Connectedness theory of relative permeability

Paul W.J. Glover [a], Wei Wei [b, a], Piroska Lorinczi [a]

[a] School of Earth and Environment, University of Leeds, Leeds LS2 9JT, UK
[b] School of Geophysics and Geomatics, China University of Geosciences, Wuhan, China

HIGHLIGHTS

- We show that permeability is equal to the connectedness *G* multiplied by the permeability of a capillary tube.
- Relative permeabilities can be expressed solely in terms of a ratio of connectednesses describing flow pathways.
- The lambda parameter is related to phase exponents in the new connectedness model, arising solely from microstructure.

ABSTRACT

The Connectedness Theory is a mathematical approach to understanding the interactions between any number of phases in a complex medium that have different physical properties. It arose from the development of an Archie`s Law for *n* phases when it is applied to fluid permeability. We have shown that Connectedness Theory allows for relative permeabilities to be expressed as ratios of connectednesses. This approach demonstrates why the sum of the non-wetting phase and wetting phase relative permeabilities is always less than unity. In its most general form the Connectedness Theory for two phase relative permeabilities has 8 independent parameters and allows both the fractions of immobile and mobile wetting phase and non-wetting phase, and the phase exponents to vary as a function of wetting phase and non-wetting phase saturation. However, if we make the common assumption that the irreducible wetting phase saturation and residual non-wetting phase saturation are constant and that the phase exponents are also constant, we can use the Connectedness Theory to prove the Brooks and Corey approach to relative permeability modelling and to relate its lamda parameters to phase exponents. In doing so, we also show that the wetting phase relative permeability endpoint is not an independent parameter but arises from variability of phase exponents and hence connectednesses as a function of fluid saturations, and that the two Brooks and Corey coefficients are interdependent. Finally, the Connectedness Theory also predicts that in principle one relative permeability curve can be calculated from the other. Since the theory upon which it is based is valid for any number of different phases, the two phase scenario followed by most of this work is easily extended to three-phase relative permeabilities.

Keywords. Relative permeability, Connectedness Theory, generalised Archie’s law, phase fraction, wetting phase saturation, non-wetting phase saturation, reservoir modelling

1. INTRODUCTION

Relative permeability is extremely important in many branches of earth science, including soil science (Jafarzadeh et al., 2025), reservoir modelling (Iqbal et al. 2025), hydrology (Lanetc et al., 2024) and the use of subsurface reservoirs in the transition to sustainable energy (Lan et al., 2024; Ahmed, 2024; Li and Horne 2004; 2005), including underground sequestration of CO_2 (Wang et al., 2025). However, it is also important in many other fields as widespread as understanding fluid dynamics in nano–biomedical applications (Abdelsalam et al., 2024; Abdelsalam and Bhatti, 2024), improving biomedical flow in arteries (El Kot et al., 2024), understanding the flow of nano-fluids (Ghoneim et al., 2025; Tliba et al., 2025), and in thermal management in engineering (Abdelsalam et al., 2025).

The measurement of relative permeability is often difficult and time consuming (e.g., Honarpour, 1988). This is because the most representative measurements are made at a steady-state which takes time to achieve, especially if carried out at temperatures and pressures for the appropriate depth. There are particular problems if the porous medium has a low permeability, or if it is undergoing precipitation or dissolution reactions. Li and Horne (2004; 2005) recognised that steam-water relative permeabilities differ importantly from conventional gas-water relative permeabilities, with implications for geothermal systems and steam-flooding procedures, while the effect of precipitation reactions on the relative permeabilities of reservoirs that are used for underground sequestration of CO_2 may also be difficult to measure in the laboratory.

A great number of mathematical models have been proposed for the wetting and non-wetting relative permeabilities of a porous medium in order that difficult experimental measurements may be avoided (e.g., Li and Horne, 2005; Chen et al., 1999; Dourado Neto et al., 2011). Most of them are purely empirical, and those that are based on fundamental physics use empirically defined parameters (Li and Horne, 2006). Hence, they may all be regarded as methods for fitting relative permeability curves to experimental data rather than an independent way of predicting relative permeability curves in the absence of experimental data.

It is beyond the scope of this paper to review them all. However, we have included a review of the six main models in Section 3 of this paper with a summary table for the full contextualisation of the results presented in this work. These models are (i) most used by earth scientists, and (ii) have a basis in petrophysics, including contributions from both fluid permeability and capillary pressure, and include combinations of the approaches of Purcell (1949), Burdine (1953) and Mualem (1976) with the water retention models of Corey (1954), Brooks and Corey (1964), van Genuchten (1980) and van Genuchten and Nielsen (1985).

However widely used these models are, their limitations are well-recognised. Demond and Roberts (1993) conducted a comparison of experimental measurements showing that the conventional methods have limited predictive capabilities, while Dury et al. (1999) restricted themselves to reviewing the models for the non-wetting phase, and again found considerable difference between predictions and experiment.

An alternative approach to predicting relative permeabilities is pore-scale network modelling. Earlier studies such as that by Rajaram et al. (1997) found that the results of pore-scale network modelling could fit the capillary-pressure saturation curves and predict the saturation-relative permeability curves with a degree of accuracy comparable to the van Genuchten (1980) relationships. More recently, Piri and Blunt (2005a; 2005b) have shown that the pore-scale network models can reproduce steady-state oil, water, and gas three-phase relative permeabilities without the need to use empirical models, while Valvatne et al. (2005), Valvatne and Blunt (2005), Nardi et al. (2009) and Raeesi and Piri (2009) have used such models to predict successfully relative permeabilities from other more easily measured parameters.

The recognised limitations of the existing empirical models using single empirically-defined parameters (Purswani et al., 2020) has led to the adoption by some of an Equation of State (EoS) approach to relative permeabilty (Khorsandi and Johns, 2017), in which relative permeability is decomposed into five measureable characteristics. The resulting empirical EoS (Purswani et al., 2020) was quadratic with respect to saturation with six coefficients that can be reduced to two on making a number of assumptions. The two remaining coefficients were estimated by linear regression to provide a very good fit to some simulation data. The approach is essentially a more flexible empirical approach with more empirical parameters than the conventional methods but takes no explicit account of the rock matrix, and it will be interesting how it performs on experimental rock data.

Some of the fundamental deficiencies in the conventional approaches to the relative permeability problem were also described by Besserer and Hilfer (2000), who proposed spliting the phases in a porous medium into five phases before calculating the energy balances and energy exchanges between the phases. This is essentially a thermodynamic approach to the relative permeability problem. It differs fundamentally from the approach using Connectedness Theory that is developed in this paper in that Connectedness Theory considers only the microstructural properties of the phases (phase fraction and phase connectedness). Connectedness Theory is a geometrical and topological approach to the relative permeability problem rather than the dynamic approach of Besserer and Hilfer (2000).

Besserer and Hilfer (2000) considered the two phase relative permeability problem, which intuitively infers the treatment of five phases (the matrix, and the mobile and immobile parts of the two miscible phases). In this paper, we also restrict ourselves to 2 phase relative permeability, and therefore the same inherent five phases. However, the connectedness approach is not limited to five phases. A three-fluid system would be described by 7 phases in Connectedness Theory calculations, and more complex systems may produce even more (Glover, 2010). In principle, Connectedness Theory could be used to contemporaneously solve both the electrical and relative permeability fluid flow in a porous medium by defining phases according to their electrical conductivity and their mobility. If we imagine a porous medium composed of three solid phases with different electrical properties, with a pore space that contains both a wetting and nonwetting phase, there will be a total of seven phases for the Connectedness Theory calculation, which will have either mobility but no conductivity, mobility and conductivity, or no mobility but be electrically conductive. This opens up the possibility that both the relative permeability and the electrical conductivity problems of a porous medium might be resolved contemporaneously by using Connectedness Theory. Consequently, the geometrical and topological nature of Connectedness Theory in principle provides huge potential for the study of porous media.

Hilfer (2006a) further used the distinction between percolating and non-percolating fluids in his treatment of microscopic capillarity, further underlying the importance of immobile phases in the definition of fluid flows in a porous medium. The concepts of capillary pressure and relative permeability are key to the conventional theoretical approach to two phase flow in porous media. It was Hilfer (2006a; 2006b; 2006c) who instituted a theoretical approach not requiring these concepts as input, but basing their approach on the mobility or immobility of the two immiscible phases. Hilfer and Doster (2010) then presented the first numerical solutions of Hilfer's (2006b) coupled nonlinear partial differential equations based on the dynamics of fluid flow.

In this paper we have developed independently the use of the same, one might say intuitive, breakdown of phases to solid, and two immiscible phases each of which can have a mobile fraction and an immobile one, but in the case of the Connectedness Theory, the similarity is restricted to the definition of phases, taking a purely geometric approach to the problem thereafter.

By contrast, other geometric approaches do exist. Minkowski functionals (MFs) can be used to describe the basic morphological properties of porous media. In three dimensional Euler space, the Minkowski functionals comprise four functions (M_0 , M_1 , M_2 and M_3), which are measurements of the volume, surface area, mean curvature and Euler characteristic of a given object, respectively (Arns et al., 2001). These four measurements uniquely define any object in 3D space according to Liu et al. (2017) who carried out a combined modelling and experimental study. Their experimental results showed that there was a strong correlation between the non-wetting phase Euler characteristic and relative permeability, but a weak correlation for the wetting phase topology.

Slotte et al. (2020) also used Minkowski functionals for the purpose of predicting permeability and electrical flow. While this approach showed some success, it concluded that Minkowski functionals alone were insufficient to characterise transport properties, and the paper did not consider the quantitative connectedness of pore space, the latter of which is a strength of Connectedness Theory.

While recognising that relative permeability shows a very strong correlation with connectivity, Alpak et al. (2017) carried out fully coupled visco-capillary simulations, using a free-energy based lattice Boltzmann approach, achieving results which "… highlight the close connection between relative permeability and fluid topology and suggest that topological measures are much more meaningful validation criteria for pore-scale simulation." (Alpak et al., 2017). In essence, our paper explores the geometrical and topological control of relative permeability with a different topological approach, which exploits the very strong correlation with connectivity recognised by Alpak et al. (2017).

Consequently, this work seeks to extend the Connectedness Theory that was developed for electrical flow in porous media containing any number of conducting and non-conducting phases to the problem of fluid flow (Glover, 2010).

Another approach, which is significantly different, but has the same goal, is to decompose the conductivity critical exponent at the percolation threshold into parts, one of which is the tortuosity exponent (Berg and Sahimi, 2024). This approach has shown significant success on network models, but differs from the approach in this paper in two important respects. The first is that the approach is based entirely upon assigning meaning to exponents that scale behaviour of the porous medium, and does not explicitly consider connectivity or connectedness of any of the phases. The second is that, conversely, Connectedness Theory does not explicitly include any percolation threshold because it is not required. Instead, any threshold is implicit in the geometrical symmetry of the approach. As a result, the two approaches are mathematically very different indeed.

In this purely theoretical work we first review a selection of the most common empirical models for the two-phase relative permeability of a porous medium, and draw attention to how the models are all forms of more general underlying equations for the relative permeability of the wetting and non-wetting phase. Subsequently, we define a conceptual framework for considering the multi-phase relative permeability problem which is very similar to that which the empirical models already use. We then use previous work of Glover et al. (2006), Glover and Walker (2009), and Glover (2009; 2010) to apply Connectedness Theory to the relative permeability problem, noting that this is a theoretical development of the previous work, where the physical parameters may be determined experimentally. The remainder of this work then examines the immediate implications of the new relative permeability model. This paper is purely theoretical and deliberately avoids comparison with experimental data. In many ways it mirrors the approach of Sakai et al. (2015) in wanting to combine electrical and flow characteristics to inform the quantitive understanding of relative permeability. It is hoped that this theoretical paper will in future be supported by new experimental measurements that allow it to be fully verified.

2. THEORETICAL BACKGROUND

The relative permeability of a phase is a dimensionless measure of the effective permeability of that phase and is given by the ratio of the effective permeability of that phase to the absolute permeability

$$\kappa_{rj} = \frac{\kappa_j}{\kappa^o}, \qquad \textbf{(1)}$$

where k_{rj} is the relative permeability of the j^{th} phase, κ_j is the absolute permeability of the j^{th} phase and κ^o is a reference permeability. Often the reference permeability is taken to be that of oil when all the oil in the porous medium is immobile, in other words at the irreducible wetting phase saturation, S_{wi}. Relative permeabilities are most often represented as a function of wetting phase saturation, S_w. The wetting phase and oil relative permeabilities depend not only on the volume fraction of each phase but the way that phase is arranged. Since any rearrangement of a fluid phase implies the movement of all fluid phases in a porous medium, and the need to include capillary effects due to wetting and non-wetting fluids, the relative permeability curves also depend upon the flow history. Hence, different relative permeability curves are measured during drainage (reduction of the wetting phase volume fraction) and imbibition (increase of the wetting phase volume fraction).

Equation (1), though simple, makes a number of important assumptions that should be considered. The approach implies that Darcy's law can be applied to each phase in the porous medium separately. The assumptions inherent in Darcy's law also apply to relative permeabilities. These are that (i) the flow is one-dimensional, and (ii) horizontal, that (iii) the fluids are immiscible, and that (iv) the medium is homogeneous. Ideally we would add a further assumption that the fluids do not interact, implying that the movement of one fluid has no effect on the movement of another. This is not true even for steady-state relative permeability measurements. Normally, each phase impedes the other phases sufficiently that the sum of all the relative permeabilities is less than unity, although there have been examples in heavy oil reservoirs where inter-phase coupling enhances flow leading to the sum of the relative permeabilities being greater than unity (Bravo and Araujo, 2008).

3. EMPIRICAL RELATIVE PERMEABILITY MODELS

It is not the aim of this paper to review the relative permeability models. However, since there are few authoritative reviews (of which the best are cited below), we feel that it is necessary to review briefly the main models so that they may be compared with the approach used in this work. The most important point is that all of the models are empirical because each assumes some form of the capillary pressure-saturation function that is based on experimental observation.

There are three historical approaches to the relative permeability problem that are discussed briefly in the following sub-sections, and are also reviewed by Li and Horne (2005) and Dury et al. (1999). All the models are summarised in Table 1, which presents the six models in the form of two generalised equations that differ only in the form of their individual coefficients. Consequently, it can be seen that, though all the models are empirical, they exhibit some underlying symmetry which hints at there being a deeper theoretical control. This paper uses the Connectedness Theory approach to show that at least one of these empirical methods (the Brooks–Corey–Purcell method) has a deeper pedigree than just being an empirical model.

Table 1. The six main relative permeability models used in this work.

Model	**Retention Model**	**Permeability Model**	**Coefficients of the Generalised models >>**						$(S_w^*)^a[1-(1-(S_w^*)^b)^c]^2$	$(1-S_w^*)^d[1-(S_w^*)^e]^f$	
			a	b	c	d	e	f	k_{rw}	k_{rnw}	**References**
BCP	Brooks and Corey (1964)	Purcell (1949)	$\frac{2+\lambda}{\lambda}$	≠0	0	0	$\frac{2+\lambda}{\lambda}$	1	$(S_w^*)^{\frac{2+\lambda}{\lambda}}$	$\left[1-(S_w^*)^{\frac{2+\lambda}{\lambda}}\right]$	Li and Horne (2005)
BCB	Brooks and Corey (1964)	Burdine (1953)	$\frac{2+3\lambda}{\lambda}$	≠0	0	2	$\frac{2+\lambda}{\lambda}$	1	$(S_w^*)^{\frac{2+3\lambda}{\lambda}}$	$(1-S_w^*)^2\left[1-(S_w^*)^{\frac{2+\lambda}{\lambda}}\right]$	Brooks and Corey (1964), Li and Horne (2005), Chen et al. (1999)
BCM	Brooks and Corey (1964)	Mualem (1976)	$\frac{4+5\lambda}{2\lambda}$	≠0	0	0.5	$\frac{1+\lambda}{\lambda}$	2	$(S_w^*)^{\frac{4+5\lambda}{2\lambda}}$	$(1-S_w^*)^{1/2}\left[1-(S_w^*)^{\frac{1+\lambda}{\lambda}}\right]^2$	Chen et al. (1999)
VGP	van Genuchten (1980)	Purcell (1949)	0	$1/m$	m	0	$1/m$	m	$\left[1-\left(1-(S_w^*)^{1/m}\right)^m\right]$	$\left[1-(S_w^*)^{1/m}\right]^m$	van Genuchten (1980), Purcell (1949)
VGB	van Genuchten (1980)	Burdine (1953)	2	$1/m$	m	2	$1/m$	m	$(S_w^*)^2\left[1-\left(1-(S_w^*)^{1/m}\right)^m\right]$	$(1-S_w^*)^2\left[1-(S_w^*)^{1/m}\right]^m$	van Genuchten (1980), Dourado Neto et al. (2011), Demond and Roberts (1993)
VGM	van Genuchten (1980)	Mualem (1976)	0.5	$1/m$	$2m$	0.5	$1/m$	$2m$	$(S_w^*)^{1/2}\left[1-\left(1-(S_w^*)^{1/m}\right)^m\right]^2$	$(1-S_w^*)^{1/2}\left[\left(1-(S_w^*)^{1/m}\right)^m\right]^2$	van Genuchten (1980), Dourado Neto et al. (2011), Parker et al. (1987)

The following sections examine each of the models in a little more detail so that their relationships can be understood.

The Purcell Approach

Purcell (1949) used capillary pressure data to calculate porous medium permeability, and in doing so created an equation that can be used to calculate the relative permeabilities in two-phase flow

$$k_{rw} = \frac{\int_0^{S_W} dS_W/[P_c(S_W)]^2}{\int_0^1 dS_W/[P_c(S_W)]^2} \text{ and } k_{rnw} = \frac{\int_{S_W}^1 dS_W/[P_c(S_W)]^2}{\int_0^1 dS_W/[P_c(S_W)]^2}\ , \qquad \textbf{(2)}$$

where k_{rw} and k_{rnw} are the relative permeabilities of the wetting and non-wetting phases, S_w is the saturation of the wetting phase and $P_c(S_w)$ is the capillary pressure as a function of S_w. Li and Horne (2005), who provide a good review of most relative permeability models, note that $k_{rw}+ k_{rnw} = 1$, which is contrary to experimental measurements on porous media. They also compare the model with experimental data and find that the equation for the wetting phase fits the data fairly well, but that for the non-wetting phase does not.

The Burdine Approach

Burdine (1953) added a tortuosity factor λ to the Purcell equations to give

$$k_{rw} = (\lambda_{rw})^2 \frac{\int_0^{S_W} dS_W/[P_c(S_W)]^2}{\int_0^1 dS_W/[P_c(S_W)]^2} \text{ and } k_{rnw} = (\lambda_{rnw})^2 \frac{\int_{S_W}^1 dS_W/[P_c(S_W)]^2}{\int_0^1 dS_W/[P_c(S_W)]^2}, \qquad \textbf{(3)}$$

where,

$$\lambda_{rw} = \frac{\tau_w(S_w\!=\!1)}{\tau_w(S_w)} = \frac{S_w - S_m}{1 - S_m} \quad \text{and} \quad \lambda_{rnw} = \frac{\tau_{nw}(S_w\!=\!1)}{\tau_{nw}(S_w)} = \frac{1 - S_w - S_e}{1 - S_m - S_e}, \tag{4}$$

where τ_w and τ_{nw} are the hydraulic tortuosities of the wetting and the non-wetting phases, respectively, S_m is the minimum wetting phase saturation from the capillary curve and S_e is the equilibrium saturation of the non-wetting phase, and the symbols have been retained from Li and Horne (2005) in order to retain consistency with their work. However, once again, the wetting phase relative permeability is a better fit to the experimental data than the non-wetting phase relative permeability from this model (e.g., Honarpour et al., 1986).

The Mualem Approach

Mualem (1976) provided an approach that has become popular amongst soil physicists. It takes the form

$$k_{rw} = (S_w^*)^{1/2}\left[\frac{\int_0^{S_w^*} dS_w/P_c(S_w)}{\int_0^1 dS_w/P_c(S_w)}\right]^2 \quad \text{and} \quad k_{rnw} = (1 - S_w^*)^{1/2}\left[\frac{\int_{S_w^*}^1 dS_w/P_c(S_w)}{\int_0^1 dS_w/P_c(S_w)}\right]^2, \tag{5}$$

where

$$S_w^* = \frac{S_w - S_{wr}}{1 - S_{wr}} \tag{6}$$

is the normalised or reduced wetting-phase saturation and S_{wr} is the residual saturation of the wetting phase.

The Corey and Brooks-Corey Relative Permeability Models

The three approaches outlined above require us to know the form of the function $P_c(S_w)$. Corey (1954) found an approximate form of this function experimentally using data measured during initial drainage,

$$P_c = A(S_w^*)^{-1/2}, \tag{7}$$

where A is a constant and S_w^* is the normalised or reduced wetting-phase saturation during initial drainage, and defined in the same way as Equation (6).

This parameter can also be defined for use during subsequent imbibition of the wetting phase as,

$$S_w^* = \frac{S_w - S_{wr}}{1 - S_{wr} - S_{nwr}}, \tag{8}$$

where S_{nwr} is the residual saturation of the non-wetting phase. Corey (1954) then obtained equations for the relative permeability of the wetting and non-wetting phases by assuming that $S_e = 0$ and $S_m = S_{wr}$, and using Equation (3) and Equation (7) to give

$$k_{rw} = (S_w^*)^4 \quad \text{and} \quad k_{rnw} = (1 - S_w^*)^2[1 - (S_w^*)^2]. \tag{9}$$

Although we often call this model the "Corey Model", it is formally the Burdine Model with the Corey capillary pressure relationship (Equation (7)), and is purely empirical. Other empirical models were developed about the same time such as that by Pirson (1958) and Wyllie (1962) for different forms of the capillary pressure relationship.

Since Equation (7) is rather restrictive, Brooks and Corey (1966) used a more general functional form

$$P_c = P_e(S_w^*)^{-1/\lambda} \quad or \quad S_w^* = \left(\frac{P_e}{P_c}\right)^{\lambda}, \tag{10}$$

where P_e is the entry capillary pressure and λ is called the pore size distribution index. This empirical model was subsequently derived theoretically from capillary pressure and fractal theory for fractal dimensions of the pore network, $D_f > 2$ (Li and Horne, 2004).

Following the same procedure as Corey (1954), the relative permeabilities become

$$k_{rw} = (S_w^*)^{\frac{2+3\lambda}{\lambda}} \text{ and } k_{rnw} = (1 - S_w^*)^2 \left[1 - (S_w^*)^{\frac{2+\lambda}{\lambda}}\right], \tag{11}$$

and noting that the Corey (1954) model is a special case of the Brooks and Corey (1966) model with λ=2. This model is called the Brooks-Corey-Burdine (BCB) Model and is also given in Table 1.

If Equation (10) is substituted into Equations (2), we obtain a relative permeability model that is based on the Purcell approach rather than the Burdine approach (i.e., the Brooks-Corey-Purcell (BCP) Model) (Li and Horne, 2005; 2006)

$$k_{rw} = (S_w^*)^{\frac{2+\lambda}{\lambda}} \text{ and } k_{rnw} = \left[1 - (S_w^*)^{\frac{2+\lambda}{\lambda}}\right]. \tag{12}$$

If Equation (10) is substituted into Equations (5), we obtain a relative permeability model that is based on the Mualem approach (i.e., the Brooks-Corey-Mualem (BCM) Model) (Chen et al., 1999)

$$k_{rw} = (S_w^*)^{\frac{4+5\lambda}{2\lambda}} \text{ and } k_{rnw} = (1 - S_w^*)^{1/2} \left[1 - (S_w^*)^{\frac{1+\lambda}{\lambda}}\right]^2. \tag{13}$$

Both the BCP and BCM models are also summarised in Table 1.

The van Genuchten Relative Permeability Model

Another popular way of describing the capillary pressure curve was developed by van Genuchten (1980) and van Genuchten and Nielsen (1985) which can be written

$$S_w^* = (1 + [\alpha P_c]^n)^m, \tag{14}$$

where α and n are fitting parameters that are related to the non-wetting fluid pore entry pressure and the width of the pore size distribution, respectively, and it is often assumed that m is a function of n. These parameters share a common physical origin with the Brooks and Corey parameters P_e and λ. At large capillary pressures $S_w^* \rightarrow \left(\frac{1}{\alpha} \cdot \frac{1}{P_c}\right)^{mn}$, while Equation (10) for the Brooks and Corey Model shows $S_w^* = \left(\frac{P_e}{P_c}\right)^{\lambda}$. Hence $P_e \leftrightarrow \frac{1}{\alpha} \;\; and \;\; \lambda \leftrightarrow mn.$

van Genuchten (1980) assumed that $m = 1 - \frac{1}{n}$ and substituted Equation (14) into Equation (5) to obtain the van Genuchten-Mualem (VGM) Model that has since been extremely popular with soil scientists (Sakai et al., 2015; Parker and Lenhard, 1987; Parker et al., 1987; Dourado Neto et al., 2011)

$$k_{rw} = (S_w^*)^{1/2}\left[1 - \left(1 - (S_w^*)^{1/m}\right)^m\right]^2 \;\; and \;\; k_{rnw} = (1 - S_w^*)^{1/2}\left[\left(1 - (S_w^*)^{1/m}\right)^m\right]^2. \tag{15}$$

Similar models can be written for the combination of the van Genuchten model with the Burdine and Purcell approaches to give

$$k_{rw} = (S_w^*)^2\left[1 - \left(1 - (S_w^*)^{1/m}\right)^m\right] \;\; \text{and} \;\; k_{rnw} = (1 - S_w^*)^2\left[1 - (S_w^*)^{1/m}\right]^m, \tag{16}$$

for the van Genuchten-Burdine (VGB) Model, and

$$k_{rw} = \left[1 - \left(1 - (S_w^*)^{1/m}\right)^m\right] \;\; \text{and} \;\; k_{rnw} = \left[1 - (S_w^*)^{1/m}\right]^m$$

, **(17)**

for the van Genuchten-Purcell (VGP) Model (Demond and Roberts, 1993; Dourado Neto et al., 2011). All of these models are summarised in Table 1.

The relationships between the models described in the section above, as exemplified by Table 1, exhibits clearly a common form given by the generalised equations and indicating some deeper theoretical symmetry in the petrophysics of relative fluid flow. Later in this work we will show that the generalised forms of these equations arise directly from the application of Connectedness Theory to relative permeabilities, and that the relationships between the various fitting parameters also arise directly from the Connectedness Theory.

Finally, it should be noted that none of the equations above require the irreducible wetting phase saturation or the residual non-wetting phase saturation to remain constant. Since these are both defined at a point where each of the fluids become immobile, retrospectively, it is a natural inference that they might be considered to be constant throughout the wetting phase saturation range. However, that is not necessarily the case, as we will see later in this paper. In addition, it should be noted that the Connectedness Theory which we will apply to relative fluid flow does not assume that the saturation of wetting or nonwetting fluid remaining immobile is constant as a function of wetting phase saturation.

4. PHASE FRACTIONS

In this work we consider a multiphase system that is solely composed of n phases, each with its volume fraction χ_j , and where

$$\sum_1^n \chi_j = 1. \tag{18}$$

There is no restriction that the phases are miscible or immiscible, and there is no dependence on the historic relative volume fraction of the phases or their distribution. The volume fractions and distributions of the phases are, hence, defined purely geometrically at a certain arbitrary instant in time. The consequence of this is that the material which follows is equally valid for any of the common two phase systems (oil-water, water-gas, gas-oil), but also less common systems (for example, oil-steam, oil-supercritical carbon dioxide, and water-supercritical carbon dioxide), with the important note that their usage would need to also take into account miscibility/dissolution if phase fractions were to change. It also implies that this approach to relative permeabilities is also valid for three phase systems, such as oil-water-gas, and so on. The approach is independent of physical properties, requiring only that the distribution of the phases is known.

Having stated the likely general validity of the approach, the rest of this paper will focus upon the well-known wetting-nonwetting phase problem, and use the indexes 'p', 'r', 'n', and 'w' to associate with pores, rock matrix, the non-wetting phase, and the wetting phase, respectively, followed by an 'm' or an 'i', to represent mobility of the phase or immobility of the phase, again respectively. The main deviation from conventional usage here is the use of 'n' for non-wetting, instead of 'nw'. This has been done to make the equations easier to read. If we assume that our multiphase system is a porous medium with a solid matrix, the volume fraction of the solid matrix χ_r encloses a pore space with a volume fraction χ_p, where the distribution of the pore space and matrix can take any geometry at any scale, and where

$$\chi_r + \chi_p = 1. \tag{19}$$

We will continue to work with volume fractions rather than saturations S or porosity ϕ for reasons of generality that will become apparent as the article progresses, but we note that $\chi_p = \phi$ and $\chi_r = 1 - \phi$.

It should be noted that commonly the two volume fractions χ_p and χ_r are considered to be constant for any given porous medium, however, if the porous medium were to undergo dissolution of precipitation reactions, these parameters would become variable, and that such a variation is therefore included implicitly in our new model.

For the remainder of this work, we assume that the pore space is completely filled with either non-wetting phase or wetting phase and that the non-wetting phase and wetting phase may be mobile or immobile. Hence we have four fluid phases, immobile non-wetting phase, immobile wetting phase, mobile non-wetting phase and mobile wetting phase, with volume fractions χ_{ni}, χ_{wi}, χ_{nm}, χ_{wm}, respectively.

The volume fractions sum as

$$\chi_r + \chi_{wi} + \chi_{ni} + \chi_{wm} + \chi_{nm} = 1, \text{ and} \tag{20}$$

$$\chi_p = \chi_{wi} + \chi_{ni} + \chi_{wm} + \chi_{nm}. \tag{21}$$

Some of these phases are miscible and some are not. The distinction is irrelevant because our two target parameters are associated with two distinct immiscible phases (mobile wetting phase and mobile non-wetting phase).

All the parameters in Equations (20) and (21) may be considered to be variable, but only any four are independently variable because of the restrictions introduced in Equation (20) that no undefined phases are also present. In our model we consider that all four fluid phases to be variable in the sense that they depend upon the overall wetting phase and non-wetting phase saturations (S_w and S_{nw}), and that a particular volume of one pore fluid that is mobile at a given wetting phase saturation may be immobile at a slightly higher or lower saturation. This implies that our samples should be homogeneous at the scale on which we apply our model. However there is no requirement for homogeneity at a smaller scale than that at which the model is applied, providing that the volume appears homogeneous when heterogeneities, and their effects, are averaged. There is no requirement for the sample to be isotropic at any scale. It should be noted that the conventional approach is rather different, treating the parameters χ_r, χ_{wi} and χ_{ni} as fixed, and letting χ_{wm} and χ_{nm} vary (i.e., just one independent variable).

In order to fully define the system and to be consistent with parameters that are used conventionally, we have

$$\chi_{wi} + \chi_{wm} = \chi_w = \phi S_w, \tag{22}$$

$$\chi_{ni} + \chi_{om} = \chi_n = \phi S_n, \tag{23}$$

$$\chi_w + \chi_n = \phi, \tag{24}$$

where each of the pore fluids has a volume fraction χ_w and χ_n and saturations S_w and S_n, respectively. If we consider that the fluid phases are completely interchangeable, the volume fraction of wetting phase varies between zero and (1-ϕ) and that of the hydrocarbon between (1-ϕ) and zero. However, there exists an immobile wetting phase fraction χ_{iw} that cannot be removed from the porous medium, which is given by

$$\chi_{wi}(S_w = S_{wi}) = \phi S_{wi}, \tag{25}$$

and a residual non-wetting phase saturation, which is given by

$$\chi_{ni}(S_w = 1 - S_{or}) = \phi S_{nr}, \tag{26}$$

where, S_{wi} is the irreducible wetting phase saturation and S_{nr} is the residual non-wetting phase saturation. Hence, the volume fraction of wetting phase varies between ϕS_{wi} and ϕ(1-S_{nr}) where

$$\phi S_{wi} \le \chi_w \le \phi(1 - S_{nr}) \ \text{ or } \ \chi_{wi} \le \chi_w \le (1 - \chi_r - \chi_{ni}), \tag{27}$$

and the analogous ranges for the other parameters are

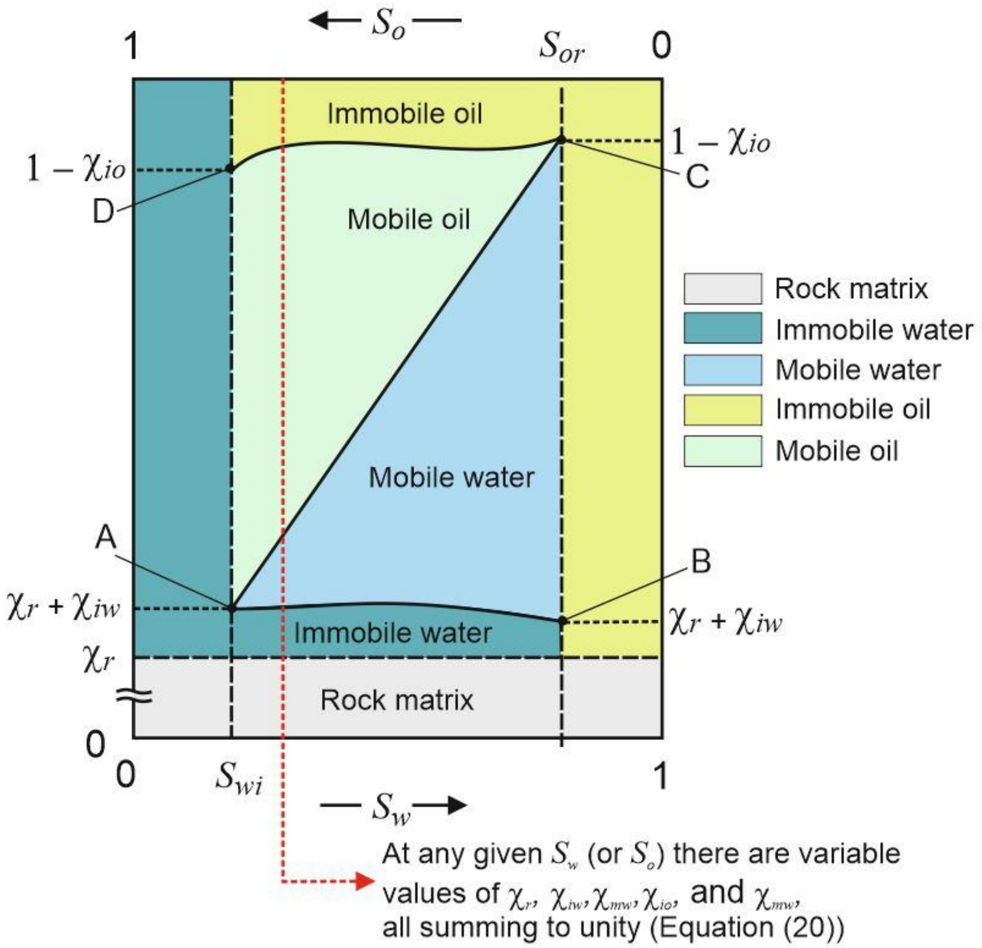
$$\phi S_{or} \le \chi_n \le \phi(1 - S_{wi}) \ \text{ or } \ \chi_{ni} \le \chi_n \le (1 - \chi_r - \chi_{wi}), \tag{28}$$

$$0 \le \chi_{wm} \le \phi(1 - S_{wi} - S_{nr}) \ \text{ or } \ 0 \le \chi_{wm} \le (1 - \chi_r - \chi_{wi} - \chi_{ni}), \tag{29}$$

$$0 \le \chi_{nm} \le \phi(1 - S_{wi} - S_{nr}) \ \text{ or } \ 0 \le \chi_{nm} \le (1 - \chi_r - \chi_{wi} - \chi_{ni}), \tag{30}$$

Figure 1 shows how the five phases may occupy the porous medium. It should be noted that the lines AB and CD, are not necessarily constant or linear, and have been drawn with an arbitrary shape in Figure 1 to express this. The functional form of these lines are ($\chi_r + \chi_{wi}$) and

$(1-\chi_{ni})$, respectively, and are non-linear if the functions $\chi_{wi}(S_w)$ and $\chi_{ni}(S_w)$ are non-linear, respectively. However, the line AC is linear because $\chi_{wi}+\chi_{wm} = S_w$ for $S_{wi} \leq S_w \leq (1-S_{nr})$.



At any given S_w (or S_o) there are variable values of χ_r, χ_{iw}, χ_{mw}, χ_{io}, and χ_{mw}, all summing to unity (Equation (20))

Figure 1. The relationship between the five phases, porous medium matrix, mobile and immobile wetting phase, and mobile and immobile non-wetting phase.

One of the perceived disadvantages of Connectedness Theory is that by its general nature can introduce a large number of parameters. The number of parameters is governed in the electrical case by the number of conducting phases, and can be large and difficult to quantify in practical applications. The same is true, but to a lesser extent, for the two phase flow application described in this paper. Nevertheless, for relative fluid flow there are eight independent parameters. Fortunately in most cases some of these can be considered to be constant. One example is the phase fraction of the matrix. In most applications one might assume that the phase fraction of the matrix would remain constant, thus simplifying the equations. However, the general nature of Connectedness Theory means that in cases where reactive fluid flow leads to dissolution or precipitation, the phase fraction of the matrix can be allowed to vary while still being described by the theory. Despite the number of parameters, this paper shows that Connectedness Theory can be used to provide some theoretical pedigree to some of the simpler conventional empirical relative permeability models described in Section 3.

5. CONNECTEDNESS AND PERMEABILITY

The permeability of a single phase flowing through a porous medium is given by the relationship (Glover et al., 2006).

$$\kappa = \frac{d^2 \phi^{3m}}{4\, a\, m^2}, \tag{31}$$

Here κ is the permeability (in m^2), d is the effective grain diameter (in m), ϕ is the porosity (unitless), m is the cementation exponent (unitless) and a is a constant that is thought to be equal to 8/3 for three dimensional samples composed of quasi-spherical grains. Equation (31) is an equation derived analytically from electro-kinetic considerations. Like many equations in electro-kinetics, it assumes that the electrical double layer is thin, which requires the value of the cementation exponent m to be derived from measurements on samples saturated with medium to high salinity fluid. The equation also assumes that O'Konski's (1960) equation is valid, and that the formation factor F $(= \phi^{-m})$ and the porosity do not approach unity.

In 2009 Glover and Walker (2009) provided an equation that allows the effective pore radius of a porous medium to be calculated if the effective grain diameter is known, and *vice versa*. The equation also requires that the porosity and cementation exponent (or formation factor) is known. The so-called theta-transformation is given by

$$d = 2\,\Theta\, r \quad \text{where} \quad \Theta = \sqrt{\frac{a\,m^2}{8\,\phi^{2m}}} = \sqrt{\frac{a\,m^2\,F^2}{8}}, \tag{32}$$

where r is the effective pore radius, and F is the formation factor (unitless). The assumptions behind Equation (32) are the same as those for Equation (31) discussed previously.

Equation (32) can be resolved for d^2 and substituted into Equation (31) to give

$$\frac{\phi}{\tau_e \kappa} = \frac{8}{r^2} \quad \text{where} \quad \tau_e = \phi^{1-m}, \tag{33}$$

and τ_e is the electrical tortuosity.

The connectedness of a porous material was defined by Glover (2009) and used in the generalized Archie`s law (Glover, 2010) as

$$G \equiv \phi^m = \phi \varsigma_e, \tag{34}$$

where ς_e is the electrical connectivity (=1/τ_e). Hence, Equation (33) can be rewritten as

$$\kappa = \frac{Gr^2}{8}. \tag{35}$$

The permeability of a tube with a radius r can be expressed by (Bernabé, 1995)

$$\kappa_t = \frac{r^2}{8}. \tag{36}$$

and is proven to scale with porosity if a bundle of capillary tubes forms an array of porosity ϕ (Adegoke and Olowofela, 2008). Hence,

$$\kappa = G\ \kappa_t. \tag{37}$$

This is a remarkable result; it is completely analogous to the equation for calculating the conductivity of a porous medium when completely saturated with a single phase

$$\sigma = G\ \sigma_f, \tag{38}$$

where σ is the conductivity of the porous medium and σ_f is the conductivity of the saturating fluid. In each case, Equations (37) and (38) express the measured transport property (κ and σ) as the product of the physical property without the presence of the pore structure (κ_t and σ_f) and the effect of the intervening pore structure given by the connectedness. The connectedness represents how the basic physical property is abated or modified by the pore structure. The corollary is that the connectedness can be said to be independent of the physical (transport) property it modifies, being solely a function of the microstructure of the porous medium.

The similarity of Equations (37) and (38) is even more surprising when one considers that electrical conduction in porous media is mediated by ions and electrons that cross phase boundaries, while fluid flow involves mass transport that by definition remains constrained to its own phase. There is no real problem for a non-conducting matrix that is saturated with a single conducting fluid, because both the conducting ions and the fluid are constrained to the same phase. The implication is that this model for permeability should only be valid for immiscible non-interacting pore fluids. The further question remains whether these equations can be generalised to three pore fluids or even n-phases as in the case of the generalised Archie's law?

In the case of conductivity the phase may have a zero conductivity for two reasons; (i) the connectedness of the phase tends to zero ($G \rightarrow 0$), or (ii) the conductivity of the material making up that phase tends to zero ($\sigma_f \rightarrow 0$) as shown by Equation (38). The case for permeability is analogous. A phase may have a zero permeability if (i) the connectedness of the phase tends to zero ($G \rightarrow 0$), or (ii) the value of κ_t for that phase tends to zero ($\kappa_t \rightarrow 0$) as shown by Equation (37). The interpretation of the connectedness is the same as for the electrical case as well as being simple and intuitive (Glover, 2009; 2010; Glover and Déry, 2010). However, the physical interpretation of the value of κ_t is not. Equation (36) depends solely upon the radius of the capillary tube that represents a characteristic radius of the phase perpendicular to flow (i.e., the characteristic pore radius of the pore space if there are only two phases; matrix and mobile fluid). Hence, both κ_t and σ_f represent the effectiveness of flow though the phase, with electrical flow being a function of the availability, charge and mobility of charge carriers, while the hydraulic flow is a function of the radius of the equivalent capillary. Hence, there are two conditions for zero permeability

$$\kappa \rightarrow 0 \quad \text{for either} \quad \left| \begin{matrix} G \rightarrow 0 \\ \kappa_t \rightarrow 0 \end{matrix} \right. \qquad \textbf{(39)}$$

In other words, there is zero permeability if either (i) the phase is not connected, or (ii) the inherent permeability of the phase approaches zero. While one might initially consider that

$$\kappa_t \to 0 \;\Rightarrow\; r \to 0 \;\Rightarrow\; \chi \to 0 \;\Rightarrow\; G \to 0, \tag{40}$$

it is possible that

$$\kappa_t \to 0 \quad \text{when} \quad r \neq 0 \;\Rightarrow\; \chi \neq 0 \;\Rightarrow\; G \neq 0 \;\; (m \neq \infty), \tag{41}$$

if there is no flow in the capillary tube due to either (a) the driving pressure being insufficient to overcome frictional forces when the capillary radius is extremely small, or (b) the driving pressure being insufficient to overcome capillary forces. There is an important mechanism associated with the second point: If fluid-fluid interfaces are successively mobilised in a porous medium from zero effective permeability at low pressure gradients, the relative permeability picture breaks down as the relation between flow rate and pressure gradient becomes a power law rather than linear. This was first seen by Tallakstad et al. (2009), and later studied by numerous authors (e.g., Gao et al., 2021). Later in this paper we will apply Connectedness Theory that was developed from Equation 38 to the analogous Equation 37 developed in this work. Connectedness Theory is concerned only with the fraction of any given phase and its connectedness. Consequently, the mobility of fluid–fluid interfaces may change both the relative fractions of the two fluid phases as well as their connectedness, so in principle the effect described by Tallakstad et al. (2009), and Gao et al. (2021), should be implicitly included in the theory.

Since Equations (36) and (37) are defined for a moment in time with specified phase fractions, there is no implicit accounting for capillary pressure, and we can discount the second mechanism (mechanism (b), above).

6. PHASE CONNECTEDNESSES

In this work we consider a multiphase system that is solely composed of n phases, each with its connectedness G_i. Glover (2010) hypothesised that there exists a law of conservation of connectedness that is given by

$$\sum_1^n G_j = 1, \tag{42}$$

which is an exact analogy of Equation (18). Glover (2010) showed using numerical modelling that for the electrical system, conservation of connectedness seemed to be valid. However, it is too early to state that such a law is valid and under what circumstances. Here we extend our hypothesis in the electrical domain to include connectedness and fluid flow.

If we continue our definition of matrix, immobile and mobile phases forward to connectednesses, we have G_r, G_{wi}, G_{ni}, G_{wm}, and G_{nm} for the connectednesses of the matrix, immobile wetting phase and non-wetting phase and mobile wetting phase and non-wetting phase, respectively. It follows that

$$G_r + G_{wi} + G_{ni} + G_{wm} + G_{nm} = 1. \tag{43}$$

If the system is linear such that Equation (39) is valid, we may also say that

$$G_p = G_{wi} + G_{ni} + G_{wm} + G_{nm}, \tag{44}$$

$$G_w = G_{wi} + G_{wm}, \text{ and} \tag{45}$$

$$G_n = G_{ni} + G_{nm}. \tag{46}$$

Since individual connectednesses are a function of the appropriate phase fraction and an exponent as defined in Glover (2009) as

$$G_j = \chi_j^{m_i} = \chi_j \cdot \chi_j^{m_j - 1}. \tag{47}$$

It is immediately obvious that each connectedness contains information both of the volume fraction of each phase χ_j and its connectivity $\chi_j^{m_j-1}$ (or tortuosity$\chi_j^{1-m_j}$).

Figure 2 shows an example of how the connectednesses of the mobile non-wetting phase and gas phases vary with wetting phase saturation for five values of both m_{nm} and m_{wm} (1, 1.5, 2, 2.5 and 3), and for which $m_{nm}=1$ *and* $m_{wm}= 1$ both provide a linear relationship. As with the cementation from the conventional Archie's Law, larger values imply lower connectednesses. The generic similarity of the curves in this figure to relatively permeability curves will be noted.

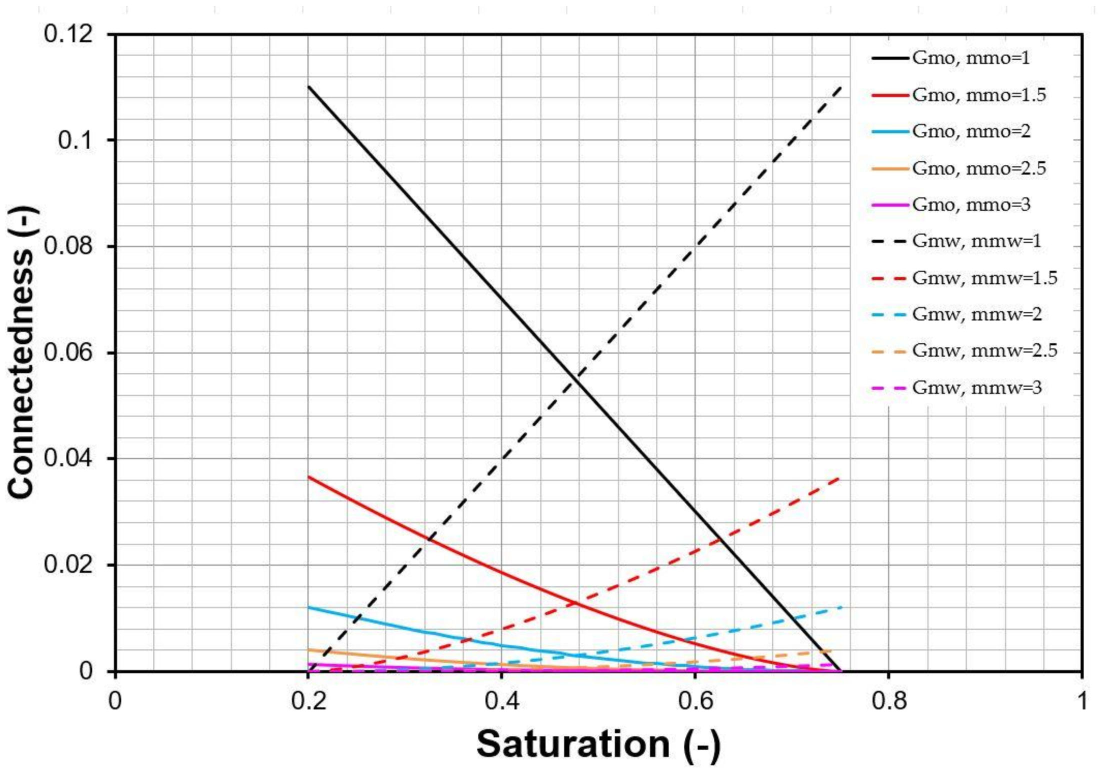


Figure 2. Variation of the connectedness of the mobile non-wetting phase and mobile wetting phases as a function of wetting phase saturation for a range of values of m_{nm} and m_{wm} given in the legend, and for S_{wi} = 0.2 (irreducible wetting phase), S_{nr} = 0.25 (residual nonwetting phase), and ϕ = 0.2.

7. CONNECTEDNESS AND RELATIVE PERMEABILITY

Let us write Equation (37) for n phases, where $1<j<n$

$$\kappa_j = G_j \kappa_{tube}. \tag{48}$$

We can generate equations for the effective permeability of both the mobile wetting and the mobile nonwetting phases

$$\kappa_{wm} = G_{wm} \kappa_{tube}, \text{ and} \tag{49}$$

$$\kappa_{nm} = G_{nm} \kappa_{tube}. \tag{50}$$

We should bear in mind that both the effective permeabilities and their respective connectednesses are a function of the volume fractions (and saturations) of each phase, respectively.

Let us define a reference effective permeability to be the effective permeability to the nonwetting phase at S_{wi} (irreducible wetting phase saturation), which is fairly common within the oil industry. We can rewrite Equations (49) and (50) in terms of relative permeabilities using the definition given by Equation (1)

$$k_{rw} = \frac{G_{wm}(\chi_w)}{G_{nm}(\chi_w=\chi_{wi})}, \text{ and} \tag{51}$$

$$k_{rnw} = \frac{G_{nm}(\chi_n)}{G_{nm}(\chi_w=\chi_{wi})}. \tag{52}$$

The sum of these relative permeabilities is

$$k_{rw} + k_{rnw} = \frac{G_{wm}(\chi_w) + G_{nm}(\chi_n)}{G_{nm}(\chi_w=\chi_{iw})} \quad . \tag{53}$$

or

$$k_{rw} + k_{rnw} = \frac{\chi_{wm}^{m_w} + \chi_{nm}^{m_n}}{\chi_{nm}^{m_n}|_{S_{wi}}}. \tag{54}$$

Let us assume that we can apply the conservation of connectedness that was developed by Glover (2010) for the generalised Archie's law, and we apply it within the reference frame of the whole porous medium sample, i.e., for all the 5 phases in the porous medium, we get

$$G_r + G_{wi} + G_{ni} + G_{wm} + G_{nm} = 1. \tag{55}$$

or

$$\chi_r^{m_r} + \chi_{wi}^{m_{wi}} + \chi_{ni}^{m_{ni}} + \chi_{wm}^{m_{wm}} + \chi_{nm}^{m_{nm}} = 1 \tag{56}$$

Now certain of these phases are immobile, but that is not the same as saying that their connectednesses are zero. Each phase present has a finite non-zero value. If that phase is immobile, it is the value of κ_{tube} that appears in Equation (37) that is zero.

The numerator of Equation (54) can be written as

$$1 - \chi_r^{m_r} - \chi_{wi}^{m_{wi}} - \chi_{ni}^{m_{ni}}. \tag{57}$$

The denominator of Equation (54) can also be written as

$$1 - \chi_r^{m_r} - \chi_{wi}^{m_{wi}} - \chi_{ni}^{m_{ni}} \tag{58}$$

and hence Equation (54) is equal to unity, which does not agree with the general experimental observation that the sum of the relative permeabilities of a 2 fluid system is less than unity (except at the point where the reference effective permeability is taken). Is this a failing of the application of Connectedness Theory to permeability? Does it mean that the conservation of connectedness according to Glover (2010),

$$\textstyle\sum_j G_j = 1 \tag{59}$$

is not followed for fluid permeability?

Perhaps the fault lies in the phase definitions. We have defined 5 phases, the porous medium, two immobile fluid phases and two mobile fluid phases. However, if there is an irreducible wetting phase saturation of say 0.2 that does not necessarily mean that this wetting phase does not contribute in some way to flow when the wetting phase saturation is greater than 0.2. This effect is not accounted for in the analysis from Equations (55) to (58).

Equation (54) should be rewritten as a

$$k_{rw} + k_{rnw} = \frac{(\chi_{wi}+\chi_{wm})^{m_w}+(\chi_{ni}+\chi_{nm})^{m_n}}{(\chi_{ni}+\chi_{nm})^{m_n}|_{S_{wi}}} \qquad \textbf{(60)}$$

The exponents now express the connectivity of the phase whether it is mobile or not. Equation (56) can be rewritten as

$$\chi_r^{m_r} + (\chi_{wi} + \chi_{wm})^{m_w} + (\chi_{ni} + \chi_{nm})^{m_n} = 1 \qquad \textbf{(61)}$$

and we can also write

$$\chi_r + (\chi_{wi} + \chi_{wm}) + (\chi_{ni} + \chi_{nm}) = 1. \qquad \textbf{(62)}$$

Using Equations (61) and (62) in Equation (60) gives

$$k_{rw} + k_{rnw} = \frac{1-\chi_r^{m_r}}{(1-\chi_r-\chi_{wi})^{m_{ni}}}. \qquad \textbf{(63)}$$

where m_{ni} is the exponent in the equation that defines the connectedness of the non-wetting phase at irreducible wetting phase saturation. This can be rewritten in terms of connectednesses as

$$k_{rw} + k_{rnw} = \frac{G_{pores}}{(1-\chi_r-\chi_{wi})^{m_{ni}}}. \qquad \textbf{(64)}$$

Since G_{pores} is the measure of the connectedness of the pore space, and the denominator can be interpreted as the connectedness of the mobile fluids, Equation (64) will be always less than unity for valid combinations of its parameters.

If we imagine the viscosity of one of the two fluids phases approaches infinity (let's say the wetting phase), that phase effectively becomes a part of the solid matrix and the permeability of that fluid phase becomes zero. The relative permeability of the other phase (here non-wetting phase) becomes unity (providing we are in a two phase system). This is the same as saying the ratio between the permeability of the porous medium where the matrix now consists of the old matrix and the immobile fluid, and the single phase permeability of the fully saturated porous medium. Applying this to Equation (64) leads to the left-hand side becoming equal to $k_{rnw} = 1$.

The numerator of the right-hand side, previously G_{pores} is now a new value because the porosity has lost that part of it which was occupied by the mobile and immobile wetting phase, it having become effectively part of the matrix. Designating the new connectedness of the pores as G'_{pores}, both values can be written down as

$$G_{pores} = (\chi_{nm} + \chi_{ni} + \chi_{wm} + \chi_{wi}\chi)^{m_{pores}} = (1 - \chi_r)^{m_{pores}} \quad \textbf{(65)}$$

$$G'_{pores} = (\chi_{nm} + \chi_{ni})^{m'_{pores}} = (1 - \chi_r - \chi_{wm} - \chi_{wi})^{m'_{pores}} \quad \textbf{(66)}$$

However, only the mobile portion of the non-wetting phase will take part in flow allowing Equation (66) to be modified to read

$$G''_{pores} = (\chi_{nm})^{m''_{pores}} = (1 - \chi_r - \chi_{wm} - \chi_{wi} - \chi_{ni})^{m''_{pores}} \quad \textbf{(67)}$$

The transformation has removed the phase fractions associated with the wetting phase, and the connectedness of the new pore space is less than that for all the old pore space because the space that was previously apportioned to contain wetting phase is now solid. This implies that $G'_{pores} < G_{pores}$ and that m_{pores} is generally not equal to m'_{pores} or m''_{pores} . The value of m'_{pores} may be larger or smaller than the value of m_{pores} depending upon whether the new pathway through the matrix has a smaller or a greater connectivity, respectively. Commonly, one would expect the process of transforming what was pore space saturated with wetting phase into matrix would result in a lower connectivity of the remaining non-wetting phase pathways, and pores saturated with mobile non-wetting phase by only accounting for part of the non-wetting phase saturation would have less connectivity than the combined non-wetting phase. Consequently, one would expect $m''_{pores} > m'_{pores} > m_{pores}$.

The denominator of the right-hand side of Equation (64) is also altered by the transformation. While it was previously

$$(1 - \chi_r - \chi_{wi})^{m_{ni}} \tag{68}$$

it becomes

$$(1 - \chi_r - \chi_{wi} - \chi_{wm} - \chi_{ni})^{m'_{om}} = (\chi_{nm})^{m'_{nm}}\,, \tag{69}$$

remembering that m_{ni} was previously defined as the exponent in the equation that defines the connectedness of the non-wetting phase at irreducible wetting phase saturation. Now, after transformation only mobile non-wetting phase is present as a mobile phase, so this exponent can be replaced with m'_{nm}, designating the exponent in the equation that defines the connectedness of the mobile nonwetting phase when that is the only phase present.

It is worth noting that this definition is the same as the definition for m''_{pores}, hence $m''_{pores} \equiv m'_{om}$. Consequently, using this result, Equation (67) and Equation (69) allows us to rewrite the right-hand side of Equation (64) to take account of the transformation, which gives

$$\frac{\chi_{nm}^{m''_{pores}}}{\chi_{nm}^{m'_{nm}}} = \frac{\chi_{nm}^{m''_{pores}}}{\chi_{nm}^{m''_{pores}}} = 1\,, \tag{70}$$

which is equal to the unity, agreeing with the left-hand side of the transformed version of Equation (64).

It should be noted in the material above that the reference permeability that was used was that of the nonwetting phase at the irreducible saturation of the wetting phase. This is common in practical usage, but not universal. A different analysis would be needed if relative permeabilities were defined relative to a different reference permeability, but the analysis would stay broadly the same. This might be useful in those circumstances when nonwetting permeability made at the irreducible wetting saturation might be unreliable, which is the case in some unconsolidated systems.

8. COMPARISON OF RELATIVE PERMEABILITY MODELS

A comparison between the results of the new connectedness approach to calculating relative permeability curves with the results of the Brookes-Corey-Purcel (BCP) approach.

In this comparison, the BCP relative permeability curves for non-wetting phase and wetting phase were first calculated using the equations in (12) as a function of wetting phase saturation values for 6 values of λ from 0.5 to 5. The model also varied porosity ϕ for the values 0.05, 0.10, 0.15, 0.20 and 0.25, for irreduscible wetting phase saturation S_{wi} for the values 0.05, 0.10, 0.15, 0.20 and 0.25 and residual non-wetting phase S_{or} for the values 0.10, 0.15, 0.2. 0.25 and 0.30, making 625 independent measurements for each of the non-wetting phase and wetting phase relative permeability curves. Figure 3a shows the results for the 6 values of λ, for $\phi = 0.20$, $S_{wi} = 0.20$, and $S_{nr} = 0.25$. Variation of the porosity, irreducible wetting phase saturation and residual non-wetting phase saturation only resulted in expected changes in the range of wetting phase saturation over which it was possible to calculate relative permeability curves, as expected.

The connectedness calculations were carried out in the following steps.

Step 1. Calculating the phase fractions for each of the five phases using the values of porosity, irreducible wetting phase saturation and residual non-wetting phase saturation (as for the conventional approach described above and for the same parameter values). We assumed that the immobile phase fractions remained constant and independent of wetting phase saturation (i.e., not as shown in the generalised diagram in Figure 1).

Step 2. Calculating the connectednesses for each phase as defined by Glover (2009) and used in the generalized Archie's law (Glover, 2010). This requires phase exponents for each phase, some of which were imposed and some of which were calculated. The exponent of the pore space m_p was set to a range of values between 0.05 and 0.30, but was found not to influence the shape of the calculated relative permeability curves. This insensitivity is not surprising as the porosity should not influence the relative permeability curves precisely because they are relative curves. Subsequently, a value of $m_p = 2$ was used in all measurements. The exponents of the immobile wetting phase (m_{wi}) and immobile non-wetting phase (m_{ni}) phases were varied between 1 and 2, and again resulted in no change in the modelled curves. The reason for this is less obvious and is here related to our assumption that χ_{wi} and χ_{ni} are independent of S_w. If χ_{wi} and χ_{ni} are a function of S_w, implementation of these parameters would be expected to produce subtle changes in both relative permeability curves as well as making the exponents of the mobile phases (m_{wm} and m_{nm}) each a function of S_w. As there was no functional dependence on m_{wm} and m_{nm}, they were set, rather arbitrarily to $m_{wm} = 1.25$, and $m_{nm} = 1.4$ for all subsequent tests. In this work the exponents of the mobile phases (m_{wm} and m_{nm}) were assumed not to be a function of S_w. Values of m_{wm} and m_{nm} were varied, taking the 5 integer values between and including 1 and 5. The exponent of the porous medium matrix m_r was calculated as the final missing parameter, using the conservation of connectedness law given by Glover (2010).

Step 3. The relative permeabilities for non-wetting phase and wetting phase were calculated for each value of wetting phase saturation using Equations (42) and (43), and no recourse to the use of a reduced effective wetting phase saturation. This approach is apparently completely independent of the methodology used in calculating the BCP relative permeability curves.

Figure 3 shows the resulting BCP and connectedness relative permeability curves. The first part of the figure (Figure 3a) shows the implementation of the BCP model for a number of different values of the pore size distribution index λ, from 0.5 to 5, which more than covers the usual range found in most porous media. Consequently, this plot shows the range of relative permeability curves for non-wetting phase and wetting phase that might be expected to occur. Figure 3b shows the implementation of the new connectedness approach to calculating relative permeability. This approach does not use the pore size distribution index λ, but uses the m_{wm} and m_{nm} exponents which describe how connected the mobile non-wetting phase and wetting phases are. In this implementation both the exponent for the mobile non-wetting phase m_{nm} and that for the mobile wetting phase m_{wm} take integer values from 1 to 5, inclusively. These exponents can be interpreted in a similar way to that of Archie's cementation exponent, from which they are ultimately derived. Consequently, a value close to unity indicates very direct connection of the phase, while values closer to 2 would be typical of those degrees of connection found in clastic porous media, and higher values would indicate less of a degree of connection, such as you might find in a carbonate that had undergone considerable diagenesis. Consequently, these values also represent the reasonable range of these parameters that you might be expected to encounter in porous media.

Comparison of the curves in Figures 3a and 3b show the calculated relative permeability curves to have a very similar shape for all the different parameters. Furthermore, curves which have lower lambda value also have higher exponent values and vice versa, which is consistent with higher lambda values being associated with more uniform pore size distributions which would give rise to more direct connection of the pores.

However, as shown in Figure 3c, the imposition of modelling with $\lambda = m_{wm} = m_{nm}$ does not result in the same relative permeability curves. At first sight this should not be surprising because the BCP model is calculated in a very different way to the connectedness approach to calculating relative permeabilities. It should be remembered that the pore size distribution index λ that is found by fitting the BCP model to data rather than used in a predictive sense. It would be a step forward if we could find the underlying meaning of the pore size distribution index λ.

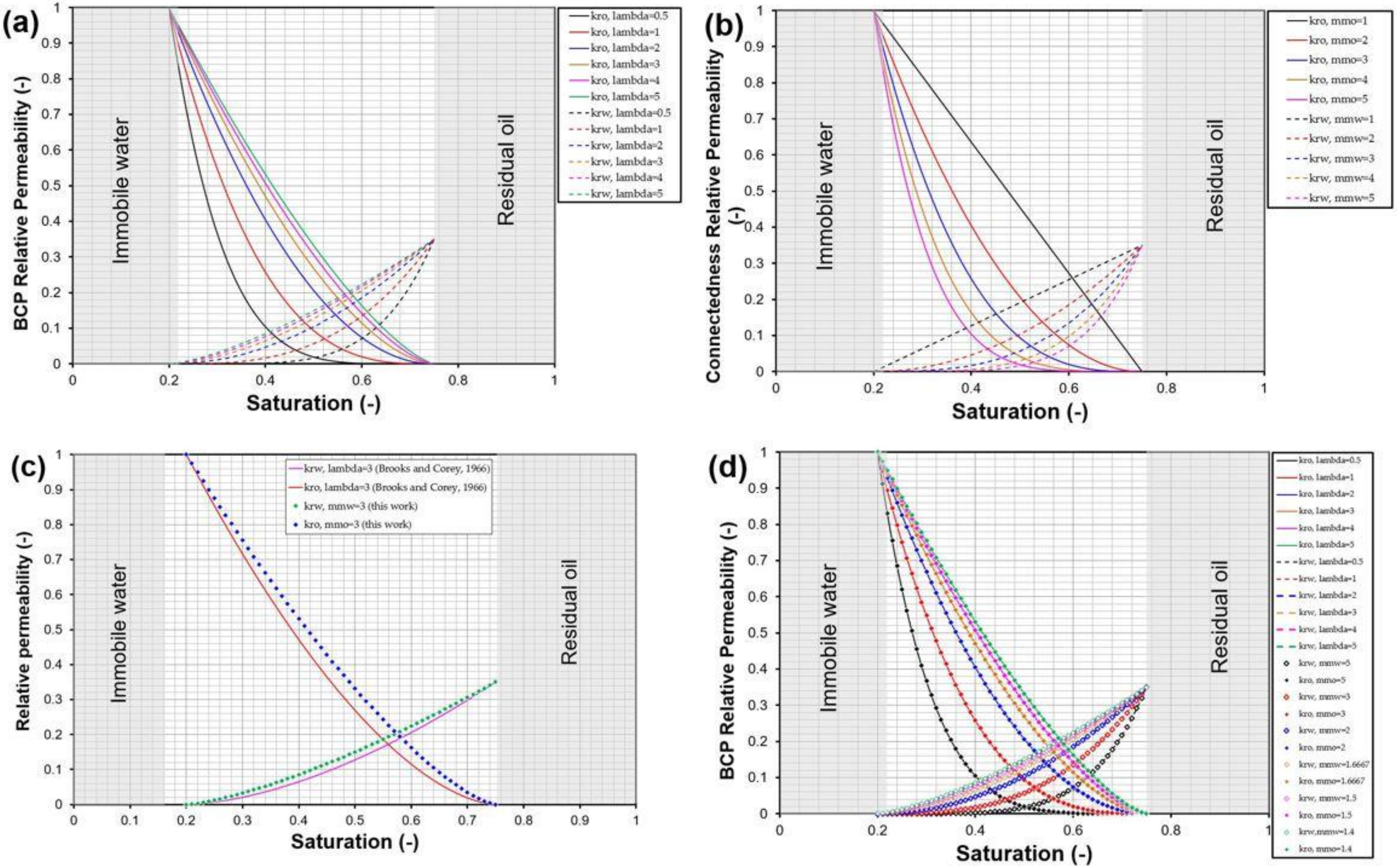

(a)
BCP Relative Permeability (-)
Saturation (-)
Immobile water
Residual oil
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1
kro, lambda=0.5
kro, lambda=1
kro, lambda=2
kro, lambda=3
kro, lambda=4
kro, lambda=5
krw, lambda=0.5
krw, lambda=1
krw, lambda=2
krw, lambda=3
krw, lambda=4
krw, lambda=5
(b)
Connectedness Relative Permeability (-)
Saturation (-)
Immobile water
Residual oil
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1
kro, mmo=1
kro, mmo=2
kro, mmo=3
kro, mmo=4
kro, mmo=5
krw, mmw=1
krw, mmw=2
krw, mmw=3
krw, mmw=4
krw, mmw=5
(c)
Relative permeability (-)
Saturation (-)
Immobile water
Residual oil
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
krw, lambda=3 (Brooks and Corey, 1966)
kro, lambda=3 (Brooks and Corey, 1966)
krw, mmw=3 (this work)
kro, mmo=3 (this work)
(d)
BCP Relative Permeability (-)
Saturation (-)
Immobile water
Residual oil
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1
kro, lambda=0.5
kro, lambda=1
kro, lambda=2
kro, lambda=3
kro, lambda=4
kro, lambda=5
krw, lambda=0.5
krw, lambda=1
krw, lambda=2
krw, lambda=3
krw, lambda=4
krw, lambda=5
krw, mmw=5
kro, mmo=5
krw, mmw=3
kro, mmo=3
krw, mmw=2
kro, mmo=2
krw, mmw=1.6667
kro, mmo=1.6667
krw, mmw=1.5
kro, mmo=1.5
krw,mmw=1.4
kro, mmo=1.4

Figure 3. (a) Relative permeability modelled with the Brooks – Corey – Purcell (BCP) (Li and Horne, 2004) model for a range of values of l from 0.5 to 5 (please see legend). **(b)** Relative permeability modelled with the new connectedness approach developed in this work for a range of mobile non-wetting phase and wetting phase exponents, each taking values from 1 to 5. **(c)** An example of the comparison between relative permeability values for non-wetting phase and wetting phase modelled with both the BCP and new approaches for both $\lambda = 3$, $m_{wm} = 3$ and $m_{nm} = 3$ showing their similarity but not exact match. **(d)** Relative permeabilities using the BCP approach (lines) and the new connectedness approach (symbols) showing their identity, together with the values of λ, m_{wm} and m_{nm} required for the matching. In all panels $S_{wi} = 0.2$, $S_{nr} = 0.25$, $\phi = 2$ and $k_{rw@Snr} = 0.35$.

Figure 3d shows the same BCP data as in Figure 3a as curves. The symbols, which follow an identical curve as each of the lines, are generated from the new connectedness approach to calculating each of the relative permeabilities. The values of m_{wm} and m_{nm} which allow this to occur conform to the equation

$$m_{wm} = m_{nm} = \frac{2+\lambda}{\lambda} \, , \tag{71}$$

which will be recognised as the exponents occuring in the BCP model. Equation (71) is not a coincidence, and indicates a deeper link between the BCP model and the fundamental geometrical basis of the Connectedness Theory in three dimensions. It also represents a validation of the BCP model.

It should also be noted that throughout this work we have taken $m_{wm} = m_{nm}$, but this need not necessarily be the case. The generalised treatment does not require $m_{wm} = m_{nm}$, and it follows that there may be a separate value of λ for each mobile phase, i.e., λ_{wm} and λ_{nm}.

The implication of Equation (65) is that the value of λ is defined solely on the basis of the same geometrical considerations that define the exponents m_{wm} and m_{nm}, and that both parameters are effectively structural parameters which are defined by the distribution of the pore space.

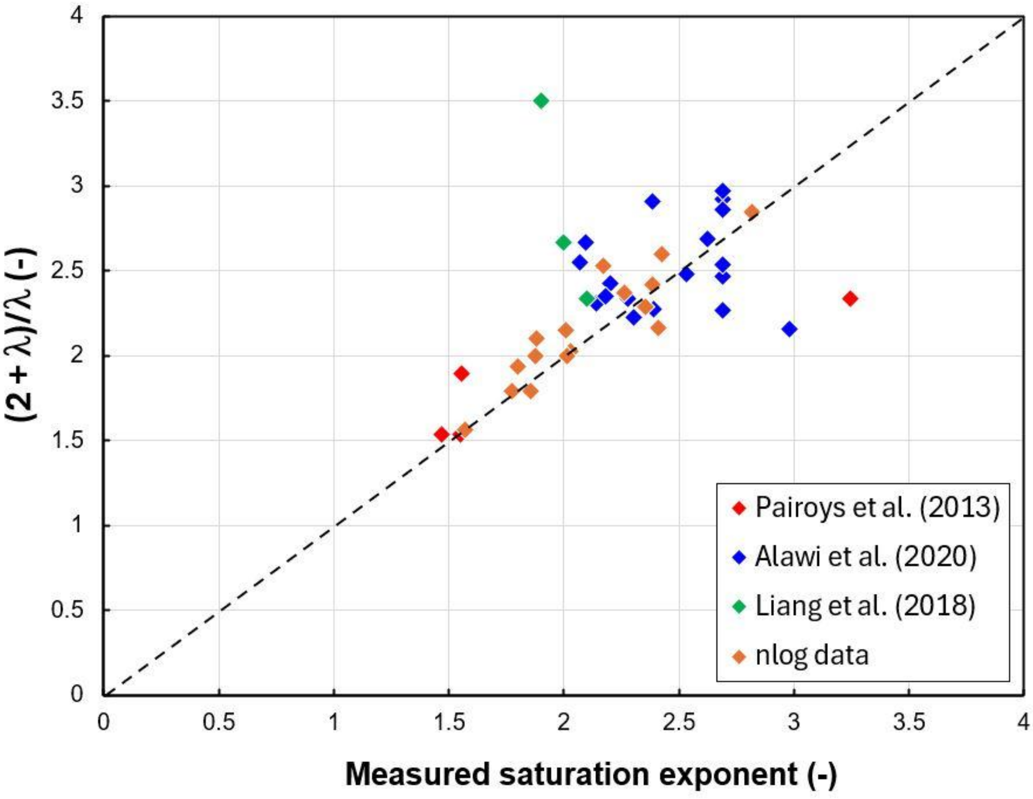


Figure 4. The variation of the parameter $(2+\lambda)/\lambda$ using values of λ from fits to experimental k_{rw} data as a function of measured saturation exponent, using data (56 points) from the Dutch Oil and Gas Portal nlog (https://www.nlog.nl/en) as well as Alawi et al. (2020), Pairoys et al. (2013), and Liang et al. (2018).

Equation (71) is testable using experimental data. The value of λ for the wetting phase from experimental relative permeability measurements can be compared with the value of the saturation exponent as a function of wetting phase saturation during drainage experiments. Figure 4 shows the result of this comparison for data from the literature. The data was collated from Alawi et al. (2020), Pairoys et al. (2013), and Liang et al. (2018), as well as from the Dutch Oil and Gas Portal nlog (https://www.nlog.nl/en), and either used explicitly quoted data or required the fitting of curves to relative permeability data. The match is not perfect, but sufficiently good to validate the link between λ, m_{wm} (and by inference m_{nm}). It should be noted that Figure 4 is essentially comparing measurements of two different types (electrical and mass flow), and in the case of Pairoys et al. (2013) used three different methods of obtaining the Brooks-Corey coefficient λ. There is a remarkable agreement considering that the measurements are derived from disparate sources.

9. VARIABLE EXPONENT VALUES

During this work we have assumed that phase exponents are not a function of fluid saturations. This is clearly an oversimplification. Indeed, one would expect the exponents, which it will be remembered express the connectivity of a phase, would increase or decrease according to how the connectivity of the phase changes as changing phase fraction opens or closes flow pathways.

Consequently, the decrease in non-wetting phase flow pathways would be expected during imbibition as the non-wetting phase is replaced by the imbibed wetting phase. This would be associated with an increase in m_{nm}, whereas the increasingly connected wetting phase would be associated with an accompanying decrease in m_{wm}. In general m_{wi} and m_{ni} are also a function of fluid saturation, though it is possible to assume that the dependance is normally weak. In most circumstances m_r and m_p will remain constant, although they may also be a function of fluid saturation if significant dissolution or precipitation was occuring in the porous medium and depended itself on wetting or nonwetting saturation.

Imposing Equation (42) from the generalised Archie's law (Glover, 2010), and assuming m_{iw}, m_{ni}, m_r and m_p remain constant we obtain

$$G_{nm} + G_{wm} = 1, \tag{72}$$

or

$$\chi_{nm}^{m_{nm}} + \chi_{wm}^{m_{wm}} = 1, \tag{73}$$

which can be rearranged to express each phase exponent

$$m_{nm} = \frac{log\left(1 - \chi_{wm}^{m_{wm}}\right)}{log(\chi_{nm})} \text{ , and} \tag{74}$$

$$m_{wm} = \frac{log\left(1 - \chi_{nm}^{m_{nm}}\right)}{log(\chi_{wm})} \text{ .} \tag{75}$$

This result, which is the analogue of that found in the modification of Archie's law for two conducting phases, is this found to have relevance to fluid flow. It also opens the possibility that it may be possible, at least in principle, that k_{rnw} might be calculable from k_{rw} and *vice versa*, providing the earlier assumptions are justified.

Figure 5 shows a plot where the k_{rnw} and k_{rw} values have been generated using the BCP model, with $\lambda = 4$ for k_{rnw} and $\lambda = 3$ for k_{rw}. There is no reason why the λ values for k_{rnw} and k_{rw} should have to be identical. The figure demonstrates a number of effects when allowing the values of the exponents to vary with wetting phase saturation. Given that there are an infinite number of functional forms for the 4 phase exponents (m_{ni}, m_{wi}, m_{nm} and m_{wm}), physical constraints ensured that sensible solutions exist.

For example, here we constrained $m_{wi} = 1.5$ and not to vary with wetting phase or non-wetting phase saturation. Assuming that the sum of the phase fractions of the immobile phases is constant, which is reasonable, this implied that the immobile wetting phase exponent values m_{wi} varied between 1.75 at S_{wi} and 1.21 at S_{nr}, which is consistent with the immobile wetting phase being in better communication with the mobile wetting phase as S_{nr} is approached. This is likely to be the case in reality, but with a smaller range of change, because in reality the m_{ni} values which we constrained to a constant value would also vary with changing wetting phase saturation such that they had higher values at and approaching S_{nr} and smaller values at and approaching S_{wi} for analogous reasons to the controls on m_{wi}.

The four connectednesses (G_{ni}, G_{wi}, G_{nm} and G_{wm}) sum to 0.04 at all values of S_w (and hence S_{nw}) as a result of Equation (42) and where the value arises from the calculation of the connectedness of the pores using a porosity of 20% and a cementation exponent $m_p = 2$, which is typical of conventional Archie's law cementation exponents describing the geometrical arrangement of pores in a sandstone.

Now, instead of fixing the mobile phase exponents to be constant, we have allowed them to vary linearly, according to the equation for each mobile phase

$$m = m_{@S_{wi}} + \frac{(m_{@S_{nr}} - m_{@S_{wi}})}{(1 - S_{wi} - S_{nr})} S_w^*, \tag{76}$$

which allows the phase exponent to increase or decrease linearly depending on the desired values at each of the end points ($S_w^* = 0$ to 1). Figure 5 shows the connectednesses G_{nm} and G_{wm} resulting from the modelling.

In the case of the connectedness of mobile non-wetting phase (G_{nm}) we obtained a perfect fit to the BCP k_{rnw} curve (λ=3 giving m_{nm} = 1.6667), as expected from similar results shown earlier in this paper. However, Figure 5 shows the connectedness calculated using $m_{at\ at\ Swi}$ = 1.6667 and $m_{at\ at\ Snr}$ = 1.7667. This seemingly small linear increase in the exponent is enough to reduce progressively the connectedness in Figure 5 and results in the calculated k_{rnw} using the connectedness method (blue diamonds) being progressively smaller that the BCP k_{rnw} (red curve) which used a fixed exponent. The inference here is that changing the phase exponent as a function of wetting phase saturation not only produces valid k_{rnw} curves, but such a change would be expected from considerations of phase connectedness as S_w changes.

In the case of k_{rw}, we have gone a step further. Previously, we have implemented a multiplier to ensure that the endpoint of the k_{rw} curve is that encountered during experiments. It was set to k_{rw} = 0.35 at S_{nr} for previous results in this paper. Here, we adjust the values of $m_{@Swi}$ and $m_{@Snr}$ for the wetting phase in order to match the BCP k_{rw} curve, which does use the $k_{rw@Sor}$ multiplier. We found that we could obtain a reasonable match through the full range of wetting phase saturations from S_{wi} to S_{nr} using the values $m_{@Swi}$ = 1.8 and $m_{@Snr}$ = 1.99, noting the sensitivity of the match to the parameters at S_{nr} (1.99 matched much better than 2.00). The inference here is that, as with the k_{rnw} test reported previously, variable phase exponents can be used to provide matching k_{rw} curves effectively (here to another model, but it would be equally the case for fitting to good experimental data). Such a fit to experimental data could be used to generate the phase exponents. Furthermore, as no multiplier was needed for the process, it is clear that the information required to fit the k_{rw} curve is contained entirely within the geometrically-defined parameters of the connectedness model, and does not need an *ad hoc* multiplier to allow for fitting.

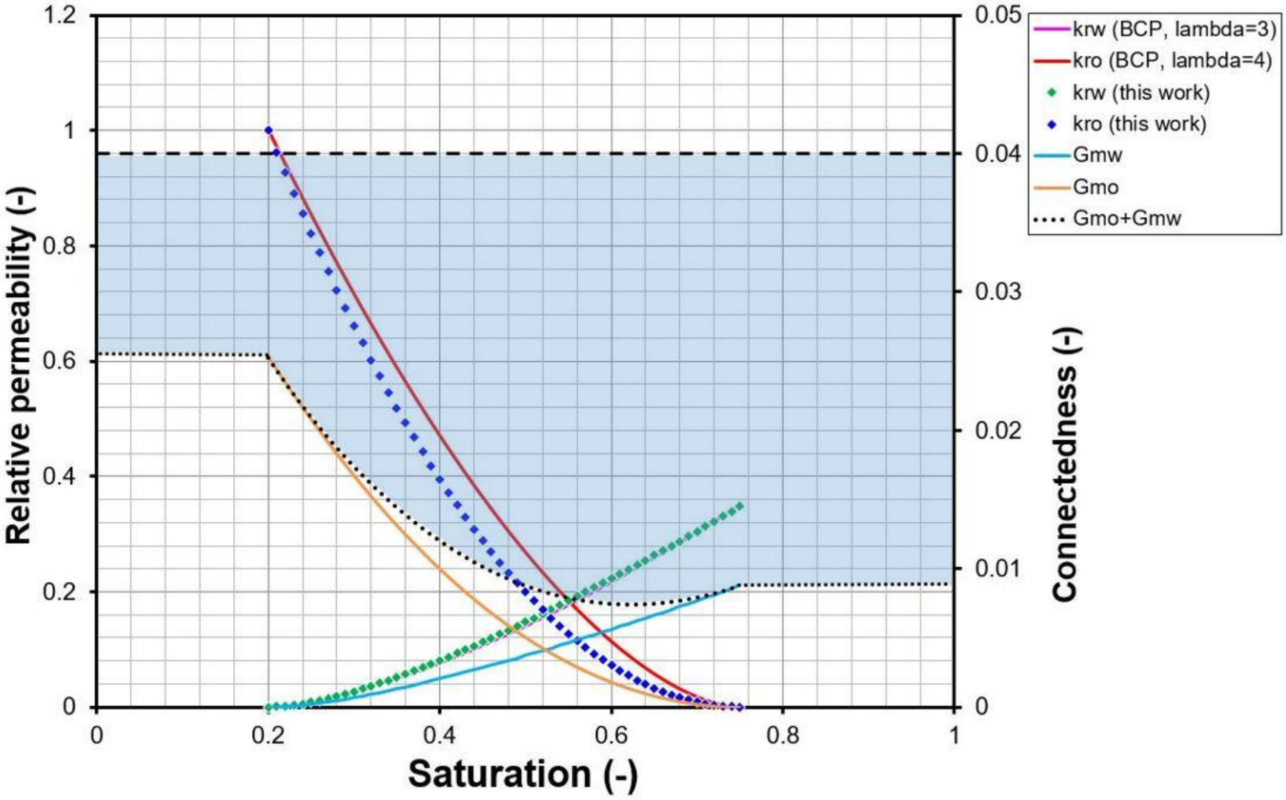


Figure 5. Relative permeability and connectedness curves for the conventional BCP model and for the connectedness model presented in this work using variable phase fractions which are a function of wetting phase saturation.

Finally, the horizontal dashed line in Figure 5 represents the connectedness of the pore space, $G_p = 0.04$. All of the mobile and immobile connectednesses must sum to this value according to the conservation of connectedness law (Glover, 2010). The dotted horizontal line represents the maximum value of the sum of the connectednesses of the mobile phases (non-wetting phase and wetting phase), inferring that the space in between the two lines, which is shaded, represents the value of the sum of the connectedness of the immobile phases (non-wetting phase and wetting phase). This is considerable, and generally becomes larger as the wetting phase progressively establishes itself in the pore space.

10. CONCLUSIONS

In this work we have recognised a fundamental similarity between electrical transport and fluid mass transport in porous media, whereby both are controlled by the amount of a conducting or flowable phase and how well it is connected. In reviewing existing relative permeability models, we recognise the fundamental similarities in their structures. We have attempted to apply the connectedness approach developed during work on the generalised Archie's law (Glover, 2010). This is a theoretical problem where the overall conductivity of a porous medium may have contributions from multiple and different conducting phases which interact with each other. In the case of relative permeabilities, there is a theoretical problem where the overall flow of each phase also may depend upon other flowing phases.

In this work, we use Connectedness Theory to understand relative permeabilities, which has allowed the formalisation of effective and relative permeability in terms of phase connectedness. Connectedness Theory can be formulated for any number of phases, and each can be mobile or not. Consequently, the approach outlined in this work can be extended to a three-phase relative permeability system easily.

In this work, we focus on the well-known and wetting/non-wetting two phase relative permeability system, though it is expected that many of its characteristics will be similar for a three-phase system. In particular, we derive the important result that the effective permeability is equal to the connectedness *G* multiplied by the permeability of a capillary tube, which is completely analogous to the equation for calculating the conductivity of a porous medium when completely saturated with a single phase. The relative permeabilities for each fluid phase in the porous medium can then be expressed solely in terms of a ratio of connectednesses, which describe the distribution of flow pathways.

Comparison of the relative permeabilities calculated with the new connectedness-based model provides identical curves to the Brooks-Corey-Purcell (BCP) model providing that the phase exponents defining each mobile phase are equivalent to $(2+\lambda)/\lambda$. This amounts to a theoretical validation of the BCP model. Comparison of this last value for the wetting phase with the saturation exponent from electrical data, which would be expected to also measure how well connected that same phase is for electrical flow.

Although commonly regarded as constant, the values of l for the wetting phase and the non-wetting phase neither have to be the same or be constant. Indeed, it would be expected that these values for wetting phase and non-wetting phase would be different and vary according to the amount of each phase present in the porous medium at a particular fluid saturation state. The connectedness approach has allowed us to examine variable phase exponents, indicating that in principle the relative permeability of one mobile phase should be calculable from the other and *vice versa*, and that the end point for the wetting phase relative permeability curve need not be imposed in a model, but arises naturally from the interplay of the mobile and immobile phase exponents.

This work examines some of the geometrical theory underlying the behaviour of relative permeabilities, putting the controlling parameters of those models on a theoretical basis and allowing electrical parameters to inform relative permeability measurements and *vice versa*. The formulation depends only on interacting geometrical considerations and is capable of taking hysteresis effects into consideration when phase connectivity changes as a function of saturation history because it utilises the connectedness of phases directly.

This paper has been purely theoretical and scope, and is consistent with existing empirical models and the data to which they have been fitted. However, it is hoped that the approach will be supported by new experimental measurements in the future. It is unlikely that existing standard measurements of relative permeability will be sufficient to take account of all the variables in the model and sufficiently validate it. Such measurements may be carried out on either rocks or physical models that imitate rocks using micro-CT or nano-CT measurements during steady-state flooding, or by modelling on digital rock models using the Lattice Boltzmann approaches. Whatever the approach, it is important that measurements or modelling are done in such a way that all the parameters of the connectedness model can be taken into account and fully tested.

Connectedness Theory puts the continuity of phases in the position of prime importance. Consequently, we predict that the development of an unconnected and dead-end pathways will have less importance than those which are continuous within the fluid, and therefore represent a latent connectedness which can be opened up by small changes in the matrix connectedness, or improving the permeability of the matrix connectedness as shown so elegantly in Ferreira and Nick (2024).

This paper also restricts itself to two phase flow. The Connectedness Theory approach can, of course, be extended to any number of phases, the most important of which is its application and three phase flow, which represents a possible extension of his work in future papers.

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Data Availability

As this paper is primarily theoretical, it has generated very minimal data. The data used in the paper is cited from other papers. A very small dataset was gathered by the authors for one of the figures and this is available from the corresponding author on reasonable request.