\mathcal{L}_\sigma$ is unbroken by the deformation. The maximal subcategory $\{\mathcal{L}_\sigma\}^{(r,s)}_{\mathrm{UV}} \subset \mathcal{V}_{(p,q)}$ of Verlinde lines commuting with the deformation is closed under fusion and generates the symmetry that is preserved along the RG flow. Using the fusion rules and Verlinde line action on the primary fields,

$$\mathcal{L}_\sigma\,|\phi_\rho\rangle =$$


$$= \frac{S_{\sigma\rho}}{S_{0\rho}}\,|\phi_\rho\rangle\,, \tag{5}$$

(4) is turned into a trigonometric equation for the label σ, at fixed (r,s). In particular, it implies that the quantum dimension of the preserved lines is an RG flow invariant,



(6)

as well as the spin content of the defect Hilbert spaces associated with the preserved lines $\mathcal{H}_{\mathcal{L}_\sigma}$ [42]. These two pieces of RG-invariant categorical data are to be considered as 't Hooft anomaly matching conditions in the realm of fusion categories. To explore the possible $\mathcal{T}_{\mathrm{IR}}$ we proceed as follows: (1) Given $\mathcal{T}_{\mathrm{UV}}$, for any relevant primary field $\phi_{(r,s)}$ of $\mathcal{T}_{\mathrm{UV}}$, we compute the fusion subcategory [43] of Verlinde lines $\{\mathcal{L}_\sigma\}^{(r,s)}_{\mathrm{UV}}$ commuting with the perturbation via Eq. (4). (2) We generate a list of the possible minimal models $\mathcal{T}_{\mathrm{IR}}$ that satisfy $c_{\mathrm{eff}}(\mathcal{T}_{\mathrm{UV}}) > c_{\mathrm{eff}}(\mathcal{T}_{\mathrm{IR}})$. This list is always finite. (3) We select among the $\mathcal{T}_{\mathrm{IR}}$ determined above, only the minimal models containing a fusion subcategory $\{\mathcal{L}_\rho\}_{\mathrm{IR}}$ of Verlinde lines coinciding with $\{\mathcal{L}_\sigma\}^{(r,s)}_{\mathrm{UV}}$. This means that all the quantum dimensions, fusion rules, and spins in the defect Hilbert spaces in these two subcategories coincide with $\{\mathcal{L}_\sigma\}^{(r,s)}_{\mathrm{UV}}$.

This, for any given (p,q) produces a list of candidate flows of the form (2) fulfilling all the anomaly-matching conditions by construction. This procedure shall be regarded as exclusive rather than inclusive, meaning that anomaly matching does not guarantee that the flow will dynamically exist. Generically more than a single relevant operator may trigger the same flow. If the set of operators triggering the flow also preserves the same fusion subcategory of lines, then along the flow all such operators (and all the Virasoro descendants thereof) may be dynamically generated. The critical point in the IR will be hit by fine-tuning a combination of the UV deformations; refer to the Appendix B in Supplemental Material [33] for a more detailed illustration. Lastly, the existence of a gapless flow triggered by a given relevant operator does not exclude the existence of gapped phases for other critical couplings. For example, the $\phi_{(1,3)}$ perturbation of the tricritical Ising model $\mathcal{M}(5,4)$ flows to either a gapped phase or to $\mathcal{M}(4,3)$ depending on the sign of the perturbation. One can also determine which fields govern the approach entering direction of the flow to the IR. Indeed, given that the topological lines are preserved along the flow, one can analogously determine which are the most relevant irrelevant operators of $\mathcal{T}_{\mathrm{IR}}$, commuting with the same subcategory of the UV theory.

Note that those may also be Virasoro descendants of relevant primaries. Often the most relevant irrelevant among the operators is $T\bar{T}$; that is always a viable direction given that it commutes with all the topological lines. In this case, the flow either enters along the $T\bar{T}$ direction [44] or along the least irrelevant descendant of a (relevant) primary that has commutations relations with the preserved topological lines and RG invariants consistent with the UV data. From this perspective, the noninvertible symmetries also provide a strong organizing principle for the effective field theory (or conformal perturbation theory) expansion around both the UV and IR. The procedure is easily automatized and implemented in *Mathematica*. In Fig. 1 we report the outcome up to $p = 8$, but it can be readily extended to any value of p [45]. Among others, we reproduce all the $\mathbb{Z}_2$ symmetric flows conjectured in [17], as well as the $\phi_{(1,2)}$, and $\phi_{(2,1)}$ flows known to be

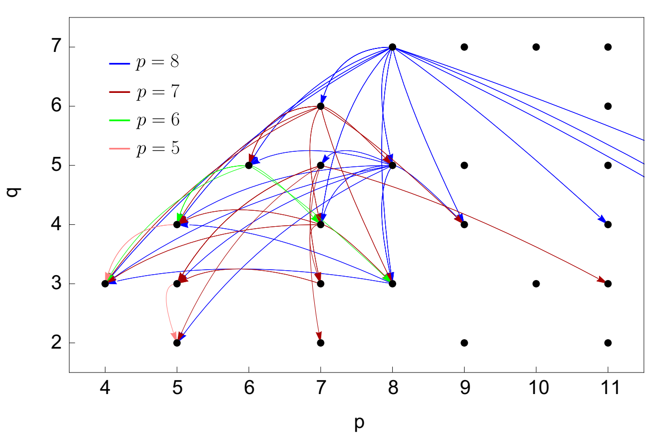


FIG. 1. Flows between minimal models determined by our algorithm $\mathcal{T}_{\mathrm{UV}} = \mathcal{M}(p \leq 8, q)$.

integrable [31,46], but not belonging to that family. In addition, we find new flows that are not part of the known families but are allowed by anomaly matching. An example is $\mathcal{M}(7,5) \xrightarrow{\phi_{(2,3)}} \mathcal{M}(11,3)$ discussed in the Appendix C in Supplemental Material [33] (together with a detailed discussion up to $p = 7$). An interesting case is the integrable flows with $\phi_{(1,5)}$, while $\phi_{(1,3)}$ could a priory be dynamically generated along the flow, the solutions of the NLIEs suggest that this is not the case. We plan to study in detail the relation between integrability and noninvertible symmetries somewhere else.

Description via NLIE—Since the seminal article [47], it was shown that certain special perturbations of minimal models could be described as *quantum reductions* of integrable quantum field theories. Specifically, perturbations controlled by the relevant field $\phi_{(1,3)}$ are obtained as quantum reductions of the sine-Gordon (sG) model [24,25,48–50], while perturbations by the fields $\phi_{(1,5)}$, $\phi_{(2,1)}$, and $\phi_{(1,2)}$ arise from the quantum reduction of the Zhiber-Mikhailov-Shabat (ZMS) model [46,51–53]. Thanks to their integrability, it has been possible to derive a nonlinear integral equation (NLIE) that *nonperturbatively* encodes the energies $E_{\mathrm{s}}(R)$ of any state s on a cylinder of radius R, which acts as an RG parameter. As functions of $r = Rm$, with m being the mass scale of the system, the energies interpolate between the UV regime

$$E_{\mathrm{s}}(R) \xrightarrow{r\to 0} -\frac{\pi(c_{\mathrm{UV}} - 24h_{\mathrm{s}})}{6R},$$

the usual Casimir behaviur [54], and the IR one

$$E_{\mathrm{s}}(R) \xrightarrow{r\to\infty} \mathrm{N}_{\mathrm{s}} \in \mathbb{Z}_{\geq 0}.$$

In [26,29,30] it was shown that the integrable structure of sG could be equally well employed to encode *massless flows* interpolating between successive unitary minimal models $\mathcal{M}(p+1,p) \xrightarrow{\phi_{(1,3)}} \mathcal{M}(p,p-1)$. Soon it became clear that this description could also address flows $\mathcal{M}(p,q) \xrightarrow{\phi_{(1,3)}} \mathcal{M}(2p-q,q)$ [27,28] and, using the integrable structure of ZMS, massless flows $\mathcal{M}(2p+I,p) \xrightarrow{\phi_{(1,5)}} \mathcal{M}(2p-I,p)$ and lastly in [31] the flows $\mathcal{M}(2p-I,p) \xrightarrow{\phi_{(2,1)}} \mathcal{M}(2p-I,p-I)$.

One of the main results of this Letter is that the NLIEs encoding the finite size spectrum of massless flows between minimal models can be extended—at the very least on a qualitative level—beyond the $\phi_{(1,3)}$, $\phi_{(1,5)}$, $\phi_{(2,1)}$, and $\phi_{(1,2)}$ cases, to the whole family of flows predicted by anomaly matching conditions associated to noninvertible symmetries.

The structure of the "massless NLIEs" is the same as for the known cases [26,31]: one first computes the solutions $f_{\mathrm{R}}(\theta)$ and $f_{\mathrm{L}}(\theta)$ to the following coupled NLIE system:

$$f_{\mathrm{R}}(\theta)=\mathrm{i}\alpha'-\mathrm{i}\frac{r}{2}e^{\theta}-\sum_{\sigma=\pm}\sigma\int_{\mathcal{C}_s^{\sigma}}d\theta'\left[\phi(\theta-\theta')L_{\mathrm{R}}^{-\sigma}(\theta')+\chi(\theta-\theta')L_{\mathrm{L}}^{\sigma}(\theta')\right],$$
$$f_{\mathrm{L}}(\theta)=-\mathrm{i}\alpha'-\mathrm{i}\frac{r}{2}e^{-\theta}+\sum_{\sigma=\pm}\sigma\int_{\mathcal{C}_s^{\sigma}}d\theta'\left[\phi(\theta-\theta')L_{\mathrm{L}}^{\sigma}(\theta')+\chi(\theta-\theta')L_{\mathrm{R}}^{-\sigma}(\theta')\right], \quad (7)$$

where $L^{\pm}_{\mathrm{R}}(\theta)=\log\left[1+\exp\left(\pm f_{\mathrm{R}}(\theta)\right)\right]$. Then, the *scaling function* $f_s(r)=6RE_s(R)/\pi$ is determined as

$$f_s(r)=\sum_{\sigma=\pm}\frac{3\mathrm{i}r\sigma}{2\pi^2}\int_{\mathcal{C}_s^{\sigma}}d\theta\left[e^{-\theta}L_{\mathrm{L}}^{\sigma}(\theta)-e^{\theta}L_{\mathrm{R}}^{-\sigma}(\theta)\right]. \quad (8)$$

In these equations, the parameter α' is known as *twist*. The kernels $\phi(\theta)$ and $\chi(\theta)$ identify the specific theory, while the contours $\mathcal{C}_s^{\pm}$ determine the state. In particular, the ground state is obtained with the choice $\mathcal{C}_s^{\pm}=\mathbb{R}\pm\mathrm{i}\eta$, with $\eta\gtrsim 0$.

The flows described in this Letter correspond to the following choice of kernels [55]:

$$\phi(\theta)=-\int_{\mathbb{R}}\frac{d\omega}{2\pi}e^{\mathrm{i}\theta\omega}\frac{\sinh\left(\frac{1}{\kappa}\pi\omega\right)\cosh\left(\frac{2\xi-\kappa}{2\kappa}\pi\omega\right)}{\sinh\left(\frac{\xi-1}{\kappa}\pi\omega\right)\cosh\left(\frac{1}{2}\pi\omega\right)},$$
$$\chi(\theta)=-\int_{\mathbb{R}}\frac{d\omega}{2\pi}e^{\mathrm{i}\theta\omega}\frac{\sinh\left(\frac{1}{\kappa}\pi\omega\right)\cosh\left(\frac{\kappa-2}{2\kappa}\pi\omega\right)}{\sinh\left(\frac{\xi-1}{\kappa}\pi\omega\right)\cosh\left(\frac{1}{2}\pi\omega\right)}, \quad (9)$$

where $\kappa>2$ and $\xi>1$, making the Fourier image integrable [56] on $\mathbb{R}$. The physical parameters of the UV and IR CFTs are determined as follows:

$$c_{\mathrm{eff}}^{\mathrm{UV}}(p,q)\equiv 1-\frac{6}{pq}=1-3\left(\frac{\alpha'}{\pi}\right)^2\frac{(\xi-1)^2}{\xi(\xi+1)},$$
$$c_{\mathrm{eff}}^{\mathrm{IR}}(p',q')\equiv 1-\frac{6}{p'q'}=1-3\left(\frac{\alpha'}{\pi}\right)^2\frac{\xi-1}{\xi},$$
$$h_{(r,s)}\equiv\frac{(pr-qs)^2-(p-q)^2}{4pq}=1-\frac{1}{z_{(r,s)}}\frac{\kappa}{\xi+1} \quad (10)$$

with $h_{(r,s)}$ being the conformal dimension of the perturbing field $\phi_{(r,s)}$ in the UV and $z_{(r,s)}=1$, 2, depending on whether the field $\phi_{(r,s)}$ is even or odd under the natural $\mathbb{Z}_2$ symmetry in the UV [57]. Fixing these three physical parameters, i.e., choosing a UV starting CFT together with an outgoing direction and a target IR CFT, uniquely fixes the form of the NLIEs (7). Consequently, any additional information extracted from (8) can be considered a nontrivial prediction. One quantity that can be analytically computed is the conformal dimension of the operator that attracts the flow in the IR CFT:

$$h_{(r',s')}=1+\frac{1}{z_{(r',s')}}\frac{\kappa}{\xi-1}. \quad (11)$$

The request that this conformal dimension appears, as it should, in the Kač table of the IR minimal model $\mathcal{M}(p',q')$ enforces a constraint on the allowed values of the integers p,q,p',q',r,s,r', and s':

$$\frac{p(r+1)-q(s-1)}{p'(r'+1)-q'(s'-1)}=-\frac{z_{(r',s')}}{z_{(r,s)}}\frac{p'(r'-1)-q'(s'+1)}{p(r-1)-q(s+1)}. \quad (12)$$

While we could not find the most general solution to the above Diophantine equation, we can verify that the special family of solutions that corresponds to the flows discovered in [17]

$$\{\mathcal{M}_{(\mu p+I,p)}\overset{\phi_{(1,2\mu+1)}}{\longrightarrow}\mathcal{M}_{(\mu p-I,p)}\} \quad (13)$$

solve all the constraints with 2μ and $\mu p-I$ being positive integers. This family includes the familiar $\phi_{(1,3)}$, $\phi_{(1,5)}$, $\phi_{(2,1)}$ and $\phi_{(1,2)}$ flows [58]. In the Appendix in Supplemental Material [33], we show how the NLIEs (7) reduce the known integrable cases for $\mu=1/2$, 1, 2 [59], where $p'/2\leq p\leq p'-2$. Further restrictions can be imposed on the solutions using the noninvertible symmetry matching.

Numerical analysis and conformal perturbation theory—Extracting analytically any further nontrivial prediction from NLIEs of the form (7), (8) is a notoriously arduous task. We can make some headway by studying them numerically. In particular, we can compare the behavior of the scaling function for large and small values of r to the predicted behavior of the ground-state energy along the flow (2). Contrary to the well-known integrable cases, we expect the general flow to be a *multifield deformation* of the UV CFT, with the IR theory only arising upon fine-tuning of the critical coupling of the various deforming fields, e.g., in the flow $\mathcal{M}(7,2)\to\mathcal{M}(5,2)$, where both UV fields $\phi_{(1,2)}$ and $\phi_{(1,3)}$ were seen to contribute by using a Hamiltonian truncation method [40,41]. Indeed, all relevant operators allowed by the preserved generalized symmetries will contribute to the flow, in agreement with the standard Wilsonian RG lore. For a flow triggered by a number M of relevant UV fields $\{\phi_{(r_i,s_i)}\}_{i=1}^{M}$, the expected small r

behavior of the scaling function (8) is

$$f(r) \overset{r\to 0}{=} \frac{3r^2/(4\pi)}{\sin\left(\frac{\pi\kappa}{\xi+1}\right)} + \sum_{\{l_i\}=0}^{\infty} a_{l_1,\ldots,l_M} r^{\sum_{i=1}^{M} l_i y_{(r_i,s_i)}},$$
$$y_{(r,s)} = 2z_{(r,s)}(1-h_{(r,s)}),$$
$$a_{0,0,\ldots,0} = c_{\rm eff}(p,q) = 1 - \frac{6}{pq}. \qquad (14)$$

Here the coefficients $a_{l_1,\ldots,l_M}$ are proportional to the correlation functions of the perturbing fields on the vacuum (see [60] for more details). While the expansion (14) is expected to have a finite radius of convergence [21,60], the situation in the IR is much less under control. There, the conformal perturbation theory (CPT) expansion

$$f(r) \overset{r\to\infty}{=} f(p',q') + \sum_{l=1}^{\infty}\left(a'_l r^{l y_{(r',s')}} + b'_l r^{-2l}\right) + \cdots \qquad (15)$$

is asymptotic, and there is little [61] control over the omitted further contributions. We performed a numerical analysis of the NLIEs (7) for several cases and found that, in all of them, the scaling function (8) agrees perfectly with the expected behaviors (14) and (15). While the parameters (10) are built in the kernel by construction, the agreement with the multiple sum for small r shall be regarded as a highly nontrivial check that our data passes with flying colors. In principle, further support can come from comparing the first few coefficients of the expansions with the estimates coming from CPT. We will report on this in a future publication. Figure 2 reports the numerical results for the flow $\mathcal{M}(10,3) \to \mathcal{M}(8,3)$, triggered by $\phi_{(1,7)}$, first proposed in [62], which has recently received a lot of attention [10,12,63]. The fit that includes contributions from all the perturbing fields, is numerically favoured, independently agreeing with the results obtained in [10] by employing Hamiltonian truncation and CPT methods.

Outlook—In this Letter, we studied RG flows between generic minimal models. Many flows can be conjectured by the matching of the global symmetries. For these flows, we propose an NLIE description encoding the ground state energy nonperturbatively. It would be interesting to confront our ground state energy with the results that can be independently obtained by conformal perturbation theory and Hamiltonian truncation. While here we focussed on gapless RG flows between minimal models, our methods extend to the ones to gapped phases. In this case, the anomalies of noninvertible symmetries predict a nontrivial structure of the vacua of the TQFT and particle-soliton degeneracies [34–37]. A direct application would be to check whether the RG flows between QCD_2 theories proposed in [64] may be obtained via matching of the anomalies associated with lines of the coset models in QCD_2 as initiated recently in [36]. Another interesting future direction is studying the IR fixed point from deformation of coupled minimal models, following the approach of [13,14,63], especially because these flows may end on compact nonrational CFT for some special deformations.

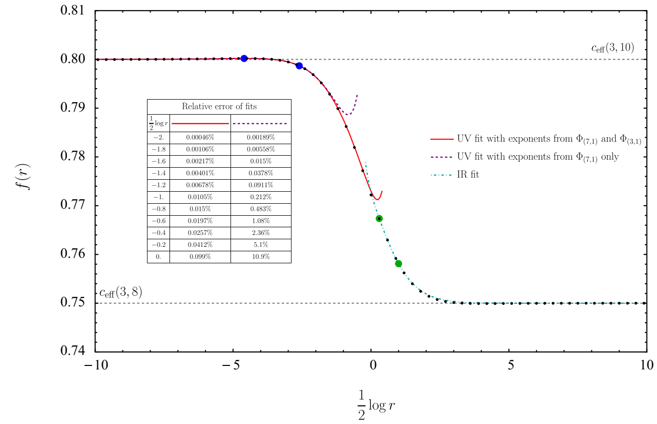


FIG. 2. Ground state scaling function (8) for the flow $\mathcal{M}(10,3) \to \mathcal{M}(8,3)$, triggered by $\phi_{(1,7)}$. We fitted the points between the pairs of larger blue and green dots against UV and IR CPT predictions, respectively. The fit performed including the contributions of all perturbing fields (red full line)—here $y_{(1,5)} = 3y_{(1,7)}$—performs much better than the fit for a single field (purple dashed line). The table collects the relative errors on points that were not used.

Our NLIEs also admit a simple extension to the massive version, similar to what happens for the $\phi_{(1,3)}$ and $\phi_{(1,5)}$, $\phi_{(1,2)}$ cases. For these integrable massive flows, the NLIE describes the ground state energy of, respectively, the sG and ZMS theories. In general, we expect the massive version of our equations to be related to the ground state of a (timelike) Liouville CFT deformed by several vertex operators. This perspective suggests the possibility of studying the one-point functions of these theories using the reflection relations proposed in [65–67]. We plan to follow this path in the near future.

Acknowledgments—We thank Patrick Dorey, Shota Komatsu, Alessio Miscioscia, Volker Schomerus, Roberto Tateo, and Igor Klebanov for useful discussions. We also thank the anonymous referees for having read carefully our manuscript and for providing detailed and precious feedback. F. A. gratefully acknowledges support from the Simons Center for Geometry and Physics, Princeton Center for Theoretical Science, and Perimeter Institute, at which some of the research for this Letter was performed. F. A. thanks the Deutsche Forschungs gemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy EXC 2121 "Quantum Universe" (390833306), and the Collaborative Research Center—SFB 1624 "Higher structures, moduli spaces and integrability" (506632645) for support.

Data availability—No data were created or analyzed in this study.

[1] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, J. High Energy Phys. 02 (2015) 172.
[2] S. Schafer-Nameki, Phys. Rep. **1063**, 1 (2024).
[3] S.-H. Shao, arXiv:2308.00747.
[4] C.-M. Chang, Y.-H. Lin, S.-H. Shao, Y. Wang, and X. Yin, J. High Energy Phys. 01 (2019) 026.
[5] J. Frohlich, J. Fuchs, I. Runkel, and C. Schweigert, Phys. Rev. Lett. **93**, 070601 (2004).
[6] J. Fuchs, M. R. Gaberdiel, I. Runkel, and C. Schweigert, J. Phys. A **40**, 11403 (2007).
[7] D. Aasen, P. Fendley, and R. S. K. Mong, arXiv:2008.08598.
[8] G. W. Moore and N. Seiberg, Commun. Math. Phys. **123**, 177 (1989).
[9] V. Benedetti, H. Casini, and J. M. Magan, Phys. Rev. D **111**, 065024 (2025).
[10] O. Delouche, J. Elias Miro, and J. Ingoldby, J. High Energy Phys. 04 (2025) 144.
[11] A. Katsevich, arXiv:2410.18069.
[12] A. Katsevich, I. R. Klebanov, and Z. Sun, J. High Energy Phys. 03 (2025) 170.
[13] A. Antunes and C. Behan, SciPost Phys. **18**, 132 (2025).
[14] A. Antunes and C. Behan, Phys. Rev. Lett. **130**, 071602 (2023).
[15] K. Kikuchi, arXiv:2212.13851.
[16] K. Kikuchi, arXiv:2412.08935.
[17] T. Tanaka and Y. Nakayama, J. High Energy Phys. 11 (2024) 137.
[18] A. B. Zamolodchikov and A. B. Zamolodchikov, Ann. Phys. (N.Y.) **120**, 253 (1979).
[19] V. A. Fateev and A. B. Zamolodchikov, Int. J. Mod. Phys. A **05**, 1025 (1990).
[20] P. Dorey, in *Eotvos Summer School in Physics: Conformal Field Theories and Integrable Models* (1996), pp. 85–125.
[21] A. B. Zamolodchikov, Nucl. Phys. **B342**, 695 (1990).
[22] C. Destri and H. J. De Vega, Nucl. Phys. **B438**, 413 (1995).
[23] A. Klumper, M. T. Batchelor, and P. A. Pearce, J. Phys. A **24**, 3111 (1991).
[24] M. J. Martins, Phys. Rev. Lett. **69**, 2461 (1992).
[25] M. J. Martins, Nucl. Phys. **B394**, 339 (1993).
[26] A. B. Zamolodchikov, Phys. Lett. B **335**, 436 (1994).
[27] D. Fioravanti, A. Mariottini, E. Quattrini, and F. Ravanini, Phys. Lett. B **390**, 243 (1997).
[28] G. Feverati, F. Ravanini, and G. Takacs, Nucl. Phys. **B570**, 615 (2000).
[29] P. Fendley, H. Saleur, and A. B. Zamolodchikov, Int. J. Mod. Phys. A **08**, 5717 (1993).
[30] P. Fendley, H. Saleur, and A. B. Zamolodchikov, Int. J. Mod. Phys. A **08**, 5751 (1993).
[31] P. Dorey, C. Dunning, and R. Tateo, Nucl. Phys. **B578**, 699 (2000).
[32] V. P. Yurov and A. B. Zamolodchikov, Int. J. Mod. Phys. A **05**, 3221 (1990).
[33] See Supplemental Material at http://link.aps.org/supplemental/10.1103/dg1s-5vp6 for a detailed discussion of all the flows with $pq \leq 42$.
[34] C. Copetti, L. Cordova, and S. Komatsu, J. High Energy Phys. 03 (2025) 204.
[35] C. Copetti, L. Cordova, and S. Komatsu, Phys. Rev. Lett. **133**, 181601 (2024).
[36] C. Cordova, D. García-Sepúlveda, and N. Holfester, arXiv:2412.21153.
[37] C. Cordova, N. Holfester, and K. Ohmori, J. High Energy Phys. 06 (2025) 001.
[38] O. A. Castro-Alvaredo, B. Doyon, and F. Ravanini, J. Phys. A **50**, 424002 (2017).
[39] A. B. Zamolodchikov, JETP Lett. **43**, 730 (1986).https://inspirehep.net/literature/240292
[40] M. Lencsés, A. Miscioscia, G. Mussardo, and G. Takács, J. High Energy Phys. 02 (2023) 046.
[41] M. Lencsés, A. Miscioscia, G. Mussardo, and G. Takács, J. High Energy Phys. 09 (2023) 052.
[42] More precisely, the defect spin content in the IR is a subset of the UV one due to possible massive decouplings.
[43] Note that this implies the closure of this category under fusion.
[44] E.g., $\mathcal{M}(3,4) \to \mathcal{M}(2,5)$, or $\mathcal{M}(4,5) \to \mathcal{M}(3,4)$. See Appendix B in Supplemental Material [33] for more examples.
[45] The known flows [40,41] $\mathcal{M}(2k+1,2) \to \mathcal{M}(2k+1,2)$ are not displayed in the figure given that these degenerate cases where all the topological lines are broken are not determined by our method.
[46] F. A. Smirnov, Int. J. Mod. Phys. A **06**, 1407 (1991).
[47] A. B. Zamolodchikov, Sov. J. Nucl. Phys. **46**, 1090 (1987).https://inspirehep.net/literature/257015
[48] F. A. Smirnov, Int. J. Mod. Phys. A **04**, 4213 (1989).
[49] A. LeClair, Phys. Lett. B **230**, 103 (1989).
[50] F. A. Smirnov, Nucl. Phys. **B337**, 156 (1990).
[51] M. J. Martins, Phys. Lett. B **262**, 39 (1991).
[52] C. J. Efthimiou, Nucl. Phys. **B398**, 697 (1993).
[53] G. Takacs, Nucl. Phys. **B489**, 532 (1997).
[54] J. L. Cardy, Nucl. Phys. **B270**, 186 (1986).
[55] We find it worth mentioning that these kernels were derived from an educated guess. We asked their Fourier transform to have the following properties: (1) be a ratio of trigonometric functions; (2) be nonvanishing at $\omega = 0$ (as this value enters the expressions of $c_{\mathrm{eff}}^{\mathrm{UV/IR}}$ and $h_{(r,s)}$); (3) be decaying as $\omega \to \pm\infty$; (4) present poles at $\omega = \pm\mathrm{i}$ (which produce the bulk term in the UV expansion of $f(r)$); and (6) present two free parameters (that, along with the twist α, allow to tune the UV and IR central charges and the UV conformal dimension at will). Asking additionally for a minimal number of trigonometric functions, we arrived at the expressions (9).
[56] The limit case $\kappa = 2$ yields the $\phi_{(1,3)}$ massless flows [26–28]. This limit is nontrivial and is discussed in details in Supplemental Material [33].
[57] Specifically, $z_{(r,s)} = 1 + [(p(r-1) - q(s-1))(\bmod 2)]$.
[58] The case $\phi_{(2,1)}$ in Eq. (3.9) in [31] is recovered from (13) by setting $\mu = 1/2$, redefining $p = 2P - J$, and $I = J/2$ and swapping the indices of the minimal models and of the primary fields.
[59] It is perhaps worth remarking that we did not ask for the known integrable cases to be included in the educated guess that yielded (9). This is a *post hoc* result that reveals our

kernels to be a *one*-parameter deformation of the known integrable ones.
[60] T. R. Klassen and E. Melzer, Nucl. Phys. **B350**, 635 (1991).
[61] A. B. Zamolodchikov, Nucl. Phys. **B358**, 497 (1991).
[62] L. Fei, S. Giombi, I. R. Klebanov, and G. Tarnopolsky, Phys. Rev. D **91**, 045011 (2015).
[63] I. R. Klebanov, V. Narovlansky, Z. Sun, and G. Tarnopolsky, J. High Energy Phys. 02 (2023) 066.
[64] D. Delmastro and J. Gomis, J. High Energy Phys. 09 (2023) 158.
[65] S. L. Lukyanov and A. B. Zamolodchikov, Nucl. Phys. **B493**, 571 (1997).
[66] V. Fateev, S. L. Lukyanov, A. B. Zamolodchikov, and A. B. Zamolodchikov, Phys. Lett. B **406**, 83 (1997).
[67] V. Fateev, S. L. Lukyanov, A. B. Zamolodchikov, and A. B. Zamolodchikov, Nucl. Phys. **B516**, 652 (1998).