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A non-local p-Kirchhoff critical problem without the Ambrosetti-Rabinowitz condition

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ABSTRACT

In this paper, we study the existence of solutions for a non-local quasi-linear partial differential equation with a critical power in the sense of the Sobolev exponent on the right-hand side plus a subcritical perturbation. After analyzing the levels where we can recover the Cerami condition, we establish some existence results without requiring the Ambrosetti-Rabinowitz condition on the subcritical term. This paper investigates the case where the dimension N of the Euclidean space satisfies $ps < N < 2ps$, and it can be seen as a continuation of some of the first author's earlier works, in which the case $N > 2ps$ was analyzed.

1. Introduction

In this paper we focus on studying the problem

$$\begin{cases} \left(a + b \int\limits_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right)(-\Delta_p)^s u = |u|^{p_s^*-2}u + \lambda g(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{$P_{a,b}^{\lambda}$}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $\mathcal{Q} = \mathbb{R}^{2N} \setminus \mathcal{O}$ and $\mathcal{O} = \Omega^c \times \Omega^c$, a and b are non-negative real numbers such that $a + b > 0$, $s \in (0,1)$, $1 < p < 2$ or $p > 2$, and $ps < N < 2ps$ and p_s^* denotes the critical exponent for the Sobolev embedding of $W^{s,p}(\mathbb{R}^N)$ into Lebesgue spaces. The p-fractional Laplacian, up to a normalization constant, can be defined as

$$(-\Delta_p)^s u(x) = 2 \lim_{\epsilon \to 0^+} \int\limits_{\mathbb{R}^N \setminus B_\epsilon(0)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\,dy.$$

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Problem ($P^{\lambda}_{a,b}$) is a higher dimensional non-local version of the classical Kirchhoff equation

$$\rho h\,\partial^2_{tt}u-\left(p_0+\frac{\mathcal{E}h}{2L}\int_0^L|\partial_x u|^2\,dx\right)\partial^2_{xx}u+\delta\,\partial_t u+g(x,u)=0 \tag{1.1}$$

for $t\geq 0$ and $0<x<L$, where $u=u(t,x)$ is the lateral displacement at time t and at position x, $\mathcal{E}$ is the Young modulus, ρ is the mass density, h is the cross section area, L the length of the string, p_0 is the initial stress tension, δ the resistance modulus and g is an external force. As highlighted by Murthy in [29], in [17], Kirchhoff's original idea was to generalize the well-known d'Alembert equation of a vibrating string by incorporating also the lateral displacement.

Afterwards, the equation introduced by Kirchhoff has found applications in different fields. In fact, as pointed out by Alves et al. in [1], solutions u of the Kirchhoff equation can also be used to model a process which depends on the average of itself such as the population density. Additionally, operators similar to the one present in (1.1) appear in phase transition phenomena, continuum mechanics, population dynamics, game theory, nonlinear optics, and minimal surfaces. The reader interested in the applications of the model can consult the survey [7], [3], [10], [11], [12], [21] and the references therein.

Starting from some ideas contained in [26], the purpose of this paper is to study the existence of solutions of ($P^{\lambda}_{a,b}$) when $\Omega\subset\mathbb{R}^N$ and $ps<N<2ps$. By using a variational approach, we will prove the existence of critical points, that correspond to weak solutions of ($P^{\lambda}_{a,b}$), of the functional

$$\mathcal{I}_\lambda(u)=\frac{a}{p}\|u\|^p+\frac{b}{2p}\|u\|^{2p}-\frac{1}{p^*_s}\|u\|^{p^*_s}_{p^*_s}-\lambda\int_\Omega G(x,u)\,dx$$

where G is the primitive of g (we refer to the next section for the precise definition of the functional framework where we will work). The geometry of the functional $\mathcal{I}_\lambda$ depends on the interplay between the parameters N, p, and s. Indeed, when $N>2ps$, it is possible to prove that the functional is coercive, and the natural approach is to look for non-trivial minimizers of the functional. The author of this paper, in collaboration with Fiscella, Molica Bisci, and Secchi in [5] and [4], by adapting some ideas developed in [14], proved that there is a non-trivial minimum for sufficiently large λ. Since this minimum is attained at a negative level, they are also able to exhibit that the functional possesses a mountain pass geometry, and they prove the existence of a second solution. On the other hand, when $ps<N<2ps$, the functional $\mathcal{I}_\lambda$ is unbounded from below, so the standard approach is to look for solutions of mountain pass type. The case $N=2ps$ is somehow special and must be treated separately since the highest power of the norm in the functional corresponding to the operator has the same order as the critical exponent.

The interest in generalizing these kinds of problems to the quasi-linear fractional case extends beyond purely mathematical purposes. For instance, Fiscella and Valdinoci in [15] constructed a model for a vibrating string in where the tension of the string is related to a non-local measurement of its displacement from the rest position. In addition to that, the study of non-local quasi-linear problems has considerably increased in recent years. Pucci et al. in [28] obtained a multiplicity result for the Kirchhoff-Schrödinger equation in $\mathbb{R}^N$, adding a potential into the Kirchhoff operator. Xiang et al. in [30] proved the existence of a nontrivial weak solution to a problem driven by a non-local operator represented by a singular integral with a generic kernel. Furthermore, Xiang et al. in [31] proved the existence of a nontrivial solution for a problem involving the fractional p-Laplace operator and a critical exponent. We would also like to mention [22], where the authors established the existence of a sequence of nontrivial solutions using the symmetric mountain pass theorem under the assumption that the nonlinear term f satisfies a superlinear growth condition.

The main mathematical difficulty in studying problem ($P^{\lambda}_{a,b}$) is the presence of a critical term in the sense of the Sobolev exponent. Indeed, the Sobolev space where it is natural to look for solutions is not compactly embedded in the Lebesgue space $L^{p_s^*}(\Omega)$. As a result of this lack of compactness, standard variational techniques cannot be applied, and the Palais-Smale condition generally fails.

The study of problems with critical exponents began with the seminal paper [9], where the author established the validity of the Palais-Smale condition under a certain threshold. They showed that the mountain pass critical value of the problem belongs to the interval where the Palais-Smale condition holds. Following a similar approach, Naimen analyzed the levels where the Palais-Smale condition is verified for a Kirchhoff-type equation with $s = 1$ and $p = 2$ in [26], proving the existence of a positive mountain pass solution. In that paper, the author assumed the so-called Ambrosetti-Rabinowitz condition on G, i.e., that there exists a $\vartheta \in (2, 6)$ such that $g(x,t)t - \vartheta G(x,t) \geq 0$ for all $x \in \Omega$ and $u \geq 0$. However, Naimen had to deal with the case $\vartheta \in (2, 4]$ using a truncated functional and relied on a less favorable Palais-Smale threshold.

In this paper, we extend Naimen's result to the case of a fractional quasi-linear operator and also address the $2p$-superlinear case, i.e., when $\vartheta = 2p$, without using a truncation argument. To achieve this, we replace the classical Ambrosetti-Rabinowitz condition with a variant and use a version of the mountain pass theorem with the Cerami condition. We refer to [18] for a survey of possible generalizations of the AR-condition. In order to study at which level the Cerami condition is valid, we will invoke the concentration-compactness principle developed by Lions in [19] and [20] and generalized to the p-fractional case by Mosconi and Squassina in [23]. Compared with Naimen's result, analyzing the validity of the Cerami (or equivalently the Palais-Smale) condition for problem $(P_{a,b}^{\lambda})$ is more challenging, as it is not possible to explicitly characterize the value under which the Cerami condition holds. Furthermore, the non-local nature of the operator introduces additional difficulties in relating the value of the candidate's critical level with the Cerami threshold. A similar result to the one proved here is present in [15], but we do not use truncation arguments to study the $2p$-superlinear case. Additionally, in the $2p$-sublinear case, the values of λ for which we have solutions do not depend on the measure of the domain Ω. We would like to emphasize that, to the best of our knowledge, the results we are going to prove are new even for the local quasi-linear case when $s = 1$ and $p \neq 2$.

To see a complete summary of the notation used we refer the reader to the next section. Our paper is structured as follows. Section 2 is devoted to introducing the notation, the functional framework we will use, and some preliminary lemmas. In Section 3 we will prove Theorem 1.2, while in Section 4 we give a proof of Theorem 1.3. To conclude the section we collect the hypothesis made on the nonlinearity, and we state the main theorems.

(H_1) $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $g(x, 0) = 0$ for almost every $x \in \Omega$, $g(x,t) \geq 0$ for $t \geq 0$ and $g(x,t) \leq 0$ for $t \leq 0$ for all $x \in \Omega$;

(H_2) $\lim_{t\to 0} g(x,t)/|t|^{p-1} = 0$ and $\lim_{|t|\to+\infty} g(x,t)/|t|^{p_s^*-1} = 0$ uniformly with respect to $x \in \Omega$;

(H_3) there exist $\theta \in [2p, p_s^*)$ such that

$$\xi(x,t_1) \leq \xi(x,t_2) \text{ for a.e } x \in \Omega \text{ and all } 0 \leq t_1 \leq t_2 \text{ or } t_2 \leq t_1 \leq 0$$

where $\xi(x,t) = g(x,t)t - \theta G(x,t)$;

(H_4) there is a nonempty open set $\omega \in \Omega$ and an interval $I \subset (0, +\infty)$ such that $g(x,t) > 0$ if $(x,t) \in \omega \times I$.

(H_5) there exist $\theta \in (p, 2p)$ such that

$$\xi(x,t_1) \leq \xi(x,t_2) \text{ for a.e } x \in \Omega \text{ and all } 0 \leq t_1 \leq t_2 \text{ or } t_2 \leq t_1 \leq 0,$$

where $\xi(x,t) = g(x,t)t - \theta G(x,t)$;

Remark 1.1. Fix $\vartheta \in (p, p_s^*)$ and consider the nonlinearities

$$g_1(x,t) = |t|^{\vartheta} t \qquad g_2(x,t) = |t|^{\vartheta} t \log(1+|t|).$$

Observe that if $\vartheta \in [2p, p_s^*)$ then g_1 and g_2 fulfill hypothesis $(H_1) - (H_4)$, whereas if $\vartheta \in (p, 2p)$ the g_1 satisfies $(H_1) - (H_2)$ and $(H_4) - (H_5)$. In particular, notice that in the first case g_2 fails to satisfy an Ambrosetti-Rabinowitz type condition introduced in [2] when $\vartheta = 2p$, that would require the existence of $q \in (2p, p_s^*)$ such that

$$g(x,t)t - qG(x,t) \geq 0$$

for all $x \in \Omega$ and $t \in \mathbb{R}$.

We end the section with the statement of the two main theorems.

Theorem 1.2. *Let $a, b \geq 0$ and $a + b > 0$. If Hypothesis $(H_1) - (H_4)$ hold, then there exists a $\tilde{\lambda}_1 > 0$ such that problem $(P_{a,b}^{\lambda})$ admits a non-trivial solution for all $\lambda \geq \tilde{\lambda}_1 > 0$.*

Theorem 1.3. *Let $a, b > 0$. If Hypothesis $(H_1) - (H_2)$ and $(H_4) - (H_5)$ hold, then there exists a $\tilde{\lambda}_2 > 0$ such that problem $(P_{a,b}^{\lambda})$ admits a non-trivial solution for all $\lambda \geq \tilde{\lambda}_2 > 0$.*

Remark 1.4. We point out that $\tilde{\lambda}_2 \geq \tilde{\lambda}_1 > 0$. In fact, these parameters only depend on Hypothesis (H_1), (H_2) and (H_4), so the candidate critical value c_λ will enter in the Cerami regime for a smaller value of λ due to larger range where the Cerami condition holds. See also Remark 4.5

2. Abstract framework and preliminary results

This section is devoted to fixing the notation we will use throughout the paper and to introducing the functional setting in which it will work. We will denote by $\|\cdot\|_p$ the standard norm of the Lebesgue space $L^p(\mathbb{R}^N)$, and we define the fractional Sobolev space

$$W^{s,p}(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N) \ : \ \int\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dx\, dy < \infty \right\},$$

endowed with the norm $\|u\|_{W^{s,p}(\mathbb{R}^N)} = \|u\|_p + \|u\|$, where

$$\|u\|^p = \int\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dx\, dy.$$

Since we aim to study the existence of solutions to $(P_{a,b}^{\lambda})$ with some boundary conditions, we will not work directly in $W^{s,p}(\mathbb{R}^N)$. Instead, we will consider functions that are zero outside of Ω. More precisely, we will work in the space

$$X_0^{s,p}(\Omega) := \left\{ u \in W^{s,p}(\mathbb{R}^N) : \ u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}.$$

Remark 2.1. The norm $\|\cdot\|$ is equivalent to $\|\cdot\|_{W^{s,p}(\mathbb{R}^N)}$ in $X_0^{s,p}(\Omega)$.

We consider the potential operator A_p associated to the functional $u \mapsto \|u\|^p/p$ on $X_0^{s,p}(\Omega)$, i.e. the operator $A_p \colon X_0^{s,p}(\Omega) \to (X_0^{s,p}(\Omega))^*$ such that

$$\langle A_p(u), v\rangle = \int\limits_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}\,(u(x)-u(y))\,(v(x)-v(y))}{|x-y|^{N+sp}} dx\,dy$$

for every u, $v \in X_0^{s,p}(\Omega)$. It is trivial to see,

$$\langle A_p(u), u\rangle = \|u\|^p, \quad |\langle A_p(u), v\rangle| \le \|u\|^{p-1}\,\|v\|\,.$$

The following lemma provides a sufficient condition under which weakly convergent sequences also converge strongly in $X_0^{s,p}(\Omega)$.

Lemma 2.2. *If a sequence $(u_n)_n$ converges weakly to u in $X_0^{s,p}(\Omega)$ and*

$$\langle A_p(u_n), u_n - u\rangle \to 0,$$

then $\|u_n - u\| \to 0$.

Proof. See [27, Proposition 1.3] for a proof. □

The following Theorem is a classical result on fractional Sobolev spaces. For a proof, we refer to [13].

Theorem 2.3. *Let $s \in (0,1)$ and $p \in [1,\infty,)$ be such that $N > ps$. Let $q \in [1, p_s^*)$, $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain and $\mathcal{F}$ a bounded subset of $L^p(\Omega)$ such that*

$$\sup_{u\in\mathcal{F}} \int\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy < \infty.$$

Then $\mathcal{F}$ is relatively compact in $L^q(\Omega)$.

We also recall that the best Sobolev constant in the Sobolev inequality can be defined as

$$S_{s,p} := \inf_{u\in X_0^{s,p}(\Omega)\setminus\{0\}} \frac{\|u\|^p}{\|u\|_{p_s^*}^p} \tag{2.1}$$

For the reader's convenience, we conclude this section by recalling the definition of the Cerami condition at level c (abbreviated as $(C)_c$ in the sequel). Additionally, we present a variant of the classic mountain pass theorem, where the classical Palais-Smale condition is replaced by the Cerami condition.

Definition 2.4. Let X be a Banach space, and let X^* be its topological dual. Let $\varphi \in C^1(X)$; we say that a sequence $(u_n)_n \subset X$ is a $(C)_c$ sequence for φ if $\varphi(u_n) \to c$ and if $(1+\|u_n\|)\|\varphi'(u_n)\| \to 0$ in X^* as $n \to \infty$. We also say that φ satisfies the $(C)_c$ condition if every $(C)_c$ sequence for φ admits a strongly convergent subsequence.

Before presenting the Mountain Pass Theorem, we state the following Proposition, which will be useful in analyzing the validity of the $(C)_c$ condition.

Proposition 2.5. *Let $(u_n)_n \subset X_0^{s,p}(\Omega)$ a bounded sequence. Suppose that $\vartheta \in C^\infty(\mathbb{R}^N)$ is such that $0 \le \vartheta \le 1$, $\vartheta = 1$ in $B(0,1)$ and $\vartheta = 0$ in $\mathbb{R}^N \setminus B(0,2)$. For $q \in \mathbb{R}^N$, let $\vartheta_\varepsilon(x) = \vartheta\left(\frac{x-q}{\varepsilon}\right)$. Then*

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^{2N}} |u_n(y)|^p \frac{|\vartheta_\varepsilon(x)-\vartheta_\varepsilon(y)|^p}{|x-y|^{N+ps}}\,dx\,dy = 0.$$

Proof. The verification of the limit is similar to [15, Theorem 2]. We omit the details. □

We conclude the section by stating a variant of the classical Mountain Pass theorem, where the classic Palais-Smale condition is replaced by the Cerami condition. This will be the main tool that will allow us to show the existence of solutions for problem ($P_{a,b}^\lambda$).

Theorem 2.6. *If X is a Banach space, $\varphi \in C^1(X)$ satisfies the C-condition, $u_0, u_1 \in X$ satisfy*

$$\max\{\varphi(u_0), \varphi(u_1)\} \le \inf\{\varphi(u) : \|u - u_0\| = \rho\} = \eta_\rho, \quad \|u_1 - u_0\| > \rho > 0,$$

set

$$\Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}$$

and

$$c := \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} \varphi(\gamma(t)),$$

then $c \ge \eta_\rho$ and c is a critical value for φ.

A proof of this Theorem can be found in [24, Theorem 5.40].

3. Existence of solutions with Hypothesis (H_3)

In this section, we present the proof of Theorem 1.2. The proof of the Theorem relies on a variational approach, so we introduce the functional

$$\mathcal{I}_\lambda(u) = \frac{a}{p}\|u\|^p + \frac{b}{2p}\|u\|^{2p} - \frac{1}{p_s^*}\|u\|_{p_s^*}^{p_s^*} - \lambda\int_\Omega G(x,u)\,dx$$

whose critical points correspond to weak solutions of ($P_{a,b}^\lambda$). We would like to emphasize that under the conditions $(H_1) - (H_2)$ the functional $\mathcal{I}_\lambda$ is well defined and continuously Fréchet differentiable on the space $X_0^{s,p}(\Omega)$. The main tool we will exploit to produce the solution is Theorem 2.6. Thus, we begin by showing that the functional $\mathcal{I}_\lambda$ possesses a mountain pass geometry.

Lemma 3.1. *If g satisfies Hypothesis $(H_1) - (H_2)$, then there exist $R > 0$ such that $\mathcal{I}_\lambda(u) > 0$ if $\|u\| = R$ and $w \in X_0^{s,p}(\Omega)$, with $\|w\| > R$, such that $\mathcal{I}_\lambda(w) \le 0$.*

Proof. We start noticing that hypothesis (H_1) and (H_2) imply that given $\varepsilon > 0$ we can find $C_\varepsilon >$ such that

$$G(x,t) \le \varepsilon|t|^p + C_\varepsilon|t|^{p_s^*}.$$

Using this and recalling that $X_0^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ continuously for $q \in [p, p_s^*]$, we obtain

$$\mathcal{I}_\lambda(u) \geq \left(\frac{a}{p} - \varepsilon\right) \|u\|^p + \frac{b}{2p}\|u\|^{2p} - \tilde{C}_\varepsilon \|u\|^{p_s^*}$$

for some $\tilde{C}_\varepsilon > 0$. From this, it is clear that the first part of the Lemma is obtained by selecting ε and R small enough, recalling that $a+b>0$ and that $p_s^* > 2p$. In order to prove the second part of the statement, take $v \in X_0^{s,p}(\Omega)$ and observe that $\mathcal{I}_\lambda(\zeta v) \to -\infty$ as $\zeta \to \infty$, so it suffices to take $w = \zeta v$ with ζ large enough. □

Now, we begin the analysis of Cerami sequences. The first step is to prove that such sequences are bounded in $X_0^{s,p}(\Omega)$, allowing us to extract a weakly convergent subsequence.

Proposition 3.2. *Assume g satisfies (H_1)-(H_3), then every $(C)_c$ sequence for the functional $\mathcal{I}_\lambda$ is bounded.*

Proof. Let $(u_n)_n \subset X_0^{s,p}(\Omega)$ be a $(C)_c$ sequence for $\mathcal{I}_\lambda$, i.e. a sequence such that

$$\mathcal{I}_\lambda(u_n) \to c \quad \text{for some } c \in \mathbb{R} \tag{3.1}$$

and

$$(1 + \|u_n\|)\mathcal{I}'_\lambda(u_n) \to 0 \quad \text{in} \quad (X_0^{s,p}(\Omega))^* \tag{3.2}$$

as $n \to \infty$. By contradiction, we suppose that $\|u_n\| \to \infty$ as $n \to \infty$, and we define a new sequence $y_n := u_n/\|u_n\|$. Obviously, this new sequence y_n is bounded in $X_0^{s,p}(\Omega)$, so it is not restrictive to assume

$$\begin{cases} y_n \rightharpoonup y & \text{in } X_0^{s,p}(\Omega) \\ y_n \to y & \text{in } L^q(\Omega) \text{ for all } q \in [1, p_s^*) \\ y_n \to y & \text{a.e in } \mathbb{R}^N \end{cases}$$

since the embedding $X_0^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for every $q \in [1, p_s^*)$. At this point, we divide our analysis separating the case $y = 0$ and $y \neq 0$. Let us first see what happens in the case $y \neq 0$. Set $\Omega_0 := \{x \in \Omega \,|\, y(x) = 0\}$, and observe that clearly

$$|y_n(x)| \to +\infty \quad \text{as } n \to \infty \text{ in } \Omega_0^c.$$

Thus, Fatou's Lemma implies that

$$\begin{aligned} \liminf_{n\to\infty} \frac{1}{\|u_n\|^{2p}} &\left(\frac{1}{p_s^*} \int_\Omega |u_n|^{p_s^*}\, dx + \lambda \int_\Omega G(x, u_n)\, dx \right) \\ &\geq \int_{\Omega_0^c} |y|^{2p} \liminf_{n\to\infty} |u_n|^{p_s^*-2p}\, dx = +\infty, \end{aligned} \tag{3.3}$$

where the last equality arises from $p_s^* - 2p > 0$ since $ps < N < 2ps$. On the other hand,

$$\begin{aligned} \frac{1}{\|u_n\|^{2p}} &\left(\frac{1}{p_s^*} \int_\Omega |u_n|^{p_s^*}\, dx + \lambda \int_\Omega G(x, u_n)\, dx \right) \\ &= \frac{a}{p} \frac{1}{\|u_n\|^p} + \frac{b}{2p} - \frac{1}{\|u_n\|^{2p}} \mathcal{I}_\lambda(u_n) \to \frac{b}{2p} \end{aligned}$$

using (3.1) as $n \to \infty$. This clearly contradicts (3.3). Now, we draw our attention to the case $y = 0$. Take $\eta > 0$, which will be determined later, set a new sequence of functions $v_n := \eta y_n$, and observe that

$$0 < \frac{\eta}{\|u_n\|} \leq 1 \tag{3.4}$$

for sufficiently large values of n. It is worth noting that, up to a subsequence, $\|y_n\|_{p_s^*}^{p_s^*} \to L \geq 0$ (notice that $L = +\infty$ is not allowed because of (2.1)). We proceed by dividing the proof into two further sub-cases based on whether $L = 0$ or $L > 0$. If $L > 0$, let ζ_n be chosen such that

$$\mathcal{I}_\lambda(\zeta_n u_n) = \min_{0 \leq \zeta \leq 1} \mathcal{I}_\lambda(\zeta u_n).$$

First of all, we emphasize that

$$\int_\Omega G(x, v_n)\, dx \to 0 \tag{3.5}$$

as $n \to \infty$, exploiting Hypothesis (H_1)-(H_2), the Lebesgue Dominated convergence Theorem, and the fact that $v_n \to 0$ in $L^q(\Omega)$ for all $q \in [1, p_s^*)$. Secondly, we see that

$$\mathcal{I}_\lambda(\zeta_n u_n) \leq \mathcal{I}_\lambda(v_n) = \frac{a}{p}\eta^p + \frac{b}{2p}\eta^{2p} - \frac{\eta^{p*_s}}{p_s^*}\|y_n\|_{p^*s}^{p_s^*} - \lambda \int_\Omega G(x, v_n)\, dx.$$

Letting $n \to \infty$, using (3.5), we get

$$\lim_{n\to\infty} \mathcal{I}_\lambda(\zeta_n u_n) \leq \frac{a}{p}\eta^p + \frac{b}{2p}\eta^{2p} - \frac{\eta^{p*_s}}{p_s^*}L.$$

From this, we infer

$$\lim_{n\to\infty} \mathcal{I}_\lambda(\zeta_n u_n) = -\infty \tag{3.6}$$

as $p_s^* - > p$ and we can select η as large as we desire. Now, we point out that (3.6) eventually implies $\zeta_n \in (0, 1)$, since $\mathcal{I}_\lambda(0) = 0$ and $\mathcal{I}_\lambda(u_n) \to c$. So, it is possible to compute the derivative with respect to ζ to obtain

$$\left.\frac{d}{d\zeta}\right|_{\zeta=\xi_n} \mathcal{I}_\lambda(\zeta u_n) = 0,$$

which implies

$$\int_\Omega g(z, \zeta_n u_n)\zeta_n u_n\, dx = a\|\zeta_n u_n\|^p + b\|\zeta_n u_n\|^{2p} - \|\zeta_n u_n\|_{p_s^*}^{p_s^*}. \tag{3.7}$$

Applying (3.7) in the expression below, and using the fact that $\xi(x, \zeta_n u_n) \geq \xi(x, 0) = 0$ a.e. in Ω by (H_3), we have

$$\begin{aligned}\theta\mathcal{I}_\lambda(\zeta_n u_n) &= a\left(\frac{\theta}{p} - 1\right)\|\zeta_n u_n\|^p + b\left(\frac{\theta}{2p} - 1\right)\|\zeta_n u_n\|^{2p} \\ &\quad + \left(1 - \frac{\theta}{p_s^*}\right)\|\zeta_n u_n\|_{p_s^*}^{p_s^*} + \int_\Omega \xi(x, \zeta_n u_n)\, dx \geq 0.\end{aligned}$$

Choosing initially $\eta > 0$ large enough such that $\mathcal{I}_\lambda(\zeta_n u_n) < 0$ leads to a contradiction.

In the case in which $L = 0$, we select ζ_n such that

$$\mathcal{I}_\lambda(\zeta_n u_n) = \max_{0\le\zeta\le1} \mathcal{I}_\lambda(\zeta u_n).$$

We have that

$$\mathcal{I}_\lambda(\zeta_n u_n) \ge \mathcal{I}_\lambda(v_n) = \frac{a}{p}\eta^p + \frac{b}{2p}\eta^{2p} - \frac{\eta^{p^*s}}{p^*_s}\|y_n\|^{p^*_s}_{p^*s} - \lambda \int_\Omega G(x, v_n)\,dx.$$

Hence, noting that (3.5) still holds, we obtain

$$\lim_{n\to\infty} \mathcal{I}_\lambda(\zeta_n u_n) \ge \frac{a}{p}\eta^p + \frac{b}{2p}\eta^{2p},$$

from which we deduce

$$\lim_{n\to\infty} \mathcal{I}_\lambda(\zeta_n u_n) = +\infty \tag{3.8}$$

since η is arbitrary. Now, similarly to the previous case, we eventually have $\xi_n \in (0, 1)$, so

$$\left.\frac{d}{d\zeta}\right|_{\zeta=\xi_n} \mathcal{I}_\lambda(\zeta u_n) = 0$$

and we get again (3.7). Finally, exploiting (3.1), (3.2), (3.7) together with hypothesis (H_3), we obtain

$$\begin{aligned}\theta\mathcal{I}_\lambda(\zeta_n u_n) &= a\left(\frac{\theta}{p} - 1\right)\|\zeta_n u_n\|^p + b\left(\frac{\theta}{2p} - 1\right)\|\zeta_n u_n\|^{2p} \\ &\quad + \left(1 - \frac{\theta}{p^*_s}\right)\|\zeta_n u_n\|^{p^*_s}_{p^*_s} + \int_\Omega \xi(x, \zeta_n u_n)\,dx \\ &\le |\theta\mathcal{I}_\lambda(u_n) - \mathcal{I}'_\lambda(u_n)[u_n]| \le C\end{aligned}$$

for some $C > 0$. This last expression contradicts (3.8) and concludes the proof. □

Once we established the boundedness of $(C)_c$ sequences, we can draw our attention on trying to understand if there are some values $c \in \mathbb{R}$ such that the $(C)_c$ condition holds. Before doing that, we need a preliminary lemma.

Lemma 3.3. *Let $a, b \ge 0$ such that $a + b > 0$ and set*

$$f(t) := a + bt - \frac{t^{\frac{p^*_s}{p}-1}}{S_{s,p}^{\frac{p^*_s}{p}}} \quad \text{and} \quad \tilde{f}(t) := aS_{s,p}t^{\frac{p}{p^*_s}} + bS^2_{s,p}t^{2\frac{p}{p^*_s}} - t.$$

The following facts hold:

1. *the equation $f(t) = 0$ has a unique positive solution that will be denoted by $K^{a,b}_{s,p}$;*
2. *the equation $\tilde{f}(t) = 0$ has a unique positive solution and it is $\left(\frac{K^{a,b}_{s,p}}{S_{s,p}}\right)^{\frac{p^*_s}{p}}$;*

3. the quantity

$$\frac{a}{p}K_{s,p}^{a,b}+\frac{b}{2p}\left(K_{s,p}^{a,b}\right)^2-\frac{S_{s,p}^{-\frac{p_s^*}{p}}}{p_s^*}\left(K_{s,p}^{a,b}\right)^{\frac{p_s^*}{p}}>0.$$

Proof. The existence of a unique positive solution of $f(t)=0$ and $\tilde f(t)=0$ is an easy exercise of calculus, taking into account that $\frac{p_s^*}{p}-1>1$ and $0<2\frac{p}{p_s^*}<1$ since $ps<N<2ps$. Also, notice that

$$\tilde f\left(\left(K_{s,p}^{a,b}S_{s,p}^{-1}\right)^{\frac{p_s^*}{p}}\right)=K_{s,p}^{a,b}f\left(K_{s,p}^{a,b}\right)=0.$$

Regarding the last point, we define the function

$$h(t):=\frac{a}{p}t^p+\frac{b}{2p}t^{2p}-\frac{S_{s,p}^{-\frac{p_s^*}{p}}}{p_s^*}t^{p_s^*}$$

whose derivative is

$$h'(t)=at^{p-1}+bt^{2p-1}-S_{s,p}^{-\frac{p_s^*}{p}}t^{p_s^*-1}.$$

It is easy to check that there is a unique maximum point $t_{\max}>0$ of h such that $h(t_{\max})>0$ and that

$$h'\left(\sqrt[p]{K_{s,p}^{a,b}}\right)=\left(K_{s,p}^{a,b}\right)^{\frac{p-1}{p}}f\left(K_{s,p}^{a,b}\right)=0. \tag{3.9}$$

Therefore, it must be

$$h\left(\sqrt[p]{K_{s,p}^{a,b}}\right)=\frac{a}{p}K_{s,p}^{a,b}+\frac{b}{2p}\left(K_{s,p}^{a,b}\right)^2-\frac{S_{s,p}^{-\frac{p_s^*}{p}}}{p_s^*}\left(K_{s,p}^{a,b}\right)^{\frac{p_s^*}{p}}>0. \quad \square$$

The next proposition allows to recover strong convergence in $L^{p_s^*}(\Omega)$ for a $(C)_c$ sequence for $\mathcal{I}_\lambda$ with some specific values of c.

Proposition 3.4. *Assume g satisfies Hypothesis (H_1)-(H_3) and that $(u_n)_n\subset X_0^{s,p}(\Omega)$ is a $(C)_c$ sequence for $\mathcal{I}_\lambda$ with*

$$c<\frac{a}{p}K_{s,p}^{a,b}+\frac{b}{2p}\left(K_{s,p}^{a,b}\right)^2-\frac{S_{s,p}^{-\frac{p_s^*}{p}}}{p_s^*}\left(K_{s,p}^{a,b}\right)^{\frac{p_s^*}{p}}$$

Then there is $u\in X_0^{s,p}(\Omega)$ such that $u_n\to u$ in $L^{p_s^*}(\Omega)$ up to a subsequence.

Proof. We begin by observing that the sequence u_n is bounded in $X_0^{s,p}(\Omega)$ as shown in Proposition 3.2. Thus, since the embedding $X_0^{s,p}(\Omega)\hookrightarrow L^q(\Omega)$ is compact for every $q\in[1,p_s^*)$ as established in [13], we have $u_n\to u$ in $L^q(\Omega)$ for all $q\in[1,p_s^*)$ and in particular $u_n(x)\to u(x)$ a.e. in $\mathbb{R}^N$ as $n\to\infty$.

We also notice, thanks to the Hölder inequality, that the sequence $(u_n)_n$ is bounded in the space of measures $\mathcal{M}(\Omega)$. So, according to [23, Theorem 2.5], there are two Borel regular measures μ and ν, and a set of indexes J at most countable corresponding to some points $(x_j)_{j\in J}\subset\Omega$ such that

$$|D^su_n|^p(x)\rightharpoonup^*\mu \quad \text{and} \quad |u_n|^{2_s^*}(x)\rightharpoonup^*\nu \quad \text{in}\,\mathcal{M}(\Omega)$$

where

$$\nu = |u|^{p_s^*} + \sum_{j \in J} \nu_j \delta_{x_j}$$

and

$$\mu \geq (-\Delta_p)^s u + \sum_{j \in J} \mu_j \delta_{x_j} \tag{3.10}$$

with

$$\nu_j = \nu\left(\{x_j\}\right) \quad \mu_j = \mu\left(\{x_j\}\right).$$

In one of the expressions above, we have used the notation

$$|D^s u|^p(x) := \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dy \quad \text{for a.e. } x \in \mathbb{R}^N.$$

In addition to that, we have the following inequality

$$\mu_j \geq S_{s,p} \nu_j^{\frac{p}{p_s^*}}. \tag{3.11}$$

At this point, we assert that the set J is empty. If this was false, then we would be able to find at least an index $j_0 \in J$ and a point x_{j_0} with $\nu_{j_0} \neq 0$ associated to it. Fix $\varepsilon > 0$, take a cut-off function such that

$$\begin{cases} 0 \leq \varphi_\varepsilon \leq 1 & \text{in } \Omega \\ \varphi_\varepsilon = 1 & \text{in } B(x_{j_0}, \varepsilon) \\ \varphi_\varepsilon = 0 & \text{in } \Omega \setminus B(x_{j_0}, 2\varepsilon). \end{cases}$$

And notice that also the sequence $(u_n \varphi_\varepsilon)_n$ is bounded in $X_0^{s,p}(\Omega)$. Hence,

$$\lim_{n\to\infty} \mathcal{I}_\lambda'(u_n)\left[u_n \varphi_\varepsilon\right] = 0.$$

Writing out the derivative in full, we get

$$\begin{aligned} o(1) &= \mathcal{I}_\lambda'(u_n)\left[u_n\varphi_\varepsilon\right] \\ &= (a + b\|u_n\|^p) \\ &\quad \times \int_{\mathcal{Q}} \frac{|u_n(x)-u_n(y)|^{p-2}\left(u_n(x)-u_n(y)\right)\left(u_n(x)\varphi_\varepsilon(x) - u_n(y)\varphi_\varepsilon(y)\right)}{|x-y|^{N+ps}}\, dx\, dy \\ &\quad - \int_\Omega |u_n|^{p_s^*} \varphi_\varepsilon\, dx - \lambda \int_\Omega g(x,u_n) u_n \varphi_\varepsilon\, dx \\ &= (a + b\|u_n\|^p) \\ &\quad \times \left[\int_{\mathcal{Q}} u_n(y) \frac{|u_n(x)-u_n(y)|^{p-2}\left(u_n(x)-u_n(y)\right)(\varphi_\varepsilon(x)-\varphi_\varepsilon(y))}{|x-y|^{N+ps}}\, dx\, dy\right. \end{aligned} \tag{3.12}$$

$$+\int_{\mathcal{Q}} \varphi_\varepsilon(x)\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Bigg] - \int_{\Omega} |u_n|^{p_s^*}\varphi_\varepsilon\,dx - \lambda\int_{\Omega} g(x,u_n)u_n\varphi_\varepsilon\,dx.$$

Now, the first term in the square brackets could be estimated with the help of the Hölder inequality obtaining

$$\begin{aligned}
&(a+b\|u_n\|^p)\\
&\times\int_{\mathcal{Q}} u_n(y)\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi_\varepsilon(x)-\varphi_\varepsilon(y))}{|x-y|^{N+ps}}\,dx\,dy\\
&\le C\left(\int_{\mathcal{Q}}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac{p-1}{p}}\left(\int_{\mathcal{Q}}|u_n(y)|^p\frac{|\varphi_\varepsilon(x)-\varphi_\varepsilon(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac{1}{p}}\\
&\le \tilde{C}\left(\int_{\mathcal{Q}}|u_n(y)|^p\frac{|\varphi_\varepsilon(x)-\varphi_\varepsilon(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac{1}{p}}
\end{aligned}$$

for some $C, \tilde{C} > 0$. In addition, Proposition 2.5 yields

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^{2N}}|u_n(y)|^p\frac{|\varphi_\varepsilon(x)-\varphi_\varepsilon(y)|^p}{|x-y|^{N+ps}}\,dx\,dy = 0,$$

hence

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{n\to\infty}(a+b\|u_n\|^p)\\
&\quad\times\int_{\mathcal{Q}} u_n(y)\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi_\varepsilon(x)-\varphi_\varepsilon(y))}{|x-y|^{N+ps}}\,dx\,dy = 0.
\end{aligned}\tag{3.13}$$

From the subcritical growth of g and the Lebesgue's dominated convergence Theorem, it follows

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\Omega} g(x,u_n)u_n\varphi_\varepsilon\,dx = 0.\tag{3.14}$$

Indeed, for any $\delta > 0$, it is standard to prove that Hypotheses (H_1)-(H_2) imply

$$g(x,t) \le \delta|t|^{p_s^*-1} + C_\delta|t|^{p-1},$$

and from this, it follows that

$$\begin{aligned}
&\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\Omega} g(x,u_n)u_n\varphi_\varepsilon\,dx\\
&\le \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\lim_{n\to\infty}\delta\int_{\Omega}|u_n|^{p_s^*}\varphi_\varepsilon\,dx + C_\delta\int_{\Omega}|u_n|^p\varphi_\varepsilon\,dx.\\
&\le \lim_{\delta\to 0}\lim_{\varepsilon\to 0}C\delta + C_\delta\int_{\Omega}|u|^p\varphi_\varepsilon\,dx = \lim_{\delta\to 0}C\delta = 0.
\end{aligned}$$

Equations (3.13) and (3.14) imply we can rewrite (3.12) as

$$o(1) = (a + b\|u_n\|^p) \int\limits_{\mathcal{Q}} \varphi_\varepsilon(x) \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}}\, dx\, dy - \int\limits_{\Omega} |u_n|^{p_s^*} \varphi_\varepsilon\, dx$$

for n large and ε small. Let us analyze what happens to the two remaining integrals separately. For the second one, an easy calculation and (3.11) show

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \int\limits_{\Omega} |u_n|^{p_s^*} \varphi_\varepsilon\, dx = \lim_{\varepsilon\to 0} \int\limits_{\Omega} |u|^{p_s^*} \varphi_\varepsilon\, dx + \nu_{j_0} = \nu_{j_0}. \tag{3.15}$$

Regarding the first one, because of (3.10), we have

$$\begin{aligned}
&\lim_{n\to\infty} (a + b\|u_n\|^p) \int\limits_{\mathcal{Q}} \varphi_\varepsilon(x) \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}}\, dx\, dy \\
&\quad \geq \lim_{n\to\infty} \left[a \int\limits_{\mathbb{R}^{2N} \setminus B(x_{j_0}, 2\varepsilon)^c \times \Omega^c} \varphi_\varepsilon(x) \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}}\, dx\, dy \right. \\
&\qquad \left. + b \left(\int\limits_{\mathcal{Q}} \varphi_\varepsilon(x) \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}}\, dx\, dy \right)^2 \right] \\
&\quad \geq a \int\limits_{\mathbb{R}^{2N} \setminus B(x_{j_0}, 2\varepsilon)^c \times \Omega^c} \varphi_\varepsilon(x) \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}}\, dx\, dy + a\mu_{j_0} \\
&\qquad + b \left(\int\limits_{\mathcal{Q}} \varphi_\varepsilon(x) \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}}\, dx\, dy \right)^2 + b\mu_{j_0}^2,
\end{aligned}$$

which implies

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} (a + b\|u_n\|^p) \int\limits_{\mathcal{Q}} \varphi_\varepsilon(x) \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}}\, dx\, dy \geq a\mu_{j_0} + b\mu_{j_0}^2. \tag{3.16}$$

Putting together (3.12), (3.15), and (3.16), we obtain

$$0 \geq a\mu_{j_0} + b\mu_{j_0}^2 - \nu_{j_0}. \tag{3.17}$$

Using (3.11) and dividing by μ_{j_0}, we get

$$a + b\mu_{j_0} - \frac{\mu_0^{\frac{p_s^*}{p} - 1}}{S_{s,p}^{\frac{p_s^*}{p}}} \leq 0,$$

which yields $\mu_0 \geq K_{s,p}^{a,b}$ by Lemma 3.3. On the other hand, exploiting (3.11), from (3.17) it also follows

$$aS_{s,p}\nu_0^{\frac{p}{p_s^*}} + bS_{s,p}^2 \nu_0^{2\frac{p}{p_s^*}} - \nu_0 \leq 0, \tag{3.18}$$

yielding $\nu_0 \geq \left(\frac{K_{s,p}^{a,b}}{S_{s,p}}\right)^{\frac{p_s^*}{p}}$ using again Lemma 3.3. Now, keeping in mind this and $\mu_0 \geq K_{s,p}^{a,b}$, recalling Hypothesis (H_3) and that $(u_n)_n$ is a $(C)_c$ sequence, we have

$$\begin{aligned}
c &= \lim_{n\to\infty} \mathcal{I}_\lambda(u_n) = \lim_{n\to\infty} \left(\mathcal{I}_\lambda(u_n) - \frac{1}{\theta}\mathcal{I}'_\lambda(u_n)\,[u_n]\right) \\
&= \lim_{n\to\infty} \Bigg[a\left(\frac{1}{p} - \frac{1}{\theta}\right)\|u_n\|^p + b\left(\frac{1}{2p} - \frac{1}{\theta}\right)\|u_n\|^{2p} + \left(\frac{1}{\theta} - \frac{1}{p_s^*}\right)\|u_n\|_{p_s^*}^{p_s^*} \\
&\quad + \frac{1}{\theta}\int_\Omega \xi(x, \zeta_n u_n)\,dx \Bigg] \\
&\geq a\left(\frac{1}{p} - \frac{1}{\theta}\right)\mu_0 + b\left(\frac{1}{2p} - \frac{1}{\theta}\right)\mu_0^2 + \left(\frac{1}{\theta} - \frac{1}{p_s^*}\right)\nu_0 \\
&\geq a\left(\frac{1}{p} - \frac{1}{\theta}\right)K_{s,p}^{a,b} + b\left(\frac{1}{2p} - \frac{1}{\theta}\right)\left(K_{s,p}^{a,b}\right)^2 + \left(\frac{1}{\theta} - \frac{1}{p_s^*}\right)\left(\frac{K_{s,p}^{a,b}}{S_{s,p}}\right)^{\frac{p}{p_s^*}} \qquad (3.19) \\
&= \frac{a}{p}K_{s,p}^{a,b} + \frac{b}{2p}\left(K_{s,p}^{a,b}\right)^2 - \frac{S_{s,p}^{-\frac{p_s^*}{p}}}{p_s^*}\left(K_{s,p}^{a,b}\right)^{\frac{p_s^*}{p}} - \frac{K_{s,p}^{a,b}}{\theta} f\left(K_{s,p}^{a,b}\right) \\
&= \frac{a}{p}K_{s,p}^{a,b} + \frac{b}{2p}\left(K_{s,p}^{a,b}\right)^2 - \frac{S_{s,p}^{-\frac{p_s^*}{p}}}{p_s^*}\left(K_{s,p}^{a,b}\right)^{\frac{p_s^*}{p}}
\end{aligned}$$

since $f\left(K_{s,p}^{a,b}\right) = 0$ reaching a contradiction. So, J must be empty, and this implies $\|u_n\|_{p_s^*} \to \|u\|_{p_s^*}$. In order to conclude, we invoke the classical Brezis-Lieb Lemma [8, Theorem 1], which states that

$$\|u_n - u\|_{p_s^*}^{p_s^*} = \|u_n\|_{p_s^*}^{p_s^*} - \|u\|_{p_s^*}^{p_s^*} + (1)$$

since u_n is bounded $L^{p_s^*}(\Omega)$ and $u_n(x) \to u(x)$ a.e. in Ω. From this, it is immediate to deduce $u_n \to u$ in $L^{p_s^*}(\Omega)$. $\square$

Remark 3.5. In the proof above, when we prove that the set J is empty, we have focused on a single point to simplify the argument. However, we would like to emphasize that even if the set J contains infinitely many indices, to reach a contradiction it suffices to localize the problem around a single point where the norm might concentrate, thus avoiding any convergence issues.

Proposition 3.6. *Assume g satisfies Hypothesis (H_1)-(H_3). Then $\mathcal{I}_\lambda$ satisfies the $(C)_c$ condition for all c such that*

$$0 < c < \frac{a}{p}K_{s,p}^{a,b} + \frac{b}{2p}\left(K_{s,p}^{a,b}\right)^2 - \frac{S_{s,p}^{-\frac{p_s^*}{p}}}{p_s^*}\left(K_{s,p}^{a,b}\right)^{\frac{p_s^*}{p}}.$$

Proof. Take a $(C)_c$ sequence for $\mathcal{I}_\lambda$ with

$$0 < c < \frac{a}{p}K_{s,p}^{a,b} + \frac{b}{2p}\left(K_{s,p}^{a,b}\right)^2 - \frac{S_{s,p}^{-\frac{p_s^*}{p}}}{p_s^*}\left(K_{s,p}^{a,b}\right)^{\frac{p_s^*}{p}}.$$

From Proposition 3.2 it follows that $(u_n)_n$ is bounded in $X_0^{s,p}(\Omega)$. As a consequence of that, by Theorem 2.3 and Proposition 3.4, we have that

$$\begin{cases} u_n \rightharpoonup u & \text{in } X_0^{s,p}(\Omega) \\ u_n \to u & \text{in } L^q(\Omega) \text{ for all } q \in [1, p_s^*] \\ u_n \to u & \text{a.e in } \mathbb{R}^N. \end{cases} \tag{3.20}$$

Now, testing the derivative of $\mathcal{I}_\lambda$ with $u_n - u$, we get

$$\begin{aligned} o(1) = \mathcal{I}_\lambda'(u_n)\,[u_n - u] &= (a + b\|u_n\|^p)\,\langle A_p(u_n), u_n - u\rangle \\ &\quad - \int_\Omega |u_n|^{p_s^*-2} u_n(u_n - u)\,dx - \lambda \int_\Omega g(x, u_n)(u_n - u)\,dx. \end{aligned} \tag{3.21}$$

At this point we claim that

$$\lim_{n\to\infty} \int_\Omega |u_n|^{p_s^*-2} u_n(u_n - u)\,dx = \lim_{n\to\infty} \int_\Omega g(x, u_n)(u_n - u)\,dx = 0. \tag{3.22}$$

Let us start with the second integral. Given $\varepsilon > 0$, by exploiting (H_2), it is possible to find $C_\varepsilon > 0$ such that $|g(x,t)| \le \varepsilon |t|^{p_s^*-1} + C_\varepsilon |t|^{p-1}$. From this and the Hölder inequality, it follows that

$$\begin{aligned} \left| \limsup_{n\to\infty} \int_\Omega g(x, u_n)(u_n - u)\,dx \right| &\le \limsup_{n\to\infty} \left(\varepsilon \int_\Omega |u_n|^{p_s^*-1}|u_n - u|\,dx \right. \\ &\qquad \left. + C_\varepsilon \int_\Omega |u_n|^{p-1}|u_n - u|\,dx \right) \\ &\le \varepsilon \limsup_{n\to\infty} \left(\int_\Omega |u_n|^{p_s^*}\,dx \right)^{\frac{p_s^*-1}{p_s^*}} \left(\int_\Omega |u_n - u|^{p_s^*}\,dx \right)^{\frac{1}{p_s^*}} \\ &\quad + C_\varepsilon \limsup_{n\to\infty} \left(\int_\Omega |u_n|^p\,dx \right)^{\frac{p-1}{p}} \left(\int_\Omega |u_n - u|^p\,dx \right)^{\frac{1}{p}} \\ &\le \tilde{C}\varepsilon, \end{aligned}$$

for some $\tilde{C} > 0$, where in the last step we used the Sobolev inequality and (3.20). Similarly,

$$\left| \limsup_{n\to\infty} \int_\Omega |u_n|^{p_s^*-2} u_n(u_n - u)\,dx \right| \le \limsup_{n\to\infty} \left(\int_\Omega |u_n|^{p_s^*}\,dx \right)^{\frac{p_s^*-1}{p_s^*}} \left(\int_\Omega |u_n - u|^{p_s^*}\,dx \right)^{\frac{1}{p_s^*}} = 0.$$

Now, replacing (3.22) in (3.21), we deduce

$$\lim_{n\to\infty} (a + b\|u_n\|^p)\,\langle A_p(u_n), u_n - u\rangle = 0.$$

Furthermore, $\|u_n\| \to 0$, is not admissible since it would contradict $c > 0$, so

$$\lim_{n\to\infty} \langle A_p(u_n), u_n - u\rangle = 0$$

recalling $a + b > 0$. At this point, the proof ends by applying Lemma 2.2. □

Remark 3.7. Hypothesis (H_3), necessary to prove the boundedness of $(C)_c$ sequences, could be a bit relaxed at the condition that we accept to get a slightly worse value in Proposition 3.4. More precisely, we can replace (H_3) assuming that there exist $\beta \in L^1(\Omega)$ non-negative, with

$$\|\beta\|_1 < \frac{a}{p} K_{s,p}^{a,b} + \frac{b}{2p} \left(K_{s,p}^{a,b}\right)^2 - \frac{S_{s,p}^{-\frac{p_s^*}{p}}}{p_s^*} \left(K_{s,p}^{a,b}\right)^{\frac{p_s^*}{p}}$$

and $\theta \in [2p, p_s^*)$ such that

$$\xi(x,t_1) \le \xi(x,t_2) + \beta(x) \text{ for a.e } x \in \Omega \text{ and all } 0 \le t_1 \le t_2 \text{ or } t_2 \le t_1 \le 0$$

where $\xi(x,t) = g(x,t)t - \theta G(x,t)$. Assuming this, it is possible to prove with a minor modification of the arguments above that the $(C)_c$ condition holds for all values c with

$$0 < c < \frac{a}{p} K_{s,p}^{a,b} + \frac{b}{2p} \left(K_{s,p}^{a,b}\right)^2 - \frac{S_{s,p}^{-\frac{p_s^*}{p}}}{p_s^*} \left(K_{s,p}^{a,b}\right)^{\frac{p_s^*}{p}} - \|\beta\|_1.$$

We point out that this condition has been introduced by Mugnai and Papageorgiou in [25] and used also in other papers such as [6] and [16].

With the last Proposition, we finished the analysis of the $(C)_c$ condition. Before proceeding with the proof of the main Theorem 1.2, we define our candidate critical level. Namely, set

$$\Gamma := \{\gamma \in C([0,1], X_0^{s,p}(\Omega)) : \gamma(0) = 0, \gamma(1) = w\}$$

where w is provided by Lemma 3.1 and

$$c_\lambda := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \mathcal{I}_\lambda(\gamma(t)).$$

We point out that the candidate critical level $c_\lambda > 0$ thanks Lemma 3.1 and has the following behavior for λ large.

Proposition 3.8. *Assume g satisfies (H_1), (H_2) and (H_4). Then $c_\lambda \to 0$ as $\lambda \to +\infty$.*

Proof. Since the problem is translation invariant, let us assume that $0 \in \omega$. Now, take a $\varphi \in C_0^\infty(\omega)$ and consider $v(x) = \varphi |x|^{-k}$ for some $k \in (0, 1/2)$. Extend v to be 0 outside its support and observe that $v \in H_0^1(\Omega)$, and so $v \in X_0^{s,p}(\Omega)$. Recalling that G is non-negative, we immediately see that

$$\mathcal{I}_\lambda(\zeta v) \le \frac{a}{p} \zeta^p \|v\|^p + \frac{b}{2p} \zeta^{2p} \|v\|^{2p} - \frac{1}{p_s^*} \zeta^{p_s^*} \|v\|_{p_s^*}^{p_s^*} - \lambda \int_\omega G(x, \zeta v)\, dx =: h(\zeta).$$

Since $p_s^* > 2p$, we have that h attains the maximum in a point $\zeta_\lambda \in (0, +\infty)$ and we see that

$$0 = \frac{h'(\zeta_\lambda)}{\zeta_\lambda^p} = a\|v\|^p + b\zeta_\lambda^p \|v\|^{2p} - \frac{1}{p_s^*} \zeta_\lambda^{p_s^* - p} \|v\|_{p_s^*}^{p_s^*} - \frac{\lambda}{\zeta_\lambda^p} \int_\omega g(x, \zeta_\lambda v) v\, dx. \tag{3.23}$$

Exploiting (H_1), we obtain

$$a\|v\|^p + b\zeta_\lambda^p \|v\|^{2p} \le \frac{1}{p_s^*} \zeta_\lambda^{p_s^* - p} \|v\|_{p_s^*}^{p_s^*}$$

from which, remembering that $N > 2ps$, we can deduce that ζ_λ remains bounded at varying of λ. Moreover, we have that $\zeta_\lambda \to 0$ as $\lambda \to \infty$. Indeed, if there exists a sequence $(\lambda_n)_n$ such that $\lambda_n \to \infty$ and $\zeta_{\lambda_n} \to \beta > 0$, using (H_1)-(H_2) and the Lebesgue dominated convergence theorem, we would have that

$$\frac{1}{\zeta_{\lambda_n}^p} \int_\omega g(x, \zeta_{\lambda_n} v) v \, dx \to \frac{1}{\beta} \int_\omega g(x, \beta v) v \, dx > 0.$$

Notice that the positivity of the right-hand side of the previous equation is ensured by (H_4) and the fact that $I \subset v(\omega)$. However, this is in contradiction with (3.23), where passing to the limit we see that the only admissible case is

$$\frac{1}{\beta} \int_\omega g(x, \beta v) v \, dx = 0$$

due to the unboundedness of $(\lambda_n)_n$. The proof of the Proposition now ends by observing that

$$c_\lambda \leq \max_{\zeta \geq 0} \mathcal{I}_\lambda(\zeta v) \leq \frac{a}{p} \zeta_\lambda^p \|v\|^p + \frac{b}{2p} \zeta_\lambda^{2p} \|v\|^{2p} \to 0$$

as $\lambda \to \infty$. □

Proof of Theorem 1.2. Thanks to Proposition 3.8, it is possible to find $\tilde{\lambda}_1 > 0$ such that

$$c_\lambda < \frac{a}{p} K_{s,p}^{a,b} + \frac{b}{2p} \left(K_{s,p}^{a,b}\right)^2 - \frac{S_{s,p}^{-\frac{p_s^*}{p}}}{p_s^*} \left(K_{s,p}^{a,b}\right)^{\frac{p_s^*}{p}}$$

for all $\lambda \geq \tilde{\lambda}_1$. At this point, Proposition 3.1 shows $\mathcal{I}_\lambda$ has a mountain pass geometry and in addition, the $(C)_c$ condition holds thanks to Proposition 3.6. Hence, all Hypotheses of Theorem 2.6 are satisfied. Thus, we have the existence of a critical point of $\mathcal{I}_\lambda$. Moreover, from $c_\lambda > 0$ it follows that this critical point is non-trivial and it corresponds to a non-trivial solution of $(P_{a,b}^\lambda)$. □

Remark 3.9. The method used in the proof of Theorem 1.2 also applies to producing solutions of constant sign. More precisely, if we restrict to the case of non-negative perturbations g, i.e., we require in (H_1) that $g(x,t) \geq 0$ for $t > 0$ and $g(x,t) = 0$ for $t \leq 0$ almost everywhere in $x \in \Omega$, then it is possible to run again all the above arguments to produce a non-trivial critical point u for $\mathcal{I}_\lambda$. At this point, we test the derivative of $\mathcal{I}_\lambda$ with u^-, where $u^- = -\min\{0, u\}$, and we obtain

$$(a + b\|u\|^p) \langle A_p(u), u^- \rangle - \int_\Omega |u^-|^{p_s^*} \, dx = 0. \tag{3.24}$$

Now, notice that

$$(u(x) - u(y)) \left(u^-(x) - u^-(y)\right) = -u^+(x)u^-(y) - u^-(x)u^+(y) - (u^-(x) - u^-(y))^2 \leq -\left|u^-(x) - u^-(y)\right|^2$$

where $u^+ = \max\{0, u\}$. Substituting this in (3.24), we get

$$\begin{aligned} 0 &= (a + b\|u\|^p) \\ &\times \int_{\mathcal{Q}} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (u^-(x) - u^-(y))}{|x-y|^{N+ps}} \, dx \, dy - \int_\Omega |u^-|^{p_s^*} \, dx \end{aligned}$$

$$\leq -\left(a+b\|u\|^p\right)\int\limits_{\mathcal{Q}}\frac{|u(x)-u(y)|^{p-2}\left|u^-(x)-u^-(y)\right|^2}{|x-y|^{N+ps}}\,dx\,dy$$

and so

$$\left|u^-(x)-u^-(y)\right|^2=0\Rightarrow u^-=const,$$

but then $u^-=0$ since $u^-\in X_0^{s,p}(\Omega)$.

4. Existence of solutions with Hypothesis (H_5)

In this section, we address the existence of solutions for problem ($P_{a,b}^{\lambda}$) under condition (H_5) on the nonlinearity. When $p<\vartheta<2p$, establishing the boundedness of $(C)_c$ sequences for the functional $\mathcal{I}_\lambda$ becomes more challenging. To overcome this difficulty, in the same spirit as in [26], we introduce a truncated functional by cutting off the higher-order part of the functional associated with the operator. More precisely, we consider a cut-off function $\psi\in C^\infty([0,\infty),[0,1])$ where $\psi=1$ if $0\leq t\leq 1$ and $\psi=0$ if $t\geq 2$. We also assume that $-2\leq\psi'\leq 0$. We take a $T>0$ that will be determined later, and we set the function $\Psi(u):=\psi(\|u\|^p/T^p)$. We define now the truncated functional

$$\mathcal{J}_\lambda(u):=\frac{a}{p}\|u\|^p+\frac{b}{2p}\|u\|^{2p}\Psi(u)-\frac{1}{p_s^*}\|u\|_{p_s^*}^{p_s^*}-\lambda\int\limits_{\Omega}G(x,u)\,dx,$$

and we observe that is Fréchet differentiable in $X_0^{s,p}(\Omega)$ and its derivative in u in the direction v, where $u,v\in X_0^{s,p}(\Omega)$, is

$$\begin{aligned}\mathcal{J}_\lambda'(u)\,[v]=&\left(a+b\|u\|^p\Psi(u)+\frac{b}{2T^p}\|u\|^{2p}\Psi'(u)\right)\times\\&\times\int\limits_{\mathcal{Q}}\frac{|u(x)-u(y)|^{p-2}\left(u(x)-u(y)\right)\left(v(x)-v(y)\right)}{|x-y|^{N+ps}}\,dx\,dy\\&-\int\limits_{\Omega}|u|^{p_s^*-2}uv\,dx-\lambda\int\limits_{\Omega}g(x,u)v\,dx.\end{aligned}$$

We point out that since we are working with a truncated functional, after we will find a critical point for $\mathcal{J}_\lambda$, we have the additional step of showing that this is also a weak solution of problem ($P_{a,b}^{\lambda}$). Now, we fix

$$T^p:=\min\left\{\frac{a}{8b},\frac{a}{4b}\frac{\theta-p}{2p-\theta}\right\}$$

and we observe that from this choice and the fact that

$$\left|\|u|^{2p}\psi'(\|u\|^p/T^p)\right|\leq 8T^{2p}$$

it follows

$$a+\frac{b}{2T^p}\|u\|^{2p}\psi'\left(\frac{\|u\|^p}{T^p}\right)\geq\frac{a}{2}\tag{4.1}$$

$$a\left(\frac{1}{p}-\frac{1}{\theta}\right)-b\left(\frac{1}{\theta}-\frac{1}{2p}\right)T^p\geq 0\tag{4.2}$$

$$a\left(\frac{1}{p}-\frac{1}{\theta}\right)T^p \geq 8b\left(\frac{1}{\theta}-\frac{1}{2p}\right)T^{2p}. \tag{4.3}$$

Now, we will prove that also this truncated functional $\mathcal{J}_\lambda$ possesses a mountain pass geometry.

Lemma 4.1. *If g satisfies Hypothesis $(H_1)-(H_2)$, then there exist $\tilde{R}>0$ such that $\mathcal{J}_\lambda(\tilde{u})>0$ if $\|\tilde{u}\|=\tilde{R}$ and $\tilde{w}\in X_0^{s,p}(\Omega)$, with $\|\tilde{w}\|>\tilde{R}$, such that $\mathcal{J}_\lambda(\tilde{w})\leq 0$.*

Proof. The proof is analogous to the one of Lemma 3.1, so we will omit the details. □

After having proved that the truncated functional $\mathcal{J}_\lambda$ has a mountain pass geometry, we can start investigating the boundedness of Cerami sequences when Hypothesis (H_3) is replaced by (H_5).

Proposition 4.2. *Assume g satisfies (H_1), (H_2) and (H_5), then every $(C)_c$ sequence for the functional $\mathcal{J}_\lambda$ is bounded.*

Proof. The proof is essentially analogous the one of Proposition 3.2, so we will highlight only the differences.

Assume by contradiction that $\|u_n\|\to\infty$ and define a new sequence $y_n:=u_n/\|u_n\|$. As done before, we can suppose

$$\begin{cases} y_n \rightharpoonup y & \text{in } X_0^{s,p}(\Omega) \\ y_n \to y & \text{in } L^q(\Omega) \text{ for all } q\in[1,p_s^*) \\ y_n\to y & \text{a.e in } \mathbb{R}^N. \end{cases}$$

The case $y\neq 0$ can be treated similarly as in Proposition 3.2, since (3.3) is still valid and

$$\frac{1}{\|u_n\|^{2p}}\left(\frac{1}{p_s^*}\int_\Omega |u_n|^{p_s^*}\,dx+\lambda\int_\Omega G(x,u_n)\,dx\right)$$
$$=\frac{a}{p}\frac{1}{\|u_n\|^p}+\frac{b}{2p}\Psi(u_n)-\frac{1}{\|u_n\|^{2p}}\mathcal{J}_\lambda(u_n)\to 0.$$

So, suppose $y=0$ and notice that up to a subsequence $\|y_n\|_{p_s^*}^{p_s^*}\to L$ as $n\to\infty$ for some $L\in[0,+\infty)$. We analyze separately the cases $L=0$ and $L>0$.

Case $L>0$: we take $\eta>0$ and we observe that for n large enough we have

$$0<\frac{\eta}{\|u_n\|}\leq 1.$$

Define $v_n:=\eta y_n$ and let ζ_n be such that $\mathcal{J}_\lambda(\zeta_n u_n)=\min_{0\leq\zeta\leq 1}\mathcal{J}_\lambda(\zeta u_n)$. We notice that

$$\mathcal{J}_\lambda(\zeta_n u_n)\leq \mathcal{J}_\lambda(v_n)=\frac{a}{p}\eta^p+\frac{b}{2p}\eta^{2p}\psi\left(\frac{\eta^p}{T^p}\right)-\frac{\eta^{p_s^*}}{p_s^*}\|y_n\|_{p_s^*}^{p_s^*}-\lambda\int_\Omega G(x,v_n)\,dx,$$

thus

$$\lim_{\eta\to\infty}\lim_{n\to\infty}\mathcal{J}_\lambda(\zeta_n v_n)\leq \lim_{\eta\to\infty}\frac{a}{p}\eta^p+\frac{b}{2p}\eta^{2p}\psi\left(\frac{\eta^p}{T^p}\right)-\frac{\eta^{p_s^*}}{p_s^*}L=-\infty$$

since $p_s^*>p$ and $\psi\left(\frac{\eta^p}{T^p}\right)\to 0$ as $\eta\to+\infty$ being ψ compactly supported. As a result of this, being $\mathcal{J}_\lambda(0)=0$ and $\mathcal{J}_\lambda(u_n)\to c$, we eventually have that $\zeta_n\in(0,1)$. Then, we get

$$\left.\frac{d}{d\zeta}\right|_{\zeta=\zeta_n} \mathcal{J}_\lambda(\zeta u_n) = 0$$

which implies

$$\lambda \int_\Omega g(z, \zeta_n u_n)\zeta_n u_n \, dx = a\|\zeta_n u_n\|^p + b\|\zeta_n u_n\|^{2p}\Psi(\zeta_n u_n) + \frac{b}{2T^p}\|\zeta_n u_n\|^{3p}\Psi'(\zeta_n u_n) - \|\zeta_n u_n\|_{p_s^*}^{p_s^*}. \tag{4.4}$$

In view of this, we see that

$$\begin{aligned}
\theta\mathcal{J}_\lambda(\zeta_n u_n) &= \theta\mathcal{J}_\lambda(\zeta_n u_n) + \lambda\int_\Omega g(x, \zeta_n u_n)\zeta_n u_n \, dx - \lambda\int_\Omega g(x, \zeta_n u_n)\zeta_n u_n \, dx \\
&\geq a\left(\frac{\theta}{p} - 1\right)\|\zeta_n u_n\|^p - b\left(1 - \frac{\theta}{2p}\right)\|\zeta_n u_n\|^{2p}\Psi(\zeta_n u_n) \\
&+ \left(1 - \frac{\theta}{p_s^*}\right)\|\zeta_n u_n\|_{p_s^*}^{p_s^*} - \frac{b}{2T^p}\|\zeta_n u_n\|^{3p}\Psi'(\zeta_n u_n) \\
&+ \lambda\int_\Omega \xi(z, \zeta_n u_n)\, dx \\
&\geq -b\left(1 - \frac{\theta}{2p}\right)\|\zeta_n u_n\|^{2p}\Psi(\zeta_n u_n) - \frac{b}{2T^p}\|\zeta_n u_n\|^{3p}\Psi'(\zeta_n u_n)
\end{aligned}$$

where we used (H_5). At this point, to reach a contradiction, it suffices to check that $\|\zeta_n u_n\|^{2p}\Psi(\zeta_n u_n)$ and $\|\zeta_n u_n\|^{3p}\Psi'(\zeta_n u_n)$ are bounded. However, verifying this is straightforward, since if $\|\zeta_n u_n\|^{3p}$ is bounded, these two terms remain bounded, whereas they are zero in case $\|\zeta_n u_n\|^{3p}$ is unbounded due to the compact support of ψ.

Case $L = 0$: as in the previous case, we pick up a parameter $\eta > 0$, we set $v_n = \eta y_n$ and we notice that

$$0 < \frac{\eta}{\|u_n\|} \leq 1$$

for n large enough. Analogously as before, let ζ_n be such that

$$\mathcal{J}_\lambda(\zeta_n u_n) = \max_{0\leq\zeta\leq 1} \mathcal{J}_\lambda(\zeta u_n).$$

Using similar arguments as the case $L = 0$ in Proposition 3.2, we see immediately that

$$\lim_{\eta\to\infty}\lim_{n\to\infty} \mathcal{J}_\lambda(\zeta_n u_n) \geq \lim_{\eta\to\infty} \frac{a}{p}\eta^p = +\infty.$$

Again, for n large enough $\zeta_n \in (0, 1)$, and we obtain

$$\left.\frac{d}{d\zeta}\right|_{\zeta=\zeta_n} \mathcal{J}_\lambda(\zeta u_n) = 0$$

which means that (4.4) is still true. To conclude, we see that

$$\begin{aligned}
\theta\mathcal{J}_\lambda(\zeta_n u_n) &= \theta\mathcal{J}_\lambda(\zeta_n u_n) + \lambda\int_\Omega g(x,\zeta_n u_n)\zeta_n u_n\,dx - \lambda\int_\Omega g(x,\zeta_n u_n)\zeta_n u_n\,dx \\
&= a\left(\frac{\theta}{p}-1\right)\|\zeta_n u_n\|^p - b\left(1-\frac{\theta}{2p}\right)\|\zeta_n u_n\|^{2p}\Psi(\zeta_n u_n) \\
&+ \left(1-\frac{\theta}{p_s^*}\right)\|\zeta_n u_n\|_{p_s^*}^{p_s^*} \\
&- \frac{b}{2T^p}\|\zeta_n u_n\|^{3p}\Psi'(\zeta_n u_n) + \lambda\int_\Omega \xi(x,\zeta_n u_n)\,dx \\
&\le |\vartheta\mathcal{J}_\lambda(u_n) - \mathcal{J}_\lambda'(u_n)u_n| - b\left(1-\frac{\theta}{2p}\right)\|\zeta_n u_n\|^{2p}\Psi(\zeta_n u_n) - \\
&- \frac{b}{2T^p}\|\zeta_n u_n\|^{3p}\Psi'(\zeta_n u_n)
\end{aligned}$$

where the right hand side of the chain of inequalities is bounded arguing similarly as in the case $L > 0$ and taking into account that $(u_n)_n$ is a $(C)_c$ sequence. □

The next proposition will identify a positive threshold value for the value c of $(C)_c$ sequences. Under this threshold, Cerami sequences will possess a convergent subsequence.

Proposition 4.3. *Assume g satisfies Hypothesis (H_1), (H_2) and (H_5). Suppose also that $(u_n)_n \subset X_0^{s,p}(\Omega)$ is a $(C)_c$ sequence for $\mathcal{J}_\lambda$ with*

$$c < \left(\frac{1}{\theta}-\frac{1}{p_s^*}\right)\left(\frac{a}{2}S_{s,p}\right)^{\frac{p_s^*}{p_s^*-p}}$$

Then there is $u \in X_0^{s,p}(\Omega)$ such that $u_n \to u$ in $L^{p_s^*}(\Omega)$ up to a subsequence.

Proof. By Proposition 4.2 the sequence $(u_n)_n$ is bounded in $X_0^{s,p}(\Omega)$ and in $\mathcal{M}(\Omega)$. As did in the previous section, invoking the second concentration-compactness principle we have that there exist two Borel regular measures μ and ν, and a set J at most countable such that

$$|D^s u_n|^p(x) \rightharpoonup^* \mu \quad \text{and} \quad |u_n|^{2_s^*}(x) \rightharpoonup^* \nu \quad \text{in}\,\mathcal{M}(\Omega)$$

where

$$\nu = |u|^{p_s^*} + \sum_{j\in J}\nu_j\delta_{x_j}$$

and

$$\mu \ge (-\Delta_p)^s u + \sum_{j\in J}\mu_j\delta_{x_j}$$

with

$$\nu_j = \nu(\{x_j\}) \quad \mu_j = \mu(\{x_j\}),$$

and

$$\mu_j \geq S_{s,p} \nu_j^{\frac{p}{p_s^*}}.$$

We claim that the set J is empty. By contradiction, we made the assumption J is non-empty e for simplicity we also assume has only one element $j_0 \in J$ and a point x_{j_0} with $\nu_{j_0} \neq 0$ corresponding to it. Fix $\varepsilon > 0$, take a smooth cut-off function such that

$$\begin{cases} 0 \leq \varphi_\varepsilon \leq 1 & \text{in } \Omega \\ \varphi_\varepsilon = 1 & \text{in } B(x_{j_0}, \varepsilon) \\ \varphi_\varepsilon = 0 & \text{in } \Omega \setminus B(x_{j_0}, 2\varepsilon), \end{cases}$$

and notice that also the sequence $(u_n \varphi_\varepsilon)_n$ is bounded in $X_0^{s,p}(\Omega)$. Hence

$$\begin{aligned} o(1) &= \mathcal{J}_\lambda'(u_n)\,[u_n \varphi_\epsilon] \\ &= \left(a + b\|u_n\|^p \Psi(u_n) + \frac{b}{2T^p}\|u_n\|^{2p}\Psi'(u)\right)\langle A_p(u_n), u_n\varphi_\epsilon\rangle \\ &\quad - \int_\Omega |u_n|^{p_s^*}\varphi_\epsilon \, dx - \lambda \int_\Omega g(x, u_n) u_n \varphi_\epsilon \, dx. \end{aligned} \tag{4.5}$$

Arguing similarly as we have done in Proposition 3.4, recalling that

$$\left| \|u|^{2p}\psi'(\|u\|^p/T^p)\right| \leq 8T^{2p}$$

to get the bound

$$\left| b\|u_n\|^p \Psi(u_n) + \frac{b}{2T^p}\|u_n\|^{2p}\Psi'(u) \right| \leq 6T^p,$$

we can easily see that

$$\langle A_p(u_n), u_n \varphi_\epsilon \rangle = \int_{\mathcal{Q}} \varphi_\varepsilon(x) \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} \, dx\, dy + o(1)$$

as $n \to \infty$ and $\varepsilon \to 0$. In view of this, (4.1) and (4.5), we have

$$\begin{aligned} \mathcal{J}_\lambda'(u_n)\,[u_n] &= \left(a + b\|u_n\|^p \Psi(u_n) + \frac{b}{2T^p}\|u_n\|^{2p}\Psi'(u)\right)\langle A_p(u_n), u_n\varphi_\epsilon\rangle \geq \\ &\geq \frac{a}{2}\int_{\mathcal{Q}} \varphi_\varepsilon(x)\frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} \, dx\, dy + o(1) \end{aligned}$$

Since the growth hypothesis and g are as in the previous section, we have that (3.14) still holds, so, letting $n \to \infty$ we have

$$\begin{aligned} 0 &\geq \lim_{n\to\infty}\left(\frac{a}{2}\int_{\mathcal{Q}} \varphi_\varepsilon(x)\frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} \, dx\, dy - \int_\Omega |u_n|^{p_s^*}\varphi_\varepsilon \, dx\right) + o(1) = \\ &= \frac{a}{2}\int_\Omega \varphi_\varepsilon \, d\mu - \int_\Omega \varphi_\varepsilon \, d\nu + o(1). \end{aligned}$$

Sending $\varepsilon \to 0$, we get

$$\frac{a}{2}\mu_{j_0} - \nu_{j_0} \leq 0,$$

and recalling (3.11), we have

$$0 \geq \frac{a}{2} S_{s,p} \nu_{j_0}^{\frac{p}{p_s^*}} - \nu_{j_0}. \tag{4.6}$$

From (4.6), we finally obtain

$$\nu_{j_0} \geq \left(\frac{a}{2} S_{s,p}\right)^{\frac{p_s^*}{p_s^*-p}}. \tag{4.7}$$

To conclude, keeping in mind that $(u_n)_n$ is a $(C)_c$ sequence, (H_1) and (H_5), we see that

$$\begin{aligned} c &= \lim_{n\to\infty} \mathcal{J}_\lambda(u_n) = \lim_{n\to\infty} \mathcal{J}_\lambda(u_n) - \frac{1}{\theta}\mathcal{J}_\lambda'(u_n)[u_n] \\ &= \lim_{n\to\infty} a\left(\frac{1}{p} - \frac{1}{\theta}\right)\|u_n\|^p - b\left(\frac{1}{\theta} - \frac{1}{2p}\right)\|u_n\|^{2p}\Psi(u_n) + \left(\frac{1}{\theta} - \frac{1}{p_s^*}\right)\|u_n\|_{p_s^*}^{p_s^*} \\ &\quad + \frac{1}{\theta}\int_\Omega \xi(x, \zeta_n u_n)\,dx - \frac{b}{2T^p}\|u_n\|^{3p}\psi'\left(\frac{\|u_n\|^p}{T^p}\right) \\ &\geq \left(\frac{1}{\theta} - \frac{1}{p_s^*}\right)\left(\frac{a}{2} S_{s,p}\right)^{\frac{p_s^*}{p_s^*-p}} \end{aligned}$$

where we also used the fact that $\psi' \leq 0$ and (4.2). At this point, the conclusion of the proof is similar as in Proposition 3.4. □

Proposition 4.4. *Assume g satisfies Hypothesis (H_1), (H_2) and (H_5). Then $\mathcal{J}_\lambda$ satisfies the $(C)_c$ condition for all c such that*

$$0 < c < \left(\frac{1}{\theta} - \frac{1}{p_s^*}\right)\left(\frac{a}{2} S_{s,p}\right)^{\frac{p_s^*}{p_s^*-p}}$$

Proof. Take a $(C)_c$ sequence as described in the statement of the Proposition. Arguing as in the proof of Proposition 3.6 and replacing Proposition 3.4 with Proposition 4.3, we get

$$o(1) = \left(a + b\|u_n\|^p\Psi(u_n) + \frac{b}{2T^p}\|u_n\|^{2p}\Psi'(u)\right)\langle A_p(u_n), u_n - u\rangle.$$

Recalling that $a > 0$ and (4.1), we can deduce that

$$\lim_{n\to\infty}\langle A_p(u_n), u_n - u\rangle = 0,$$

and the conclusion follows by applying Lemma 2.2. □

Remark 4.5. It is clear that the validity of (3.18) implies (4.6), thus it must be

$$\left(\frac{K_{s,p}^{a,b}}{S_{s,p}}\right)^{\frac{p}{p_s^*}} \geq \left(\frac{1}{\theta} - \frac{1}{p_s^*}\right)\left(\frac{a}{2} S_{s,p}\right)^{\frac{p_s^*}{p_s^*-p}}.$$

As a consequence of that, from (3.19) it follows that

$$\frac{a}{p}K^{a,b}_{s,p} + \frac{b}{2p}\left(K^{a,b}_{s,p}\right)^2 - \frac{S_{s,p}^{-\frac{p_s^*}{p}}}{p_s^*}\left(K^{a,b}_{s,p}\right)^{\frac{p_s^*}{p}} \geq \left(\frac{1}{\theta} - \frac{1}{p_s^*}\right)\left(\frac{a}{2}S_{s,p}\right)^{\frac{p_s^*}{p_s^*-p}},$$

showing that Proposition 3.6 identify a larger threshold than Proposition 4.4 for the validity of the $(C)_c$ condition.

As done in the previous section, we introduce the candidate critical level. More precisely, we define

$$\tilde{\Gamma} := \{\gamma \in C([0,1], X_0^{s,p}(\Omega)) : \gamma(0) = 0, \gamma(1) = \tilde{w}\}$$

where $\tilde{w}$ is provided by Lemma 4.1 and

$$\tilde{c}_\lambda := \inf_{\gamma\in\tilde{\Gamma}} \sup_{t\in[0,1]} \mathcal{J}_\lambda(\gamma(t)).$$

Similarly as Proposition 3.8, next result shows that for λ sufficiently large, the candidate critical level belongs to a range where the Cerami condition holds.

Proposition 4.6. *Assume g satisfies (H_1), (H_2) and (H_4). Then $\tilde{c}_\lambda \to 0$ as $\lambda \to +\infty$.*

Proof. The proof is a minor modification of the one carried out in Proposition 3.8. □

Now, we are ready to give the proof of Theorem 1.3.

Proof of Theorem 1.3. Take λ large enough so that we have

$$\tilde{c}_\lambda < \min\left\{\left(\frac{1}{\theta} - \frac{1}{p_s^*}\right)\left(\frac{a}{2}S_{s,p}\right)^{\frac{p_s^*}{p_s^*-p}}, 4b\left(\frac{1}{\theta} - \frac{1}{2p}\right)T^{2p}\right\}. \tag{4.8}$$

Thanks to Propositions 4.1 and 4.4, we can apply Theorem 2.6 and we can produce a non-trivial critical point u for the functional $\mathcal{J}_\lambda$ in $X_0^{s,p}(\Omega)$. In order to conclude the proof, it is enough to show that the solution we obtained has the property $\|u\| \leq T$. Let us assume $\|u\| > T$. To reach a contradiction, on the one hand, observe that (H_5) and $\mathcal{J}_\lambda(u) = c_\lambda$ imply

$$\begin{aligned}\frac{a}{p}\|u\|^p + \frac{b}{2p}\|u\|^{2p}\Psi(u) &= \frac{1}{p_s^*}\|u\|_{p_s^*}^{p_s^*} + \lambda\int_\Omega G(x,u)\,dx + c_\lambda \\ &\leq \frac{1}{\theta}\|u\|_{p_s^*}^{p_s^*} + \frac{\lambda}{\theta}\int_\Omega g(x,u)u\,dx + c_\lambda.\end{aligned}$$

On the other hand, by writing in full $\mathcal{J}'_\lambda(u)\,[u] = 0$ and multiplying by $1/\theta$ we get

$$\begin{aligned}&\frac{1}{\theta}\left[a + b\|u\|^p\Psi(u) + \frac{b}{2T^p}\|u\|^{2p}\psi'\left(\frac{\|u\|^p}{T^p}\right)\right]\|u\|^p \\ &\qquad = \frac{1}{\theta}\int_\Omega |u|^{p_s^*}\,dx - \frac{\lambda}{\theta}\int_\Omega g(x,u)u\,dx.\end{aligned}$$

Putting the last two expressions together, recalling that $\|u\|^{2p}\Psi(u) \leq 4T^{2p}$ and $\psi' \leq 0$, we obtain

$$
\begin{aligned}
a\left(\frac{1}{p}-\frac{1}{\theta}\right)\|u\|^p &\le b\left(\frac{1}{\theta}-\frac{1}{2p}\right)\|u\|^{2p}\Psi(u)+\frac{b}{2\theta T^p}\|u\|^{3p}\psi'\left(\frac{\|u\|^p}{T^p}\right)+c_\lambda \\
&\le 4b\left(\frac{1}{\theta}-\frac{1}{2p}\right)T^{2p}+c_\lambda.
\end{aligned}
$$

Now, using $\|u\| > T$, it follows that

$$
a\left(\frac{1}{p}-\frac{1}{\theta}\right)T^p \le 4b\left(\frac{1}{\theta}-\frac{1}{2p}\right)T^{2p}+c_\lambda.
$$

Coupling this with (4.3), we finally get

$$
\begin{aligned}
c_\lambda &\ge a\left(\frac{1}{p}-\frac{1}{\theta}\right)T^p - 4b\left(\frac{1}{\theta}-\frac{1}{2p}\right)T^{2p} \\
&\ge 4b\left(\frac{1}{\theta}-\frac{1}{2p}\right)T^{2p}
\end{aligned}
$$

which contradicts (4.8). □

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