This is a repository copy of *On the images of Galois representations attached to low weight Siegel modular forms*.

White Rose Research Online URL for this paper:
https://eprints.whiterose.ac.uk/id/eprint/230675/

Version: Published Version

RESEARCH ARTICLE

On the images of Galois representations attached to low weight Siegel modular forms

Ariel Weiss

Department of Mathematics, Ben-Gurion
University of the Negev, Be'er Sheva,
Israel

Correspondence
Ariel Weiss, Department of Mathematics,
Ben-Gurion University of the Negev, Be'er
Sheva 8410501, Israel.
Email: weiss.ariel@mail.huji.ac.il

Abstract

Let π be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A_Q})$, whose archimedean component is a holomorphic discrete series or limit of discrete series representation. If π is not CAP or endoscopic, then we show that its associated ℓ-adic Galois representations are irreducible and crystalline for 100% of primes ℓ. If, moreover, π is neither an automorphic induction nor a symmetric cube lift, then we show that, for 100% of primes ℓ, the image of its mod ℓ Galois representation contains $\mathrm{Sp}_4(\mathbf{F}_\ell)$.

1 | INTRODUCTION

Under the Langlands correspondence, where automorphic representations of GL_n should correspond to n-dimensional Galois representations, *cuspidal* automorphic representations should correspond to *irreducible* Galois representations. More generally, one expects that the image of an automorphic Galois representation should be as large as possible, unless there is an automorphic reason for it to be small.

In this paper, we study the images of Galois representations attached to low weight, genus 2 Siegel modular forms. These automorphic forms are the genus 2 analogues of weight 1 modular forms, and are of particular interest due to their conjectural relationship with abelian surfaces. Our main result is the following theorem:

Theorem 1.1. *Let π be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A_Q})$ such that π_∞ is a holomorphic discrete series or limit of discrete series representation. Assume that the weak functorial lift*

of π to GL_4 exists and is cuspidal. Let E be the coefficient field of π. For each prime λ of E, of residue characteristic ℓ, let

$$\rho_\lambda : \ \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GSp}_4(\overline{E}_\lambda)$$

be the λ-adic Galois representation associated to π. Then:

(i) If $\rho_\lambda|_{\mathbf{Q}_\ell}$ is de Rham, and if $\ell \geqslant 5$, then ρ_λ is irreducible.
(ii) $\rho_\lambda|_{\mathbf{Q}_\ell}$ is crystalline for all $\lambda \mid \ell$ for a set of primes ℓ of Dirichlet density 1.

In particular, ρ_λ is irreducible for all $\lambda \mid \ell$ for a set of primes ℓ of Dirichlet density 1.

The automorphic representations π in the statement of the theorem are exactly those that arise from classical genus 2 vector-valued Siegel modular forms. The assumption that the functorial lift is cuspidal amounts to demanding that π arises from a Siegel modular form that is not CAP or endoscopic; in these cases, the associated Galois representation is known to be reducible. In general, the existence of this lift follows from Arthur's classification [2]; see Section 2.4 for further discussion.

Beyond irreducibility, we also prove a big image theorem for the images of the mod λ representations attached to π:

Theorem 1.2. *Let π be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_{\mathbf{Q}})$ such that π_∞ is a holomorphic discrete series or limit of discrete series representation. Assume that the weak functorial lift of π to GL_4 exists and is cuspidal. Let E be the coefficient field of π. For each prime λ of E, of residue characteristic ℓ, let $\mathbf{F}_\lambda = \mathcal{O}_E/\lambda$ and let*

$$\overline{\rho}_\lambda : \ \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GSp}_4(\mathbf{F}_\lambda)$$

be the mod λ Galois representation associated to π. Let $\mathcal{L}$ be the set of primes λ of E for which $\rho_\lambda|_{\mathbf{Q}_\ell}$ is crystalline. Then,

(i) *for all but finitely many primes $\lambda \in \mathcal{L}$, $\overline{\rho}_\lambda$ is irreducible;*
(ii) *if π is neither an automorphic induction nor a symmetric cube lift, then, for all but finitely many primes $\lambda \in \mathcal{L}$, the image of $\overline{\rho}_\lambda$ contains $\mathrm{Sp}_4(\mathbf{F}_\ell)$.*

The corresponding results for elliptic modular forms were proven by Deligne–Serre, Ribet and Momose [12, 32, 39, 41]. For high weight Siegel modular forms, irreducibility for all but finitely many primes follows from the work of Ramakrishnan [38], while the analogue of Theorem 1.2 follows from [5, Prop. 5.3.2] and from the work of Dieulefait–Zenteno [16], but only for a set of primes λ of residual Dirichlet density 1. Conjecturally, ρ_λ should be irreducible for all primes λ, and $\overline{\rho}_\lambda$ should be irreducible for all but finitely many primes λ. Since, in the high weight case, the set $\mathcal{L}$ contains all but finitely many primes, Theorem 1.2 gives an improvement on existing results even in the cohomological case.

These previous results depend crucially on the facts that the associated Galois representations are geometric, satisfy the Ramanujan conjectures and, in the high weight Siegel modular form case, are Hodge–Tate regular. All other recent results proving the irreducibility of automorphic Galois representations rely on these inputs [5, 9, 35, 38, 53].

The novelty of this paper is that we prove a big image theorem in a situation where these key inputs are not available. In the case of low weight Siegel modular forms, the Hodge–Tate–Sen weights of ρ_λ are irregular, purity is an open problem and, a priori, crystallinity is not known. Indeed, a priori, we do not even know that ρ_λ is Hodge–Tate.

1.1 | Methods

Our proof of Theorem 1.1 proceeds in two steps. First, in Theorem 4.1, we prove, with no assumptions on the prime λ, that either ρ_λ is irreducible or it decomposes as a direct sum of two irreducible two-dimensional representations that are Hodge–Tate regular and odd. Our key input is Theorem 4.2, which demonstrates that, in many cases, the Jacquet–Shalika bounds [24], can be used to rule out the existence of certain subrepresentations. These bounds are weaker than those predicted by the generalised Ramanujan conjecture, and are known in general for cuspidal automorphic representations of GL_n,

If these two-dimensional subrepresentations of ρ_λ were modular, then a routine L-functions argument would lead to a contradiction. By recent work of Pan [33], these representations are modular if they are de Rham and if $\ell \geqslant 5$, but, a priori, these representations need not even be Hodge–Tate. On the other hand, a criterion of Jorza [25] shows that, if π is unramified at ℓ and if $\lambda \mid \ell$, then ρ_λ is crystalline if the four Satake parameters of π_ℓ are distinct.

Our second step is to observe that Jorza's criterion can be translated into a condition on the image of ρ_λ for a single prime λ. Indeed, for all unramified primes $p \neq \ell$, the Satake parameters of π_p are (up to normalisation) exactly the eigenvalues $\rho_\lambda(\mathrm{Frob}_p)$. Combining work of Rajan [37] with the restrictions on the decomposition of ρ_λ proven in Theorem 4.1, we prove that the characteristic polynomial of $\rho_\lambda(\mathrm{Frob}_p)$ has distinct roots for a set of primes p of Dirichlet density 1. Thus, ρ_λ is crystalline for all $\lambda \mid \ell$ for a set of primes ℓ of Dirichlet density 1, and we can apply Pan's result to reach a contradiction.

This distinctness of Satake parameters is a key input in the proof of Theorem 1.2. Indeed, in Lemma 7.3, we use it to prove that $\overline{\rho}_\lambda$ cannot have an even two-dimensional subrepresentation for infinitely many $\lambda \in \mathcal{L}$. In the cohomological case, this argument allows us to strengthen the result of [16] to apply to all but finitely many primes, rather than just a density 1 set of primes.

1.2 | The structure of this paper

In Section 2, we review key properties of automorphic representations of $\mathrm{GSp}_4(\mathbf{A_Q})$. In particular, we define holomorphic (limit of) discrete series representations in terms of their L-parameters, and we discuss the transfer map from GSp_4 to GL_4. In Section 3, we survey existing results on the construction of Galois representations associated to Siegel modular forms. In addition, we prove that, for low weight forms, the Galois representations are symplectic.

In Section 4, we prove Theorem 4.1 and the key input Theorem 4.2. In Section 5, we use the results of Section 4 to prove that the Satake parameters of π_p are distinct for a set of primes p of Dirichlet density 1. Applying a theorem of Jorza [25], we deduce that ρ_λ is crystalline for all primes $\lambda \mid \ell$ for a set of primes ℓ of Dirichlet density 1. Using this result, in Section 6, we apply a recent result of Pan [33] to complete the proof of Theorem 1.1. Finally, in Section 7, we prove Theorem 1.2.

2 | AUTOMORPHIC REPRESENTATIONS OF $\mathbf{GSp_4(A_Q)}$

In this section, we review aspects of the local and global theory of automorphic representations of $\mathrm{GSp}_4(\mathbf{A_Q})$.

Definition 2.1. For a ring R, let

$$\mathrm{GSp}_4(R) = \left\{ \gamma \in \mathrm{GL}_4(R) : \gamma^t J \gamma = \nu J, \ \nu \in R^\times \right\},$$

where

$$
J = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}.
$$

For $\gamma \in \mathrm{GSp}_4(R)$, the constant ν is called the *similitude* of γ and is denoted $\mathrm{sim}(\gamma)$. Let $\mathrm{Sp}_4(R)$ be the subgroup of elements for which $\mathrm{sim}(\gamma) = 1$.

2.1 | Archimedean L-parameters

If π is a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A_Q})$, then π can be decomposed as a restricted tensor product $\pi = \otimes'_p \pi_p$, with each π_p an admissible representation of $\mathrm{GSp}_4(\mathbf{Q}_p)$. Throughout this paper, π will denote a unitary cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A_Q})$ whose archimedean component π_∞ is a holomorphic discrete series or limit of discrete series representation. In this subsection, we define these representations via their associated L-parameters. Our exposition follows [31, § 3.1].

Let $W_{\mathbf{C}} = \mathbf{C}^\times$ denote the Weil group of $\mathbf{C}$ and let $W_{\mathbf{R}}$ be the Weil group of $\mathbf{R}$. Then

$$W_{\mathbf{R}} = \mathbf{C}^\times \sqcup \mathbf{C}^\times j,$$

where $j^2 = -1$ and $jzj^{-1} = \overline{z}$ for $z \in \mathbf{C}^\times$. We define a map

$$|\cdot| : W_{\mathbf{R}} \to \mathbf{R}^\times$$

by sending $z \in \mathbf{C}^\times$ to $z\overline{z} = |z|^2$ and $j \mapsto -1$. This map induces an isomorphism $W_{\mathbf{R}}^{ab} \cong \mathbf{R}^\times$.

2.1.1 | L-parameters for $\mathrm{GL}_2(\mathbf{R})$

We begin by defining the L-parameters that correspond to the discrete series and limit of discrete series representations of $\mathrm{GL}_2(\mathbf{R})$. Let w, n be integers, with $n \geqslant 0$ and $n \equiv w + 1 \pmod{2}$. Define

$$\phi_{(w;n)} : W_{\mathbf{R}} \to \mathrm{GL}_2(\mathbf{C})$$

$$z \mapsto |z|^{-w} \begin{pmatrix} (z/\overline{z})^{n/2} & \\ & (z/\overline{z})^{-n/2} \end{pmatrix} \quad \text{for } z \in \mathbf{C}^{\times}$$

$$j \mapsto \begin{pmatrix} 1 & (-1)^n \end{pmatrix}.$$

Here, if $z = re^{i\theta} \in \mathbf{C}^{\times}$, then we let $(z/\overline{z})^{n/2} = e^{in\theta}$. When $n \geqslant 1$, $\phi_{(w;n)}$ corresponds to the weight $n + 1$ discrete series representation of $\mathrm{GL}_2(\mathbf{R})$ with central character $a \mapsto a^{-w}$, for $a \in \mathbf{R}^{\times}$. When $n = 0$, $\phi_{(w;n)}$ corresponds to the limit of discrete series representation of $\mathrm{GL}_2(\mathbf{R})$ with central character $a \mapsto a^{-w}$. The parity condition on w ensures that, in both cases, $\phi_{(w;n)}$ is the archimedean L-parameter attached to a classical modular form of weight $n + 1$.

2.1.2 | L-parameters for $\mathrm{GSp}_4(\mathbf{R})$

Next, we define the L-parameters whose L-packets contain the (limit of) discrete series representations of $\mathrm{GSp}_4(\mathbf{R})$. For integers w, m_1, m_2, with $m_1 > m_2 \geqslant 0$ and $m_1 + m_2 \equiv w + 1 \pmod 2$, we define an L-parameter

$$\phi_{(w; m_1, m_2)} : W_{\mathbf{R}} \to \mathrm{GSp}_4(\mathbf{C})$$

by

$$z \mapsto |z|^{-w} \begin{pmatrix} (z/\overline{z})^{(m_1+m_2)/2} & & & \\ & (z/\overline{z})^{(m_1-m_2)/2} & & \\ & & (z/\overline{z})^{-(m_1-m_2)/2} & \\ & & & (z/\overline{z})^{-(m_1+m_2)/2} \end{pmatrix}$$

for $z \in \mathbf{C}^{\times}$, and

$$j \mapsto \begin{pmatrix} & & & 1 \\ & & 1 & \\ & (-1)^{m_1+m_2} & & \\ (-1)^{m_1+m_2} & & & \end{pmatrix}.$$

The image of $\phi_{(w; m_1, m_2)}$ lies in $\mathrm{GSp}_4(\mathbf{C})$ and has similitude character given by

$$z \mapsto |z|^{-2w}$$

$$j \mapsto (-1)^w = (-1)^{m_1+m_2+1}.$$

If we compose $\phi_{(w; m_1, m_2)}$ with the inclusion $\mathrm{GSp}_4(\mathbf{C}) \hookrightarrow \mathrm{GL}_4(\mathbf{C})$, the resulting representation of $W_{\mathbf{R}}$ is isomorphic to the direct sum $\phi_{(w; m_1+m_2)} \oplus \phi_{(w; m_1-m_2)}$ of parameters of discrete series representations of $\mathrm{GL}_2(\mathbf{R})$.

The L-packet corresponding to $\phi_{(w; m_1, m_2)}$ has two elements:

$$\left\{ \pi^H_{(w; m_1, m_2)}, \pi^W_{(w; m_1, m_2)} \right\}.$$

Both $\pi^H_{(w;m_1,m_2)}$ and $\pi^W_{(w;m_1,m_2)}$ have a central character given by $a \mapsto a^{-w}$ for $a \in \mathbf{R}^{\times}$. When $m_2 \geqslant 1$ they are (up to twist) discrete series representations: $\pi^H_{(w;m_1,m_2)}$ is a holomorphic discrete series and $\pi^W_{(w;m_1,m_2)}$ is a generic discrete series representation. When $m_2 = 0$, $\pi^H_{(w;m_1,0)}$ is a holomorphic limit of discrete series and $\pi^W_{(w;m_1,0)}$ is a generic limit of discrete series representation.

2.2 | The 'weight' of a holomorphic (limit of) discrete series representation

The Blattner parameter (k_1, k_2) of the holomorphic (limit of) discrete series representation $\pi^H_{(w;m_1,m_2)}$ is defined to be

$$k_1 = m_1 + 1, \quad k_2 = m_2 + 2.$$

If π is the automorphic representation corresponding to a classical Siegel modular form of weight (k_1, k_2) — that is, a vector-valued Siegel modular form with weight $\mathrm{Sym}^{k_1-k_2} \det^{k_2}$ — then π_∞ is a holomorphic (limit of) discrete series representation with Blattner parameter (k_1, k_2). Hence, if π is a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A_Q})$ such that π_∞ is a holomorphic (limit of) discrete series representation, then we refer to the Blattner parameter (k_1, k_2) of π_∞ as the weight of π.

We say that π is *cohomological* or has *high weight* if $k_2 \geqslant 3$. If $k_2 = 2$, then we say that π is *non-cohomological* or has *low weight*. Cohomological automorphic representations can be realised in the étale cohomology of a local system on a suitable Shimura variety, while non-cohomological automorphic representations can only be realised in coherent cohomology.

2.3 | Other non-degenerate limits of discrete series for $\mathrm{GSp}_4(\mathbf{R})$

Finally, for completeness, we note that there is another L-packet of non-degenerate limits of discrete series. Following [42], if m is a positive integer, then this L-packet corresponds, up to twist, to the L-parameter $W_{\mathbf{R}} \to \mathrm{GSp}_4(\mathbf{C})$ that sends

$$z \mapsto \begin{pmatrix} (z/\bar{z})^{m/2} & & & \\ & 1 & & \\ & & 1 & \\ & & & (z/\bar{z})^{-m/2} \end{pmatrix}, \qquad j \mapsto \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}.$$

If π is a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_F)$, whose archimedean components are all either discrete series or non-degenerate limits of discrete series, then, by [20, Thm. 10.5.1], we can attach Galois representations to π. Assuming local–global compatibility results for these Galois representations as in Theorem 3.3, it is likely that the arguments of this paper can be extended to prove a big image theorem for these Galois representations.

2.4 | The transfer map

The principle of Langlands functoriality predicts that there is a global transfer map from automorphic representations of GSp_4 to automorphic representations of GL_4. Indeed, $^L\mathrm{GSp}_4 = \mathrm{GSp}_4(\mathbf{C})$ and $^L\mathrm{GL}_4 = \mathrm{GL}_4(\mathbf{C})$, and the existence of the embedding $\mathrm{GSp}_4(\mathbf{C}) \hookrightarrow \mathrm{GL}_4(\mathbf{C})$ of L-groups means that there should be a corresponding lifting of automorphic representations. This lifting should be compatible with the local Langlands correspondence: if an automorphic representation π of $\mathrm{GSp}_4(\mathbf{A_Q})$ lifts to an automorphic representation Π of $\mathrm{GL}_4(\mathbf{A_Q})$, then, for almost all primes p, the Weil–Deligne representation

$$W_{\mathbf{Q}_p} \times \mathrm{SL}_2(\mathbf{C}) \to \mathrm{GL}_4(\mathbf{C})$$

corresponding to Π_p via the local Langlands correspondence for GL_4 should be isomorphic to the Weil–Deligne representation

$$W_{\mathbf{Q}_p} \times \mathrm{SL}_2(\mathbf{C}) \to \mathrm{GSp}_4(\mathbf{C}) \hookrightarrow \mathrm{GL}_4(\mathbf{C})$$

corresponding to π_p via the local Langlands correspondence for GSp_4. Moreover, an automorphic representation Π should be in the image of this lifting if and only if it is of *symplectic type*, that is, if there is a Hecke character χ such that $\Pi \cong \Pi^\vee \otimes \chi$ and the partial L-function $L^*(\bigwedge^2 \Pi \otimes \chi^{-1}, s)$ (which exists by [27]) has a pole at $s = 1$.

This lifting has been achieved for *globally generic* cuspidal automorphic representations of $\mathrm{GSp}_4(\mathbf{A_Q})$ by Asgari–Shahidi [4]. However, the automorphic representations that correspond to Siegel modular forms are not globally generic: their archimedean components are holomorphic (limit of) discrete series representations, which are not generic.

If π is cohomological, then Weissauer [52, Thm. 1.1] has shown that π is weakly equivalent to a globally generic automorphic representation of $\mathrm{GSp}_4(\mathbf{A_Q})$. Combined with the result of Asgari–Shahidi, we obtain a weak lift of π to GL_4. In particular, the results of this paper apply unconditionally to cohomological Siegel modular forms.

More generally, the existence of a lift to GL_4 follows from Arthur's endoscopic classification [2]. Indeed, the case of PGSp_4, that is, where π has trivial central character, has been spelled out in detail in [1], and this work has been generalised by Gee–Taïbi [19] to cover all automorphic representations of GSp_4. However, these works are all dependent on several papers that have been announced, but have yet to appear: see the discussions in [19, p. 3] and in [31, p. 527].

3 | GALOIS REPRESENTATIONS ASSOCIATED TO SIEGEL MODULAR FORMS

In this section, we review the construction of Galois representations associated to Siegel modular forms. In Sections 3.1 and 3.2, we discuss the construction of Galois representations in the cohomological and non-cohomological cases. In Section 3.3, we prove that, in the non-cohomological case, the Galois representations are symplectic.

3.1 | The case of cohomological weight

We review the construction of Galois representations attached to high weight Siegel modular forms.

Theorem 3.1. *Let π be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A_Q})$ of weight (k_1, k_2), $k_1 \geqslant k_2 \geqslant 3$. Let S denote the set of primes at which π is ramified. Let E denote the coefficient field of π. Then, for every place λ of E, of residue characteristic ℓ, there exists a continuous, semisimple, symplectic Galois representation*

$$\rho_\lambda : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GSp}_4(\overline{E}_\lambda)$$

that satisfies the following properties:

(i) *The representation is unramified at all primes $p \notin S \cup \{\ell\}$.*

(ii) *The similitude character $\mathrm{sim}\,\rho_\lambda$ is odd and $\rho_\lambda \simeq \rho_\lambda^\vee \otimes \mathrm{sim}\,\rho_\lambda$, where $\rho_\lambda^\vee$ is the dual representation.*

(iii) *The representation $\rho_\lambda|_{\mathbf{Q}_\ell}$ is de Rham for all primes λ, and crystalline if $\ell \notin S$.*

(iv) *Local–global compatibility is satisfied up to semisimplification: for any prime $p \neq \ell$,*

$$\mathrm{WD}(\rho_\lambda|_{\mathbf{Q}_p})^{ss} \cong \mathrm{rec}_p(\pi_p \otimes |\mathrm{sim}|_p^{(3-k_1-k_2)/2})^{ss},$$

where rec_p denotes the local Langlands reciprocity map [18].

(v) *The set of Hodge–Tate weights of $\rho_\lambda|_{\mathbf{Q}_\ell}$ is $\{0, k_2 - 2, k_1 - 1, k_1 + k_2 - 3\}$.*

Moreover, if π is not CAP (that is, of Saito–Kurokawa type), then ρ_λ satisfies the following stronger properties:

(vii) *Local–global compatibility is satisfied up to Frobenius semisimplification: for any prime $p \neq \ell$,*

$$\mathrm{WD}(\rho_\lambda|_{\mathbf{Q}_p})^{F\text{-}ss} \cong \mathrm{rec}_p(\pi_p \otimes |\mathrm{sim}|_p^{(3-k_1-k_2)/2}).$$

(viii) *The representation ρ_λ is pure of weight $k_1 + k_2 - 3$. In particular, if $p \notin S \cup \{\ell\}$ and if $\alpha \in \mathbf{C}$ is a root of the characteristic polynomial of $\rho_\lambda(\mathrm{Frob}_p)$, then $|\alpha| = p^{\frac{k_1+k_2-3}{2}}$.*

Finally, if π is neither CAP nor endoscopic and if $\ell \geqslant 5$, then ρ_λ is irreducible.

Remark 3.2. If $p \notin S \cup \{\ell\}$, a_p is the (suitably normalised) eigenvalue of the Hecke operator T_p and χ is the Galois character associated to the central character of π, then condition (iv) implies that

$$\mathrm{Tr}\,\rho_\lambda(\mathrm{Frob}_p) = a_p, \qquad \mathrm{sim}\,\rho_\lambda = \chi\epsilon_\ell^{k_1+k_2-3},$$

where ϵ_ℓ is the ℓ-adic cyclotomic character.

Proof. When π is CAP or endoscopic, the construction of ρ_λ follows from the construction of Galois representations for elliptic modular forms. A discussion of these cases is given in [31, pp. 537–538]. In these cases, ρ_λ is reducible.

When π is of general type (type **(G)** in the notation of [43]), there are two different constructions of the compatible system of λ-adic Galois representations attached to π:

- The original construction, due to Laumon [29] and Weissauer [51], builds on previous work of Taylor [46], and works directly with a symplectic Shimura variety. The Galois representations are constructed from the étale cohomology of Siegel threefolds. The fact that the Galois representations are valued in GSp_4 was proven by Weissauer in [52].
- The second construction, due to Sorensen [44], utilises the transfer map from GSp_4 to GL_4 in combination with Harris–Taylor's construction of Galois representations for automorphic representations of GL_4, which uses unitary Shimura varieties [21]. Sorensen's costruction works in the more general setting of automorphic representations π of $\mathrm{GSp}_4(\mathbf{A}_F)$, with F totally real, so long as there exists a weak functorial lift of π to GL_4. For cohomological automorphic representations of $\mathrm{GSp}_4(\mathbf{A}_\mathbf{Q})$, the existence of this lift is due to Weissauer [52, Thm. 1]. Using this construction, Mok [31, Thm. 3.5] proves local–global compatibility at ramified primes.

Irreducibility when ℓ is sufficiently large is [38, Thm. B]. Using [33, Thm. 1.0.4] in place of [48] in Ramakrishnan's proof, the condition on ℓ can be taken to be $\ell \geqslant 5$. $\qquad\square$

3.2 | The case of non-cohomological weight

The situation for low weight automorphic representations is much less comprehensive. Since the automorphic representations are non-cohomological, the associated Galois representations cannot be constructed directly from the étale cohomology of symplectic or unitary Shimura varieties. Instead, they are constructed as limits of cohomological Galois representations. The process of taking a limit of Galois representations loses information, in particular about local–global compatibility and geometricity at ℓ.

Theorem 3.3. *Let π be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_\mathbf{Q})$ of weight $(k, 2)$. Suppose that π is not CAP or endoscopic. Let S denote the set of primes at which π is ramified. Let E denote the coefficient field of π. Then, for every place λ of E, of residue characteristic ℓ, there exists a continuous, semisimple, symplectic Galois representation*

$$\rho_\lambda : \ \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GSp}_4(\overline{E}_\lambda)$$

that satisfies the following properties:

(i) *The representation is unramified at all primes $p \notin S \cup \{\ell\}$.*

(ii) *If $p \notin S \cup \{\ell\}$, a_p is the (suitably normalised) eigenvalue of the Hecke operator T_p and χ is the Galois character associated to the central character of π, then*

$$\mathrm{Tr}\,\rho_\lambda(\mathrm{Frob}_p) = a_p, \qquad \mathrm{sim}\,\rho_\lambda = \chi \epsilon_\ell^{k_1+k_2-3},$$

where ϵ_ℓ is the ℓ-adic cyclotomic character.

(iii) *The similitude character $\mathrm{sim}\,\rho_\lambda$ is odd and $\rho_\lambda \simeq \rho_\lambda^\vee \otimes \mathrm{sim}\,\rho_\lambda$.*

(iv) *Local–global compatibility is satisfied up to semisimplification: for any prime $p \neq \ell$,*

$$\mathrm{WD}(\rho_\lambda|_{\mathbf{Q}_p})^{ss} \cong \mathrm{rec}_p(\pi_p \otimes |\,\mathrm{sim}\,|_p^{(3-k_1-k_2)/2})^{ss}.$$

(v) *The Hodge–Tate–Sen weights (that is, the eigenvalues of the Sen operator) of $\rho_\lambda|_{\mathbf{Q}_\ell}$ are $\{0, 0, k - 1, k - 1\}$.*

(vi) *If $\ell \notin S$ and the roots of the ℓ-th Hecke polynomial of π are pairwise distinct, then $\rho_\lambda|_{\mathbf{Q}_\ell}$ is crystalline.*

Proof. As in the cohomological case, there are two different constructions of the compatible system of λ-adic Galois representations attached to π. In both cases, ρ_λ is constructed, via its pseudorepresentation, as a limit of cohomological Galois representations.

- The original construction, due to Taylor [45], uses the Hasse invariant to find congruences between the Hecke eigenvalue system of π and mod ℓ^n cohomological eigenforms π_n. The associated Galois pseudorepresentation is constructed as a limit of the Galois pseudorepresentations attached to the π_n. This construction is sufficient to prove the existence of the compatible system of Galois representations and parts (i)–(iii) of the theorem.
- A second construction, due to Mok [31], extends the work of Sorensen [44] and constructs an eigencurve for GSp_4. As in the cohomological case, this construction generalises to automorphic representations of GSp_4 over totally real fields. However, the construction only requires the existence of a functorial lift from GSp_4 to GL_4 for cohomological forms, so the construction is still unconditional for automorphic representations of $\mathrm{GSp}_4(\mathbf{A}_{\mathbf{Q}})$. Using this construction, Mok [31, Thm. 3.5] proves local–global compatibility at ramified primes up to semisimplification.

Part (vi) is due to Jorza [25, Thm. 4.1]. Finally, the fact that the Galois representations are valued in $\mathrm{GSp}_4(\overline{E}_\lambda)$ is Theorem 3.6. $\qquad\square$

Remark 3.4. While it should always be true that $\rho_\lambda|_{\mathbf{Q}_\ell}$ is crystalline when $\ell \notin S$, without the condition in part (vi), we do not even know that $\rho_\lambda|_{\mathbf{Q}_\ell}$ is Hodge–Tate.

Corollary 3.5. *Let π be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A}_{\mathbf{Q}})$ of weight $(k, 2)$, which is not CAP or endoscopic. Then there exists an integer N such that, for all primes λ of E, the Serre conductor of ρ_λ divides N.*

Proof. Fix a prime λ, and let N_ρ be the Serre conductor of ρ_λ, where ρ_λ is viewed as a representation valued in GL_4. Let S be the set of primes at which π is ramified and let

$$N_\pi = \prod_{p \in S} \mathrm{cond}(\mathrm{rec}_p(\pi_p \otimes |\sim|_p^{(3-k_1-k_2)/2}))$$

be the conductor of the transfer of π to GL_4. Since S is finite, we can assume, without loss of generality, that $\ell \notin S$. By definition,

$$N_\rho = \prod_{p \in S} \mathrm{cond}(\mathrm{WD}(\rho_\lambda|_{\mathbf{Q}_p})^{F\text{-}ss}).$$

A Weil–Deligne representation (V, ρ, N) of $W_{\mathbb{Q}_p}$ has conductor

$$\mathrm{cond}(\rho)p^{\dim(V^I) - \dim(V_N^I)},$$

where V^I is the subspace of V fixed by the inertia group and $V_N^I = \ker(N)^I$. If (V, ρ, N) is a Weil–Deligne representation, then, by definition, $(V, \rho, N)^{ss} = \rho^{ss}$. If, moreover, (V, ρ, N) is Frobenius semisimple then $\rho^{ss} = \rho$, and it follows that

$$\mathrm{cond}(V, \rho, N) \mid \mathrm{cond}(\rho) p^{\dim(\rho)}.$$

Hence, N_ρ divides $\prod_{p \in S} \mathrm{cond}(\mathrm{WD}(\rho_\lambda|_{\mathbf{Q}_p})^{ss}) p^4$. By part (iv) of Theorem 3.3,

$$\prod_{p \in S} \mathrm{cond}\left(\mathrm{WD}(\rho_\lambda|_{\mathbf{Q}_p})^{ss}\right) p^4 = \prod_{p \in S} \mathrm{cond}\left(\mathrm{rec}_p(\pi_p \otimes |\sim|_p^{(3-k_1-k_2)/2})^{ss}\right) p^4,$$

which divides

$$\prod_{p \in S} \mathrm{cond}(\mathrm{rec}_p(\pi_p \otimes |\sim|_p^{(3-k_1-k_2)/2})) p^4.$$

Since

$$\prod_{p \in S} \mathrm{cond}(\mathrm{rec}_p(\pi_p \otimes |\sim|_p^{(3-k_1-k_2)/2})) p^4 = N_\pi \prod_{p \in S} p^4,$$

we deduce that $N_\rho \mid N_\pi \prod_{p \in S} p^4$. $\qquad\square$

3.3 | Galois representations valued in GSp_4

The goal of this section is to prove the following theorem:

Theorem 3.6. *Let π be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A_Q})$ of weight $(k, 2)$, with associated λ-adic Galois representation $\rho_\lambda \colon \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_4(\overline{E}_\lambda)$. Then ρ_λ is isomorphic to a representation that factors through $\mathrm{GSp}_4(\overline{E}_\lambda)$.*

The idea of the proof is to reformulate Taylor's original construction of ρ_λ, using V. Lafforgue's G-pseudorepresentations [28] in place of Taylor's pseudorepresentations [45]. We are grateful to B. Stroh for providing a broad outline of the proof. The details of the proof are tangential to the remainder of the paper, so the reader should feel free to skip to the next section.

Remark 3.7. This theorem has long been known to experts, however, prior to the appearance of this paper, no proof had appeared in the literature. Since the first appearance of this paper, an ostensibly simpler proof of Theorem 3.6 was given in [8, Thm. 4.3.4]. Their proof is morally equivalent to the one we provide: the key fact in both proofs is that a representation valued in GSp_4 is determined by its characteristic polynomial and its similitude. However, the argument given here is more robust, and does not require the low weight Siegel modular form to be a p-adic limit of cohomological eigenforms in characteristic 0. Moreover, we expect that the arguments given here can be generalised to broader settings. For example, see [6] for a different application of these ideas.

3.3.1 | Taylor's construction and the limitations of pseudorepresentations

In [45], Taylor gives a blueprint for constructing Galois representations attached to low weight Siegel modular forms by utilising congruences with Siegel modular forms of cohomological weight. This subsection gives an overview of Taylor's construction.

Recall that π is the cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A_Q})$ corresponding to a cuspidal Siegel modular eigenform of weight $(k, 2)$ and level $\Gamma(N)$. Let E be the finite extension of $\mathbf{Q}$ spanned by the Hecke eigenvalues of π, and fix a prime λ of E with residue characteristic ℓ. Following [45, p. 316], let $\mathcal{H}_p$ denote the Hecke algebra of $\mathrm{GSp}_4(\mathbf{Z}_p)$ bi-invariant functions $\mathrm{GSp}_4(\mathbf{Q}_p) \to \mathbf{Z}$ that are supported in $\mathrm{M}_4(\mathbf{Z}_p)$, and let $\mathcal{H}^N = \widehat{\bigotimes}_{p \nmid N} \mathcal{H}_p$. For each tuple $\vec{k} = (k_1, k_2)$ of weights, let $\mathbf{T}_{\vec{k}}$ denote the quotient of $\mathcal{H}^N$ acting on the space of cuspidal Siegel modular forms of weight $\vec{k}$ and level N. Then $\mathbf{T}_{\vec{k}} \otimes \mathbf{Q}$ is a semisimple algebra.

Associated to π is a character $\theta : \mathcal{H}^N \to \mathbf{T}_{(k,2)} \to \mathcal{O}_{E_\lambda}$. Moreover, for each integer $i \geqslant 1$, the automorphic analogue of multiplying a classical form by the Hasse invariant [45, Prop. 3] gives a commutative diagram

$$
\begin{array}{ccc}
\mathcal{H}^N & \xrightarrow{\ \theta\ } & \mathcal{O}_{E_\lambda} \\
\ \downarrow{\scriptstyle \theta_i} & & \downarrow \\
\mathbf{T}_{\vec{k}_i} \otimes_{\mathbf{Z}} \mathcal{O}_{E_\lambda} & \xrightarrow{\ r_i\ } & \mathcal{O}_{E_\lambda}/\lambda^i
\end{array}
$$

where $\vec{k}_i = (k + a_\ell \ell^{i-1}(\ell - 1), 2 + a_\ell \ell^{i-1}(\ell - 1))$, with $a_\ell \in \mathbf{N}$ a constant depending on ℓ. In the classical language, for each i, π is congruent to a mod λ^i eigenform of cohomological weight.

For every i, by Theorem 3.1 and the fact that $\mathbf{T}_{\vec{k}_i} \otimes \mathbf{Q}$ is semisimple, there is a finite extension E_i/E_λ and a Galois representation

$$
\rho_i : \ \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GSp}_4(\mathbf{T}_{\vec{k}} \otimes_{\mathbf{Z}} E_i)
$$

such that $\mathrm{Tr}\, \rho_i(\mathrm{Frob}_p) = \theta_i(T_p)$ whenever $p \notin S_i$, for some finite set of places S_i.

If we could compose ρ_i with r_i to construct a representation $\overline{\rho}_i : \ \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GSp}_4(\mathcal{O}_{E_\lambda}/\lambda^i)$, then we would be able to construct ρ_λ as the limit $\varprojlim_i \overline{\rho}_i$. However, while $\mathrm{Tr}\, \rho_i(\mathrm{Frob}_p) \in \mathbf{T}_{\vec{k}} \otimes_{\mathbf{Z}} \mathcal{O}_{E_i}$ for all $p \notin S_i$, it is not necessarily true that ρ_i can be chosen to be valued in $\mathrm{GSp}_4(\mathbf{T}_{\vec{k}} \otimes_{\mathbf{Z}} \mathcal{O}_{E_i})$. The solution to this problem is to work with pseudorepresentations. Associated to ρ_i is a pseudorepresentation

$$
T_i = \mathrm{Tr}\, \rho_i : \ \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathbf{T}_{\vec{k}_i} \otimes_{\mathbf{Z}} E_i
$$

and, since $T_i(\mathrm{Frob}_p) \in \mathbf{T}_{\vec{k}_i} \otimes_{\mathbf{Z}} \mathcal{O}_{E_i}$ for all $p \notin S_i$ and since $\mathbf{T}_{\vec{k}_i} \otimes_{\mathbf{Z}} \mathcal{O}_{E_i}$ is flat, it is clear that

$$
T_i : \ \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathbf{T}_{\vec{k}_i} \otimes_{\mathbf{Z}} \mathcal{O}_{E_i}
$$

is valued in $\mathbf{T}_{\vec{k}_i} \otimes_{\mathbf{Z}} \mathcal{O}_{E_i}$. Composing with r_i, we obtain a pseudorepresentation

$$
\overline{T}_i : \ \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathcal{O}_{E_i}/\lambda^i.
$$

A computation shows that each $\overline{T}_i$ is in fact valued in $\mathcal{O}_{E_\lambda}/\lambda^i$ and that, for $i \geqslant m$, $\overline{T}_m \equiv \overline{T}_i$ (mod λ^m). Hence, there is a pseudorepresentation

$$T = \varprojlim_i \overline{T}_i : \ \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathcal{O}_{E_\lambda} \subset \overline{E}_\lambda.$$

It follows from the theory of pseudorepresentations [45, Thm. 1] that there is a semisimple Galois representation

$$\rho_\lambda : \ \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_4(\overline{E}_\lambda)$$

associated to T, which is, by construction, the Galois representation associated to π.

Taylor's construction via pseudorepresentations shows that ρ_λ is valued in $\mathrm{GL}_4(\overline{E}_\lambda)$, but is insufficient to show that the representation is isomorphic to one that is valued in $\mathrm{GSp}_4(\overline{E}_\lambda)$: taking the trace of ρ_i 'forgets' the fact that ρ_i is symplectic. The proof of Theorem 3.6 follows the same structure as Taylor's proof, replacing pseudorepresentations with Lafforgue's G-pseudorepresentations.

3.3.2 | Lafforgue pseudorepresentations

In this subsection, we define Lafforgue pseudorepresentations and state their key properties. Most of these are lifted directly from [28, Sec. 11] and [7, Sec. 4].

Let G be a split reductive group over $\mathbf{Z}$, and let $\mathbf{Z}[G^n]^G$ denote the ring of regular functions of G^n that are invariant under conjugation by G.

Definition 3.8. Let A be a topological ring, let Γ be a topological group and let $C(\Gamma^n, A)$ denote the algebra of continuous functions $\Gamma^n \to A$. A (continuous) G-*pseudorepresentation* $\Theta = (\Theta_n)_{n \geqslant 1}$ of Γ over A is a collection of continuous algebra homomorphisms

$$\Theta_n : \ \mathbf{Z}[G^n]^G \to C(\Gamma^n, A)$$

for each integer $n \geqslant 1$, which are functorial in the following sense:

(i) If $n, m \geqslant 1$ are integers and if $\zeta : \{1, \dots, m\} \to \{1, \dots, n\}$, then, for every $f \in \mathbf{Z}[G^m]^G$ and $\gamma_1, \dots, \gamma_n \in \Gamma$, we have

$$\Theta_n(f^\zeta)(\gamma_1, \dots, \gamma_n) = \Theta_m(f)(\gamma_{\zeta(1)}, \dots, \gamma_{\zeta(m)}),$$

where $f^\zeta(\gamma_1, \dots, \gamma_n) := f(\gamma_{\zeta(1)}, \dots, \gamma_{\zeta(m)})$.
(ii) For every integer $n \geqslant 1$, $f \in \mathbf{Z}[G^n]^G$ and $\gamma_1, \dots, \gamma_{n+1} \in \Gamma$, we have

$$\Theta_{n+1}(\hat{f})(\gamma_1, \dots, \gamma_{n+1}) = \Theta_n(f)(\gamma_1, \dots, \gamma_{n-1}, \gamma_n\gamma_{n+1}),$$

where $\hat{f}(\gamma_1, \dots, \gamma_{n+1}) := f(\gamma_1, \dots, \gamma_{n-1}, \gamma_n\gamma_{n+1})$.

For a category-theoretic interpretation of this definition, see [49].

As with classical pseudorepresentations, we can change the ring A. The following facts are immediate from the definitions:

Lemma 3.9. *Let A, A' be topological rings.*

(i) *If $h : A \to A'$ is a morphism of topological rings and if $\Theta = (\Theta_n)_{n \geqslant 1}$ is a G-pseudorepresentation over A, then $h_*(\Theta) = (h \circ \Theta_n)_{n \geqslant 1}$ is a G-pseudorepresentation over A'.*
(ii) *Let $h : A \hookrightarrow A'$ be an injection of topological rings and let Θ' be a G-pseudorepresentation over A'. Suppose that, for every $f \in \mathbf{Z}[G^n]^G$, $\Theta'_n(f) = h \circ g$ for some $g \in C(\Gamma^n, A)$. Then the collection $\Theta = (\Theta_n)_{n \geqslant 1}$ given by $\Theta_n(f) = g$ is a G-pseudorepresentation over A, and $\Theta' = h_*(\Theta)$.*

The connection between G-pseudorepresentations and G-valued representations is encapsulated in the following lemma:

Lemma 3.10. *Let $\rho : \Gamma \to G(A)$ be a continuous homomorphism. For each integer $n \geqslant 1$, let*

$$\Theta_n : \mathbf{Z}[G^n]^G \to C(\Gamma^n, A)$$

be given by

$$\Theta(f)(\gamma_1, \dots, \gamma_n) = f(\rho(\gamma_1), \dots, \rho(\gamma_n)).$$

Then the collection $(\Theta_n)_{n \geqslant 1}$ is a G-pseudorepresentation, which we denote by $\mathrm{Tr}\,\rho$.

Remark 3.11. Fix an embedding $\iota : G \hookrightarrow \mathrm{GL}_r$ for some r, and let χ denote the composition of ι with the usual trace function. Then $\chi \in \mathbf{Z}[G]^G$. Suppose that $\rho : \Gamma \to G(A)$ is a representation with corresponding G-pseudorepresenation $\mathrm{Tr}\,\rho = (\Theta_n)_{n \geqslant 1}$. Then $\Theta_1(\chi) : \Gamma \to A$ is the classical pseudorepresentation associated to the representation $\iota \circ \rho$. Indeed, we have

$$\Theta_1(\chi)(\gamma) = \chi(\rho(\gamma)) = \mathrm{Tr}(\iota \circ \rho(\gamma)),$$

and the properties of this classical pseudorepresentation follow from the properties of $\mathrm{Tr}\,\rho$ [28, Rmk. 11.8].

As in the case of classical pseudorepresentations, if A is in fact an algebraically closed field, then every G-pseudorepresentation arises as the trace of a G-valued representation.

Theorem 3.12 [28, Prop. 11.7], [7, Thm. 4.5]. *Let A be an algebraically closed field and let Θ be a G-pseudorepresentation of Γ over A. Then there is a completely reducible representation $\rho : \Gamma \to G(A)$ such that $\Theta = \mathrm{Tr}(\rho)$.*

Being completely reducible generalises the notion of a GL_n-representation being semisimple. Since we will not use this notion, we refer the reader to [7, Def. 3.3] for the definition.

3.3.3 | Lafforgue pseudorepresentations and Galois representations

A key step in Taylor's construction is to show that the pseudorepresentation T_i is valued in $\mathbf{T}_{\vec{k}_i} \otimes_{\mathbf{Z}} \mathcal{O}_{E_i}$. The following lemma will enable us to prove the analogue of this fact when using GSp_4-pseudorepresentations in place of pseudorepresentations.

Lemma 3.13. *Let $\chi_1, \dots, \chi_r \in \mathbf{Z}[G]^G$. Suppose that, for each integer $n \geqslant 1$, $\mathbf{Z}[G^n]^G$ is generated by functions of the form*

$$(\gamma_1, \dots, \gamma_n) \mapsto \chi_j(\gamma_{\zeta(1)}^{a_1} \gamma_{\zeta(2)}^{a_2} \cdots \gamma_{\zeta(m)}^{a_m}),$$

where $1 \leqslant j \leqslant r$, $m \geqslant 1$, $\zeta : \{1, \dots, m\} \to \{1, \dots, n\}$ and $a_j \in \mathbf{Z}$. Let

$$\rho : \Gamma \to G(A)$$

be a continuous representation. Then $\Theta = \mathrm{Tr}\,\rho$ is completely determined by $\Theta_1(\chi_1), \dots, \Theta_1(\chi_r)$.

Proof. Let Θ be a G-pseudorepresentation, $n \geqslant 1$ be an integer, $\gamma_1, \dots, \gamma_n \in \Gamma$ and $f \in \mathbf{Z}[G^n]^G$. Since each Θ_n is an algebra homomorphism, we may assume that f is of the form

$$f : (\gamma_1, \dots, \gamma_n) \mapsto \chi(\gamma_{\zeta(1)}^{a_1} \gamma_{\zeta(2)}^{a_2} \cdots \gamma_{\zeta(m)}^{a_m}),$$

where $\chi = \chi_j$ for some j, $\zeta : \{1, \dots, m\} \to \{1, \dots, n\}$ and $a_j \in \mathbf{Z}$.

First note that $f = g^{\zeta}$ where $g \in \mathbf{Z}[G^m]^G$ is given by

$$g : (\gamma_1, \dots, \gamma_m) \mapsto \chi(\gamma_1^{a_1} \cdots \gamma_m^{a_m}).$$

It follows that

$$\Theta_n(f)(\gamma_1, \dots, \gamma_n) = \Theta_m(g)(\gamma_{\zeta(1)}, \dots, \gamma_{\zeta(m)}).$$

Since $\Theta = \mathrm{Tr}(\rho)$, we observe that

$$\begin{aligned}
\Theta_m(g)(\gamma_{\zeta(1)}, \dots, \gamma_{\zeta(m)}) &= g\big(\rho(\gamma_{\zeta(1)}), \dots, \rho(\gamma_{\zeta(m)})\big) \\
&= \chi\big(\rho(\gamma_{\zeta(1)})^{a_1} \cdots \rho(\gamma_{\zeta(m)})^{a_m}\big) \\
&= \Theta_m(g')(\gamma_{\zeta(1)}^{a_1}, \dots, \gamma_{\zeta(m)}^{a_m}),
\end{aligned}$$

where

$$g' : (\gamma_1, \dots, \gamma_m) \mapsto \chi(\gamma_1 \cdots \gamma_m).$$

If $m \geqslant 2$, then $g' = \hat{h}$, where $h \in \mathbf{Z}[G^{m-1}]^G$ is given by

$$h : (\gamma_1, \dots, \gamma_{m-1}) \mapsto \chi(\gamma_1 \cdots \gamma_{m-1}),$$

so that

$$\Theta_m(g')(\gamma_{\zeta(1)}^{a_1}, \dots, \gamma_{\zeta(m)}^{a_m}) = \Theta_{m-1}(h)(\gamma_{\zeta(1)}^{a_1}, \dots, \gamma_{\zeta(m-1)}^{a_{m-1}} \gamma_{\zeta(m)}^{a_m}).$$

It follows by induction on m that

$$\Theta_n(f)(\gamma_1, \dots, \gamma_n) = \Theta_1(\chi)(\gamma_{\zeta(1)}^{a_1} \cdots \gamma_{\zeta(m)}^{a_m}). \qquad \square$$

Example 3.14. If $G = \mathrm{GL}_n$, then by work of Processi [11, 36], the ring $\mathbf{Z}[\mathrm{GL}_n^m]^{\mathrm{GL}_n}$ is generated by $\det^{-1}$ and the functions $X \mapsto s_i(X)$ that map $X \in \mathrm{GL}_n$ to the i-th coefficient of its characteristic polynomial, and $\mathbf{Q}[\mathrm{GL}_n^m]^{\mathrm{GL}_n}$ is generated just by the trace map and by $\det^{-1}$. In particular, a Lafforgue GL_n-pseudorepresentation over a characteristic 0 field is completely determined by its associated Taylor pseudorepresentation (cf. [28, Rmk. 11.8]).

In order to prove Theorem 3.6, we show that GSp_4 satisfies the conditions of Lemma 3.13:

Lemma 3.15. *For an element $X \in \mathrm{GSp}_4$, let $t^4 + \sum_{i=1}^{4}(-1)^i s_i(X)t^{4-i}$ be its characteristic polynomial. Then $\mathbf{Z}[\mathrm{GSp}_4^m]^{\mathrm{GSp}_4}$ is generated by $\mathrm{sim}^{\pm 1}$ and the functions $X \mapsto s_i(X)$, $i = 1, 2$, where $s_i(X)$ is the i-th coefficient of the characteristic polynomial.*

Proof. Let M_4 denote the algebra of 4×4 matrices. If K is any infinite field, then, by [54, Thm. 1], the ring $K[\mathrm{M}_4^m]^{\mathrm{Sp}_4} = K[\mathrm{M}_4^m]^{\mathrm{GSp}_4}$ is generated by functions of the form

$$(X_1, \dots, X_m) \mapsto s_i(Y_{j_1} \cdots Y_{j_s}),$$

where each matrix Y_i is either X_i or its symplectic transpose X_i^t. By [54, Prop. 3.2], the ring $K[\mathrm{Sp}_4^m]^{\mathrm{Sp}_4}$ is generated by the images of these functions under the canonical map $K[\mathrm{M}_4^m] \to K[\mathrm{Sp}_4^m]$. If $X \in \mathrm{Sp}_4$, then, by definition, $X^t = X^{-1}$. It follows that $K[\mathrm{Sp}_4^m]^{\mathrm{Sp}_4}$ is generated by functions of the form

$$(X_1, \dots, X_m) \mapsto s_i(Y_{j_1} \cdots Y_{j_s}),$$

where each matrix Y_i is either X_i or X_i^{-1}.

Now, the natural grading on $\mathbf{Z}[\mathrm{M}_4^m]$ by multi-homogeneous degree restricts to a filtration on $\mathbf{Z}[\mathrm{Sp}_4^m]^{\mathrm{Sp}_4}$, and each filtered piece is a finite $\mathbf{Z}$-module. Hence, by the same argument as [11, 15.2.1], it follows that $\mathbf{Z}[\mathrm{Sp}_4^m]^{\mathrm{Sp}_4}$ is generated by functions of the above form as well.

To conclude, we argue as in [54, p. 316]. The natural surjection $\mathrm{Sp}_4^m \times \mathrm{GL}_1^m \to \mathrm{GSp}_4^m$ induces an embedding

$$\mathbf{Z}[\mathrm{GSp}_4^m]^{\mathrm{GSp}_4} \hookrightarrow (\mathbf{Z}[\mathrm{Sp}_4^m] \otimes \mathbf{Z}[\mathrm{GL}_1^m])^{\mathrm{Sp}_4 \times \mathrm{GL}_1} \simeq \mathbf{Z}[\mathrm{Sp}_4^m]^{\mathrm{Sp}_4} \otimes \mathbf{Z}[\mathrm{GL}_1^m].$$

But, by the above computations, every element of $\mathbf{Z}[\mathrm{Sp}_4^m]^{\mathrm{Sp}_4}$ extends to an element of $\mathbf{Z}[\mathrm{GSp}_4^m]^{\mathrm{GSp}_4}$. Hence, the above map is surjective. It follows that $\mathbf{Z}[\mathrm{GSp}_4^m]^{\mathrm{GSp}_4}$ is generated by $\mathrm{sim}^{\pm 1}$ and by functions of the above form, as required. $\qquad \square$

3.3.4 | Proof of Theorem 3.6

Proof of Theorem 3.6. We use the notation from the beginning of the section. Consider the GSp_4-pseudorepresentation $\Theta^{(i)} = \mathrm{Tr}\,\rho_i$ associated to

$$\rho_i: \ \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GSp}_4(\mathbf{T}_{\vec{k}_i} \otimes_{\mathbf{Z}} E_i).$$

By Lemmas 3.13 and 3.15, $\mathrm{Tr}\,\rho_i$ is completely determined by

$$\Theta_1^{(i)}(s_1) = T_i: \ \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathbf{T}_{\vec{k}_i} \otimes_{\mathbf{Z}} E_i,$$

$$\Theta_1^{(i)}(s_2): \ \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathbf{T}_{\vec{k}_i} \otimes_{\mathbf{Z}} E_i$$

and

$$\Theta_1^{(i)}(\mathrm{sim}^{\pm 1}): \ \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathbf{T}_{\vec{k}_i} \otimes_{\mathbf{Z}} E_i.$$

Since each of these maps factors through $\mathbf{T}_{\vec{k}_i} \otimes_{\mathbf{Z}} \mathcal{O}_{E_i}$, it follows from Lemma 3.9(ii) that we can view each $\Theta^{(i)}$ as a GSp_4-pseudorepresentation over $\mathbf{T}_{\vec{k}_i} \otimes_{\mathbf{Z}} \mathcal{O}_{E_i}$. By Lemma 3.9(i) we may compose $\Theta^{(i)}$ with the map $r_i: \mathbf{T}_{\vec{k}_i} \otimes_{\mathbf{Z}} \mathcal{O}_{E_i} \to \mathcal{O}_{E_i}/\lambda^i$ to produce a new GSp_4-pseudorepresentation $\overline{\Theta}^{(i)}$ of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ over $\mathcal{O}_{E_i}/\lambda^i$. Since each $\Theta^{(i)}$ is determined by $\Theta_1^{(i)}(s_i)$, $i = 1, 2$ and $\Theta_1^{(i)}(\mathrm{sim}^{\pm 1})$, it follows that $\overline{\Theta}^{(i)}$ is too. Hence, the arguments of Taylor summarised above show that these maps actually land in $\mathcal{O}_{E_\lambda}/\lambda^i$, so that each $\overline{\Theta}^{(i)}$ is actually a GSp_4-pseudorepresentation over $\mathcal{O}_{E_\lambda}/\lambda^i$. Therefore, we can form a GSp_4-pseudorepresentation

$$\Theta = \varprojlim_i \overline{\Theta}^{(i)}$$

of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ over $\mathcal{O}_{E_\lambda}$. Finally, viewing $\mathcal{O}_{E_\lambda}$ as a subalgebra of $\overline{E}_\lambda$, we may view Θ as a GSp_4-pseudorepresentation over $\overline{E}_\lambda$ and, by Theorem 3.12, there is a representation

$$\rho: \ \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GSp}_4(\overline{E}_\lambda),$$

such that $\Theta = \mathrm{Tr}(\rho)$. This is the Galois representation associated to π. Indeed,

$$\Theta_1(s_1) = T: \ \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \overline{E}_\lambda$$

is exactly the classical pseudorepresentation constructed by Taylor. □

4 | RESTRICTIONS ON THE DECOMPOSITION OF ρ_λ

Recall that π is a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbf{A_Q})$ of weight $(k, 2)$ in the notation of Section 2.2, and that the weak functorial lift Π of π to GL_4 exists and is cuspidal. The goal of this section is to prove the following theorem:

Theorem 4.1. *Keep the assumptions of Theorem 1.1. For each prime λ, either:*

- ρ_λ *is irreducible or*
- ρ_λ *decomposes as a direct sum $\tau_1 \oplus \tau_2$ of distinct, irreducible, two-dimensional representations, each with determinant $\mathrm{sim}\, \rho_\lambda$. Moreover, for each i, $\tau_i|_{\mathbf{Q}_\ell}$ is Hodge–Tate with Hodge–Tate weights $\{0, k-1\}$.*

When π is cohomological, Theorem 4.1 is due to Weissauer [51, Thm. II] and Ramakrishnan [38, Thm. A]. In the cohomological case, since ρ_λ has distinct Hodge–Tate weights, the fact that the two representations are distinct and Hodge–Tate regular is obvious.

4.1 | Irreducibility without the Ramanujan bounds

The proof of [51, Thm. II] makes crucial use of the Ramanujan bounds, which are not available in the non-cohomological case. In place of the Ramanujan bounds, we prove the following theorem, which requires only the Jacquet–Shalika bounds [24], which are known for all cuspidal automorphic representations of GL_n.

Theorem 4.2. *Let $\rho :\ \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$ be a Galois representation. Assume that ρ is automorphic: there exists a unitary cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbf{A_Q})$ and an integer w such that $L^*(\Pi, s - \frac{w}{2}) = L^*(\rho, s)$.*

Let τ be an m-dimensional subrepresentation of ρ, and suppose that $\det \tau|_{\mathbf{Q}_\ell}$ is Hodge–Tate with Hodge–Tate weight h. Then

$$\left| h - \frac{mw}{2} \right| < \frac{m}{2}.$$

Remark 4.3. The generalised Ramanujan conjecture would give $h = \frac{mw}{2}$.

Proof. Fix an embedding $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ as well as a prime $p \neq \ell$ at which both ρ and Π are unramified. Let $\alpha_1, \dots, \alpha_n \in \mathbf{C}$ be the Satake parameters of Π_p. Then, by assumption and via our fixed embedding, the eigenvalues of $\rho(\mathrm{Frob}_p)$ are $\alpha_1 p^{\frac{w}{2}}, \dots, \alpha_n p^{\frac{w}{2}}$.

By [24, Cor. 2.5], for each i, we have

$$p^{\frac{w-1}{2}} < |\alpha_i p^{\frac{w}{2}}| < p^{\frac{w+1}{2}}.$$

Since τ is an m-dimensional subrepresentation of ρ, it follows that $\det \tau(\mathrm{Frob}_p)$ is a product of m of the eigenvalues of $\rho(\mathrm{Frob}_p)$. Thus

$$p^{\frac{m(w-1)}{2}} < |\det \tau(\mathrm{Frob}_p)| < p^{\frac{m(w+1)}{2}}.$$

On the other hand, the assumption on the Hodge–Tate weight of $\det \tau|_{\mathbf{Q}_\ell}$ means that $\det \tau \simeq \chi \epsilon_\ell^h$, where χ is a finite order character. Thus, $|\det \tau(\mathrm{Frob}_p)| = p^h$ and hence, $\frac{m(w-1)}{2} < h < \frac{m(w+1)}{2}$. $\qquad\square$

4.2 | Proof of Theorem 4.1

Lemma 4.4. *Suppose that Π is an automorphic induction. Then ρ_λ is irreducible.*

Proof. Suppose that Π is automorphically induced from an automorphic representation Π' of $\mathrm{GL}_2(\mathbf{A}_F)$ for some quadratic extension $F/\mathbf{Q}$, and let $\rho_{\Pi',\lambda} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\overline{E}_\lambda)$ be the λ-adic Galois representation associated to Π'. It is well known that $\rho_{\Pi',\lambda}$ is irreducible: when F is real quadratic, this is due to Ribet [40], while if F is imaginary quadratic, Taylor [47, Sec. 3] proved irreducibility in many cases. See [50, Thm. 1.2.6] for a complete proof.

Since, by assumption, Π is cuspidal, we have $\Pi' \not\cong (\Pi')^\sigma$, where σ is the non-trivial element of $\mathrm{Gal}(F/\mathbf{Q})$ [3]. By local–global compatibility and the strong multiplicity one theorem for GL_n, $\rho_\lambda \simeq \mathrm{Ind}_F^{\mathbf{Q}} \rho_{\Pi',\lambda}$ and $\rho_{\Pi',\lambda} \not\simeq \rho_{\Pi',\lambda}^\sigma$. It follows that ρ_λ is irreducible. $\qquad\square$

Lemma 4.5. *Suppose that τ is a proper subrepresentation of ρ_λ. Then τ is two-dimensional and $\tau|_{\mathbf{Q}_\ell}$ is Hodge–Tate, with Hodge–Tate weights $\{0, k-1\}$.*

Proof. If τ is three-dimensional, then ρ_λ also has a one-dimensional subrepresentation. Hence, we can assume that τ is m-dimensional, with $m = 1$ or 2. By Theorem 3.3, the Hodge–Tate–Sen weights of $\rho_\lambda|_{\mathbf{Q}_\ell}$ are $\{0, 0, k-1, k-1\}$. Thus, the Hodge–Tate–Sen weights $a_1, \dots, a_m$ of $\tau|_{\mathbf{Q}_\ell}$ are contained in $\{0, 0, k-1, k-1\}$ and, by Theorem 4.2, $a_1 + \cdots + a_m = \frac{1}{2}m(k-1)$. It follows that $m = 2$ and that $\tau|_{\mathbf{Q}_\ell}$ has Hodge–Tate–Sen weights $\{0, k-1\}$. Since the Hodge–Tate–Sen weights are distinct integers, it follows that the Sen operator is semisimple [31, Thm. 5.17], and hence that $\tau|_{\mathbf{Q}_\ell}$ is Hodge–Tate. $\qquad\square$

Lemma 4.6. *Suppose that $\rho_\lambda \simeq \tau_1 \oplus \tau_2$, with τ_1, τ_2 irreducible and two-dimensional. Then there does not exist a finite order character χ with $\tau_1 \simeq \tau_2 \otimes \chi$.*

Proof. Suppose that $\tau_1 \simeq \tau_2 \otimes \chi$ for a finite order character χ. Then χ cuts out a finite cyclic extension K of $\mathbf{Q}$, and $\rho_\lambda|_K \simeq \tau_1|_K \oplus \tau_1|_K$. We have

$$\bigwedge{}^2 \rho_\lambda|_K \simeq (\tau_1|_K \otimes \tau_1|_K) \oplus \det \tau_1|_K \oplus \det \tau_1|_K \simeq \mathrm{Sym}^2 \tau_1|_K + \det \tau_1|_K^{\oplus 3}.$$

Now, $\det \tau_1|_K$ is Hodge–Tate at all places above ℓ by Lemma 4.5, so, by class field theory, $L(\det \tau_1|_K, s)$ has meromorphic continuation to the whole of $\mathbf{C}$. Since $K/\mathbf{Q}$ is cyclic, by cyclic base change [3] and by [27], $\bigwedge^2 \rho_\lambda|_K$ is the Galois representation associated to an isobaric automorphic representation of $\mathrm{GL}_6(\mathbf{A}_K)$. Thus, $L(\bigwedge^2 \rho_\lambda|_K, s)$ has meromorphic continuation to the whole of $\mathbf{C}$. It follows that $L(\mathrm{Sym}^2 \tau_1|_K, s)$ does too and, by twisting, so does $L(\mathrm{Sym}^2 \tau_1 \otimes \det \tau_1^{-1}|_K, s)$.

On the other hand,

$$\rho_\lambda|_K \otimes \rho_\lambda|_K^\vee \simeq (\tau_1 \otimes \tau_1^\vee)^{\oplus 4} \simeq (\mathrm{Sym}^2 \tau_1 \otimes \det \tau_1^{-1})^{\oplus 4} \oplus \mathbf{1}^{\oplus 4}.$$

Let Π be the transfer of π to $\mathrm{GL}_4(\mathbf{A}_\mathbf{Q})$. By assumption, Π is cuspidal. Moreover, by Lemma 4.4, we may assume that Π is not an automorphic induction. Hence, the cyclic base change Π_K of Π is also cuspidal [3, Thm. 4.2] and, by [23, Prop. 3.6], the L-function

$$L^*(\Pi_K \otimes \Pi_K^\vee, s) = L^*(\rho_\lambda|_K \otimes \rho_\lambda|_K^\vee, s)$$

has a simple pole at $s = 1$. Since $L(\mathbf{1}, s) = \zeta(s)$ also has a simple pole at $s = 1$, it follows that

$$\mathrm{ord}_{s=1} L(\mathrm{Sym}^2 \tau_1 \otimes \det \tau_1^{-1}|_K, s)^4 = 3.$$

But $L(\mathrm{Sym}^2 \tau_1 \otimes \det \tau_1^{-1}|_K, s)$ is meromorphic, so its order of vanishing at $s = 1$ must be an integer. This is a contradiction. $\square$

Proof of Theorem 4.1. After Lemma 4.5, it remains to show that, if $\rho_\lambda \simeq \tau_1 \oplus \tau_2$, with τ_1, τ_2 irreducible and two-dimensional, then $\det \tau_1 \simeq \det \tau_2 \simeq \mathrm{sim}\, \rho_\lambda$.

Let $\chi = \det \tau_1^{-1} \otimes \mathrm{sim}\, \rho_\lambda$. By Lemma 4.5, $\chi|_{\mathbf{Q}_\ell}$ has Hodge–Tate weight 0, that is, χ is a finite order character. We need to show that χ is the trivial character. First, since

$$(\mathrm{sim}\, \rho_\lambda)^2 \simeq \det \rho_\lambda \simeq \det \tau_1 \det \tau_2,$$

it follows that $\chi \simeq \det \tau_2 \otimes \mathrm{sim}\, \rho_\lambda^{-1}$. Moreover,

$$\begin{aligned}
\tau_1 \oplus \tau_2 &\simeq \rho_\lambda \simeq \rho_\lambda^\vee \otimes \mathrm{sim}\, \rho_\lambda \\
&\simeq \left(\tau_1^\vee \otimes \mathrm{sim}\, \rho_\lambda\right) \oplus \left(\tau_2^\vee \otimes \mathrm{sim}\, \rho_\lambda\right) \\
&\simeq \left(\tau_1 \otimes \det \tau_1^{-1} \otimes \mathrm{sim}\, \rho_\lambda\right) \oplus \left(\tau_2 \otimes \det \tau_2^{-1} \otimes \mathrm{sim}\, \rho_\lambda\right) \\
&\simeq (\tau_1 \otimes \chi) \oplus \left(\tau_2 \otimes \chi^{-1}\right).
\end{aligned}$$

By Schur's lemma and by Lemma 4.6, it follows that $\tau_i \otimes \chi \simeq \tau_i$ for each $i = 1, 2$. Thus, $\rho_\lambda \otimes \chi \simeq \rho_\lambda$. Let Π be the transfer of π to GL_4. By class field theory, we may view χ as a Hecke character, and it follows that Π and $\Pi \otimes \chi$ have the same Hecke polynomials at almost all primes. By the strong multiplicity one theorem for GL_4, we have $\Pi \simeq \Pi \otimes \chi$. Hence, by [3, Thm. 4.2], if χ is nontrivial, then Π is an automorphic induction, in which case, ρ_λ is already irreducible by Lemma 4.4. Hence, χ must be trivial. $\square$

5 | CRYSTALLINITY AND DISTINCTNESS OF SATAKE PARAMETERS

In the previous section, we proved that, if ρ_λ is reducible, then it decomposes as a direct sum $\tau_1 \oplus \tau_2$ of distinct, irreducible, odd, two-dimensional representations. In this section, we will apply that result to prove the following theorem:

Theorem 5.1. *Keep the assumptions of Theorem 1.1, and assume that the transfer Π of π to GL_4 is not an automorphic induction. For a set of primes p of Dirichlet density 1, the roots of the p-th Hecke polynomial of π are pairwise distinct.*

Applying part Theorem 3.3(vi), that is, [25, Thm. 4.1], we immediately deduce:

Corollary 5.2. *The representation $\rho_\lambda|_{\mathbf{Q}_\ell}$ is crystalline for all primes $\lambda \mid \ell$ for a set of primes ℓ of Dirichlet density 1.*

Remark 5.3. If Π is automorphically induced from an automorphic representation Π' of $\mathrm{GL}_2(\mathbf{A}_F)$, then Corollary 5.2 still holds for π. Indeed, let $\rho_{\Pi',\lambda}$ be the λ-adic Galois representation attached to Π'. Then $\rho_\lambda \simeq \mathrm{Ind}_F^{\mathbf{Q}} \rho_{\Pi',\lambda}$. If F is real quadratic, then Π' arises from a cohomological Hilbert modular form. Hence, $\rho_{\Pi',\lambda}$ is crystalline, from which it is clear that ρ_λ is too. If $F/\mathbf{Q}$ is imaginary quadratic, then one can prove Corollary 5.2 by slightly amending the arguments of this section. In both these cases, ρ_λ is irreducible by Lemma 4.4.

5.1 | The Lie algebra of ρ_λ

Fix a prime λ, and let F/E_λ be a finite extension such that

$$\rho_\lambda : \ \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GSp}_4(F)$$

is defined over F. Let G_λ be the Zariski closure of the image of ρ_λ in $\mathrm{GSp}_{4/F}$, let G_λ° be its identity connected component and let $\mathfrak{g}_\lambda$ be its Lie algebra. In this subsection, we determine the Lie algebra $\mathfrak{g}_\lambda$ and the group G_λ when ρ_λ is irreducible.

Proposition 5.4. *Assume that the weight of π is not of the form $(2k-1, k+1)$ for some $k \geqslant 2$. If ρ_λ is irreducible, then $\mathfrak{g}_\lambda \simeq \mathfrak{gsp}_4(F)$.*

Remark 5.5. If the weight of π is of the form $(2k-1, k+1)$ for some $k \geqslant 2$ — so, in particular, π is cohomological — then we cannot rule out the case that $\mathfrak{g}_\lambda \simeq \mathrm{Sym}^3 \mathfrak{sl}_2(F)$ in complete generality. However, when $\lambda \mid \ell$ with $\ell \geqslant 5$, then one can rule out this case by combining [10, Thm. 3.8] with [33, Thm. 1.0.4].

Corollary 5.6. *Assume that the weight of π is not of the form $(2k-1, k+1)$ for some $k \geqslant 2$. If ρ_λ is irreducible, then $G_\lambda \simeq \mathrm{GSp}_4(F)$.*

Proof. G_λ is a Zariski closed subgroup of $\mathrm{GSp}_4(F)$. Moreover, $\mathrm{Lie}(G_\lambda) = \mathrm{Lie}(\mathrm{GSp}_4(F))$, so G_λ is also a Zariski open subgroup of $\mathrm{GSp}_4(F)$. Since $\mathrm{GSp}_4(F)$ is Zariski-connected, it follows that $G_\lambda = \mathrm{GSp}_4(F)$. □

Definition 5.7. Let G be a group, and k be a field. We say that a representation

$$\rho : G \to \mathrm{GL}_n(k)$$

is *Lie irreducible* if $\rho|_H$ is irreducible for all finite index subgroups $H \leqslant G$.

Definition 5.8. Let G be a group, and k be a field. We say that a representation

$$\rho : G \to \mathrm{GL}_n(k)$$

is *imprimitive* if it is absolutely irreducible, but there is a finite index subgroup H of G and a $\bar{k}$-representation τ of H such that $\rho \otimes \bar{k} \simeq \mathrm{Ind}_H^G \tau$. Otherwise, we say that ρ is *primitive*.

Lemma 5.9. *If ρ_λ is irreducible, then it is Lie irreducible.*

Proof. First suppose that $\rho_\lambda \simeq \mathrm{Ind}_K^{\mathbf{Q}} \tau$ is imprimitive. We first note that we may assume that $K/\mathbf{Q}$ is quadratic. If not, then, by counting dimensions, $[K : \mathbf{Q}] = 4$. If K contains a quadratic subextension K', then $\rho_\lambda = \mathrm{Ind}_{K'}^{\mathbf{Q}}(\mathrm{Ind}_{K'}^K \tau)$. If K does not contain a quadratic subfield, the proof of [17, Lem. 5.3] shows that ρ_λ is induced from a different quadratic extension. It follows from Clifford theory that

$$\rho_\lambda \simeq \rho_\lambda \otimes \chi_{K/\mathbf{Q}},$$

where $\chi_{K/\mathbf{Q}}$ is the quadratic character that cuts out the extension $K/\mathbf{Q}$. Let Π be the lift of π to GL_4 and recall that, by assumption, Π is cuspidal and not an automorphic induction. By class field theory, we may view $\chi_{K/\mathbf{Q}}$ as a Hecke character, and the above isomorphism implies that Π and $\Pi \otimes \chi_{K/\mathbf{Q}}$ are weakly equivalent. Hence, by the strong multiplicity one theorem for GL_4,

$$\Pi \simeq \Pi \otimes \chi_{K/\mathbf{Q}}.$$

By [3, Thm. 4.2], this isomorphism is equivalent to Π being an automorphic induction, a contradiction.

Hence, we may assume that ρ_λ is imprimitive. By [34, Prop. 3.4.1], we can write

$$\rho_\lambda \simeq \tau \otimes \omega,$$

where τ is a Lie irreducible representation of dimension d with $d \mid 4$, and ω is an Artin representation of dimension $\frac{4}{d}$. The fact that the Hodge–Tate–Sen weights of ρ_λ are not all equal shows that ρ_λ is not a twist of an Artin representation. Hence, $d \neq 1$. Suppose that $d = 2$. If ω is imprimitive — say $\omega \simeq \mathrm{Ind}_K^{\mathbf{Q}} \chi$ for some quadratic extension $K/\mathbf{Q}$ and character χ of $\mathrm{Gal}(\overline{\mathbf{Q}}/K)$ — then

$$\rho_\lambda \simeq \mathrm{Ind}_K^{\mathbf{Q}}(\tau|_K \otimes \chi)$$

is also imprimitve, a contradiction. Hence, we may assume that both τ and ω are primitive. It follows that $\mathrm{Sym}^2 \tau$ and $\mathrm{Sym}^2 \omega$ are both irreducible. Taking exterior squares, we find that

$$\bigwedge\nolimits^2 \rho_\lambda \simeq \bigwedge\nolimits^2 (\tau \otimes \omega) \simeq \left(\bigwedge\nolimits^2 \tau \otimes \mathrm{Sym}^2 \omega \right) \oplus \left(\bigwedge\nolimits^2 \omega \otimes \mathrm{Sym}^2 \tau \right)$$

does not contain a one-dimensional subrepresentation, contradicting the fact that $\rho_{\pi,\ell}$ is symplectic.

The only remaining possibility is that $d = 4$, that is, that ρ_λ is Lie irreducible. $\square$

Proof of Proposition 5.4. Let G'_λ be the commutator subgroup of G_λ and let $\mathfrak{g}'_\lambda$ be its Lie algebra. Since ρ_λ is a semisimple representation, it follows that G_λ is a reductive group, that G'_λ is semisimple and, hence, that $\mathfrak{g}'_\lambda$ is a semisimple Lie subalgebra of $\mathfrak{sp}_4(F)$. Moreover, since the similitude of ρ_λ does not have finite image, we have $\mathfrak{g}_\lambda \simeq \mathfrak{g}'_\lambda \oplus \mathfrak{gl}_1$.

Fix an embedding $F \hookrightarrow \overline{\mathbf{Q}}_\ell$. Then, by the classification of semisimple algebras, the Lie algebra $\mathfrak{g}'_\lambda \otimes_F \overline{\mathbf{Q}}_\ell$ is one of the following Lie algebras [22, 9.3.1]:

(i) $\mathfrak{sp}_4(\overline{\mathbf{Q}}_\ell)$;
(ii) $\mathfrak{sl}_2(\overline{\mathbf{Q}}_\ell) \times \mathfrak{sl}_2(\overline{\mathbf{Q}}_\ell)$;
(iii) $\mathfrak{sl}_2(\overline{\mathbf{Q}}_\ell)$ embedded in a Klingen parabolic subalgebra;
(iv) $\mathfrak{sl}_2(\overline{\mathbf{Q}}_\ell)$ embedded in a Siegel parabolic subalgebra;
(v) $\mathfrak{sl}_2(\overline{\mathbf{Q}}_\ell)$ embedded via the symmetric cube representation $SL_2 \to Sp_4$;
(vi) $\{1\}$.

Let $d\rho_\lambda : \mathfrak{g}_\lambda \to \mathfrak{gsp}_4(F)$ be the Lie algebra representation associated to the map $G_\lambda \hookrightarrow GSp_4(F)$

We need to show that $\mathfrak{g}'_\lambda \simeq \mathfrak{sp}_4(F)$. Since ρ_λ is irreducible, by Lemma 5.9, ρ_λ is Lie irreducible, which exactly says that $d\rho_\lambda$ is irreducible. Moreover, since $\mathfrak{g}_\lambda$ is semisimple, we can write $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{a}$, where $\mathfrak{a}$ is abelian. It follows that $d\rho_\lambda$ is irreducible if and only its restriction ot $\mathfrak{g}'_\lambda$ is. Thus, $\mathfrak{g}'_\lambda \otimes_F \overline{\mathbf{Q}}_\ell$ cannot be as in cases (ii), (iii), (iv) or (vi).

Suppose that $\mathfrak{g}'_\lambda \otimes_F \overline{\mathbf{Q}}_\ell$ is as in case (v), that is, that $\mathfrak{g}'_\lambda \otimes_F \overline{\mathbf{Q}}_\ell \simeq Sym^3 \mathfrak{sl}_2(\overline{\mathbf{Q}}_\ell)$. Then $(G^\circ_\lambda)' \times_F \overline{\mathbf{Q}}_\ell \simeq Sym^3 SL_2(\overline{\mathbf{Q}}_\ell)$. Since the similitude of ρ_λ does not have finite image, it follows that $G^\circ \times_F \overline{\mathbf{Q}}_\ell \simeq Sym^3 GL_2(\overline{\mathbf{Q}}_\ell)$. There is a finite Galois extension $K/\mathbf{Q}$ such that $\rho_\lambda(Gal(\overline{\mathbf{Q}}/K)) \subseteq G^\circ$. It follows that $\rho_\lambda|_K \simeq Sym^3 \tau$ for some two-dimensional representation τ. If the Hodge–Tate–Sen weights of τ at any place $v \mid \infty$ are $\{a, b\}$, then the Hodge–Tate–Sen weights of $\rho_\lambda|_K$ at v are $\{3a, 2a + b, a + 2b, 3b\}$, which contradicts the assumption on the weights of π.

It follows that $\mathfrak{g}'_\lambda \otimes_F \overline{\mathbf{Q}}_\ell \simeq \mathfrak{sp}_4(\overline{\mathbf{Q}}_\ell)$. Now, $\mathfrak{g}'_\lambda$ is a vector subspace of $\mathfrak{sp}_4(F)$. The fact that $\mathfrak{g}'_\lambda \otimes_F \overline{\mathbf{Q}}_\ell \simeq \mathfrak{sp}_4(F) \otimes_F \overline{\mathbf{Q}}_\ell$ shows that $\mathfrak{g}'_\lambda$ and $\mathfrak{sp}_4(F)$ have the same dimension, and hence are equal. $\square$

5.2 | Distinctness of Satake parameters

We recall the following theorem, due to Rajan:

Theorem 5.10 [37, Thm. 3]. *Let F be a finite extension of $\mathbf{Q}_\ell$ and let $\mathcal{G}$ be an algebraic group defined over F. Let X be an algebraic subscheme of $\mathcal{G}$, defined over F, that is stable under the adjoint action of $\mathcal{G}$. Let*

$$\rho : Gal(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathcal{G}(F)$$

be a Galois representation, and let

$$C = X(F) \cap \rho(Gal(\overline{\mathbf{Q}}/\mathbf{Q})).$$

Let G denote the Zariski closure of $\rho(Gal(\overline{\mathbf{Q}}/\mathbf{Q}))$ in $\mathcal{G}_{/F}$, with identity connected component G° and component group $\Phi = G/G^\circ$. For each $\phi \in \Phi$, let G^ϕ denote the corresponding connected component,

and let

$$\Psi = \big\{\phi \in \Phi : G^\phi \subseteq X\big\}.$$

Then the set of primes p such that $\rho(\mathrm{Frob}_p) \in C$ has Dirichlet density $\frac{|\Psi|}{|\Phi|}$.

Corollary 5.11. *Suppose that ρ_λ is irreducible and not a symmetric cube lift. Then the roots of the p-th Hecke polynomial of π are pairwise distinct for a set of primes p of Dirichlet density 1.*

Proof. Up to multiplication by a normalisation factor, if ρ_λ is unramified at p, then the roots of the p-th Hecke polynomial are exactly the eigenvalues of $\rho_\lambda(\mathrm{Frob}_p)$.

Let $X \subseteq \mathrm{GSp}_4$ be the set of elements $g \in \mathrm{GSp}_4$ whose characteristic polynomials have indistinct eigenvalues. Then X is a closed subscheme of GSp_4 — it is the vanishing set of the discriminant of the characteristic polynomial — that is stable under the conjugation action of GSp_4. By Corollary 5.6, the Zariski closure of ρ_λ is $\mathrm{GSp}_4(F)$, which is connected, and clearly is not contained in X. The result follows from Theorem 5.10. $\square$

Lemma 5.12. *Suppose that $\rho_\lambda \simeq \mathrm{Sym}^3 \tau$ is a symmetric cube lift from an irreducible two-dimensional representation τ. Then the roots of the p-th Hecke polynomial of π are pairwise distinct for a set of primes p of Dirichlet density 1.*

Proof. By Lemma 5.9, ρ_λ is Lie irreducible, and hence τ is as well. Arguing as in Proposition 5.4 (or as in [41]), we see that the Zariski closure of the image of τ is $\mathrm{GL}_2(F)$. Thus, as in Corollary 5.11, it follows that the eigenvalues α_p, β_p of $\tau(\mathrm{Frob}_p)$ are distinct for a set of primes p of Dirichlet density 1. Moreover, by [34, Prop. 3.4.9], $\alpha_p \neq \beta_p$ for a set of primes p of Dirichlet density 1. Lastly, if $\alpha_p = \zeta_3 \beta_p$, where ζ_3 is a cube root of unity, then $\mathrm{Tr}\,\mathrm{Sym}^2 \tau(\mathrm{Frob}_p) = 0$. Thus, if $\alpha_p = \zeta_3 \beta_p$ for a positive density of primes p, it follows from [34, Prop. 3.4.9] that $\mathrm{Sym}^2 \tau$ is irreducible, but not Lie irreducible. But if $\mathrm{Sym}^2 \tau|_K$ is reducible, then $\tau|_K$ is induced from a quadratic extension L/K, so $\tau|_L$ is reducible. Hence, $\rho_\lambda|_L$ is reducible, contradicting Lemma 5.9.

It follows that the eigenvalues $\alpha_p^3, \alpha_p^2 \beta_p, \alpha_p \beta_p^2, \beta_p^3$ of $\rho_\lambda(\mathrm{Frob}_p)$ are distinct for a set of primes p of Dirichlet density 1. $\square$

Lemma 5.13. *Suppose that $\rho_\lambda \simeq \tau_1 \oplus \tau_2$, with τ_1, τ_2 irreducible and two-dimensional. Then, up to reordering, τ_1 is Lie irreducible.*

Proof. Suppose that τ_1 and τ_2 are both irreducible, but not Lie irreducible. Then, for each i, there exists a quadratic extension $K_i/\mathbf{Q}$ such that $\tau_i|_{K_i}$ is reducible. Let $K = K_1 \cdot K_2$. Then K is a solvable extension of $\mathbf{Q}$. Recall that there is a cuspidal automorphic representation Π of $\mathrm{GL}_4(\mathbf{A}_\mathbf{Q})$ associated to π. By [3] and the assumption that Π is not an automorphic induction, the solvable base change Π_K of Π is cuspidal. On the other hand, $\rho_\lambda|_K$ is a direct sum of four Hodge–Tate characters, $\chi_1 \oplus \chi_2 \oplus \chi_3 \oplus \chi_4$. By class field theory, we may view the characters χ_i as Hecke characters. By the strong multiplicity one theorem for GL_4, it follows that Π_K is isomorphic to the isobaric sum of four Hecke characters, contradicting the fact that Π_K is cuspidal. $\square$

Proof of Theorem 5.1. After Theorem 4.1, Corollary 5.11 and Lemma 5.12, it remains to consider the case that $\rho_\lambda \simeq \tau_1 \oplus \tau_2$, with τ_1, τ_2 distinct, irreducible two-dimensional Hodge–Tate representations, each with Hodge–Tate weights $\{0, k-1\}$ and with determinant sim ρ_λ.

For each $i = 1, 2$ and for each prime p at which ρ_λ is unramified, let $\alpha_{p,i}, \beta_{p,i}$ be the roots of the characteristic polynomial of $\tau_i(\mathrm{Frob}_p)$. By Lemma 5.13, we may assume that τ_1 is Lie irreducible. Hence, as in the proof of Lemma 5.12, we see that $\alpha_{p,1} \neq \beta_{p,1}$ for a set of primes p of Dirichlet density 1.

If τ_2 is Lie irreducible, then the same argument shows that $\alpha_{p,1} \neq \beta_{p,2}$ for a set of primes p of Dirichlet density 1. Otherwise, since τ_2 is not an Artin representation, it follows that there is a quadratic extension $K/\mathbf{Q}$ and a character χ of $\mathrm{Gal}(\overline{\mathbf{Q}}/K)$ such that

$$\tau_2 \simeq \mathrm{Ind}_K^{\mathbf{Q}}(\chi).$$

If p is a prime that is inert in K, then $\mathrm{Tr}\,\tau_2(\mathrm{Frob}_p) = \alpha_{p,1} + \beta_{p,1} = 0$, from which it follows that $\alpha_{p,1} \neq \beta_{p,2}$. If p splits as vv^c in K, then the eigenvalues of $\tau_2(\mathrm{Frob}_p)$ are $\chi(\mathrm{Frob}_v)$ and $\chi(\mathrm{Frob}_{v^c}) = \chi^c(\mathrm{Frob}_v)$, where c is the non-trivial element of $\mathrm{Gal}(K/\mathbf{Q})$. Note that $\chi \not\simeq \chi^c$, since τ_2 is irreducible. Moreover, since τ_2 is Hodge–Tate regular, it follows that $K/\mathbf{Q}$ is imaginary quadratic, that χ and χ^c have infinite image and that they have different Hodge–Tate weights at the two complex places. If $\chi(\mathrm{Frob}_v) = \chi^c(\mathrm{Frob}_v)$ for a positive density of primes p, then, by [37, Thm. 2], there exists a finite order character ω such that $\chi \simeq \chi^c \otimes \omega$, which contradicts the fact that χ and χ^c have distinct Hodge–Tate weights.

It follows that for a set of primes p of Dirichlet density 1, we have $\alpha_{p,i} \neq \beta_{p,i}$ for each $i = 1, 2$. If the eigenvalues of $\rho_\lambda(\mathrm{Frob}_p)$ are indistinct for a positive density of primes p, relabelling, we may assume that $\alpha_{p,1} = \alpha_{p,2}$. Since $\det \tau_1 \simeq \det \tau_2$, it follows that $\beta_{p,1} = \beta_{p,2}$ as well. Hence, for a set of primes p of positive density, we have $\mathrm{Tr}\,\tau_1(\mathrm{Frob}_p) = \mathrm{Tr}\,\tau_2(\mathrm{Frob}_p)$. By [37, Thm. 2], there exists a finite order character χ such that $\tau_1 \simeq \tau_2 \otimes \chi$, contradicting Lemma 4.6. $\qquad\square$

6 | PROOF OF THEOREM 1.1

Proof of Theorem 1.1. After Corollary 5.2, it remains to show that, if $\lambda \mid \ell$ with $\ell \geqslant 5$ and if $\rho_\lambda|_{\mathbf{Q}_\ell}$ is de Rham, then ρ_λ is irreducible.

Suppose that ρ_λ is reducible. Then, by Theorem 4.1, we can write $\rho_\lambda \simeq \tau_1 \oplus \tau_2$ where the representations τ_1, τ_2 are distinct, two-dimensional, irreducible, Hodge–Tate regular and odd. Since $\rho_\lambda|_{\mathbf{Q}_\ell}$ is de Rham, so are $\tau_1|_{\mathbf{Q}_\ell}$ and $\tau_2|_{\mathbf{Q}_\ell}$. Thus, since $\ell \geqslant 5$, by [33, Thm. 1.0.4], there exist distinct cuspidal automorphic representations π_1, π_2 of $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})$ such that, for each i, τ_i is the λ-adic Galois representation associated to π_i.

Consider the representation

$$\rho_\lambda \otimes \rho_\lambda^\vee \simeq (\tau_1 \otimes \tau_1^\vee) \oplus (\tau_1 \otimes \tau_2^\vee) \oplus (\tau_2 \otimes \tau_1^\vee) \oplus (\tau_2 \otimes \tau_2^\vee).$$

Since ρ_λ, τ_1 and τ_2 are automorphic, we obtain an equality of (partial) L-functions

$$L^*(\pi \otimes \pi^\vee, s) = L^*(\pi_1 \otimes \pi_1^\vee, s) L^*(\pi_1 \otimes \pi_2^\vee, s) L^*(\pi_2 \otimes \pi_1^\vee, s) L^*(\pi_2 \otimes \pi_2^\vee, s).$$

Since the transfer of π to GL_4 is cuspidal, by [23, Prop. 3.6], the left-hand side has a simple pole at $s = 1$. Similarly, for each $i = 1, 2$, $\mathrm{ord}_{s=1} L^*(\pi_i \otimes \pi_i^\vee, s) = -1$. However, by [23, Thm. 3.7], the L-functions $L^*(\pi_1 \otimes \pi_2^\vee, s)$ and $L^*(\pi_2 \otimes \pi_1^\vee, s)$ are non-zero at $s = 1$. It follows that the right-hand side has a pole of order 2 at $s = 1$, a contradiction. Hence, ρ_λ is irreducible. $\square$

7 | RESIDUAL IRREDUCIBILITY AND THE IMAGE OF GALOIS

In this section, we prove Theorem 1.2. Our arguments generalise those of [14–16] to the case that π is non-cohomological. Moreover, Lemma 7.3 allows us to strengthen the results of [16] even in the cohomological case (see [16, Rmk. 3.4]). Although the results in this section apply to automorphic representations of arbitrary weight, for ease of notation, we will assume that π has non-cohomological weight $(k, 2)$, for some integer $k \geqslant 2$.

Our key tool is following proposition, which is a simple consequence of Fontaine–Laffaille theory. Recall that $\mathcal{L}$ is the set of primes λ for which $\rho_\lambda|_{\mathbf{Q}_\ell}$ is crystalline. By Theorem 1.1, $\mathcal{L}$ contains all primes $\lambda \mid \ell$ for a set of rational primes ℓ of Dirichlet density 1.

Proposition 7.1. *Suppose that $\lambda \in \mathcal{L}$ has residue characteristic $\ell > k$. Then we have the following possibilities for the action of the inertia group I_ℓ at ℓ (cf. [13]):*

$$\begin{pmatrix} 1 & * & * & * \\ 0 & \bar{\epsilon}_\ell^{k-1} & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & \bar{\epsilon}_\ell^{k-1} \end{pmatrix}, \begin{pmatrix} \psi_2^{k-1} & 0 & * & * \\ 0 & \psi_2^{\ell(k-1)} & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & \bar{\epsilon}_\ell^{k-1} \end{pmatrix},$$

$$\begin{pmatrix} \psi_2^{k-1} & 0 & * & * \\ 0 & \psi_2^{\ell(k-1)} & * & * \\ 0 & 0 & \psi_2^{\ell(k-1)} & 0 \\ 0 & 0 & 0 & \psi_2^{k-1} \end{pmatrix}, \begin{pmatrix} \psi_4^{(\ell+\ell^2)(k-1)} & 0 & 0 & 0 \\ 0 & \psi_4^{(\ell^2+\ell^3)(k-1)} & 0 & 0 \\ 0 & 0 & \psi_4^{(\ell^3+1)(k-1)} & 0 \\ 0 & 0 & 0 & \psi_4^{(1+\ell)(k-1)} \end{pmatrix},$$

where $\bar{\epsilon}_\ell$ is the mod ℓ cyclotomic character, and ψ_i is the fundamental character of level i.

Lemma 7.2. *Suppose that, for infinitely many $\lambda \in \mathcal{L}$, $\bar{\rho}_\lambda$ contains a subrepresentation $\bar{\tau}_\lambda$. Then for all but finitely many such λ, $\bar{\tau}_\lambda$ is irreducible and two-dimensional, and $\det \bar{\tau}_\lambda|_{I_\ell} = \bar{\epsilon}_\ell^{k-1}$.*

Proof. First suppose that, for infinitely many $\lambda \in \mathcal{L}$, $\bar{\tau}_\lambda$ is one-dimensional. By Proposition 7.1, it follows that, for almost all such λ, $\bar{\tau}_\lambda \simeq \bar{\chi}_\lambda \bar{\epsilon}_\ell^{n_\lambda}$, where $\bar{\chi}_\lambda$ is unramified at ℓ and $n_\lambda = 0$ or $k - 1$. By Corollary 3.5, the conductor of $\bar{\chi}_\lambda$ is bounded independently of λ. Since there are only finitely many characters of bounded conductor, there exists a character $\chi : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \overline{\mathbf{Q}}^\times$ such that, for infinitely many λ, $\bar{\tau}_\lambda = \chi \epsilon_\ell^n \pmod{\lambda}$, where $n = 0$ or $k - 1$ is independent of λ.

Fix a prime p at which π is unramified. Then, for infinitely many λ, $\rho_\lambda(\mathrm{Frob}_p)$ has an eigenvalue α_p such that $\alpha_p \equiv \chi \epsilon_\ell^n(\mathrm{Frob}_p) = \chi(p) p^n \pmod{\lambda}$. It follows that $\alpha_p = \chi(p) p^n$. Since $n \neq \frac{k-1}{2}$, this contradicts the Jacquet–Shalika bounds [24, Cor. 2.5], as in Theorem 4.2.

Hence, if $\overline{\rho}_\lambda$ contains a subrepresentation $\overline{\tau}_\lambda$ for infinitely many $\lambda \in \mathcal{L}$, then $\overline{\tau}_\lambda$ must be two-dimensional for all but finitely many such λ. If $\det \overline{\tau}_\lambda|_{I_\ell} \neq \overline{\epsilon}_\ell^{k-1}$, then, by Proposition 7.1, we must have $\det \overline{\tau}_\lambda = \overline{\chi}_\lambda \overline{\epsilon}_\ell^{n_\lambda}$, where $\overline{\chi}_\lambda$ is unramified at ℓ and $n_\lambda = 0$ or $2(k-1)$. As above, it follows that for a fixed prime p, $\rho_\lambda(\mathrm{Frob}_p)$ has two eigenvalues α_p, β_p that satisfy $|\alpha_p \beta_p| = p^n$ with $n = 0$ or $2(k-1)$, which contradicts the Jacquet–Shalika bounds. $\qquad\square$

Lemma 7.3. *Suppose that, for infinitely many $\lambda \in \mathcal{L}$, $\overline{\rho}_\lambda \simeq \overline{\tau}_\lambda \oplus \overline{\tau}'_\lambda$, with $\overline{\tau}_\lambda, \overline{\tau}'_\lambda$ irreducible and two-dimensional. Then for all but finitely many such λ, $\overline{\tau}_\lambda, \overline{\tau}'_\lambda$ are odd.*

Proof. Write $\overline{\chi}_\lambda = \mathrm{sim}\,\overline{\rho}_\lambda \otimes \det \overline{\tau}_\lambda^{-1}$ and suppose that $\det \overline{\tau}_\lambda$ is even. Since $\mathrm{sim}\,\overline{\rho}_\lambda$ is odd, it follows that $\overline{\chi}_\lambda$ is non-trivial. Moreover, since $\det \overline{\rho}_\lambda \simeq (\mathrm{sim}\,\overline{\rho}_\lambda)^2 \simeq \det \overline{\tau}_\lambda \det \overline{\tau}'_\lambda$, it follows that $\overline{\chi}_\lambda = \det \overline{\tau}'_\lambda \,\mathrm{sim}\,\overline{\rho}_\lambda^{-1}$.

Now, by Lemma 7.2, we may assume that $\overline{\chi}_\lambda$ is unramified at ℓ and, by Corollary 3.5, the conductor of $\overline{\chi}_\lambda$ is bounded independently of λ. Hence, there exists a Dirichlet character χ such that $\overline{\chi}_\lambda = \chi \pmod{\lambda}$ for infinitely many λ.

By duality, we see that

$$\overline{\tau}_\lambda \oplus \overline{\tau}'_\lambda \simeq \overline{\rho}_\lambda \simeq \overline{\rho}_\lambda^\vee \otimes \mathrm{sim}\,\overline{\rho}_\lambda \simeq (\overline{\tau}_\lambda^\vee \otimes \mathrm{sim}\,\overline{\rho}_\lambda) \oplus (\overline{\tau}'^{\,\vee}_\lambda \otimes \mathrm{sim}\,\overline{\rho}_\lambda) \simeq (\overline{\tau}_\lambda \otimes \overline{\chi}_\lambda) \oplus (\overline{\tau}'_\lambda \otimes \overline{\chi}_\lambda^{-1}).$$

By Schur's lemma, it follows that either $\overline{\tau}_\lambda \simeq \overline{\tau}_\lambda \otimes \overline{\chi}_\lambda$ and $\overline{\tau}'_\lambda \simeq \overline{\tau}'_\lambda \otimes \overline{\chi}_\lambda$ or $\overline{\tau}'_\lambda \simeq \overline{\tau}_\lambda \otimes \overline{\chi}_\lambda$.

In the first case, $\overline{\rho}_\lambda \otimes \overline{\chi}_\lambda \simeq \overline{\rho}_\lambda$. If this case occurs for infinitely many λ, then $\rho_\lambda \simeq \rho_\lambda \otimes \chi$. Thus, ρ_λ is not Lie irreducible, contradicting Lemma 5.9.

In the second case, $\overline{\rho}_\lambda|_{\ker \chi} \simeq \overline{\tau}_\lambda|_{\ker \chi}^{\oplus 2}$. Thus, if p splits in the number field $\overline{\mathbf{Q}}^{\ker \chi}$, then the eigenvalues of $\overline{\rho}_\lambda(\mathrm{Frob}_p)$ are indistinct. If this case occurs for infinitely many λ, then for the positive density of primes p that split in $\overline{\mathbf{Q}}^{\ker \chi}$, the p-th Hecke polynomial has indistinct roots modulo λ for infinitely many λ. Hence, for a positive density of primes p, the p-th Hecke polynomial has indistinct roots in $\overline{\mathbf{Q}}$, contradicting Theorem 5.1. $\qquad\square$

Proof of Theorem 1.2. We first show that $\overline{\rho}_\lambda$ is irreducible for all but finitely many $\ell \in \mathcal{L}$. After the previous lemmas, it remains to consider the case that, for infinitely many $\ell \in \mathcal{L}$, $\overline{\rho}_\lambda \simeq \overline{\tau}_\lambda \oplus \overline{\tau}'_\lambda$, with $\overline{\tau}_\lambda, \overline{\tau}'_\lambda$ irreducible, two-dimensional and odd. In this case, we can argue as in [14] and apply Serre's conjecture [26].

By Corollary 3.5, the conductors of $\overline{\tau}_\lambda, \overline{\tau}'_\lambda$ are bounded independently of λ, and by Lemma 7.2, they each have Serre weight k. Thus, there is an integer N, independent of λ, and modular forms $f_\lambda, f'_\lambda \in S_k(\Gamma_1(N))$ with associated residual representations $\overline{\tau}_\lambda, \overline{\tau}'_\lambda$.

If this case occurs for infinitely any λ, then, since $S_k(\Gamma_1(N))$ is finite dimensional, there exist fixed modular forms $f, f' \in S_k(\Gamma_1(N))$ such that $f = f_\lambda$ and $f' = f'_\lambda$ for infinitely many λ. Thus, for infinitely many λ and for almost all p, $\mathrm{Tr}\,\rho_\lambda(\mathrm{Frob}_p) \equiv a_f(p) + a_{f'}(p) \pmod{\lambda}$ and, since this congruence holds for infinitely many λ, it must be an equality. By the Chebotarev density theorem and the fact that a semisimple representation in characteristic 0 is determined by its trace, it follows that ρ_λ is reducible for all λ, contradicting Theorem 1.1. Hence, $\overline{\rho}_\lambda$ is irreducible for all but finitely many $\lambda \in \mathcal{L}$.

The remainder of the proof of Theorem 1.2 is exactly the same as for the cohomological case [16, 3.2–3.5]. By the classification of the maximal subgroups of $\mathrm{GSp}_4(\mathbf{F}_{\ell^n})$ [30], if $\overline{\rho}_\lambda$ is irreducible and does not contain $\mathrm{Sp}_4(\mathbf{F}_\ell)$, then one of the following cases must hold:

(i) the image contains a reducible index two subgroup, that is, $\overline{\rho}_\lambda$ is induced from a quadratic extension;

(ii) $\overline{\rho}_\lambda$ is isomorphic to the symmetric cube of a two-dimensional representation;

(iii) the image is a small exceptional group.

By using the description of the image of inertia as in Proposition 7.1, Dieulefait–Zenteno show that each of these cases can only occur for finitely many $\ell \in \mathcal{L}$. $\qquad\square$

JOURNAL INFORMATION

The *Journal of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

REFERENCES

1. J. Arthur, *Automorphic representations of GSp(4), Contributions to automorphic forms, geometry, and number theory*, 2004, pp. 65–81. MR2058604.
2. J. Arthur, *The endoscopic classification of representations*, American Mathematical Society Colloquium Publications, vol. 61, American Mathematical Society, Providence, RI, 2013. MR3135650.
3. J. Arthur and L. Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*, Annals of Mathematics Studies, vol. 120, Princeton University Press, Princeton, NJ, 1989. MR1007299.
4. M. Asgari and F. Shahidi, *Generic transfer from GSp(4) to GL(4)*, Compos. Math. **142** (2006), no. 3, 541–550. MR2231191.
5. T. Barnet-Lamb, T. Gee, D. Geraghty, and R. Taylor, *Potential automorphy and change of weight*, Ann. of Math. (2) **179** (2014), no. 2, 501–609. MR3152941.
6. T. Berger and A. Weiss, *Lafforgue pseudocharacters and parities of limits of Galois representations*, Manuscripta Math. (2021), 1–32.
7. G. Böckle, M. Harris, C. Khare, and J. A. Thorne, *$\hat{G}$-local systems on smooth projective curves are potentially automorphic*, Acta Math. **223** (2019), no. 1, 1–111. MR4018263.
8. A. Brumer, A. Pacetti, C. Poor, G. Tornaría, J. Voight, and D. S. Yuen, *On the paramodularity of typical abelian surfaces*, Algebra Number Theory **13** (2019), no. 5, 1145–1195. MR3981316.
9. F. Calegari and T. Gee, *Irreducibility of automorphic Galois representations of GL(n), n at most 5*, Ann. Inst. Fourier (Grenoble) **63** (2013), no. 5, 1881–1912. MR3186511.
10. A. Conti, *Galois level and congruence ideal for p-adic families of finite slope Siegel modular forms*, Compos. Math. **155** (2019), no. 4, 776–831. MR3932577.
11. C. De Concini and C. Procesi, *The invariant theory of matrices*, University Lecture Series, vol. 69, American Mathematical Society, Providence, RI, 2017. MR3726879.
12. P. Deligne and J.-P. Serre, *Formes modulaires de poids 1*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 507–530 (1975). MR0379379.

13. L. V. Dieulefait, *Explicit determination of the images of the Galois representations attached to abelian surfaces with* End(*A*) = $\mathbb{Z}$, Experiment. Math. **11** (2002), no. 4, 503–512 (2003). MR1969642.

14. L. V. Dieulefait, *On the images of the Galois representations attached to genus 2 Siegel modular forms*, J. Reine Angew. Math. **553** (2002), 183–200. MR1944811.

15. L. V. Dieulefait, *Uniform behavior of families of Galois representations on Siegel modular forms and the endoscopy conjecture*, Bol. Soc. Mat. Mexicana (3) **13** (2007), no. 2, 243–253. MR2472504.

16. L. Dieulefait and A. Zenteno, *On the images of the Galois representations attached to generic automorphic representations of* GSp(4), Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **20** (2020), no. 2, 635–655. MR4105913.

17. W. T. Gan and S. Takeda, *The local Langlands conjecture for Sp(4)*, Int. Math. Res. Not. (2010), no. 15, 2987–3038. MR2673717.

18. W. T. Gan and S. Takeda, *The local Langlands conjecture for* GSp$_4$(4), Ann. of Math. (2) **173** (2011), no. 3, 1841–1882. MR2800725.

19. T. Gee and O. Taïbi, *Arthur's multiplicity formula for GSp4 and restriction to S p$_4$*, J. Éc. polytech. Math. **6** (2019), 469–535. MR3991897.

20. W. Goldring and J.-S. Koskivirta, *Strata Hasse invariants, Hecke algebras and Galois representations*, Invent. Math. **217** (2019), no. 3, 887–984. MR3989256.

21. M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001, With an appendix by Vladimir G. Berkovich. MR1876802.

22. H. Hida and J. Tilouine, *Big image of Galois representations and congruence ideals*, Arithmetic and geometry, London Math. Soc. Lecture Note Ser., vol. 420, Cambridge Univ. Press, Cambridge, 2015, pp. 217–254. MR3467125.

23. H. Jacquet and J. A. Shalika, *On Euler products and the classification of automorphic forms. II*, Amer. J. Math. **103** (1981), no. 4, 777–815. MR623137.

24. H. Jacquet and J. A. Shalika, *On Euler products and the classification of automorphic representations. I*, Amer. J. Math. **103** (1981), no. 3, 499–558. MR618323.

25. A. Jorza, *p-adic families and Galois representations for* GSp(4) *and* GL(2), Math. Res. Lett. **19** (2012), no. 5, 987–996. MR3039824.

26. C. Khare and J.-P. Wintenberger, *Serre's modularity conjecture. II*, Invent. Math. **178** (2009), no. 3, 505–586. MR2551764.

27. H. H. Kim, *Functoriality for the exterior square of* GL$_4$ *and the symmetric fourth of* GL$_2$, J. Amer. Math. Soc. **16** (2003), no. 1, 139–183, With appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak. MR1937203.

28. V. Lafforgue, *Chtoucas pour les groupes réductifs et paramétrisation de Langlands globale*, J. Amer. Math. Soc. **31** (2018), no. 3, 719–891. MR3787407.

29. G. Laumon, *Fonctions zêtas des variétés de Siegel de dimension trois*, Astérisque (2005), no. 302, 1–66, Formes automorphes. II. Le cas du groupe GSp$_4$(4). MR2234859.

30. H. H. Mitchell, *The subgroups of the quaternary abelian linear group*, Trans. Amer. Math. Soc. **15** (1914), no. 4, 379–396. MR1500986.

31. C. P. Mok, *Galois representations attached to automorphic forms on* GL$_2$ *over CM fields*, Compos. Math. **150** (2014), no. 4, 523–567. MR3200667.

32. F. Momose, *On the l-adic representations attached to modular forms*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1981), no. 1, 89–109. MR617867.

33. L. Pan, *The Fontaine-Mazur conjecture in the residually reducible case*, To appear in J. Amer. Math. Soc., arXiv:1901.07166, 2019.

34. S. Patrikis, *Variations on a theorem of Tate*, Mem. Amer. Math. Soc. **258** (2019), no. 1238, viii+156. MR3915584.

35. S. Patrikis and R. Taylor, *Automorphy and irreducibility of some l-adic representations*, Compos. Math. **151** (2015), no. 2, 207–229. MR3314824.

36. C. Procesi, *The invariant theory of n × n matrices*, Adv. Math. **19** (1976), no. 3, 306–381. MR0419491.

37. C. S. Rajan, *On strong multiplicity one for l-adic representations*, Internat. Math. Res. Notices (1998), no. 3, 161–172. MR1606395.

38. D. Ramakrishnan, *Decomposition and parity of Galois representations attached to* GL(4), Automorphic representations and L-functions, Tata Inst. Fundam. Res. Stud. Math., vol. 22, Tata Inst. Fund. Res., Mumbai, 2013, pp. 427–454. MR3156860.

39. K. A. Ribet, *Galois representations attached to eigenforms with Nebentypus*, Modular functions of one variable, V (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976), Springer, Berlin, 1977, pp. 17–51. Lecture Notes in Math., Vol. 601. MR0453647.

40. K. Ribet, *this result was proven by Ribet in an (unpublished) letter to Carayol* (1984).

41. K. A. Ribet, *On l-adic representations attached to modular forms. II*, Glasgow Math. J. **27** (1985), 185–194. MR819838.

42. R. Schmidt, *Archimedean aspects of Siegel modular forms of degree 2*, Rocky Mountain J. Math. **47** (2017), no. 7, 2381–2422. MR3748235.

43. R. Schmidt, *Paramodular forms in CAP representations of* GSp(4), Acta Arith. **194** (2020), no. 4, 319–340. MR4103275.

44. C. M. Sorensen, *Galois representations attached to Hilbert-Siegel modular forms*, Doc. Math. **15** (2010), 623–670. MR2735984.

45. R. Taylor, *Galois representations associated to Siegel modular forms of low weight*, Duke Math. J. **63** (1991), no. 2, 281–332. MR1115109.

46. R. Taylor, *On the l-adic cohomology of Siegel threefolds*, Invent. Math. **114** (1993), no. 2, 289–310. MR1240640.

47. R. Taylor, *l-adic representations associated to modular forms over imaginary quadratic fields. II*, Invent. Math. **116** (1994), no. 1-3, 619–643. MR1253207.

48. R. Taylor, *On the meromorphic continuation of degree two L-functions*, Doc. Math. (2006), no. Extra Vol., 729–779. MR2290604.

49. M. Weidner, *Pseudocharacters of homomorphisms into classical groups*, Transform. Groups **25** (2020), no. 4, 1345–1370. MR4166690.

50. A. Weiss, *On Galois representations associated to low weight Hilbert–Siegel modular forms*, PhD thesis, The University of Sheffield, 2019.

51. R. Weissauer, *Four dimensional Galois representations*, Astérisque (2005), no. 302, 67–150, Formes automorphes. II. Le cas du groupe $GSp_4(4)$. MR2234860.

52. R. Weissauer, *Existence of Whittaker models related to four dimensional symplectic Galois representations*, Modular forms on Schiermonnikoog, Cambridge Univ. Press, Cambridge, 2008, pp. 285–310. MR2530981.

53. Y. Xia, *Irreducibility of automorphic Galois representations of low dimensions*, Math. Ann. **374** (2019), no. 3-4, 1953–1986. MR3985128.

54. A. N. Zubkov, *Invariants of an adjoint action of classical groups*, Algebra Log. **38** (1999), no. 5, 549–584, 639. MR1766702.