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Weak saturation in graphs: A combinatorial approach

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ABSTRACT

The weak saturation number $\mathrm{wsat}(n, F)$ is the minimum number of edges in a graph on n vertices such that all the missing edges can be activated sequentially so that each new edge creates a copy of F. In contrast to previous algebraic approaches, we present a new combinatorial approach to prove lower bounds for weak saturation numbers that allows to establish worst-case tight (up to constant additive terms) general lower bounds as well as to get exact values of the weak saturation numbers for certain graph families. It is known (Alon, 1985) that, for every F, there exists c_F such that $\mathrm{wsat}(n, F) = c_F n(1 + o(1))$. Our lower bounds imply that all values in the interval $\left[\frac{\delta}{2} - \frac{1}{\delta+1}, \delta - 1\right]$ with step size $\frac{1}{\delta+1}$ are achievable by c_F for graphs F with minimum degree δ (while any value outside this interval is not achievable).

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1. Introduction

Given a graph F, an *F-bootstrap percolation process* is a sequence of graphs $H_0 \subset H_1 \subset \cdots \subset H_m$ such that, for $i = 1, \ldots, m$, H_i is obtained from H_{i-1} by adding an edge that belongs to a copy of F in H_i. The F-bootstrap percolation process was introduced by Bollobás [3] and can be seen as a special case of cellular automata. This notion is also related to the r-neighborhood bootstrap percolation model having applications in physics; see, for example, [1,6,12]. Given $n \in \mathbb{N}$ and a graph F, we call a graph H on $[n] := \{1, \ldots, n\}$ *weakly F-saturated*, if there exists an F-bootstrap percolation process $H = H_0 \subset H_1 \subset \cdots \subset H_m = K_n$. The minimum number of edges in a weakly F-saturated graph is called the *weak F-saturation number* and is denoted by $\mathrm{wsat}(n, F)$. We also denote by $\mathrm{wSAT}(n, F)$ the set of all weakly F-saturated graphs and by $\underline{\mathrm{wSAT}}(n, F)$ the set of those of them that have exactly $\mathrm{wsat}(n, F)$ edges.

Throughout the paper we consider graphs F without isolated vertices since, obviously, for $n \geq |V(F)|$, the value of $\mathrm{wsat}(n, F)$ coincides with the weak saturation number of the graph that is obtained from F be removing the isolated vertices. Everywhere below, we denote by v, ℓ and δ the number of vertices in F, the number of edges in F, and the minimum degree of F respectively.

Note that a graph obtained from an $H \in \mathrm{wSAT}(v, F)$ by drawing from every vertex of $[n] \setminus [v]$ exactly $\delta - 1$ edges to the vertices of H belongs to $\mathrm{wSAT}(n, F)$. This observation immediately gives the upper bound (this bound was also observed in [5, Theorem 6])

$$\mathrm{wsat}(n, F) \leq \mathrm{wsat}(v, F) + (n - v)(\delta - 1) \leq \binom{v}{2} - 1 + (n - v)(\delta - 1). \tag{1}$$

This bound is sharp since $\mathrm{wsat}(n, K_v) = \binom{v}{2} - 1 + (n - v)(v - 2)$ [11] (alternative proofs of this famous result have been obtained in [2,7–9]). Another example that shows optimality of the first inequality in (1) is a star graph $F = K_{1,v-1}$ (see, e.g., [10]).

The best known general lower bound for $\mathrm{wsat}(n, F)$ is due to Faudree, Gould and Jacobson [5]. They showed that for graphs F with minimum degree δ

$$\mathrm{wsat}(n, F) \geq \left(\frac{\delta}{2} - \frac{1}{\delta + 1}\right) n \tag{2}$$

for n sufficiently large. This bound is true (even for all $n \geq v$). However, the argument in the form presented in the paper seems to be false. It is very short, and so, for convenience, we duplicate it in the Appendix as well as explain the issue in the proof.

In this paper, we prove (2) by giving a new general lower bound expressed in terms of an invariant of the graph F that equals to the vector $(e_i, i \in [v])$, where

$$e_i := \min_{S \in \binom{V(F)}{i}} |E(F) \setminus E(F \setminus S)| - 1. \tag{3}$$

Hereinafter $F \setminus S$ is a subgraph of F induced on $|V(F)| \setminus S$. Below, we state its more explicit corollary for certain graph families. In the worst case our bound is slightly stronger than (2). Also, it implies even better (and almost sharp) bounds for connected graphs H with $\delta > 1$. Note that, from the upper bound (1), it immediately follows that $\mathrm{wsat}(n, F) = O(1)$ if $\delta = 1$ (also, the lower bound (2) becomes trivial). Therefore, we restrict ourselves with $\delta > 1$.

Let us state the new lower bound. Recall that a graph is called *k-edge-connected*, if it remains connected whenever at most $k - 1$ edges are removed.

Theorem 1.1. *Let F be a graph with v vertices, ℓ edges, and minimum degree $\delta > 1$. Then, for all $n \geq v$,*

$$\mathrm{wsat}(n, F) \geq \left(\frac{\delta}{2} - \frac{1}{\delta + 1}\right)(n - v) + \ell - 1.$$

If δ is odd and F is connected, then, for all $n \geq v$,

$$\mathrm{wsat}(n, F) \geq \left(\frac{\delta}{2} - \frac{1}{2(\delta + 2)}\right)(n - v) + \ell - 1.$$

If δ is even and F is connected, or δ is arbitrary and F is 2-edge-connected, then, for all $n \geq v$,

$$\mathrm{wsat}(n, F) \geq \frac{\delta}{2}(n - v) + \ell - 1.$$

We also prove that all our bounds are sharp up to a constant additive term. Note that, since $\ell \geq \frac{1}{2}v\delta$, the first bound implies $\mathrm{wsat}(n, F) \geq \left(\frac{\delta}{2} - \frac{1}{\delta+1}\right)n + \frac{v}{\delta+1} - 1$ which is better than (2). Indeed, $\frac{v}{\delta+1} - 1 \geq 0$, and the equality holds only when $F = K_v$ for which the upper bound is the answer.

It was observed by Alon [2] that, for every graph F, $\mathrm{wsat}(n, F) = c_F n(1 + o(1))$ for some constant $c_F \geq 0$. Though the possible values of $c_F := \lim_{n\to\infty} \frac{\mathrm{wsat}(n,F)}{n}$ are unknown, we have that, for all F with $\delta \geq 2$, $\frac{\delta}{2} - \frac{1}{\delta+1} \leq c_F \leq \delta - 1$, and both bounds are achievable. If $\delta = 1$, then the only possible value of c_F is 0. It is natural to ask, how are the values of c_F distributed in this interval, if $\delta \geq 2$ is fixed? We have proved that they are not concentrated around the endpoints.

Theorem 1.2. *For every integer $\delta \geq 2$, every $k \in \{0, 1, \ldots, (\delta/2 - 1)(\delta + 1)\}$, and every integer N there exists a connected graph F with the minimum degree δ and $|V(F)| \geq N$ such that*

$$\mathrm{wsat}(n, F) = \left(\frac{\delta}{2} + \frac{k}{\delta + 1}\right)n + O(1).$$

In other words, all values between $\frac{\delta}{2} - \frac{1}{\delta+1}$ and $\delta - 1$ with step size $\frac{1}{\delta+1}$ are achievable by c_F. In [13] Tuza conjectured that, for every graph F, $\mathrm{wsat}(n, F) = c_F n + O(1)$. The conjecture clearly holds for graphs with minimum degree $\delta = 1$. The last part of Theorem 1.1 together with the upper bound (1) implies that the conjecture is also true for all connected graphs with $\delta = 2$. In the paper, we suggest a set of possible values of the parameter c_F and prove that the conjecture is true for $c_F = \delta - 1$.

Theorem 1.3. *If a graph F with the minimum degree δ satisfies $\lim_{n\to\infty} \frac{\mathrm{wsat}(n,F)}{n} = \delta - 1$, then $\mathrm{wsat}(n, F) = (\delta - 1)n + O(1)$. Moreover, if for some F, $|\mathrm{wsat}(n, F) - c_F n|$ is not bounded, then there exists an increasing sequence of positive integers $\{n_k, k \in \mathbb{N}\}$ such that $\mathrm{wsat}(n_k, F) - c_F n_k \to +\infty$ as $k \to \infty$.*

In [5], the authors also tried to go beyond $F = K_v$ and considered $F_{v,\delta}$ obtained from K_v by removing $(v-1-\delta)$ edges adjacent to the same vertex. They conjectured that the upper bound is tight for these graphs, i.e. $\mathrm{wsat}(n, F_{v,\delta}) = \binom{v-1}{2} + (n - v + 1)(\delta - 1)$ and prove the conjecture only for $v = 5$ and $\delta = 3$. In this paper, we prove the conjecture for all v and δ.

In the last section of the paper, we demonstrate that our method is very powerful for certain graph families, so that it allows to find *exact* values of weak saturation numbers. We make a deeper analysis of our techniques and refine our upper bounds that imply tightness of the bounds in Theorem 1.1. In particular, these refined bounds imply exact values of the weak saturation numbers for families of graphs F that are obtained from two cliques by drawing several edges between them. A motivation for considering these graphs is that these are, probably, the most straightforward examples of graphs having weak saturation numbers rather close to the lower bounds. Moreover, we show that the refined bounds imply the exact value of the weak saturation number for all connected graphs F that are not 4-edge-connected and satisfy $\mathrm{wsat}(v, F) = \ell - 1$. These bounds also imply that $\mathrm{wsat}(n, K_4) = 2n - 3$ (though a combinatorial proof of this fact was known [3]). The formulations of all these results appear in Section 6.2 but not in the Introduction since we do not want to overload it with massive notations needed for that.

Organization of the paper In Section 2.1, we state and prove the new general lower bound on $\mathrm{wsat}(n, F)$. We derive Theorem 1.1 from this bound in Section 3.1. Its tightness is proven in Section 3.2. The proof is based on a general upper bound on the weak saturation number that we prove in Section 2.2. Theorem 1.2 is proven in Section 4. In Section 5, we prove Theorem 1.3. In Section 6.1, we prove the conjecture of Faudree, Gould, and Jacobson. In Sections 6.3 and 6.4, we prove the refined bounds from Section 6.2.

Notations For a graph G and a set of vertices $U \subset V(G)$, we denote by $G|_U$ the subgraph of G induced on the set U. For a set X and its subset Y, we denote by $I(Y)$ the characteristic function of Y, that is $I(Y)(x) = 1$ for all $x \in Y$ and $I(Y)(x) = 0$ for all $x \in X \setminus Y$. The domain set X is always clear from the context. In the paper, we usually describe set Y as the property of all elements that belong to it. For example, $I(F \text{ is connected})$ is defined on the set of all graphs and equals 1 if and only if F is connected. Everywhere in the paper, we denote by v, ℓ and δ the number of vertices, the number of edges, and the minimum degree of a fixed graph F, which is always clear from the context.

2. General bounds

2.1. Lower bound

As was noted in [2], the existence of $\lim_{n\to\infty} \frac{\mathrm{wsat}(n,F)}{n}$ is immediate due to the fact that $\mathrm{wsat}(n,F) + (v-2)^2$ is subadditive. Indeed, divide $[n]$ into parts $[m]$ and $[n] \setminus [m]$ and draw two graphs isomorphic to some graphs from $\mathrm{wSAT}(m,F)$ and $\mathrm{wSAT}(n-m,F)$ on $[m]$ and $[n] \setminus [m]$ respectively. Then, draw all edges between fixed $(v-2)$-sets in $[m]$ and $[n] \setminus [m]$ to make the final graph weakly F-saturated. This implies

$$\mathrm{wsat}(n,F) \leq \mathrm{wsat}(m,F) + \mathrm{wsat}(n-m,F) + (v-2)^2.$$

Therefore, it is natural to construct a lower bound in the form $g(n) + O(1)$, where $g(n)$ is a subadditive function. We show that any such subadditive function is suitable unless, for some $i \in \{0,1,\ldots,v\}$, $g(i)$ exceeds the value of e_i defined in (3). Note that $e_1 = \delta - 1$. Set $e_0 = 0$.

Theorem 2.1. *Let F be a graph with v vertices and ℓ edges. Let $g : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ satisfy the following conditions:*

- *for every $i,j \in \mathbb{Z}_{\geq 0}$, $g(i+j) \leq g(i) + g(j)$;*
- *for every $i \in \{0,1,\ldots,v-1\}$, $g(i) \leq e_i$,*

where e_i are defined in (3). Then, for every integer $n \geq v$,

$$\mathrm{wsat}(n,F) \geq g(n-v) + \ell - 1.$$

Proof. Let $n \geq v$. For $i \geq v$, set $f(i) = g(i-v) + \ell - 1$. Let $H \in \mathrm{wSAT}(n,F)$. Let $\mathcal{O}_H$ be the set of all vectors $(B_1,\ldots,B_k)$ such that

- for every $\kappa \in [k]$, $B_\kappa \subset [n]$,
- for $\kappa_1 \neq \kappa_2$, $B_{\kappa_1} \cap B_{\kappa_2} = \varnothing$,
- for every $\kappa \in [k]$, $|E(H|_{B_\kappa})| \geq f(|B_\kappa|)$,
- $|B_1| \geq |B_2| \geq \ldots \geq |B_k| \geq v$.

Note that $\mathcal{O}_H$ is non-empty. Indeed, consider the first edge added to H in an F-bootstrap percolation process. This edge creates a copy $\tilde{F}$ of F. Then, clearly, $(V(\tilde{F})) \in \mathcal{O}_H$. Indeed

$$\left|E(H|_{V(\tilde{F})})\right| \geq |E(\tilde{F})| - 1 = |E(F)| - 1.$$

Also note that if, for every H, $([n]) \in \mathcal{O}_H$, then we get the statement of Theorem 2.1 immediately. For $\mathcal{B}_\alpha = (B_1^\alpha, \ldots, B_{k_\alpha}^\alpha) \in \mathcal{O}_H$, $\alpha \in \{1,2\}$, set $\mathcal{B}_1 \leq \mathcal{B}_2$, if $(|B_1^1|, \ldots, |B_{k_1}^1|) \leq (|B_1^2|, \ldots, |B_{k_2}^2|)$ in the lexicographical order. Let $\mathcal{B}^*$ be a maximal element of $\mathcal{O}_H$. The following lemma concludes the proof of Theorem 2.1. □

Lemma 2.2. $\mathcal{B}^* = ([n])$.

Proof. Assume the contrary: let $\mathcal{B}^* = (B_1, \ldots, B_k) \neq ([n])$. Take $v \notin B_1$. If v is adjacent to all vertices in B_1, then, due to the properties of g,

$$\begin{aligned}
|E(H|_{B_1 \cup \{v\}})| &= |E(H|_{B_1})| + |B_1| \geq f(|B_1|) + v \\
&= g(|B_1| - v) + v + \ell - 1 \\
&> g(|B_1| - v) + e_1 + \ell - 1 \\
&\geq g(|B_1| - v) + g(1) + \ell - 1 \\
&\geq g(|B_1| - v + 1) + \ell - 1 = f(|B_1| + 1) = f(|B_1 \cup \{v\}|).
\end{aligned}$$

Therefore $(B_1 \cup \{v\}) \in \mathcal{O}_H$ — a contradiction with the maximality of $\mathcal{B}^*$.

Therefore, there exists a pair of different vertices $\{u, v\} \notin E(H)$ such that u and v do not belong to the same B_i. Let $\{u, v\}$ be the first such edge in an F-bootstrap percolation process that starts on H. Let this edge, when added, create a copy $\tilde{F}$ of F. If $\tilde{F}$ does not meet any of B_κ, $\kappa \in [k]$, then $(B_1, \ldots, B_k, V(\tilde{F})) \in \mathcal{O}_H$, that contradicts the maximality of $\mathcal{B}^*$. Let $\mathcal{I}$ be the non-empty set of all $\kappa \in [k]$ such that $B_\kappa \cap V(\tilde{F}) \neq \varnothing$. For every $\kappa \in \mathcal{I}$, set $W_\kappa = B_\kappa \cap V(\tilde{F})$. Let $\tilde{B} = \bigcup_{\kappa \in \mathcal{I}} B_\kappa \cup V(\tilde{F})$. Let $\kappa^* = \min \mathcal{I}$. Let us prove that $(B_1, \ldots, B_{\kappa^*-1}, \tilde{B}) \in \mathcal{O}_H$ and reach a contradiction with the maximality of $\mathcal{B}^*$. Since all the edges of $\tilde{F} \setminus \{u, v\}$ that do not lie inside any of B_κ belong to the initial graph H, we get that there are

$$\left|E\left(H|_{V(\tilde{F})}\right)\right| - \sum_{\kappa \in \mathcal{I}} |E\left(H|_{W_\kappa}\right)| \geq \ell - 1 - \sum_{\kappa \in \mathcal{I}} |E(\tilde{F}|_{W_\kappa})| \geq \ell - 1 - \sum_{\kappa \in \mathcal{I}} (\ell - (e_{v - |W_\kappa|} + 1))$$

such edges. Therefore,

$$\begin{aligned}
|E(H|_{\tilde{B}})| &= \sum_{\kappa \in \mathcal{I}} |E\left(H|_{B_\kappa}\right)| + \left[\left|E\left(H|_{V(\tilde{F})}\right)\right| - \sum_{\kappa \in \mathcal{I}} |E\left(H|_{W_\kappa}\right)|\right] \\
&\geq \sum_{\kappa \in \mathcal{I}} f(|B_\kappa|) + \left[\ell - 1 - \sum_{\kappa \in \mathcal{I}} (\ell - (e_{v - |W_\kappa|} + 1))\right]
\end{aligned}$$

$$\geq \sum_{\kappa\in\mathcal{I}}(f(|B_\kappa|)-\ell+1+g(v-|W_\kappa|))+\ell-1$$
$$=\sum_{\kappa\in\mathcal{I}}(g(|B_\kappa|-v)+g(v-|W_\kappa|))+\ell-1$$
$$\geq g\left(\sum_{\kappa\in\mathcal{I}}(|B_\kappa|-|W_\kappa|)\right)+\ell-1=g(|\tilde{B}|-v)+\ell-1=f(|\tilde{B}|).\quad\square$$

Remark 2.3. If, for every $i\in\mathbb{Z}_{\geq 0}$, $g^*(i)$ is defined as the supremum of $g(i)$ over all g satisfying the conditions of Theorem 2.1, then the function $g^*:\mathbb{Z}_{\geq 0}\to\mathbb{R}$ satisfies these conditions as well. Therefore, $g^*(n-v)+\ell-1$ is the best possible bound that follows from Theorem 2.1. Let us show that it is not hard to define g^* explicitly. Set

$$g^*(0)=0,\quad g^*(i)=\min_{s\in[i],\,1\leq i_1,\ldots,i_s\leq v-1:\,i_1+\ldots+i_s=i}(e_{i_1}+\ldots+e_{i_s}).$$

Obviously the conditions in Theorem 2.1 hold for g^*. Moreover, if g satisfies these conditions and, for some i, $g(i)>g^*(i)$, then find $s\in[i]$ and $i_1,\ldots,i_s$ such that $g^*(i)=e_{i_1}+\ldots+e_{i_s}$. Since g is subadditive, we get

$$g(i)\leq g(i_1)+\ldots+g(i_s)\leq e_{i_1}+\ldots+e_{i_s}=g^*(i)$$

— a contradiction.

2.2. Upper bound

Here we prove an upper bound on the weak saturation number that we use in Section 3.2 to prove tightness of the assertions in Theorem 1.1 as well as in Section 4 to prove Theorem 1.2.

Claim 2.4. *Let F be a graph with v vertices, ℓ edges, and minimum degree δ. If* $\mathrm{wsat}(v,F)=\ell-1$ *and $P\subset V(F)$ is such that $|V(F)\setminus P|\geq\delta-1$, then*

$$\mathrm{wsat}(n,F)\leq\frac{\ell-|E(F|_{V(F)\setminus P})|-1}{|P|}n+O(1).$$

Proof. Denote $k=|P|$. We are going to show by induction that, for every $m\in\mathbb{Z}_{\geq 0}$,

$$\mathrm{wsat}(v+km,F)\leq(\ell-|E(F|_{V(F)\setminus P})|-1)m+\ell-1.\qquad(4)$$

Note that (4) immediately implies the statement of Claim 2.4 since, if r is a remainder of the division of $n-v$ by k, then $(\delta-1)r=O(1)$ edges are sufficient to restore all edges of K_n from K_{v+km}.

The base of induction is straightforward. Let m be a positive integer and assume that (4) is proven for $m-1$. Let $H\in\underline{\mathrm{wSAT}}(v+k(m-1),F)$. Let G be obtained from H by adding a copy of $F|_P$ to H (on a set of k vertices $\tilde{P}$ disjoint with $V(H)$), distinguishing a subset $K \subset V(H)$ of $v-k$ vertices, drawing edges between K and $\tilde{P}$ in the same way as they appear between $V(F) \setminus P$ and P, and deleting one of these edges. All missing edges between K and $\tilde{P}$ in G can be restored since $\mathrm{wsat}(v, F) = \ell - 1$. After that, every vertex from $\tilde{P}$ has at least $|K| \geq \delta - 1$ neighbors in $V(H)$. Therefore, all edges between $\tilde{P}$ and $V(H) \setminus K$ can be restored as well. This finishes the proof. □

3. Lower bounds for graph families

In this section we prove Theorem 1.1 and, after that, show that the bounds are best possible, up to an additive constant.

3.1. Proof of Theorem 1.1

Recall that $\delta > 1$.

We will use Theorem 2.1. Within the notations of Section 2.1, let $\gamma = \min_{1 \leq i \leq v-1} \frac{e_i}{i}$. Set $g(n) = \gamma n$. Clearly, g satisfies the conditions in Theorem 2.1. Therefore, $\mathrm{wsat}(n, F) \geq \gamma(n - v) + \ell - 1$. It remains to apply the claim stated below.

Claim 3.1. *The following lower bounds on γ hold.*

1. $\gamma \geq \frac{\delta}{2} - \frac{1}{\delta+1}$.
2. *If δ is even and F is connected, or δ is arbitrary and F is 2-edge-connected, then* $\gamma \geq \frac{\delta}{2}$.
3. *If δ is odd and F is connected, then* $\gamma \geq \frac{\delta}{2} - \frac{1}{2(\delta+2)}$.

Proof. Let $i \in [v-1]$. Since δ is the minimum degree of F, for $S \in \binom{V(F)}{i}$, the summation of degrees of all vertices from S is at least δi. On the other hand, it is exactly $2e[S] + e[S, V(F) \setminus S]$, where $e[S] = |E(F|_S)|$ is the number of edges inside S, and $e[S, V(F) \setminus S]$ is the number of edges between the vertices of S and the vertices of $V(F) \setminus S$. Therefore,

$$e[S] \geq \left\lceil \frac{\delta i - e[S, V(F) \setminus S]}{2} \right\rceil \tag{5}$$

and

$$|E(F) \setminus E(F \setminus S)| = e[S] + e[S, V(F) \setminus S] \geq \delta i - e[S] \geq \delta i - \binom{i}{2}. \tag{6}$$

Let $\lambda = I(F \text{ is connected}) + I(F \text{ is 2-edge-connected})$. Clearly, $e[S, V(F) \setminus S] \geq \lambda$. From (5), we get

$$|E(F) \setminus E(F \setminus S)| = e[S] + e[S, V(F) \setminus S]$$

$$\geq \frac{\delta i+\lambda}{2} I(\delta i-\lambda \text{ is even})+\frac{\delta i+\lambda+1}{2} I(\delta i-\lambda \text{ is odd}). \tag{7}$$

Combining (6) with (7), we get that

$$\gamma \geq \min\left\{\min_{1\leq i\leq \delta}\left[\delta-\frac{i-1}{2}-\frac{1}{i}\right], \min_{\delta+1\leq i\leq v-1}\frac{\delta i+\lambda-2+I(\delta i-\lambda \text{ is odd})}{2i}\right\}$$

Therefore, for odd δ,

$$\gamma \geq \min\left\{\frac{\delta}{2}+\frac{1}{2}-\frac{1}{\delta},\frac{\delta}{2}+\left[-\frac{I(\lambda=0)}{\delta+1}-\frac{I(\lambda=1)}{2(\delta+2)}\right]\right\}=\frac{\delta}{2}-\frac{I(\lambda=0)}{\delta+1}-\frac{I(\lambda=1)}{2(\delta+2)}.$$

For even δ,

$$\gamma \geq \min\left\{\frac{\delta}{2}+\frac{1}{2}-\frac{1}{\delta},\frac{\delta}{2}-\frac{I(\lambda=0)}{\delta+1}\right\}=\frac{\delta}{2}-\frac{I(\lambda=0)}{\delta+1}. \quad \square$$

3.2. Optimality

Here we show that the bounds in Theorem 1.1 are optimal up to an additive constant term.

Theorem 3.2. *There exists $C>0$ such that, for every $x\in\mathbb{Z}_{\geq 0}$ and every integer $\delta\geq 2$,*

1. *there exists a graph F with minimum degree δ and at least x vertices such that*

$$\mathrm{wsat}(n,F)\leq\left(\frac{\delta}{2}-\frac{1}{\delta+1}\right)n+C;$$

2. *there exists a 2-edge-connected graph F with minimum degree δ and at least x vertices such that*

$$\mathrm{wsat}(n,F)\leq\frac{\delta}{2}n+C;$$

3. *if δ is odd, then there exists a connected graph F with minimum degree δ and at least x vertices such that*

$$\mathrm{wsat}(n,F)\leq\left(\frac{\delta}{2}-\frac{1}{2(\delta+2)}\right)n+C.$$

Proof. We construct the desired graph sequences for each item of Theorem 3.2 separately. Each time, due to Claim 2.4, it is sufficient to find a sequence of graphs F_m with v_m vertices, ℓ_m edges, and minimum degree $\delta\geq 2$ such that $v_m\to\infty$ as $m\to\infty$, $\mathrm{wsat}(v_m,F)=\ell_m-1$, and each F_m contains $P_m\subset V(F_m)$ satisfying $|V(F_m)\setminus P_m|\geq\delta-1$ with the value of $(\ell_m-|E(F|_{V(F_m)\setminus P_m})|-1)/|P_m|$ required in the respective item in Theorem 3.2.

1. Here, we find a sequence of graphs F_m as above so that

$$\frac{\ell_m - |E(F|_{V(F_m)\setminus P_m})| - 1}{|P_m|} = \frac{\delta}{2} - \frac{1}{\delta+1}.$$

Consider a disjoint union of $m \geq 3$ cliques $K_{\delta+1}$. F_m is obtained by drawing one edge between one pair of these cliques. Clearly, F_m is the desired sequence with P_m chosen to be one of the disjoint cliques (that does not have an edge joining it with another clique). Indeed, $|V(F_m) \setminus P_m| \geq 2(\delta+1)$, $\ell_m = m\frac{\delta(\delta+1)}{2} + 1$, $|E(F|_{V(F_m)\setminus P_m})| = (m-1)\frac{\delta(\delta+1)}{2} + 1$. It is also clear that $\mathrm{wsat}(v_m = m(\delta+1), F_m) = \ell_m - 1$.

2. Here, we require

$$\frac{\ell_m - |E(F|_{V(F_m)\setminus P_m})| - 1}{|P_m|} = \frac{\delta}{2}$$

and that all F_m are 2-edge connected. Consider a disjoint union of two $(\delta+1)$-cliques A_1, A_2 and an m-clique B, $m \geq \delta+5$. F_m is obtained by drawing two disjoint edges $\{u_1, w_1\}$ and $\{u_2, w_2\}$ between A_1 and B (u_1, u_2 are vertices of A_1), two disjoint edges between A_2 and B that also do not meet vertices w_1, w_2, and deleting the edge $\{u_1, u_2\}$. It is easy to verify that $\mathrm{wsat}(v_m = m + 2(\delta+1), F_m) = \ell_m - 1$, where $\ell_m = \frac{m(m-1)}{2} + \delta(\delta+1) + 3$. The desired set P_m is the set of vertices of A_1. Indeed, $|V(F_m) \setminus P_m| = m + \delta + 1$, $|E(F|_{V(F_m)\setminus P_m})| = \frac{m(m-1)}{2} + \frac{\delta(\delta+1)}{2} + 2$.

3. Finally, we construct connected F_m with odd $\delta \geq 3$ and $P_m \subset V(F_m)$ satisfying

$$\frac{\ell_m - |E(F|_{V(F_m)\setminus P_m})| - 1}{|P_m|} = \frac{\delta}{2} - \frac{1}{2(\delta+2)}.$$

Consider a disjoint union of two $(\delta+1)$-cliques A_1, A_2 and an m-clique B, $m \geq \delta+3$, with distinguished vertices $w_1 \neq w_2$. F_m is obtained by
- for every $j \in \{1, 2\}$, choosing a perfect matching in A_j arbitrarily and deleting it from A_j,
- for every $j \in \{1, 2\}$, selecting a vertex $u_j \in V(A_j)$ and drawing an edge between u_j and w_j,
- adding two vertices x_1, x_2 and drawing edges between x_2 and all vertices of $V(A_2)$, between x_1 and all vertices from $V(A_1) \setminus \{u_1\}$.

Note that $\mathrm{wsat}(v_m = m+2(\delta+1)+2, F_m) = \ell_m - 1$, where $\ell_m = \frac{m(m-1)}{2} + (\delta+1)^2 + 1$. Indeed, the missing edges can be added one by one, say, in the following order: start with missing $\{x_1, u_1\}$, then draw all edges between $V(A_1) \cup V(A_2)$ and $V(B) \setminus \{w_1, w_2\}$, proceed with the missing matchings in A_1, A_2, then draw edges between $\{x_1, x_2\}$ and $V(B)$, draw all the remaining edges between $V(A_1) \cup V(A_2)$ and $V(B)$, and, finally, restore the missing edges between $V(A_1) \cup \{x_1\}$ and $V(A_2) \cup \{x_2\}$. It remains to set $P_m = V(A_1) \cup \{x_1\}$. Indeed, $|V(F_m) \setminus P_m| = m + \delta + 2$, $|E(F|_{V(F_m)\setminus P_m})| = \frac{m(m-1)}{2} + \frac{(\delta+1)^2}{2} + 1$. □

4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2.

Proof. Let $\delta \geq 2$ and $k \leq \frac{(\delta-2)(\delta+1)}{2}$ be a non-negative integer. The proof strategy is similar to the proof of Theorem 3.2. We are going to construct a sequence of graphs F such that γ defined in Section 3.1 coincides with the minimum value of $\frac{\ell-|E(F|_{V(F)\setminus P})|-1}{|P|}$ over all P satisfying the requirements in Claim 2.4, and this minimum value equals to $\frac{\delta}{2}+\frac{k}{\delta+1}=:\rho_k$. We start with constructing the desired graph sequence.

Fix a sequence of integers $s_1 \leq s_2 \leq \ldots \leq s_{\delta+1}$ such that $s_1 = s_2 = 0$, for every $i \in [\delta]$, $s_{i+1}-s_i \leq 1$, $\sum_{i=1}^{\delta+1} s_i = k+1$. Such a sequence clearly exists. Let $m \geq \max\{2(k+1), 2\delta+1\}$ be an integer. Consider disjoint cliques $A \cong K_{\delta+1}$ and $B \cong K_m$. Let $[\delta+1]$ be the vertex set of A. Let $M_1, \ldots, M_{\delta+1}$ be disjoint subsets of $V(B)$ satisfying $|M_i| = s_i$, $i \in [\delta+1]$. The graph F_m is obtained from A and B by drawing edges between the vertex i and every vertex from M_i for all $i \in [\delta+1]$. Let $P_m := V(A)$. The graph F_m has $v_m = m+\delta+1$ vertices and $\ell_m = \binom{m}{2}+\binom{\delta+1}{2}+k+1$ edges. Note that $|V(F_m)\setminus P_m| = m \geq 2\delta+1$ and

$$\frac{\ell_m-|E(F_m|_{V(F_m)\setminus P_m})|-1}{|P_m|} = \frac{\binom{\delta+1}{2}+k}{\delta+1} = \frac{\delta}{2}+\frac{k}{\delta+1} = \rho_k.$$

Due to the definition of γ and its properties described in Section 3.1, by Theorem 2.1 and Claim 2.4, it remains to prove the following:

1. $\mathrm{wsat}(v_m, F_m) = \ell_m - 1$;
2. for every $Q \subset V(F_m)$, we have

$$\frac{\ell_m-|E(F_m|_{V(F_m)\setminus P_m})|-1}{|P_m|} \leq \frac{\ell_m-|E(F_m|_{V(F_m)\setminus Q})|-1}{|Q|}. \tag{8}$$

Indeed, on the one hand, (8) yields $\gamma = \rho_k$ due to the definition of γ, and then $\mathrm{wsat}(n, F) \geq \rho_k n + O(1)$ by Theorem 2.1. On the other hand, due to Claim 2.4, we get $\mathrm{wsat}(n, F) \leq \rho_k n + O(1)$.

Verification of the requirements from Claim 2.4. The first condition holds since the missing edges of K_{v_m} can be added one by one, say, in the following order: first join vertices $1, 2$ (note that these vertices do not have neighbors in B initially) with all the vertices of $V(B)\setminus(\cup_i M_i)$, then join all the other vertices of A with all the vertices of $V(B)\setminus(\cup_i M_i)$, and, finally, restore all the rest.

Now, fix $Q \subset V(F_m)$, $|Q| = x$. Let us prove the inequality (8).

Assume that $Q \subset V(A)$. We have

$$\frac{\ell_m - |E(F_m|_{V(F_m)\setminus Q})| - 1}{|Q|} = \frac{x(\delta - \frac{x-1}{2}) + \sum_{i\in Q} s_i - 1}{x} \geq \frac{x(\delta - \frac{x-1}{2}) + \sum_{i=1}^{x} s_i - 1}{x}, \tag{9}$$

while

$$\frac{\ell_m - |E(F_m|_{V(F_m)\setminus P_m})| - 1}{|P_m|} = \frac{(\delta+1)(\delta - \frac{\delta}{2}) + \sum_{i=1}^{\delta+1} s_i - 1}{\delta+1}.$$

It remains to notice that the right hand part of (9) (we denote it by $\xi(x)$) decreases in x. Indeed,

$$[\xi(x+1)-\xi(x)]x(x+1) = -\frac{x(x+1)}{2}+1-\sum_{i=1}^{x} s_i + x s_{x+1} = \sum_{i=1}^{x}(s_{x+1}-s_i)+1-\frac{x(x+1)}{2} \leq 0.$$

If $Q \subset V(B)$, then

$$\frac{\ell_m - |E(F_m|_{V(F_m)\setminus Q})| - 1}{|Q|} \geq \frac{x\frac{m-1}{2} - 1}{x} \geq \frac{m-1}{2} - 1 \geq \delta - 1 \geq \rho_k.$$

Finally, if Q has non-empty intersections both with A and B, then, letting $W = Q \cap V(A)$, we get

$$\begin{aligned} &\frac{\ell_m - |E(F_m|_{V(F_m)\setminus Q})| - 1}{|Q|} \\ &\geq \frac{\ell_m - |E(F_m|_{V(F_m)\setminus W})| - 1 + (|Q|-|W|)\frac{m-1}{2}}{|Q|} \\ &\geq \frac{|W|\rho_k + (|Q|-|W|)\frac{m-1}{2}}{|Q|} > \frac{|W|\rho_k + (|Q|-|W|)\rho_k}{|Q|} = \rho_k. \quad \square \end{aligned}$$

5. On the conjecture of Tuza

Let us recall that Tuza conjectured that, for every graph F and some $c_F \geq 0$, $\mathrm{wsat}(n, F) = c_F n + O(1)$. In the proof of Theorem 1.1 we introduced the parameter γ that equals to the minimum value of $\gamma(S) := (|E(F)| - |E(F \setminus S)| - 1)/|S|$ over all proper $S \subset V(F)$. Note that γ is exactly the constant in front of n in our lower bounds from Theorem 1.1 derived from Theorem 2.1 and Claim 3.1. It is natural to ask whether, for any F, there exists an $S \subset V(F)$ such that $c_F = \gamma(S)$. This turns out to be true for all F that we are familiar of. In particular, this is true not only for graphs with $\delta = 1$, for connected graphs with $\delta = 2$, and for cliques, but also for a (not necessarily disjoint) union of arbitrary number of cliques of sizes at least $\delta + 1$ (see Remark 6.2).

Theorem 1.3 follows directly from the upper bound (1) and the lower bound given in the claim below.

Claim 5.1. *For every graph F, $\mathrm{wsat}(n,F) \geq c_F(n-v)+\ell-1$, where v and ℓ are the number of vertices and the number of edges in F respectively.*

Proof. The assertion of the claim follows immediately from

- the fact that the function $g_F : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ defined as $g_F(i) = \mathrm{wsat}(v+i,F)-(\ell-1)$ is subadditive (see Claim 6.10 in Section 6.4), and
- the obvious bound $g(i) \geq (\lim_{x\to\infty} g(x)/x)i$ that holds for any subadditive function $g : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ (indeed, if the opposite inequality holds for some $i > 0$, then $\frac{g(ji)}{ji} \leq \frac{g(i)}{i} < \lim_{x\to\infty} \frac{g(x)}{x}$ — a contradiction). □

6. Tight results for other graph families

6.1. The conjecture of Faudree, Gould and Jacobson

Here we prove the conjecture of Faudree, Gould and Jacobson [5]. Let $F_{v,\delta}$ be obtained from K_v by removing $v-1-\delta$ edges adjacent to the same vertex.

Theorem 6.1. *For all integer $v \geq 3$ and $\delta \in [v-2]$,*

$$\mathrm{wsat}(n, F_{v,\delta}) = \binom{v-1}{2} + (n-v+1)(\delta-1).$$

Proof. Recall that, for every graph F, $\mathrm{wsat}(n,F) \leq (\delta-1)(n-v)+\mathrm{wsat}(v,F)$ (see (1) in the Introduction). Since $\mathrm{wsat}(v,F_{v,\delta}) = \ell-1 = \binom{v}{2}-(v-\delta)$, it immediately gives the upper bound.

Now we prove the lower bound. Let $H \in \underline{\mathrm{wSAT}}(n, F_{v,\delta})$. Consider an $F_{v,\delta}$-bootstrap percolation process that starts on H and stops on K_n. Note that every new edge in this process creates a copy of $F_{v,\delta}$ and thus creates a copy of $K_{\delta+1}$. Let F_0 be a copy of $F_{v,\delta}$ that is created together with the first edge e added to H. Let x be the vertex of F_0 of degree δ, and X be a $(\delta-1)$-subset in its neighborhood in $F_0 \setminus \{e\}$. Delete from H all the edges of $F_0 \setminus \{e\}$ that have both vertices outside X. Clearly, the obtained graph is weakly $K_{\delta+1}$-saturated in K_n. Therefore,

$$\mathrm{wsat}(n, K_{\delta+1}) \leq \mathrm{wsat}(n, F_{v,\delta}) - \binom{v-\delta}{2}.$$

Then the equality $\mathrm{wsat}(n,K_{\delta+1}) = n(\delta-1)-\binom{\delta}{2}$ (see [11]) immediately gives the desired lower bound. □

Remark 6.2. If F is an arbitrary (not necessarily disjoint) union of any number of cliques of size at least $\delta+1$, then the same argument as in the proof of Theorem 6.1 implies that $\mathrm{wsat}(n,F) = n(\delta-1)+O(1)$ certifying the positive answer to the question about possible values of c_F asked in Section 5. We shall also note that Theorem 6.1 follows immediately from (1), Claim 5.1 and the inequality $c_F \geq \delta - 1$ that holds true since every edge appearing in an F-bootstrap percolation process creates a copy of $K_{\delta+1}$. However, we decided to present the above proof since it shows another neat and quite general approach (a similar idea was used in [4, Theorem 2]). Moreover, Claim 5.1 also implies that the upper bound (1) is tight for all F being unions of cliques of size at least $\delta + 1$ with $\mathrm{wsat}(v, F) = \ell - 1$.

6.2. General tight bounds

In this section, we present an improvement of Theorem 2.1 that gives the exact value of the weak saturation number for a certain family of graphs F.

Everywhere in this section, F is a fixed graph with v vertices and ℓ edges. For $r \in \mathbb{Z}_{\geq 0}$, set

$$g_r^*(0) = 0, \quad g_r^*(i) = \min_{s \in [i],\, 1 \leq i_1, \dots, i_s \leq v-r:\, i_1 + \ldots + i_s = i} (e_{i_1} + \ldots + e_{i_s}),$$

where e_i are defined in (3). Thus, $g^* = g_1^*$. Note that, for every fixed i, $g_r^*(i)$ is a nondecreasing function of r. The following improvement of Theorem 2.1 is valid when there is a small enough weakly F-saturated graph.

Theorem 6.3. *If, for all $n \geq v$, $\mathrm{wsat}(n, F) \leq g_2^*(n - v) + \ell - 1$, then, for all $n \geq v$, $\mathrm{wsat}(n, F) = g_2^*(n - v) + \ell - 1$.*

We prove Theorem 6.3 in Section 6.3. Note that it immediately implies that $\mathrm{wsat}(n, K_4) = 2n - 3$ (though a combinatorial proof of this fact was known [3]). A family of graphs F such that Theorem 6.3 gives an exact value of the weak saturation number can be distilled using Theorem 6.4 which we state below and prove in Section 6.4.

Consider the set $\mathcal{K}$ of all $i \in [v]$ such that there are no positive integers $i_1, \dots, i_s$, $s \geq 2$, satisfying $i_1 + \ldots + i_s = i$ and $e_i \geq e_{i_1} + \ldots + e_{i_s}$. Let $\mathcal{K}_r = \mathcal{K} \cap [v - r]$ for $r \in \{0, 1, \dots, v - 1\}$. Let $\beta = \beta(F)$ be the minimum number of vertices that can be deleted from F in such a way that the remaining graph contains a cut-edge (i.e. its deletion increases the number of connected components).

Theorem 6.4. *Let $r \in \{0, 1, \dots, v-1\}$. Then the following three properties are equivalent:*

1. *For every $n \geq v$, $\mathrm{wsat}(n, F) \leq g_r^*(n - v) + \ell - 1$.*
2. *$\mathrm{wsat}(v, F) = \ell - 1$, and the maximum element of $\mathcal{K}_r$ is at most $v - \beta$.*
3. *$\mathrm{wsat}(v, F) = \ell - 1$, and, for every $i \geq 0$, $g_\beta^*(i) \leq g_r^*(i)$.*

Note that, in particular, if we let $r = \beta$, then we immediately get the following corollary.

Corollary 6.5. *If* $\mathrm{wsat}(v,F)=\ell-1$, *then, for every* $n\geq v$, $\mathrm{wsat}(n,F)\leq g^*_\beta(n-v)+\ell-1$.

Let us stress that any other possible upper bound of the form $g^*_r(n-v)+\ell-1$ could not be better than the bound provided in Corollary 6.5. Moreover,

Corollary 6.6. *If* $\mathrm{wsat}(v,F)=\ell-1$ *and the maximum element of* $\mathcal{K}_2$ *is at most* $v-\beta$, *then, for every* $n\geq v$, $\mathrm{wsat}(n,F)=g^*_2(n-v)+\ell-1$.

We immediately get that $\mathrm{wsat}(n,F)=g^*_2(n-v)+\ell-1$ for *all connected graphs* F *that are not 4-edge-connected and satisfy* $\mathrm{wsat}(v,F)=\ell-1$, since for such graphs $\beta\leq 2$ due to the following claim.

Claim 6.7. *Let* k *be the edge-connectivity of* F. *Then* $\beta\leq k-1$.

Proof. Let $\mathcal{E}$ be a set of k edges such that their deletion makes F disconnected. Let $\{a,b\}\in\mathcal{E}$. Let us follow the edges of $\mathcal{E}\setminus\{a,b\}$ one by one and delete from F, at each step, a single vertex of the considered edge other than both a and b (if such a vertex is already deleted, then we just move to the next edge without any deletion). Eventually we get a graph with the cut-edge $\{a,b\}$. The number of vertices that were deleted is at most $k-1$. □

We are also able to get the sharp value for a family of graphs F that are obtained by drawing several *disjoint* edges between two cliques. For positive integers $1\leq c\leq a\leq b$, let $F_{a,b,c}$ be obtained by drawing c disjoint edges between disjoint K_a and K_b. It is clear that $\mathrm{wsat}(v,F)=\ell-1$. Therefore, the exact value for $c\leq 3$ follows from Corollary 6.6 and Claim 6.7: $\mathrm{wsat}(n,F_{a,b,c})=g^*_2(n-v)+\ell-1$, where g^*_2 can be easily computed directly. In particular, for i divisible by a, we get $g^*_2(i)=\frac{i}{a}\left(\binom{a}{2}+c-1\right)$. It could be generalized to larger values of c.

Theorem 6.8. *If* $a<b$, *then, for every* $n\geq v$, $\mathrm{wsat}(n,F_{a,b,c})=g^*(n-v)+\ell-1$. *If* $a=b=5$, $c=4$, *then* $\mathrm{wsat}(n,F_{a,b,c})=g^*_2(n-v)+\ell-1$.

Proof. First, let $a<b$. Due to Theorem 6.4, it is sufficient to prove that the maximum element of $\mathcal{K}_1$ is at most a, since $a\leq v-\beta$. Assume the contrary: there exists $i\in[a+1,a+b-1]$ such that $i\in\mathcal{K}_1$. Let $\tilde{a}\leq a$ and $\tilde{b}\leq b$ be such that the union of some $\tilde{a}$ vertices from the K_a-part of $F_{a,b,c}$ with some $\tilde{b}$ vertices from the K_b-part has exactly i vertices and e_i+1 edges with endpoints in this union. Since $\tilde{a}+\tilde{b}=i$ and $\tilde{a}\leq a$, we get $\tilde{b}>0$.

If $\tilde{b}<a$, then

$$e_i\geq\tilde{b}(b-1)-\binom{\tilde{b}}{2}+e_{\tilde{a}}=\tilde{b}\left(b-1-\frac{\tilde{b}-1}{2}\right)+e_{\tilde{a}}\geq\tilde{b}\left(a-\frac{\tilde{b}-1}{2}\right)+e_{\tilde{a}}>e_{\tilde{b}}+e_{\tilde{a}}$$

— a contradiction.

If $a \le \tilde{b} < b$, then consider integers $s \ge 1$ and $0 \le r < a$ such that $\tilde{b} = sa + r$. We shall prove that $e_i \ge e_{\tilde{a}} + se_a + e_r$. In the same way as above, it is sufficient to show that $\tilde{b}(b-1) - \binom{\tilde{b}}{2} \ge se_a + e_r$. Note that $e_a \le \binom{a}{2} + a - 1$, $e_r \le ra - \binom{r}{2}$. We get

$$\begin{aligned}\tilde{b}(b-1) - \binom{\tilde{b}}{2} &= (sa+r)\left(b - \frac{\tilde{b}+1}{2}\right) \ge (sa+r)\frac{\tilde{b}+1}{2} \\ &\ge (sa+r)\frac{a+r+1}{2} \\ &\ge se_a + s + \frac{sar}{2} + r\frac{a+r+1}{2} > se_a + ar \ge se_a + e_r\end{aligned}$$

as needed.

Finally, let $\tilde{b} = b$. Then $\tilde{a} < a$. We let $\tilde{b} + \tilde{a} = sa + r$, where $s \ge 1$ and $0 \le r < a$. Note that $e_a \le \binom{a}{2} + c - 1$ and $e_r \le r(a-1) - \binom{r}{2} + \min\{r, c\}$. We get

$$\begin{aligned}e_i &= \binom{b}{2} + c + \tilde{a}(a-1) - \binom{\tilde{a}}{2} - 1 \\ &= \frac{(sa+r-\tilde{a})(sa+r-\tilde{a}-1)}{2} + c + \tilde{a}(a-1) - \binom{\tilde{a}}{2} - 1 \\ &\ge se_a + e_r + \frac{s^2-s}{2}a^2 + s(ar - a\tilde{a} - c + 1) + r^2 - r(a+\tilde{a}) + a\tilde{a} + c - \min\{r, c\} - 1.\end{aligned}$$

If $s = 1$, then $r = \tilde{b} + \tilde{a} - a \ge \tilde{a} + 1$. In this case,

$$e_i \ge se_a + e_r + r^2 - \tilde{a}r - \min\{r, c\} \ge se_a + e_r + r - \min\{r, c\} \ge se_a + e_r.$$

If $s \ge 2$, then

$$\begin{aligned}e_i &\ge se_a + e_r + (s-1)a^2 + r^2 + sar - (s-1)a\tilde{a} - \tilde{a}r - sc + s - ra + c - r - 1 \\ &\ge se_a + e_r + (s-1)a + r^2 - sc + s + c - r - 1 \\ &\ge se_a + e_r + r^2 + s - r - 1 \ge se_a + e_r.\end{aligned}$$

The first part of Theorem 6.8 follows.

Let $a = b = 5$, $c = 4$. Computing directly all e_i, $i \le 7$, we get that the maximum element in $\mathcal{K}_2$ equals 5. Since $\beta = 3$, the second part of Theorem 6.8 follows from Theorem 6.4. □

Unfortunately, we can not generalize Theorem 6.8 to the case $a = b > 5$, $c \ge 4$, or $a = b = c = 5$. Also, note that, when $a < b$, we get the upper bound g_1^*, and not g_2^*, as everywhere before in this section. Actually, $g_1^* = g_2^*$ in this case since g_r^* is non-decreasing. Indeed, if $g_1^*(i) < g_2^*(i)$ for some i, then $\text{wsat}(v+i, F_{a,b,c}) < g_2^*(i) + \ell - 1$ whereas $\text{wsat}(n, F_{a,b,c}) \le g_2^*(n-v) + \ell - 1$ for all n, that contradicts Theorem 6.3.

We shall conclude this section by noting that it is not always true that $\mathrm{wsat}(n,F) = g^*_\beta(n-v)+\ell-1$ even when $\mathrm{wsat}(v,F)=\ell-1$. To see this, consider F obtained by the deletion of a maximal matching (consisting of 4 edges) from K_9. It is obvious that $\mathrm{wsat}(v=9,F)=\ell-1=31$, that $\beta=6$ and that $g^*_6(i)=\frac{17}{3}i+O(1)$. Let us now show that actually $\mathrm{wsat}(n,F)\leq\frac{11}{2}n+O(1)$ implying $\mathrm{wsat}(n,F)<g^*_\beta(n-v)+\ell-1$ for n large enough.

Consider a 4-set $P\subset V(F)$ of single ends of all 4 edges of the missing matching in F. Clearly P induces a clique, and there are exactly 22 edges in F adjacent to P. Let us show that from any weakly saturated H we may get another weakly saturated graph $\tilde{H}$ by adding 4 vertices and 22 edges, that clearly implies the desired claim. Let $\tilde{H}$ be obtained from H by adding the 4-clique P together with the 16 edges going from P to some 5 vertices $v_1,\dots,v_5$ in H exactly as in F. Delete one of the edges with both ends in P, and add an edge from P to some $v_6\in V(H)\setminus\{v_1,\dots,v_5\}$ instead. It is easy to see that the final graph $\tilde{H}$ is indeed weakly saturated. First of all, the deleted edge in P and all missing edges between P and $v_1,\dots,v_5$ can be added since $\mathrm{wsat}(v,F)=\ell-1$. Secondly, we may add all edges from v_6 to P, and then add all the other edges.

6.3. Proof of Theorem 6.3

The proof strategy is actually similar to those in the proof of Theorem 2.1, the main difference is that we will not force sets of vertices B_κ to be disjoint. However, instead we will require sets of edges induced by B_κ to be disjoint. That would actually imply that each pair of vertex sets has at most 1 vertex in common.

So, let $n\geq v$. For $i\geq v$, set $f_2(i)=g^*_2(i-v)+\ell-1$. Let $H\in\underline{\mathrm{wSAT}}(n,F)$. Let $\mathcal{O}_H$ be the set of all vectors $(B_1,\dots,B_k)$ such that

- for every $\kappa\in[k]$, $B_\kappa\subset[n]$, $|B_\kappa|\geq v$,
- for $\kappa_1\neq\kappa_2$, $E(H|_{B_{\kappa_1}})\cap E(H|_{B_{\kappa_2}})=\varnothing$,
- for every $\kappa\in[k]$, $|E(H|_{B_\kappa})|\geq f_2(|B_\kappa|)$,
- $|B_1|\geq|B_2|\geq\ldots\geq|B_k|$.

Note that the inequality $|E(H|_{B_\kappa})|\geq f_2(|B_\kappa|)$ immediately implies that $|E(H|_{B_\kappa})|=f_2(|B_\kappa|)$ since, by the assumption of Theorem 6.3, $f_2(|B_\kappa|)$ edges are enough to saturate a clique on B_κ. Moreover, for $\kappa_1\neq\kappa_2$, $|B_{\kappa_1}\cap B_{\kappa_2}|\leq 1$, since otherwise we may find a weakly F-saturated $\tilde{H}$ in K_n with the number of edges less than in H. Indeed, since there are no edges in $B:=B_{\kappa_1}\cap B_{\kappa_2}$, we may renew the $f_2(|B_{\kappa_1}|)$ edges induced by B_{κ_1} in such a way that at least 1 edge is entirely inside B, and the new set of edges on B_{κ_1} is weakly F-saturated in the clique K_1 on B_{κ_1}. Next, in the same way, it is also possible to renew the set of $f_2(|B_{\kappa_2}|)$ edges induced by B_{κ_2} in such a way that at least 1 edge is entirely inside B, and this set of edges E_2 is weakly F-saturated in the clique K_2 on B_{κ_2}. Add to the constructed at the previous step graph all the edges from E_2 that are not entirely in B. We get the graph with less than $f_2(|B_{\kappa_1}|)+f_2(|B_{\kappa_2}|)$ edges which is weakly F-saturated in $K_1 \cup K_2$. In particular, this graph saturates all the edges of H in $B_{\kappa_1} \cup B_{\kappa_2}$ with less number of edges than H has in this set — a contradiction with the minimality of H.

Let us also note that $\mathcal{O}_H$ is obviously non-empty since we may take a single B which is the vertex set of the first copy of F that appears in the bootstrap percolation process that starts at H. We order $\mathcal{O}_H$ lexicographically in the same way as in the proof of Theorem 2.1, and define $\mathcal{B}^*_H$ as a maximal element of $\mathcal{O}_H$. Thus, it remains to prove the following lemma.

Lemma 6.9. $\mathcal{B}^* := \max_{H \in \underline{\mathrm{wSAT}}(n,F)} \mathcal{B}^*_H = ([n])$.

Proof. Assume the contrary. Let H be a weakly F-saturated subgraph of K_n such that $\mathcal{B}^*_H = \mathcal{B}^*$. Let us first prove that there exists a pair of non-adjacent in H vertices u and v that do not belong to any common B_κ from $\mathcal{B}^*$. Let $B := B_1$ be the first set in $\mathcal{B}^*$. Due to the assumption, there exists a vertex v outside B. Assume that every $u \in B$ is either adjacent to v in H, or belongs together with v to the same B_u from $\mathcal{B}^*$. Take any $u \in B$ non-adjacent to v and satisfying the second condition. As above, we may renew edges inside B_u in order to make u and v adjacent while keeping the same amount of edges inside B_u. Note that we have not changed adjacencies in all the other B_κ from $\mathcal{B}^*$ since any two sets share at most 1 vertex. Thus, proceeding in this way, we may actually make v adjacent to all vertices of B. Denote the renewed graph by $\tilde{H}$. It is clear from the definition of g^*_2 that $g^*_2(|B|+1-v) \leq g^*_2(|B|-v)+|B|$ implying that $f_2(|B|+1) \leq f_2(|B|)+|B|$. Then, $(B \cup \{v\}) \in \mathcal{O}_{\tilde{H}}$ — contradiction with the maximality of $\mathcal{B}^*$.

We then take a pair of non-adjacent vertices u and v that do not belong to any common B_κ from $\mathcal{B}^*$ and that is activated first in a bootstrap percolation process initiated at H. Let $\tilde{F}$ be a copy of F that appears together with $\{u,v\}$ in this process. Consider the set S of all κ such that $|B_\kappa \cap V(\tilde{F})| \geq 2$. Note that S is non-empty since otherwise we may add $V(\tilde{F})$ to $\mathcal{B}^*$ — contradiction with maximality. Let $q \in S$ be such that $|B_q|$ is maximal. In the usual way, we may renew edges in every B_κ, $\kappa \in S \setminus \{q\}$, so that $B_\kappa \cap V(\tilde{F})$ induces in H exactly the same graph as in $\tilde{F}$. Thus, we may assume that all edges of $\tilde{F}$ other than $\{u,v\}$ and those that belong to B_q initially belong to H. Note that the set of edges induced by $B := B_q \cup V(\tilde{F})$ equals the union of the set of edges induced by B_q and $E(\tilde{F}) \setminus \{u,v\}$ since otherwise there exists a graph on B which is weakly F-saturated and has less number of edges — contradiction with the minimality of H.

Let us show that the tuple $\mathcal{B}$, composed of sets B_κ, $\kappa \notin S$, and B placed in the right order, belongs to $\mathcal{O}_H$, contradicting the maximality of $\mathcal{B}^*$. The only not so trivial condition from the definition of $\mathcal{O}_H$ is the third one (in particular, note that the second one holds since any of B_κ, $\kappa \notin S$, does not contain edges from $E(\tilde{F}) \setminus \{u,v\}$). It remains thus to verify this third condition. Denote $i := v - |B_q \cap V(\tilde{F})|$. Then

$$\begin{aligned}|E(H|_B)| &\geq |E(H|_{B_q})| + e_i \geq f_2(|B_q|) + e_i = g_2^*(|B_q| - v) + e_i + \ell - 1\\ &\geq g_2^*(|B_q| + i - v) + \ell - 1 = f_2(|B|)\end{aligned}$$

completing the proof. □

6.4. Proof of Theorem 6.4

We need several claims. First of all, let us prove that $\mathrm{wsat}(i+v, F) - (\ell - 1)$ is subadditive.

Claim 6.10. *For every graph F and every $i, j \in \mathbb{Z}_{\geq 0}$, we have that*

$$\mathrm{wsat}(i+j+v, F) - (\ell - 1) \leq [\mathrm{wsat}(i+v, F) - (\ell - 1)] + [\mathrm{wsat}(j+v, F) - (\ell - 1)].$$

Proof. Let $A \in \underline{\mathrm{wSAT}}(v+i, F)$, $B \in \underline{\mathrm{wSAT}}(v+j, F)$. Let U be a set of vertices in A of size v, and let $\tilde{F}$ be the first copy of F that appears in an F-bootstrap percolation process initiated at B. Note that the number of edges in $B|_{V(\tilde{F})}$ is at least $\ell - 1$.

Consider the graph H on $V(A) \cup [V(B) \setminus V(\tilde{F})]$ constructed as follows. Add to A disjointly the set of vertices $W := V(B) \setminus V(\tilde{F})$ and preserve those edges of B that have at least one end in W in the following way. The edges that are entirely inside W just remain the same, and the edges between W and $V(\tilde{F})$ are moved to form a bipartite graph between W and U: consider any bijection $\varphi : V(\tilde{F}) \to U$, and draw an edge $\{x \in W, \varphi(y)\}$ if and only if $\{x, y\} \in E(B)$. It is obvious that H is a weakly F-saturated graph with at most $\mathrm{wsat}(i+v, F) + \mathrm{wsat}(j+v, F) - (\ell - 1)$ edges, completing the proof. □

Next we claim that, for a subadditive function $g(i)$ not to exceed $g_r^*(i)$ for all i, it is sufficient to satisfy $g(i) \leq g_r^*(i)$ for $i \in \mathcal{K}_r$.

Claim 6.11. *Let $g : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ be a subadditive function such that $g(0) = 0$. If $g(i) \leq g_r^*(i)$ for all $i \in \mathcal{K}_r$, then the same inequality holds true for all $i \in \mathbb{Z}_{\geq 0}$.*

Proof. Note that $g(0) = g_r^*(0)$ by the definition of g_r^*. Let us then take any $i \in \mathbb{N}$, $i \notin \mathcal{K}_r$. First of all, by the definition of g_r^*, there exist $i_1, \ldots, i_h \in [v-r]$ such that $i_1 + \ldots + i_h = i$ and $g_r^*(i) = e_{i_1} + \ldots + e_{i_h}$. Secondly, by the definition of $\mathcal{K}_r$, for every $j \in [h]$ such that $i_j \notin \mathcal{K}_r$, there exist $i_1^j, \ldots, i_{s_j}^j \in \mathcal{K}_r$ such that $i_1^j + \ldots + i_s^j = i_j$ and $e_{i_j} \geq e_{i_1^j} + \ldots + e_{i_s^j}$. We conclude that there exist $i^1, \ldots, i^s \in \mathcal{K}_r$ such that $i = i^1 + \ldots + i^s$ and

$$g_r^*(i) \geq e_{i^1} + \ldots + e_{i^s} \geq g_r^*(i^1) + \ldots + g_r^*(i^s) \geq g(i^1) + \ldots + g(i^s) \geq g(i),$$

as needed. □

Let $r_1, r_2 \in [v-1]$. Assume that the maximum element of $\mathcal{K}_{r_1}$ is at most $v - r_2$. It immediately implies that $\mathcal{K}_{r_1} \subseteq \mathcal{K}_{r_2}$. Then, for all $i \in \mathcal{K}_{r_1}$, $g_{r_2}^*(i) = e_i$. Therefore, for all $i \in \mathbb{Z}_{\geq 0}$, $g^*_{r_2}(i) \leq g^*_{r_1}(i)$ due to Claim 6.11. On the other hand, assuming that $g^*_{r_2}(i) \leq g^*_{r_1}(i)$ for all $i \in \mathbb{Z}_{\geq 0}$ and that $i > v - r_2$ for some $i \in \mathcal{K}_{r_1}$, we get that there exist $i_1, \ldots, i_s \in [v - r_2]$ satisfying $i_1 + \ldots + i_s = i$ and $g^*_{r_2}(i) = e_{i_1} + \ldots + e_{i_s}$. But then $e_i \geq g^*_{r_1}(i) \geq e_{i_1} + \ldots + e_{i_s}$ contradicting the fact that $i \in \mathcal{K}_{r_1}$. Setting $r_1 = r$ and $r_2 = \beta$, we immediately get the equivalence of the second and the third properties in Theorem 6.4.

To prove the equivalence of the first and the second property, let us fix a graph F with $\mathrm{wsat}(v, F) = \ell - 1$, and note that Claim 6.10 and Claim 6.11 imply that the inequality $\mathrm{wsat}(n, F) \leq g^*_r(n - v) + \ell - 1$ is true for every n if and only if the inequality $\mathrm{wsat}(i + v, F) \leq e_i + (\ell - 1)$ is true for ever $i \in \mathcal{K}_r$. Then, the following lemma finishes the proof of Theorem 6.4.

Lemma 6.12. *Let F be a graph such that* $\mathrm{wsat}(v, F) = \ell - 1$. *Then* $\mathcal{K}_\beta = \{i \in \mathcal{K} : \mathrm{wsat}(i + v, F) \leq e_i + (\ell - 1)\}$.

Proof. Let us first proof Corollary 6.5 without relying on the statement of Theorem 6.4. It would imply $\mathcal{K}_\beta \subseteq \{i \in \mathcal{K} : \mathrm{wsat}(i + v, F) \leq e_i + (\ell - 1)\}$.

So, we fix $n \geq v$ and construct a weakly F-saturated graph H with $g^*_\beta(n - v) + \ell - 1$ edges. First of all, let us find $i_1, \ldots, i_s \in [v - \beta]$ such that $i_1 + \ldots + i_s = n - v$ and $g^*_\beta(n - v) = e_{i_1} + \ldots + e_{i_s}$. Then, we construct H as follows. Start from H_0 which is a copy of F missing a single edge. Clearly, H_0 is weakly F-saturated. Then, for every $j = 1, \ldots, s$, assuming that a weakly F-saturated graph H_{j-1} is constructed, set $V(H_j) = V(H_{j-1}) \sqcup U_j$, where U_j has size i_j, and $E(H_j) = E(H_{j-1}) \sqcup E_j$, where E_j has size e_{i_j}. The edges of E_j are drawn in the following way:

- let $\tilde{F}_j$ be a copy of F with $U_j \subset V(\tilde{F}_j)$ such that $e_{i_j} + 1$ is exactly the number of edges in $E(\tilde{F}_j) \setminus E(\tilde{F}_j \setminus U_j)$;
- E_j restricted on U_j coincides with $E(\tilde{F}_j|_{U_j})$;
- take an arbitrary set $W_j \subset V(H_{j-1})$ of size $|V(\tilde{F}_j)| - |U_j| = v - i_j$ and draw edges between U_j and W_j exactly in the same way as they appear between U_j and $V(\tilde{F}_j) \setminus U_j$ in $\tilde{F}_j$;
- remove a single edge from the final set of edges defined above.

It remains to prove that H_j is weakly F-saturated, and then conclude that $H := H_s$ is the desired weakly F-saturated graph on n vertices. Note that there exists an F-bootstrap percolation process initiated at H_j that ends up at the union K of a clique on $V(H_{j-1})$ and $U_j \sqcup W_j$. Note that the intersection of these two sets W_j has cardinality at least β and both sets have cardinality at least v. Thus, due to the definition of β, for every choice of $x \in V(H_{j-1}) \setminus W_j$ and $y \in U_j$, a copy of F can be placed inside $V(H_j)$ in such a way that $\{x, y\}$ is the only edge that does not belong to the union of cliques K. Thus, all the remaining edges of the clique on $V(H_j)$ can be activated, completing the proof.

It remains to prove that $\{i \in \mathcal{K} : \text{wsat}(i+v, F) \leq e_i + (\ell - 1)\} \subseteq \mathcal{K}_\beta$. Assume the contrary: take $i \in [v-\beta+1, v] \cap \mathcal{K}$ such that $\text{wsat}(i+v, F) \leq e_i + (\ell-1)$. Let $H \in \underline{\text{wSAT}}(v+i, F)$. Consider an F-bootstrap percolation process $H = H_0 \subset \ldots \subset H_M = K_{v+i}$, each single edge from $H_j \setminus H_{j-1}$ creates $F_j \cong F$ in H_j. For every $j \in [M]$, let $U_j = V(F_1) \cup \ldots \cup V(F_j)$. Note that $U_M = V(H)$ since otherwise there exists a vertex $v \in V(H)$ adjacent to all the other vertices in H and such that $H \setminus \{v\}$ is weakly F-saturated. But then deleting all but $\delta - 1$ edges going from v keeps the graph H weakly F-saturated — a contradiction with its minimality. Now, consider an inclusion-maximum sequence $U_1 \subset U_{j_1} \subset \ldots \subset U_{j_h}$ such that each successive set is strictly bigger than its predecessor. Set $S_t = V(F_{j_t}) \setminus U_{j_{t-1}}$, $t \in [h]$, where $j_0 = 1$. Note that

$$i_t := |S_t| \leq v + i - |U_{j_{t-1}}| \leq v + i - |U_1| = i,$$

that $e_{i_t} \leq E(H|_{U_{j_t}}) \setminus E(H|_{U_{j_{t-1}}})$, and that $i_1 + \ldots + i_h = i$. Then $\text{wsat}(v+i, F) \geq \ell - 1 + \sum_{j=1}^{h} e_{i_j}$, and we get that

$$e_i + (\ell - 1) \geq \text{wsat}(v+i, F) \geq \ell - 1 + \sum_{j=1}^{h} e_{i_j}.$$

The obtained inequality $e_i \geq \sum_{j=1}^{h} e_{i_j}$ may only happen if $h = 1$ and $i_1 = i$ since $i \in \mathcal{K}$. But then $\text{wsat}(v+i, F) = \ell - 1 + e_i$. It means that there is a bootstrap F-percolation process initiated at H such that, firstly, a clique on $V_1 := V(F_1)$ is activated, secondly, a clique on $V_2 := V(F_{j_1})$ is activated, and, finally, all edges between $V_1 \setminus V_2$ and $V_2 \setminus V_1$ are activated. Let us take the first edge $\{x, y\}$ that appears between $V_1 \setminus V_2$ and $V_2 \setminus V_1$ and note that

$$|V_1 \cap V_2| = |V_2| - |V_2 \setminus V_1| = v - i_1 = v - i \leq \beta - 1.$$

But then we get that there exists a copy of F such that the deletion of at most $|V_1 \cap V_2| \leq \beta - 1$ vertices from it creates the cut-edge $\{x, y\}$ — a contradiction with the definition of β. □

Appendix A. The proof of Faudree, Gould and Jacobson

Let us recall the proof of the lower bound (2) due to Faudree, Gould and Jacobson [5]:

Assume that $G \in \text{wSAT}(n, F)$ for n sufficiently large. Partition the vertices of G into A and B, with B being the vertices of degree $\delta - 1$ and A the remaining vertices. Let $|B| = k$, and so $|A| = n - k$. The vertices in B form an independent set, since the addition of a first edge to $v \in B$ must result in v and all of its neighbors having degree at least δ. Likewise, each vertex in A must have degree at least $\delta - 2$ relatively to A to be able to add edges between B and A, since at most 2 vertices of B can be used. This gives the inequality $(\delta-1)k+(\delta-2)(n-k)\le\delta(n-k)$. This implies $k\le 2n/(\delta+1)$, and so $\mathrm{wsat}(n,F)\ge\frac{\delta n}{2}-\frac{n}{\delta+1}$.

First, the counting argument $(\delta-1)k+(\delta-2)(n-k)\le\delta(n-k)$ is unclear and seems to be false since k could be greater than $2n/(\delta+1)$. In particular, the set B of $H_{v,n}\in\underline{\mathrm{wSAT}}(n,K_v)$ obtained by drawing all edges from vertices on $[n]\setminus[v-2]$ to the clique on $[v-2]$ has cardinality $n-\delta+1$ which is bigger than $2n/(\delta+1)$ for all $\delta>1$. Second, the lower bound for $\mathrm{wsat}(n,F)$ may follow only from a lower bound on k (we have the bound $\mathrm{wsat}(n,F)\ge(\delta-1)k+\frac{(\delta-2)(n-k)}{2}=\frac{\delta-2}{2}n+k\frac{\delta}{2}$ that increases in k). So, as might appear at first sight, the authors wrote the opposite inequality — it might be $k\ge 2n/(\delta+1)$. However, B could be even empty (take $K_n\in\mathrm{wSAT}(n,K_v)$) and then $k=0$.

Data availability

No data was used for the research described in the article.

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