Larmor radiation as a witness to the Unruh effect

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We discuss the emission of radiation from general sources in quantum scalar, electromagnetic, and gravitational fields using the Rindler coordinate frame, which is suitable for uniformly accelerated observers, in the Minkowski vacuum. In particular, we point out that to recover, from the point of view of uniformly accelerated observers in the interaction picture, the usual Larmor radiation, which is independent of the choice of the vacuum state, it is necessary to incorporate the Unruh effect assuming the Minkowski vacuum. Thus, the observation of classical Larmor radiation in the Minkowski vacuum could be seen as vindicating the Unruh effect in the sense that it is not correctly recovered in the uniformly accelerated frame unless the Unruh effect is taken into account.

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I. INTRODUCTION

Quantum mechanics (QM) and special relativity (SR), each in their own way have profoundly changed our understanding of nature. While classic determinism had to give way to the inherent uncertainties of quantum superpositions, space and time were eventually understood as just parts of a more fundamental stage, the spacetime, on which mass-energy equivalence opened the way for "matter" to no longer be conserved. In view of these shifts in the paradigm separately promoted by QM and SR, it should not come as a surprise that their combination into the successful framework of quantum field theory (QFT) would lead to novel effects which defy our intuition. The relative (hence, nonfundamental) nature of the quantum particle concept is possibly one of the most underappreciated revelations of QFT. It probably finds its utmost realization in the Unruh effect [1], according to which the usual vacuum state of a QFT in Minkowski spacetime—i.e., the state that inertial observers describe as the absence of real particles—is "seen" as a thermal bath of particles by uniformly accelerated observers. This effect is closely related to the Hawking effect, which leads to the evaporation of black holes [2,3].

Despite the rigorous derivations of the Unruh effect in QFT using different approaches [see Refs. [4–7]], its reality (or consequences) are frequently put into question [see, e.g., Refs. [8–14]]—not rarely due to misconceptions about its precise meaning. These debates have motivated several proposals for observing signatures of the acceleration radiation in the laboratory [see, e.g., Refs. [15–25] and references therein]. In considering these proposals, it is important to keep in mind that the Unruh effect is *not* an additional ingredient which is assumed on top of QFT in the inertial frame; this is an effect that must be taken into account when interpreting standard inertial-frame QFT results from the perspective of uniformly accelerated observers. In other words, without the Unruh thermal bath in the uniformly accelerated frame, uniformly accelerated observers would not be able to explain the phenomena that inertial observers in the usual vacuum successfully describe with the standard QFT. The Unruh effect in QFT is analogous to *inertial* forces, also known as *fictitious forces*, in Newtonian mechanics: one needs to take the inertial

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forces into account in the non-inertial frame in order to correctly describe the motion of particles which inertial observers successfully describe using Newton's laws without these "extra" forces. Similarly, one needs to take the Unruh effect into account to use QFT in a uniformly accelerated frame in order to correctly describe the results of QFT in an inertial frame. This analogy clarifies what constitutes an "observation" of the Unruh effect in *any* inertial lab-based experiment: it means measuring an effect in the inertial frame which uniformly accelerated observers can only account for by using the Unruh thermal bath.

The main purpose of this paper is to explain further the fact that usual *classical* radiation (as measured in inertial frames), such as Larmor radiation [26,27], can be considered as such an observation of the Unruh effect. This has already been pointed out in the literature with particular sources and worldlines [28–35]; here, we present a general proof valid not only for the electromagnetic field with general charge distributions but also for the scalar and graviton fields.

Building on the observation of Unruh and Wald that the emission of a quantum in the inertial frame corresponds to either the emission or absorption of a quantum to or from the Unruh thermal bath in the uniformly accelerated frame [36], we show here that, at first order in perturbation theory, the *interaction probability* (i.e., the sum of emission and absorption probabilities) of a classical source according to uniformly accelerated observers, in the presence of the Unruh thermal bath, gives exactly the emission probability needed to reproduce classical radiation from the inertial perspective. The quantum nature of the Unruh effect is reflected in the fact that this agreement is only possible by assuming that radiation is made of *quanta*, whose energy E and frequency ν are related by Planck's formula, $E = h\nu$, with h being Planck's constant.

The rest of the paper is organized as follows. In Sec. II, we expand the massless Klein-Gordon scalar field using the Unruh modes and their complex conjugates, which are eigenfunctions of the boost Killing vector field and are positive- and negative-frequency solutions with respect to inertial time translations—and, as such, can be used to define the vacuum state, representing absence of particles according to inertial observers. In Sec. III, we relate the Unruh modes with the Rindler modes, which are eigenfunctions of the boost Killing vector field with support in a Rindler wedge—hence, related to quantization in the uniformly accelerated frame. Then, in Sec. IV, we calculate the following two quantities: (i) the probability $P^{(\mathrm{M})}_{\mathrm{em}}$ for a classical source to emit a quantum of the field, from the inertial-frame perspective, and (ii) the probability $P^{(\mathrm{R})}_{\mathrm{int}}$ for the same classical source to absorb or emit a quantum from or to the Unruh thermal bath, from the uniformly accelerated perspective. These two quantities turn out to be exactly the same, provided the source is completely contained in the Rindler wedge (so that it can be fully described in the uniformly accelerated frame). Section V is devoted to extending the result in Sec. IV to the electromagnetic and graviton fields. Finally, in Sec. VI, we present our final considerations and discussion. In Appendix A we confirm for the scalar case that the classical radiation formula is found in the Heisenberg picture in any vacuum state, and in Appendix B we show, also for the scalar case, how the classical radiation formula is reproduced in the Fulling vacuum state as well in the interaction picture. We adopt the metric signature $(+,-,-,-)$ and natural units, in which $\hbar = c = 1$.

II. THE UNRUH MODES AS THE BOOST EIGENFUNCTIONS

We begin by considering the Unruh effect for the free massless scalar field. This quantum field is naturally expanded in terms of the momentum eigenstates as [37]

$$\hat{\phi}(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k}[f^{\mathbf{k}}(x)\hat{a}_{\mathbf{k}} + \overline{f^{\mathbf{k}}(x)}\hat{a}^{\dagger}_{\mathbf{k}}], \tag{1}$$

with $x = (t, \mathbf{x}) = (t, z, \mathbf{x}_\perp)$, $\mathbf{k} = (k_z, \mathbf{k}_\perp)$, and $k = \|\mathbf{k}\|$, where

$$f^{\mathbf{k}}(x) = e^{-ikt+i\mathbf{k}\cdot\mathbf{x}} = e^{-ikt+ik_z z+i\mathbf{k}_\perp\cdot\mathbf{x}_\perp}. \tag{2}$$

The annihilation and creation operators, $\hat{a}_{\mathbf{k}}$ and $\hat{a}^{\dagger}_{\mathbf{k}}$, satisfy

$$[\hat{a}_{\mathbf{k}}, \hat{a}^{\dagger}_{\mathbf{k}'}] = (2\pi)^3 2k\delta^{(3)}(\mathbf{k} - \mathbf{k}'), \tag{3}$$

with all other commutators among them vanishing. The Minkowski vacuum, $|0_{\mathrm{M}}\rangle$, is defined by

$$\hat{a}_{\mathbf{k}}|0_{\mathrm{M}}\rangle = 0 \quad \text{for all } \mathbf{k}. \tag{4}$$

The Unruh modes, introduced in Unruh's original paper [1], can be characterized as the superpositions of $f^{\mathbf{k}}(x)$ that are eigenfunctions of the boost transformation in the z-direction. To see this, we first define the rapidity in the z-direction by

$$\vartheta = \frac{1}{2a}\ln\frac{k + k_z}{k - k_z}, \tag{5}$$

where $a(>0)$ and ϑ have the dimensions of acceleration and time, respectively. Then,

$$k = k_\perp \cosh a\vartheta, \tag{6}$$

$$k_z = k_\perp \sinh a\vartheta, \tag{7}$$

where $k_\perp = \|\mathbf{k}_\perp\|$. The commutator (3) can be written as

$$[\hat{a}_{\mathbf{k}}, \hat{a}^{\dagger}_{\mathbf{k}'}] = 16\pi^3 a^{-1}\delta(\vartheta - \vartheta')\delta^{(2)}(\mathbf{k}_\perp - \mathbf{k}'_\perp), \tag{8}$$

where ϑ' is the rapidity for $\mathbf{k}'$, and the measure for the integration over $\mathbf{k}$ can be written as

$$\frac{d^3\mathbf{k}}{(2\pi)^3 2k} = \frac{a}{16\pi^3} d\vartheta d^2\mathbf{k}_\perp. \tag{9}$$

Now, we define the Unruh modes by [38,39]

$$u^{(\omega,\mathbf{k}_\perp)}(x) = a\int_{-\infty}^{\infty} e^{-i\omega\vartheta} f^{\mathbf{k}}(x) d\vartheta, \tag{10}$$

which can be inverted as

$$f^{\mathbf{k}}(x) = \frac{1}{2\pi a}\int_{-\infty}^{\infty} e^{i\omega\vartheta} u^{(\omega,\mathbf{k}_\perp)}(x) d\omega. \tag{11}$$

[See Ref. [40] for a similar formula for the Unruh modes for the spinor field.] Under the boost transformation parametrized by β,

$$(t,z) \mapsto (t\cosh a\beta - z\sinh a\beta, -t\sinh a\beta + z\cosh a\beta), \tag{12}$$

we have $f^{\mathbf{k}}(x) \mapsto f^{\mathbf{k}'}(x)$, where

$$\mathbf{k}'_\perp = \mathbf{k}_\perp \quad \text{and} \quad k'_z = k_\perp \sinh[a(\vartheta+\beta)].$$

This transformation can be undone in the integral in Eq. (10) by letting $\vartheta \mapsto \vartheta - \beta$. Thus, under the boost transformation (12), the Unruh modes transform as $u^{(\omega,\mathbf{k}_\perp)}(x) \mapsto e^{i\omega\beta} u^{(\omega,\mathbf{k}_\perp)}(x)$. That is, the Unruh modes are indeed eigenfunctions of the boost transformation in the z-direction.

It is clear from Eqs. (10) and (11) that the Unruh modes form a (generalized) basis for the space of positive-frequency solutions to the Klein-Gordon equation. Hence, one can expand the quantum field $\hat{\phi}(x)$ in terms of the Unruh modes:

$$\hat{\phi}(x) = \int \frac{d^2\mathbf{k}_\perp}{16\pi^3 a}\int_{-\infty}^{\infty} d\omega \Big[u^{(\omega,\mathbf{k}_\perp)}(x)\hat{b}_{(\omega,\mathbf{k}_\perp)} + \overline{u^{(\omega,\mathbf{k}_\perp)}(x)}\hat{b}^\dagger_{(\omega,\mathbf{k}_\perp)}\Big]. \tag{13}$$

Substituting Eq. (11) into Eq. (1) with Eq. (9) and identifying the coefficients of $u^{(\omega,\mathbf{k}_\perp)}(x)$ as $\hat{b}_{(\omega,\mathbf{k}_\perp)}$ in Eq. (13), we find

$$\hat{b}_{(\omega,\mathbf{k}_\perp)} = \frac{a}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\vartheta}\hat{a}_{\mathbf{k}} d\vartheta. \tag{14}$$

Then, we find from Eq. (8),

$$[\hat{b}_{(\omega,\mathbf{k}_\perp)}, \hat{b}^\dagger_{(\omega',\mathbf{k}'_\perp)}] = 8\pi^2 a\delta(\omega-\omega')\delta^{(2)}(\mathbf{k}_\perp - \mathbf{k}'_\perp). \tag{15}$$

Since the operators $\hat{b}_{(\omega,\mathbf{k}_\perp)}$ are superpositions of $\hat{a}_{\mathbf{k}}$, the Minkowski vacuum state is annihilated by them:

$$\hat{b}_{(\omega,\mathbf{k}_\perp)}|0_{\mathrm{M}}\rangle = 0 \quad \text{for all } \omega \quad \text{and} \quad \mathbf{k}_\perp. \tag{16}$$

III. THE UNRUH MODES IN RINDLER COORDINATES

In this section we show that the definition (10) of the Unruh modes agrees with the standard definition as linear combinations of the Rindler modes [41]. Then we review some aspects of the Unruh effect. First we define the Rindler coordinates and Rindler wedges. Right Rindler coordinates, τ and ξ, are defined by the relations

$$t = a^{-1}e^{a\xi}\sinh a\tau, \tag{17}$$

$$z = a^{-1}e^{a\xi}\cosh a\tau. \tag{18}$$

The constant a is the proper acceleration of the worldline defined by $\xi = 0$ and $\mathbf{x}_\perp = \text{constant}$. These coordinates, together with $\mathbf{x}_\perp$, cover the right Rindler wedge defined by $z > |t|$. Left Rindler coordinates, $\tilde{\tau}$ and $\tilde{\xi}$, are defined by

$$t = a^{-1}e^{a\tilde{\xi}}\sinh a\tilde{\tau}, \tag{19}$$

$$z = -a^{-1}e^{a\tilde{\xi}}\cosh a\tilde{\tau}. \tag{20}$$

These coordinates, together with $\mathbf{x}_\perp$, cover the left Rindler wedge defined by $z < -|t|$.

The Unruh modes (10) can be expressed in Rindler coordinates given by Eqs. (17)–(20), using Eqs. (6) and (7), as

$$u^{(\omega,\mathbf{k}_\perp)}(x) = ae^{-i\omega\tau}\int_{-\infty}^{\infty} e^{-i\omega s + i(k_\perp/a)\exp(a\xi)\sinh as + i\mathbf{k}_\perp\cdot\mathbf{x}_\perp} ds, \tag{21}$$

in the right Rindler wedge, and

$$u^{(\omega,\mathbf{k}_\perp)}(x) = ae^{i\omega\tilde{\tau}}\int_{-\infty}^{\infty} e^{-i\omega s - i(k_\perp/a)\exp(a\tilde{\xi})\sinh as + i\mathbf{k}_\perp\cdot\mathbf{x}_\perp} ds, \tag{22}$$

in the left Rindler wedge. Using the formula [[42], Eq. 10.32.7]

$$\int_{-\infty}^{\infty} e^{-i\omega s} e^{ix\sinh as} ds = \frac{2}{a} e^{\pi\omega/2a} K_{i\omega/a}(x) \quad \text{for } x > 0, \tag{23}$$

and its complex conjugate, where $K_\nu(x)$ is the modified Bessel function of the second kind, and recalling that $K_\nu(x) = K_{-\nu}(x)$, we find

$$u^{(\omega,\mathbf{k}_\perp)}(x) = \begin{cases} 2e^{\pi\omega/2a}K_{i\omega/a}\left(\frac{k_\perp}{a}e^{a\xi}\right)e^{-i\omega\tau+i\mathbf{k}_\perp\cdot\mathbf{x}_\perp} & \text{in the right Rindler wedge,} \\ 2e^{-\pi\omega/2a}K_{i\omega/a}\left(\frac{k_\perp}{a}e^{a\tilde{\xi}}\right)e^{i\omega\tilde{\tau}+i\mathbf{k}_\perp\cdot\mathbf{x}_\perp} & \text{in the left Rindler wedge.} \end{cases} \tag{24}$$

This equation is valid for all real values of ω. Now, we define the right and left Rindler modes, $v^{(\mathrm{R}:\omega,\mathbf{k}_\perp)}(x)$ and $v^{(\mathrm{L}:\omega,\mathbf{k}_\perp)}(x)$, respectively, for $\omega > 0$ by

$$v^{(\mathrm{R}:\omega,\mathbf{k}_\perp)}(x) = \frac{u^{(\omega,\mathbf{k}_\perp)}(x) - e^{-\pi\omega/a}\overline{u^{(-\omega,-\mathbf{k}_\perp)}(x)}}{\sqrt{1-e^{-2\pi\omega/a}}}, \tag{25}$$

$$v^{(\mathrm{L}:\omega,\mathbf{k}_\perp)}(x) = \frac{u^{(-\omega,\mathbf{k}_\perp)}(x) - e^{-\pi\omega/a}\overline{u^{(\omega,-\mathbf{k}_\perp)}(x)}}{\sqrt{1-e^{-2\pi\omega/a}}}. \tag{26}$$

Then,

$$v^{(\mathrm{R}:\omega,\mathbf{k}_\perp)}(x) = \begin{cases} \sqrt{8\sinh(\pi\omega/a)}K_{i\omega/a}\left(\frac{k_\perp}{a}e^{a\xi}\right)e^{-i\omega\tau+i\mathbf{k}_\perp\cdot\mathbf{x}_\perp} & \text{in the right Rindler wedge,} \\ 0 & \text{in the left Rindler wedge,} \end{cases} \tag{27}$$

$$v^{(\mathrm{L}:\omega,\mathbf{k}_\perp)}(x) = \begin{cases} 0 & \text{in the right Rindler wedge,} \\ \sqrt{8\sinh(\pi\omega/a)}K_{i\omega/a}\left(\frac{k_\perp}{a}e^{a\tilde{\xi}}\right)e^{-i\omega\tilde{\tau}+i\mathbf{k}_\perp\cdot\mathbf{x}_\perp} & \text{in the left Rindler wedge.} \end{cases} \tag{28}$$

The right Rindler modes are positive-frequency with respect to the Killing vector field $\partial/\partial\tau$, whereas the left Rindler modes are positive-frequency with respect to the Killing vector field $\partial/\partial\tilde{\tau}$. The Unruh modes can be expressed in terms of the Rindler modes by inverting the relations (25) and (26) for $\omega > 0$ as

$$u^{(\omega,\mathbf{k}_\perp)}(x) = \frac{v^{(\mathrm{R}:\omega,\mathbf{k}_\perp)}(x) + e^{-\pi\omega/a}\overline{v^{(\mathrm{L}:\omega,-\mathbf{k}_\perp)}(x)}}{\sqrt{1-e^{-2\pi\omega/a}}}, \tag{29}$$

$$u^{(-\omega,\mathbf{k}_\perp)}(x) = \frac{v^{(\mathrm{L}:\omega,\mathbf{k}_\perp)}(x) + e^{-\pi\omega/a}\overline{v^{(\mathrm{R}:\omega,-\mathbf{k}_\perp)}(x)}}{\sqrt{1-e^{-2\pi\omega/a}}}. \tag{30}$$

The scalar field $\hat{\phi}(x)$ can be written as

$$\hat{\phi}(x) = \hat{\phi}_{\mathrm{R}}(x) + \hat{\phi}_{\mathrm{L}}(x), \tag{31}$$

where

$$\hat{\phi}_{\mathrm{R}}(x) = \int \frac{d^2\mathbf{k}_\perp}{16\pi^3 a}\int_0^\infty d\omega\Big[v^{(\mathrm{R}:\omega,\mathbf{k}_\perp)}(x)\hat{b}_{(\mathrm{R}:\omega,\mathbf{k}_\perp)} + \overline{v^{(\mathrm{R}:\omega,\mathbf{k}_\perp)}(x)}\hat{b}^\dagger_{(\mathrm{R}:\omega,\mathbf{k}_\perp)}\Big], \tag{32}$$

and similarly for $\hat{\phi}_{\mathrm{L}}(x)$. By substituting Eqs. (29) and (30) into Eq. (13) and comparing the resulting expression with Eq. (32), the annihilation operators $\hat{b}_{(\mathrm{R}:\omega,\mathbf{k}_\perp)}$ and $\hat{b}_{(\mathrm{L}:\omega,\mathbf{k}_\perp)}$ can be expressed as

$$\hat{b}_{(\mathrm{R}:\omega,\mathbf{k}_\perp)} = \frac{\hat{b}_{(\omega,\mathbf{k}_\perp)} - e^{-\pi\omega/a}\hat{b}^\dagger_{(-\omega,-\mathbf{k}_\perp)}}{\sqrt{1-e^{-2\pi\omega/a}}}, \tag{33}$$

$$\hat{b}_{(\mathrm{L}:\omega,\mathbf{k}_\perp)} = \frac{\hat{b}_{(-\omega,\mathbf{k}_\perp)} - e^{-\pi\omega/a}\hat{b}^\dagger_{(\omega,-\mathbf{k}_\perp)}}{\sqrt{1-e^{-2\pi\omega/a}}}. \tag{34}$$

All right Rindler operators, $\{\hat{b}_{(\mathrm{R}:\omega,\mathbf{k}_\perp)}, \hat{b}^\dagger_{(\mathrm{R}:\omega,\mathbf{k}_\perp)}\}$, commute with all left Rindler operators, $\{\hat{b}_{(\mathrm{L}:\omega,\mathbf{k}_\perp)}, \hat{b}^\dagger_{(\mathrm{L}:\omega,\mathbf{k}_\perp)}\}$, and these operators satisfy

$$[\hat{b}_{(\mathrm{R}:|\omega|,\mathbf{k}_\perp)}, \hat{b}^\dagger_{(\mathrm{R}:|\omega'|,\mathbf{k}'_\perp)}] = [\hat{b}_{(\mathrm{L}:|\omega|,\mathbf{k}_\perp)}, \hat{b}^\dagger_{(\mathrm{L}:|\omega'|,\mathbf{k}'_\perp)}] = 8\pi^2 a\delta(|\omega|-|\omega'|)\delta^{(2)}(\mathbf{k}_\perp-\mathbf{k}'_\perp). \tag{35}$$

The Fulling vacuum, $|0_{\mathrm{F}}\rangle$, is defined by

$$\hat{b}_{(\mathrm{R}:\omega,\mathbf{k}_\perp)}|0_{\mathrm{F}}\rangle = \hat{b}_{(\mathrm{L}:\omega,\mathbf{k}_\perp)}|0_{\mathrm{F}}\rangle = 0. \tag{36}$$

Thus, there are no “Rindler particles,” i.e., particles created by the operators $\hat{b}^\dagger_{(\mathrm{R}:\omega,\mathbf{k}_\perp)}$ or $\hat{b}^\dagger_{(\mathrm{L}:\omega,\mathbf{k}_\perp)}$, in this state. Then we find, from Eq. (33),

$$\langle 0_{\mathrm{M}}|\hat{b}^\dagger_{(\mathrm{R}:\omega,\mathbf{k}_\perp)}\hat{b}_{(\mathrm{R}:\omega',\mathbf{k}'_\perp)}|0_{\mathrm{M}}\rangle = \frac{8\pi^2 a}{e^{2\pi\omega/a}-1}\delta(\omega-\omega')\delta^{(2)}(\mathbf{k}_\perp-\mathbf{k}'_\perp). \tag{37}$$

Now, let us define the operator $\hat{A}$ by

$$\hat{A} = \frac{1}{\sqrt{8\pi^2 a}} \int_0^\infty d\omega \int d^2\mathbf{k}_\perp F(\omega, \mathbf{k}_\perp) \hat{b}_{(\mathrm{R}:\omega,\mathbf{k}_\perp)}, \quad (38)$$

where $F(\omega, \mathbf{k}_\perp)$ is a continuous and square-integrable function. Then

$$[\hat{A}, \hat{A}^\dagger] = \int_0^\infty d\omega \int d^2\mathbf{k}_\perp |F(\omega, \mathbf{k}_\perp)|^2, \quad (39)$$

and

$$\langle 0_\mathrm{M}|\hat{A}^\dagger \hat{A}|0_\mathrm{M}\rangle = \int_0^\infty d\omega \int d^2\mathbf{k}_\perp \frac{|F(\omega, \mathbf{k}_\perp)|^2}{e^{2\pi\omega/a} - 1}. \quad (40)$$

Thus, the expected number of the "Rindler particles" with given Rindler energy (i.e., the energy associated with the boost Killing vector field $\partial/\partial\tau$) is given by the Bose-Einstein distribution function with temperature $a/2\pi$. This is a manifestation of the well-known Unruh effect: the Minkowski vacuum, $|0_\mathrm{M}\rangle$, is a thermal state with temperature $a/2\pi$ with respect to the energy corresponding to the boost Killing vector field, $\partial/\partial\tau$, i.e., the Rindler energy, in the right Rindler wedge.

IV. RADIATION BY A CLASSICAL SOURCE THROUGH THE UNRUH EFFECT

In the Heisenberg picture, analysis of radiation from a classical source does not depend on the quantum state because it reduces to solving the classical field equation (see Appendix A). Thus, the radiation can be discussed in the purely classical context, as is well known. However, perhaps surprisingly, to analyze such a process of radiation in the interaction picture in the Rindler frame, it is crucial to take the Unruh effect into account as the present authors have emphasized in the context of the uniformly accelerated source. In this section we explain this fact for a general classical source [43].

The classical source is introduced by adding the following term in the Lagrangian density:

$$\hat{\mathcal{L}}_\mathrm{int}(x) = j(x)\hat{\phi}(x), \quad (41)$$

with the corresponding interaction action,

$$\hat{S}_\mathrm{int} = \int \hat{\mathcal{L}}_\mathrm{int}(x) d^4x. \quad (42)$$

Then, in the interaction picture, this interaction adds to the final state the following one-particle part:

$$|1\mathrm{p}\rangle = i \int j(x)\hat{\phi}(x)|0_\mathrm{M}\rangle d^4x \quad (43)$$

$$= i \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k} \tilde{j}(k) \hat{a}^\dagger_\mathbf{k} |0_\mathrm{M}\rangle, \quad (44)$$

where $\tilde{j}(k)$ is the 4-dimensional Fourier transform of $j(x)$:

$$\tilde{j}(k) = \int j(x) e^{ikt - i\mathbf{k}\cdot\mathbf{x}} d^4x. \quad (45)$$

The final state to first order is

$$|f\rangle = (1 + i\mathcal{A}_\mathrm{for})|0_\mathrm{M}\rangle + |1\mathrm{p}\rangle, \quad (46)$$

where the forward-scattering amplitude $\mathcal{A}_\mathrm{for}$ satisfies

$$2\ \mathrm{Im}\ \mathcal{A}_\mathrm{for} = \langle 1\mathrm{p}|1\mathrm{p}\rangle, \quad (47)$$

so that $\langle f|f\rangle = 1$ to first order.

The emission probability can be found as an integral over $\mathbf{k}$:

$$P^{(\mathrm{M})}_\mathrm{em} = \langle 1\mathrm{p}|1\mathrm{p}\rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k} |\tilde{j}(k)|^2. \quad (48)$$

The Minkowski particle number operator is defined as

$$\hat{N} = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k} \hat{a}^\dagger_\mathbf{k} \hat{a}_\mathbf{k} = \frac{1}{8\pi^2 a} \int_{-\infty}^\infty d\omega \int d^2\mathbf{k}_\perp \hat{b}^\dagger_{(\omega,\mathbf{k}_\perp)} \hat{b}_{(\omega,\mathbf{k}_\perp)}. \quad (49)$$

The first expression for $\hat{N}$ can be used to show that $\langle f|\hat{N}|f\rangle = \langle 1\mathrm{p}|1\mathrm{p}\rangle = P^{(\mathrm{M})}_\mathrm{em}$. That is, the one-particle emission probability is equal to the expected Minkowski-particle number in the final state at first order in perturbation theory. Equation (49) can be regarded as a classical result in the following sense: If we Fourier-transform the energy density of the emitted wave, divide each $\mathbf{k}$-component by k, and integrate the result over $\mathbf{k}$, then we obtain this expression [44].

Now, if the classical source $j(x)$ has support only in the right Rindler wedge, then the same process can be described as follows. The field $\hat{\phi}(x)$ restricted to the right Rindler wedge is the field $\hat{\phi}_\mathrm{R}(x)$ given by Eq. (32). Recall that the Minkowski vacuum is seen as a thermal bath of temperature $a/2\pi$ with respect to the energy associated with the Killing vector field $\partial/\partial\tau$. We define the emission and absorption amplitudes by

$$\mathcal{A}^{(\omega,\mathbf{k}_\perp)}_\mathrm{em} = \langle 0_\mathrm{F}|\hat{b}_{(\mathrm{R}:\omega,\mathbf{k}_\perp)} \hat{S}_\mathrm{int}|0_\mathrm{F}\rangle = \frac{1}{2\pi} \int j(x) \overline{v^{(\mathrm{R}:\omega,\mathbf{k}_\perp)}(x)} \sqrt{-g}\, d^4x, \quad (50)$$

$$\mathcal{A}_{\rm abs}^{(\omega,\mathbf{k}_\perp)} = \langle 0_{\rm F}|\hat{S}_{\rm int}\hat{b}^\dagger_{({\rm R}:\omega,\mathbf{k}_\perp)}|0_{\rm F}\rangle = \frac{1}{2\pi}\int j(x)v^{({\rm R}:\omega,\mathbf{k}_\perp)}(x)\sqrt{-g}d^4x, \quad (51)$$

respectively, where g is the determinant of the metric $g_{\mu\nu}$ in Minkowski (and Rindler) spacetime. If the initial state was the Fulling vacuum, then the emission probability would be found using the expansion (32) of the field $\hat{\phi}_{\rm R}$ and the commutation relation (15), with $\hat{b}_{(\omega,\mathbf{k}_\perp)}$ replaced by $\hat{b}_{(R:\omega,\mathbf{k}_\perp)}$:

$$P_{\rm em}^{({\rm R},0)} = \|\hat{S}_{\rm int}|0_{\rm F}\rangle\|^2 = \frac{1}{8\pi^2 a}\int_0^\infty d\omega\int d^2\mathbf{k}_\perp|\mathcal{A}_{\rm em}^{(\omega,\mathbf{k}_\perp)}|^2. \quad (52)$$

Since the Minkowski vacuum is the thermal state of temperature $a/2\pi$ with respect to the Rindler energy, the interaction probability for the Minkowski vacuum is

$$P_{\rm int}^{({\rm R})} = \frac{1}{8\pi^2 a}\int_0^\infty d\omega\int d^2\mathbf{k}_\perp\left(\frac{|\mathcal{A}_{\rm em}^{(\omega,\mathbf{k}_\perp)}|^2}{1-e^{-2\pi\omega/a}} + \frac{|\mathcal{A}_{\rm abs}^{(\omega,\mathbf{k}_\perp)}|^2}{e^{2\pi\omega/a}-1}\right). \quad (53)$$

The first and second terms in the integrand above represent the (spontaneous and induced) emission [with $(1-e^{-2\pi\omega/a})^{-1} = 1 + (e^{2\pi\omega/a}-1)^{-1}$] and absorption, respectively.

Now, as Unruh and Wald have shown [36], both the emission and absorption of a particle in the thermal bath of temperature $a/2\pi$ in the Rindler wedge are seen as emission of a particle in Minkowski spacetime. Hence, we expect that $P_{\rm int}^{({\rm R})} = P_{\rm em}^{({\rm M})}$. To demonstrate this equality, we use the expansion of $\hat{\phi}(x)$ in terms of the Unruh modes, i.e., Eq. (13), in Eq. (43). Thus, we find

$$|1{\rm p}\rangle = \frac{i}{16\pi^3 a}\int_{-\infty}^{\infty} d\omega\int d^2\mathbf{k}_\perp \tilde{\jmath}^{({\rm R})}(\omega,\mathbf{k}_\perp)\hat{b}^\dagger_{(\omega,\mathbf{k}_\perp)}|0_{\rm M}\rangle, \quad (54)$$

where

$$\tilde{\jmath}^{({\rm R})}(\omega,\mathbf{k}_\perp) = \int j(x)\overline{u^{(\omega,\mathbf{k}_\perp)}(x)}\sqrt{-g}\,d^4x. \quad (55)$$

Since the classical source $j(x)$ has support only in the right Rindler wedge by assumption, we have from Eqs. (29) and (30),

$$\tilde{\jmath}^{({\rm R})}(\omega,\mathbf{k}_\perp) = \frac{2\pi}{\sqrt{1-e^{-2\pi\omega/a}}}\mathcal{A}_{\rm em}^{(\omega,\mathbf{k}_\perp)}, \quad (56)$$

$$\tilde{\jmath}^{({\rm R})}(-\omega,\mathbf{k}_\perp) = \frac{2\pi}{\sqrt{e^{2\pi\omega/a}-1}}\mathcal{A}_{\rm abs}^{(\omega,-\mathbf{k}_\perp)}, \quad (57)$$

with $\omega > 0$. Thus, we find

$$|1{\rm p}\rangle = \frac{i}{8\pi^2 a}\int_0^\infty d\omega\int d^2\mathbf{k}_\perp\left[\frac{\mathcal{A}_{\rm em}^{(\omega,\mathbf{k}_\perp)}}{\sqrt{1-e^{-2\pi\omega}}}\hat{b}^\dagger_{(\omega,\mathbf{k}_\perp)} + \frac{\mathcal{A}_{\rm abs}^{(\omega,-\mathbf{k}_\perp)}}{\sqrt{e^{2\pi\omega}-1}}\hat{b}^\dagger_{(-\omega,\mathbf{k}_\perp)}\right]|0_{\rm M}\rangle. \quad (58)$$

Then, we have $\langle 1{\rm p}|1{\rm p}\rangle = P_{\rm int}^{({\rm R})}$ by Eq. (15), where $P_{\rm int}^{({\rm R})}$ is given by Eq. (53). Thus, we indeed find $P_{\rm int}^{({\rm R})} = P_{\rm em}^{({\rm M})}$ given by Eq. (48).

V. THE ELECTROMAGNETIC AND GRAVITATIONAL CASES

In this section we briefly discuss the electromagnetic and gravitational fields coupled to a classical source and show that the results for the massless scalar field presented in the previous sections hold for these fields as well. This will be the generalization of some results for the uniformly accelerated sources coupled to the electromagnetic field [33] and gravitational field [35] [see also Ref. [34]].

A. The electromagnetic case

The Lagrangian density with the interaction term between the quantum field and a classical current is given by

$$\mathcal{L}_{\rm int} = \sqrt{-g}\left[-\frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} - j^\mu(x)\hat{A}_\mu(x)\right], \quad (59)$$

where $\hat{F}_{\mu\nu} = \nabla_\mu\hat{A}_\nu - \nabla_\nu\hat{A}_\mu$. We define the anti-symmetric tensors $\epsilon_{\mu\nu}^{(\|)}$ and $\epsilon_{\mu\nu}^{(\perp)}$ by

$$\epsilon_{tz}^{(\|)} = -\epsilon_{zt}^{(\|)} = 1, \quad (60)$$

$$\epsilon_{xy}^{(\perp)} = -\epsilon_{yx}^{(\perp)} = 1, \quad (61)$$

with all other components vanishing. In Rindler coordinates, we have

$$\epsilon_{\tau\xi}^{(\|)} = -\epsilon_{\xi\tau}^{(\|)} = e^{2a\xi}, \quad (62)$$

with all other components vanishing.

Two physical modes with momentum $\mathbf{k}$ can be chosen as

$$A_\mu^{({\rm I}:\mathbf{k})}(x) = \frac{1}{k_\perp}\epsilon_{\mu\nu}^{(\|)}\nabla^\nu f^{\mathbf{k}}(x), \quad (63)$$

$$A_\mu^{(\mathrm{II}:\mathbf{k})}(x) = \frac{1}{k_\perp}\epsilon_{\mu\nu}^{(\perp)}\nabla^\nu f^{\mathbf{k}}(x). \tag{64}$$

These modes are transverse, $\nabla^\mu A_\mu^{(\mathrm{P}:\mathbf{k})} = 0$, and satisfy the orthonormality condition for the transverse modes,

$$-i\int_\Sigma \overline{A^{(\mathrm{P}':\,\mathbf{k}')\mu}}\overleftrightarrow{\nabla}_\nu A_\mu^{(\mathrm{P}:\mathbf{k})} d\Sigma^\nu = (2\pi)^3 2k\delta^{\mathrm{P}'\mathrm{P}}\delta^{(3)}(\mathbf{k}'-\mathbf{k}), \tag{65}$$

where $\overleftrightarrow{\nabla}_\nu = \overleftarrow{\nabla}_\nu - \overrightarrow{\nabla}_\nu$ and where Σ is a $t = \text{constant}$ Cauchy surface. Then, the quantum electromagnetic field $\hat{A}_\mu(x)$ with complete gauge fixing is

$$\hat{A}_\mu(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k}\sum_{\mathrm{P}=\mathrm{I},\mathrm{II}}\left[A_\mu^{(\mathrm{P}:\mathbf{k})}(x)\hat{a}_{(\mathrm{P}:\mathbf{k})} + \overline{A_\mu^{(\mathrm{P}:\mathbf{k})}(x)}\hat{a}^\dagger_{(\mathrm{P}:\mathbf{k})}\right], \tag{66}$$

where the operators $\hat{a}_{(\mathrm{P}:\mathbf{k})}$ and $\hat{a}^\dagger_{(\mathrm{P}:\mathbf{k})}$, $\mathrm{P} = \mathrm{I}, \mathrm{II}$, satisfy

$$[\hat{a}_{(\mathrm{P}:\mathbf{k})}, \hat{a}^\dagger_{(\mathrm{P}':\,\mathbf{k}')}] = (2\pi)^3 2k\delta_{\mathrm{PP}'}\delta^{(3)}(\mathbf{k}-\mathbf{k}'), \tag{67}$$

with all other commutators among them vanishing, which is a consequence of the orthonormality condition (65). The Minkowski vacuum state $|0_\mathrm{M}\rangle$ is annihilated by $\hat{a}_{(\mathrm{P}:\mathbf{k})}$, i.e., $\hat{a}_{(\mathrm{P}:\mathbf{k})}|0_\mathrm{M}\rangle = 0$, for all P and $\mathbf{k}$.

The one-photon state which the classical current $j^\mu(x)$ generates is

$$\begin{aligned}|1\mathrm{p}\rangle &= -i\int j^\mu(x)\hat{A}_\mu(x)|0_\mathrm{M}\rangle d^4x\\ &= -\int \frac{d^3\mathbf{k}}{(2\pi)^2 2k}\tilde{\jmath}^\mu(k)k_\perp^{-1}[\epsilon_{\mu\nu}^{(\parallel)}k^\nu\hat{a}^\dagger_{(\mathrm{I}:\mathbf{k})} + \epsilon_{\mu\nu}^{(\perp)}k^\nu\hat{a}^\dagger_{(\mathrm{II}:\mathbf{k})}]|0_\mathrm{M}\rangle,\end{aligned} \tag{68}$$

where $\tilde{\jmath}^\mu(k)$ is the Fourier transform of $j^\mu(x)$ defined in the same way as $\tilde{\jmath}(k)$ is defined from $j(x)$ in Eq. (45). Then, the emission probability is

$$\begin{aligned}P_{\mathrm{em}}^{(\mathrm{M})} &= \langle 1\mathrm{p}|1\mathrm{p}\rangle\\ &= \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k}\overline{\tilde{\jmath}^\mu(k)}k_\perp^{-2}[\epsilon_{\mu\alpha}^{(\parallel)}k^\alpha\epsilon_{\nu\beta}^{(\parallel)}k^\beta + \epsilon_{\mu\alpha}^{(\perp)}k^\alpha\epsilon_{\nu\beta}^{(\perp)}k^\beta]\tilde{\jmath}^\nu(k).\end{aligned} \tag{69}$$

One can verify the following formula (for $k_\perp \neq 0$), e.g., in the Lorentz frame with $k^t = k^x = k_\perp$ and $k^z = k^y = 0$:

$$\epsilon_{\mu\alpha}^{(\parallel)}k^\alpha\epsilon_{\nu\beta}^{(\parallel)}k^\beta + \epsilon_{\mu\alpha}^{(\perp)}k^\alpha\epsilon_{\nu\beta}^{(\perp)}k^\beta = -k_\perp^2 g_{\mu\nu} + \frac{1}{2}(k_\mu\check{k}_\nu + k_\nu\check{k}_\mu), \tag{70}$$

where the vector $\check{k}^\mu$ is obtained by multiplying the x- and y-components of k^μ by -1. Since $k_\mu\tilde{\jmath}^\mu(k) = 0$, we find

$$\langle 1\mathrm{p}|1\mathrm{p}\rangle = -\int \frac{d^3\mathbf{k}}{(2\pi)^3 2k}\overline{\tilde{\jmath}^\mu(k)}\tilde{\jmath}_\mu(k), \tag{71}$$

as is well known.

We define the Rindler modes by

$$A_\mu^{(\mathrm{RI}:\omega,\mathbf{k}_\perp)}(x) = \frac{1}{k_\perp}\epsilon_{\mu\lambda}^{(\parallel)}\nabla^\lambda v^{(\mathrm{R}:\omega,\mathbf{k}_\perp)}(x), \tag{72}$$

$$A_\mu^{(\mathrm{RII}:\omega,\mathbf{k}_\perp)}(x) = \frac{1}{k_\perp}\epsilon_{\mu\lambda}^{(\perp)}\nabla^\lambda v^{(\mathrm{R}:\omega,\mathbf{k}_\perp)}(x). \tag{73}$$

Since the differential operators $\epsilon_{\mu\lambda}^{(\parallel)}\nabla^\lambda$ and $\epsilon_{\mu\lambda}^{(\perp)}\nabla^\lambda$ commute with the operations for defining the Unruh and Rindler modes from the modes $f^{\mathbf{k}}(x)$, the relationship between the right Rindler modes $A_\mu^{(\mathrm{RP}:\omega,\mathbf{k}_\perp)}(x)$, $\mathrm{P} = \mathrm{I}, \mathrm{II}$, and the Minkowski modes $A_\mu^{(\mathrm{P}:\mathbf{k})}(x)$ is exactly the same as that between $v^{(\mathrm{R}:\omega,\mathbf{k}_\perp)}(x)$ and $f^{\mathbf{k}}(x)$. Hence, the quantum field $\hat{A}_\mu(x)$ in the right Rindler wedge can be expanded in the same way as in the scalar case (32):

$$\begin{aligned}\hat{A}_{\mathrm{R}\mu}(x) = \int \frac{d^2\mathbf{k}_\perp}{16\pi^3 a}\int_0^\infty d\omega \sum_{\mathrm{P}=\mathrm{I},\mathrm{II}}\Big[&A_\mu^{(\mathrm{RP}:\omega,\mathbf{k}_\perp)}(x)\hat{b}_{(\mathrm{RP}:\omega,\mathbf{k}_\perp)}\\ &+ \overline{A_\mu^{(\mathrm{RP}:\omega,\mathbf{k}_\perp)}(x)}\hat{b}^\dagger_{(\mathrm{RP}:\omega,\mathbf{k}_\perp)}\Big],\end{aligned} \tag{74}$$

where the operators $\hat{b}_{(\mathrm{RP}:\omega,\mathbf{k}_\perp)}$ and $\hat{b}^\dagger_{(\mathrm{RP}:\omega,\mathbf{k}_\perp)}$ satisfy the commutation relations

$$\left[\hat{b}_{(\mathrm{RP}:\omega,\mathbf{k}_\perp)}, \hat{b}^\dagger_{(\mathrm{RP}':\omega',\mathbf{k}'_\perp)}\right] = 8\pi^2 a\delta_{\mathrm{PP}'}\delta(\omega-\omega')\delta^{(2)}(\mathbf{k}_\perp - \mathbf{k}'_\perp), \tag{75}$$

with all other commutators among them vanishing. Then, exactly as in the scalar case, we find

$$\langle 1\mathrm{p}|1\mathrm{p}\rangle = \int \frac{d^2\mathbf{k}_\perp}{8\pi^2 a}\int_0^\infty d\omega \sum_{\mathrm{P}=\mathrm{I},\mathrm{II}} \times\left(\frac{|\mathcal{A}_{\mathrm{em}}^{(\mathrm{P}:\omega,\mathbf{k}_\perp)}|^2}{1-e^{-2\pi\omega/a}} + \frac{|\mathcal{A}_{\mathrm{abs}}^{(\mathrm{P}:\omega,\mathbf{k}_\perp)}|^2}{e^{2\pi\omega/a}-1}\right), \tag{76}$$

where

$$\mathcal{A}_{\mathrm{em}}^{(\mathrm{P}:\omega,\mathbf{k}_\perp)} = \frac{1}{2\pi}\int j^\mu(x)\overline{A_\mu^{(\mathrm{RP}:\omega,\mathbf{k}_\perp)}(x)}\sqrt{-g}\,d^4x, \tag{77}$$

$$\mathcal{A}_{\mathrm{abs}}^{(\mathrm{P}:\omega,\mathbf{k}_\perp)} = \frac{1}{2\pi}\int j^\mu(x)A_\mu^{(\mathrm{RP}:\omega,\mathbf{k}_\perp)}(x)\sqrt{-g}d^4x, \tag{78}$$

with the same interpretation of Eq. (76) in terms of the Unruh effect as in the scalar case.

B. The gravitational case

The Lagrangian density $\mathcal{L}_{\rm EH}^{(2)}$ for the linearized gravitational field coupled to a classical stress-energy tensor is

$$\frac{1}{\sqrt{-g}}\mathcal{L}_{\rm EH}^{(2)} = \frac{1}{2}\nabla_\alpha h_{\mu\nu}\nabla^\alpha h^{\mu\nu} - \nabla_\alpha h_{\beta\mu}\nabla^\beta h^{\alpha\mu} + \left(\nabla_\alpha h^{\mu\alpha} - \frac{1}{2}\nabla^\mu h\right)\nabla_\mu h + \kappa T^{\mu\nu}h_{\mu\nu}, \tag{79}$$

where $h = h^\alpha{}_\alpha$ and $\kappa = \sqrt{8\pi G}$. Two physical modes with momentum $\mathbf{k}$ can be chosen as

$$h_{\mu\nu}^{(\mathrm{I}:\mathbf{k})}(x) = \frac{1}{\sqrt{2}}[g_{\mu\nu} + 2q_{\mu\nu}(\mathbf{k}_\perp)]f^{\mathbf{k}}(x), \tag{80}$$

where

$$q_{\mu\nu}(\mathbf{k}_\perp) = \begin{cases} -g_{\mu\nu} - k_\mu k_\nu/k_\perp^2 & \text{if } \mu,\nu = x \text{ or } y, \\ 0 & \text{otherwise,} \end{cases} \tag{81}$$

and

$$h_{\mu\nu}^{(\mathrm{II}:\mathbf{k})}(x) = \frac{1}{\sqrt{2}k_\perp^2}\left[\epsilon_{\mu\alpha}^{(\parallel)}\epsilon_{\nu\beta}^{(\perp)} + \epsilon_{\nu\alpha}^{(\parallel)}\epsilon_{\mu\beta}^{(\perp)}\right]\nabla^\alpha\nabla^\beta f^{\mathbf{k}}(x). \tag{82}$$

These modes satisfy the de Donder condition,

$$\nabla^\mu h_{\mu\nu}^{(\mathrm{P}:\mathbf{k})} - \frac{1}{2}\nabla_\nu h^{(\mathrm{P}:\mathbf{k})} = 0, \tag{83}$$

and the normalization condition for the modes satisfying Eq. (83),

$$i\int\left[\overline{h^{(\mathrm{P}':\,\mathbf{k}')\mu\nu}}\overleftrightarrow{\nabla}_\alpha h_{\mu\nu}^{(\mathrm{P}:\mathbf{k})} - \frac{1}{2}\overline{h^{(\mathrm{P}':\mathbf{k}')}}\overleftrightarrow{\nabla}_\alpha h^{(\mathrm{P}:\mathbf{k})}\right]d\Sigma^\alpha = (2\pi)^3 2k\delta^{\mathrm{P}'\mathrm{P}}\delta^{(3)}(\mathbf{k}'-\mathbf{k}). \tag{84}$$

Then, the emission probability from the classical stress-energy tensor $T^{\mu\nu}(x)$ is

$$\langle 1\mathrm{p}|1\mathrm{p}\rangle = \int\frac{d^3\mathbf{k}}{(2\pi)^3 2k}\sum_{\mathrm{P=I,II}}|\mathcal{A}^{(\mathrm{P}:\mathbf{k})}|^2, \tag{85}$$

where

$$\mathcal{A}^{(\mathrm{P}:\mathbf{k})} = \kappa\int T^{\mu\nu}(x)\overline{h_{\mu\nu}^{(\mathrm{P}:\mathbf{k})}(x)}\sqrt{-g}\,d^4x. \tag{86}$$

Define the Fourier transform of $T^{\mu\nu}(x)$ by

$$\mathcal{T}^{\mu\nu}(k) = \int T^{\mu\nu}(x)e^{ik\cdot x}d^4x. \tag{87}$$

(Note that $\sqrt{-g} = 1$ here.) Then,

$$\langle 1\mathrm{p}|1\mathrm{p}\rangle = \kappa^2\int\frac{d^3\mathbf{k}}{(2\pi)^3 2k}\overline{\mathcal{T}^{\mu\nu}(k)}S_{\mu\nu\lambda\sigma}(k)\mathcal{T}^{\lambda\sigma}(k), \tag{88}$$

where

$$S_{\mu\nu\lambda\sigma} = \frac{1}{2}(g_{\mu\nu} + 2q_{\mu\nu})(g_{\lambda\sigma} + 2q_{\lambda\sigma}) + \frac{2}{k_\perp^4}\epsilon_{\mu\alpha}^{(\parallel)}k^\alpha\epsilon_{\nu\beta}^{(\perp)}k^\beta\epsilon_{\lambda\gamma}^{(\parallel)}k^\gamma\epsilon_{\sigma\delta}^{(\perp)}k^\delta. \tag{89}$$

By working in the Lorentz frame where $k^z = k^y = 0$ so that $k^t = k^x = k_\perp$ (with the assumption that $k_\perp \neq 0$) and using $k_\mu\mathcal{T}^{\mu\nu}(k) = 0$, we find

$$\overline{\mathcal{T}^{\mu\nu}}(k)S_{\mu\nu\lambda\sigma}(k)\mathcal{T}^{\lambda\sigma}(k) = \overline{\mathcal{T}^{\mu\nu}(k)}\mathcal{T}_{\mu\nu}(k) - \frac{1}{2}\overline{\mathcal{T}(k)}\mathcal{T}(k), \tag{90}$$

where $\mathcal{T}(k) = \mathcal{T}^\mu{}_\mu(k)$. Thus, the emission probability is

$$\langle 1\mathrm{p}|1\mathrm{p}\rangle = \kappa^2\int\frac{d^3\mathbf{k}}{(2\pi)^3 2k}\left[\mathcal{T}^{\mu\nu}(k)\mathcal{T}_{\mu\nu}(k) - \frac{1}{2}\overline{\mathcal{T}}(k)\mathcal{T}(k)\right], \tag{91}$$

as is well known.

The physical right Rindler modes can be chosen as

$$h_{\mu\nu}^{(\mathrm{IR}:\omega,\mathbf{k}_\perp)}(x) = \frac{1}{\sqrt{2}}[g_{\mu\nu} + 2q_{\mu\nu}(\mathbf{k}_\perp)]v^{(\mathrm{R}:\omega,\mathbf{k}_\perp)}(x), \tag{92}$$

$$h_{\mu\nu}^{(\mathrm{IIR}:\omega,\mathbf{k}_\perp)}(x) = \frac{1}{\sqrt{2}k_\perp^2}\left[\epsilon_{\mu\alpha}^{(\parallel)}\epsilon_{\nu\beta}^{(\perp)} + \epsilon_{\nu\alpha}^{(\parallel)}\epsilon_{\mu\beta}^{(\perp)}\right]\nabla^\alpha\nabla^\beta v^{(\mathrm{R}:\omega,\mathbf{k}_\perp)}(x). \tag{93}$$

The modes $h_{\mu\nu}^{(\mathrm{IIR}:\omega,\mathbf{k}_\perp)}(x)$ were found in Ref. [45] up to a normalization factor whereas the modes $h_{\mu\nu}^{(\mathrm{IR}:\omega,\mathbf{k}_\perp)}(x)$ were obtained in Ref. [35] by a gauge transformation from the modes given in Ref. [45] up to a normalization factor. The tensors in Eq. (92) and the differential operator in Eq. (93) commute with the operations for defining the Unruh and Rindler modes from the Minkowski modes. Hence, the relationship between the description of the emission process in Minkowski spacetime and the interaction of the classical source with the thermal bath is exactly the same as in the scalar case. Thus, just like in the scalar case, if we define the emission and absorption amplitudes in the right Rindler wedge as

$$\mathcal{A}_{\mathrm{em}}^{(\mathrm{P}:\omega,\mathbf{k}_\perp)} = \frac{\kappa}{2\pi}\int T^{\mu\nu}(x)\overline{h_{\mu\nu}^{(\mathrm{RP}:\omega,\mathbf{k}_\perp)}(x)}\,\sqrt{-g}\,d^4x, \quad (94)$$

$$\mathcal{A}_{\mathrm{abs}}^{(\mathrm{P}:\omega,\mathbf{k}_\perp)} = \frac{\kappa}{2\pi}\int T^{\mu\nu}(x)h_{\mu\nu}^{(\mathrm{RP}:\omega,\mathbf{k}_\perp)}(x)\sqrt{-g}d^4x, \quad (95)$$

respectively, then Eq. (76) for $\langle 1\mathrm{p}|1\mathrm{p}\rangle$ holds as in the electromagnetic case together with its interpretation in terms of the Unruh effect.

VI. DISCUSSION

We have shown that what uniformly accelerated observers interpret as interaction of an *arbitrary* charge distribution with the Unruh thermal bath is interpreted by inertial observers as emission of radiation by the same charge distribution. All calculations were performed in first-order perturbation theory, which means that the inertial-frame result corresponds to *classical* radiation—once one accepts that the latter consists of *quanta* of energy satisfying Planck's energy–frequency relation. This result, obtained for the scalar, electromagnetic, and graviton fields, corroborates the claim that classical radiation observed in the inertial frame can already be considered as an "observation" of the Unruh effect in the same sense that, in Newtonian mechanics, the planetary motion in the inertial frame can be considered to be providing evidence of the inertial forces, e.g., the centrifugal force, in the rotating frame. This observation has already been made in the literature for the case of pointlike sources in specific trajectories [28–35]. Here, the observation is extended to arbitrary currents for the scalar, electromagnetic, and gravitational fields. In our view, the analysis presented here proves, beyond any doubt, that classical radiation seen by inertial observers, with the only extra assumption that it is constituted by *quanta* of energy, is a testimony to the existence of the Unruh thermal bath in the uniformly accelerated frame. Those who dispute this observation would have to reproduce the radiation from the classical source in a uniformly accelerated frame without using the Unruh effect.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: RADIATION IN THE HEISENBERG PICTURE

In this appendix, we discuss radiation of massless scalar field in four dimensions from a classical source in the Heisenberg picture. This confirms that the radiation formula from a classical source does not depend on the quantum state.

The field equation for the Heisenberg operator for a massless scalar field, $\hat{\phi}_{\mathrm{H}}(x)$, with a classical source term $j(x)$, is

$$\Box\hat{\phi}_{\mathrm{H}}(x) = j(x), \quad (\mathrm{A1})$$

where we assume $j(x)$ to be compactly supported. The retarded Green's function $G_R(x,x')$ satisfying

$$\Box_x G_R(x,x') = \delta^{(4)}(x-x') \quad (\mathrm{A2})$$

is given by

$$G_R(x,x') = i\theta(t-t')\int\frac{d^3\mathbf{k}}{(2\pi)^3 2k}[e^{-ik\cdot(x-x')} - e^{ik\cdot(x-x')}], \quad (\mathrm{A3})$$

where θ is the Heaviside step function.

Let $\hat{\phi}(x)$ be the quantum field without the source $j(x)$ that agrees with $\hat{\phi}_{\mathrm{H}}(x)$ in the past of the source. Then the field $\hat{\phi}_{\mathrm{H}}(x)$ at a time in the future of the source is

$$\hat{\phi}_{\mathrm{H}}(x) = \hat{\phi}(x) + \phi^{(\mathrm{cl})}(x), \quad (\mathrm{A4})$$

where

$$\phi^{(\mathrm{cl})}(x) = \int G_R(x,x')j(x')d^4x' = i\int\frac{d^3\mathbf{k}}{(2\pi)^3 2k}[\tilde{j}(k)e^{-ik\cdot x} - \overline{\tilde{j}(k)}e^{ik\cdot x}], \quad (\mathrm{A5})$$

where $\tilde{j}(k)$ is the Fourier transform of $j(x)$ defined by Eq. (45). Here we have used the fact that the source $j(x)$ is real. Thus, if we expand the fields $\hat{\phi}_{\rm H}(x)$ and $\hat{\phi}(x)$ as

$$\hat{\phi}_{\rm H}(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k}[\hat{a}^{\rm H}_{\mathbf{k}} e^{-ik\cdot x} + \hat{a}^{\rm H}_{\mathbf{k}} e^{ik\cdot x}], \quad \text{(A6)}$$

$$\hat{\phi}(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k}[\hat{a}_{\mathbf{k}} e^{-ik\cdot x} + \hat{a}_{\mathbf{k}} e^{ik\cdot x}], \quad \text{(A7)}$$

in the future of the source, then we have

$$\hat{a}^{\rm H}_{\mathbf{k}} = \hat{a}_{\mathbf{k}} + i\tilde{j}(k). \quad \text{(A8)}$$

We assume that the Heisenberg state $|{\rm H}\rangle$ satisfies $\langle {\rm H}|a_{\mathbf{k}}|{\rm H}\rangle = 0$ for all $\mathbf{k}$. The initial and final number operators, $\hat{N}^{(i)}$ and $\hat{N}^{(f)}$, are defined by

$$\hat{N}^{(i)} = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k}\hat{a}^{\dagger}_{\mathbf{k}} a_{\mathbf{k}}, \quad \text{(A9)}$$

$$\hat{N}^{(f)} = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k}\hat{a}^{\rm H\dagger}_{\mathbf{k}} a^{\rm H}_{\mathbf{k}}. \quad \text{(A10)}$$

Then, the initial and final particle numbers, $\langle {\rm H}|\hat{N}^{(i)}|{\rm H}\rangle$ and $\langle {\rm H}|\hat{N}^{(f)}|{\rm H}\rangle$, are ill-defined in general, but their difference, i.e., the increase in the particle number, is well-defined and state independent. It is given by

$$\Delta N = \langle {\rm H}|\hat{N}^{(f)}|{\rm H}\rangle - \langle {\rm H}|\hat{N}^{(i)}|{\rm H}\rangle = \int \frac{d^3{\rm k}}{(2\pi)^3 2k}|\tilde{j}(k)|^2. \quad \text{(A11)}$$

Thus, the increase in the particle number is independent of the state and agrees with the classical result.

APPENDIX B: EMISSION FROM A CLASSICAL SOURCE IN THE FULLING VACUUM

In this appendix, we analyze the emission process from a classical source in the Fulling vacuum state in the interaction picture. We find that the classical result is reproduced differently compared to the case with the Minkowski vacuum state.

The final state to first order in perturbation theory is

$$|f\rangle = (1 + i\mathcal{A}_{\rm for})|0_{\rm F}\rangle + |1{\rm p}^{({\rm R})}\rangle, \quad \text{(B1)}$$

where the final 1-particle state is

$$|1{\rm p}^{({\rm R})}\rangle = \frac{i}{8\pi^2 a}\int_0^\infty d\omega \int d^2\mathbf{k}_\perp \mathcal{A}^{(\omega,\mathbf{k}_\perp)}_{\rm em}\hat{b}^{\dagger}_{({\rm R}:\omega,\mathbf{k}_\perp)}|0_{\rm F}\rangle, \quad \text{(B2)}$$

and where the forward-scattering amplitude $\mathcal{A}_{\rm for}$ satisfies

$$2\,{\rm Im}\,\mathcal{A}_{\rm for} = \langle 1{\rm p}^{({\rm R})}|1{\rm p}^{({\rm R})}\rangle. \quad \text{(B3)}$$

[See Eq. (46) for the case with the Minkowski vacuum.] We find

$$\langle 1{\rm p}^{({\rm R})}|1{\rm p}^{({\rm R})}\rangle = \frac{1}{8\pi^2 a}\int_0^\infty d\omega \int d^2\mathbf{k}_\perp |\mathcal{A}^{(\omega,\mathbf{k}_\perp)}_{\rm em}|^2. \quad \text{(B4)}$$

The right Rindler particle number operator is

$$\hat{N}^{({\rm R})} = \int_0^\infty d\omega \int d^2\mathbf{k}_\perp \hat{b}^{\dagger}_{({\rm R}:\omega,\mathbf{k}_\perp)}\hat{b}_{({\rm R}:\omega,\mathbf{k}_\perp)}, \quad \text{(B5)}$$

and similarly for the left Rindler particle number $\hat{N}^{({\rm L})}$. Then, we have $\langle 1{\rm p}^{({\rm R})}|\hat{N}^{(R)}|1{\rm p}^{({\rm R})}\rangle = \langle 1{\rm p}^{({\rm R})}|1{\rm p}^{({\rm R})}\rangle$ and $\langle 1{\rm p}^{({\rm R})}|\hat{N}^{(L)}|1{\rm p}^{({\rm R})}\rangle = 0$. Then, by Eq. (B1), we find

$$\langle f|\hat{N}^{({\rm R}))}|f\rangle = \langle 1{\rm p}^{({\rm R})}|1{\rm p}^{({\rm R})}\rangle, \quad \text{(B6)}$$

$$\langle f|\hat{N}^{({\rm L}))}|f\rangle = 0. \quad \text{(B7)}$$

Thus, the particle-emission probability is equal to the expected Rindler particle number in the final state rather than the expected Minkowski particle number.

Now let us find the increase in the expected Minkowski particle number in the interaction picture and verify that it agrees with the Heisenberg picture, which gives the classical result as seen in Appendix A. The Fulling vacuum is not the Minkowski vacuum, and hence, the expected Minkowski particle number is nonzero. (In fact it is infinite.) First from Eqs. (33) and (34) we find that the Fulling vacuum $|0_{\rm F}\rangle$ has the following expectation values for the Unruh modes:

$$\langle 0_{\rm F}|b^{\dagger}_{(\omega,\mathbf{k}_\perp)}b_{(\omega',\mathbf{k}'_\perp)}|0_{\rm F}\rangle = \frac{8\pi^2 a}{e^{2\pi|\omega|/a}-1}\delta(\omega-\omega')\delta^{(2)}(\mathbf{k}_\perp - \mathbf{k}'_\perp). \quad \text{(B8)}$$

Notice the similarity of this equation with Eq. (37). We also find, using Eqs. (33), (34), and (35),

$$\begin{aligned}\langle 1{\rm p}^{({\rm R})}|b^{\dagger}_{(|\omega|,\mathbf{k}_\perp)}b_{(|\omega'|,\mathbf{k}'_\perp)}|1{\rm p}^{({\rm R})}\rangle &= \frac{\langle 1{\rm p}^{({\rm R})}|b^{\dagger}_{({\rm R}:|\omega|,\mathbf{k}_\perp)}b_{({\rm R}:|\omega'|,\mathbf{k}'_\perp)}|1{\rm p}^{({\rm R})}\rangle}{\sqrt{(1-e^{-2\pi|\omega|/a})(1-e^{-2\pi|\omega'|/a})}}\\ &\quad + \frac{8\pi^2 a\delta(|\omega|-|\omega'|)\delta^{(2)}(\mathbf{k}_\perp - \mathbf{k}'_\perp)}{\sqrt{(e^{2\pi|\omega|/a}-1)(e^{2\pi|\omega'|/a}-1)}}\langle 1{\rm p}^{({\rm R})}|1{\rm p}^{({\rm R})}\rangle, \end{aligned} \quad \text{(B9)}$$

$$\langle 1\mathrm{p}^{(\mathrm{R})}|b^{\dagger}_{(-|\omega|,\mathbf{k}_{\perp})}b_{(-|\omega'|,\mathbf{k}'_{\perp})}|1\mathrm{p}^{(\mathrm{R})}\rangle = \frac{\langle 1\mathrm{p}^{(\mathrm{R})}|b^{\dagger}_{(\mathrm{R}:|\omega'|,\mathbf{k}'_{\perp})}b_{(\mathrm{R}:|\omega|,\mathbf{k}_{\perp})}|1\mathrm{p}^{(\mathrm{R})}\rangle}{\sqrt{(e^{2\pi|\omega|/a}-1)(e^{2\pi|\omega'|/a}-1)}} + \frac{8\pi^2 a\delta(|\omega|-|\omega'|)\delta^{(2)}(\mathbf{k}_{\perp}-\mathbf{k}'_{\perp})}{\sqrt{(e^{2\pi|\omega|/a}-1)(e^{2\pi|\omega'|/a}-1)}}\langle 1\mathrm{p}^{(\mathrm{R})}|1\mathrm{p}^{(\mathrm{R})}\rangle. \tag{B10}$$

Using the expression (B2) for the state $|1\mathrm{p}^{(\mathrm{R})}\rangle$ and the commutation relations (35), we obtain

$$\langle 1\mathrm{p}^{(\mathrm{R})}|b^{\dagger}_{(\mathrm{R}:|\omega|,\mathbf{k}_{\perp})}b_{(\mathrm{R}:|\omega'|,\mathbf{k}'_{\perp})}|1\mathrm{p}^{(\mathrm{R})}\rangle = \overline{\mathcal{A}^{(|\omega|,\mathbf{k}_{\perp})}_{\mathrm{em}}}\mathcal{A}^{(|\omega'|,\mathbf{k}'_{\perp})}_{\mathrm{em}}. \tag{B11}$$

Substituting this equation into Eqs. (B9) and (B10) and using the expression (B1), we find to lowest nontrivial order,

$$\langle f|b^{\dagger}_{(|\omega|,\mathbf{k}_{\perp})}b_{(|\omega'|,\mathbf{k}'_{\perp})}|f\rangle - \langle 0_{\mathrm{F}}|b^{\dagger}_{(|\omega|,\mathbf{k}_{\perp})}b_{(|\omega'|,\mathbf{k}'_{\perp})}|0_{\mathrm{F}}\rangle = \frac{\overline{\mathcal{A}^{(|\omega|,\mathbf{k}_{\perp})}_{\mathrm{em}}}\mathcal{A}^{(|\omega'|,\mathbf{k}'_{\perp})}_{\mathrm{em}}}{\sqrt{(1-e^{-2\pi|\omega|/a})(1-e^{-2\pi|\omega'|/a})}}, \tag{B12}$$

$$\langle f|b^{\dagger}_{(-|\omega|,\mathbf{k}_{\perp})}b_{(-|\omega'|,\mathbf{k}'_{\perp})}|f\rangle - \langle 0_{\mathrm{F}}|b^{\dagger}_{(-|\omega|,\mathbf{k}_{\perp})}b_{(-|\omega'|,\mathbf{k}'_{\perp})}|0_{\mathrm{F}}\rangle = \frac{\mathcal{A}^{(|\omega|,\mathbf{k}_{\perp})}_{\mathrm{em}}\overline{\mathcal{A}^{(|\omega'|,\mathbf{k}'_{\perp})}_{\mathrm{em}}}}{\sqrt{(e^{2\pi|\omega|/a}-1)(e^{2\pi|\omega'|/a}-1)}}. \tag{B13}$$

Then, from the second expression in Eq. (49) for the Minkowski particle number operator, $\hat{N}$, we find, using $|\mathcal{A}^{(|\omega|,\mathbf{k}_{\perp})}_{\mathrm{em}}| = |\mathcal{A}^{(|\omega|,\mathbf{k}_{\perp})}_{\mathrm{abs}}|$,

$$\langle f|\hat{N}|f\rangle - \langle 0_{\mathrm{F}}|\hat{N}|0_{\mathrm{F}}\rangle = P^{(\mathrm{R})}_{\mathrm{int}}, \tag{B14}$$

where $P^{(\mathrm{R})}_{\mathrm{int}}$ is given by Eq. (53), which was shown to be equal to $P^{(\mathrm{M})}_{\mathrm{em}}$, which in turn is equal to the classical result [see Eq. (48)]. Thus, inertial observers in the Fulling vacuum will witness usual Larmor radiation being emitted from classical sources over the nontrivial particle content they experience. The corresponding Minkowski particles, as described by inertial observers, should be associated with the emission of Rindler particles, as described by uniformly accelerated observers at 0 K.

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