**Article:**

Bowman-Scargill, Chris orcid.org/0000-0001-6046-8930, Hazi, Amit, De Visscher, Maud et al. (1 more author) (2025) The anti-spherical Hecke categories for Hermitian symmetric pairs. Advances in Mathematics. 110501. ISSN: 0001-8708

https://doi.org/10.1016/j.aim.2025.110501

# THE ANTI-SPHERICAL HECKE CATEGORIES FOR HERMITIAN SYMMETRIC PAIRS

CHRIS BOWMAN, MAUD DE VISSCHER, AMIT HAZI, AND EMILY NORTON

Abstract. We calculate the $p$-Kazhdan–Lusztig polynomials for Hermitian symmetric pairs and prove that the corresponding anti-spherical Hecke categories categories are standard Koszul. We prove that the combinatorial invariance conjecture can be lifted to the level of graded Morita equivalences between subquotients of these Hecke categories.

## Introduction

Anti-spherical Hecke categories first rose to mathematical celebrity as the centrepiece of the proof of the Kazhdan–Lusztig positivity conjecture and its anti-spherical counterpart [EW14, LW22]. Understanding the $p$-Kazhdan–Lusztig polynomials of these categories subsumes the problem of determining prime divisors of Fibonacci numbers [Wil17]; this is a notoriously difficult problem in number theory, for which a combinatorial solution is highly unlikely. As $p \to \infty$ the situation simplifies and we encounter the classical Kazhdan–Lusztig polynomials; these are important combinatorial objects which can be calculated via a recursive algorithm. We seek to understand this gulf between the combinatorial and non-combinatorial realms within $p$-Kazhdan–Lusztig theory.

Over fields of infinite characteristic, the families of anti-spherical Kazhdan–Lusztig polynomials which are best understood combinatorially are those for Hermitian symmetric pairs, $P \leqslant W$. These polynomials admit inexplicably simple combinatorial formulae in terms of Dyck paths or Temperley–Lieb diagrams [Boe88, Bre07, Bre09, BS11a, ES16a]. Their importance derives from their universality: these polynomials control the structure of parabolic Verma modules for Lie algebras [ES87]; algebraic supergroups [Bru03]; Khovanov arc algebras [BS12b, ES17, BW, BDHS24, BDHS, BDD$^+$a, BDD$^+$b]; Brauer and walled Brauer algebras [BS12a, ES16b, Mar15, CD11]; categories $\mathcal{O}$ for Grassmannians [LS81, BS11b]; topological and algebraic Springer fibres, Slodowy slices, and $W$-algebras [ES16a].

**Koszulity and $p$-Kazhdan–Lusztig theory.** The first main result of this paper extends our understanding of the Kazhdan–Lusztig theory for Hermitian symmetric pairs to all fields.

**Theorem A.** *Let $\Bbbk$ be a field of characteristic $p \geqslant 0$ and $(W, P)$ a Hermitian symmetric pair. The Hecke category, $\mathcal{H}_{(W,P)}$, is standard Koszul (in the sense of* [ADL03, Introduction]*) and the $p$-Kazhdan–Lusztig polynomials are $p$-independent (and hence admit closed combinatorial interpretations).*

It is very unusual that our infinite families of $p$-Kazhdan–Lusztig polynomials are independent of $p \geqslant 0$. Indeed, it was pointed out to us by both Pramod Achar and the anonymous referee that our result is surprising from a geometric perspective: in the non-parabolic setting, a typical geometric explanation for characteristic-free behaviour is the existence of small resolutions for the relevant Schubert varieties. In contrast, for our parabolic setting of Hermitian symmetric pairs Theorem A provides infinite families of characteristic-free $p$-Kazhdan–Lusztig polynomials for which it is known that many *(but not all!)* Schubert varieties admit small resolutions [Zel83, Per07].

Over a field of characteristic zero, Elias–Williamson and Libedinsky–Williamson have shown that these 0-Kazhdan–Lusztig polynomials are the classical anti-spherical Kazhdan–Lusztig polynomials [EW14, LW22]. Thus our Theorem A implies that the $p$-Kazhdan–Lusztig polynomials are equal to the (classical) anti-spherical Kazhdan–Lusztig polynomials of [Deo87, Soe97]. Thus the combinatorial interpretations of these polynomials alluded to in Theorem A can be found in terms of the tiling language of this paper in [Bre09] (building on earlier work of [LS81, Boe88, ES87]). Combinatorial interpretations in terms of oriented Temperley–Lieb algebras can be found in [BDF+25] (building on earlier work [BS12a, CD11]).

**Tetris presentations and combinatorial invariance.** The first step towards proving Theorem A is to reduce to simply laced types. The graph automorphisms of Coxeter graphs of type $A$ and $D$ give rise to fixed point subgroups of types $B$ and $C$, respectively. A crucial step in the proof of Theorem A is to lift this to the level of the corresponding Hecke categories of Hermitian symmetric pairs of types $(B_n, B_{n-1})$ and $(A_{2n-1}, A_{2n-2})$ and of type $(C_n, A_{n-1})$ and $(D_{n+1}, A_n)$ — thus categorifying an observation of Boe, namely that the Kazhdan–Lusztig polynomials for these pairs coincide. This is an example of Lusztig–Dyer–Marietti's combinatorial invariance conjecture: which states that anti-spherical Kazhdan–Lusztig polynomials depend only on local isomorphisms of the strong Bruhat graphs (and proven by Brenti [Bre09] for Hermitian symmetric pairs).

**Theorem B.** *Let $\Pi = [\lambda, \mu]$ and $\Pi = [\lambda', \mu']$ be subquotients of the Bruhat graphs of Hermitian symmetric pairs $\lambda, \mu \in (W, P)$ and $\lambda', \mu' \in (W', P')$. If $\Pi$ and $\Pi'$ are isomorphic as partially ordered sets, then the corresponding subquotients $\mathcal{H}^{\Pi}_{(W,P)}$ and $\mathcal{H}^{\Pi'}_{(W',P')}$ are Morita equivalent (in the sense of* [Ben98, Section 2.2]*) and this equivalence preserves the grading, cellular, and highest-weight structures of these algebras.*

In other words, Theorem B says that all important representation theoretic information is preserved. In order to prove Theorems A and B we must provide new presentations of the $\mathcal{H}_{(W,P)}$ for $(W, P)$ a Hermitian symmetric pair, see Theorem 4.14. While the original presentations have many advantages, they are ill-equipped for tackling the combinatorial invariance conjecture. This is because these are "too local" and therefore cannot possibly hope to reflect the wider structure of the Bruhat graph. Defining these new presentations requires the full power of Soergel diagrammatics and the development of new "Tetris style" closed combinatorial formulas for manipulation of diagrams in $\mathcal{H}_{(W,P)}$. This provides an extremely thorough understanding of these Hecke categories, and we expect that it will serve as a springboard for further combinatorial analysis of more general Hecke categories.

**Singular Soergel diagrammatics and proof of Koszulity.** The standard Koszulity property is a particularly beautiful property which is characteristic of *complex* Lie theory ([BGS96, ADL03]); a much-loved consequence of this property is that we can explicitly calculate the radical filtrations of projective and cell modules by way of the grading structure. Many well-loved objects in Lie theory are Koszul over the complex field, for example the quantum Schur algebras [Sha12], extended Khovanov arc algebras [BS10], and the (diagrammatic) Cherednik algebras [RSVV16, Los16, Web17].

Koszulity of Lie theoretic objects is usually difficult to prove and it is an incredibly rare attribute over fields of characteristic $p > 0$. Our proof of Koszulity explicitly constructs linear projective resolutions of standard modules using the following theorem, which recasts the results of Enright–Shelton's monograph [ES87] in the setting of Hecke categories and generalises their results to fields of positive characteristic. We hence make headway on the difficult problem of constructing singular Soergel diagrammatics.

**Theorem C.** *Let $(W, P)$ be a simply laced Hermitian symmetric pair, and suppose $\tau$ is a simple reflection in $W$. We explicitly construct the $\tau$-singular Hecke category*[1] *$\mathcal{H}^{\tau}_{(W,P)}$ as a subcategory of $\mathcal{H}_{(W,P)}$. We prove that $\mathcal{H}^{\tau}_{(W,P)}$ is isomorphic to the Hecke category of a Hermitian symmetric pair $(W, P)^{\tau}$ of smaller rank.*

The combinatorial shadow of Theorem C is a graded bijection between paths in the smaller Bruhat graph of $(W, P)^{\tau}$ and paths in a truncation of the larger Bruhat graph of $(W, P)$. This can be categorified to the level of "dilation" homomorphisms between the anti-spherical Hecke categories. Sections 5 and 6 are dedicated to constructing these dilation maps and proving that they are indeed homomorphisms. We depict an example of the embedding of Bruhat graphs and the effect of the homomorphism on the fork generator in Figure 1.

[1]This is a diagrammatic, anti-spherical analogue of the category $J\mathbb{BS}\mathrm{Bim}$ introduced in [Eli16] for $J = \{\tau\}$. Repeated applications of our construction will give an analogue of $J\mathbb{BS}\mathrm{Bim}$ for $J$ arbitrary.

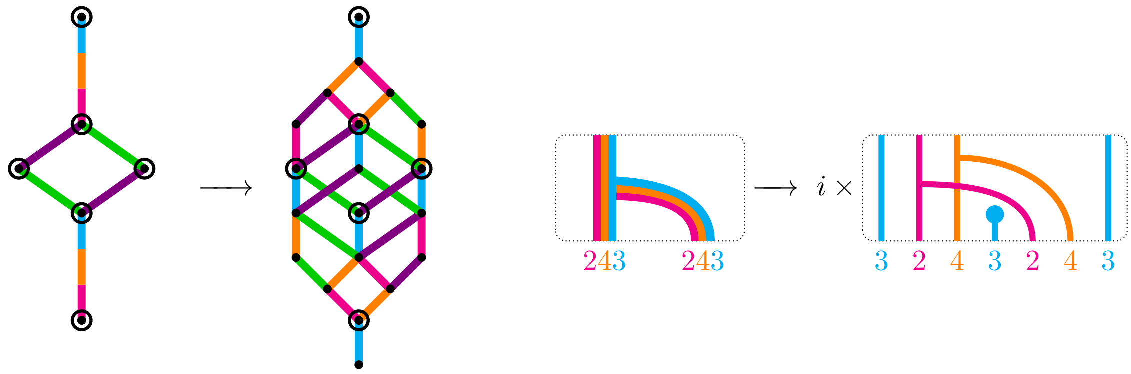



FIGURE 1. On the left we depict the embedding of the Bruhat graph of $(A_3, A_1 \times A_1)$ into $(A_5, A_2 \times A_2)$. On the right we depict the corresponding dilation map on the fork generator. Here $i$ is a primitive 4th root of 1. The tri-colouring 243 of single edges in the Bruhat graph and Soergel diagram comes from a single tricoloured node in the truncated Coxeter diagram.

**Structure of the paper.** The paper is organised as follows. Section 1 contains the basic definitions needed in this paper, namely, the tile combinatorics of Hermitian symmetric pairs and the original definition of the Hecke category for an arbitrary parabolic Coxeter system. In Section 2, we prove that in the case of Hermitian symmetric pairs, the presentation of the Hecke category can be simplified dramatically, lifting to the Hecke category a result of Stembridge which states that these parabolic quotients are fully commutative. In Section 3, we recall the construction of the light leaves basis for these Hecke categories. Section 4 constructs Tetris style presentations and proves Theorem B. In particular, we show that the Hecke categories corresponding to non-simply laced Hermitian symmetric pairs are graded Morita equivalent to Hecke categories of simply laced types. Section 5 constructs $\boldsymbol{\tau}$-singular Hecke categories for simply-laced types by truncating the original categories and identifies them with Hecke categories for Hermitian symmetric pairs of smaller ranks. (The proof is given in Section 6.) This construction of $\boldsymbol{\tau}$-singular Hecke categories allows us to prove results by induction on the rank. In Section 7, we use this, together with the reduction to simply-laced types, to give a description of the graded decomposition numbers and prove Koszulity of the Hecke categories for Hermitian symmetric pairs.

## 1. The Hecke categories for Hermitian symmetric pairs

Let $(W, S_W)$ be a Coxeter system: $W$ is the group generated by the finite set $S_W$ subject to the relations $(\boldsymbol{\sigma\tau})^{m_{\boldsymbol{\sigma\tau}}} = 1$ for $\boldsymbol{\sigma}, \boldsymbol{\tau} \in S_W$, $m_{\boldsymbol{\sigma\tau}} \in \mathbb{N} \cup \{\infty\}$ satisfying $m_{\boldsymbol{\sigma\tau}} = m_{\boldsymbol{\tau\sigma}}$, and $m_{\boldsymbol{\sigma\tau}} = 1$ if and only if $\boldsymbol{\sigma} = \boldsymbol{\tau}$. Let $\ell : W \to \mathbb{N}$ be the corresponding length function. Consider $S_P \subseteq S_W$ a subset and $(P, S_P)$ its corresponding Coxeter system. We say that $P$ is the parabolic subgroup corresponding to $S_P \subseteq S_W$. Let ${}^P W \subseteq W$ denote a set of minimal coset representatives in $P \backslash W$. For $\underline{w} = \sigma_1 \sigma_2 \cdots \sigma_\ell$ an expression, we define a subword to be a sequence $\underline{t} = (t_1, t_2, \ldots, t_\ell) \in \{0, 1\}^\ell$ and we set $\underline{w}^{\underline{t}} := \sigma_1^{t_1} \sigma_2^{t_2} \cdots \sigma_\ell^{t_\ell}$. We let $\leqslant$ denote the strong Bruhat order on ${}^P W$: namely $y \leqslant w$ if for some reduced expression $\underline{w}$ there exists a subword $\underline{t}$ and a reduced expression $\underline{y}$ such that $\underline{w}^{\underline{t}} = \underline{y}$. The Hasse diagram of this partial ordering is called the Bruhat graph of $(W, P)$. For the remainder of this paper we will assume that $W$ is a Weyl group and indeed that $(W, P)$ is a Hermitian symmetric pair, which are classified as follows:

Let $W$ be a finite Coxeter group and $P$ a parabolic subgroup. The $(W, P)$ corresponding to Hermitian symmetric spaces were first studied by Cartan [Car35] and have been classified (see for example [Boe88]). There are five infinite families $(A_n, A_{k-1} \times A_{n-k})$ with $1 \leqslant k \leqslant n$, $(D_n, A_{n-1})$, $(D_n, D_{n-1})$, $(B_n, B_{n-1})$, $(C_n, A_{n-1})$ and two exceptional ones for $n \geqslant 2$, $(E_6, D_5)$, or $(E_7, E_6)$. Our main interest in these pairs $(W, P)$ stems from the rich combinatorial and representation theoretic structures associated to them (see for example [CIS88, ES87, Boe88, Bre07, Bre09, EHP14]); and the fact that they are tractable and diverse enough to serve as milestones of our understanding of Lie theoretic objects. This paper extends the work above on Hermitian symmetric pairs in order to provide a new milestone in our understanding of anti-spherical Hecke categories.

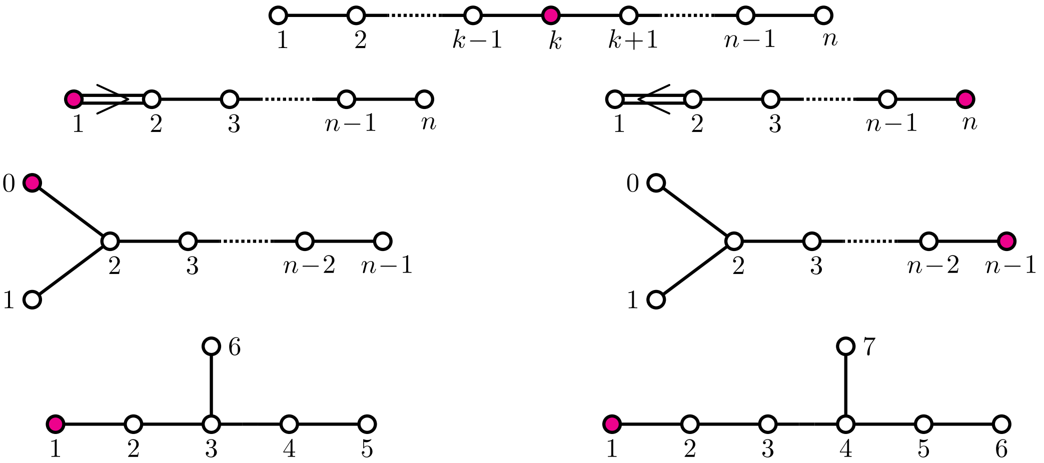



FIGURE 2. Enumeration of nodes in the parabolic Dynkin diagram of types of type $(A_n, A_{k-1} \times A_{n-k})$, $(C_n, A_{n-1})$ and $(B_n, B_{n-1})$, $(D_n, A_{n-1})$ and $(D_n, D_{n-1})$ and $(E_6, D_5)$ and $(E_7, E_6)$ respectively. The single node not belonging to the parabolic is highlighted in pink in each case.

In Figure 2, we recall the Dynkin diagrams of Hermitian symmetric pairs explicitly. For type $D$, we use a slightly unusual labelling of nodes, which will allow us to pass between types $C$ and $D$ more easily. The remainder of this section is dedicated to the combinatorics of Hermitian symmetric pairs. This has been lifted from [EHP14, Appendix: diagrams of Hermitian type], but has been translated into a more diagrammatic language.

1.1. **Tile partitions.** The Bruhat graphs of Hermitian symmetric pairs can be encapsulated in terms of tilings of "admissible regions" of the plane, which we now define. In type $(A_n, A_{k-1} \times A_{n-k})$, the admissible region is simply a $(k \times (n-k+1))$-rectangle, and the tilings governing the combinatorics are Young diagrams which fit in this rectangle. The general picture is as follows:

**Definition 1.1.** *Let $(W, P)$ be a Hermitian symmetric pair of classical type. We call a point $[r, c] \in \mathbb{N}^2$ a* tile. *The* admissible region $\mathscr{A}_{(W,P)}$ *is a certain finite subset of tiles defined as follows:*

- *for type $(W, P) = (A_n, A_{k-1} \times A_{n-k})$, the admissible region is the subset of tiles*
$$\{[r, c] \mid r \leqslant n - k + 1, c \leqslant k\}.$$
- *for types $(W, P) = (C_n, A_{n-1})$ and $(D_n, A_{n-1})$, the admissible region is the subset of tiles*
$$\{[r, c] \mid r, c \leqslant n \text{ and } r - c \geqslant 0\}.$$
- *for type $(W, P) = (B_n, B_{n-1})$, the admissible region is the subset of tiles*
$$\{[r, c] \mid r = 1 \text{ and } c < n\} \sqcup \{[r, c] \mid c = n \text{ and } r \leqslant n\}.$$
- *for type $(W, P) = (D_n, D_{n-1})$, the admissible region is the subset of tiles*
$$\{[r, c] \mid r = 1 \text{ and } c < n\} \sqcup \{[r, c] \mid c = n \text{ and } r \leqslant n\} \sqcup \{[2, n-1]\}.$$

*We draw tiles and admissible regions in the "Russian" style, with rows (i.e. fixed values of $r$) pointing northwest and columns (i.e. fixed values of $c$) pointing northeast.*

**Example 1.2.** *We illustrate the admissible region for type $(A_8, A_4 \times A_3)$ in Figure 3, for types $(D_6, A_5)$ and $(C_6, A_5)$ in Figure 4, and for types $(B_6, B_5)$ and $(D_7, D_6)$ in Figure 5. For the two exceptional types $(E_6, D_5)$ and $(E_7, E_6)$, the admissible region consists of the subset of tiles pictured in Figure 6.*

Each tile $[r, c] \in \mathscr{A}_{(W,P)}$ carries a coloured label, inherited from the Dynkin diagram of $W$. This is explained in detail in [EHP14, Appendix], but can be deduced easily from Figures 3 to 6. Given $[r, c] \in \mathscr{A}_{(W,P)}$, we let $s_{[r,c]}$ denote the corresponding simple reflection in $S$. For example, in types $(A_n, A_{k-1} \times A_{n-k})$ and $(C_n, A_{n-1})$ the reflection $s_{[r,c]}$ is determined simply by the $x$-coordinate of the tile $[r, c] \in \mathscr{A}_{(W,P)}$ (i.e. it is determined by $c - r$). Given $\boldsymbol{\tau} \in W$ a label of the Dynkin diagram,

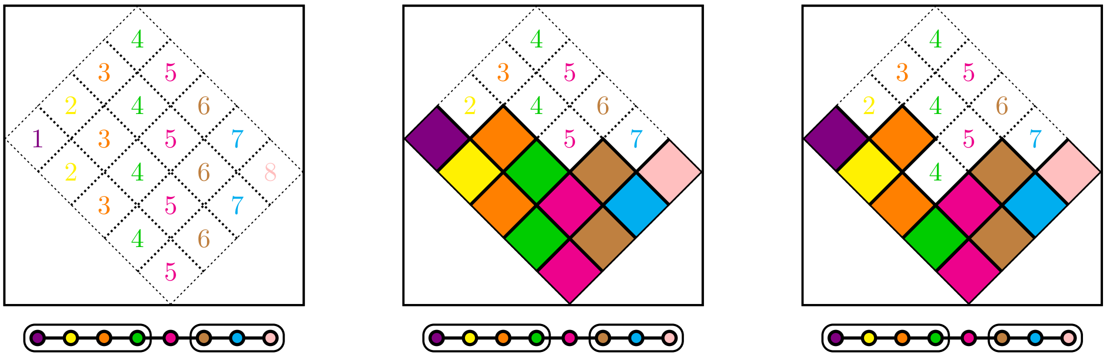



FIGURE 3. On the left we picture the admissible region for $(A_8, A_4 \times A_3)$. We then picture two tilings; the first of which is a tile partition, but the latter is not (the tile $[2, 4]$ is not supported).

we refer to a tile $[r, c] \in \mathscr{A}_{(W,P)}$ as a $\boldsymbol{\tau}$-tile if $s_{[r,c]} = \boldsymbol{\tau}$. We emphasise this connection by colouring the tile, $[r, c] \in \mathscr{A}_{(W,P)}$, when appropriate.

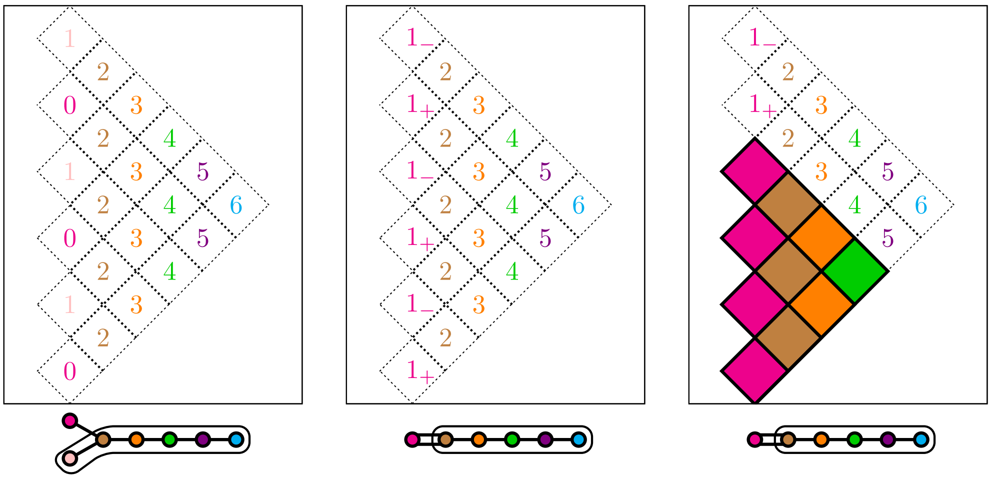



FIGURE 4. We picture two admissible regions $\mathscr{A}_{(W,P)}$ for types $(D_6, A_5)$ and $(C_6, A_5)$ and a tile-partition $\lambda = (1, 2, 3, 4) \in \mathscr{P}_{(W,P)}$ for types $(C_6, A_5)$. The $\pm$ signs are explained in Subsection 1.3.

We say that a pair of tiles are **neighbouring** if they meet at an edge (which necessarily has an angle of 45° or 135° to the horizontal axis, by construction). Given a pair of neighbouring tiles $X$ and $Y$, we write $Y < X$ if $X$ appears above $Y$ (i.e. the $y$-coordinate of $X$ is strictly larger than that of $Y$). We extend $<$ to a partial order on the tiles in $\mathscr{A}_{(W,P)}$ by transitivity and we say that $Y$ **supports** $X$ if $Y < X$ in this ordering. We say that a collection of tiles, $\lambda \subseteq \mathscr{A}_{(W,P)}$ is a **tile-partition** if for every tile $X \in \lambda$ and every $Y \in \mathscr{A}_{(W,P)}$ such that $Y < X$, we have that $Y \in \lambda$. We let $\mathscr{P}_{(W,P)}$ denote the set of all tile-partitions. We depict a tile-partition $\lambda$ by colouring the tiles of $\lambda$. See Figure 3 for examples and non-examples of tile-partitions.

We define the **length** of a tile-partition $\lambda$ to be the total number of tiles $[r, c] \in \lambda$. We let $\mathscr{P}^{\ell}_{(W,P)}$ denote the subset of all tile partitions of length $\ell$. There is a natural bijection between ${}^PW$ and $\mathscr{P}_{(W,P)}$ (see [EHP14, Appendix]) under which the length functions coincide. For $\lambda, \mu \in \mathscr{P}_{(W,P)}$, we define the **Bruhat order** on tile partitions by $\lambda \leqslant \mu$ if

$$\{[r, c] \mid [r, c] \in \lambda\} \subseteq \{[r, c] \mid [r, c] \in \mu\}.$$

Given $\lambda \leqslant \nu$, we define the **skew tile-partition** $\nu \setminus \lambda$ to be the set difference of $\lambda$ and $\nu$.

**Definition 1.3.** *Let $[r, c] \in \mathscr{A}_{(W,P)}$ denote any tile. We define $\lambda_{[r,c]}$ to be the tile partition*

$$\lambda_{[r,c]} := \{[x, y] \in \mathscr{A}_{(W,P)} \mid x \leqslant r, y \leqslant c\}$$

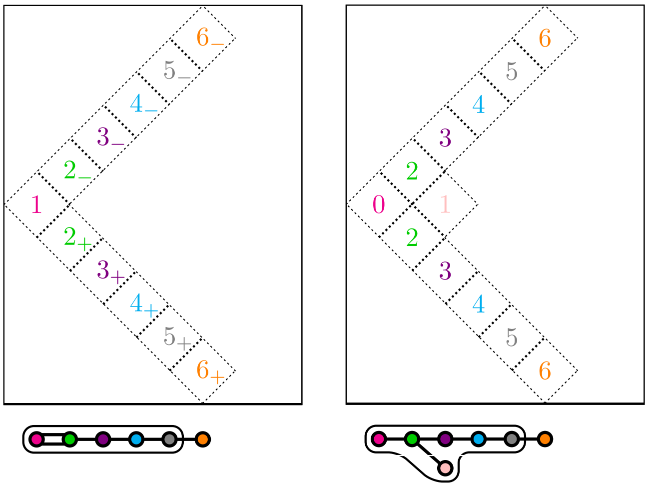



FIGURE 5. The admissible regions for types $(B_6, B_5)$ and $(D_7, D_6)$. The $\pm$ signs are explained in Subsection 1.3.

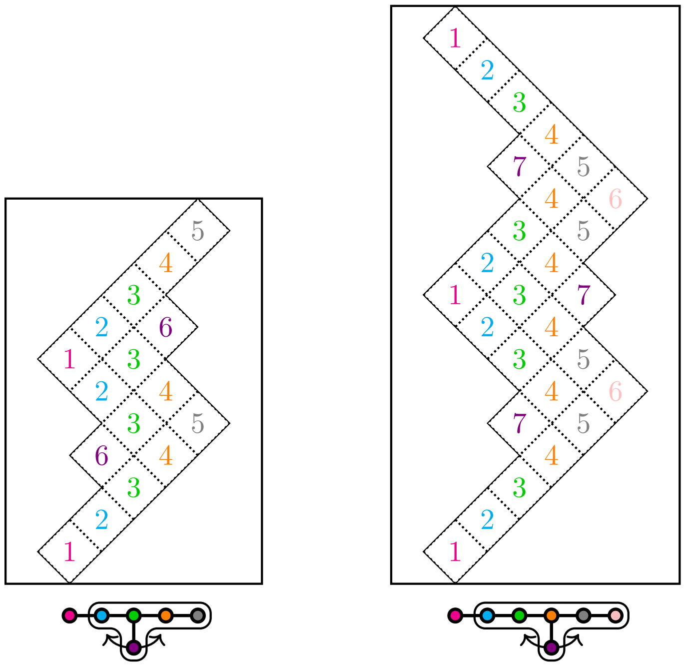


FIGURE 6. The admissible regions $\mathscr{A}_{(W,P)}$ for the pairs $(E_6, D_5)$ and $(E_7, E_6)$ respectively. We think of the violet node as being free to swing from left to right; in this way we continue to associate tiles with the $x$-coordinates of nodes in the Coxeter graph.

**Remark 1.4.** *In type $(A_n, A_{k-1} \times A_{n-k})$ a tile-partition $\lambda$ is the Young diagram (in Russian notation) of a classical partition with at most $k$ columns and $(n-k+1)$ rows. In this case $\lambda_{[r,c]}$ is the $(r \times c)$-rectangle. In other types, it is this rectangle intersected with the region $\mathscr{A}_{(W,P)}$.*

1.2. **Tile tableaux.** The combinatorics of reduced and non-reduced words for Hermitian symmetric pairs can be encapsulated in terms of *tile-paths* or *tile-tableaux*, which we now define. Given $\mu \in \mathscr{P}_{(W,P)}$, we define the set of all addable and removable tiles to be $\mathrm{Add}(\mu) = \{[r,c] \mid \mu \cup [r,c] \in \mathscr{P}_{(W,P)}\}$ and $\mathrm{Rem}(\mu) = \{[r,c] \mid \mu \setminus [r,c] \in \mathscr{P}_{(W,P)}\}$ respectively. Abusing notation, we will write $\mu + [r,c]$ for $\mu \cup [r,c]$ and $\mu - [r,c]$ for $\mu \setminus [r,c]$. Any tile-partition $\mu$ has at most one addable or removable tile of any given colour $\boldsymbol{\tau} \in S_W$. Thus given $[r,c] \in \mathscr{A}_{(W,P)}$ with $\boldsymbol{\tau} = s_{[r,c]}$, we often write $\boldsymbol{\tau} \in \mathrm{Add}(\mu)$ or $\boldsymbol{\tau} \in \mathrm{Rem}(\mu)$ and we write $\mu + \boldsymbol{\tau}$ or simply $\mu\boldsymbol{\tau}$ for $\mu\boldsymbol{\tau} := \mu + [r,c]$; we write or $\mu - \boldsymbol{\tau} := \mu - [r,c]$.

**Definition 1.5.** *For $\lambda \in \mathscr{P}_{(W,P)}$ we define a* tile-tableau *of length $\ell$ and shape $\lambda$ to be a path*

$$\mathsf{T} : \varnothing = \lambda_0 \xrightarrow{[r_1,c_1]} \lambda_1 \xrightarrow{[r_2,c_2]} \lambda_2 \xrightarrow{[r_3,c_3]} \cdots \xrightarrow{[r_\ell,c_\ell]} \lambda_\ell = \lambda$$

*such that for each $k = 1, \ldots, \ell$, $\lambda_k \in \mathscr{P}_{(W,P)}$ and $\lambda_k$ satisfies one of the following*

*(i)* $\lambda_k = \lambda_{k-1} + [r_k, c_k]$ *with* $[r_k, c_k] \in \mathrm{Add}(\lambda_{k-1})$*; or*
*(ii)* $\lambda_k = \lambda_{k-1} - [r_k, c_k]$ *with* $[r_k, c_k] \in \mathrm{Rem}(\lambda_{k-1})$*; or*
*(iii)* $\lambda_k = \lambda_{k-1}$ *with* $[r_k, c_k] \in \mathrm{Rem}(\lambda_{k-1})$ *or* $\mathrm{Add}(\lambda_{k-1})$*;*

*We let $\mathrm{Path}_\ell(\lambda)$ denote the set of all tile-tableaux of shape $\lambda$ and length $\ell$. We say that a tile tableau, $\mathsf{T}_\lambda \in \mathrm{Path}_\ell(\lambda)$, is* reduced *if $\lambda \in \mathscr{P}^\ell_{(W,P)}$ and we denote the set of all such tableaux by $\mathrm{Std}(\lambda)$. We use uppercase (respectively lowercase) san-serif letters for general tile-tableaux (respectively reduced tile-tableaux).*

**Remark 1.6.** *In type $(A_n, A_{k-1} \times A_{n-k})$, the notions of addable and removable tiles correspond to the familiar notions of addable and removable boxes for Young diagrams. The set of reduced tile tableaux coincides with the usual notion of standard Young tableaux.*

For $\lambda \in \mathscr{P}^\ell_{(W,P)}$ we identify a reduced tableau $\mathsf{t} \in \mathrm{Std}(\lambda)$ with a bijective map $\mathsf{t} : \lambda \to \{1, \ldots, \ell\}$ and we record this by placing the entry $\mathsf{t}^{-1}(k)$ in the $[r, c]$th tile, in the usual manner. In this fashion, we can identify $\mathrm{Std}(\lambda)$ with the set of all possible fillings of $\lambda$ with the numbers $\{1, \ldots, \ell\}$ in such a way that these numbers increase alongs the rows and columns of $\lambda$. For $\nu \setminus \lambda$ a skew tile partition, we can define $\mathrm{Std}(\nu \setminus \lambda)$ in the obvious fashion. Given $1 \leqslant k \leqslant \ell$, we let $\mathsf{t}{\downarrow}_{\{1,\ldots,k\}}$ denote the restriction of the map to the pre-image of $\{1, \ldots, k\}$. Examples are depicted in Figure 7.

**Definition 1.7.** *Let $\lambda, \mu \in \mathscr{P}_{(W,P)}$ and fix $\mathsf{t} \in \mathrm{Std}(\mu)$ such that $\mathsf{t}([x_k, y_k]) = k$ for $1 \leqslant k \leqslant \ell$. We say that a tile-tableau*

$$\mathsf{T} : \varnothing = \lambda_0 \xrightarrow{[r_1,c_1]} \lambda_1 \xrightarrow{[r_2,c_2]} \lambda_2 \xrightarrow{[r_3,c_3]} \cdots \xrightarrow{[r_\ell,c_\ell]} \lambda_\ell = \lambda$$

*is obtained by* folding-up $\mathsf{t} \in \mathrm{Std}(\mu)$ *if $s_{[r_k,c_k]} = s_{[x_k,y_k]}$ for $1 \leqslant k \leqslant \ell$. We let $\mathrm{Path}(\lambda, \mathsf{t})$ denote the set of all paths obtained in this manner.*

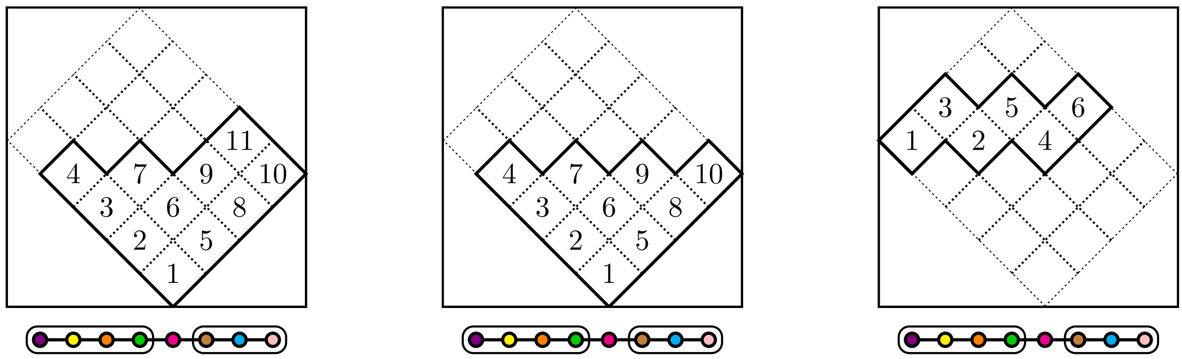



FIGURE 7. Tableaux $\mathsf{s}, \mathsf{t}, \mathsf{u}$ of shape $(4, 3, 2^2)$, $(4, 3, 2, 1)$ and $(5^2, 4, 3) \setminus (4, 3, 2^2)$ respectively, in type $(A_8, A_4 \times A_3)$. We have that $\mathsf{s} = \mathsf{t} \otimes \boldsymbol{\tau}$ for $\boldsymbol{\tau} = s_7$ and $\mathsf{t} = \mathsf{s}{\downarrow}_{\{1,\ldots,10\}}$.

**Definition 1.8.** *Given $\lambda \in \mathscr{P}^\ell_{(W,P)}$, $\mathsf{t} \in \mathrm{Std}(\lambda)$ and $[r, c] \in \mathrm{Add}(\lambda)$, we let $\mathsf{t} \otimes \boldsymbol{\tau} \in \mathrm{Std}(\lambda\boldsymbol{\tau})$ denote the reduced tableau uniquely determined by $(\mathsf{t} \otimes \boldsymbol{\tau})[x, y] = \mathsf{t}[x, y]$ for $[x, y] \neq [r, c]$.*

**Example 1.9.** *We now provide an example of Definition 1.7. An element of $\mathrm{Path}((3, 2), \mathsf{s})$ for $\mathsf{s} \in \mathrm{Std}(4, 3, 2^2)$ as in Figure 7 is given as follows:*

$$\varnothing \to (1) \to (2) \to (3) \to (3) \to (3,1) \to (3,2) \to (3,2) \to (3,2,1) \to (3,2,1) \to (3,2,1) \to (3,2)$$

*where the tile-labels of the arrows can be deduced from their colours (which match with the colours of the tiles for $\mathsf{s}$ pictured in Figure 7). The 1st, 2nd, 3rd, 5th, 6th, 8th steps are all of the form (i) Definition 1.5; the 11th step is of the form (ii) Definition 1.5; the 4th, 7th, 9th, and 10th steps are of the form (iii) Definition 1.5.*

**Remark 1.10.** *For $\lambda \in {}^PW$, the set of reduced tableaux* $\mathrm{Std}(\lambda)$ *is in bijection with the set of reduced expressions for $\lambda$ in a natural fashion. For example, the tableau* $\mathsf{t} \in \mathrm{Std}(4,3,2,1)$ *in Figure 7 corresponds to the reduced word* $s_5 s_4 s_3 s_2 s_6 s_5 s_4 s_7 s_6 s_8$.

1.3. **Parity conditions for non-simply laced tiles.** Finally, we are now ready to explain the existence of $\pm$ signs in Figures 4 and 5. For non-simply-laced Weyl groups (that is, $(W,P) = (C_n, A_{n-1})$ or $(B_n, B_{n-1})$ we allow a simple reflection $s_i \in S_W$ to carry a parity label, $s_{\pm i}$). Given $\lambda = \sigma_1\sigma_2\cdots\sigma_\ell \in {}^PW$ we use this parity label to record the odd/even number of prior appearances of this reflection (read from left-to-right). For example, in Figure 4 we picture the element $\lambda = s_{+1}s_2s_{-1}s_3s_2s_4s_{+1}s_3s_5s_6 \in {}^PW$ for $(W,P) = (C_6, A_5)$. We note that this $\pm$ label is well-defined for elements of ${}^PW$ (as these cosets are all "fully commuting" in the sense of [Ste96]). We record this parity label in terms of the $y$-coordinate of tiles in $\mathscr{A}_{(W,P)}$ as in Figures 4 and 5. Given $\lambda, \mu \in \mathscr{P}_{(W,P)}$ and $\mathsf{s} \in \mathrm{Std}(\mu)$, we let $\mathrm{Path}^{\pm}(\lambda, \mathsf{s}) \subseteq \mathrm{Path}(\lambda, \mathsf{s})$ denote the subset of paths which preserve this parity condition. More precisely, using the notation from Definition 1.7, we insist that if $s_{[x_k,y_k]} = s_{\pm i}$ then we also have $s_{[r_k,c_k]} = s_{\pm i}$ (with the same parity).

**Example 1.11.** *Let $(W,P) = (C_6, A_5)$ as in Figure 4, we have that*

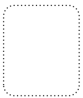
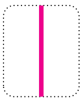
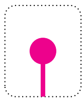
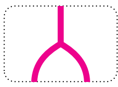
$$\varnothing \to (1) \to (1^2) \to (1^3) \to (1^4) \to (1,2,1^2) \to (1,2^2,1) \to (1,2^2,1) \to (1,2^2,1) \to (1,2,1^2) \to (1^4)$$

*and*

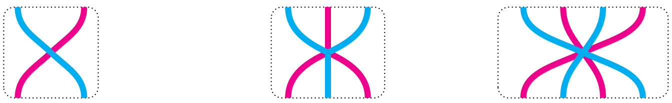
$$\varnothing \to (1) \to (1^2) \to (1^3) \to (1^4) \to (1,2,1^2) \to (1,2^2,1) \to (1,2,3,1) \to (1,2,3,2) \to (1,2,3,2) \to (1,2^3)$$

*are both elements of* $\mathrm{Path}(\lambda, \mathsf{s})$ *for some $\lambda = (1^4), (1,2^3) \in {}^PW$ and for the same* $\mathsf{s} \in \mathrm{Path}(1,2,3,4)$*. The former of these paths belongs to* $\mathrm{Path}^{\pm}(1,2,3,4)$ *whereas the latter does not as $s_{[x_{10},y_{10}]} = s_{[4,4]} = s_{-1}$ but $s_{[r_{10},c_{10}]} = s_{[3,3]} = s_{+1}$.*

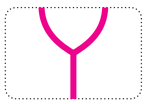
1.4. **The diagrammatic Hecke categories.** Almost everything from this section is lifted from Elias–Williamson's original paper [EW16] or is an extension of their results to the parabolic setting [LW22]. Let $(W,S)$ denote a Coxeter system for $W$ a Weyl group. Given $\boldsymbol{\sigma} \in S_W$ we define the monochrome Soergel generators to be the framed graphs

$\mathsf{1}_\emptyset =$  $\mathsf{1}_{\boldsymbol{\sigma}} =$  $\mathsf{spot}^{\emptyset}_{\boldsymbol{\sigma}} =$  $\mathsf{fork}^{\boldsymbol{\sigma}}_{\boldsymbol{\sigma}\boldsymbol{\sigma}} =$  (G1)

and given any $\boldsymbol{\sigma}, \boldsymbol{\tau} \in S_W$ with $m_{\boldsymbol{\sigma}\boldsymbol{\tau}} = m = 2, 3$ or $4$ we have the bi-chrome generator $\mathsf{braid}^{\boldsymbol{\tau}\boldsymbol{\sigma}}_{\boldsymbol{\sigma}\boldsymbol{\tau}}(m)$ which is pictured as follows

 (G2)

for $m$ equal to $2, 3$ or $4$ respectively. (We will also sometimes write $\mathsf{braid}^{\boldsymbol{\tau}\boldsymbol{\sigma}}_{\boldsymbol{\sigma}\boldsymbol{\tau}}$, $\mathsf{braid}^{\boldsymbol{\tau}\boldsymbol{\sigma}\boldsymbol{\tau}}_{\boldsymbol{\sigma}\boldsymbol{\tau}\boldsymbol{\sigma}}$, and $\mathsf{braid}^{\boldsymbol{\tau}\boldsymbol{\sigma}\boldsymbol{\tau}\boldsymbol{\sigma}}_{\boldsymbol{\sigma}\boldsymbol{\tau}\boldsymbol{\sigma}\boldsymbol{\tau}}$ for $\mathsf{braid}^{\boldsymbol{\tau}\boldsymbol{\sigma}}_{\boldsymbol{\sigma}\boldsymbol{\tau}}(m)$ with $m = 2$, $m = 3$, and $m = 4$ respectively.) Pictorially, we define the duals of these generators to be the graphs obtained by reflection through their horizontal axes. Non-pictorially, we simply swap the sub- and superscripts. We sometimes denote duality by $*$. For example, the dual of the fork generator is pictured as follows

$\mathsf{fork}^{\boldsymbol{\sigma}\boldsymbol{\sigma}}_{\boldsymbol{\sigma}} =$  .

We define the northern/southern reading word of a Soergel generator (or its dual) to be word in the alphabet $S$ obtained by reading the colours of the northern/southern edge of the frame respectively. Given two (dual) Soergel generators $D$ and $D'$ we define $D \otimes D'$ to be the diagram obtained by horizontal concatenation (and we extend this linearly). The northern/southern colour sequence of $D \otimes D'$ is the concatenation of those of $D$ and $D'$ ordered from left to right. Given any two (dual) Soergel generators, we define their product $D \circ D'$ (or simply $DD'$) to be the vertical concatenation of $D$ on top of $D'$ if the southern reading word of $D$ is equal to the northern reading word of $D'$ and zero otherwise. We define a Soergel diagram to be any graph obtained by repeated horizontal and vertical concatenation of (dual) Soergel generators.

For $\underline{w} = \sigma_1 \dots \sigma_\ell$ an expression, we define $1_{\underline{w}} = 1_{\sigma_1} \otimes 1_{\sigma_2} \otimes \cdots \otimes 1_{\sigma_\ell}$ and given $k > 1$ and $\boldsymbol{\sigma}, \boldsymbol{\tau} \in S_W$ we set $1^k_{\boldsymbol{\sigma\tau}} = 1_{\boldsymbol{\sigma}} \otimes 1_{\boldsymbol{\tau}} \otimes 1_{\boldsymbol{\sigma}} \otimes 1_{\boldsymbol{\tau}} \dots$ to be the alternately coloured idempotent on $k$ strands (so that the final strand is $\boldsymbol{\sigma}$- or $\boldsymbol{\tau}$-coloured if $k$ is odd or even respectively). Given $\boldsymbol{\sigma}, \boldsymbol{\tau} \in S_W$ with $m_{\boldsymbol{\sigma\tau}} = 2$, let $\underline{w} = \rho_1 \cdots \rho_k (\boldsymbol{\sigma\tau}) \rho_{k+3} \cdots \rho_\ell$ and $\underline{\underline{w}} = \rho_1 \cdots \rho_k (\boldsymbol{\tau\sigma}) \rho_{k+3} \cdots \rho_\ell$ be two reduced expressions for $w \in W$. We say that $\underline{w}$ and $\underline{\underline{w}}$ are adjacent and we set

$$\mathsf{braid}^{\underline{w}}_{\underline{\underline{w}}} = 1_{\rho_1} \otimes \cdots \otimes 1_{\rho_k} \otimes \mathsf{braid}^{\boldsymbol{\sigma\tau}}_{\boldsymbol{\tau\sigma}}(2) \otimes 1_{\rho_{k+3}} \otimes \cdots \otimes 1_{\rho_\ell}.$$

Given a *fixed* sequence of adjacent reduced expressions, $\underline{w} = \underline{w}^{(1)}, \underline{w}^{(2)}, \dots, \underline{w}^{(q)} = \underline{\underline{w}}$ and the value $q$ is minimal such that this sequence exists, then we set

$$\mathsf{braid}^{\underline{w}}_{\underline{\underline{w}}} = \prod_{1 \leqslant p < q} \mathsf{braid}^{\underline{w}^{(p)}}_{\underline{w}^{(p+1)}}.$$

Given $\boldsymbol{\sigma}$, we define the corresponding "barbell" and "gap" diagrams to be the elements

$$\mathsf{bar}(\boldsymbol{\sigma}) = \mathsf{spot}^{\emptyset}_{\boldsymbol{\sigma}} \mathsf{spot}^{\boldsymbol{\sigma}}_{\emptyset} \qquad \mathsf{gap}(\boldsymbol{\sigma}) = \mathsf{spot}^{\boldsymbol{\sigma}}_{\emptyset} \mathsf{spot}^{\emptyset}_{\boldsymbol{\sigma}},$$

respectively. Let $\lambda, \mu \in \mathscr{P}_{(W,P)}$ with $\ell = \ell(\mu) - \ell(\lambda)$ and $\mathsf{t} \in \mathrm{Std}(\mu \setminus \lambda)$ such that $\mathsf{t}([x_k, y_k]) = k$ for $1 \leqslant k \leqslant \ell$. We let

$$1_{\mathsf{t}} = 1_{s_{[x_1,y_1]}} \otimes 1_{s_{[x_2,y_2]}} \otimes \cdots \otimes 1_{s_{[x_\ell,y_\ell]}}$$

and for $\boldsymbol{\sigma} = s_{[x_k,y_k]}$ we set

$$\mathsf{gap}(\mathsf{t} - [x_k, y_k]) = 1_{\mathsf{t}\downarrow_{\{1,\dots,k-1\}}} \otimes \mathsf{gap}(\boldsymbol{\sigma}) \otimes 1_{\mathsf{t}\downarrow_{\{k+1,\dots,\ell\}}}.$$

We also define the corresponding "double fork" diagram to be the element

$$\mathsf{dork}^{\boldsymbol{\sigma\sigma}}_{\boldsymbol{\sigma\sigma}} = \mathsf{fork}^{\boldsymbol{\sigma\sigma}}_{\boldsymbol{\sigma}} \mathsf{fork}^{\boldsymbol{\sigma}}_{\boldsymbol{\sigma\sigma}}.$$

It is standard (in Soergel diagrammatics) to draw the element $\mathsf{cap}^{\emptyset}_{\boldsymbol{\tau\tau}} := \mathsf{spot}^{\emptyset}_{\boldsymbol{\tau}} \mathsf{fork}^{\boldsymbol{\tau}}_{\boldsymbol{\tau\tau}}$ simply as a strand which starts and ends on the southern edge of the frame. (We define $\mathsf{cup}^{\boldsymbol{\tau\tau}}_{\emptyset} := (\mathsf{cap}^{\emptyset}_{\boldsymbol{\tau\tau}})^*$.) For $\underline{x} = \sigma_1 \sigma_2 \cdots \sigma_\ell$ a word, we define $\underline{x}_{\mathsf{rev}} = \sigma_\ell \cdots \sigma_2 \sigma_1$. Then we inductively define

$$\mathsf{cap}^{\emptyset}_{\underline{x}\,\underline{x}_{\mathsf{rev}}} := \mathsf{cap}^{\emptyset}_{\underline{y}\,\underline{y}_{\mathsf{rev}}} (1_{\underline{y}} \otimes \mathsf{cap}^{\emptyset}_{\sigma_\ell \sigma_\ell} \otimes 1_{\underline{y}_{\mathsf{rev}}})$$

where $\underline{y} = \sigma_1 \sigma_2 \cdots \sigma_{\ell-1}$. This diagram can be visualised as a rainbow of concentric arcs (with $\mathsf{cap}^{\emptyset}_{\sigma_\ell \sigma_\ell}$ the innermost arc). Since we have $\underline{x}\underline{x}_{\mathsf{rev}} = 1_W$ when evaluated in the group $W$, we will simply write $\mathsf{cap}^{\emptyset}_{\underline{x}\underline{x}^{-1}}$ for $\mathsf{cap}^{\emptyset}_{\underline{x}\,\underline{x}_{\mathsf{rev}}}$.

In order to make our notation less dense, we will often suppress mention of idempotents by including them in the sub- and super-scripts of other generators. This is made possible by recording where the edits to the underlying words are with emptysets. For example

$$\mathsf{spot}^{\emptyset\boldsymbol{\beta}\emptyset}_{\boldsymbol{\alpha\beta\tau}} := \mathsf{spot}^{\emptyset}_{\boldsymbol{\alpha}} \otimes 1_{\boldsymbol{\beta}} \otimes \mathsf{spot}^{\emptyset}_{\boldsymbol{\tau}} \quad \mathsf{fork}^{\boldsymbol{\gamma\beta\alpha}}_{\boldsymbol{\gamma\gamma\beta\alpha\alpha}} := \mathsf{fork}^{\boldsymbol{\gamma}}_{\boldsymbol{\gamma\gamma}} \otimes 1_{\boldsymbol{\beta}} \otimes \mathsf{fork}^{\boldsymbol{\alpha}}_{\boldsymbol{\alpha\alpha}} \quad \mathsf{bar}(\boldsymbol{\alpha\gamma}) = \mathsf{bar}(\boldsymbol{\alpha}) \otimes \mathsf{bar}(\boldsymbol{\gamma}).$$

We make use of all of the above notational shorthands (even within the same equation). Finally, for distinct $\boldsymbol{\sigma}, \boldsymbol{\tau} \in S_W$ we recall the entries of the Cartan matrix corresponding to the Dynkin diagram of $(W, P)$:

$$\langle \alpha^\vee_{\boldsymbol{\sigma}}, \alpha_{\boldsymbol{\tau}} \rangle = \begin{cases} 0 & \text{if } \boldsymbol{\sigma} \not\!\!-\, \boldsymbol{\tau}, \\ -1 & \text{if } \boldsymbol{\sigma} - \boldsymbol{\tau}, \\ -1 & \text{if } \boldsymbol{\sigma} \Longrightarrow \boldsymbol{\tau}, \\ -2 & \text{if } \boldsymbol{\sigma} \Longleftarrow \boldsymbol{\tau}. \end{cases}$$

**Definition 1.12.** *Let $W$ be a Weyl group, $P$ be a parabolic subgroup, and let $\Bbbk$ be a field. We define $\mathcal{H}_{(W,P)}$ to be the locally-unital associative $\Bbbk$-algebra spanned by all Soergel-graphs with multiplication given by $\circ$-concatenation modulo the following local relations and their vertical and horizontal flips.*

$$1_{\boldsymbol{\sigma}} 1_{\boldsymbol{\tau}} = \delta_{\boldsymbol{\sigma},\boldsymbol{\tau}} 1_{\boldsymbol{\sigma}}, \qquad 1_{\emptyset} 1_{\boldsymbol{\sigma}} = 0, \qquad 1^2_{\emptyset} = 1_{\emptyset}, \tag{R1}$$

$$1_{\emptyset} \mathsf{spot}^{\emptyset}_{\boldsymbol{\sigma}} 1_{\boldsymbol{\sigma}} = \mathsf{spot}^{\emptyset}_{\boldsymbol{\sigma}}, \qquad 1_{\boldsymbol{\sigma}} \mathsf{fork}^{\boldsymbol{\sigma}}_{\boldsymbol{\sigma\sigma}} 1_{\boldsymbol{\sigma\sigma}} = \mathsf{fork}^{\boldsymbol{\sigma}}_{\boldsymbol{\sigma\sigma}}, \qquad 1^m_{\boldsymbol{\tau\sigma}} \mathsf{braid}^{\boldsymbol{\tau\sigma}}_{\boldsymbol{\sigma\tau}}(m) 1^m_{\boldsymbol{\sigma\tau}} = \mathsf{braid}^{\boldsymbol{\tau\sigma}}_{\boldsymbol{\sigma\tau}}(m), \tag{R2}$$

*For each $\boldsymbol{\sigma} \in S$ we have the fork-spot, double-fork, circle-annihilation relations*

$$(\mathsf{spot}^{\emptyset}_{\boldsymbol{\sigma}} \otimes 1_{\boldsymbol{\sigma}}) \mathsf{fork}^{\boldsymbol{\sigma\sigma}}_{\boldsymbol{\sigma}} = 1_{\boldsymbol{\sigma}}, \quad (1_{\boldsymbol{\sigma}} \otimes \mathsf{fork}^{\boldsymbol{\sigma}}_{\boldsymbol{\sigma\sigma}})(\mathsf{fork}^{\boldsymbol{\sigma\sigma}}_{\boldsymbol{\sigma}} \otimes 1_{\boldsymbol{\sigma}}) = \mathsf{fork}^{\boldsymbol{\sigma\sigma}}_{\boldsymbol{\sigma}} \mathsf{fork}^{\boldsymbol{\sigma}}_{\boldsymbol{\sigma\sigma}}, \quad \mathsf{fork}^{\boldsymbol{\sigma}}_{\boldsymbol{\sigma\sigma}} \mathsf{fork}^{\boldsymbol{\sigma\sigma}}_{\boldsymbol{\sigma}} = 0, \tag{R3}$$

*pictured in Figure 8 together with the cinching relation*

$$1_{\sigma} \otimes 1_{\sigma} = \mathsf{spot}^{\emptyset}_{\sigma} \otimes \mathsf{fork}^{\sigma\sigma}_{\sigma} + \mathsf{spot}^{\sigma}_{\emptyset} \otimes \mathsf{fork}^{\sigma}_{\sigma\sigma} - \mathsf{bar}(\sigma) \otimes \mathsf{dork}^{\sigma\sigma}_{\sigma\sigma} \tag{R4}$$

*pictured in Figure 9. For every ordered pair* $(\sigma, \tau) \in S^2_W$ *with* $\sigma \neq \tau$*, the bi-chrome relations: The* $\sigma\tau$*-barbell,*

$$\mathsf{bar}(\tau) \otimes 1_{\sigma} - 1_{\sigma} \otimes \mathsf{bar}(\tau) = \langle \alpha^{\vee}_{\sigma}, \alpha_{\tau} \rangle (\mathsf{gap}(\sigma) - 1_{\sigma} \otimes \mathsf{bar}(\sigma)). \tag{R5}$$

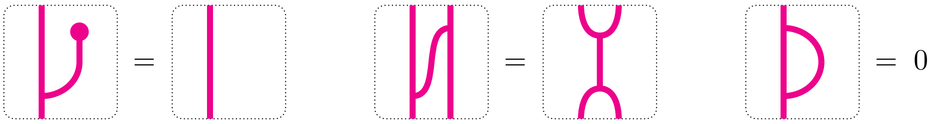


FIGURE 8. The fork-spot, double-fork, and circle-annihilation relations of (R3).

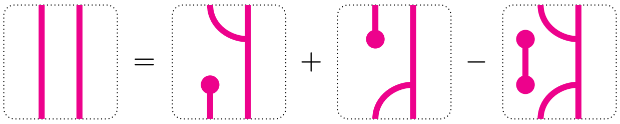


FIGURE 9. The cinching relation of (R4).

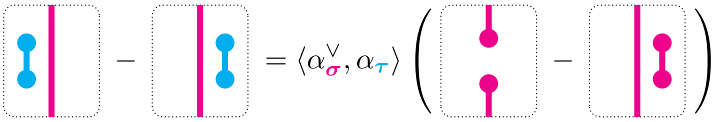


FIGURE 10. The two-colour barbell relation of (R5).

*For* $m = m_{\sigma\tau} \in \{2, 3, 4\}$ *we also have the fork-braid relations*

$$\begin{aligned} \mathsf{braid}^{\tau\sigma\cdots\sigma\tau}_{\sigma\tau\cdots\tau\sigma}(\mathsf{fork}^{\sigma}_{\sigma\sigma} \otimes 1^{m-1}_{\tau\sigma})(1_{\sigma} \otimes \mathsf{braid}^{\sigma\tau\cdots\tau\sigma}_{\tau\sigma\cdots\sigma\tau}) &= (1^{m-1}_{\tau\sigma} \otimes \mathsf{fork}^{\tau}_{\tau\tau})(\mathsf{braid}^{\tau\sigma\cdots\sigma\tau}_{\sigma\tau\cdots\tau\sigma} \otimes 1_{\tau}) \\ \mathsf{braid}^{\tau\sigma\cdots\tau\sigma}_{\sigma\tau\cdots\sigma\tau}(\mathsf{fork}^{\sigma}_{\sigma\sigma} \otimes 1^{m-1}_{\tau\sigma})(1_{\sigma} \otimes \mathsf{braid}^{\sigma\tau\cdots\sigma\tau}_{\tau\sigma\cdots\tau\sigma}) &= (1^{m-1}_{\tau\sigma} \otimes \mathsf{fork}^{\sigma}_{\sigma\sigma})(\mathsf{braid}^{\tau\sigma\cdots\tau\sigma}_{\sigma\tau\cdots\sigma\tau} \otimes 1_{\sigma}) \end{aligned} \tag{R6}$$

*for* $m$ *odd and even, respectively — these are pictured in Figure 11. We require the cyclicity relation,*

$$\begin{aligned} (1^{m}_{\tau\sigma} \otimes (\mathsf{cap}^{\emptyset}_{\sigma\sigma}))(1_{\tau} \otimes \mathsf{braid}^{\sigma\tau}_{\tau\sigma}(m) \otimes 1_{\sigma})(\mathsf{cup}^{\tau\tau}_{\emptyset} \otimes 1^{m}_{\sigma\tau}) &= \mathsf{braid}^{\tau\sigma\ldots\sigma\tau}_{\sigma\tau\cdots\tau\sigma} \\ (1^{m}_{\tau\sigma} \otimes (\mathsf{cap}^{\emptyset}_{\tau\tau}))(1_{\tau} \otimes \mathsf{braid}^{\sigma\tau}_{\tau\sigma}(m) \otimes 1_{\tau})(\mathsf{cup}^{\tau\tau}_{\emptyset} \otimes 1^{m}_{\sigma\tau}) &= \mathsf{braid}^{\tau\sigma\ldots\tau\sigma}_{\sigma\tau\cdots\sigma\tau} \end{aligned} \tag{R7}$$

*for* $m$ *odd or even, respectively.*

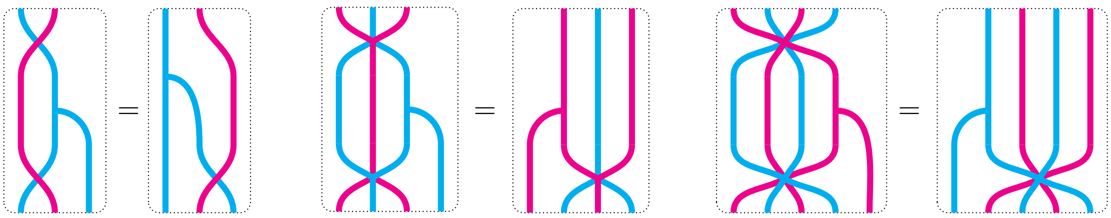


FIGURE 11. The fork braid relations of (R6) for $m(\sigma, \tau) = 2$ and 3 and 4 respectively.

*For* $m = 2, 3,$ *or* 4 *we have the double-braid relations*[2]

$$1_{\tau\sigma} = \mathsf{braid}^{\tau\sigma}_{\sigma\tau}\mathsf{braid}^{\sigma\tau}_{\tau\sigma} \qquad 1_{\sigma\tau\sigma} = \mathsf{braid}^{\sigma\tau\sigma}_{\tau\sigma\tau}\mathsf{braid}^{\tau\sigma\tau}_{\sigma\tau\sigma} - \mathsf{spot}^{\sigma\tau\sigma}_{\sigma\emptyset\sigma}\mathsf{fork}^{\sigma\sigma}_{\sigma}\mathsf{fork}^{\sigma}_{\sigma\sigma}\mathsf{spot}^{\sigma\emptyset\sigma}_{\sigma\tau\sigma} \tag{R8}$$

$$1_{\tau\sigma\tau\sigma} = \mathsf{braid}^{\tau\sigma\tau\sigma}_{\sigma\tau\sigma\tau}\mathsf{braid}^{\sigma\tau\sigma\tau}_{\tau\sigma\tau\sigma} + \langle \alpha^{\vee}_{\sigma}, \alpha_{\tau} \rangle \mathsf{spot}^{\tau\sigma\tau\sigma}_{\tau\emptyset\tau\sigma}\mathsf{fork}^{\tau\tau\sigma}_{\tau\sigma}\mathsf{fork}^{\tau\sigma}_{\tau\tau\sigma}\mathsf{spot}^{\tau\emptyset\sigma\tau}_{\tau\sigma\tau\sigma}$$

[2]The double-braid relations can replace the usual Jones–Wenzl relations, by [EMTW20, Exercise 9.39(1)].

$$
\begin{aligned}
&+\langle \alpha_{\tau}^{\vee}, \alpha_{\sigma}\rangle \mathsf{spot}^{\tau\sigma\tau\sigma}_{\tau\sigma\emptyset\sigma}\mathsf{fork}^{\tau\sigma\sigma}_{\tau\sigma}\mathsf{fork}^{\tau\sigma}_{\tau\sigma\sigma}\mathsf{spot}^{\tau\sigma\emptyset}_{\tau\sigma\tau\sigma} \\
&-\mathsf{spot}^{\tau\sigma\tau\sigma}_{\tau\emptyset\tau\sigma}\mathsf{fork}^{\tau\tau\sigma}_{\tau\sigma}\mathsf{fork}^{\tau\sigma}_{\tau\sigma\sigma}\mathsf{spot}^{\tau\sigma\emptyset\sigma}_{\tau\sigma\tau\sigma} - \mathsf{spot}^{\tau\sigma\tau\sigma}_{\tau\sigma\emptyset\sigma}\mathsf{fork}^{\tau\sigma\sigma}_{\tau\sigma}\mathsf{fork}^{\tau\sigma}_{\tau\tau\sigma}\mathsf{spot}^{\tau\emptyset\tau\sigma}_{\tau\sigma\tau\sigma}
\end{aligned} \tag{R9}
$$

*respectively, pictured in Figures 12 and 13. For* $(\sigma, \tau, \rho) \in S_W^3$ *with* $m_{\sigma\rho} = m_{\rho\tau} = 2$ *and* $m_{\sigma\tau} = m$*, we have the commutating-braids relation*

$$
\begin{aligned}
(\mathsf{braid}^{\tau\sigma\dots\sigma\tau}_{\sigma\tau\dots\tau\sigma} \otimes 1_{\rho})\mathsf{braid}^{\sigma\tau\cdots\tau\sigma\rho}_{\rho\sigma\tau\cdots\tau\sigma} &= \mathsf{braid}^{\tau\sigma\cdots\sigma\tau\rho}_{\rho\tau\sigma\cdots\sigma\tau}(1_{\rho} \otimes \mathsf{braid}^{\tau\sigma\dots\sigma\tau}_{\sigma\tau\dots\tau\sigma}) \\
(\mathsf{braid}^{\tau\sigma\dots\tau\sigma}_{\sigma\tau\dots\sigma\tau} \otimes 1_{\rho})\mathsf{braid}^{\sigma\tau\cdots\sigma\tau\rho}_{\rho\sigma\tau\cdots\sigma\tau} &= \mathsf{braid}^{\tau\sigma\cdots\sigma\tau\rho}_{\rho\tau\sigma\cdots\sigma\tau}(1_{\rho} \otimes \mathsf{braid}^{\tau\sigma\dots\sigma\tau}_{\sigma\tau\dots\tau\sigma}).
\end{aligned} \tag{R10}
$$

*for* $m$ *odd or even respectively, this is pictured in Figure 14.*

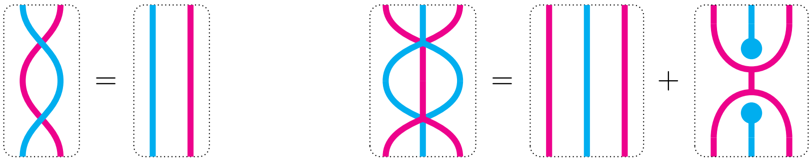


FIGURE 12. Double braid relations for $m(\sigma, \tau) = 2$ and 3 of (R8) respectively.

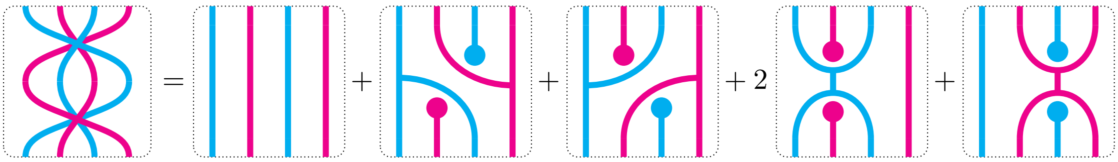



FIGURE 13. Double braid relations for $\sigma = s_2$ and $\tau = s_1$ in type $W = C_n$ as in (R9).

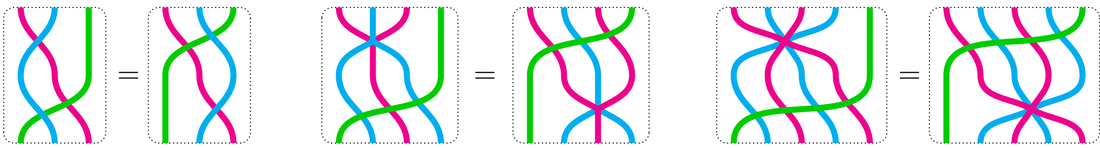


FIGURE 14. The commuting braids relation of (R10) for $m(\sigma, \rho) = 2 = m(\tau, \rho)$ and $m(\sigma, \tau) = 2$ and 3 and 4 respectively.

*Finally, we have the Zamolodchikov relations: for a triple* $\sigma, \tau, \rho \in S_W$ *with* $m_{\sigma\tau} = 3 = m_{\sigma\gamma}$ *and* $m_{\sigma\gamma} = 2$ *we have that*

$$
\begin{aligned}
&\mathsf{braid}^{\gamma\sigma\gamma\tau\sigma\gamma}_{\sigma\gamma\sigma\tau\sigma\gamma}\mathsf{braid}^{\sigma\gamma\sigma\tau\sigma\gamma}_{\sigma\gamma\tau\sigma\tau\gamma}\mathsf{braid}^{\sigma\gamma\tau\sigma\tau\gamma}_{\sigma\tau\gamma\sigma\gamma\tau}\mathsf{braid}^{\sigma\tau\gamma\sigma\gamma\tau}_{\sigma\tau\sigma\gamma\sigma\tau}\mathsf{braid}^{\sigma\tau\sigma\gamma\sigma\tau}_{\tau\sigma\tau\gamma\sigma\tau}\mathsf{braid}^{\tau\sigma\tau\gamma\sigma\tau}_{\tau\sigma\gamma\tau\sigma\tau} \\
=\;&\mathsf{braid}^{\gamma\sigma\gamma\tau\sigma\gamma}_{\gamma\sigma\tau\gamma\sigma\gamma}\mathsf{braid}^{\gamma\sigma\tau\gamma\sigma\gamma}_{\gamma\sigma\tau\sigma\gamma\sigma}\mathsf{braid}^{\gamma\sigma\tau\sigma\gamma\sigma}_{\gamma\tau\sigma\tau\gamma\sigma}\mathsf{braid}^{\gamma\tau\sigma\tau\gamma\sigma}_{\tau\gamma\sigma\gamma\tau\sigma}\mathsf{braid}^{\tau\gamma\sigma\gamma\tau\sigma}_{\tau\sigma\gamma\sigma\tau\sigma}\mathsf{braid}^{\tau\sigma\gamma\sigma\tau\sigma}_{\tau\sigma\gamma\tau\sigma\tau}.
\end{aligned} \tag{R11}
$$

*For a triple* $\sigma, \tau, \gamma \in S$ *such that* $m_{\sigma\gamma} = 4$*,* $m_{\tau\gamma} = 2$*,* $m_{\sigma\tau} = 3$*, we have that*

$$
\begin{aligned}
&\mathsf{braid}^{\gamma\sigma\gamma\tau\sigma\tau\gamma\sigma\tau}_{\gamma\sigma\gamma\sigma\tau\sigma\gamma\sigma\tau}\mathsf{braid}^{\gamma\sigma\gamma\sigma\tau\sigma\gamma\sigma\tau}_{\sigma\gamma\sigma\gamma\tau\sigma\gamma\sigma\tau}\mathsf{braid}^{\sigma\gamma\sigma\gamma\tau\sigma\gamma\sigma\tau}_{\sigma\gamma\sigma\tau\gamma\sigma\gamma\sigma\tau}\mathsf{braid}^{\sigma\gamma\sigma\tau\gamma\sigma\gamma\sigma\tau}_{\sigma\gamma\sigma\tau\sigma\gamma\sigma\gamma\tau}\mathsf{braid}^{\sigma\gamma\sigma\tau\sigma\gamma\sigma\gamma\tau}_{\sigma\gamma\tau\sigma\tau\gamma\sigma\gamma\tau}\times \\
&\mathsf{braid}^{\sigma\gamma\tau\sigma\tau\gamma\sigma\gamma\tau}_{\sigma\tau\gamma\sigma\gamma\tau\sigma\tau\gamma}\mathsf{braid}^{\sigma\tau\gamma\sigma\gamma\tau\sigma\tau\gamma}_{\sigma\tau\gamma\sigma\gamma\sigma\tau\sigma\gamma}\mathsf{braid}^{\sigma\tau\gamma\sigma\gamma\sigma\tau\sigma\gamma}_{\sigma\tau\sigma\gamma\sigma\gamma\tau\sigma\gamma}\mathsf{braid}^{\sigma\tau\sigma\gamma\sigma\gamma\tau\sigma\gamma}_{\tau\sigma\tau\gamma\sigma\gamma\tau\sigma\gamma}\mathsf{braid}^{\tau\sigma\tau\gamma\sigma\gamma\tau\sigma\gamma}_{\tau\sigma\gamma\tau\sigma\tau\gamma\sigma\gamma} \\
=\;&\mathsf{braid}^{\gamma\sigma\gamma\tau\sigma\tau\gamma\sigma\tau}_{\gamma\sigma\gamma\tau\sigma\gamma\tau\sigma\tau}\mathsf{braid}^{\gamma\sigma\gamma\tau\sigma\gamma\tau\sigma\tau}_{\gamma\sigma\tau\gamma\sigma\gamma\sigma\tau\sigma}\mathsf{braid}^{\gamma\sigma\tau\gamma\sigma\gamma\sigma\tau\sigma}_{\gamma\sigma\tau\sigma\gamma\sigma\gamma\tau\sigma}\mathsf{braid}^{\gamma\sigma\tau\sigma\gamma\sigma\gamma\tau\sigma}_{\gamma\tau\sigma\tau\gamma\sigma\gamma\tau\sigma}\mathsf{braid}^{\gamma\tau\sigma\tau\gamma\sigma\gamma\tau\sigma}_{\tau\gamma\sigma\gamma\tau\sigma\tau\gamma\sigma}\times \\
&\mathsf{braid}^{\tau\gamma\sigma\gamma\tau\sigma\tau\gamma\sigma}_{\tau\gamma\sigma\gamma\sigma\tau\sigma\gamma\sigma}\mathsf{braid}^{\tau\gamma\sigma\gamma\sigma\tau\sigma\gamma\sigma}_{\tau\sigma\gamma\sigma\gamma\tau\sigma\gamma\sigma}\mathsf{braid}^{\tau\sigma\gamma\sigma\gamma\tau\sigma\gamma\sigma}_{\tau\sigma\gamma\sigma\tau\gamma\sigma\gamma\sigma}\mathsf{braid}^{\tau\sigma\gamma\sigma\tau\gamma\sigma\gamma\sigma}_{\tau\sigma\gamma\sigma\tau\sigma\gamma\sigma\gamma}\mathsf{braid}^{\tau\sigma\gamma\sigma\tau\sigma\gamma\sigma\gamma}_{\tau\sigma\gamma\tau\sigma\tau\gamma\sigma\gamma}.
\end{aligned} \tag{R12}
$$

*Further, we require the interchange law*

$$(\mathsf{D}_1 \otimes \mathsf{D}_2) \circ (\mathsf{D}_3 \otimes \mathsf{D}_4) = (\mathsf{D}_1 \circ \mathsf{D}_3) \otimes (\mathsf{D}_2 \circ \mathsf{D}_4) \tag{R13}$$

*and the monoidal unit relation*

$$1_{\emptyset} \otimes \mathsf{D}_1 = \mathsf{D}_1 = \mathsf{D}_1 \otimes 1_{\emptyset} \tag{R14}$$

*for all diagrams* $\mathsf{D}_1, \mathsf{D}_2, \mathsf{D}_3, \mathsf{D}_4$*. Finally, we require the following* non-local *cyclotomic relations*

$$\mathsf{bar}(\boldsymbol{\sigma}) \otimes D = 0 \qquad \text{for all } \boldsymbol{\sigma} \in S_W \text{ and } D \text{ any diagram,} \tag{R15}$$

$$1_{\boldsymbol{\tau}} \otimes D = 0 \qquad \text{for all } \boldsymbol{\tau} \in S_P \subset S_W \text{ and } D \text{ any diagram.} \tag{R16}$$

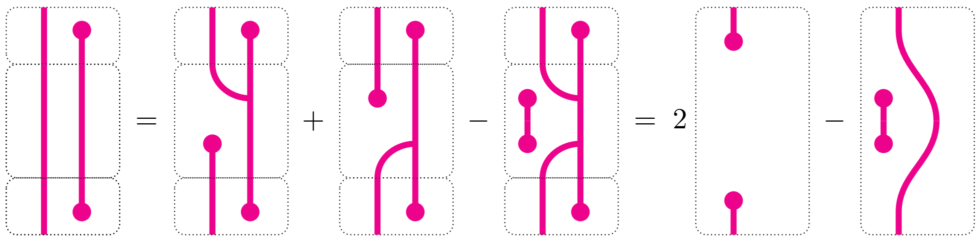
The algebra $\mathcal{H}_{(W,P)}$ can be equipped with a $\mathbb{Z}$-grading which preserves the duality $*$. The degrees of the generators under this grading are defined as follows:

$$\mathsf{deg}(1_\emptyset) = 0 \quad \mathsf{deg}(1_{\boldsymbol{\sigma}}) = 0 \quad \mathsf{deg}(\mathsf{spot}^{\emptyset}_{\boldsymbol{\sigma}}) = 1 \quad \mathsf{deg}(\mathsf{fork}^{\boldsymbol{\sigma}}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}) = -1 \quad \mathsf{deg}(\mathsf{braid}^{\boldsymbol{\sigma}\boldsymbol{\tau}}_{\boldsymbol{\tau}\boldsymbol{\sigma}}(m)) = 0$$

for $\boldsymbol{\sigma}, \boldsymbol{\tau} \in S_W$ arbitrary and $m \geqslant 2$. We will define the $p$-Kazhdan–Lusztig polynomials of these categories in (3.4).

**Remark 1.13.** *Of the defining relations of Definition 1.12, we have only diagrammatically depicted those which will explicitly appear in the arguments of this paper; the remaining relations can be found for example in* [EW16, Bow25]*).*

**Remark 1.14.** *We can pre- and post-multiply the relation depicted in Figure 9 with spot generators. We hence obtain the usual "one colour Demazure relation" as follows:*


(1.1)

**Remark 1.15.** *The Hecke category is usually defined in the literature using the one-colour Demazure relation* (1.1) *instead of the cinching relation* (R4)*, plus an additional technical assumption called* Demazure surjectivity [EW16, Assumption 3.7]*. In this setting Demazure surjectivity is necessary to prove the cinching relation, which is essential for the Hecke category to be well behaved (more precisely, for the light leaves construction in Section 3 to yield a basis). However a careful analysis of the proof in* [EW16, §7] *shows that Demazure surjectivity is* not *necessary for the Hecke category to be well behaved if one assumes the cinching relation to begin with! In other words, our definition of* $\mathcal{H}_{(W,P)}$ *is* always *well behaved, and is equivalent to the usual definition of the Hecke category in the literature when the latter is well behaved. We believe this trivial observation has been overlooked until now due to historical motivation of the diagrammatic Hecke category from Soergel bimodules, which rely on Demazure surjectivity much more heavily.*

## 2. Lifting full-commutativity to the Hecke categories of Hermitian symmetric pairs

Stembridge proved that the parabolic quotients for Hermitian symmetric pairs are fully commutative [Ste96, Theorem 6.1]. In other words, in ${}^PW$ the non-commuting braid *relations* are redundant. (This is discussed in more detail in terms of Temperley–Lieb diagrammatics in the companion paper [BDF+25].) We now lift this idea to the 2-categorical level; in $\mathcal{H}_{(W,P)}$ the non-commuting braid *generators* are redundant.

**Remark 2.1.** *By a "local relation" of the algebra* $\mathcal{H}_{(W,P)}$ *we mean any relation that can be applied in* an arbitrary *local neighbourhood of a diagram (as opposed to the relations in* (R15) *and* (R16) *which can only be applied to the leftmost edge of a diagram). In this section we state and prove new* **local relations** *of* $\mathcal{H}_{(W,P)}$ *— our proofs will make use of the* **non-local** *relations of* (R15) *and* (R16)*. To see how this works, we first observe that any element of* $\mathcal{H}_{(W,P)}$ *can be obtained by vertical concatenation of diagrams of the form* $1_{\underline{x}} \otimes D \otimes 1_{\underline{y}}$ *for* $\underline{x}$ *and* $\underline{y}$ *two expressions for* $x, y \in W$ *and* $D$ *a diagram as in* (G1) *or* (G2)*. Thus proving a local relation* $D_1 \circ D_2 = D_3$ *is equivalent to proving the non-local relation*

$$1_{\underline{x}} \otimes D_1 \circ D_2 \otimes 1_{\underline{y}} = 1_{\underline{x}} \otimes D_3 \otimes 1_{\underline{y}}$$

*for all possible* $\underline{x}$ *and* $\underline{y}$ *for* $x, y \in W$*. In fact, by* [LW22, Theorem 5.3]*, it is enough to consider only* reduced *expressions* $\underline{x}, \underline{y}$ *for* $x, y \in W$*.*

**Theorem 2.2.** *In* $\mathcal{H}_{(W,P)}$ *we have the local relation* $\mathsf{braid}^{\sigma\tau}_{\tau\sigma}(m) = 0$ *for any* $m = m(\sigma, \tau) > 2$*.*

**Proposition 2.3.** *Let* $w \in {}^P W$ *and let* $\underline{w}, \underline{w}'$ *be a pair of reduced expressions for* $w$*. We have that*

$$1_{\underline{w}} = \mathsf{braid}^{\underline{w}}_{\underline{w}'} \mathsf{braid}^{\underline{w}'}_{\underline{w}} \qquad 1_{\underline{w}'} = \mathsf{braid}^{\underline{w}'}_{\underline{w}} \mathsf{braid}^{\underline{w}}_{\underline{w}'}$$

*Proof.* For $(W, P)$ a Hermitian symmetric pair, [Ste96, Theorem 6.1] implies that any two words $\underline{w}$ and $\underline{w}'$ in ${}^P W$ differ only by application of the commuting braid relations of the Coxeter group. These lift to commuting braid generators in $\mathcal{H}_{(W,P)}$ and the result follows. □

We are now ready to prove the main result of this section.

*Proof of Theorem 2.2.* As in Remark 2.1, it will suffice to prove that

$$1_{\mathsf{s}} \otimes \mathsf{braid}^{\sigma\tau}_{\tau\sigma}(m) = 0,$$

for all $\mathsf{s} \in \mathrm{Std}(\lambda)$ and all $\lambda \in \mathscr{P}_{(W,P)}$. We will proceed by induction on the $\ell = \ell(\lambda)$. If $\lambda = \varnothing$, then $\mathsf{braid}^{\sigma\tau}_{\tau\sigma}(m) = 0$ for all $m \geqslant 2$ by relation R16. In what follows, we only explicitly consider the cases for which neither of $1_{\mathsf{s}} \otimes 1_{\sigma}$ or $1_{\mathsf{s}} \otimes 1_{\tau}$ is equal to zero by application of the commutativity and cyclotomic relations (as these cases are trivial). Now assume that $\ell(\lambda) \geqslant 1$. We have two cases to consider.

**Case 1.** Either $\sigma$ or $\tau \in \mathsf{Rem}(\lambda)$. We consider the former case, as the latter is identical and we set $\mu = \lambda - \sigma$ and we can assume that $\mathsf{t} \in \mathrm{Std}(\mu)$ so that $\mathsf{t} \otimes \sigma = \mathsf{s}$ (by Proposition 2.3). For $m > 2$, we have that

$$\begin{aligned}
&1_{\mathsf{s}} \otimes \mathsf{braid}^{\sigma\tau}_{\tau\sigma}(m)\\
=&1_{\mathsf{t}} \otimes 1_{\sigma} \otimes \mathsf{braid}^{\sigma\tau}_{\tau\sigma}(m)\\
=&1_{\mathsf{t}} \otimes (1_{\sigma} \otimes 1_{\sigma} \otimes 1^{m-1}_{\tau\sigma})(1_{\sigma} \otimes \mathsf{braid}^{\sigma\tau}_{\tau\sigma}(m))\\
=&1_{\mathsf{t}} \otimes ((\mathsf{spot}^{\emptyset}_{\sigma} \otimes \mathsf{fork}^{\sigma\sigma}_{\sigma} + \mathsf{spot}^{\sigma}_{\emptyset} \otimes \mathsf{fork}^{\sigma}_{\sigma\sigma} - \mathsf{bar}(\sigma) \otimes \mathsf{dork}^{\sigma\sigma}_{\sigma\sigma}) \otimes 1^{m-1}_{\tau\sigma})(1_{\sigma} \otimes \mathsf{braid}^{\sigma\tau}_{\tau\sigma}(m)).
\end{aligned}$$

where the first two equalities are trivial and the final equality is an application of the cinching relation (R4) visualised in Figure 9. This is depicted in diagrammatically in Figure 15. The first term is zero by induction, since it factors through $1_{\mathsf{t}} \otimes \mathsf{braid}^{\sigma\tau}_{\tau\sigma}(m)$ with $\ell(\mathsf{t}) = \ell(\mathsf{s}) - 1$. The other two terms factor through a diagram of the form

$$1_{\mathsf{t}} \otimes (\mathsf{fork}^{\sigma}_{\sigma\sigma} \otimes 1^{m-1}_{\tau\sigma})(1_{\sigma} \otimes \mathsf{braid}^{\sigma\tau}_{\tau\sigma}(m)),$$

and we can apply the fork-braid relation (R6) visualised in Figure 11 and hence obtain

$$1_{\mathsf{t}} \otimes \mathsf{braid}^{\sigma\tau\sigma}_{\tau\sigma\tau}(1_{\tau\sigma} \otimes \mathsf{fork}^{\tau}_{\tau\tau})(\mathsf{braid}^{\tau\sigma\tau}_{\sigma\tau\sigma} \otimes 1_{\tau})$$
$$1_{\mathsf{t}} \otimes \mathsf{braid}^{\sigma\tau\sigma\tau}_{\tau\sigma\tau\sigma}(1_{\tau\sigma\tau} \otimes \mathsf{fork}^{\sigma}_{\sigma\sigma})(\mathsf{braid}^{\tau\sigma\tau\sigma}_{\sigma\tau\sigma\tau} \otimes 1_{\sigma})$$

for $m = 3$ or $4$ respectively. In both cases, this is zero by induction.

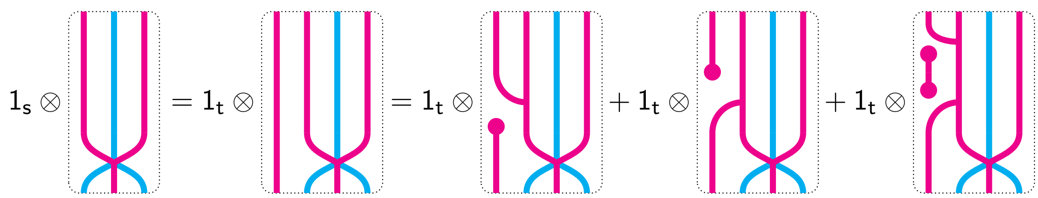


FIGURE 15. Case 1 of the proof of Theorem 2.2 with $\mathsf{s} = \mathsf{t} \otimes \sigma$ and $m(\sigma, \tau) = 3$. The first term after the second equality is zero by induction, the latter two terms require an application of the fork-braid relation pictured in Figure 11 before they can be deduced to be zero.

**Case 2.** It remains to consider the case that $\sigma, \tau \notin \mathsf{Rem}(\lambda)$. We first note that if there exists $\rho \in \mathsf{Rem}(\lambda)$ with $m(\sigma,\rho) = 2 = m(\tau,\rho)$ then we can assume (by Proposition 2.3) that $\mathsf{s} = \mathsf{t} \otimes \rho$ for $\mathsf{t} \in \mathrm{Std}(\lambda - \rho)$ and

$$1_{\mathsf{s}} \otimes \mathsf{braid}^{\sigma\tau}_{\tau\sigma}(m) = 1_{\mathsf{t}} \otimes (\mathsf{braid}^{\rho\tau\sigma\cdots}_{\tau\sigma\ldots\rho}(\mathsf{braid}^{\sigma\tau}_{\tau\sigma}(m) \otimes 1_\rho)\mathsf{braid}^{\tau\sigma\ldots\rho}_{\rho\tau\sigma\ldots}) = 0$$

where the second equality follows by the commuting braid relation (R10) pictured in Figure 14. and the third follows by induction (as $\ell(\mathsf{t}) = \ell(\mathsf{s}) - 1$). Thus for the remainder of Case 2 we can assume without loss of generality that $\mathsf{s} = \mathsf{t} \otimes \rho$ such that $\rho \neq \tau, \sigma$ and that $m(\sigma,\rho) > 2$ (and that $\sigma, \tau \notin \mathsf{Rem}(\lambda)$).

**Subcase when $\sigma \notin \mathsf{Add}(\lambda)$.** We first consider the "generic" case in which $\sigma \notin \mathsf{Add}(\lambda)$. Our assumptions (and inspection of the $\mathscr{A}_{(W,P)}$) imply that $\sigma \in \mathsf{Rem}(\lambda - \rho)$; furthermore if $m(\rho,\sigma) = 4$, then $\rho \in \mathsf{Rem}(\lambda - \rho - \sigma)$. By Proposition 2.3 we can assume that

$$\mathsf{s} = \begin{cases} \mathsf{t} \otimes \sigma \otimes \rho & \text{for } m(\rho,\sigma) = 3 \text{ and some } \mathsf{s} \in \mathrm{Std}(\lambda - \rho - \sigma) \\ \mathsf{t} \otimes \rho \otimes \sigma \otimes \rho & \text{for } m(\rho,\sigma) = 4 \text{ and some } \mathsf{s} \in \mathrm{Std}(\lambda - \rho - \sigma - \rho) \end{cases} \tag{2.1}$$

this covers all instances of types $(A, A \times A)$, $(C, A)$, and $(D, A)$ cases. We consider the first generic case in which $m(\rho,\sigma) = 3$ (and $m = m(\tau,\sigma) > 2$). We have

$$\begin{aligned} &1_{\mathsf{t}} \otimes 1_{\sigma\rho} \otimes \mathsf{braid}^{\sigma\tau}_{\tau\sigma}(m) \\ =&1_{\mathsf{t}} \otimes ((\mathsf{braid}^{\sigma\rho\sigma}_{\rho\sigma\rho}\mathsf{braid}^{\rho\sigma\rho}_{\sigma\rho\sigma} \otimes 1^{m-1}_{\tau\sigma} - \mathsf{spot}^{\sigma\rho\sigma}_{\sigma\emptyset\sigma}\mathsf{dork}^{\sigma\sigma}_{\sigma\sigma}\mathsf{spot}^{\sigma\emptyset\sigma}_{\sigma\rho\sigma}))(1_{\sigma\rho} \otimes \mathsf{braid}^{\sigma\tau}_{\tau\sigma}(m)) \end{aligned} \tag{2.2}$$

using the double-braid relation. We now observe that

$$1_{\mathsf{t}} \otimes \mathsf{braid}^{\sigma\rho\sigma}_{\rho\sigma\rho} \otimes 1^{m-1}_{\tau\sigma} = 0 \qquad\qquad 1_{\mathsf{t}} \otimes 1_\sigma \otimes \mathsf{spot}^{\emptyset}_{\rho} \otimes \mathsf{braid}^{\sigma\tau}_{\tau\sigma}(m) = 0$$

by induction (since $\ell(\mathsf{t}), \ell(\mathsf{t} \otimes \sigma) < \ell(\mathsf{s})$) and so both terms in (2.2) are zero, as required. We now consider the second generic case in which $m(\rho,\sigma) = 4$ (which implies that $m = m(\tau,\sigma) = 3$). We have that

$$\begin{aligned} &1_{\mathsf{t}} \otimes 1_{\rho\sigma\rho} \otimes \mathsf{braid}^{\sigma\tau\sigma}_{\tau\sigma\tau} \\ =&1_{\mathsf{t}} \otimes (\mathsf{braid}^{\rho\sigma\rho\sigma}_{\sigma\rho\sigma\rho}\mathsf{braid}^{\sigma\rho\sigma\rho}_{\rho\sigma\rho\sigma} \otimes 1_{\tau\sigma} + \langle\alpha^\vee_\sigma, \alpha_\rho\rangle\mathsf{spot}^{\rho\sigma\rho}_{\rho\emptyset\rho}\mathsf{dork}^{\rho\rho}_{\rho\rho}\mathsf{spot}^{\rho\emptyset\rho}_{\rho\sigma\rho} \otimes 1_{\sigma\tau\sigma} \\ &+ \langle\alpha^\vee_\rho, \alpha_\sigma\rangle\mathsf{spot}^{\rho\sigma\rho\sigma}_{\rho\sigma\emptyset\sigma}\mathsf{fork}^{\rho\sigma\sigma}_{\rho\sigma}\mathsf{fork}^{\rho\sigma}_{\rho\sigma\sigma}\mathsf{spot}^{\rho\sigma\emptyset\sigma}_{\rho\sigma\rho\sigma} \otimes 1_{\tau\sigma} \\ &- \mathsf{spot}^{\rho\sigma\rho\sigma\tau\sigma}_{\rho\emptyset\rho\sigma\tau\sigma}\mathsf{fork}^{\rho\rho\sigma\tau\sigma}_{\rho\sigma\sigma\tau\sigma}\mathsf{spot}^{\rho\sigma\emptyset\sigma\tau\sigma}_{\rho\sigma\rho\sigma\tau\sigma}) - \mathsf{spot}^{\rho\sigma\rho\sigma\tau\sigma}_{\rho\sigma\emptyset\sigma\tau\sigma}\mathsf{fork}^{\rho\sigma\sigma\tau\sigma}_{\rho\rho\sigma\tau\sigma}\mathsf{spot}^{\rho\emptyset\rho\sigma\tau\sigma}_{\rho\sigma\rho\sigma\tau\sigma})(1_{\rho\sigma\rho} \otimes \mathsf{braid}^{\sigma\tau\sigma}_{\tau\sigma\tau}) \end{aligned}$$

and all of these terms are zero by induction on length (similarly to the $m(\rho,\sigma) = 3$ case, above). The righthand-side of this equation is depicted in Figure 16 in type $(C, A)$.

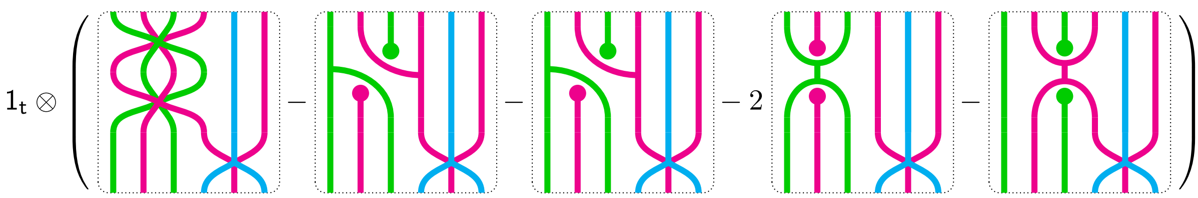


FIGURE 16. Rewriting the diagram $1_{\mathsf{t}} \otimes 1_{\rho\sigma\rho} \otimes \mathsf{braid}^{\sigma\tau\sigma}_{\tau\sigma\tau}$ in type $(C, A)$. The type $(B, B)$ case is similar, but with the coefficients on the last two terms switched.

**Exceptional subcases where $\sigma \in \mathsf{Add}(\lambda)$.** If $\sigma \in \mathsf{Add}(\lambda)$, this implies that $\tau \notin \mathsf{Add}(\lambda)$, because the addable tiles for a tile partition must commute with each other. (This is a general fact about fully commutative elements of Coxeter groups; for the analogous statement regarding removable tiles see e.g. the proof of [Ste96, Theorem 4.2].) It remains to consider this case. We remark that our assumptions on $\lambda$ and the fact that $\sigma \in \mathsf{Add}(\lambda)$ implies that we must be in one of types $(D_n, D_{n-1})$, $(B_n, B_{n-1})$ and exceptional type and so we refer to this as the "exceptional" case. In this case, we can write $\mu \subseteq \lambda$ where $|\lambda| - |\mu| = L$ is maximal such that $\mathsf{t}_{\mu\setminus\lambda} = s_{i_1} \ldots s_{i_L}$ and $m(s_{i_k}, \tau) = 2$ for all $1 \leqslant k \leqslant L$ and such that $1_{\mathsf{s}\otimes\tau} = 0$ for any $\mathsf{s} \in \mathrm{Std}(\mu)$ by (possibly repeated application of) the double-braid (pictured in Figure 12) and cyclotomic relations (R16).

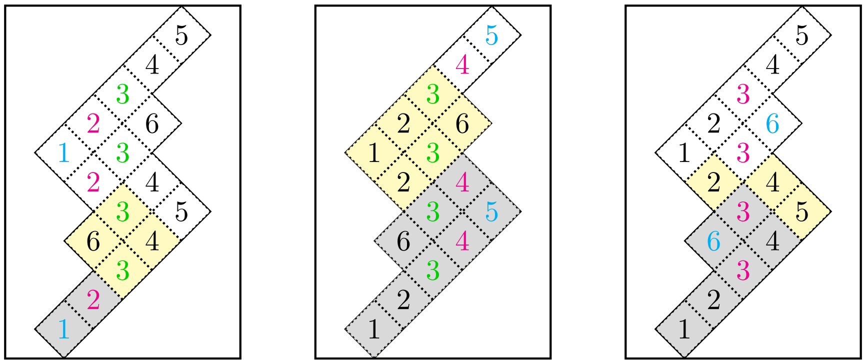
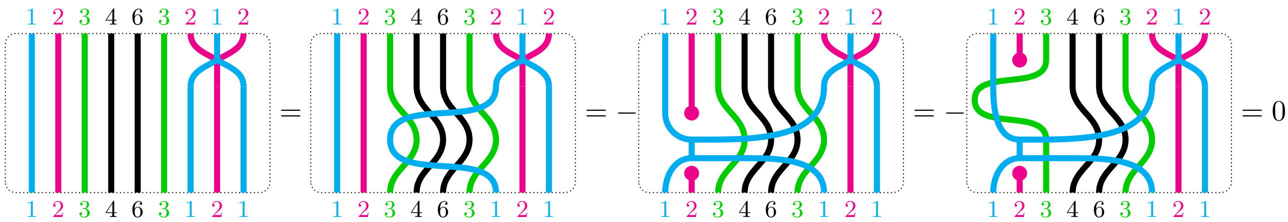


FIGURE 17. Some examples of the "exceptional" examples for which $\sigma \in \mathsf{Add}(\lambda)$ in Case 2 of the proof of Figure 17. The region $\mu \subseteq \lambda$ is pictured in yellow (with $\mu$ pictured in grey). In the first case $(\sigma, \tau) = (s_2, s_1)$ and $\rho = s_3$. In the second case, $(\sigma, \tau) = (s_4, s_5)$ and $\rho = s_3$. In the third case $(\sigma, \tau) = (s_3, s_6)$ and $\rho \in \{s_2, s_4\}$. The colouring of nodes is chosen to emphasise the roles of $\sigma, \tau, \rho$ in each case (and so is inconsistent with the colouring of Figure 6).

Rather than go into the word combinatorics for each exceptional case in detail (as they are all very similar) we simply check one of these exceptional cases here. Further illustrative examples are in Figure 17 (and we leave these as an exercise for the reader). Let $\lambda = (1^2, 2^2) \in \mathscr{P}_{(W,P)}$ for $(W, P) = (E_6, D_5)$ with $\sigma = s_2 \in \mathsf{Add}(\lambda)$ and $\rho = s_3 \in \mathsf{Rem}(\lambda)$ with $m(\sigma, \rho) = 3$ as pictured in Figure 17. We have that $(1^2) = \mu \subseteq \lambda = (1^2, 2^2)$ and setting $\tau = s_1$ we note that $m(\tau, s_i) = 2$ for $s_i \in \mu \setminus \lambda = s_3 s_4 s_6 s_3$. Using the colouring of the leftmost diagram in Figure 17, we have that

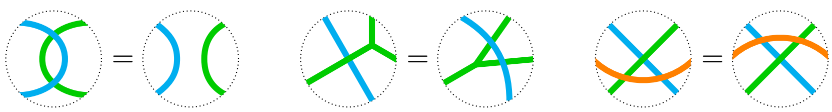



The first equality follows by the double braid relation for $m(s_i, \tau) = 2$ of (R8) (pictured on the left in Figure 12) for $i = 3, 4, 6$. The second equality follows by the double-braid relation (R8) (pictured in Figure 12) and the cyclotomic relation (R15); the third equality follows by the commutation relation (R8); the fourth by the cyclotomic relation (R15). $\square$

We now state the obvious corollary, for ease of reference.

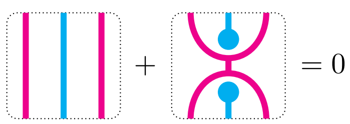
**Definition 2.4.** *We define a* simple Soergel diagram *to be any Soergel diagram which does not contain any barbells or* $\mathsf{braid}^{\sigma\tau}_{\tau\sigma}(m)$ *for* $m = m(\sigma, \tau) > 2$.

**Corollary 2.5.** *Let* $(W, P)$ *be a Hermitian symmetric pair. We can define* $\mathcal{H}_{(W,P)}$ *to be the locally-unital associative* $\Bbbk$*-algebra spanned by all simple Soergel diagrams with multiplication given by vertical concatenation of diagrams modulo relations R1, R2, R3, R4, R5, R13, R14, R15, R16, for* $(\gamma, \tau, \rho) \in S^3$ *with* $m(\gamma, \rho) = m(\rho, \tau) = m(\gamma, \tau) = 2$*, we have the commutation relations*

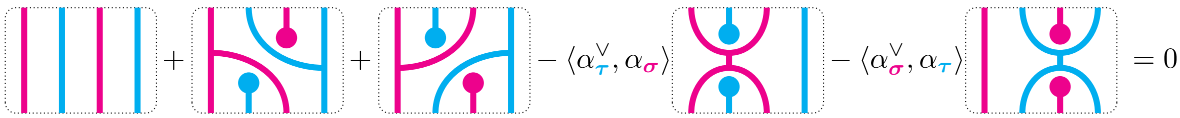


(2.3)

*and for* $m(\sigma, \tau) = 3$*, we have the null-braid relation*



*and for $m(\sigma,\tau)=4$, we have the null-braid relation*



*and their horizontal flips.*

## 3. Light leaves for the Hecke categories of Hermitian symmetric pairs

In this section, we recall Libedinsky–Williamson's construction of the light leaves basis in the case of Hermitian symmetric pairs. This could have been done in Subsection 1.4, however we delayed until now so that we could simplify the presentation of this material by virtue of Theorem 2.2. We regard $\mathcal{H}_{(W,P)}$ as a locally unital associative algebra in the sense of [BS24, Section 2.2] via the following idempotent decomposition

$$\mathcal{H}_{(W,P)} = \bigoplus_{\substack{\underline{x}\in \mathsf{exp}(x)\\ \underline{y}\in \mathsf{exp}(y)\\ x,y\in {}^PW}} 1_{\underline{x}}\mathcal{H}_{(W,P)}1_{\underline{y}}$$

**Remark 3.1.** *Given $\mathsf{s},\mathsf{t}\in \mathrm{Std}(\lambda)$, by Proposition 2.3 we have that*

$$\mathsf{braid}^{\mathsf{s}}_{\mathsf{t}} \circ 1_{\mathsf{t}} \circ \mathsf{braid}^{\mathsf{t}}_{\mathsf{s}} = 1_{\mathsf{s}} \qquad \mathsf{braid}^{\mathsf{t}}_{\mathsf{s}} \circ 1_{\mathsf{s}} \circ \mathsf{braid}^{\mathsf{s}}_{\mathsf{t}} = 1_{\mathsf{t}}$$

*Thus from now on, we may fix any preferred choice of $\mathsf{t}_\lambda\in \mathrm{Std}(\lambda)$, for each $\lambda\in\mathscr{P}_{(W,P)}$.*

By Remark 3.1, we can truncate the set of weights to be "as small as possible".

**Definition 3.2.** *We set*

$$1_{(W,P)} = \textstyle\sum_{\mu\in\mathscr{P}_{(W,P)}} 1_{\mathsf{t}_\mu} \qquad \textit{and} \qquad h_{(W,P)} = 1_{(W,P)}\mathcal{H}_{(W,P)}1_{(W,P)}.$$

Let $\lambda,\mu\in\mathscr{P}_{(W,P)}$ and $\mathsf{T}\in \mathrm{Path}_\ell(\lambda,\mathsf{t}_\mu)$ be a path of the form

$$\mathsf{T}: \varnothing=\lambda_0\to\lambda_1\to\lambda_2\to\cdots\to\lambda_\ell=\lambda$$

and we let $\tau\in\mathrm{Add}(\mu)$. If $\tau\in\mathrm{Add}(\lambda)$, we set $\lambda^+=\lambda\tau=\lambda+\tau$ and $\lambda^-=\lambda$. If $\tau\in\mathrm{Rem}(\lambda)$, we set $\lambda^+=\lambda$ and $\lambda^-=\lambda-\tau$. We set $\mathsf{T}^+$ and $\mathsf{T}^-$ to be the paths

$$\mathsf{T}^+: \varnothing=\lambda_0\to\lambda_1\to\lambda_2\to\cdots\to\lambda_\ell\to\lambda^+ \qquad \mathsf{T}^-: \varnothing=\lambda_0\to\lambda_1\to\lambda_2\to\cdots\to\lambda_\ell\to\lambda^-.$$

For the empty path $\mathsf{T}^\varnothing$ we set $c_{\mathsf{T}^\varnothing}=1_\varnothing$ to be empty diagram. We now inductively define the basis via certain "add" and "remove" operators (denoted $A^\pm_\tau$ and $R^\pm_\tau$ respectively). If $\tau\in\mathrm{Add}(\lambda)$, then we define

$$A^+_\tau(c_\mathsf{T}) := \mathsf{braid}^{\mathsf{t}_{\lambda+\tau}}_{\mathsf{t}_\lambda\otimes\tau}(c_\mathsf{T}\otimes 1_\tau) \qquad A^-_\tau(c_\mathsf{T}) := c_\mathsf{T}\otimes \mathsf{spot}^{\emptyset}_{\tau}.$$

and we set $c_{\mathsf{T}^+}=A^+_\tau(c_\mathsf{T})$ and $c_{\mathsf{T}^-}=A^-_\tau(c_\mathsf{T})$. If $\tau\in\mathrm{Rem}(\lambda)$, then $\lambda=\lambda'\tau$ and we define

$$R^+_\tau(c_\mathsf{T}) := \mathsf{braid}^{\mathsf{t}_\lambda}_{\mathsf{t}_{\lambda'}\otimes\tau}(1_{\mathsf{t}_{\lambda'}}\otimes\mathsf{fork}^{\tau}_{\tau\tau})(\mathsf{braid}^{\mathsf{t}_{\lambda'}\otimes\tau}_{\mathsf{t}_\lambda}c_\mathsf{T}\otimes 1_\tau)$$

$$R^-_\tau(c_\mathsf{T}) := (1_{\mathsf{t}_{\lambda'}}\otimes\mathsf{cap}^{\emptyset}_{\tau\tau})(\mathsf{braid}^{\mathsf{t}_{\lambda'}\otimes\tau}_{\mathsf{t}_\lambda}c_\mathsf{T}\otimes 1_\tau)$$

and we set $c_{\mathsf{T}^+}=R^+_\tau(c_\mathsf{T})$ and $c_{\mathsf{T}^-}=R^-_\tau(c_\mathsf{T})$. An example is given in the rightmost diagram in Figure 18.

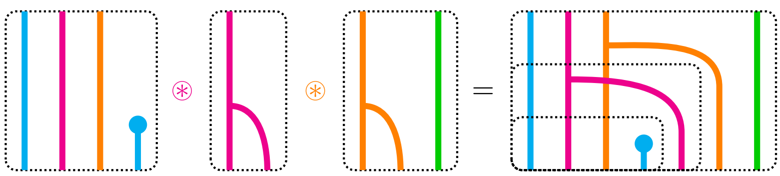


Figure 18. Construction of a light leaves basis element (for $(A_5, A_2\times A_2)$, whose Bruhat graph is the leftmost of Figure 1) using the diagrammatic composition defined in Definition 3.9. The rightmost diagram is equal to $A^+_{s_1}R^+_{s_4}R^+_{s_2}A^-_{s_3}A^+_{s_4}A^+_{s_2}A^+_{s_3}(1_\emptyset)$

We are hence able to equip the algebras $h_{(W,P)}$ with powerful "light leaves" graded cellular bases which encode a great deal of representation theoretic information.

**Theorem 3.3** ([EW16, Section 6.4] and [LW22, Theorem 5.3])**.** *Let $\lambda_1 \prec \lambda_2 \prec \cdots \prec \lambda_t$ be any total refinement of the Bruhat order $\leqslant$ on $\mathscr{P}_{(W,P)}$. The algebra $h_{(W,P)}$ has a chain of two-sided ideals*

$$0 \subset h_{(W,P)}1_{\lambda_1}h_{(W,P)} \subset h_{(W,P)}(1_{\lambda_1}+1_{\lambda_2})h_{(W,P)} \subset \cdots \subset h_{(W,P)}(1_{\lambda_1}+1_{\lambda_2}+\cdots+1_{\lambda_t})h_{(W,P)} = h_{(W,P)}$$

*such that*

$$\{c^{\lambda_k}_{\mathsf{ST}} := c^*_{\mathsf{S}}c_{\mathsf{T}} \mid \mathsf{S} \in \mathrm{Path}_{(W,P)}(\lambda_k, \mathsf{t}_\mu), \mathsf{T} \in \mathrm{Path}_{(W,P)}(\lambda_k, \mathsf{t}_\nu), \mu, \nu \in \mathscr{P}_{(W,P)}\} \tag{3.1}$$

*is a $\Bbbk$-basis of $h_{(W,P)}1_{\lambda_k}h_{(W,P)}/h_{(W,P)}1_{\lambda_{k-1}}h_{(W,P)}$. Thus $h_{(W,P)}$ is is a graded cellular algebra in the sense of* [HM10] *and a quasi-hereditary algebra in the sense of* [CPS88].

**Remark 3.4.** *The careful reader will notice that we have* not *assumed the* parabolic property *from* [LW22, §2.3]*, as this can fail in positive characteristic. It turns out that this condition is not always necessary to obtain a light leaves basis as in Theorem 3.3 above. In our setting, the trick in* [BHN22, Example 1.11(1)] *is always enough to show that the light leaves construction yields a basis in any characteristic (including* p = 2*, by way of Remark 1.15). In more detail: let $(\mathbb{K}, \mathcal{O}, \Bbbk)$ be a p-modular system. In other words, suppose $\mathcal{O}$ is a complete discrete valuation ring whose residue field is $\Bbbk$ and whose field of fractions $\mathbb{K}$ is of characteristic 0. Since $W$ is a Weyl group there is a standard $\mathcal{O}$-form $\mathcal{H}^{\mathcal{O}}_{(W,P)}$ of the algebra $\mathcal{H}_{(W,P)}$, which arises from the $\mathcal{O}$-form of the geometric realisation of $W$. This realisation is faithful (and thus satisfies the parabolic property) because the fraction field $\mathbb{K}$ is of characteristic 0. By* [LW22, Theorem 5.3] *the light leaves construction yields a basis for $\mathcal{H}^{\mathcal{O}}_{(W,P)}$, which descends to a light leaves basis for $\mathcal{H}_{(W,P)}$.*

**Remark 3.5.** *Whilst the quasi-heredity property is not mentioned explicitly in* [EW16, Section 6.4] *and* [LW22, Theorem 5.3]*, one of the first theorems in the literature on cellular algebras was that they are quasi-hereditary if and only if each layer of the cell-filtration has an idempotent, as above (see* [KX98, Proposition 4.1]*). We refer to a quasi-hereditary algebra as having a* highest weight structure *as in* [CPS88].

When it cannot result in confusion, we write $c_{\mathsf{ST}}$ for $c^{\lambda}_{\mathsf{ST}}$. For $\mu \in \mathscr{P}_{(W,P)}$, we define one-sided ideals

$$h^{\leqslant\mu}_{(W,P)} = 1_{\leqslant\mu}h_{(W,P)} \qquad\qquad h^{<\mu}_{(W,P)} = h^{\leqslant\mu}_{(W,P)} \cap \Bbbk\{c^{\lambda}_{\mathsf{ST}} \mid \mathsf{S},\mathsf{T} \in \mathrm{Path}(\lambda), \lambda < \mu\}$$

and we hence define the **standard** or **cell** modules of $h_{(W,P)}$ as follows

$$\Delta(\mu) = \{c^{\mu}_{\mathsf{S}} := c^{\mu}_{\mathsf{t}^\mu\mathsf{S}} + h^{<\mu}_{(W,P)} \mid \mathsf{S} \in \mathrm{Path}(\mu, \mathsf{t}_\nu), \nu \in \mathscr{P}_{(W,P)}\}. \tag{3.2}$$

We recall that the cellular structure allows us to define, for each $\mu \in \mathscr{P}_{(W,P)}$, a bilinear form $\langle\ ,\ \rangle^\mu$ on $\Delta(\mu)$ which is determined by

$$c^{\mu}_{\mathsf{ST}}c^{\mu}_{\mathsf{UV}} \equiv \langle c_{\mathsf{T}}, c_{\mathsf{U}}\rangle^\mu c^{\mu}_{\mathsf{SV}} \pmod{h^{<\mu}_{(W,P)}} \tag{3.3}$$

for any $\mathsf{S},\mathsf{T},\mathsf{U},\mathsf{V} \in \mathrm{Path}(\mu, -)$. We obtain a complete set of non-isomorphic simple modules for $h_{(W,P)}$ as follows

$$L(\mu) = \Delta(\mu)/\mathrm{rad}(\langle\ ,\ \rangle^\mu)$$

for $\mu \in \mathscr{P}_{(W,P)}$. The projective indecomposable $h_{(W,P)}$-modules are the direct summands

$$1_{\mathsf{t}_\mu}h_{(W,P)} = \bigoplus_{\lambda\leqslant\mu} \dim_q(L(\lambda)1_{\mathsf{t}_\mu})P(\lambda).$$

For $\Bbbk$ a field of characteristic $p \geqslant 0$, the **anti-spherical $p$-Kazhdan–Lusztig polynomials** are defined as follows,

$$^{p}n_{\lambda,\mu}(q) := \dim_q(\mathrm{Hom}_{h_{(W,P)}}(P(\lambda), \Delta(\mu)) = \sum_{k\in\mathbb{Z}}[\Delta(\mu) : L(\lambda)\langle k\rangle]q^k \tag{3.4}$$

for any $\lambda, \mu \in \mathscr{P}_{(W,P)}$. These polynomials were first defined via the diagrammatic character of [EW16, Definition 6.23] and [LW22, Section 8] and rephrased as above in [Pla17, Theorem 4.8].

Elias–Williamson and Libedinsky–Williamson [EW14, LW22] proved that over a field of characteristic $p = 0$, the (anti-spherical) $p$-Kazhdan–Lusztig polynomials are, in fact, equal to the classical (anti-spherical) Kazhdan–Lusztig polynomials of [Deo87, Soe97] thus justifying the nomenclature. Interesting families of $p$-Kazhdan–Lusztig polynomials calculated in the literature include Williamson's torsion explosion examples [Wil17] and examples of Fiebig, Lanini–McNamara and Libedinsky–Williamson, [Fie12, Fie10, LM21, LW17]; algorithms for calculating the $p$-Kazhdan–Lusztig polynomials are considered in [GJW23, JW17].

**Remark 3.6.** *We note that $\mathcal{H}_{(W,P)}$ and $h_{(W,P)}$ are graded Morita equivalent (as the latter is obtained from the former by a truncation which does not kill any simple module).*

**Remark 3.7.** *The trivial and sign representations of the Hecke algebra give rise to the spherical and anti-spherical Hecke categories, respectively. The latter of which are the focus of this paper and are more well-studied than their spherical counterparts (see for example [RW18, LW18, GJW23, BCHM22, BHN22, BCH23]). The spherical Hecke categories are much more mysterious: constructing presentations of these categories (generalising Definition 1.12) is an important open problem. The problem of constructing bases of these categories for type $(A_n, A_{k-1} \times A_{n-k})$ was solved in [Pat22] using tiling combinatorics akin to this paper; generalising this to all parabolic Coxeter systems has been the subject of much recent work [EK23, EKLP, EKLP24]. The p-(in)dependence of spherical p-Kazhdan–Lusztig polynomials is the focus of a recent preprint by Baine [Bai].*

3.1. **A "singular" horizontal concatenation.** Singular Soergel bimodules were first considered in Williamson's thesis, where it was proven that they categorify the Hecke algebroid [Wil11]. At present, we do not have a diagrammatic construction of the category of singular Soergel bimodules (although some progress has been made, see [EMTW20, Chapter 24]). In this paper, we will give a complete realisation of singular Soergel bimodules within the diagrammatic Hecke category for $(W, P)$ a Hermitian symmetric pair. In order to accomplish this goal, we first need to provide a Soergel-diagrammatic analogue of the tensor product (denoted $\otimes_{R^{\tau}}$) for singular Soergel bimodules "on a $\tau$-hyperplane" for $\tau \in S_W$ (which we will denote by $\circledast$).

**Definition 3.8.** *We suppose that a diagram $D \in 1_{\underline{x}\tau}\mathcal{H}_{(W,P)}1_{\underline{y}\tau}$ is such that (i) the rightmost $\tau$ in the northern boundary is connected to the rightmost $\tau$ in the southern boundary by a strand and (ii) there are no barbells to the right of this strand. We say that such a diagram has a* a final exposed propagating $\tau$-strand. *Similarly, we define a* first exposed propagating $\tau$-strand *by reflecting this definition through the vertical axis.*

**Definition 3.9.** *We let $D_1 \in 1_{\underline{x}\tau}\mathcal{H}_{(W,P)}1_{\underline{y}\tau}$ and $D_2 \in 1_{\tau\underline{u}}\mathcal{H}_{(W,P)}1_{\tau\underline{v}}$. We suppose that $D_1$ (respectively $D_2$) has a final (respectively first) exposed propagating $\tau$-strand. We define*

$$D_1 \circledast D_2 = (D_1 \otimes 1_{\underline{u}})(1_{\underline{y}} \otimes D_2) = (1_{\underline{x}} \otimes D_2)(D_1 \otimes 1_{\underline{v}}) \tag{3.5}$$

*Now suppose that $D_1' = \mathsf{braid}^{\underline{w}}_{\underline{x}\tau} D_1 \mathsf{braid}^{\underline{y}\tau}_{\underline{z}} \neq 0$. We extend the above definition as follows*

$$D_1' \circledast D_2 = (\mathsf{braid}^{\underline{w}}_{\underline{x}\tau} \otimes 1_{\underline{u}})(D_1 \circledast D_2)(\mathsf{braid}^{\underline{y}\tau}_{\underline{z}} \otimes 1_{\underline{v}}). \tag{3.6}$$

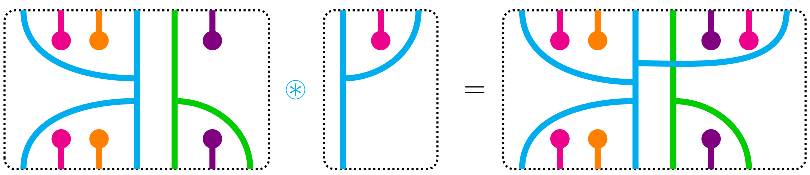


FIGURE 19. An example of $\circledast$ merging the rightmost and leftmost $\tau$-strands.

**Remark 3.10.** *Diagrammatically, we can think of $\circledast$ as identifying the rightmost $\tau$-strand of $D_1^{(\prime)}$ with the leftmost $\tau$-strand of $D_2$. For examples, see Figures 18, 19 and 56.*

**Proposition 3.11.** *The operation* $\circledast$ *satisfies the interchange law:*

$$(D_1 \circledast D_2) \circ (D_3 \circledast D_4) = (D_1 \circ D_3) \circledast (D_2 \circ D_4).$$

*Proof.* For equation (3.5), the result follows immediately by diagram chasing using the fact that

$$(D_1 \otimes 1_{\underline{u}})(1_{\underline{y}} \otimes D_2) = (1_{\underline{x}} \otimes D_2)(D_1 \otimes 1_{\underline{v}}) \tag{3.7}$$

The case of equation (3.6) follows by applying commuting braid generators. □

We will abuse notation and use $\circledast$ as a shorthand as follows.

**Definition 3.12.** *We let* $D_1 \in 1_{\underline{x}\tau}\mathcal{H}_{(W,P)}1_{\underline{y}\tau}$ *and* $D_2 \in 1_{\tau\underline{u}}\mathcal{H}_{(W,P)}1_{\tau\underline{v}}$. *We suppose that* $D_1$ *(respectively* $D_2$*) has a final (respectively first) exposed propagating* $\tau$*-strand. We define*

$$D_1 \circledast (\mathsf{spot}^{\emptyset}_{\tau} \otimes 1_{\underline{u}})D_2) = (1_{\underline{x}} \otimes \mathsf{spot}^{\emptyset}_{\tau} \otimes 1_{\underline{u}})(D_1 \circledast D_2). \tag{3.8}$$

*We extend this via commuting braid generators in an analogous fashion to equation* (3.6).

**Remark 3.13.** *We note that the operation in equation* (3.8) *considers non-propagating* $\tau$*-strands and therefore does not satisfy the interchange law (as equation* (3.7) *no longer holds).*

## 4. Categorical combinatorial invariance

In this section we provide new presentations of the anti-spherical Hecke categories $\mathcal{H}_{(W,P)}$ for $(W,P)$ a simply laced Hermitian symmetric pair. Our presentations encode the combinatorics of the Bruhat graph more effectively than those of Definition 1.12 and Corollary 2.5.

**Definition 4.1.** *We let* $\lambda, \mu \in {}^{P}W$ *and we set* $\Pi = [\lambda, \mu] := \{\nu \mid \lambda \leqslant \nu \leqslant \mu\}$. *We let*

$$e = \sum_{\lambda \nleqslant \nu} 1_\nu \quad \text{and} \quad f = \sum_{\nu \leqslant \mu} 1_\nu \tag{4.1}$$

*in* $\mathcal{H}_{(W,P)}$. *We let* $\mathcal{H}^{\Pi}_{(W,P)}$ *denote the subquotient of* $\mathcal{H}_{(W,P)}$ *given by*

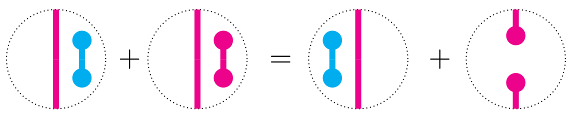
$$\mathcal{H}^{\Pi}_{(W,P)} = f(\mathcal{H}_{(W,P)}/(\mathcal{H}_{(W,P)}e\mathcal{H}_{(W,P)}))f.$$

*We set* $h_\Pi = 1_{(W,P)}\mathcal{H}^{\Pi}_{(W,P)}1_{(W,P)}$ *as in Definition 3.2.*

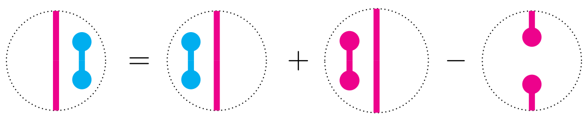
**Theorem 4.2.** *Let* $\Pi = [\lambda, \mu]$ *and* $\Pi = [\lambda', \mu']$ *be subquotients of the Bruhat graphs of Hermitian symmetric pairs* $\lambda, \mu \in (W,P)$ *and* $\lambda', \mu' \in (W', P')$. *If* $\Pi$ *and* $\Pi'$ *are isomorphic as partially ordered sets, then the corresponding subquotients* $\mathcal{H}^{\Pi}_{(W,P)}$ *and* $\mathcal{H}^{\Pi'}_{(W',P')}$ *are Morita equivalent (in the sense of* [Ben98, Section 2.2]*) and this equivalence preserves the grading, cellular, and highest-weight structures of these algebras.*

This section is dedicated to the proof of Theorem 4.2. In order to do this, we must first provide "Tetris-style" presentations of these categories.

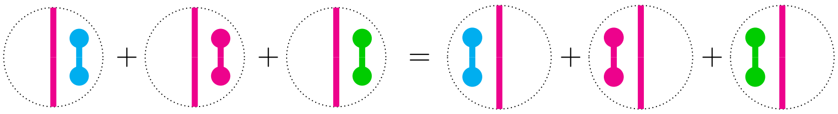
4.1. **Tetris combinatorics for "gaps" in reduced words.** In what follows we let $\sigma, \tau \in S_W$ with $m(\sigma,\tau) = 3$ or 4. If $m(\sigma,\tau) = 4$ and $(W,P) = (C_n, A_{n-1})$, then suppose that $(\sigma,\tau) = (s_1, s_2)$ or if that $(W,P) = (B_n, B_{n-1})$, then $(\sigma,\tau) = (s_2, s_1)$. Then $\langle \alpha^\vee_\sigma, \alpha_\tau \rangle = -1$, and the two-colour $\sigma\tau$-barbell relation is


(4.2)

Using the one-colour barbell relation we then obtain


(4.3)

Finally, for $m(\sigma,\tau)=3=m(\sigma,\rho)$, by summing over the one and two colour barbell relations we obtain the following



(4.4)

We now provide inductive versions of the equation (4.2) and (4.3). First, we define a type $A$ string $T\subseteq S_W$ to be an ordered set of reflections $s_{i_1},\dots,s_{i_t}$ such that $m(s_{i_j},s_{i_k})=2+\delta_{j,k+1}+\delta_{j,k-1}$ for $1\leqslant j,k\leqslant t$. By induction on equation (4.3) we have the following:

**Lemma 4.3.** *Let $T\subseteq S_W$ be a type $A$ string. We have the following local relation*

$$1_{s_{i_1}s_{i_2}\dots s_{i_{t-1}}}\otimes \mathsf{bar}(s_{i_t})=\sum_{k=1}^{t}\mathsf{bar}(s_{i_k})\otimes 1_{s_{i_1}s_{i_2}\dots s_{i_{t-1}}}-\sum_{k=1}^{t-1}1_{s_{i_1}s_{i_2}\dots s_{i_{k-1}}}\otimes \mathsf{gap}(s_{i_k})\otimes 1_{s_{i_{k+1}}\dots s_{i_{t-1}}}$$

By induction on equation (4.2) we have the following:

**Lemma 4.4.** *Let $T\subseteq S_W$ be a type $A$ string. We have the following local relation*

$$\sum_{k=1}^{t}1_{s_{i_2}s_{i_3}\dots s_{i_t}}\otimes \mathsf{bar}(s_{i_k})=\mathsf{bar}(s_{i_1})\otimes 1_{s_{i_2}s_{i_3}\dots s_{i_t}}+\sum_{k=2}^{t}1_{s_{i_2}s_{i_3}\dots s_{i_{k-1}}}\otimes \mathsf{gap}(s_{i_k})\otimes 1_{s_{i_{k+1}}\dots s_{i_t}}$$

**Lemma 4.5.** *Let $\beta,\gamma\in S_W$ be such that $m(\beta,\gamma)=3$. We have the following local relation*

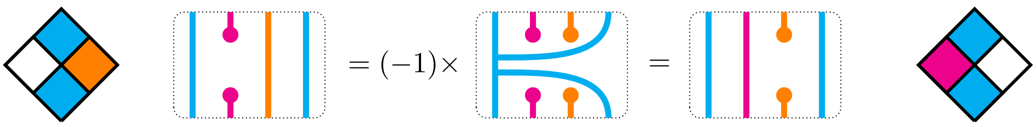
$$1_\gamma\otimes(\mathsf{bar}(\beta)+\mathsf{bar}(\gamma))\otimes 1_\beta=0. \tag{4.5}$$

*Proof.* We have that

$$\begin{aligned}1_\gamma\otimes(\mathsf{bar}(\beta)+\mathsf{bar}(\gamma))\otimes 1_\beta&=1_\gamma\otimes 1_\beta\otimes \mathsf{bar}(\gamma)+1_\gamma\otimes \mathsf{gap}(\beta)\\&=\mathsf{spot}^{\gamma\beta\emptyset}_{\gamma\beta\gamma}\left(1_{\gamma\beta\gamma}+\mathsf{spot}^{\gamma\beta\gamma}_{\gamma\emptyset\gamma}\mathsf{dork}^{\gamma\gamma}_{\gamma\gamma}\mathsf{spot}^{\gamma\emptyset\gamma}_{\gamma\beta\gamma}\right)\mathsf{spot}^{\gamma\beta\gamma}_{\gamma\beta\emptyset}\end{aligned}$$

and so the result follows from the $\beta\gamma$-null-braid relation. □

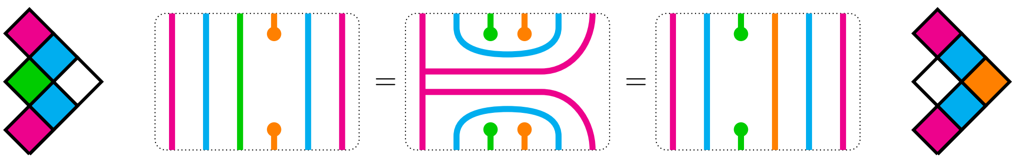
**Lemma 4.6.** *Suppose that $m(\sigma,\tau)=3=m(\gamma,\tau)$ and $m(\sigma,\gamma)=2$. We have that*



(4.6)

*Proof.* We apply the $\tau\gamma$- and $\sigma\tau$-null-braid relation to the left and righthand-sides of (4.6) respectively. The result follows. □

**Lemma 4.7.** *Suppose that $m(\sigma,\tau)=m(\gamma,\tau)=m(\rho,\tau)=3$ and $m(\sigma,\gamma)=m(\rho,\gamma)=m(\sigma,\rho)=2$. We have that*



(4.7)

*Proof.* We apply the $\tau\rho$- (respectively $\tau\gamma$-) and $\sigma\tau$-null-braid relation to the lefthand-side (respectively righthand-side) of equation (4.7) respectively. The result follows. □

**Proposition 4.8.** *Suppose that we are in one of the following cases: (i) $(W,P)=(D_n,A_{n-1})$ and $\{\sigma,\tau\}=\{s_0,s_1\}$ (ii) $(W,P)=(A_n,A_{n-1})$ and $m(\sigma,\tau)=2$ (iii) $(W,P)=(D_n,D_{n-1})$ and $m(\sigma,\tau)=2$ with $\{\sigma,\tau\}\neq\{s_0,s_1\}$. Then we have the following local relation: $1_{\sigma\tau}=0$ in $\mathcal{H}_{(W,P)}$.*

*Proof.* As in Remark 2.1, it will suffice to prove that $1_{\mathsf{s}}\otimes 1_{\sigma\tau}=0$, for all $\mathsf{s}\in\mathrm{Std}(\mu)$ and all $\mu\in\mathscr{P}_{(W,P)}$. We will proceed by induction on $\ell(\mu)=\ell\geqslant 0$ with the base case $1_{\sigma\tau}=0$ being trivial for any $m(\sigma,\tau)=2$ (by the commuting relations (2.3) and the cyclotomic relation (R16)). Now assume that $\ell(\mu)\geqslant 1$. We let $\rho\in\mathsf{Rem}(\mu)$ and set $\mu'=\mu-\rho$. If $\rho=\sigma$ (or similarly $\tau$), then

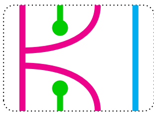
$$1_\mu=1_{\mu'}\otimes 1_\rho\otimes 1_{\sigma\tau}=1_{\mu'}\otimes 1_\sigma\otimes 1_{\sigma\tau}=1_{\mu'}\otimes 1_\sigma\otimes \mathsf{braid}^{\sigma\tau}_{\tau\sigma}1_{\tau\sigma}\mathsf{braid}^{\tau\sigma}_{\sigma\tau}=0$$

and similarly, if $m(\boldsymbol{\rho},\boldsymbol{\sigma}) = 2 = m(\boldsymbol{\rho},\boldsymbol{\tau})$ then

$$1_\mu = 1_{\mu'} \otimes 1_{\boldsymbol{\rho}} \otimes 1_{\boldsymbol{\sigma\tau}} = 1_{\mu'} \otimes \mathsf{braid}^{\boldsymbol{\rho\sigma\tau}}_{\boldsymbol{\sigma\tau\rho}} 1_{\boldsymbol{\sigma\tau\rho}} \mathsf{braid}^{\boldsymbol{\rho\sigma\tau}}_{\boldsymbol{\sigma\tau\rho}} = 0$$

by induction (as $\ell(\mu') < \ell(\mu)$) by the leftmost and rightmost equations in (2.3) respectively. Thus we can assume, without loss of generality, that $m(\boldsymbol{\sigma},\boldsymbol{\rho}) = 3$. We consider each type in turn.

For $(W,P) = (D_n, A_{n-1})$ it remains to consider the case that $\rho = s_2 \in \mathsf{Rem}(\mu)$ is the unique removable node. We will assume that $\mathsf{t}_\mu = \mathsf{t}_{\mu''} \otimes s_0 \otimes s_2$ (the case $\mathsf{t}_\mu = \mathsf{t}_{\mu''} \otimes s_1 \otimes s_2$ is identical). We have that

$$1_{\mathsf{t}_\mu} \otimes 1_{s_0 s_1} = 1_{\mathsf{t}_{\mu''}} \otimes 1_{s_0 s_2 s_0 s_1} = -1_{\mathsf{t}_{\mu''}} \otimes$$  $= 0$

by induction on $\ell(\mu'') < \ell(\mu)$. (We note that $1_{\mathsf{t}_\mu} \otimes 1_{s_1 s_0} = 0$ by further application of the commutativity relation (2.3)).

For $(W,P) = (A_n, A_{n-1})$ we can assume without loss of generality that $\boldsymbol{\sigma} = s_i$ and $\boldsymbol{\tau} = s_j$ for $i \leqslant j-2$. Let $\mu = s_1 \dots s_k$ and $j \leqslant k+1$, we have that $1_{\mathsf{t}_\mu} \otimes 1_{s_i} \otimes 1_{s_j} = 0$ by the $(s_{i-1}, s_i)$-null-braid relation, the leftmost commutativity relation of (2.3), and the cyclotomic relation (R16). For $j \geqslant k+2$ if follows immediately from the commutativity and cyclotomic relations.

For $(W,P) = (D_n, D_{n-1})$ we can assume without loss of generality that $\boldsymbol{\sigma} = s_i$ and $\boldsymbol{\tau} = s_j$ for $i \leqslant j-2$. If $\mu \subseteq s_{n-1}s_{n-2}\dots s_2 s_0$ or $s_{n-1}s_{n-2}\dots s_2 s_1$ the result follows as in type $A$. It remains to consider $s_{n-1}s_{n-2}\dots s_2 s_1 s_0 \subseteq \mu$ and we assume without loss of generality that $j \geqslant 3$ if $i = 0$. If $\mu = s_{n-1}s_{n-2}\dots s_2 s_1 s_0$ then $j-2 \geqslant i \geqslant 2$ and we can apply the $(s_j, s_{j-1})$-null-braid relation, the leftmost commutativity relation of (2.3), and the cyclotomic relation (R16).

Finally, if $\mu = s_{n-1}s_{n-2}\dots s_2 s_1 s_0 s_2 \dots s_k$ for $k \geqslant 2$. If $j \geqslant k+2$ (respectively $j \leqslant k+1$), we apply the $(s_j, s_{j+1})$-null-braid (respectively $(s_i, s_{i+1})$-null-braid) relation and the commutativity and cyclotomic relations. □

**Definition 4.9.** *For $(W,P)$ and $\boldsymbol{\sigma},\boldsymbol{\tau}$ as in Proposition 4.8, we will call the corresponding braid generator* $\mathsf{braid}^{\boldsymbol{\sigma\tau}}_{\boldsymbol{\tau\sigma}}$ *a* zero braid generator.

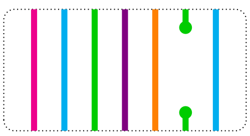
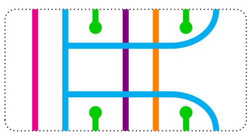
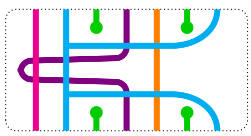
**Proposition 4.10.** *(1) Let $W$ be simply laced and $[r,c] \in \mu \in \mathscr{P}_{(W,P)}$. If $[r,c-1] \notin \mathscr{A}_{(W,P)}$ (respectively $[r-1,c] \notin \mathscr{A}_{(W,P)}$) then $\mathsf{gap}(\mathsf{t}_\mu - [r,c-1]) = 0$ (respectively $\mathsf{gap}(\mathsf{t}_\mu - [r-1,c]) = 0$).*

*(2) For $(W,P) = (C_n, A_{n-1})$, $\mu \in \mathscr{P}_{(W,P)}$ and $[r,c] \in \mu$ with $[r,c-1] \notin \mathscr{A}_{(W,P)}$ we have $\mathsf{gap}(\mathsf{t}_\mu - [r-1,c]) = 0$.*

*(3) For $(W,P) = (B_n, B_{n-1})$ and $\mu \in \mathscr{P}_{(W,P)}$, if $[1,c] \in \mu$ then $\mathsf{gap}(\mathsf{t}_\mu - [1,c-1]) = 0$ and if $[r,n] \in \mu$ with $r \geqslant 3$ then $\mathsf{gap}(\mathsf{t}_\mu - [r-1,n]) = 0$.*

*Proof.* (1) First assume that $r = 1$ or $c = 1$. In either case the result follows by the leftmost commutativity relation of (2.3), and the cyclotomic relation (R16). We now consider the types in turn. Note that $(W,P) = (A_n, A_k \times A_{n-k-1})$ follows from the $r = 1$ and $c = 1$ cases. For $(W,P) = (D_n, A_{n-1})$ we can assume that $s_{[r,c-1]} = s_2$ and $s_{[r,c]} = s_0$ or $s_1$ and the result follows from Proposition 4.8. Type $(D_n, D_{n-1})$ also follows from Proposition 4.8. For the exceptional Weyl groups, there are two types of subcase to consider. If $[r,c]$ is a ♠ tile in Figure 20 then then $\mathsf{gap}(\mathsf{t}_\mu - [r,c-1])$ or $\mathsf{gap}(\mathsf{t}_\mu - [r-1,c])$ is zero immediately by the leftmost commuting relation of (2.3) and the cyclotomic relation (R16).

We now consider the more interesting subcases where $[r,c]$ is a ♣ tile in Figure 20. The ♣ cases can all be treated uniformly using some applications of the null-braid relations and the leftmost commuting relation of (2.3) and the cyclotomic relation (R16); rather than checking them all explicitly, we will just provide an illustrative example. For $(W,P) = (E_6, D_5)$ with colouring as in Figures 6 and 20 and $[r,c] = [4,3]$, with $\lambda_{[4,3]} = (1^2,2,3)$ we have that

$\mathsf{gap}(\mathsf{t}_{(1^2,2,3)} - [4,2]) =$  $=$  $=$  $= 0$

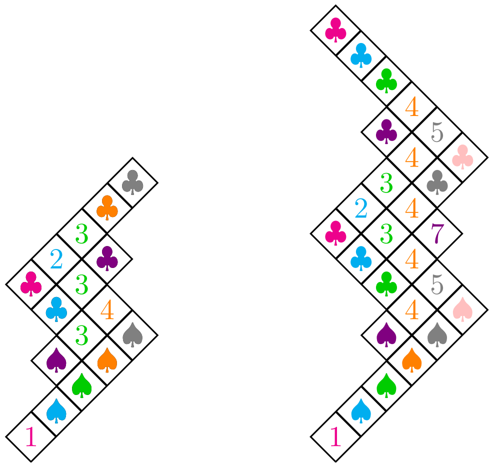


FIGURE 20. The tilings for $(W, P)$ of type $(E_6, D_5)$ or $(E_7, E_6)$. The cases that $[r, c]$ is a ♠ tile all follow from commutativity and cyclotomic relations. The ♣ cases further require some applications of the null-braid relations.

he first equality is by definition, the second follows from the $(s_2, s_3)$-null braid and the leftmost commutativity relation of (2.3), the penultimate equality is again by commutativity, and the final equality holds by the cyclotomic relation (R16).

Statement (2) and the first part of statement (3) are immediate from the commutation and cyclotomic relation (R16). The second part of statement (3) follows by commutativity, repeated application of the nullbraid relation and cyclotomic relation. □

4.2. **Gaps as basis elements.** Fix $\mu \in \mathscr{P}_{(W,P)}$ for $(W, P) = (A_n, A_{k-1} \times A_{n-k})$ or $(D_n, A_{n-1})$ and let $[r, c] \in \mu$. Let $l, k$ be the maximal non-negative integer such that $[r-i, c+i] \in \mu$ for all $0 \leqslant i \leqslant l$, $[r-i+1, c+i] \in \mu$ for all $1 \leqslant i \leqslant l$ and $[r+j, c-j] \in \mu$ for all $0 \leqslant j \leqslant k$, $[r+j, c-j+1] \in \mu$ for all $1 \leqslant j \leqslant k$. Then define the path generated by the tile $[r, c] \in \mu$, denoted by $\langle r, c\rangle_\mu$, to be the collection of tiles

$$[r-k, c+k], [r-k+1, c+k], \dots [r-1, c+1], [r, c+1], [r, c], [r+1, c], [r+1, c-1], \dots, [r+l, c-l]$$

see for example, the pink regions in Figure 21. Now, assume that $[r-k, c+k+1] \notin \mu$ and $[r+l+1, c-l] \notin \mu$. Suppose first that $[r-k+1, c+k+1] \notin \mu$. In this case we choose $\lambda \subseteq \nu \subseteq \mu$ such that $\nu \setminus \lambda = \langle r, c\rangle_\mu$ and $\mathsf{t}_\mu$ such that $\mathsf{t}_\mu = \mathsf{t}_\lambda \circ \mathsf{t}_{\nu\setminus\lambda} \circ \mathsf{t}_{\mu\setminus\nu}$ and we let $s_{[r-k,c+k]} = s_i$ and $s_{[r+l,c-l]} = s_j$ with $i \leqslant j$. We define

$$c_{\mathsf{S}} = ((R^+_{j-1}A^-_j) \dots (R^+_{i+3}A^-_{i+4})(R^+_{i+1}A^-_{i+2})A^-_i(1_{\mathsf{t}_\lambda})) \otimes 1_{\mathsf{t}_{\mu\setminus\nu}}.$$

Now suppose $[r-k+1, c+k+1] \in \mu$. Note that, as we assumed $[r-k, c+k+1] \notin \mu$, this case can only happen when $(W, P) = (D, A)$ and $s_{[r-k,c+k]} = s_0$ or $s_1$. Now take $m \geqslant 1$ maximal such that

$$[r-k+1, c+k+1], [r-k+2, c+k], \dots [r-k+m, c+k-m+2] \in \mu.$$

We let $\langle\langle r, c\rangle\rangle_\mu$ denote the collection of tiles

$$[r-k+1, c+k+1], [r-k, c+k-1], \dots [r-k+m, c+k-m+2], [r-k+m-1, c+k-m]$$

see for example, the blue region in Figure 21.

Now we choose $\lambda \subseteq \nu \subseteq \mu$ such that $\nu \setminus \lambda = \langle r, c\rangle_\mu \sqcup \langle\langle r, c\rangle\rangle_\mu$ and $\mathsf{t}_\mu$ such that $\mathsf{t}_\mu = \mathsf{t}_\lambda \circ \mathsf{t}_{\nu\setminus\lambda} \circ \mathsf{t}_{\mu\setminus\nu}$. By assumption, $s_{[r-k,c+k]} = s_0$ or $s_1$ and in the former case we define

$$c_{\mathsf{S}} = ((R^+_{2k+2l}A^-_{2k+2l+1}) \dots (R^+_{2m+2}A^-_{2m+3}))((R^+_{2m-1}R^-_{2m}A^-_{2m+1}A^+_{2m}) \dots (R^+_1R^-_2A^-_3A^+_2))A^-_0(1_{\mathsf{t}_\lambda}))\otimes 1_{\mathsf{t}_{\mu\setminus\nu}}.$$

and in the latter case we define

$$c_{\mathsf{S}} = ((R^+_{2k+2l}A^-_{2k+2l+1}) \dots (R^+_{2m+2}A^-_{2m+3}))((R^+_{2m-1}R^-_{2m}A^-_{2m+1}A^+_{2m}) \dots (R^+_0R^-_2A^-_3A^+_2))A^-_1(1_{\mathsf{t}_\lambda}))\otimes 1_{\mathsf{t}_{\mu\setminus\nu}}.$$

Examples are depicted in Figures 22 and 23.

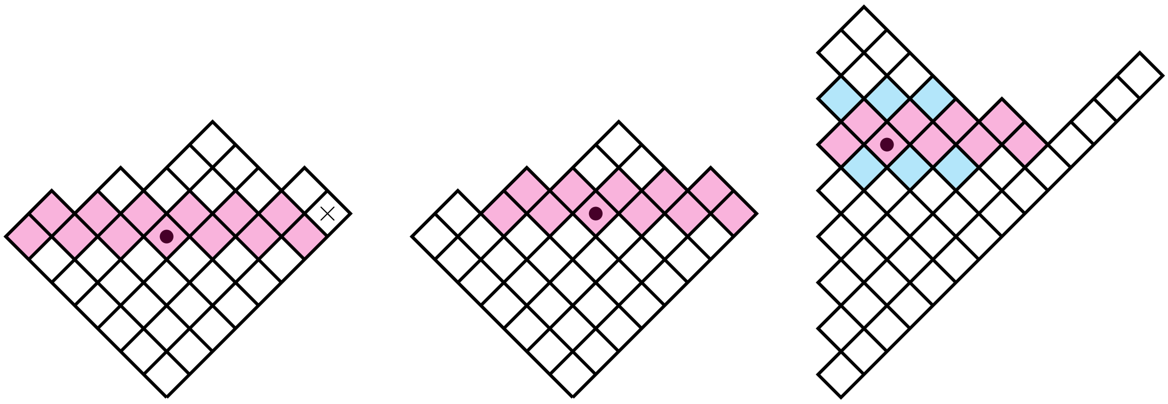


FIGURE 21. The first two cases depict $\langle r,c\rangle_\mu$ for $[r,c]=[4,4],[5,4]\in\mu=(7^2,6^2,5^3,2)\in\mathscr{P}_{(W,P)}$ with $(W,P)=(A,A\times A)$. In the first case we have $k=3$ and $l=3$. In the second case we have $k=2$ and $l=3$. Notice that in the first case the $\times$ denotes the box $[r+l+1,c-l]\in\mu$. On the right we depict $\langle r,c\rangle_\mu$ and $\langle\langle r,c\rangle\rangle_\mu$ for $[r,c]=[7,5]\in(1,2,3,4,5,6,7,8^2,3,1^4)$, we note that in this final case $k=1,l=3$ and $m=3$.

Now, if $[r-k,c+k+1]\in\mu$ or $[r-l+1,c-l]\in\mu$ we claim that

$$\mathsf{gap}(\mathsf{t}_\mu-[r,c])=\mathsf{gap}(\mathsf{t}_\mu-[r+l,c-l])=\mathsf{gap}(\mathsf{t}_\mu-[r-k,c+k])=0. \tag{4.8}$$

The first two equalities follow by repeated applications of Lemma 4.6. The final equality is immediate from the leftmost commuting relation of (2.3) and the cyclotomic relation (R16). Otherwise,

$$\mathsf{gap}(\mathsf{t}_\mu-[r,c])=\mathsf{gap}(\mathsf{t}_\mu-[r+l,c-l])=\mathsf{gap}(\mathsf{t}_\mu-[r-k,c+k])=(-1)^{k+l-m}c^*_{\mathsf{S}}c_{\mathsf{S}}. \tag{4.9}$$

where $c_{\mathsf{S}}$ is defined above. The first two equalities follow directly from Lemma 4.6. To verify the $m=0$ case of the third equality, we simply apply Lemma 4.6 $(k+l)$ times from left-to-right as in Figure 22. For $m>0$, we apply Lemma 4.7 $m$ times, and then apply Lemma 4.6 $k+l-m$ times.

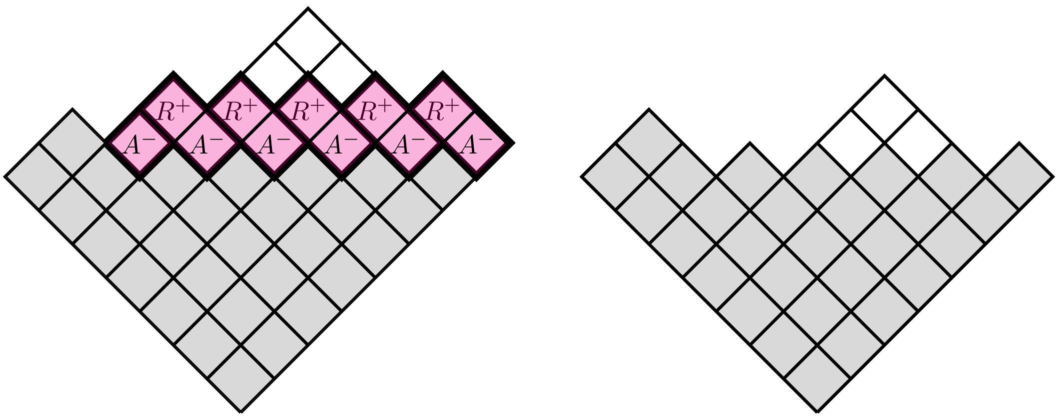



FIGURE 22. For $\mu=(7^2,6^2,5^3,2)$ and $[r,c]=[3,6]$ we we depict on the left the operators applied in the definition of $c_{\mathsf{S}}$. The thick lines break up the operators according to the bracketed terms in the definition of $c_{\mathsf{S}}$. On the right we depict $\mu-\langle 3,6\rangle_\mu$, obtained by deleting the pink tiles and letting the white tiles fall under gravity.

4.3. **Tetris combinatorics for barbells.** We now provide closed combinatorial formulas for removing barbells from diagrams.

**Definition 4.11.** *Let $\mu\in\mathscr{P}_{(W,P)}$. Given $[x,y]$ a (possibly non-admissible) tile, we set $SW[x,y]=[x-1,y]$ and $SE[x,y]=[x,y-1]$. For a pair of such tiles $[x,y]$ and $[x',y']$ we define a* trail*, denoted $\mathsf{T}_{[x,y]\to[x',y']}$, to be a (possibly empty) set of tiles*

$$[x,y]=T_1,T_2,\dots,T_{x+y-x'-y'+1}=[x',y'] \tag{4.10}$$

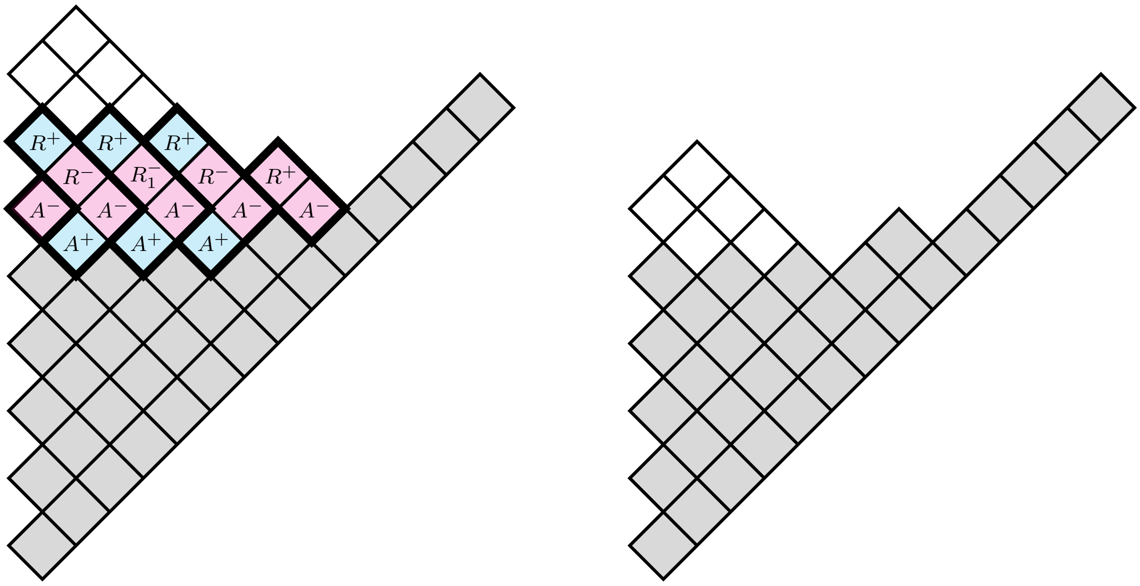



FIGURE 23. For $\mu = (1,2,3,4,5,6,7,8^2,3,1^4)$ and $[r,c]=[6,6]$ we depict on the left the operators applied in the definition of $c_{\mathsf{S}}$. The thick lines break up the operators according to the bracketed terms in the definition of $c_{\mathsf{S}}$. On the right we depict $\mu - \langle 6,6\rangle_\mu - \langle\langle 6,6\rangle\rangle_\mu$, obtained by deleting the pink and blue tiles and letting the white tiles fall under gravity.

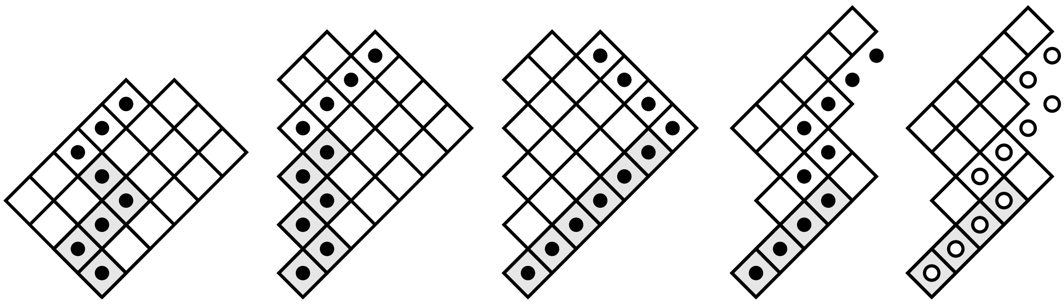


FIGURE 24. Examples of trails $\mathtt{T}^\mu_{[x,y]\to[1,1]}$, the first 4 of which are maximal length (the 5th is not). Here $[x,y]=[5,4],[7,4],[8,3]$ and $(W,P)=(A_9, A_3\times A_5)$, $(D_8,A_7)$, $(E_6,D_5)$ with $\mu=(4^5,3)$, $(1,2,3,4,5^2,4)$, and $(1^2,2,4^2,2,1^2)$ respectively (only the distinct cases are listed). The 5th trail is non-maximal length as it has length 6, whereas the 4th trail has length 8. The grey tiles $[a,b]$ are those such that $\mathsf{gap}(\mathsf{t}_\mu - [a,b])=0$ by Lemma 4.6 and Proposition 4.10

*such that $T_{i+1}=SW(T_i)$ or $T_{i+1}=SE(T_i)$ for $1<i\leqslant x+y-x'-y'+1$. We write*

$$\mathtt{T}^\mu_{[x,y]\to[x',y']} := \mu \cap \mathtt{T}_{[x,y]\to[x',y']}$$

*and we define the $\mu$-*length *of the trail to be $|\mathtt{T}^\mu_{[x,y]\to[x',y']}|$. Given $\tau=[r,c]\in \mathrm{Add}(\mu)$, we let $\mathrm{Hook}_\tau(\mu)$ denote any multiset of the form*

$$\mathrm{Hook}_\tau(\mu) = \begin{cases} 2\cdot \mathtt{T}^\mu_{[r,c-1]\to[1,1]} & \text{if } [r-1,c]\notin \mathscr{A}_{(W,P)} \text{ in type } (C_n,A_{n-1}) \\ \mathtt{T}^\mu_{[r-1,c]\to[1,1]} \sqcup \mathtt{T}^\mu_{[1,n]\to[1,1]} & \text{if } [r,c-1]\notin \mathscr{A}_{(W,P)} \text{ in type } (B_n,B_{n-1}) \\ \mathtt{T}^\mu_{[r-1,c]\to[1,1]} \sqcup \mathtt{T}^\mu_{[r,c-1]\to[1,1]} & \text{otherwise} \end{cases}$$

*for any preferred choices of maximal $\mu$-length trails on the right-hand side.*

This allows us to provide a closed combinatorial formula for rewriting barbells in diagrams, as follows. This formula is essential to our proof of combinatorial invariance. Finding such formulas for general $(W,P)$ seems to be an impossible task.

**Proposition 4.12.** *Let $(W, P)$ be a Hermitian symmetric pair. Let $\mu \in \mathscr{P}_{(W,P)}$ and $\boldsymbol{\tau} = [r, c] \in \mathrm{Add}(\mu)$. We have that*

$$1_{\mathsf{t}_\mu} \otimes \mathsf{bar}(\boldsymbol{\tau}) = -\sum_{[x,y]\in \mathsf{Hook}_{\boldsymbol{\tau}}(\mu)} \mathsf{gap}(\mathsf{t}_\mu - [x, y]). \tag{4.11}$$

Before embarking on the proof, we emphasise that equation (4.11) has a lot of redundancy. Many of the terms on the righthand-side of this sum are zero, using the results of Subsection 4.1 (some of these are highlighted in grey in Figure 24). We can pick preferred choices of the maximal length trails in the definition of $\mathsf{Hook}_{\boldsymbol{\tau}}(\mu)$ and delete some of these redundant terms. In particular, in classical types we have the following simplification.

**Lemma 4.13.** *Let $\mu \in \mathscr{P}_{(W,P)}$ and $\boldsymbol{\tau} = [r, c] \in \mathrm{Add}(\mu)$. For $(W, P) = (A_n, A_{k-1} \times A_{n-k})$, $(D_n, A_{n-1})$, or $(D_n, D_{n-1})$, or finally $(W, P) = (C_n, A_{n-1})$ and $r \neq c$, we set*

$$\underline{\mathsf{Hook}}_{\boldsymbol{\tau}}(\mu) = \mathsf{T}^{\mu}_{[r-1,c]\to[r-1,2]} \sqcup \mathsf{T}^{\mu}_{[r,c-1]\to[r,1]}.$$

*If $(W, P) = (C_n, A_{n-1})$ with $r = c$, we set*

$$\underline{\mathsf{Hook}}_{\boldsymbol{\tau}}(\mu) = 2\mathsf{T}^{\mu}_{[r,r-1]\to[r,1]}.$$

*If $(W, P) = (B_n, B_{n-1})$ then we set*

$$\underline{\mathsf{Hook}}_{\boldsymbol{\tau}}(\mu) = \begin{cases} [1, c-1] & \text{if } r = 1 \\ 2[1, n] & \text{if } [r, c] = [2, n] \\ 2[1, n] \sqcup [r-1, n] & \text{if } [r, c] = [r, n] \text{ with } r \geqslant 3. \end{cases}$$

*We have that*

$$\sum_{[x,y]\in \mathsf{Hook}_{\boldsymbol{\tau}}(\mu)} \mathsf{gap}(\mathsf{t}_\mu - [x, y]) = \sum_{[x,y]\in \underline{\mathsf{Hook}}_{\boldsymbol{\tau}}(\mu)} \mathsf{gap}(\mathsf{t}_\mu - [x, y]).$$

*Proof.* We need only to show that $\mathsf{gap}(\mathsf{t}_\mu - [x, y]) = 0$ for $[x, y] \in \mathsf{Hook}_{\boldsymbol{\tau}}(\mu) \setminus \underline{\mathsf{Hook}}_{\boldsymbol{\tau}}(\mu)$. This follows from Lemma 4.6 and Proposition 4.10. □

Examples of the multisets $\underline{\mathsf{Hook}}_{\boldsymbol{\tau}}(\mu)$ are provided in Figure 25.

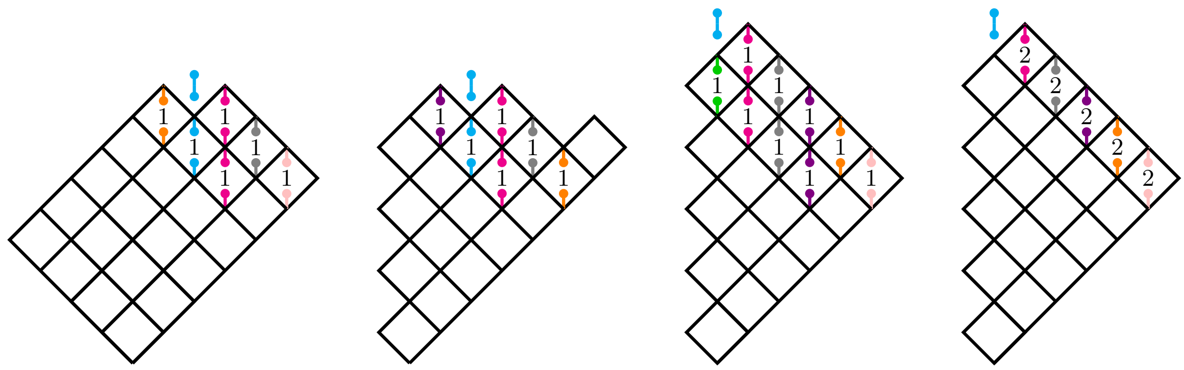


FIGURE 25. The multisets $\underline{\mathsf{Hook}}_{\boldsymbol{\tau}}(\mu)$. The first case is $\boldsymbol{\tau} = [6, 4]$ and $\mu = (5^2, 4^3, 3, 1)$ in type $(A_{11}, A_4 \times A_6)$. The second case is $\boldsymbol{\tau} = [6, 4]$ and $\mu = (1, 2, 3, 4^2, 3, 1)$ in type $(D_8, A_7)$. The third and fourth cases are for $\mu = (1, 2, 3, 4, 5^2)$ and $\boldsymbol{\tau} = s_{[6,6]}$ in types $(D, A)$ and $(C, A)$ respectively. We highlight the tiles in $\underline{\mathsf{Hook}}_{\boldsymbol{\tau}}(\mu)$ by placing a gap diagram in the tile and the multiplicity of that tile within the multiset.

*Proof of Proposition 4.12.* We consider the cases in which the parabolic is of type $A$. The other cases are left as an exercise for the reader. By the commutativity relations, it is enough to prove the result for $\boldsymbol{\tau}$ and $\mu$ such that $\boldsymbol{\tau} = s_{[r,c]}$ and $\mu\boldsymbol{\tau} = \lambda_{[r,c]}$ for some $r, c \geqslant 1$ (in the notation of Definition 1.3). If $r = c = 1$ then the result is immediate from the cyclotomic relation. If $r > 1$ and $c = 1$ or $r = 1$ and $c > 1$, then the result follows by Lemma 4.3 and the cyclotomic relation. Thus by Lemma 4.13 it is enough to show that $1_{\mathsf{t}_\mu} \otimes \mathsf{bar}(\boldsymbol{\tau}) = -\sum_{[x,y]\in \underline{\mathsf{Hook}}[r,c]} \mathsf{gap}(\mathsf{t}_\mu - [x, y])$ for $r, c > 1$.

**Case 1.** We now assume that $r, c > 1$, and consider the case where $[r, c]$ is such that $[r, c-1], [r-1, c] \in \mu$ as this is uniform across all types. We set $\nu$ to be the partition obtained by removing the final box from each column of $\mu$, that is $\nu\boldsymbol{\sigma} = \lambda_{[r-1,c]}$. We have that

$$1_{\mathsf{t}_\mu} \otimes \mathsf{bar}([r,c]) = 1_{\mathsf{t}_{\nu\boldsymbol{\sigma}}} \otimes \mathsf{bar}([r,c]) \otimes 1_{\mathsf{t}_{\mu\setminus\nu\boldsymbol{\sigma}}} + \sum_{y<c} 1_{\mathsf{t}_{\nu\boldsymbol{\sigma}}} \otimes \mathsf{bar}([r,y]) \otimes 1_{\mathsf{t}_{\mu\setminus\nu\boldsymbol{\sigma}}} - \sum_{1\leqslant y<c} \mathsf{gap}(\mathsf{t}_\mu - [r,y])$$

by Lemma 4.3 and applying equation (4.2) we obtain

$$1_{\mathsf{t}_\mu} \otimes \mathsf{bar}([r,c]) = 1_{\mathsf{t}_\nu} \otimes \mathsf{bar}([r-1,c]) \otimes 1_{\mathsf{t}_{\mu\setminus\nu}} + \sum_{y\leqslant c} 1_{\mathsf{t}_\nu} \otimes \mathsf{bar}([r,y]) \otimes 1_{\mathsf{t}_{\mu\setminus\nu}} - \sum_{[x,y]\in\mu\setminus\nu} \mathsf{gap}(\mathsf{t}_\mu - [x,y]).$$

We set $\pi$ to be the partition obtained by removing the final two rows of $\mu$, that is $\pi = \lambda_{[r-2,c]}$. By Lemma 4.4, we have that

$$\sum_{y\leqslant c} 1_{\mathsf{t}_\nu} \otimes \mathsf{bar}([r,y]) \otimes 1_{\mathsf{t}_{\mu\setminus\nu}} = 1_{\mathsf{t}_\pi} \otimes \mathsf{bar}([r,1]) \otimes 1_{\mathsf{t}_{\mu\setminus\pi}} + \sum_{y<c} \mathsf{gap}(\mathsf{t}_\mu - [r-1,y])$$

and we note that the first term after the equality is zero by the commutativity and cyclotomic relations. Putting these two equations above together, we have that

$$1_{\mathsf{t}_\mu} \otimes \mathsf{bar}([r,c]) = 1_{\mathsf{t}_\nu} \otimes \mathsf{bar}([r-1,c]) \otimes 1_{\mathsf{t}_{\mu\setminus\nu}} + \Big(\sum_{y<c} \mathsf{gap}(\mathsf{t}_\mu - [r-1,y]) - \sum_{[x,y]\in\mu\setminus\nu} \mathsf{gap}(\mathsf{t}_\mu - [x,y])\Big)$$

and so the result follows by induction and Lemma 4.6 and Proposition 4.10. This inductive step is visualised in Figure 26; we have bracketed the latter two terms above in order to facilitate comparison with the rightmost diagram in Figure 26.

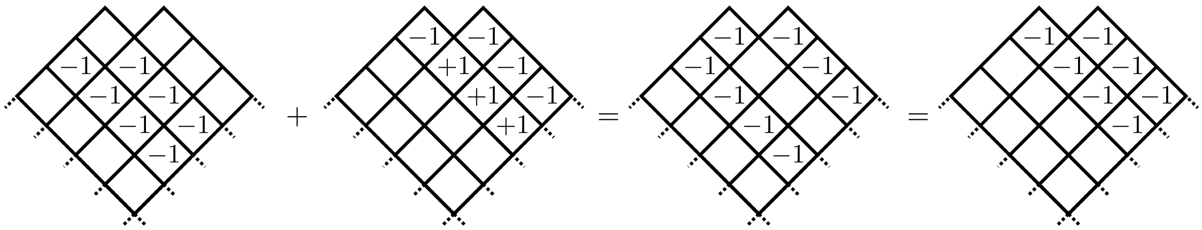



FIGURE 26. The first term on the left-hand side depicts $1_{\mathsf{t}_\nu} \otimes \mathsf{bar}([r-1,c]) \otimes 1_{\mathsf{t}_{\mu\setminus\nu}}$ (known by induction) the second term depicts the coefficients of the gap terms in the inductive step in the proof. The first equality records the cancellations; the second equality follows from Lemma 4.6 and Proposition 4.10. The rightmost diagram depicts $1_{\mathsf{t}_\mu} \otimes \mathsf{bar}([r,c])$ (for $r \neq c$ in types $C$ and $D$).

**Case 2.** Now consider the type $C$ and $D$ cases for $\boldsymbol{\tau} = s_{[r,r]}$ with $r > 1$ and we let $\boldsymbol{\sigma} = s_2 \in W$.

$$1_{\mathsf{t}_\mu} \otimes \mathsf{bar}([r,r]) = \begin{cases} 1_{\mathsf{t}_{\mu-\boldsymbol{\sigma}}} \otimes (\mathsf{bar}(\boldsymbol{\tau}) + 2\mathsf{bar}(\boldsymbol{\sigma})) \otimes 1_{\boldsymbol{\sigma}} - 2\mathsf{gap}(\mathsf{t}_\mu - [r,r-1]) & \text{in type } C \\ 1_{\mathsf{t}_{\mu-\boldsymbol{\sigma}}} \otimes (\mathsf{bar}(\boldsymbol{\tau}) + \mathsf{bar}(\boldsymbol{\sigma})) \otimes 1_{\boldsymbol{\sigma}} - \mathsf{gap}(\mathsf{t}_\mu - [r,r-1]) & \text{in type } D \end{cases}$$

and the $r = 2$ case now follows by the cyclotomic relation.

Now suppose $r > 2$. We first consider the type $C$ case. We set $\nu$ to be the partition such that $\nu\boldsymbol{\tau} = \lambda_{[r-1,r-1]}$. By the commutativity and one-colour barbell relations, we have that

$$1_{\mathsf{t}_{\mu-\boldsymbol{\sigma}}} \otimes \mathsf{bar}(\boldsymbol{\tau}) \otimes 1_{\boldsymbol{\sigma}} = -1_{\mathsf{t}_\nu} \otimes \mathsf{bar}(\boldsymbol{\tau}) \otimes 1_{\mathsf{t}_{\mu\setminus\nu}} + 2\mathsf{gap}(\mathsf{t}_\mu - [r-1,r-1])$$

and so

$$\begin{aligned} 1_{\mathsf{t}_\mu} \otimes \mathsf{bar}([r,r]) =& 2 \times 1_{\mathsf{t}_{\mu-\boldsymbol{\sigma}}} \otimes \mathsf{bar}([r,r-1]) \otimes 1_{\boldsymbol{\sigma}} - 1_{\mathsf{t}_\nu} \otimes \mathsf{bar}([r-1,r-1]) \otimes 1_{\mathsf{t}_{\mu\setminus\nu}} \\ &+ 2(\mathsf{gap}(\mathsf{t}_\mu - [r-1,r-1]) - 2\mathsf{gap}(\mathsf{t}_\mu - [r,r-1])) \end{aligned}$$

and so the result follows by induction. This inductive step is visualised in Figure 27; we have bracketed the latter two terms above in order to facilitate comparison with the rightmost diagram in Figure 27.

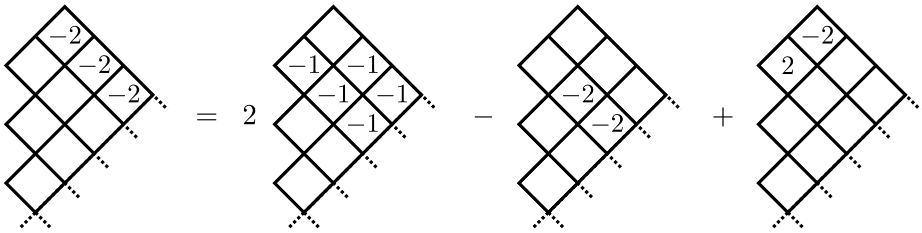



FIGURE 27. The left-hand side depicts $1_{\mathsf{t}_\mu} \otimes \mathsf{bar}([r,r])$ in type $C$. The first term on the right-hand side depicts $2 \times 1_{\mathsf{t}_{\mu-\sigma}} \otimes \mathsf{bar}([r, r-1]) \otimes 1_{\sigma}$; the second term depicts $-1_{\mathsf{t}_\nu} \otimes \mathsf{bar}([r-1, r-1]) \otimes 1_{\mathsf{t}_{\mu \setminus \nu}}$; the third term depicts the coefficients of the gap terms in the inductive step.

We now consider type $D$. We colour $s_{[r-1,r-1]}$ violet and set $\nu$, $\pi$, and $\rho$ to be the partitions $\nu + [r-1, r-1] = \lambda_{[r-1,r-1]}$, $\pi + [r-2, r-2] = \lambda_{[r-2,r-2]}$, and $\rho = \lambda_{[r-3,r-3]}$. We have that

$$1_{\mathsf{t}_\mu} \otimes \mathsf{bar}([r,r]) = 1_{\mathsf{t}_\nu} \otimes \mathsf{bar}([r-1,r-1]) \otimes 1_{\mathsf{t}_{\mu\setminus\nu}} + \sum_{y=1}^{r} 1_{\mathsf{t}_\nu} \otimes \mathsf{bar}([r,y]) \otimes 1_{\mathsf{t}_{\mu\setminus\nu}} - \sum_{[x,y]\in\mu\setminus\nu} \mathsf{gap}(\mathsf{t}_\mu - [x,y])$$

by Lemma 4.3. The second term can be rewritten as follows

$$\begin{aligned}
\sum_{y=1}^{r} 1_{\mathsf{t}_\nu} \otimes \mathsf{bar}([r,y]) \otimes 1_{\mathsf{t}_{\mu\setminus\nu}} &= \sum_{y=1}^{r} 1_{\mathsf{t}_{\pi\tau}} \otimes \mathsf{bar}([r,y]) \otimes 1_{\mathsf{t}_{\mu-\pi\tau}} \\
&= \sum_{y=1}^{r-1} 1_{\mathsf{t}_\pi} \otimes \mathsf{bar}([r,y]) \otimes 1_{\mathsf{t}_{\mu-\pi}} + \mathsf{gap}(\mathsf{t}_\mu - [r-2, r-2]) \\
&= \sum_{y=1}^{r-2} \mathsf{gap}(\mathsf{t}_\mu - [r-2,y]) + 1_{\mathsf{t}_\rho} \otimes (\mathsf{bar}([r,1]) + \mathsf{bar}([r,2])) \otimes 1_{\mathsf{t}_{\mu-\rho}} \\
&= \sum_{y=1}^{r-2} \mathsf{gap}(\mathsf{t}_\mu - [r-2,y])
\end{aligned}$$

where the first equality follows by repeated applications of equation (4.4); the second from equation (4.2); the third from Lemma 4.4 and the commutation relations; the fourth from the commutation and cyclotomic relations (notice that no tile in $\pi$ has colour label corresponding to the reflections $s_{[r,1]}$ or $s_{[r,2]}$). Substituting this into the above, we obtain

$$1_{\mathsf{t}_\mu} \otimes \mathsf{bar}([r,r]) = 1_{\mathsf{t}_\nu} \otimes \mathsf{bar}([r-1,r-1]) \otimes 1_{\mathsf{t}_{\mu\setminus\nu}} + \sum_{y=1}^{r-2} \mathsf{gap}(\mathsf{t}_\mu - [r-2,y]) - \sum_{[x,y]\in\mu\setminus\nu} \mathsf{gap}(\mathsf{t}_\mu - [x,y])$$

The result follows by induction (see Figure 28 for a visualisation of this step). □

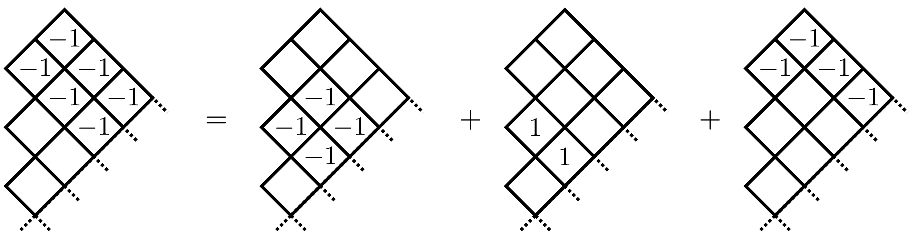



FIGURE 28. The lefthand-side depicts $1_{\mathsf{t}_\mu} \otimes \mathsf{bar}([r,r])$. The first term on the righthand-side depicts $1_{\mathsf{t}_\nu} \otimes \mathsf{bar}([r-1, r-1]) \otimes 1_{\mathsf{t}_{\mu\setminus\nu}}$ (known by induction); the second and third terms depict $+\sum_{y \leqslant r} \mathsf{gap}(\mathsf{t}_\mu - [r-2, y])$ and $-\sum_{[x,y]\in\mu\setminus\nu} \mathsf{gap}(\mathsf{t}_\mu - [x,y])$ respectively, which provide the gap terms in the inductive step in the proof.

4.4. **The Tetris-style presentation.** We are now ready to provide a new presentation for the Hecke categories of simply laced Hermitian symmetric pairs. One should notice that this presentation is mainly given in terms of the tiling combinatorics and *not* the usual Dynkin diagram combinatorics (the exception to this being discussion of commuting relations which are "far apart" in the Dynkin diagram).

**Theorem 4.14.** *Let $(W,P)$ denote a simply laced Hermitian symmetric pair. The algebra $\mathcal{H}_{(W,P)}$ can be defined as the locally-unital associative $\Bbbk$-algebra spanned by simple Soergel diagrams with multiplication given by vertical concatenation of diagrams modulo the following local relations and their horizontal and vertical flips. Firstly, for any $\boldsymbol{\sigma} \in S_W$ we have the relations*

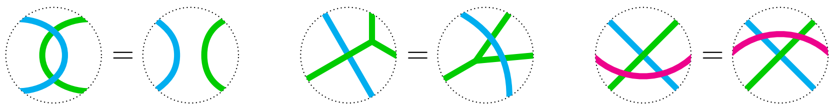
$$1_{\sigma}1_{\tau} = \delta_{\sigma,\tau}1_{\sigma} \qquad 1_{\emptyset}1_{\sigma} = 0 \qquad 1_{\emptyset}^2 = 1_{\emptyset}$$

$$1_{\emptyset}\mathsf{spot}_{\sigma}^{\emptyset}1_{\sigma} = \mathsf{spot}_{\sigma}^{\emptyset} \qquad 1_{\sigma}\mathsf{fork}_{\sigma\sigma}^{\sigma}1_{\sigma\sigma} = \mathsf{fork}_{\sigma\sigma}^{\sigma} \qquad 1_{\tau\sigma}\mathsf{braid}_{\sigma\tau}^{\tau\sigma}1_{\sigma\tau} = \mathsf{braid}_{\sigma\tau}^{\tau\sigma}$$

*where the final relation holds for all ordered pairs $(\boldsymbol{\sigma},\boldsymbol{\tau}) \in S_W^2$ with $m(\boldsymbol{\sigma},\boldsymbol{\tau}) = 2$. For each $\boldsymbol{\sigma} \in S_W$ we have fork-spot contraction, the double-fork, and circle-annihilation relations:*

$$(\mathsf{spot}_{\sigma}^{\emptyset} \otimes 1_{\sigma})\mathsf{fork}_{\sigma}^{\sigma\sigma} = 1_{\sigma}, \quad (1_{\sigma} \otimes \mathsf{fork}_{\sigma\sigma}^{\sigma})(\mathsf{fork}_{\sigma}^{\sigma\sigma} \otimes 1_{\sigma}) = \mathsf{fork}_{\sigma}^{\sigma\sigma}\mathsf{fork}_{\sigma\sigma}^{\sigma}, \quad \mathsf{fork}_{\sigma\sigma}^{\sigma}\mathsf{fork}_{\sigma}^{\sigma\sigma} = 0,$$

*For $(\boldsymbol{\sigma},\boldsymbol{\tau},\boldsymbol{\rho}) \in S^3$ with $m_{\sigma\rho} = m_{\rho\tau} = m_{\sigma\tau} = 2$, we have the commutation relations*



*For $\mu$ any partition tiling and $\boldsymbol{\sigma} \in \mathrm{Add}(\mu)$, we have the monochrome Tetris relation*

$$1_{\mathsf{t}_{\mu\sigma}} \otimes 1_{\sigma} = 1_{\mathsf{t}_{\mu}} \otimes (\mathsf{spot}_{\sigma}^{\emptyset} \otimes \mathsf{fork}_{\sigma}^{\sigma\sigma} + \mathsf{spot}_{\emptyset}^{\sigma} \otimes \mathsf{fork}_{\sigma\sigma}^{\sigma}) + \sum_{[r,c]\in \mathrm{Hook}_{\sigma}(\mu)} \mathsf{gap}(\mathsf{t}_{\mu} - [r,c]) \otimes \mathsf{dork}_{\sigma\sigma}^{\sigma\sigma}.$$

*For any $\boldsymbol{\sigma},\boldsymbol{\tau} \in S_W$ with $m(\boldsymbol{\sigma},\boldsymbol{\tau}) = 3$, we have the null-braid relation*

$$1_{\sigma\tau\sigma} + (1_{\sigma} \otimes \mathsf{spot}_{\emptyset}^{\tau} \otimes 1_{\sigma})\mathsf{dork}_{\sigma\sigma}^{\sigma\sigma}(1_{\sigma} \otimes \mathsf{spot}_{\tau}^{\emptyset} \otimes 1_{\sigma}) = 0$$

*For $\mu \in \mathscr{P}_{(W,P)}$ and $\boldsymbol{\tau} \in \mathrm{Add}(\mu)$, we have the bi-chrome Tetris relation*

$$1_{\mathsf{t}_{\mu}} \otimes \mathsf{bar}(\tau) = -\sum_{[r,c]\in \mathrm{Hook}_{\tau}(\mu)} \mathsf{gap}(\mathsf{t}_{\mu} - [r,c]).$$

*Further, we require the interchange law and the monoidal unit relation*

$$\big(\mathsf{D}_1 \otimes \mathsf{D}_2\big) \circ \big(\mathsf{D}_3 \otimes \mathsf{D}_4\big) = (\mathsf{D}_1 \circ \mathsf{D}_3) \otimes (\mathsf{D}_2 \circ \mathsf{D}_4) \qquad 1_{\emptyset} \otimes \mathsf{D}_1 = \mathsf{D}_1 = \mathsf{D}_1 \otimes 1_{\emptyset}$$

*for all diagrams $\mathsf{D}_1, \mathsf{D}_2, \mathsf{D}_3, \mathsf{D}_4$. Finally, we require the non-local cyclotomic relations*

$$\mathsf{bar}(\sigma) \otimes \mathsf{D} = 0 \qquad \text{for all } \boldsymbol{\sigma} \in S_W \text{ and } \mathsf{D} \text{ any diagram}$$

$$1_{\tau} \otimes \mathsf{D} = 0 \qquad \text{for all } \boldsymbol{\tau} \in S_P \subseteq S_W \text{ and } \mathsf{D} \text{ any diagram.}$$

*Proof.* In light of Corollary 2.5, we need only show that the one and two colour barbell relations can be replaced by the monochrome and bi-chrome Tetris relations.

Given two simple Soergel diagrams $\mathsf{D}_1$ and $\mathsf{D}_2$, the one and two colour barbell relations allow us to inductively move leftwards any barbell *anywhere* in $\mathsf{D}_1\mathsf{D}_2$; once all barbells are at the leftmost edge of the diagram these are zero by the cyclotomic relation. Thus the one and two colour barbell relations allow us to rewrite a product of simple Soergel diagrams as a linear combination of simple Soergel diagrams.

We now show that we can rewrite, using only the relations of Theorem 4.14, any diagram $\mathsf{D}_1\mathsf{D}_2$ as a linear combination of simple Soergel diagrams. Any diagram can be rewritten in terms of the cellular basis and the cellular basis elements are all of the form $c_{\mathsf{S}}^{*}1_{\mathsf{t}_{\lambda}}c_{\mathsf{T}}$. Thus it suffices to have a list of rules which rewrites $1_{\mathsf{t}_{\lambda}} \otimes \mathsf{bar}(\tau)$ as a linear combination of simple Soergel diagrams. This is precisely what the monochrome and bi-chrome Tetris relation do. The result follows. □

4.5. **Combinatorial invariance for simply laced types.** Equipped with our Tetris style presentations, we are now ready to prove Theorem 4.2. We begin by restricting our explicit attention to simply laced types only, as this case follows easily from our Tetris-style presentation.

**Proposition 4.15.** *Let $(W,P)$ and $(W',P')$ be simply laced Hermitian symmetric pairs. Let $\Pi = [\lambda,\mu]$ and $\Pi' = [\lambda',\mu']$ be closed subsets of $\mathscr{P}_{(W,P)}$ and $\mathscr{P}_{(W',P')}$ respectively. Given a map $\jmath : \Pi \to \Pi'$, $\jmath(\alpha) = \alpha'$ for $\lambda \leqslant \alpha \leqslant \mu$, we have that $\jmath$ is a poset isomorphism if and only if $\jmath$ sends like-coloured tiles in $\alpha$ to like-coloured tiles in $\alpha'$.*

*Proof.* We consider the classical types, as the exceptional cases can be verified by exhaustion. We identify $\lambda \subseteq \mu$ and $\lambda' \subseteq \mu'$ with their tilings $\lambda \subseteq \mu$ and $\lambda' \subseteq \mu'$. If the tiling $\lambda \subseteq \mu$ can appear as a subregion of $\mathscr{A}_{(W,P)}$ for $W$ of type $A$ (that is, $[r,c],[r+1,c+1] \in (\mu \setminus \lambda)$ implies $[r,c+1],[r+1,c] \in (\mu\setminus\lambda)$) then the colouring on each $\alpha \in \Pi$, $\alpha' \in \Pi'$ is simply given by shifting the $x$-coordinates of the diagonals (up to reflecting through the $y-$axis, see the first pair in Figure 29). Otherwise, $\lambda \subseteq \mu$ and $\lambda' \subseteq \mu'$ each have a single diagonal in which the colouring alternates (either alternating between $s_0$ and $s_1$ in type $(D,A)$ as in the rightmost examples in Figure 29, or alternating between $s_1$ and $s_3$ in type $(D,D)$). If such a $\Pi$ fits within a $3\times 3$ rectangle in type $(D,A)$, then one can check that there exists an isomorphic $\Pi'$ in type $(D,D)$ and that the colourings match-up in these small cases. Otherwise, if $\Pi$ and $\Pi'$ are two distinct isomorphic posets, then they are both of type $(D,A)$ and can be obtained by vertical translation (perhaps swapping $s_0$ and $s_1$ in the process, depending on the parity of the translation). □

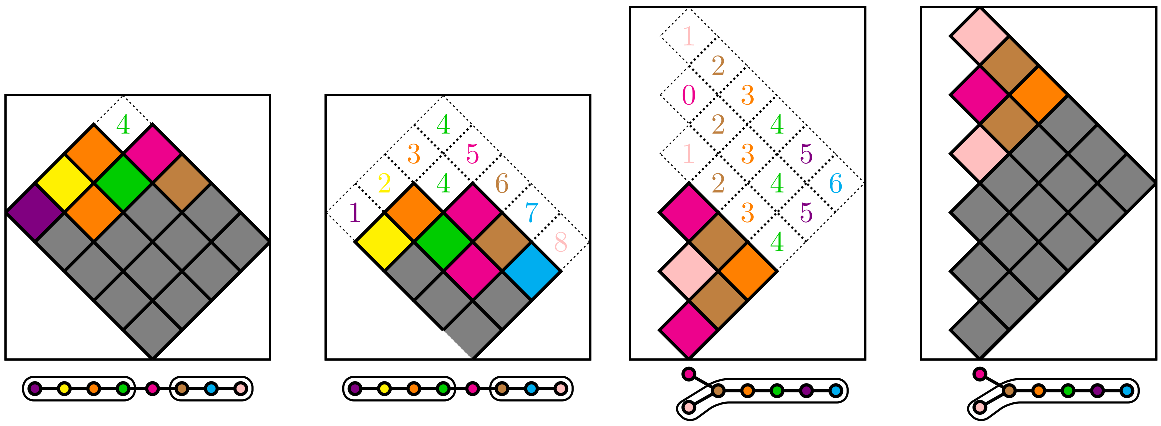



FIGURE 29. Two pairs of isomorphic tilings. The pair $\Pi = \{\alpha \mid \varnothing \leqslant \alpha \leqslant (1,2,3)\}$ and $\Pi' = \{\alpha \mid (1,2,3^4) \leqslant \alpha \leqslant (1,2,3,4,5,6)\}$ on the right can only appear in type $D$ (note the jagged edge on the left) and the recolouring of tiles is given by $\iota(s_0) = s_1$, $\iota(s_1) = s_0$, $\iota(s_2) = s_2$, $\iota(s_3) = s_3$). The pair on the left are $\Pi = \{\alpha \mid (4,3^2,2) \leqslant \alpha \leqslant (5^3,4)\}$ and $\Pi' = \{\alpha \mid (3,1) \leqslant \alpha \leqslant (4^2,3)\}$ and the recolouring of tiles is given by $\iota(s_1) = s_7$, $\iota(s_2) = s_6$, $\iota(s_3) = s_5$, $\iota(s_4) = s_4$, $\iota(s_5) = s_3$, $\iota(s_6) = s_2$ (note the flip through the vertical axis).

**Definition 4.16.** *Let $\Gamma$ (respectively $\Gamma'$) be the set of colours of the tiles in $\Pi$ (respectively $\Pi'$). We let $\imath : \Gamma \to \Gamma'$, be a surjective map. We lift this to a recolouring map on Soergel diagrams as follows. For $\gamma,\delta \in \Gamma$ with $\imath(\gamma) = \gamma'$ and $\imath(\delta) = \delta'$ we set*

$$\imath(1_\gamma) = 1_{\gamma'} \quad \imath(\mathsf{spot}^{\emptyset}_{\gamma}) = \mathsf{spot}^{\emptyset}_{\gamma'} \quad \imath(\mathsf{fork}^{\gamma}_{\gamma\gamma}) = \mathsf{fork}^{\gamma'}_{\gamma'\gamma'} \quad \imath(\mathsf{braid}^{\gamma\delta}_{\delta\gamma}) = \mathsf{braid}^{\gamma'\delta'}_{\delta'\gamma'}$$

*and we set $\imath(\mathsf{D}^*) = (\imath(\mathsf{D}))^*$. We then inductively define*

$$\imath(\mathsf{D}_1 \otimes \mathsf{D}_2) = \imath(\mathsf{D}_1) \otimes \imath(\mathsf{D}_2) \qquad \imath(\mathsf{D}_1 \circ \mathsf{D}_2) = \imath(\mathsf{D}_1) \circ \imath(\mathsf{D}_2)$$

*and extend this map $\Bbbk$-linearly.*

**Proposition 4.17.** *Suppose we have a poset isomorphism $\jmath : \Pi \to \Pi'$ for $\Pi = [\lambda, \mu]$ and $\Pi' = [\lambda', \mu']$ as in Proposition 4.15 inducing a recolouring map $\imath$. Then $\jmath$ extends to a unique graded isomorphism of $\Bbbk$-algebras $\jmath : h_\Pi \to h_{\Pi'}$ satisfying*

$$\jmath(1_{\mathsf{t}_\lambda} \otimes D) = 1_{\mathsf{t}_{\lambda'}} \otimes \imath(\mathsf{D})$$

*for* $\mathsf{D}$ *any Soergel diagram , and*

$$\jmath(1_{\mathsf{t}_\lambda} \circledast_\gamma D) = 1_{\mathsf{t}_{\lambda'}} \circledast_{\gamma'} \imath(\mathsf{D})$$

*for $\gamma \in \mathrm{Rem}(\lambda)$, and $\mathsf{D}$ any Soergel diagram for which the $\circledast_\gamma$- and $\circledast_{\gamma'}$-products makes sense.*

**Remark 4.18.** *We can regard the diagram $\mathsf{D}$ as being "coloured by" the initial Coxeter system $W$ and the effect of the map is to "recolour" this diagram according to the Coxeter system $W'$. The effect of changing $\mathsf{t}_\lambda$ for $\mathsf{t}_{\lambda'}$ is merely to identify the different regions $\Pi$ and $\Pi'$ within our Bruhat graphs ${}^PW$ and ${}^{P'}W'$. An example is given in Figure 30.*

We are now almost ready to prove Proposition 4.17. We first observe that the Tetris relations are compatible with restriction to a closed subregion.

**Lemma 4.19.** *Let $\Pi = [\lambda, \mu]$. Given $[x, y] \in \lambda$, we have that $\mathsf{gap}(\mathsf{t}_\lambda - [x, y]) = 0$ in $h_\Pi$.*

*Proof.* We suppose $W$ is classical as the general case is similar. By (4.8) and (4.9) we can rewrite $\mathsf{gap}(\mathsf{t}_\lambda - [x, y]) = \pm c^{\alpha}_{\mathsf{SS}}$ (or zero) for $\alpha \subset \lambda$ and the result follows as $1_\alpha = 0 \in h_\Pi$ by definition. $\square$

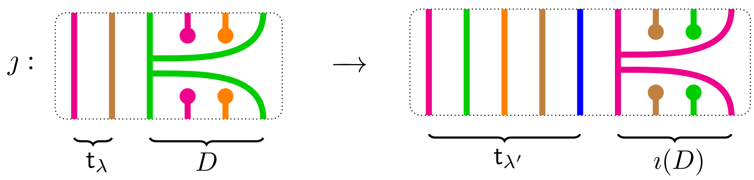



Figure 30. The left diagram, $D$, is an element from $h_\Pi$ for $\Pi = \{\alpha \mid (1^2) \leqslant \alpha \leqslant (3^2)\} \subset \mathscr{A}_{(A_8, A_4 \times A_3)}$ and the right diagram is the corresponding, $\imath(D)$, in $h_{\Pi'}$ for $\Pi' = \{\alpha' \mid (3, 1^2) \leqslant \alpha \leqslant (3^3)\} \subset \mathscr{A}_{(A_8, A_4 \times A_3)}$. Compare the colouring with that of $(A_8, A_4 \times A_3)$ in Figure 3.

*Proof of Proposition 4.17.* Let $\alpha \in \Pi = [\lambda, \mu]$ for $\lambda, \mu \in \mathscr{P}_{(W,P)}$. Each cellular basis element $c^{\alpha}_{\mathsf{ST}}$ in $h_\Pi$ can be written in the form $1_{\mathsf{t}_\lambda} \otimes \mathsf{D}$ or $1_{\mathsf{t}_{\lambda'}} \circledast_\gamma \mathsf{D}$ for some simple Soergel diagram $\mathsf{D}$, so $\jmath$ is well defined. The monochrome and idempotent relations are trivially preserved by the map $\jmath$. Two commuting reflections, $\boldsymbol{\tau}, \boldsymbol{\sigma} \in S_W$ correspond to tiles $[x, y]$, $[r, c]$ from $\Pi$ if and only if $(x - y) - (r - c) \neq \pm 1$; this distance is preserved by the map $\jmath : \Pi \to \Pi'$ (by Proposition 4.15) and so the commuting relations are preserved.

The Tetris relations for $\mathscr{H}_\Pi$ are written entirely in terms of the addable and removable nodes of tilings and the sets $\mathrm{Hook}_{\boldsymbol{\tau}}(\alpha)$ for $\lambda \leqslant \alpha \leqslant \mu$ and this is compatible with restriction to $\Pi$ (using Lemma 4.19). The sets $\mathrm{Hook}_{\boldsymbol{\tau}}(\alpha)$ depend only on information which is preserved under $\jmath : \Pi \to \Pi'$ (using Lemma 4.6 to flip left versus right in the definition of $\mathsf{Hook}_{\boldsymbol{\tau}}(\mu)$, if necessary). Therefore the Tetris relations go through $\jmath$. Finally, we note that the cyclotomic relations follow from the Tetris relations and Lemma 4.19. Thus the map $\jmath$ is an algebra homomorphism.

One can similarly define $\jmath^{-1}$ as the recolouring map in the opposite direction. We have that $\jmath \circ \jmath^{-1}$ and $\jmath^{-1} \circ \jmath$ are both identity maps (as they amount to recolouring and recolouring again) and so the map is indeed an algebra isomorphism. $\square$

4.6. **Fixed point subgroups and non-simply laced types.** We now consider the group automorphisms, $\sharp$, for type $A_{2n-1}$ and $D_{n+1}$ given by flipping the Coxeter diagrams through the horizontal and vertical axes, respectively. Explicitly, the map $\sharp$ is determined by $\sharp(s_i) = s_{2n-1-i}$ for the group of type $A_{2n-1}$. The map $\sharp$ is determined by $\sharp(s_0) = s_1$, $\sharp(s_1) = s_0$ and $\sharp(s_i) = s_i$ for the group of type $D_{n+1}$. The fixed point groups of these automorphism are the groups $\langle s_i s_{2n-1-i}, s_n \mid 1 \leqslant i < n \rangle$ of type $B_n$ and $\langle s_0 s_1, s_i \mid 2 \leqslant i \leqslant n \rangle$ of type $C_n$. By restricting our attention from the group to its fixed point subgroup, we obtain a surjective map $\imath$ on the tile-colourings. These are depicted in Figures 31 and 32.

**Remark 4.20.** *For the remainder of this section, we will fix* $\boldsymbol{\sigma} = s_2$ *and* $\boldsymbol{\tau} = s_1$ *for* $(W, P) = (C_n, A_{n-1})$ *and* $\boldsymbol{\sigma} = s_1$ *and* $\boldsymbol{\tau} = s_2$ *for* $(W, P) = (B_n, B_{n-1})$.

We extend our colouring convention from Remark 4.20 by setting $\boldsymbol{\sigma} = s_n$ in type $A_{2n-1}$ and $\boldsymbol{\sigma} = s_2$ in type $D_{n+1}$. We similarly set $\boldsymbol{\rho} = s_{n-1}, \boldsymbol{\pi} = s_{n+1}$ in type $A_{2n-1}$ and $\boldsymbol{\rho} = s_0, \boldsymbol{\pi} = s_1$ in type $D_{2n+1}$ so that green and purple map to blue in both cases. We can easily extend this colouring map to a map, $\imath$, on the level of paths and moreover we have the following:

**Lemma 4.21.** *The colouring maps on paths map bijectively onto the subsets of parity preserving paths from Subsection 1.3. In other words,*

$$\imath : \mathrm{Path}_{(D_{n+1}, A_n)}(\lambda, \mathsf{t}_\mu) \to \mathrm{Path}^{\pm}_{(C_n, A_{n-1})}(\lambda, \mathsf{t}_\mu), \quad \imath : \mathrm{Path}_{(A_{2n-1}, A_{2n-2})}(\lambda, \mathsf{t}_\mu) \to \mathrm{Path}^{\pm}_{(B_n, B_{n-1})}(\lambda, \mathsf{t}_\mu)$$

*are both grading-preserving bijections.*

We now prove that $h_{(C_n, A_{n-1})}$ and $h_{(B_n, B_{n-1})}$ are graded Morita equivalent to $h_{(D_{n+1}, A_n)}$ and $h_{(A_{2n-1}, A_{2n-2})}$ respectively.

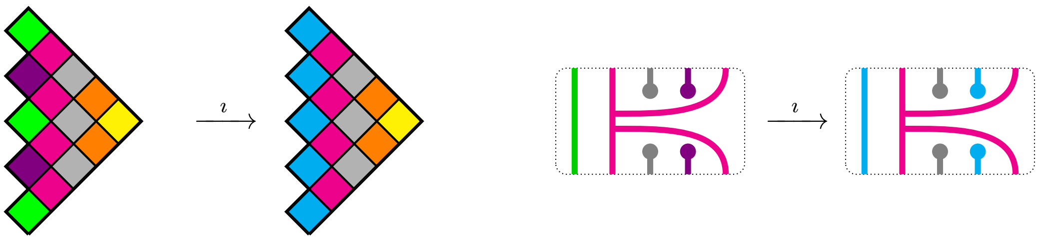


FIGURE 31. An example of the colouring map $\imath$ from type $D_{n+1}$ to type $C_n$.

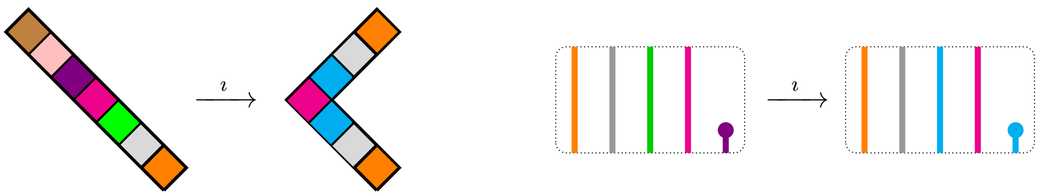


FIGURE 32. An example of the colouring map $\imath$ from type $A_{2n-1}$ to type $B_n$.

**Lemma 4.22.** *Let* $m(\boldsymbol{\beta}, \boldsymbol{\gamma}) = 3$ *or* $m(\boldsymbol{\beta}, \boldsymbol{\gamma}) = 4$. *If* $m(\boldsymbol{\beta}, \boldsymbol{\gamma}) = 4$, *then suppose that* $(\boldsymbol{\beta}, \boldsymbol{\gamma}) = (\boldsymbol{\tau}, \boldsymbol{\sigma})$ *as in Remark 4.20. We have that*

$$\mathsf{fork}^{\boldsymbol{\beta}}_{\boldsymbol{\beta}\boldsymbol{\beta}}(1_{\boldsymbol{\beta}} \otimes \mathsf{bar}(\boldsymbol{\gamma}) \otimes 1_{\boldsymbol{\beta}})\mathsf{fork}^{\boldsymbol{\beta}\boldsymbol{\beta}}_{\boldsymbol{\beta}} = -1_{\boldsymbol{\beta}} \tag{4.12}$$

$$\mathsf{cap}^{\emptyset}_{\boldsymbol{\beta}\boldsymbol{\beta}}(1_{\boldsymbol{\beta}} \otimes \mathsf{bar}(\boldsymbol{\gamma}) \otimes 1_{\boldsymbol{\beta}})\mathsf{cup}^{\boldsymbol{\beta}\boldsymbol{\beta}}_{\emptyset} = -\mathsf{bar}(\boldsymbol{\beta}) \tag{4.13}$$

$$\mathsf{fork}^{\boldsymbol{\beta}}_{\boldsymbol{\beta}\boldsymbol{\beta}}(1_{\boldsymbol{\beta}} \otimes \mathsf{bar}(\boldsymbol{\gamma}) \otimes \mathsf{bar}(\boldsymbol{\gamma}) \otimes 1_{\boldsymbol{\beta}})\mathsf{fork}^{\boldsymbol{\beta}\boldsymbol{\beta}}_{\boldsymbol{\beta}} = -1_{\boldsymbol{\beta}} \otimes (2\mathsf{bar}(\boldsymbol{\gamma}) + \mathsf{bar}(\boldsymbol{\beta})) \tag{4.14}$$

*Proof.* Equation (4.12) follows by applying the $\boldsymbol{\gamma\beta}$-barbell relation, followed by the $\boldsymbol{\beta}$-circle annihilation relation and $\boldsymbol{\beta}$-fork-spot contraction relation. Equation (4.13) follows from equation (4.12) by apply the spot generator on top and bottom. Equation (4.14) follows by applying the $\boldsymbol{\gamma\beta}$-barbell relation to the lefthand-side, followed by equation (4.12). □

We will find the following shorthand useful,

$$\mathsf{trid}^{\boldsymbol{\tau}}_{\boldsymbol{\tau\sigma\tau}} = \mathsf{fork}^{\boldsymbol{\tau}}_{\boldsymbol{\tau\tau}}(1_{\boldsymbol{\tau}} \otimes \mathsf{spot}^{\emptyset}_{\boldsymbol{\sigma}} \otimes 1_{\boldsymbol{\tau}}) \qquad \mathsf{trid}^{\emptyset}_{\boldsymbol{\tau\sigma\tau}} = \mathsf{spot}^{\emptyset}_{\boldsymbol{\tau}}\mathsf{trid}^{\boldsymbol{\tau}}_{\boldsymbol{\tau\sigma\tau}}$$

the former of which can be pictured as a "trident". We set

$$\mathsf{trid}^{\boldsymbol{\tau\sigma\tau}}_{\boldsymbol{\tau}} = (\mathsf{trid}^{\boldsymbol{\tau}}_{\boldsymbol{\tau\sigma\tau}})^* \qquad \mathsf{trid}^{\boldsymbol{\tau\sigma\tau}}_{\boldsymbol{\tau\sigma\tau}} = \mathsf{trid}^{\boldsymbol{\tau\sigma\tau}}_{\boldsymbol{\tau}}\mathsf{trid}^{\boldsymbol{\tau}}_{\boldsymbol{\tau\sigma\tau}}.$$

By equation (4.12), we have that $-\mathsf{trid}^{\tau\sigma\tau}_{\tau\sigma\tau}$ is an idempotent and that

$$(1_{\tau\sigma\tau} + \mathsf{trid}^{\tau\sigma\tau}_{\tau\sigma\tau})\mathsf{trid}^{\tau\sigma\tau}_{\tau\sigma\tau} = 0. \tag{4.15}$$

**Definition 4.23.** *Let $(W,P) = (C_n, A_{n-1})$ or $(B_n, B_{n-1})$ and suppose $\sigma, \tau \in S_W$ satisfy the assumptions of Remark 4.20. For $\mu \in \mathscr{P}_{(W,P)}$ and $1 < k \leqslant \ell_\tau(\mu)$, we set*

$$\rho_k = \begin{cases} (1,2,3,\dots,k) \in \mathscr{P}_{(W,P)} & \text{if } W = C_n \\ (n,1) \in \mathscr{P}_{(W,P)} & \text{if } W = B_n \end{cases}$$

*and we set $\kappa = \rho_k - \tau - \sigma - \tau$ and define*

$$\mathsf{e}^k_\mu = \mathsf{braid}^{\mathsf{t}_\mu}_{\mathsf{t}_\kappa \circ \mathsf{t}_{\mu-\kappa}}\left(1_{\mathsf{t}_\kappa} \otimes \mathsf{trid}^{\tau\sigma\tau}_{\tau\sigma\tau} \otimes 1_{\mathsf{t}_{\mu-\rho_k}}\right)\mathsf{braid}^{\mathsf{t}_\kappa \circ \mathsf{t}_{\mu-\kappa}}_{\mathsf{t}_\mu}. \tag{4.16}$$

*We define the idempotent*

$$\mathsf{e}_\mu = \textstyle\prod_{1<k\leqslant \ell_\tau(\mu)}(1_{\mathsf{t}_\mu} + \mathsf{e}^k_\mu)$$

*and set $\mathsf{e}_\mu = 1_{\mathsf{t}_\mu}$ if $\ell_\tau(\mu) \leqslant 1$. We define $\mathsf{e} = \sum_{\mu \in \mathscr{P}_{(W,P)}} \mathsf{e}_\mu$.*

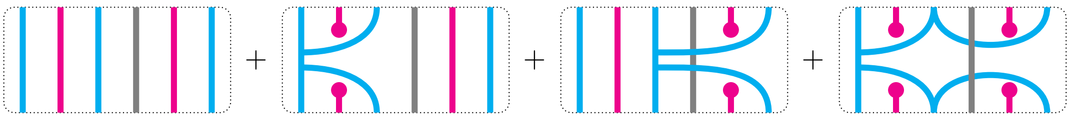


FIGURE 33. The element $\mathsf{e}_\mu$ for $\mu = (1,2,3)$ in type $C$. The colouring is the same as that of Figure 31. The summands are $1_\mu$, $\mathsf{e}^2_\mu$, $\mathsf{e}^3_\mu$, and $\mathsf{e}^2_\mu\mathsf{e}^3_\mu$ respectively.

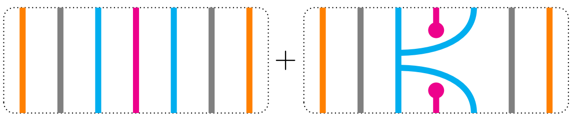


FIGURE 34. We picture the element $\mathsf{e}_\mu$ for $\mu = (4, 1^3)$ for type $(B_4, B_3)$ as in Definition 4.23. The colouring is the same as that of Figure 32.

We can extend the maps $\imath$ from Lemma 4.21 to injective $\Bbbk$-linear maps $\imath : h_{(D_{n+1}, A_n)} \hookrightarrow h_{(C_n, A_{n-1})}$ and $\imath : h_{(A_{2n-1}, A_{2n-2})} \hookrightarrow h_{(B_n, B_{n-1})}$ by setting $\imath(c_{\mathsf{ST}}) = c_{\imath(\mathsf{S})\imath(\mathsf{T})}$. We note that $\imath$ is not a $\Bbbk$-algebra homomorphism, but we will prove the following:

**Theorem 4.24.** *The maps $\Theta : h_{(D_{n+1}, A_n)} \to \mathsf{e}h_{(C_n, A_{n-1})}\mathsf{e}$ and $\Theta : h_{(A_{2n-1}, A_{2n-2})} \to \mathsf{e}h_{(B_n, B_{n-1})}\mathsf{e}$ defined by $\Theta(a) = e \circ \imath(a) \circ e$ are graded $\Bbbk$-algebra isomorphisms. Moreover, as $\mathsf{e}$ is a full idempotent these maps give rise to graded Morita equivalences between $h_{(D_{n+1}, A_n)}$ and $h_{(C_n, A_{n-1})}$ and between $h_{(A_{2n-1}, A_{2n-2})}$ and $h_{(B_n, B_{n-1})}$.*

We note that the first isomorphism categorifies an observation of Boe in [Boe88]. This section is dedicated to the proof. We begin with the simpler result for orthogonal groups.

4.6.1. *The orthogonal case, type $(B_n, B_{n-1})$.* We first consider the case of the orthogonal group. We can simplify the proof by focussing on the cellular basis. We prove that if $c_{\mathsf{ST}}c_{\mathsf{UV}} = \sum a_{\mathsf{XY}}c_{\mathsf{XY}}$ for coefficients $a_{\mathsf{XY}} \in \Bbbk$, then we have that

$$\Theta(c_{\mathsf{ST}})\Theta(c_{\mathsf{UV}}) = \sum a_{\mathsf{XY}}\Theta(c_{\mathsf{XY}}) \tag{4.17}$$

for $\mathsf{S}, \mathsf{T}, \mathsf{U}, \mathsf{V}, \mathsf{X}, \mathsf{Y} \in \mathrm{Path}_{(A_{2n-1}, A_{2n-2})}$ and hence deduce that Theorem 4.24 holds for type $(B_n, B_{n-1})$. For $\mu = (c-1)$ with $[1,c] \in \mathrm{Add}(\mu)$ with $\gamma = s_{[1,c]}$ and $1 \leqslant c \leqslant n$ the elements $c^\mu_{\mathsf{ST}}$ are of the form

$$1_{\mathsf{t}_\mu}, \quad 1_{\mathsf{t}_\mu} \otimes \mathsf{spot}^{\emptyset}_{\gamma}, \quad 1_{\mathsf{t}_\mu} \otimes \mathsf{spot}^{\gamma}_{\emptyset}, \quad \text{and} \quad 1_{\mathsf{t}_\mu} \otimes \mathsf{gap}(\gamma).$$

Thus rewriting products in equation (4.17) requires only the idempotent, bi-chrome Tetris, commutativity and cyclotomic relations. We consider the bi-chrome Tetris relation as the others are trivial. By Proposition 4.12 and Lemma 4.13, we have that

$$c_{\mathsf{T}} c_{\mathsf{T}}^{*} = 1_{\mathsf{t}_\mu} \otimes \mathsf{bar}(\gamma) = -\mathsf{gap}(\mathsf{t}_\mu - [1, c-1])$$

for $\mathsf{T} \in \mathrm{Path}(\mu, \mathsf{t}_{\mu+\gamma})$. For $c \leqslant n$, we have that $\mathsf{e}_{\imath(\mu\gamma)} = 1_{\imath(\mathsf{t}_{\mu\gamma})} = 1_{(c)}$ and

$$\Theta(1_{\mathsf{t}_{(c-1)}} \otimes \mathsf{bar}(\gamma)) = -1_{(c-1)} \otimes \mathsf{gap}(\imath(\gamma)) = \Theta(-\mathsf{gap}(\mathsf{t}_\mu - [1, c-1]))$$

as required. For $c = n+1$ we have that $\imath(\gamma) = \tau$ and

$$\begin{aligned}
\Theta(1_{\mathsf{t}_{(n)}} \otimes \mathsf{bar}(\gamma)) &= \imath(c_{\mathsf{T}}) \circ (1_{(n,1)} + 1_{(n-2)} \otimes \mathsf{trid}_{\tau\alpha\tau}^{\tau\alpha\tau}) \circ \imath(c_{\mathsf{T}}^{*}) \\
&= 1_{(n)} \otimes \mathsf{bar}(\tau) + 1_{(n-1)} \otimes \mathsf{gap}(\sigma) \\
&= -2 \cdot 1_{(n-1)} \otimes \mathsf{gap}(\sigma) + 1_{(n-1)} \otimes \mathsf{gap}(\sigma) \\
&= \Theta(-\mathsf{gap}(\mathsf{t}_\mu - [1, n]))
\end{aligned}$$

where as required. Here the first equality follows from the definition of $\mathsf{e}$; the second from the $\tau$-fork-spot contraction relation; the third equality from Proposition 4.12 and Lemma 4.13; and the fourth is trivial. For $c = n+2$, we have that

$$\begin{aligned}
\Theta(1_{\mathsf{t}_{(n+1)}} \otimes \mathsf{bar}(\gamma)) &= 1_{\mathsf{t}_{(n,1)}} \otimes \mathsf{bar}(\gamma) + \mathsf{e}_{\mathsf{t}_{(n-2)}} \otimes \mathsf{trid}_{\tau\sigma\tau}^{\tau\sigma\tau} \otimes \mathsf{bar}(\gamma) \\
&= -\mathsf{gap}(\mathsf{t}_{(n,1)} - [n,2]) - 2\mathsf{gap}(\mathsf{t}_{(n,1)} - [n,1]) - \mathsf{e}_{\mathsf{t}_{(n-2)}} \otimes \mathsf{trid}_{\emptyset}^{\tau\sigma\tau} \mathsf{trid}_{\tau\sigma\tau}^{\emptyset} \\
&= -\mathsf{gap}(\mathsf{t}_{(n,1)} - [n,2]) - \mathsf{e}_{\mathsf{t}_{(n-2)}} \otimes \mathsf{trid}_{\emptyset}^{\tau\sigma\tau} \mathsf{trid}_{\tau\sigma\tau}^{\emptyset}
\end{aligned}$$

where the first equality is trivial; the second follows by applying Proposition 4.12 and Lemma 4.13 to the first term and applying the $(\tau, \imath(\gamma))$-null-braid and $\imath(\gamma)$-fork-spot-contraction to the second term; the third follows by applying (9) to the $\tau$-strands in the second term, followed by Proposition 4.12 and the cyclotomic and commutativity relations. On the other hand,

$$\begin{aligned}
&\Theta(1_{\mathsf{t}_{(n)}} \otimes \mathsf{gap}(\gamma)) \\
= &- (1_{(n,1)} + 1_{(n-2)} \otimes \mathsf{trid}_{\tau\alpha\tau}^{\tau\alpha\tau})(1_{\mathsf{t}_{(n)}} \otimes \mathsf{gap}(\tau))(1_{(n,1)} + 1_{(n-2)} \otimes \mathsf{trid}_{\tau\alpha\tau}^{\tau\alpha\tau}) \\
= &- 1_{(n-2)} \otimes (1_{\tau\sigma} \otimes \mathsf{gap}(\tau) + \mathsf{trid}_{\tau}^{\tau\sigma\tau} \mathsf{spot}_{\tau\sigma\tau}^{\tau\emptyset\emptyset} + \mathsf{spot}_{\tau\emptyset\emptyset}^{\tau\sigma\tau} \mathsf{trid}_{\tau\sigma\tau}^{\tau} + \mathsf{trid}_{\tau\sigma\tau}^{\tau\sigma\tau} \otimes \mathsf{bar}(\sigma)) \\
= &- 1_{(n-2)} \otimes (1_{\tau\sigma} \otimes \mathsf{gap}(\tau) - \mathsf{trid}_{\emptyset}^{\tau\sigma\tau} \mathsf{trid}_{\tau\sigma\tau}^{\emptyset} - \mathsf{bar}(\tau) \otimes \mathsf{dork}_{\tau\tau}^{\tau\tau} + 1_{\tau} \otimes \mathsf{gap}(\sigma) \otimes 1_{\tau}) \\
= &- \mathsf{gap}(\mathsf{t}_{(n,1)} - [n,2]) - \mathsf{e}_{\mathsf{t}_{(n-2)}} \otimes \mathsf{trid}_{\emptyset}^{\tau\sigma\tau} \mathsf{trid}_{\tau\sigma\tau}^{\emptyset}
\end{aligned}$$

as required. Here the penultimate equality follows by applying 9 to the middle two terms and applying Proposition 4.12 and Lemma 4.13 to the final term; the final equality follows from Proposition 4.12 and Lemma 4.13 and the commutativity and cyclotomic relations. Finally, we suppose that $c > n+2$. We have that

$$\begin{aligned}
\Theta(1_{\mathsf{t}_{(c-1)}} \otimes \mathsf{bar}(\gamma)) &= (1_{\mathsf{t}_{(n,1^{c-1-n})}} + \mathsf{e}_{(n,1^{c-1-n})}) \otimes \mathsf{bar}(\gamma) \\
&= -2\mathsf{gap}(\mathsf{t}_{(n,1^{c-1-n})} - [1,n]) - \mathsf{gap}(\mathsf{t}_{(n,1^{c-1-n})} - [c-1-n, n]) \\
&= -\mathsf{gap}(\mathsf{t}_{(n,1^{c-1-n})} - [c-1-n, n]) \\
&= \Theta(-\mathsf{gap}(\mathsf{t}_{(c+1)}) - [1, c+1]))
\end{aligned}$$

as required. For the second equality, we apply Proposition 4.12 and Lemma 4.13 to the first term and observe that the second term is zero by applying the bull-braid relations followed by Proposition 4.12 and Lemma 4.13 and the commutativity and cyclotomic relations. The other equalities are trivial. Thus the bi-chrome Tetris relation holds in all cases and we are done.

4.6.2. *The symplectic case, type* $(C_n, A_{n-1})$. We now consider the, more difficult, case of the symplectic group.

**Lemma 4.25.** *For* $\mathsf{e}_{\tau\sigma\tau\sigma\tau} = (1_{\tau\sigma\tau\sigma\tau} + \mathsf{trid}_{\tau\sigma\tau}^{\tau\sigma\tau} \otimes 1_{\sigma\tau})(1_{\tau\sigma\tau\sigma\tau} + 1_{\tau\sigma} \otimes \mathsf{trid}_{\tau\sigma\tau}^{\tau\sigma\tau})$*, we have that*

$$\begin{aligned}
(1_{\tau} \otimes \mathsf{trid}_{\sigma\tau\sigma}^{\sigma} \otimes 1_{\tau})\mathsf{e}_{\tau\sigma\tau\sigma\tau} &= -\mathsf{trid}_{\tau\sigma\tau}^{\tau\sigma\tau}(1_{\tau} \otimes \mathsf{trid}_{\sigma\tau\sigma}^{\sigma} \otimes \tau) - \mathsf{trid}_{\tau\sigma\tau}^{\tau\sigma\tau} \circledast \mathsf{trid}_{\tau\sigma\tau}^{\tau} \\
&= -\mathsf{trid}_{\tau\sigma\tau}^{\tau\sigma\tau}(1_{\tau} \otimes \mathsf{trid}_{\sigma\tau\sigma}^{\sigma} \otimes \tau)\mathsf{e}_{\tau\sigma\tau\sigma\tau}
\end{aligned}$$

*Proof.* We prove the first equality, the second follows as $\mathsf{e}_{\tau\sigma\tau\sigma\tau}$ is an idempotent which kills the second term. Consider the $m(\sigma,\tau)=4$ braid relation and tensor it on the left by $1_\tau$. Vertically concatenating $\mathsf{trid}^{\tau}_{\tau\sigma\tau}\otimes 1_{\sigma\tau}$ on top of this combination of diagrams, we obtain

$$1_{\tau\sigma}\otimes\mathsf{trid}^{\tau}_{\tau\sigma\tau}+\mathsf{trid}^{\tau}_{\tau\sigma\tau}\otimes 1_{\sigma\tau}+\mathsf{trid}^{\tau\sigma\tau}_{\tau\sigma\tau}(1_\tau\otimes\mathsf{trid}^{\sigma}_{\sigma\tau\sigma}\otimes 1_\tau)+2\mathsf{trid}^{\tau\sigma\tau}_{\tau\sigma\tau}\circledast\mathsf{trid}^{\tau}_{\tau\sigma\tau}+1_\tau\otimes\mathsf{trid}^{\sigma}_{\sigma\tau\sigma}\otimes 1_\sigma=0.$$

Moving the third term and one copy (of the two available) of the fourth term to the right, the result follows. □

We will split the proof of (the symplectic case of) Theorem 4.24 into two propositions. the first one, Proposition 4.26, shows that $\Theta$ is an isomorphism of graded vector spaces. The second one, Proposition 4.30, shows that $\Theta$ is an algebra homomorphism.

**Proposition 4.26.** *We have that the map* $\Theta: h_{(D_{n+1},A_n)}\to \mathsf{e}h_{(C_n,A_{n-1})}\mathsf{e}$ *given by* $\Theta(c_{\mathsf{ST}})=\mathsf{e}\circ\imath(c_{\mathsf{ST}})\circ\mathsf{e}$ *is an isomorphism of graded* $\Bbbk$*-spaces.*

*Proof.* We will show that the set

$$\{\mathsf{e}\circ c_{\mathsf{ST}}\circ\mathsf{e}\mid \mathsf{S},\mathsf{T}\in \mathrm{Path}^{\pm}_{(C_n,A_{n-1})}(\lambda,\mathsf{t}_\mu)\}\tag{4.18}$$

form a basis of $\mathsf{e}h_{(C_n,A_{n-1})}\mathsf{e}$ and thus deduce the result. We do this by considering $\Delta(\lambda)\mathsf{e}$ for all $\lambda\in\mathscr{P}_{(C_n,A_{n-1})}$. We will prove the following claim:

$$(c_{\mathsf{S}}+h^{<\lambda}_{(C_n,A_{n-1})})\mathsf{e}=\begin{cases} c_{\mathsf{S}}+\sum_{\mathsf{T}\notin\mathrm{Path}^{\pm}_{(C_n,A_{n-1})}(\lambda,\mathsf{t}_\mu)}a_{\mathsf{T}}c_{\mathsf{T}}+h^{<\lambda}_{(C_n,A_{n-1})} & \text{if } \mathsf{S}\in\mathrm{Path}^{\pm}_{(C_n,A_{n-1})}(\lambda,\mathsf{t}_\mu)\\ 0 & \text{otherwise}\end{cases}$$

from which we will immediately deduce the result. We first note that we can choose our $\mathsf{t}_\mu$ for each $\mu\in\mathscr{P}_{(C_n,A_{n-1})}$ in such a way that $\sigma$ always occurs immediately prior to a $\tau$. We prove this for $\mathsf{t}_{\mu+\gamma}$ assuming it holds for $\mathsf{t}_\mu$ (with the $\ell(\mathsf{t}_\mu)=0$ case being trivial). Let $\mathsf{S}\in\mathrm{Path}(\lambda,\mathsf{t}_\mu)$. For $\gamma\neq\tau$, we have that

$$(A^{\pm}_{\gamma}(c_{\mathsf{S}}))\mathsf{e}=A^{\pm}_{\gamma}(c_{\mathsf{S}}\mathsf{e})\qquad (R^{\pm}_{\gamma}(c_{\mathsf{S}}))\mathsf{e}=R^{\pm}_{\gamma}(c_{\mathsf{S}}\mathsf{e})$$

for $\gamma\in\mathrm{Add}(\lambda)$ or $\gamma\in\mathrm{Add}(\lambda)$, respectively. (Whence $\ell_\tau(\mu+\gamma)=\ell_\tau(\mu)$ implies $\mathsf{e}_{\mu+\gamma}=\mathsf{e}_\mu\otimes 1_\gamma$.)

Thus we may now assume that $\gamma=\tau\in\mathrm{Add}(\mu)$. In which case $\sigma=s_2\in\mathrm{Rem}(\mu)$ by our choice of $\mathsf{t}_\mu$. We let $\mu'=\mu-\sigma$. We suppose $\ell_\tau(\mu)$ is odd (the even case is identical) so that $\pi\in\mathrm{Add}(\imath^{-1}(\mu))$ and $\rho\in\mathrm{Rem}(\imath^{-1}(\mu'))$. Given $\lambda'\subseteq\mu'$, we let $c_{\mathsf{S}'}\in\mathrm{Path}_{(C_n,A_{n-1})}(\lambda',\mathsf{t}_{\mu'})$. We construct $c_{\mathsf{S}}$ for $\mathsf{S}\in\mathrm{Path}_{(C_n,A_{n-1})}(\lambda,\mathsf{t}_{\mu\tau})$ by applying the inductive process twice: once for $\sigma$ and once for $\tau$ as follows,

$$c_{\mathsf{S}}=X^{\pm}_{\tau}X^{\pm}_{\sigma}(c_{\mathsf{S}'})$$

for $X\in\{A,R\}$. Note that

$$c_{\mathsf{S}}\mathsf{e}=(X^{\pm}_{\tau}X^{\pm}_{\sigma}(c_{\mathsf{S}'}\mathsf{e}))(1_{\mathsf{t}_{\mu\tau}}+\mathsf{e}^{\ell_\tau(\mu)+1}_{\mu+\tau}).\tag{4.19}$$

We assume, by induction, that the claim holds for $c_{\mathsf{S}'}$. So we have

$$c_{\mathsf{S}'}\mathsf{e}=c_{\mathsf{S}'}+\textstyle\sum_{\mathsf{T}'\notin\mathrm{Path}^{\pm}_{(C_n,A_{n-1})}(\lambda',\mathsf{t}_{\mu'})}a_{\mathsf{T}'}c_{\mathsf{T}'}+h^{<\lambda'}_{(C_n,A_{n-1})}.\tag{4.20}$$

Since $\mathsf{T}'\notin\mathrm{Path}^{\pm}_{(C_n,A_{n-1})}(\lambda',\mathsf{t}_{\mu'})$, this implies by definition $X^{\pm}_{\tau}X^{\pm}_{\sigma}(c_{\mathsf{T}'})\notin\mathrm{Path}^{\pm}_{(C_n,A_{n-1})}(\lambda,\mathsf{t}_\mu)$. We will now consider

$$c_{\mathsf{S}}=X^{\pm}_{\tau}X^{\pm}_{\sigma}(c_{\mathsf{S}'})$$

for $\mathsf{S}'\in\mathrm{Path}^{\pm}_{(C_n,A_{n-1})}(\lambda',\mathsf{t}_{\mu'})$. Before considering the above case-wise, we remark that either $\rho\in\mathrm{Rem}(\imath^{-1}(\lambda'))$ or $\mathrm{Add}(\imath^{-1}(\lambda'))$ (because it appears at the edge of the region).

**Case 1.** Suppose $\sigma\in\mathrm{Add}(\lambda')$. This implies that $\rho\in\mathrm{Rem}(\imath^{-1}(\lambda'))$. The first two subcases which we consider simultaneously are

$$A^{+}_{\tau}A^{+}_{\sigma}(c_{\mathsf{S}'})=c_{\mathsf{S}'}\otimes 1_\sigma\otimes 1_\tau\qquad A^{-}_{\tau}A^{+}_{\sigma}(c_{\mathsf{S}'})=c_{\mathsf{S}'}\otimes 1_\sigma\otimes\mathsf{spot}^{\emptyset}_{\tau}.$$

Here we have that $c_{\mathsf{S}}=A^{\pm}_{\tau}A^{\pm}_{\sigma}(c_{\mathsf{S}'})$ satisfies $\mathsf{S}\in\mathrm{Path}^{\pm}_{(C_n,A_{n-1})}$. We have that

$$(A^{+}_{\tau}A^{+}_{\sigma}(c_{\mathsf{S}'})+h^{<\lambda}_{(C,A)})(1+\mathsf{e}^{\ell_\tau(\mu)+1}_{\mu+\tau})=c_{\mathsf{S}}+c_{\mathsf{S}'}\circledast\mathsf{trid}^{\tau\sigma\tau}_{\tau\sigma\tau}+h^{<\lambda}_{(C,A)}=c_{\mathsf{S}}+h^{<\lambda}_{(C,A)}$$

$$(A^{-}_{\tau}A^{+}_{\sigma}(c_{\mathsf{S}'})+h^{<\lambda}_{(C,A)})(1+\mathsf{e}^{\ell_\tau(\mu)+1}_{\mu+\tau})=c_{\mathsf{S}}+c_{\mathsf{S}'}\circledast(\mathsf{trid}^{\tau}_{\tau\sigma\tau}\otimes\mathsf{spot}^{\sigma}_{\emptyset})+h^{<\lambda}_{(C,A)}=c_{\mathsf{S}}+h^{<\lambda}_{(C,A)}$$

where in both cases the diagram $c_{\mathsf{S}'} \circledast \dots$ factors through the idempotent labelled by $\mathsf{t}_{\lambda'}$ and so belongs to the ideal $h^{<\lambda}_{(C,A)}$. The final two subcases which we will consider simultaneously are

$$R^+_{\tau} A^-_{\sigma}(c_{\mathsf{S}'}) = c_{\mathsf{S}'} \circledast \mathsf{trid}^{\tau}_{\tau\sigma\tau} \qquad R^-_{\tau} A^-_{\sigma}(c_{\mathsf{S}'}) = c_{\mathsf{S}'} \circledast \mathsf{trid}^{\emptyset}_{\tau\sigma\tau}.$$

Here $c_{\mathsf{S}} = R^{\pm}_{\tau} A^-_{\sigma}(c_{\mathsf{S}'})$ satisfies $\mathsf{S} \notin \mathrm{Path}^{\pm}_{(C_n,A_{n-1})}$. We note that $\mathsf{e}^{\ell_\tau(\mu)+1}_{\mu+\tau} = 1_{\mathsf{t}_{\mu-\sigma}} \circledast (1_{\tau\sigma\tau} + \mathsf{trid}^{\tau\sigma\tau}_{\tau\sigma\tau})$ and hence applying equation (4.12) we obtain

$$(R^{\pm}_{\tau} A^-_{\sigma}(c_{\mathsf{S}'}))(1 + \mathsf{e}^{\ell_\tau(\mu)+1}_{\mu+\tau}) = 0$$

as required.

**Case 2.** Suppose $\sigma \in \mathrm{Rem}(\lambda')$. This implies that $\rho \in \mathrm{Add}(\imath^{-1}(\lambda'))$. Since any two $\sigma$-tiles in $\mu'$ are separated by some $\tau$-tile (and $\tau \notin \mathrm{Rem}(\lambda')$ but $\sigma \in \mathrm{Rem}(\lambda')$) we have that $c_{\mathsf{S}'} = c_{\mathsf{S}''} \circledast \mathsf{spot}^{\emptyset}_{\tau}$ for some $\mathsf{S}'' \in \mathrm{Path}^{\pm}_{(C_n,A_{n-1})}(\lambda', \mu' - \tau)$. Here we have that

$$R^+_{\alpha}(c_{\mathsf{S}'}) = c_{\mathsf{S}'} \circledast \mathsf{fork}^{\alpha}_{\alpha\alpha} = c_{\mathsf{S}''} \circledast \mathsf{spot}^{\emptyset}_{\tau} \circledast \mathsf{fork}^{\alpha}_{\alpha\alpha} \quad R^-_{\alpha}(c_{\mathsf{S}'}) = c_{\mathsf{S}'} \circledast \mathsf{cap}^{\emptyset}_{\sigma\sigma} = c_{\mathsf{S}''} \circledast \mathsf{spot}^{\emptyset}_{\tau} \circledast \mathsf{cap}^{\emptyset}_{\sigma\sigma}$$

We start with

$$(A^{\pm}_{\tau} R^+_{\sigma}(c_{\mathsf{S}'}))(1_{\mathsf{t}_{\mu\tau}} + \mathsf{e}^{\ell_\tau(\mu)}_{\mu\tau})(1_{\mathsf{t}_{\mu\tau}} + \mathsf{e}^{\ell_\tau(\mu)+1}_{\mu\tau}).$$

and we first consider $A^+_{\tau} R^+_{\sigma}(c_{\mathsf{S}'})$. We note that $\tau \in \mathrm{Add}(\lambda')$ but that $\rho \notin \mathrm{Add}(\imath^{-1}(\lambda'))$ (rather, the "wrong" colour $\rho$ is). Therefore $\mathsf{S} \notin \mathrm{Path}^{\pm}_{(C_n,A_{n-1})}(\lambda, \mathsf{t}_\mu)$ and using Lemma 4.25 we have

$$(A^+_{\tau} R^+_{\sigma}(c_{\mathsf{S}'})) \circ (1_{\mathsf{t}_{\mu\tau}} + \mathsf{e}^{\ell_\tau(\mu)}_{\mu\tau}) \circ (1_{\mathsf{t}_{\mu\tau}} + \mathsf{e}^{\ell_\tau(\mu)+1}_{\mu\tau}) \in h^{<\lambda}_{(C,A)}$$

since the terms in the sum factor through the idempotent $\lambda' - \tau$. Therefore $c_{\mathsf{S}}\mathsf{e} = 0$ modulo $h^{<\lambda}_{(C,A)}$ as required. Arguing in an identical manner, (or by simply "putting a blue spot on top of the above calculation") we have that

$$(A^-_{\tau} R^+_{\sigma}(c_{\mathsf{S}'}))(1_{\mathsf{t}_{\mu\tau}} + \mathsf{e}^{\ell_\tau(\mu)}_{\mu\tau})(1_{\mathsf{t}_{\mu\tau}} + \mathsf{e}^{\ell_\tau(\mu)+1}_{\mu\tau}) \in h^{<\lambda}_{(C,A)}$$

as required. The final two subcases which we will consider simultaneously are

$$c_{\mathsf{S}} = R^+_{\tau} R^-_{\sigma}(c_{\mathsf{S}'}) = c_{\mathsf{S}''} \circledast \mathsf{spot}^{\emptyset}_{\tau} \circledast \mathsf{cap}^{\emptyset}_{\sigma\sigma} \circledast \mathsf{fork}^{\tau}_{\tau\tau}$$
$$c_{\mathsf{S}} = R^-_{\tau} R^-_{\sigma}(c_{\mathsf{S}'}) = c_{\mathsf{S}''} \circledast \mathsf{spot}^{\emptyset}_{\tau} \circledast \mathsf{cap}^{\emptyset}_{\sigma\sigma} \circledast \mathsf{cap}^{\emptyset}_{\tau\tau}$$

In both cases, $\mathsf{S} \in \mathrm{Path}^{\pm}_{(C_n,A_{n-1})}$. We have that

$$(R^+_{\tau} R^-_{\sigma}(c_{\mathsf{S}'}))(1_{\mathsf{t}_{\mu\tau}} + \mathsf{e}^{\ell_\tau(\mu)+1}_{\mu\tau}) = c_{\mathsf{S}''} \circledast \mathsf{spot}^{\emptyset}_{\tau} \circledast \mathsf{cap}^{\emptyset}_{\sigma\sigma} \circledast \mathsf{fork}^{\tau}_{\tau\tau} + c_{\mathsf{S}''} \circledast \mathsf{spot}^{\emptyset}_{\sigma} \circledast (\mathsf{fork}^{\tau}_{\tau\tau} \circledast \mathsf{trid}^{\tau}_{\tau\sigma\tau})$$
$$(R^-_{\tau} R^-_{\sigma}(c_{\mathsf{S}'}))(1_{\mathsf{t}_{\mu\tau}} + \mathsf{e}^{\ell_\tau(\mu)+1}_{\mu\tau}) = c_{\mathsf{S}''} \circledast \mathsf{spot}^{\emptyset}_{\tau} \circledast \mathsf{cap}^{\emptyset}_{\sigma\sigma} \circledast \mathsf{cap}^{\tau}_{\tau\tau} + c_{\mathsf{S}''} \circledast \mathsf{spot}^{\emptyset}_{\sigma} \circledast (\mathsf{fork}^{\tau}_{\tau\tau} \circledast \mathsf{trid}^{\emptyset}_{\tau\sigma\tau})$$

In each case the former term on the righthand-side of the equality is equal to $c_{\mathsf{S}}$ and the latter term is equal to $c_{\mathsf{T}}$ for $\mathsf{T} \notin \mathrm{Path}^{\pm}_{(C_n,A_{n-1})}(\lambda, \mathsf{t}_\mu)$. The result follows. □

**Lemma 4.27.** *Let* $\mathsf{e}_{\tau\sigma\tau\sigma} = (1_{\tau\sigma\tau\sigma} + \mathsf{trid}^{\tau\sigma\tau}_{\tau\sigma\tau} \otimes 1_{\sigma})$*, then we have*

$$\mathsf{e}_{\tau\sigma\tau\sigma} 1_{\tau\sigma\tau\sigma} \mathsf{e}_{\tau\sigma\tau\sigma} = -\mathsf{e}_{\tau\sigma\tau\sigma}(1_{\tau} \otimes \mathsf{trid}^{\sigma\tau\sigma}_{\sigma\tau\sigma})\mathsf{e}_{\tau\sigma\tau\sigma}$$

*Proof.* Applying $\mathsf{e}_{\tau\sigma\tau\sigma}$ to both sides of the $m(\sigma,\tau) = 4$ null-braid relations and using equation (4.15) immediately gives the result. □

**Corollary 4.28.** *Let* $\mu \in \mathscr{P}_{(C_{n+1},A_n)}$ *and* $[r,c] \in \mu$. *Define* $k, l, m$ *as in Subsection 4.2. If* $[r-k, c+k+1] \notin \mu$ *or* $[r+l+1, c-l] \notin \mu$ *then we have* $\mathsf{e} \circ \mathsf{gap}(\mathsf{t}_\mu - [r,c]) \circ \mathsf{e} = 0$. *Otherwise we have*

$$\mathsf{e} \circ \mathsf{gap}(\mathsf{t}_\mu - [r,c]) \circ \mathsf{e} = (-1)^{k+l-m} \mathsf{e} \circ \iota(c_{\mathsf{SS}}) \circ \mathsf{e}$$

*where* $c_{\mathsf{SS}}$ *is defined in Subsection 4.2.*

*Proof.* The proof follows exactly the same arguments as for the proof the corresponding statement in type $(D_{n+1}, A_n)$ given in equation (4.8) and (4.9). There are only two additional things to check. First we need to prove that for $[r,r] \in \mu$ we have $\mathsf{e} \circ \mathsf{gap}(\mathsf{t}_\mu - [r, r-1]) \circ \mathsf{e} = 0$. This follows directly from Figure 9 and equation (4.15). The second thing is that the $(s_0, s_2)$ and $(s_1, s_2)$-nullbraid relation is preserved under the map $\Theta$. This is precisely the statement of Lemma 4.27. □

**Lemma 4.29.** *For* $\mathsf{S} \in \mathrm{Path}_{(D_{n+1},A_n)}(\lambda, \mathsf{t}_\mu)$*, we have that* $\imath(c_\mathsf{S}) \circ e = e \circ \imath(c_\mathsf{S}) \circ e$*.*

*Proof.* For $1 \leqslant k \leqslant \ell_{\tau}(\imath(\lambda))$, we will show that $\mathsf{e}^k_\lambda \imath(c_\mathsf{S}) = \imath(c_\mathsf{S})\mathsf{e}^j_\mu$ for some $1 \leqslant j \leqslant \ell_\tau(\imath(\mu))$ and hence deduce the result. Assume $k$ is even (the odd case is identical). We can assume that $\mathsf{t}_\mu$ is such that each $\boldsymbol{\rho}$-strand is immediately preceded by a $\boldsymbol{\sigma}$-strand, so that $\mathsf{t}_\lambda = \underline{v_0}\boldsymbol{\rho}\underline{v_1}\boldsymbol{\sigma}\boldsymbol{\pi}\underline{v_2}$ where $\ell(\imath(\underline{v_0}\boldsymbol{\rho}\underline{v_1}\boldsymbol{\sigma}\boldsymbol{\pi})) = k$ and where $\underline{v_1}$ does not contain $\boldsymbol{\rho}, \boldsymbol{\sigma}, \boldsymbol{\pi}$. We can write $\mathsf{t}_\mu$ in the form $\underline{w_0}\boldsymbol{\rho}\underline{w_1}\boldsymbol{\sigma}\underline{w_2}\boldsymbol{\pi}\underline{w_3}$ such that the $\boldsymbol{\rho}, \boldsymbol{\sigma}, \boldsymbol{\pi}$ in this expression are connected to the $\boldsymbol{\rho}, \boldsymbol{\sigma}, \boldsymbol{\pi}$ in the expression $\underline{v_0}\boldsymbol{\rho}\underline{v_1}\boldsymbol{\sigma}\boldsymbol{\pi}\underline{v_2}$ by strands in the Soergel diagram. Moreover, we can assume that $\ell(\underline{w_0})$ and $\ell(\underline{w_3})$ are maximal with respect to this property. We claim that $\underline{w_1}$ and $\underline{w_2}$ do not contain any occurrences of $\boldsymbol{\rho}, \boldsymbol{\sigma}, \boldsymbol{\pi}$. Thus the specified $\boldsymbol{\rho}$ and $\boldsymbol{\beta}$ strands commute with all strands lying between them, except for the specified $\boldsymbol{\sigma}$-strand. Under $\imath$ these correspond to $\boldsymbol{\tau}$-strands which commute with all strands lying between them, except for the specified $\boldsymbol{\sigma}$-strand. Thus applying the trident on top/bottom of these strands we get the same result, as required.

It only remains to verify the claim. Suppose one of the three colours does occur in $\underline{w_1}$. The first colour to appear must be $\boldsymbol{\sigma}$ (because it follows $\boldsymbol{\rho}$) and this must be a $A^-_{\boldsymbol{\sigma}}$ step in the basis (because $\underline{v_1}$ has no $\boldsymbol{\sigma}$ and so it cannot be a $X^+_{\boldsymbol{\sigma}}$ step and the prior $\boldsymbol{\rho}$ step was an $X^+_{\boldsymbol{\rho}}$ and so it cannot be an $R^-_{\boldsymbol{\sigma}}$ step). After this $\boldsymbol{\sigma}$, there must be a $\boldsymbol{\pi}$ but this cannot be an $R^\pm_{\boldsymbol{\pi}}$ (as the prior $\boldsymbol{\rho}$ step was an $X^+_{\boldsymbol{\rho}}$) or be $A^\pm_{\boldsymbol{\pi}}$ (because the prior $\boldsymbol{\sigma}$ was a $A^-_{\boldsymbol{\sigma}}$). Thus the claims follows for $\underline{w_1}$. The case of $\underline{w_3}$ is similar. □

**Proposition 4.30.** *The map* $\Theta : h_{(D_{n+1},A_n)} \to \mathsf{e}h_{(C_n,A_{n-1})}\mathsf{e}$ *given by* $\Theta(c_{\mathsf{ST}}) = \mathsf{e} \circ \imath(c_{\mathsf{ST}}) \circ \mathsf{e}$ *is a* $\Bbbk$*-algebra homomorphism.*

*Proof.* We check this on the cellular basis by showing that

$$\mathsf{e} \circ \imath(c_{\mathsf{ST}}c_{\mathsf{UV}}) \circ \mathsf{e} = \Theta(c_{\mathsf{ST}}c_{\mathsf{UV}}) = \Theta(c_{\mathsf{ST}})\Theta(c_{\mathsf{UV}}) = \mathsf{e} \circ \imath(c_{\mathsf{ST}}) \circ \mathsf{e} \circ \imath(c_{\mathsf{UV}}) \circ \mathsf{e}$$

for $\mathsf{S} \in \mathrm{Path}_{(D_{n+1},A_n)}(\nu, -)$, $\mathsf{T} \in \mathrm{Path}_{(D_{n+1},A_n)}(\nu, \mathsf{t}_\mu)$, $\mathsf{U} \in \mathrm{Path}_{(D_{n+1},A_n)}(\eta, \mathsf{t}_\mu)$, $\mathsf{V} \in \mathrm{Path}_{(D_{n+1},A_n)}(\eta, -)$. By Lemma 4.29, we have that

$$\mathsf{e} \circ \imath(c_{\mathsf{ST}}) \circ \mathsf{e} \circ \imath(c_{\mathsf{UV}}) \circ \mathsf{e} = \mathsf{e} \circ \imath(c^*_\mathsf{S}) \circ \mathsf{e} \circ \imath(c_\mathsf{T}) \circ \mathsf{e} \circ \imath(c^*_\mathsf{U}) \circ \mathsf{e} \circ \imath(c_\mathsf{V}) \circ \mathsf{e}.$$

We proceed by induction on $\ell(\mu)$, the base case $\ell(\mu) = 0$ is trivial. We can assume $\ell(\pi), \ell(\rho) < \ell(\mu)$ as if $\ell(\pi) = \ell(\rho) = \ell(\mu)$ then $1_\pi = 1_\rho = 1_\mu$ and this product becomes $\mathsf{e} \circ \imath(c_{\mathsf{SV}}) \circ \mathsf{e}$ as required. Similarly, if $\ell(\pi) = \ell(\mu)$ and $\ell(\rho) < \ell(\mu)$ (or vice versa) this product becomes

$$\mathsf{e} \circ \imath(c^*_\mathsf{S}) \circ \mathsf{e} \circ \imath(c^*_\mathsf{U}) \circ \mathsf{e} \circ \imath(c_\mathsf{V}) = (\mathsf{e} \circ \imath(c^*_\mathsf{S}) \circ \mathsf{e} \circ \imath(c^*_\mathsf{U}) \circ \mathsf{e})(\mathsf{e} \circ \imath(c_\mathsf{V}) \circ \mathsf{e}) = \Theta(c^*_\mathsf{S}c^*_\mathsf{U})\Theta(c_\mathsf{V})$$

and so we can again appeal to our inductive assumption. We will focus on the middle of the product and prove that

$$\mathsf{e} \circ \imath(c_\mathsf{T}) \circ \mathsf{e} \circ \imath(c^*_\mathsf{U}) \circ \mathsf{e} = \mathsf{e} \circ \imath(c_{\mathsf{TU}}) \circ \mathsf{e}. \tag{4.21}$$

As $\ell(\eta), \ell(\nu) \leqslant \ell(\mu)$ we can then apply induction to deal with the products with $\mathsf{e} \circ \imath(c^*_\mathsf{S})$ and $\imath(c_\mathsf{V}) \circ \mathsf{e}$. Now, the basis elements $c_\mathsf{T}$ and $c^*_\mathsf{U}$ are constructed inductively and we will consider cases depending on the last step in this inductive procedure.

**Case 1.** We first consider the case that $c_\mathsf{T} = A^+_\alpha(c_{\mathsf{T}'})$ and $c_\mathsf{U} = A^+_\alpha(c_{\mathsf{U}'})$. By induction, we can assume that

$$\mathsf{e} \circ \imath(c_{\mathsf{T}'}) \circ \mathsf{e} \circ \imath(c^*_{\mathsf{U}'}) \circ \mathsf{e} = \textstyle\sum_{\mathsf{X},\mathsf{Y}} a_{\mathsf{X},\mathsf{Y}} \mathsf{e} \circ \imath(c_{\mathsf{XY}}) \circ \mathsf{e} \quad \text{where} \quad c_{\mathsf{T}'}c^*_{\mathsf{U}'} = \textstyle\sum_{\mathsf{X},\mathsf{Y}} a_{\mathsf{X},\mathsf{Y}} c_{\mathsf{XY}}$$

If $\alpha \neq \boldsymbol{\pi}, \boldsymbol{\rho}$, then $\ell_{\boldsymbol{\tau}}(\imath(\mu)) = \ell_{\boldsymbol{\tau}}(\imath(\mu - \alpha))$ and therefore

$$\mathsf{e} \circ \imath(c_\mathsf{T}) \circ \mathsf{e} \circ \imath(c^*_\mathsf{U}) \circ \mathsf{e} = \textstyle\sum_{\mathsf{X},\mathsf{Y}} a_{\mathsf{X},\mathsf{Y}} (\mathsf{e}(\imath(c_{\mathsf{XY}}) \otimes 1_\alpha)\mathsf{e}).$$

If $\alpha = \boldsymbol{\pi}$ (the $\boldsymbol{\rho}$ case is identical) then $c_\mathsf{T} = A^+_{\boldsymbol{\pi}}A^+_{\boldsymbol{\sigma}}(c_{\mathsf{T}''})$, $c_\mathsf{U} = A^+_{\boldsymbol{\pi}}A^+_{\boldsymbol{\sigma}}(c_{\mathsf{U}''})$ and

$$\begin{aligned}
\mathsf{e} \circ \imath(c_\mathsf{T}) \circ \mathsf{e} \circ \imath(c^*_\mathsf{U}) \circ \mathsf{e} &= \mathsf{e}\left((\imath(c_{\mathsf{T}''}) \circ \mathsf{e}_{\mu - \boldsymbol{\tau} - \boldsymbol{\sigma}} \circ \imath(c^*_{\mathsf{U}''})) \circledast (1_{\boldsymbol{\tau}\boldsymbol{\sigma}\boldsymbol{\tau}} + \mathsf{trid}^{\boldsymbol{\tau}\boldsymbol{\sigma}\boldsymbol{\tau}}_{\boldsymbol{\tau}\boldsymbol{\sigma}\boldsymbol{\tau}})\right)\mathsf{e} \\
&= \mathsf{e}\left((\imath(c_{\mathsf{T}''}) \circ \mathsf{e}_{\mu - \boldsymbol{\tau} - \boldsymbol{\sigma}} \circ \imath(c^*_{\mathsf{U}''})) \circledast 1_{\boldsymbol{\tau}\boldsymbol{\sigma}\boldsymbol{\tau}}\right)\mathsf{e} \\
&= \mathsf{e}\left((\imath(c_{\mathsf{T}'}) \circ \mathsf{e}_{\mu - \boldsymbol{\tau}} \circ \imath(c^*_{\mathsf{U}'})) \otimes 1_{\boldsymbol{\tau}}\right)\mathsf{e} \\
&= \textstyle\sum_{\mathsf{X},\mathsf{Y}} a_{\mathsf{X},\mathsf{Y}} \mathsf{e}(\imath(c_{\mathsf{XY}}) \otimes 1_{\boldsymbol{\tau}})\mathsf{e}
\end{aligned}$$

the first equality follows from the definition of the idempotents; for the second equality, we note that the trident term in the sum is zero by equation (4.15); the third equality follows by definition of the cellular basis elements and the idempotents; the final equality holds by induction. Thus in all cases, we have that

$$\mathsf{e} \circ \imath(c_{\mathsf{T}}) \circ \mathsf{e} \circ \imath(c^*_{\mathsf{U}}) \circ \mathsf{e} = \sum_{\mathsf{X},\mathsf{Y}} a_{\mathsf{X},\mathsf{Y}} \mathsf{e}(\imath(c_{\mathsf{XY}}) \otimes 1_{\imath(\alpha)})\mathsf{e}. \tag{4.22}$$

It remains to show that every $\mathsf{e}(\imath(c_{\mathsf{XY}}) \otimes 1_{\imath(\alpha)})\mathsf{e} = \mathsf{e}(\imath(c_{\mathsf{XY}} \otimes 1_\alpha))\mathsf{e}$ for every $\mathsf{X}, \mathsf{Y}$ appearing in the above sum. We set $\lambda := \mathrm{Shape}(\mathsf{X}) = \mathrm{Shape}(\mathsf{Y})$.

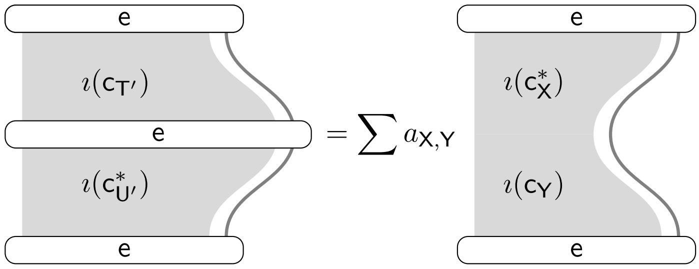



FIGURE 35. Case 1: a diagrammatic version of equation (4.22).

*Subcase 1.1.* If $\alpha \in \mathrm{Add}(\lambda)$ and $\imath(\alpha) \in \mathrm{Add}(\imath(\lambda))$ then $\imath(c_{\mathsf{XY}}) \otimes 1_{\imath(\alpha)}$ and $c_{\mathsf{XY}} \otimes 1_\alpha$ are both cellular basis elements and we are done.

*Subcase 1.2.* If $\alpha \notin \mathrm{Add}(\lambda)$ and $\imath(\alpha) \in \mathrm{Add}(\imath(\lambda))$ then we can assume that $\alpha = \boldsymbol{\pi}$ (the $\alpha = \boldsymbol{\rho}$ case is identical). This implies that $\boldsymbol{\rho} \in \mathrm{Add}(\lambda)$ (but by assumption $\boldsymbol{\sigma} \in \mathrm{Rem}(\lambda)$ and $\boldsymbol{\pi} \in \mathrm{Rem}(\lambda - \boldsymbol{\sigma})$) this implies that we can write $c_{\mathsf{X}}$ and $c_{\mathsf{Y}}$ as

$$c_{\mathsf{X}} = c_{\mathsf{X}'} \circledast \mathsf{spot}^{\emptyset}_{\boldsymbol{\rho}} \circledast \mathsf{fork}^{\boldsymbol{\sigma}}_{\boldsymbol{\sigma}\boldsymbol{\sigma}} \qquad c_{\mathsf{Y}} = c_{\mathsf{Y}'} \circledast \mathsf{spot}^{\emptyset}_{\boldsymbol{\rho}} \circledast \mathsf{fork}^{\boldsymbol{\sigma}}_{\boldsymbol{\sigma}\boldsymbol{\sigma}} \tag{4.23}$$

for some $\mathsf{X}', \mathsf{Y}' \in \mathrm{Path}_{(D_{n+1},A_n)}(\lambda + \boldsymbol{\rho}, \mathsf{t}_{\nu-\boldsymbol{\rho}-\boldsymbol{\sigma}})$. Now, as $\boldsymbol{\pi} \in \mathrm{Rem}(\lambda - \boldsymbol{\sigma})$, using the $\boldsymbol{\pi\sigma}$-bull-braid relations we get

$$c_{\mathsf{XY}} \otimes 1_{\boldsymbol{\pi}} = -(c_{\mathsf{X}} \circledast \mathsf{spot}^{\emptyset}_{\boldsymbol{\sigma}} \circledast \mathsf{fork}^{\boldsymbol{\pi}}_{\boldsymbol{\pi}\boldsymbol{\pi}})^*(c_{\mathsf{Y}} \circledast \mathsf{spot}^{\emptyset}_{\boldsymbol{\sigma}} \circledast \mathsf{fork}^{\boldsymbol{\pi}}_{\boldsymbol{\pi}\boldsymbol{\pi}}) \tag{4.24}$$

On the other hand, using equation (4.23) and Lemma 4.25 we have

$$(\imath(c_{\mathsf{X}} \otimes 1_{\boldsymbol{\tau}})\mathsf{e} = -(1_{\mathsf{t}_\lambda} \circledast \mathsf{spot}^{\emptyset}_{\boldsymbol{\sigma}} \circledast \mathsf{fork}^{\boldsymbol{\tau}}_{\boldsymbol{\tau}\boldsymbol{\tau}})^*(\imath(c_{\mathsf{X}}) \circledast \mathsf{spot}^{\emptyset}_{\boldsymbol{\sigma}} \circledast \mathsf{fork}^{\boldsymbol{\tau}}_{\boldsymbol{\tau}\boldsymbol{\tau}})\mathsf{e}$$

and similarly for $(\imath(c_{\mathsf{Y}}) \otimes 1_{\boldsymbol{\tau}})\mathsf{e}$. Thus we get

$$\begin{aligned} &\mathsf{e}(\imath(c_{\mathsf{XY}}) \otimes 1_{\boldsymbol{\tau}})\mathsf{e} \\ &= \mathsf{e}(\imath(c_{\mathsf{X}}) \circledast \mathsf{spot}^{\emptyset}_{\boldsymbol{\sigma}} \circledast \mathsf{fork}^{\boldsymbol{\tau}}_{\boldsymbol{\tau}\boldsymbol{\tau}})^*(1_{\mathsf{t}_\lambda} \circledast \mathsf{spot}^{\emptyset}_{\boldsymbol{\sigma}} \circledast \mathsf{fork}^{\boldsymbol{\tau}}_{\boldsymbol{\tau}\boldsymbol{\tau}})(1_{\mathsf{t}_\lambda} \circledast \mathsf{spot}^{\emptyset}_{\boldsymbol{\sigma}} \circledast \mathsf{fork}^{\boldsymbol{\tau}}_{\boldsymbol{\tau}\boldsymbol{\tau}})^*(\imath(c_{\mathsf{Y}}) \circledast \mathsf{spot}^{\emptyset}_{\boldsymbol{\sigma}} \circledast \mathsf{fork}^{\boldsymbol{\tau}}_{\boldsymbol{\tau}\boldsymbol{\tau}})\mathsf{e} \\ &= -\mathsf{e}(\imath(c_{\mathsf{X}}) \circledast \mathsf{spot}^{\emptyset}_{\boldsymbol{\sigma}} \circledast \mathsf{fork}^{\boldsymbol{\tau}}_{\boldsymbol{\tau}\boldsymbol{\tau}})^*(\imath(c_{\mathsf{Y}}) \circledast \mathsf{spot}^{\emptyset}_{\boldsymbol{\sigma}} \circledast \mathsf{fork}^{\boldsymbol{\tau}}_{\boldsymbol{\tau}\boldsymbol{\tau}})\mathsf{e} \end{aligned} \tag{4.25}$$

comparing equation (4.24) and (4.25) we are done.

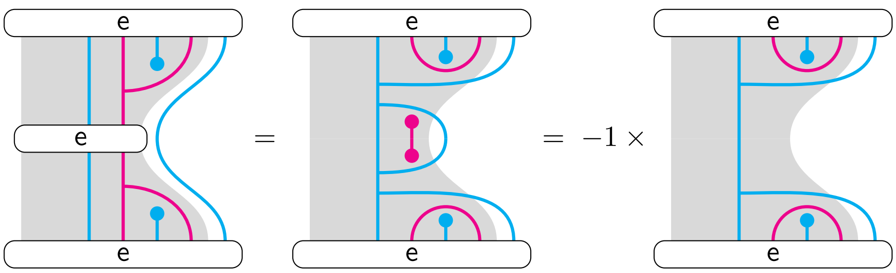



FIGURE 36. Subcase 1.2

*Subcase 1.3.* If $\alpha \in \mathrm{Rem}(\lambda)$ and $\imath(\alpha) \in \mathrm{Rem}(\imath(\lambda))$ then the monochrome Tetris relation implies that

$$c_{\mathsf{XY}} \otimes 1_\alpha = (c_{\mathsf{X}} \circledast \mathsf{fork}^{\alpha}_{\alpha\alpha})^*(c_{\mathsf{Y}} \circledast \mathsf{spot}^{\emptyset}_{\alpha} \otimes 1_\alpha) + (c_{\mathsf{X}} \circledast \mathsf{spot}^{\emptyset}_{\alpha} \otimes 1_\alpha)^*(c_{\mathsf{Y}} \circledast \mathsf{fork}^{\alpha}_{\alpha\alpha})$$

$$+ (c_{\mathsf{X}} \circledast \mathsf{fork}^{\alpha}_{\alpha\alpha})^*(\textstyle\sum_{[x,y]\in \underline{\mathsf{Hook}}_\alpha(\lambda-\alpha)} \mathsf{gap}(\mathsf{t}_\lambda - [x,y]))(c_{\mathsf{Y}} \circledast \mathsf{fork}^{\alpha}_{\alpha\alpha})$$

Similarly, we obtain

$$\begin{aligned} &\mathsf{e}(\imath(c_{\mathsf{XY}}) \otimes 1_{\imath(\alpha)})\mathsf{e} \\ &= \mathsf{e}(\imath(c_{\mathsf{X}}) \circledast \mathsf{fork}^{\imath(\alpha)}_{\imath(\alpha)\imath(\alpha)})^*(\imath(c_{\mathsf{Y}}) \circledast \mathsf{spot}^{\emptyset}_{\imath(\alpha)} \otimes 1_{\imath(\alpha)})\mathsf{e} \\ &\quad + \mathsf{e}(\imath(c_{\mathsf{X}}) \circledast \mathsf{spot}^{\emptyset}_{\imath(\alpha)} \otimes 1_{\imath(\alpha)})^*(\imath(c_{\mathsf{Y}}) \circledast \mathsf{fork}^{\imath(\alpha)}_{\imath(\alpha)\imath(\alpha)})\mathsf{e} \\ &\quad + \mathsf{e}(\imath(c_{\mathsf{X}}) \circledast \mathsf{fork}^{\imath(\alpha)}_{\imath(\alpha)\imath(\alpha)})^*\mathsf{e}(\textstyle\sum_{[x,y]\in \underline{\mathsf{Hook}}_{\imath(\alpha)}(\imath(\lambda-\alpha))} \mathsf{gap}(\mathsf{t}_{\imath(\lambda)} - [x,y]))\mathsf{e}(\imath(c_{\mathsf{Y}}) \circledast \mathsf{fork}^{\imath(\alpha)}_{\imath(\alpha)\imath(\alpha)})\mathsf{e} \end{aligned}$$

where we have inserted extra idempotents $\mathsf{e}$ in the final summand using Lemma 4.29. Recall that we need to check that

$$\mathsf{e}(\imath(c_{\mathsf{XY}} \otimes 1_\alpha))\mathsf{e} = \mathsf{e}(\imath(c_{\mathsf{XY}}) \otimes 1_{\imath(\alpha)})\mathsf{e}$$

where the first two terms in each of the above equations obviously agree. For the final term, note that if $\alpha \neq \boldsymbol{\rho}, \boldsymbol{\pi}$ we have

$$\underline{\mathsf{Hook}}_\alpha(\lambda - \alpha) = \underline{\mathsf{Hook}}_{\imath(\alpha)}(\imath(\lambda - \alpha)).$$

If $\alpha = \boldsymbol{\rho}$ then $\imath(\alpha) = \boldsymbol{\tau}$ (the $\alpha - \boldsymbol{\pi}$ case is identical). Say $s_{[r,r]} = \boldsymbol{\rho} \in \mathrm{Rem}(\lambda)$. We note that

$$\underline{\mathsf{Hook}}_{\boldsymbol{\rho}}(\lambda - \boldsymbol{\rho}) \sqcup \{[r, r-1]\} = \underline{\mathsf{Hook}}_{\boldsymbol{\tau}}(\imath(\lambda) - \boldsymbol{\tau}) \tag{4.26}$$

where $s_{[r,r-1]} = \boldsymbol{\sigma}$. Note that $\mathsf{e} \circ \mathsf{gap}(\mathsf{t}_\mu - [r, r-1]) \circ \mathsf{e} = 0$ using Corollary 4.28 so the result holds in subcase 1.3.

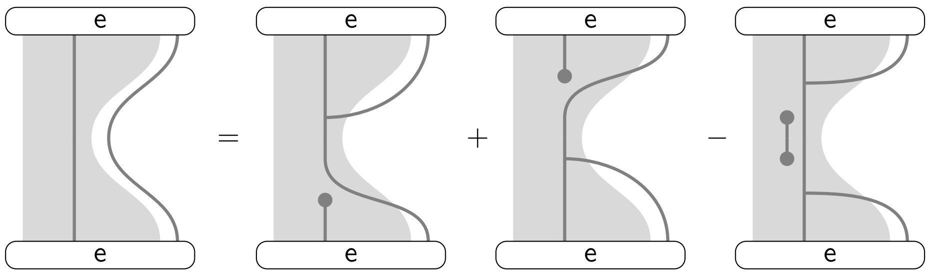


FIGURE 37. Subcase 1.3.

*Subcase 1.4.* If $\alpha \notin \mathrm{Rem}(\lambda)$ and $\imath(\alpha) \in \mathrm{Rem}(\imath(\lambda))$ then we can assume that $\alpha = \boldsymbol{\pi}$ (the $\alpha = \boldsymbol{\rho}$ is identical). Then we must have that $s_{[r,r]} = \boldsymbol{\rho} \in \mathrm{Rem}(\lambda)$ and $s_{[r+1,r]} = \boldsymbol{\sigma}$ and $s_{[r+1,r+1]} = \boldsymbol{\pi}$. We have that

$$c_{\mathsf{X}} = c_{\mathsf{X}'} \circledast \mathsf{spot}^{\emptyset}_{\boldsymbol{\sigma}} \quad c_{\mathsf{Y}} = c_{\mathsf{Y}'} \circledast \mathsf{spot}^{\emptyset}_{\boldsymbol{\sigma}}$$

for $\mathsf{X}', \mathsf{Y}' \in \mathrm{Path}_{(D_{n+1},A_n)}(\lambda + \boldsymbol{\sigma}, -)$. Therefore

$$c_{\mathsf{XY}} \otimes 1_{\boldsymbol{\pi}} = (c^*_{\mathsf{X}'} \otimes 1_{\boldsymbol{\pi}})\mathsf{gap}(\mathsf{t}_{\lambda+\boldsymbol{\sigma}+\boldsymbol{\pi}} - [r+1, r])(c_{\mathsf{Y}'} \otimes 1_{\boldsymbol{\pi}}) = 0$$

using Lemma 4.7. On the other hand

$$\mathsf{e} \circ (\imath(c_{\mathsf{XY}}) \otimes 1_{\boldsymbol{\tau}}) \circ \mathsf{e} = \mathsf{e} \circ (\imath(c^*_{\mathsf{X}'}) \otimes 1_{\boldsymbol{\tau}}) \circ \mathsf{e} \circ \mathsf{gap}(\mathsf{t}_{\imath(\lambda)+\boldsymbol{\sigma}+\boldsymbol{\tau}} - [r+1, r]) \circ \mathsf{e} \circ (\imath(c_{\mathsf{Y}'}) \otimes 1_{\boldsymbol{\tau}}) \circ \mathsf{e} = 0$$

where the first equality follows from Lemma 4.29 (inserting extra idempotents, $\mathsf{e}$) and the second follows by Corollary 4.28.

*Subcase 1.5.* If $\imath(\alpha) \notin \mathrm{Add}(\imath(\lambda))$ nor $\mathrm{Rem}(\imath(\lambda))$, then $\alpha \notin \mathrm{Add}(\lambda)$ or $\mathrm{Rem}(\lambda)$. Take $[x, y]$ with $x + y$ minimal such that $s_{[x,y]} = \alpha$ and $[x, y] \notin \lambda$. Then precisely one of $[x, y-1]$ or $[x-1, y] \in \lambda$. We assume $[x, y-1] \in \lambda$ and we set $\gamma = s_{[x,y-1]}$. We have

$$c_{\mathsf{XY}} \otimes 1_\alpha = -(c_{\mathsf{X}} \circledast \mathsf{spot}^{\emptyset}_{\gamma} \circledast \mathsf{fork}^{\alpha}_{\alpha\alpha})^*(c_{\mathsf{Y}} \circledast \mathsf{spot}^{\emptyset}_{\gamma} \circledast \mathsf{fork}^{\alpha}_{\alpha\alpha}).$$

by the $\gamma\alpha$-null-braid relation. This might not be a cellular basis diagram, but can be rewritten as such using equation (4.8) and (4.9). Similarly $\mathsf{e}(\imath(c_{\mathsf{XY}}) \otimes 1_{\imath(\alpha)})\mathsf{e}$ can be rewritten in the same form using Corollary 4.28. Subcase 1.5 follows.

**Case 2.** We now consider the case that $c_{\mathsf{T}} = A^+_\alpha(c_{\mathsf{T}'})$ and $c_{\mathsf{U}} = A^-_\alpha(c_{\mathsf{U}'})$ (the dual case with $\mathsf{T}$ and $\mathsf{U}$ swapped is similar). If $\alpha \neq \boldsymbol{\pi}, \boldsymbol{\rho}$, then

$$\begin{aligned}\mathsf{e} \circ \imath(c_{\mathsf{T}'}) \otimes 1_{\imath(\alpha)} \circ \mathsf{e} \circ \imath(c^*_{\mathsf{U}}) \otimes \mathsf{spot}^{\imath(\alpha)}_{\emptyset} \circ \mathsf{e} &= \mathsf{e}((\imath(c_{\mathsf{T}'}) \circ \mathsf{e} \circ \imath(c^*_{\mathsf{U}})) \otimes \mathsf{spot}^{\alpha}_{\emptyset})\mathsf{e} \\ &= \mathsf{e}\big(\imath(1_{\mathsf{t}_{\nu-\alpha}} \otimes \mathsf{spot}^{\alpha}_{\emptyset})\big)\mathsf{e} \circ \mathsf{e}\big(\imath(c_{\mathsf{T}'}) \circ \mathsf{e} \circ \imath(c^*_{\mathsf{U}})\big)\mathsf{e}\end{aligned}$$

and so the result follows by induction on $\ell(\mu)$.

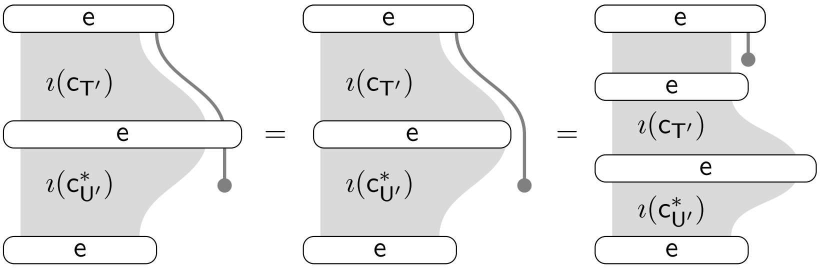



FIGURE 38. Case 2 for $\sigma \neq \boldsymbol{\pi}$

If $\alpha = \boldsymbol{\pi}$, then $c_{\mathsf{T}} = A^+_{\boldsymbol{\pi}} A^+_{\boldsymbol{\sigma}}(c_{\mathsf{T}''})$ and $c_{\mathsf{U}} = A^-_{\boldsymbol{\pi}} A^+_{\boldsymbol{\sigma}}(c_{\mathsf{U}''})$ and we also set $c_{\mathsf{T}'} = A^+_{\boldsymbol{\sigma}}(c_{\mathsf{T}''})$ and $c_{\mathsf{U}'} = A^+_{\boldsymbol{\sigma}}(c_{\mathsf{U}''})$. Expanding out the final term of the middle idempotent $\mathsf{e}$ and applying equation (4.15), we obtain

$$\begin{aligned}&\mathsf{e} \circ (\imath(c_{\mathsf{T}''}) \otimes 1_{\boldsymbol{\sigma}\boldsymbol{\tau}}) \circ \mathsf{e} \circ (\imath(c^*_{\mathsf{U}''}) \otimes 1_{\boldsymbol{\sigma}} \otimes \mathsf{spot}^{\boldsymbol{\tau}}_{\emptyset}) \circ \mathsf{e} \\ =&\mathsf{e} \circ \big((\imath(c_{\mathsf{T}'}) \circ \mathsf{e} \circ \imath(c_{\mathsf{U}'})^*) \otimes \mathsf{spot}^{\boldsymbol{\tau}}_{\emptyset}\big) \circ \mathsf{e} + \mathsf{e} \circ (\imath(c_{\mathsf{T}''}) \circledast \mathsf{trid}^{\boldsymbol{\tau}\boldsymbol{\sigma}\boldsymbol{\tau}}_{\boldsymbol{\tau}}) \circ \mathsf{e} \circ (\imath(c_{\mathsf{U}''})^* \otimes \mathsf{spot}^{\emptyset}_{\boldsymbol{\sigma}}) \circ \mathsf{e} \\ =&\mathsf{e} \circ \big((\imath(c_{\mathsf{T}'}) \circ \mathsf{e} \circ \imath(c_{\mathsf{U}'})^*) \otimes \mathsf{spot}^{\boldsymbol{\tau}}_{\emptyset}\big) \circ \mathsf{e}\end{aligned}$$

and so the result again follows by induction on length, as above.

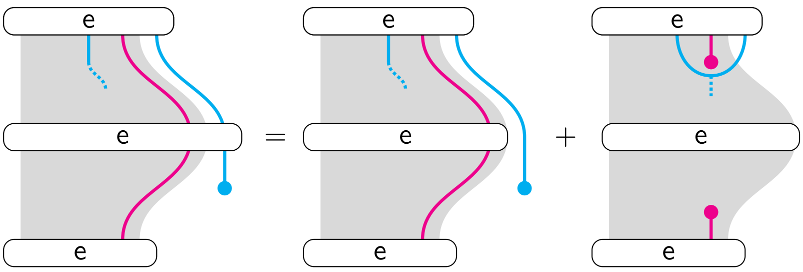



FIGURE 39. Case 2 for $\boldsymbol{\pi}$.

**Case 3.** We now consider the case that $c_{\mathsf{T}} = R^+_\alpha(c_{\mathsf{T}'})$ and $c_{\mathsf{U}} = A^+_\alpha(c_{\mathsf{U}'})$ (the dual case with $\mathsf{T}$ and $\mathsf{U}$ swapped is identical). By the same inductive argument as we used in Case 1 (in order to deduce equation (4.22)), we have that

$$\mathsf{e} \circ \imath(c_{\mathsf{T}}) \circ \mathsf{e} \circ \imath(c^*_{\mathsf{U}}) \circ \mathsf{e} = \sum_{\mathsf{X},\mathsf{Y}} a_{\mathsf{X},\mathsf{Y}} \mathsf{e}(\imath(c^*_{\mathsf{X}}) \circledast \mathsf{fork}^{\imath(\alpha)}_{\imath(\alpha)\imath(\alpha)})(\imath(c_{\mathsf{Y}}) \otimes 1_{\imath(\alpha)})\mathsf{e}. \tag{4.27}$$

We set $\lambda = \mathrm{Shape}(\mathsf{X}) = \mathrm{Shape}(\mathsf{Y})$. Observe that $\alpha \in \mathrm{Add}(\lambda)$ or $\mathrm{Rem}(\lambda)$. If $\alpha \in \mathrm{Add}(\lambda)$, then

$$c^*_{\mathsf{X}} \circledast \mathsf{fork}^{\alpha}_{\alpha\alpha} = (X^-_\alpha(c_{\mathsf{X}'}))^* \circledast \mathsf{fork}^{\alpha}_{\alpha\alpha} = (X^+_\alpha(c_{\mathsf{X}'}))^*$$

for $X \in \{R, A\}$ and some $\mathsf{X}'$. We have that $c_{\mathsf{Y}} \otimes 1_\alpha = A^+_\alpha(c_{\mathsf{Y}})$. In particular, both diagrams are light leaves basis elements. If $\alpha \in \mathrm{Rem}(\lambda)$ then we move the fork through the centre of the product $c_{\mathsf{XY}}$ and notice that

$$c_{\mathsf{Y}} \circledast \mathsf{fork}^{\alpha}_{\alpha\alpha} = R^+_\alpha(c_{\mathsf{Y}})$$

and $c_{\mathsf{X}}$ are both light leaves basis elements. Exactly the same is true replacing $c_{\mathsf{X}}$, $c_{\mathsf{Y}}$, and $\alpha$ with their images under $\imath$. The result follows.

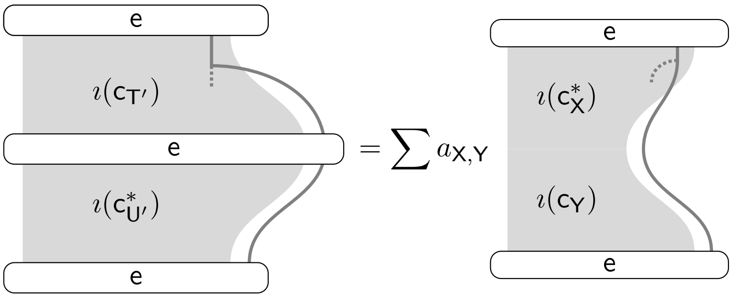



FIGURE 40. Case 3: a diagrammatic version of equation (4.27).

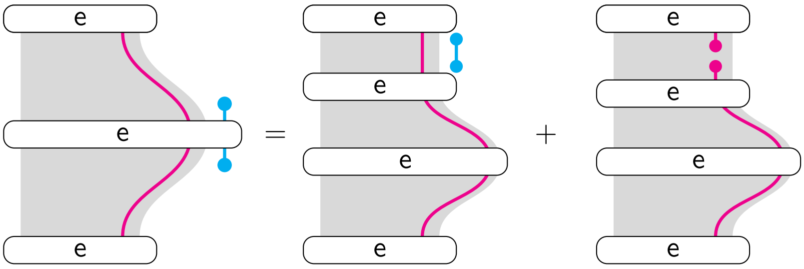



FIGURE 41. Case 5 for $\alpha = \sigma$.

**Case 4.** Now suppose $c_T = R^-_\alpha(c_{T'})$ and $c_U = A^+_\alpha(c_{U'})$. In this case, we are simply placing a dot on the diagrams from case 3 and so the result follows from case 3 and induction on $\ell(\mu)$.

**Case 5.** Let $c_T = A^-_\alpha(c_{T'})$ and $c_U = A^-_\alpha(c_{U'})$. If $\imath(\alpha) \neq \tau$ then $\ell_\tau(\imath(\mu)) = \ell_\tau(\imath(\mu - \alpha))$ and so

$$\mathsf{e} \circ \imath(c_T) \circ \mathsf{e} \circ \imath(c_U)^* \circ \mathsf{e} = (\mathsf{e} \circ \imath(c_{T'}) \circ \mathsf{e} \circ \imath(c_{U'})^* \circ \mathsf{e}) \otimes \mathsf{bar}(\imath(\alpha))$$
$$= \mathsf{e} \circ (1_{\mathsf{t}_{\imath(\nu)}} \otimes \mathsf{bar}(\imath(\alpha)) \circ \mathsf{e} \circ \imath(c_{T'}) \circ \mathsf{e} \circ \imath(c_{U'})^* \circ \mathsf{e}$$

and if $\imath(\alpha) = \tau$ (say $\alpha = \rho$ as the $\pi$ case is identical) we have

$$\begin{aligned}\mathsf{e} \circ \imath(c_T) \circ \mathsf{e} \circ \imath(c_U)^* \circ \mathsf{e} = {}& \mathsf{e} \circ (1_{\mathsf{t}_{\imath(\nu)}} \otimes \mathsf{bar}(\imath(\alpha)) \circ \mathsf{e} \circ \imath(c_{T'}) \circ \mathsf{e} \circ \imath(c_{U'})^* \mathsf{e} \\ & + \mathsf{e} \circ (1_{\mathsf{t}_{\imath(\nu)-\sigma}} \otimes \mathsf{gap}(\sigma)) \circ \mathsf{e} \circ \imath(c_{T'}) \circ \mathsf{e} \circ \imath(c_{U'})^* \mathsf{e}\end{aligned} \tag{4.28}$$

this is picture in Figure 41. As observed in subcase 1.3, the rules for resolving $1_{\mathsf{t}_\nu} \otimes \mathsf{bar}(\alpha)$ in type $(D_{n+1}, A_n)$ and $1_{\mathsf{t}_{\imath(\nu)}} \otimes \mathsf{bar}(\imath(\alpha))$ in type $(C_n, A_{n-1})$ are identical *except* when $\imath(\alpha) = \tau$ in which case we get an extra term; this term cancels with the second summand on the right of equation (4.28). Using equation (4.8) and (4.9) versus Corollary 4.28 we see that

$$\mathsf{e} \circ \imath(c_T) \circ \mathsf{e} \circ \imath(c_U)^* \circ \mathsf{e} = \mathsf{e} \circ \imath(c_{TU}) \circ \mathsf{e}$$

by induction on $\ell(\mu)$.

**Case 6.** Let $c_T = R^+_\alpha(c_{T'})$ and $c_U = A^-_\alpha(c_{U'})$ (the dual case is similar). If $\imath(\alpha) \neq \tau$ then $\ell_\tau(\imath(\mu)) = \ell_\tau(\imath(\mu - \alpha))$ and using the fork-spot relation we have

$$\mathsf{e} \circ \imath(c_T) \circ \mathsf{e} \circ \imath(c_U)^* \circ \mathsf{e} = \mathsf{e} \circ \imath(c_{T'}) \circ \mathsf{e} \circ \imath(c_{U'})^* \mathsf{e}$$

and so the result follows by induction. If $\imath(\alpha) = \tau$ then $\ell_\tau(\imath(\mu)) = \ell_\tau(\imath(\mu - \alpha)) + 1$ and we set $\alpha = \rho$ (the $\alpha = \pi$ case is identical) and we must have $c_T = R^+_\rho R^-_\sigma(c_{T''})$. We have that

$$\imath(c_T)\mathsf{e} = \imath(c_T)e_{\imath(\mu)} = \imath(c_T)(1_{\mathsf{t}_{\imath(\mu)}} + e^{\ell_\tau(\imath(\mu))}_{\imath(\mu)})(e_{\imath(\mu)-\tau} \otimes 1_\tau) = \imath(c_T)(e_{\imath(\mu)-\tau} \otimes 1_\tau)$$

as illustrated in Figure 42. So we have

$$\mathsf{e} \circ \imath(c_T) \circ e_{\imath(\mu)} \circ \imath(c_U)^* \circ \mathsf{e} = \mathsf{e} \circ \imath(c_T) \circ (e_{\imath(\mu)-\tau} \otimes 1_\tau) \circ \imath(c_U)^* \circ \mathsf{e} = \mathsf{e} \circ \imath(c_{T'}) \circ \mathsf{e} \circ \imath(c_{U'})^* \circ \mathsf{e}$$

using the fork-spot relation as above. Again we are done by induction.

**Case 7.** The case $c_T = A^-_\alpha(c_{T'})$ and $c_U = R^-_\alpha(c_{U'})$ follows from case 6 (in the manner that case 4 followed from case 3).

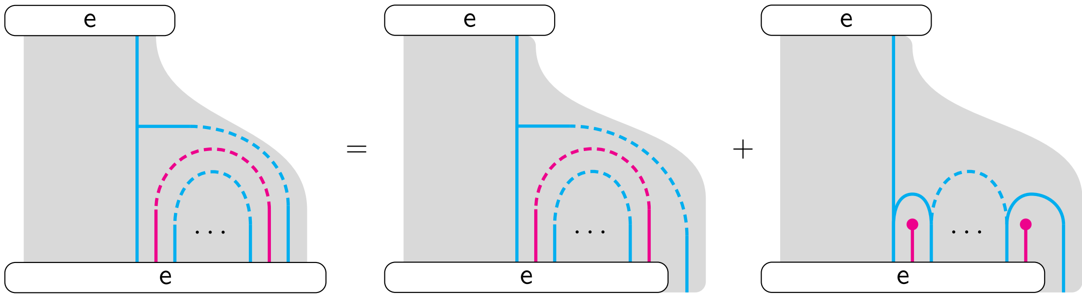



FIGURE 42. Case 6 for $\alpha = \boldsymbol{\sigma}$.

**Case 8.** We now consider the case that $c_{\mathsf{T}} = R^+_\alpha(c_{\mathsf{T}'})$ and $c_{\mathsf{U}} = R^+_\alpha(c_{\mathsf{U}'})$. By the same inductive argument as we used in Case 1 (in order to deduce equation (4.22)), we have that

$$\mathsf{e}\circ\imath(c_{\mathsf{T}})\circ\mathsf{e}\circ\imath(c^*_{\mathsf{U}})\circ\mathsf{e} = \sum_{\mathsf{X},\mathsf{Y}} a_{\mathsf{X},\mathsf{Y}}\mathsf{e}(\imath(c^*_{\mathsf{X}}) \circledast \mathsf{fork}^{\imath(\alpha)}_{\imath(\alpha)\imath(\alpha)})(\imath(c_{\mathsf{Y}}) \circledast \mathsf{fork}^{\imath(\alpha)\imath(\alpha)}_{\imath(\alpha)})\mathsf{e}.$$

We set $\lambda = \mathrm{Shape}(\mathsf{X}) = \mathrm{Shape}(\mathsf{Y})$. Note that either $\alpha \in \mathrm{Add}(\lambda)$ or in $\mathrm{Rem}(\lambda)$. If $\alpha \in \mathrm{Add}(\lambda)$, then arguing as in case 3 we get that

$$c^*_{\mathsf{X}} \otimes \mathsf{fork}^{\alpha}_{\alpha\alpha} \quad \imath(c_{\mathsf{X}})^* \otimes \mathsf{fork}^{\imath(\alpha)}_{\imath(\alpha)\imath(\alpha)} \quad c_{\mathsf{Y}} \otimes \mathsf{fork}^{\alpha}_{\alpha\alpha} \quad \imath(c_{\mathsf{Y}}) \otimes \mathsf{fork}^{\imath(\alpha)}_{\imath(\alpha)\imath(\alpha)}$$

are cellular basis elements and so we are done. If $\alpha \in \mathrm{Rem}(\lambda)$, then we have

$$(c^*_{\mathsf{X}} \circledast \mathsf{fork}^{\alpha}_{\alpha\alpha})(c_{\mathsf{Y}} \circledast \mathsf{fork}^{\alpha\alpha}_{\alpha}) = c_{\mathsf{XY}} \circledast \mathsf{fork}^{\alpha}_{\alpha\alpha}\mathsf{fork}^{\alpha\alpha}_{\alpha} = 0$$

$$(\imath(c_{\mathsf{X}})^* \circledast \mathsf{fork}^{\imath(\alpha)}_{\imath(\alpha)\imath(\alpha)})(\imath(c_{\mathsf{Y}}) \circledast \mathsf{fork}^{\imath(\alpha)\imath(\alpha)}_{\imath(\alpha)}) = \imath(c_{\mathsf{XY}}) \circledast \mathsf{fork}^{\imath(\alpha)}_{\imath(\alpha)\imath(\alpha)}\mathsf{fork}^{\imath(\alpha)\imath(\alpha)}_{\imath(\alpha)} = 0$$

and so we are done.

**Cases 9 and 10.** The case in which $c_{\mathsf{T}} = R^+_\alpha(c_{\mathsf{T}'})$ and $c_{\mathsf{U}} = R^-_\alpha(c_{\mathsf{U}'})$ and the case in which $c_{\mathsf{T}} = R^-_\alpha(c_{\mathsf{T}'})$ and $c_{\mathsf{U}} = R^-_\alpha(c_{\mathsf{U}'})$ both follow from case 8 (in the manner that case 4 followed from case 3). □

4.7. **Proof of Theorem 4.2.** Using Theorem 4.24, it is enough to consider the simply laced cases. Now the result follows from Proposition 4.17.

## 5. Coxeter truncation

In this section we prove one of the main results of this paper: that $\boldsymbol{\tau}$-singular Hecke categories for Hermitian symmetric pairs (defined *à la* [Eli16]) are graded Morita equivalent to (regular) Hecke categories for smaller rank Hermitian symmetric pairs. For the underlying Kazhdan–Lusztig polynomials, this was first observed by Enright–Shelton [ES87]. Our result lifts theirs to the 2-categorical level and to positive characteristic. By Theorem 4.24 we can focus on the simply laced case without loss of generality. Let $(W, P)$ a simply laced Hermitian symmetric pair of rank $n$ and fix $\boldsymbol{\tau} \in S_W$. We define

$$\mathscr{P}^{\boldsymbol{\tau}}_{(W,P)} = \{\mu \in \mathscr{P}_{(W,P)} \mid \boldsymbol{\tau} \in \mathrm{Rem}(\mu)\}.$$

We will show that the subalgebra of $h_{(W,P)}$ spanned by

$$\{c_{\mathsf{ST}} \mid \mathsf{S} \in \mathrm{Path}(\lambda, \mathsf{t}_\mu), \mathsf{T} \in \mathrm{Path}(\lambda, \mathsf{t}_\nu), \lambda, \mu, \nu \in \mathscr{P}^{\boldsymbol{\tau}}_{(W,P)}\}$$

is isomorphic to $h_{(W,P)^{\boldsymbol{\tau}}}$ for some Hermitian symmetric pair $(W, P)^{\boldsymbol{\tau}}$ of strictly smaller rank.

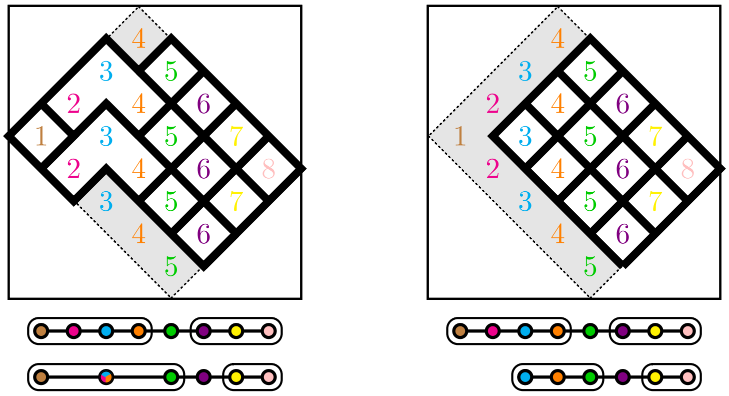


Figure 43. The contraction tilings for $(A_8, A_4 \times A_3)^\tau$ with $\tau = s_3$ and $\tau = s_1$ respectively. The tiling is discussed in Subsection 5.1 and the Dynkin diagrams are discussed in Subsection 5.2.

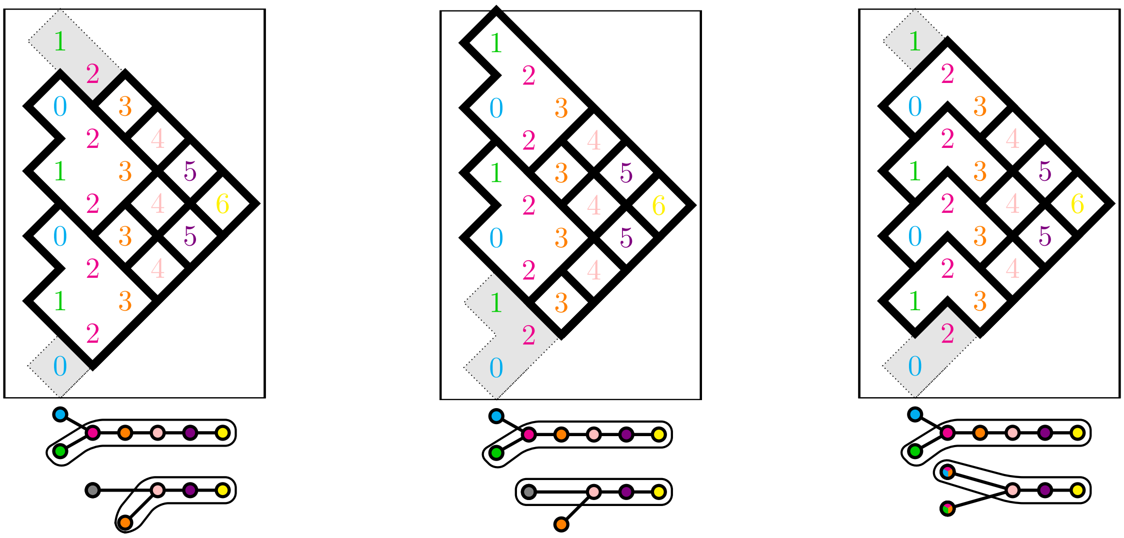


Figure 44. The contraction tilings $(D_6, A_5)^\tau$ for $\tau = s_0, s_1, s_2 \in W$ respectively. In the first two cases, the grey node is labelled by 21320 and 20321.

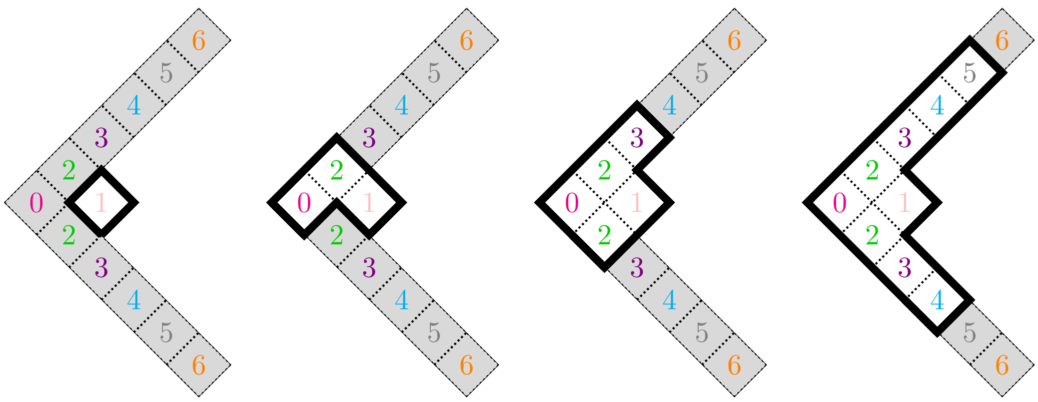


Figure 45. The contraction tilings of $(D_7, D_6)^\tau$ for $\tau = s_0, s_2, s_3, s_5$ in order.

5.1. **The $\tau$-contraction tilings.** In what follows, we let $(W, P)$ be a simply laced Hermitian symmetric pair and $\tau \in S_W$. We now introduce a contraction map which will allow us to work by induction on the rank.

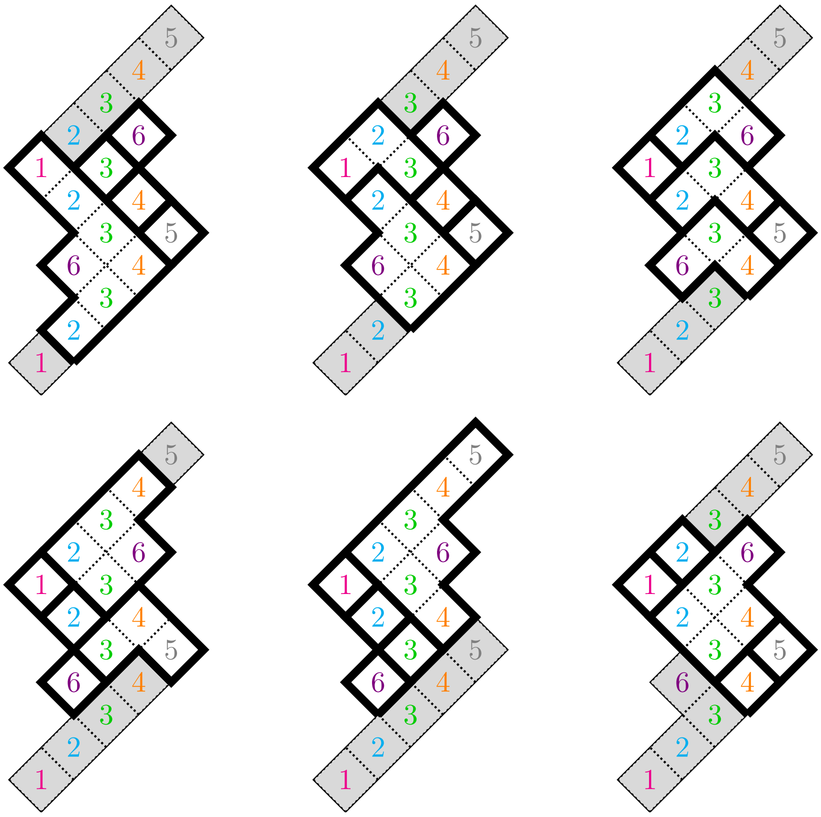


FIGURE 46. The contraction tilings of $(E_6, D_5)$ in order.

**Definition 5.1.** *Given $(W, P)$ a Hermitian symmetric pair and $\tau \in S_W$, we let $[r_1, c_1] < [r_2, c_2] < \cdots < [r_k, c_k]$ denote the completely ordered set (according to the natural ordering on $r_i + c_i \in \mathbb{N}$) of all $\tau$-tiles in $\mathscr{A}_{(W,P)}$. Given two $\tau$-tiles, $[r_i, c_i]$ and $[r_{i+1}, c_{i+1}]$ that are adjacent in this ordering, we define*

$$T^{\tau}_{i\to i+1} = \{[x, y] \mid r_i \leqslant x \leqslant r_{i+1} \text{ and } c_i \leqslant y \leqslant c_{i+1} \text{ and } [x, y] \neq [r_i, c_i]\} \cap \mathscr{A}_{(W,P)}.$$

*Associated to the minimal tile $[r_1, c_1]$ we define a corresponding null-region*

$$T^{\tau}_{0\to 1} = \{[x, y] \mid x \leqslant r_1 \text{ and } y \leqslant c_1\} \cap \mathscr{A}_{(W,P)}$$

*and for the maximal tile $[r_k, c_k]$, we define the maximal null-region*

$$T^{\tau}_{\infty} = \{[x, y] \mid r_k \leqslant x \text{ and } c_k \leqslant y \text{ and } [x, y] \neq [r_k, c_k]\} \cap \mathscr{A}_{(W,P)}$$

*and we set $\mathcal{N}^{\tau} = T^{\tau}_{0\to 1} \cup T^{\tau}_{\infty}$. We define the $\tau$-contraction tiling to be the disjoint union of the $T^{\tau}_{i\to i+1}$-tiles and all remaining tiles in $\mathscr{A}_{(W,P)} \setminus \mathcal{N}^{\tau}$. We refer to any tile in this overall tiling as a $\tau$-contraction tile. Given $\mathbb{T}$ a $\tau$-contraction tile. We define a reading word of $\mathbb{T}$ by recording the constituent tiles $[r, c]$ within $\mathbb{T}$ from bottom to top (that is, by the natural order on $r + c \in \mathbb{Z}_{\geqslant 0}$).*

**Remark 5.2.** *We note that any tile-partition $\lambda \in \mathscr{P}^{\tau}_{(W,P)}$ can be obtained by stacking $\tau$-contraction tiles on top of $T^{\tau}_{0\to 1}$.*

**Example 5.3.** *In the leftmost diagram in Figure 44 the large contraction tiles are all identical. These identical tiles both have two distinct choices for their reading word (as we can order the tiles of the same height freely); explicitly, these reading words are 21320 and 23120. Notice that these words differ only by the commuting relations in the Coxeter groups.*

There is, in essence, only one type of large contraction tile: this is the contraction tiles of type $(A, A \times A)$ and their *augmentations* pictured in Figure 49. We will see in Subsection 5.3 that these augmentations merely "bulk out" the corresponding Soergel diagram (using degree zero strands) without changing its substance. In more detail, we define the tricorne to be formed from three tiles in a formation $\mathbb{T} = \{[r, c], [r-1, c], [r, c-1]\}$. Here the tile $[r, c]$ is the only $\tau$-tile in $\mathbb{T}$. We augment

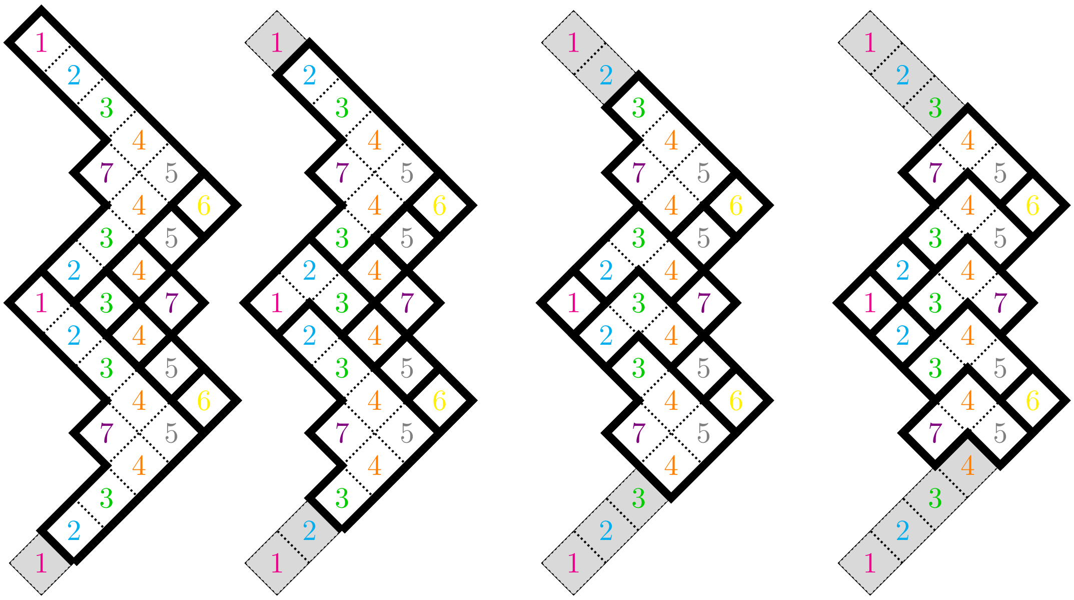


FIGURE 47. The first 4 of 7 contraction tilings for $(E_7, E_6)$, in order.

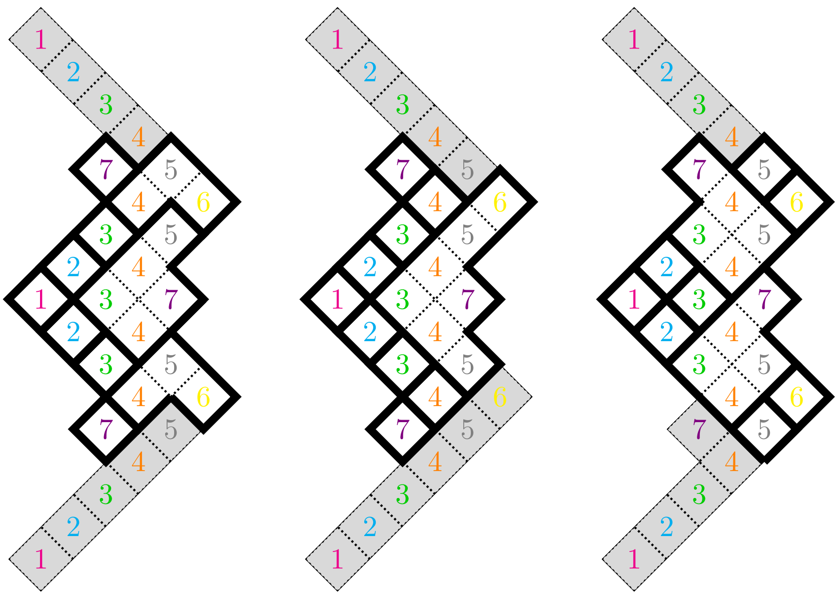


FIGURE 48. The final 3 contraction tilings for $(E_7, E_6)$, in order.

this picture by adding $k$ tiles symmetrically above and below the brim (thus displacing the $\tau$-tile) to obtain an augmented tile $\mathrm{aug}_k(\mathbb{T})$ as depicted in the rightmost diagram of Figure 49.

5.2. **The Dynkin types of $\tau$-contraction tilings.** We now identify the $\tau$-tilings of $\mathscr{A}_{(W,P)}$ with the tilings of the admissible region of a Hermitian symmetric pair of smaller rank. The nodes of $(W,P)^\tau$ will be labelled by the reading words of the tiles in the $\tau$-contraction tiling.

**Proposition 5.4.** *Let $(W,P)$ be a simply laced Hermitian symmetric pair and let $\tau \in S_W$. There is an order preserving bijection $\varphi_\tau : \mathscr{P}_{(W,P)^\tau} \to \mathscr{P}^\tau_{(W,P)}$ where $(W,P)^\tau = (W^\tau, P^\tau)$ is defined by*

- $(A_n, A_k \times A_{n-k})^\tau = (A_{n-2}, A_{k-1} \times A_{n-k-1})$;
- $(D_n, A_{n-1})^\tau = (D_{n-2}, A_{n-3})$;
- $(D_n, D_{n-1})^\tau = (A_1, A_0)$;
- $(E_6, D_5)^\tau = (A_5, A_4)$;
- $(E_7, E_6)^\tau = (D_6, D_5)$.

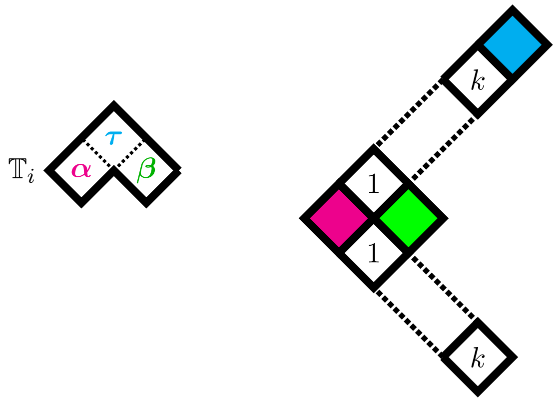



FIGURE 49. The first diagram depicts the construction of the type $A$ "tricorne-like" $\tau$-tile. The second diagram depicts the augmented tiles $\mathrm{aug}_k(\mathbb{T})$ for $k \geqslant 0$; the original tiles are coloured. (The right diagram is sometimes flipped through the vertical axis.)

*Moreover, fixing a reading word for each $\tau$-contraction tile, this defines a reduced path* $\varphi_\tau(\mathsf{t}) \in \mathrm{Std}(\varphi_\tau(\lambda))$ *for each* $\mathsf{t} \in \mathrm{Std}(\lambda)$ *and* $\lambda \in \mathscr{P}_{(W,P)^\tau}$.

*Proof.* This follows by inspection, comparing Figures 43 to 48 with Figures 3 to 6. □

We label the nodes of the smaller rank Coxeter system with the reading words of the $\tau$-contraction tiles of the larger Coxeter system. The positions of these labels can easily be deduced from the $\tau$-tilings, see for example Figures 50 to 52.

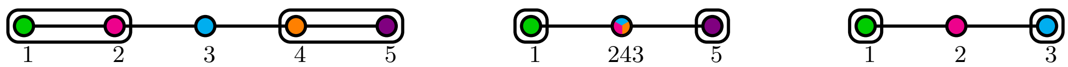



FIGURE 50. An example of a graph and its contraction at the vertices 3 and 5 respectively.

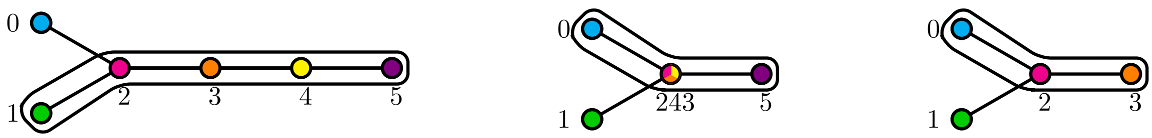



FIGURE 51. An example of a graph and its contractions at vertices 3 and 5, respectively.

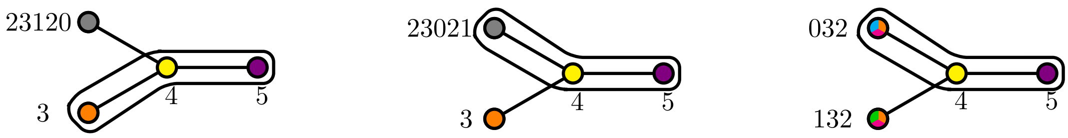



FIGURE 52. The graphs obtained from the leftmost graph in Figure 51 by contraction at the type $D$ vertices 0, 1 and 2, respectively.

In fact, it can be shown that the map $\varphi_\tau$ can be extended from reduced paths (where the observation is trivial) to *non-reduced* paths. The proof of this involves much more substantial combinatorics and so is postponed to the companion paper, [BDF+25].

**Proposition 5.5** ([BDF+25, Proposition 8.1])**.** *We have a graded bijection*

$$\varphi_\tau : \mathrm{Path}_{(W,P)^\tau}(\lambda, \mathsf{t}_\mu) \longrightarrow \mathrm{Path}_{(W,P)}(\varphi_\tau(\lambda), \mathsf{t}_{\varphi_\tau(\mu)}).$$

**Example 5.6.** *In Figure 50 we depict the pair* $(A_5, A_2 \times A_2)$ *and the truncation at the node* $\tau = s_3 \in S_W$*. We have that* $\varphi_\tau(243) = 2 \otimes 4 \otimes 3$*. On the right of Figure 1 we depict the Bruhat graph for* $(A_5 \setminus A_2 \times A_2)^\tau$*. The bottommost (and topmost) edge of this graph is tricoloured by* $2 \otimes 4 \otimes 3$ *and maps to a concatenate of three distinct edges in the leftmost graph.*

**5.3. The dilation homomorphism.** We now lift the map $\varphi_\tau$ of Proposition 5.5 to the level of a graded $\Bbbk$-algebra isomorphism. We let $i$ denote a square root of $-1$. We first define the dilation maps on the monoidal generators. Let $\sigma = s_1 s_2 \dots s_\ell$ be a (composite) label of a node of the Coxeter graph $(W,P)^\tau$. (Note that $s_1, s_2, \dots, s_\ell \in S_W$ belong to the dilated Coxeter group, whereas $\sigma \in S_{W^\tau}$.) We define the dilation map on the idempotent generators as follows

$$\mathsf{dil}_\tau(1_\sigma) = 1_\tau \otimes 1_{s_1} \otimes 1_{s_2} \otimes \dots \otimes 1_{s_\ell}.$$

For a non-zero braid generator (see Definition 4.9 for the list of zero braid generators) we define the dilation map as follows

$$\mathsf{dil}_\tau(\mathsf{braid}^{\alpha\sigma}_{\sigma\alpha}) = 1_\tau \otimes \mathsf{braid}^{\alpha s_1 s_2 \dots s_\ell}_{s_1 s_2 \dots s_\ell \alpha}.$$

Examples are depicted in Figure 53.


FIGURE 53. Examples of the $\mathsf{dil}_\tau$ map on idempotent and braid generators. The colouring corresponds to that of Figure 43. The leftmost $\tau$-strand is drawn as a dotted strand in order to remind the reader that we horizontally concatenate these diagrams using $\circledast$ (which identifies this blue strand with an earlier blue strand in the diagram).

We now define the dilation map on the fork and spot generators. For $\sigma = s_i$ a tile labelled by a singleton $s_i \in S_{W^\tau}$ (which necessarily commutes with $\tau \in S_W$) we define

$$\mathsf{dil}_\tau(\mathsf{fork}^\sigma_{\sigma\sigma}) = 1_\tau \otimes \mathsf{fork}^{s_i}_{s_i s_i} \qquad \mathsf{dil}_\tau(\mathsf{spot}^\emptyset_\sigma) = 1_\tau \otimes \mathsf{spot}^\emptyset_{s_i}.$$

We set $\sigma = \alpha\gamma\tau$, the reading word of a tricorne tile. We define

$$\mathsf{dil}_\tau(\mathsf{fork}^\sigma_{\sigma\sigma}) = i \times 1_\tau \otimes (\mathsf{fork}^\alpha_{\alpha\alpha} \otimes \mathsf{fork}^\gamma_{\gamma\gamma} \otimes 1_\tau)(1_\alpha \otimes \mathsf{braid}^{\alpha\gamma}_{\gamma\alpha} \otimes 1_{\gamma\tau})(1_{\alpha\gamma} \otimes \mathsf{spot}^\emptyset_\tau \otimes 1_{\alpha\gamma\tau})$$

$$\mathsf{dil}_\tau(\mathsf{spot}^\emptyset_\sigma) = -i \times \mathsf{fork}^\tau_{\tau\tau}(1_\tau \otimes \mathsf{spot}^\emptyset_\alpha \otimes \mathsf{spot}^\emptyset_\gamma \otimes 1_\tau).$$

Examples are depicted in Figure 54.

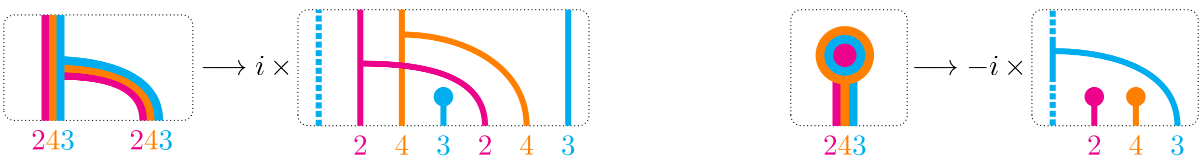



FIGURE 54. Let $(W,P) = (A_5, A_2 \times A_2)$ and $\tau = s_3 \in W$ (see also Figures 1 and 50). We depict $\varphi_\tau(\mathsf{fork}^\sigma_{\sigma\sigma})$ and $\varphi_\tau(\mathsf{spot}^\emptyset_\sigma)$ for $\sigma = s_2 s_4 s_3$. The leftmost $\tau$-strand is drawn as a dotted strand in order to remind the reader that we horizontally concatenate these diagrams using $\circledast$ (which identifies this blue strand with an earlier blue strand in the diagram).

We now describe how one can "augment" the diagrams of tricornes to obtain arbitrary diagrams. Let $\sigma$ be a label of a vertex in the graph $(W,P)^\tau$ of the form $\sigma = \underline{x}\alpha\gamma\underline{x}^{-1}\tau$. (That is, $\sigma$ is the reading word of an augmented tricorne.) We define $\mathsf{dil}_\tau(\mathsf{fork}^\sigma_{\sigma\sigma})$ to be the element

$$i^{\ell(x)+1}(1_{\tau\underline{x}} \otimes (\mathsf{fork}^{\alpha\gamma}_{\alpha\alpha\gamma\gamma}\mathsf{braid}^{\alpha\alpha\gamma\gamma}_{\alpha\gamma\alpha\gamma}(1_{\alpha\gamma} \otimes \mathsf{cap}^\emptyset_{\underline{x}\underline{x}^{-1}} \otimes 1_{\alpha\gamma})(1_{\alpha\gamma\underline{x}^{-1}} \otimes \mathsf{spot}^\emptyset_\tau \otimes 1_{\underline{x}\alpha\gamma})) \otimes 1_{\underline{x}^{-1}\tau})$$

and we define $\mathsf{dil}_\tau(\mathsf{spot}^\emptyset_\sigma)$ to be the element

$$(-i)^{\ell(x)+1}\mathsf{fork}^\tau_{\tau\tau}(1_\tau \otimes \mathsf{cap}^\emptyset_{\underline{x}\underline{x}^{-1}} \otimes 1_\tau)(1_{\tau\underline{x}} \otimes \mathsf{spot}^\emptyset_\alpha \otimes \mathsf{spot}^\emptyset_\gamma \otimes 1_{\underline{x}^{-1}\tau}).$$

Examples are depicted in Figure 55.

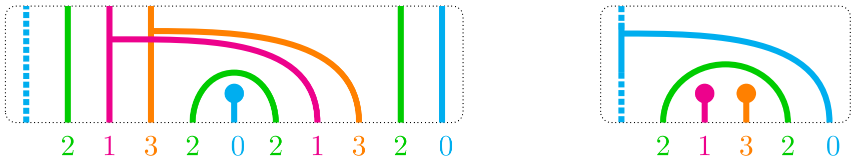



FIGURE 55. Let $(W,P) = (D_6, A_5)$ and $\tau = s_0 \in W$ (see also Figures 51 and 52). We depict $\mathsf{dil}_\tau(\mathsf{fork}^{\sigma}_{\sigma\sigma})$ and $\mathsf{dil}_\tau(\mathsf{spot}^{\emptyset}_{\sigma})$ for $\sigma = s_2 s_1 s_3 s_2 s_0$. These diagrams are obtained from Figure 54 by adding strands of degree zero.

Having defined $\mathsf{dil}_\tau$ on all Soergel generators, we set $\mathsf{dil}_\tau(D^*) = (\mathsf{dil}_\tau(D))^*$. We now extend this definition to arbitrary Soergel diagrams and hence define our contraction homomorphisms.

**Definition 5.7.** *Given diagrams $D_1, D_2 \in \mathcal{H}_{(W,P)}$, we inductively define*

$$\mathsf{dil}_\tau(D_1 \otimes D_2) = \mathsf{dil}_\tau(D_1) \circledast \mathsf{dil}_\tau(D_2) \qquad \mathsf{dil}_\tau(D_1 \circ D_2) = \mathsf{dil}_\tau(D_1) \circ \mathsf{dil}_\tau(D_2)$$

*and we extend this map $\Bbbk$-linearly. We hence define $\varphi_\tau : \mathcal{H}_{(W,P)^\tau} \hookrightarrow \mathcal{H}_{(W,P)}$ as follows,*

$$\varphi_\tau(D) = 1_{T_{0\to1}} \circledast \mathsf{dil}_\tau(D)$$

*where we recall that $T_{0\to1}$ is the null region at the bottom of the $\tau$-contraction tiling of $(W,P)$.*

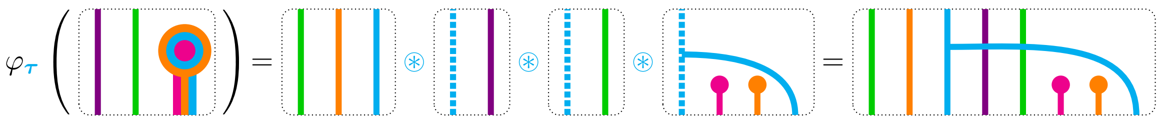


FIGURE 56. The map $\varphi_\tau$ on a diagram, for the leftmost contraction tiling in Figure 43.

**Remark 5.8.** *Each null-region tile $T_{0\to1}$ has a unique reading word and so there is no ambiguity here. That the map $\mathsf{dil}_\tau$ is well-defined on diagrams follows from the interchange law.*

The map $\varphi_\tau$ preserves the light leaves basis (because our map is defined on monoidal generators) and thus lifts the map of Proposition 5.5 to an isomorphism of graded $\Bbbk$-modules between

$$h_{(W,P)^\tau} = \Bbbk\{c_{\mathsf{ST}} \mid \mathsf{S} \in \mathrm{Path}(\lambda, \mathsf{t}_\mu), \mathsf{T} \in \mathrm{Path}(\lambda, \mathsf{t}_\nu), \lambda, \mu, \nu \in \mathscr{P}_{(W,P)^\tau}\}$$

and $h^\tau_{(W,P)} \subset h_{(W,P)}$ which we define to be the subspace with basis

$$\Bbbk\{c_{\mathsf{ST}} \mid \mathsf{S} \in \mathrm{Path}(\varphi_\tau(\lambda), \mathsf{t}_{\varphi_\tau(\mu)}), \mathsf{T} \in \mathrm{Path}(\varphi_\tau(\lambda), \mathsf{t}_{\varphi_\tau(\nu)}), \varphi_\tau(\lambda), \varphi_\tau(\mu), \varphi_\tau(\nu) \in \mathscr{P}^\tau_{(W,P)}\}.$$

In fact we will now lift this to the level of graded $\Bbbk$-algebras.

**Theorem 5.9.** *Let $(W,P)$ be a Hermitian symmetric pair and $\tau \in S_W$. We have a graded $\Bbbk$-algebra isomorphism $\varphi_\tau(h_{(W,P)^\tau}) \cong h^\tau_{(W,P)}$.*

## 6. Proof of the Coxeter dilation homomorphism

This section is dedicated to the proof that the map of $\varphi_\tau$ is a homomorphism. This amounts to checking the relations for these algebras. By Definition 5.7, we have that

$$\varphi_\tau(D)\varphi_\tau(D') = (1_{T_{0\to1}} \circledast \mathsf{dil}_\tau(D))(1_{T_{0\to1}} \circledast \mathsf{dil}_\tau(D')) = 1_{T_{0\to1}} \circledast \mathsf{dil}_\tau(D \circ D') \tag{6.1}$$

using the interchange law and moreover

$$\varphi_\tau(D \circ D') = 1_{T_{0\to1}} \circledast \mathsf{dil}_\tau(D \circ D'). \tag{6.2}$$

Therefore for the local relations, it suffices to show that

$$\mathsf{dil}_\tau(D \circ D') = \mathsf{dil}_\tau(D) \circ \mathsf{dil}_\tau(D'). \tag{6.3}$$

Most of this section is dedicated to the proof that the relations of Corollary 2.5 are preserved under equation (6.3) (but replacing $\otimes$ with $\circledast$). For the non-local relations, we check that equation (6.1) and (6.2) coincide at the end of the section. We now turn to the local relations. Relations R1 and R2, and R14 are all trivial. Relation R13 is satisfied by Proposition 3.11.

In what follows, we let $\sigma$ be a reading word of some $\boldsymbol{\tau}$-contraction tile. In the case that $\sigma$ is an augmented tricorne $\sigma = \underline{x}\boldsymbol{\alpha}\boldsymbol{\gamma}\underline{x}^{-1}\boldsymbol{\tau}$, we set and $\underline{x} = \boldsymbol{\sigma}_k \dots \boldsymbol{\sigma}_1$. In diagrams we put a gradient which reflects the ordering of the purple strands (with $k$ the lightest and $1$ the darkest). We label strands in a diagram simply by $1 \leqslant j \leqslant k$ (rather than by $\boldsymbol{\sigma}_j$) for brevity.

6.1. **The dilated fork-spot relation.** We first consider the leftmost relation in R3, namely the fork-spot relation

$$(\mathsf{dil}_{\boldsymbol{\tau}}(1_\sigma) \circledast \mathsf{dil}_{\boldsymbol{\tau}}(\mathsf{spot}_\sigma^{\emptyset})) \circ \mathsf{dil}_{\boldsymbol{\tau}}(\mathsf{fork}_\sigma^{\sigma\sigma}) = \mathsf{dil}_{\boldsymbol{\tau}}((1_\sigma) \tag{6.4}$$

For $\sigma = s_i \in S_W$, it follows trivially. For $\sigma = \underline{x}\boldsymbol{\alpha}\boldsymbol{\gamma}\underline{x}^{-1}\boldsymbol{\tau}$, the equality follows by one application of each of the $\boldsymbol{\alpha}$- $\boldsymbol{\gamma}$- and $\boldsymbol{\tau}$-fork-spot contractions and the $\boldsymbol{\alpha}\boldsymbol{\gamma}$-commutativity relation (and monoidal unit relation); and by "straightening out" the $\underline{x}$-strands via application of the fork-spot and double-fork relations (this is sometimes referred to simply as "isotopy" in the literature). For a tricorne, this relation is depicted in Figure 57. For an augmented tricorne, one side of this relation is depicted in Figure 58 and it is easy to see that the argument goes through unchanged. The argument for the horizontal and vertical flips of equation (6.4) is similar.

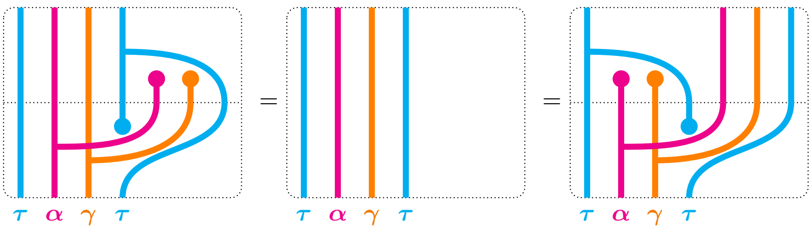



FIGURE 57. The dilation of the fork-spot relation and its flip through the vertical axis, for a tricorne $\sigma = \boldsymbol{\alpha}\boldsymbol{\gamma}\boldsymbol{\tau}$. (We note that the scalar coefficient for both these products is $i \times -i = 1$.)

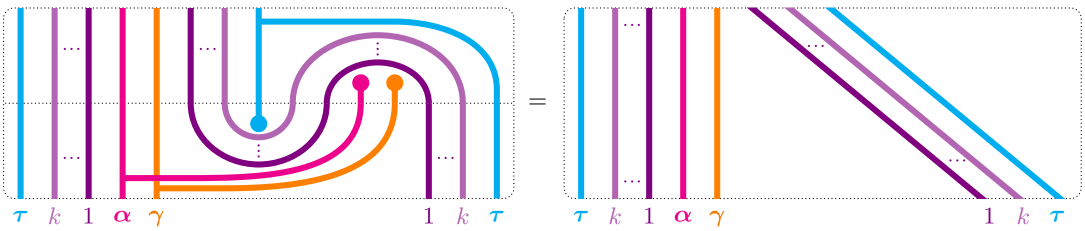



FIGURE 58. The lefthand-side of the dilated fork-spot relation for an augmented tricorne $\sigma = \underline{x}\boldsymbol{\alpha}\boldsymbol{\gamma}\underline{x}^{-1}\boldsymbol{\tau}$. (We note that the scalar coefficient for both these products is $i \times -i = 1$ or $1 \times 1 = 1$ depending on the parity of $k \geqslant 1$.)

6.2. **The dilated double-fork relation.** We now consider the rightmost relation in R3, namely, the double-fork relation

$$(\mathsf{dil}_{\boldsymbol{\tau}}(1_\sigma) \circledast \mathsf{dil}_{\boldsymbol{\tau}}(\mathsf{fork}_{\sigma\sigma}^{\sigma})) \circ (\mathsf{dil}_{\boldsymbol{\tau}}(\mathsf{fork}_\sigma^{\sigma\sigma}) \circledast \mathsf{dil}_{\boldsymbol{\tau}}(1_\sigma)) = \mathsf{dil}_{\boldsymbol{\tau}}(\mathsf{fork}_\sigma^{\sigma\sigma}) \circ \mathsf{dil}_{\boldsymbol{\tau}}(\mathsf{fork}_{\sigma\sigma}^{\sigma}).$$

We apply the double-fork relation to every constituent doubly-forked strand in the diagram in turn, and the result follows. See Figure 59 for the corresponding picture for tricornes, the augmented tricorne picture can be obtained in a similar fashion to Figure 58.

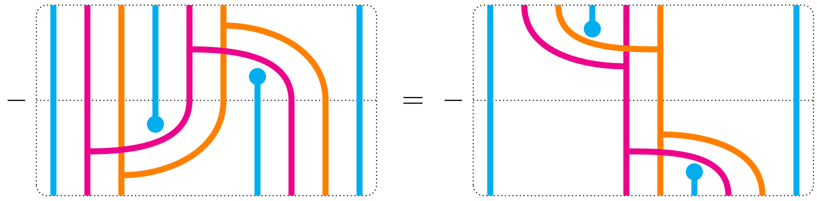


FIGURE 59. The dilated double-fork relation for $\sigma = \alpha\gamma\tau$ the reading word of a tricorne. The equality follows by applying the double-fork relation to the $\alpha$- and $\gamma$-strands.

6.3. **The dilated circle annihilation relation.** We now verify the leftmost relation in R4, namely, the circle-annihilation relation

$$\mathsf{dil}_\tau(\mathsf{fork}^{\sigma}_{\sigma\sigma})\mathsf{dil}_\tau(\mathsf{fork}^{\sigma\sigma}_{\sigma}) = 0. \tag{6.5}$$

For a tricorne $\sigma = \alpha\gamma\tau$ we have that $\mathsf{dil}_\tau(\mathsf{fork}^{\sigma}_{\sigma\sigma})\mathsf{dil}_\tau(\mathsf{fork}^{\sigma\sigma}_{\sigma})$ is equal to

$$-1_\tau \otimes \mathsf{fork}^{\alpha\gamma}_{\alpha\alpha\gamma\gamma}\mathsf{braid}^{\alpha\alpha\gamma\gamma}_{\alpha\gamma\alpha\gamma}(1_{\alpha\gamma} \otimes \mathsf{bar}(\tau) \otimes 1_{\alpha\gamma})\mathsf{braid}^{\alpha\gamma\alpha\gamma}_{\alpha\alpha\gamma\gamma}\mathsf{fork}^{\alpha\alpha\gamma\gamma}_{\alpha\gamma}) \otimes 1_\tau \tag{6.6}$$

by definition (note $(-i)^2 = -1$). Applying equation (4.3) to the $\tau\gamma$-strands in equation (6.6) we obtain

$$\begin{aligned}
&- 1_\tau \otimes \mathsf{fork}^{\alpha\gamma}_{\alpha\alpha\gamma\gamma}\mathsf{braid}^{\alpha\alpha\gamma\gamma}_{\alpha\gamma\alpha\gamma}(1_{\alpha} \otimes \mathsf{bar}(\tau) \otimes 1_{\gamma\alpha\gamma})\mathsf{braid}^{\alpha\gamma\alpha\gamma}_{\alpha\alpha\gamma\gamma}\mathsf{fork}^{\alpha\alpha\gamma\gamma}_{\alpha\gamma} \otimes 1_\tau\\
&- 1_\tau \otimes \mathsf{fork}^{\alpha\gamma}_{\alpha\alpha\gamma\gamma}\mathsf{braid}^{\alpha\alpha\gamma\gamma}_{\alpha\gamma\alpha\gamma}(1_{\alpha} \otimes \mathsf{bar}(\gamma) \otimes 1_{\gamma\alpha\gamma})\mathsf{braid}^{\alpha\gamma\alpha\gamma}_{\alpha\alpha\gamma\gamma}\mathsf{fork}^{\alpha\alpha\gamma\gamma}_{\alpha\gamma} \otimes 1_\tau\\
&+ 1_\tau \otimes \mathsf{fork}^{\alpha\gamma}_{\alpha\alpha\gamma\gamma}\mathsf{braid}^{\alpha\alpha\gamma\gamma}_{\alpha\gamma\alpha\gamma}(1_{\alpha} \otimes \mathsf{gap}(\gamma) \otimes 1_{\alpha\gamma})\mathsf{braid}^{\alpha\gamma\alpha\gamma}_{\alpha\alpha\gamma\gamma}\mathsf{fork}^{\alpha\alpha\gamma\gamma}_{\alpha\gamma} \otimes 1_\tau
\end{aligned}$$

(these three terms are depicted in Figure 60). Now, the first term is zero by the $\alpha\gamma$-commutativity relation and the $\gamma$-circle annihilation relation. The second and third terms are zero by the $\alpha\gamma$-commutativity relation and the $\alpha$-circle annihilation relation.

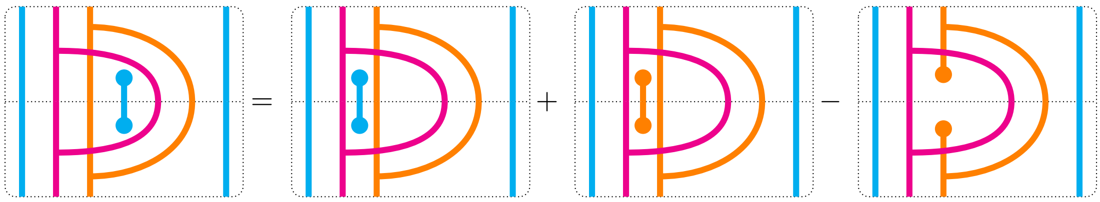


FIGURE 60. The circle annihilation relation for $\sigma = \alpha\gamma\tau$. We apply the $\gamma\tau$-barbell relation to the lefthand-side. The first term (respectively latter two terms) on the righthand-side is zero by the circle annihilation relation for $\gamma$ (respectively $\alpha$) and the $\alpha\gamma$-commutativity relations.

We now consider the case of an augmented tricorne $\sigma = \underline{x}\alpha\gamma\underline{x}^{-1}\tau$, with $\underline{x} = \sigma_k \dots \sigma_1$. The diagram $\mathsf{dil}_\tau(\mathsf{fork}^{\sigma}_{\sigma\sigma})\mathsf{dil}_\tau(\mathsf{fork}^{\sigma\sigma}_{\sigma})$ has a $\tau$-barbell in the centre of $k$ concentric circles with the innermost circle labelled by $\sigma_k$ and the outermost labelled by $\sigma_1$ (as pictured in the diagram on the lefthand-side of Figure 61). We pull this barbell through these $k$ circles using $k$ applications of equation (4.13) and hence obtain

$$\mathsf{cap}^{\emptyset}_{\underline{x}\underline{x}^{-1}}(1_{\underline{x}} \otimes \mathsf{bar}(\tau) \otimes 1_{\underline{x}^{-1}})\mathsf{cup}^{\underline{x}\underline{x}^{-1}}_{\emptyset} = (-1)^k \mathsf{bar}(\boldsymbol{\sigma_1}).$$

We therefore have that $\mathsf{dil}_\tau(\mathsf{fork}^{\sigma}_{\sigma\sigma})\mathsf{dil}_\tau(\mathsf{fork}^{\sigma\sigma}_{\sigma})$ is equal to

$$1_\tau \otimes 1_{\underline{x}} \otimes (\mathsf{fork}^{\alpha\gamma}_{\alpha\alpha\gamma\gamma}\mathsf{braid}^{\alpha\alpha\gamma\gamma}_{\alpha\gamma\alpha\gamma}(1_{\alpha\gamma} \otimes \mathsf{bar}(\boldsymbol{\sigma_1}) \otimes 1_{\alpha\gamma})\mathsf{braid}^{\alpha\gamma\alpha\gamma}_{\alpha\alpha\gamma\gamma}\mathsf{fork}^{\alpha\alpha\gamma\gamma}_{\alpha\gamma}) \otimes 1_{\underline{x}^{-1}} \otimes 1_\tau.$$

We can now apply the $\gamma\boldsymbol{\sigma}_1$-barbell relation and show that the three resulting terms are zero exactly as in the case of the tricorne, above. (See also Figure 61.)

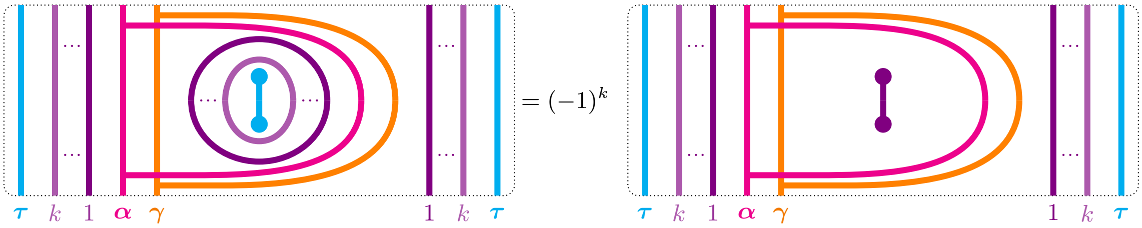



Figure 61. Simplifying the lefthand-side of the circle annihilation relation (equation (6.5)) using equation (4.13). Compare the righthand-side of the equation above with the lefthand-side of the equation pictured in Figure 60.

**6.4. The dilated null-braid relations.** Let $\boldsymbol{\beta} \in S_{W^{\tau}}$ and $m(\sigma, \boldsymbol{\beta}) = 3$. By inspection of Figures 43 to 48, we see that $\boldsymbol{\beta}$ must be a singleton label and either $(i)$ $m(\boldsymbol{\beta}, \boldsymbol{\alpha}) = 3$, $m(\boldsymbol{\beta}, \boldsymbol{\gamma}) = m(\beta, \boldsymbol{\tau}) = m(\boldsymbol{\beta}, \boldsymbol{\sigma}_i) = 2$ for all $1 \leqslant i \leqslant k$ $(ii)$ $m(\boldsymbol{\beta}, \boldsymbol{\gamma}) = 3$, $m(\boldsymbol{\beta}, \boldsymbol{\alpha}) = m(\beta, \boldsymbol{\tau}) = m(\boldsymbol{\beta}, \boldsymbol{\sigma}_i) = 2$ for all $1 \leqslant i \leqslant k$. We assume without loss of generality that $m(\boldsymbol{\beta}, \boldsymbol{\alpha}) = 3$. We must prove that

$$\mathsf{dil}_{\tau}(1_{\beta\sigma\beta}) = -\mathsf{dil}_{\tau}((\mathsf{spot}^{\beta\sigma\beta}_{\beta\emptyset\beta})\mathsf{dil}_{\tau}(\mathsf{dork}^{\beta\beta}_{\beta\beta})\mathsf{dil}_{\tau}(\mathsf{spot}^{\beta\emptyset\beta}_{\beta\sigma\beta}), \tag{6.7}$$

$$\mathsf{dil}_{\tau}(1_{\sigma\beta\sigma}) = -\mathsf{dil}_{\tau}(\mathsf{spot}^{\sigma\beta\sigma}_{\sigma\emptyset\sigma})\mathsf{dil}_{\tau}(\mathsf{dork}^{\sigma\sigma}_{\sigma\sigma})\mathsf{dil}_{\tau}(\mathsf{spot}^{\sigma\emptyset\sigma}_{\sigma\beta\sigma}). \tag{6.8}$$

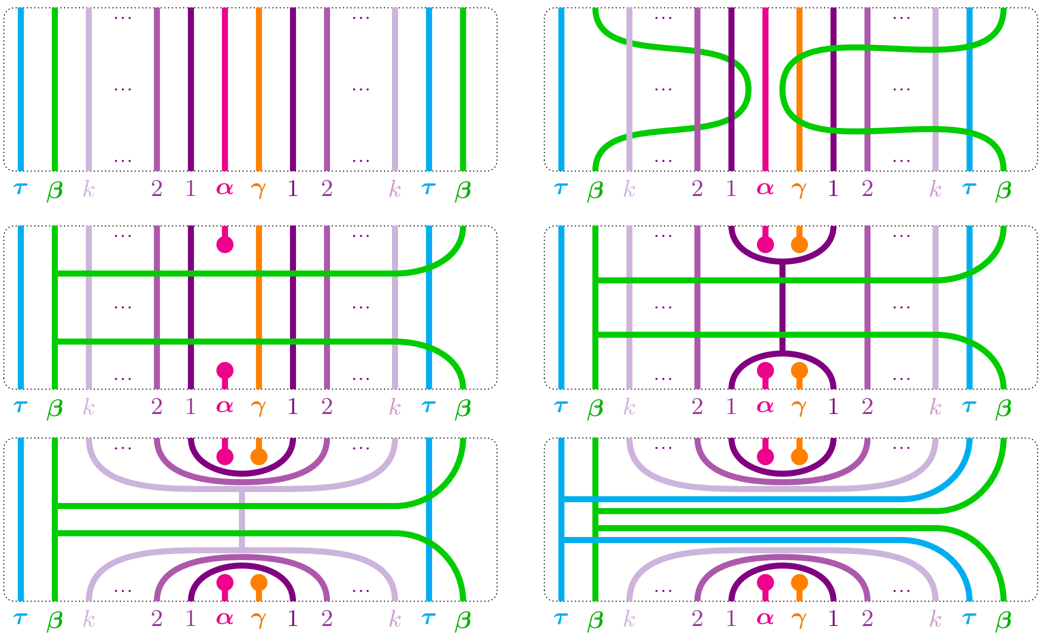



Figure 62. The six steps in proving equation (6.7). Read from left to right, one row at a time.

We first prove equation (6.7). We first apply the commutativity relations to the two $\boldsymbol{\beta}$-strands in $\mathsf{dil}_{\tau}(1_{\beta\sigma\beta})$ in order to bring them as close to the $\boldsymbol{\alpha}$-strand as possible (to obtain the top-right diagram of Figure 62) and we then apply the $\boldsymbol{\alpha\beta}$-null-braid (to obtain $-1$ times the middle-left diagram of Figure 62). We then apply the $\boldsymbol{\gamma\sigma}_1$-null-braid (to obtain the middle-right diagram of Figure 62) followed by the $\boldsymbol{\sigma}_i\boldsymbol{\sigma}_{i+1}$-null-braids for $1 \leqslant i < k$ in turn (to obtain $(-1)^{k+1}$ times the bottom-left diagram of Figure 62). Finally, we apply the $\boldsymbol{\tau\sigma}_k$-null-braid (to obtain $(-1)^{k+2}$ times the bottom-right diagram of Figure 62) and hence obtain $-\mathsf{dil}_{\tau}((\mathsf{spot}^{\beta\sigma\beta}_{\beta\emptyset\beta})\mathsf{dil}_{\tau}(\mathsf{dork}^{\beta\beta}_{\beta\beta})\mathsf{dil}_{\tau}(\mathsf{spot}^{\beta\emptyset\beta}_{\beta\sigma\beta})$ as required.

We now prove equation (6.8) in a similar fashion. We first apply the $\boldsymbol{\tau\sigma}_k$-null-braid relation to $\mathsf{dil}_{\tau}(1_{\beta\sigma\beta})$ followed by the $\boldsymbol{\sigma}_i\boldsymbol{\sigma}_{i+1}$-null-braid relations for $k > i \geqslant 1$ (to obtain $(-1)^k$ times the second diagram of Figure 63). We then apply the $\boldsymbol{\gamma\beta}$-null-braid and $\boldsymbol{\alpha\sigma}_1$-null-braid relations (to obtain

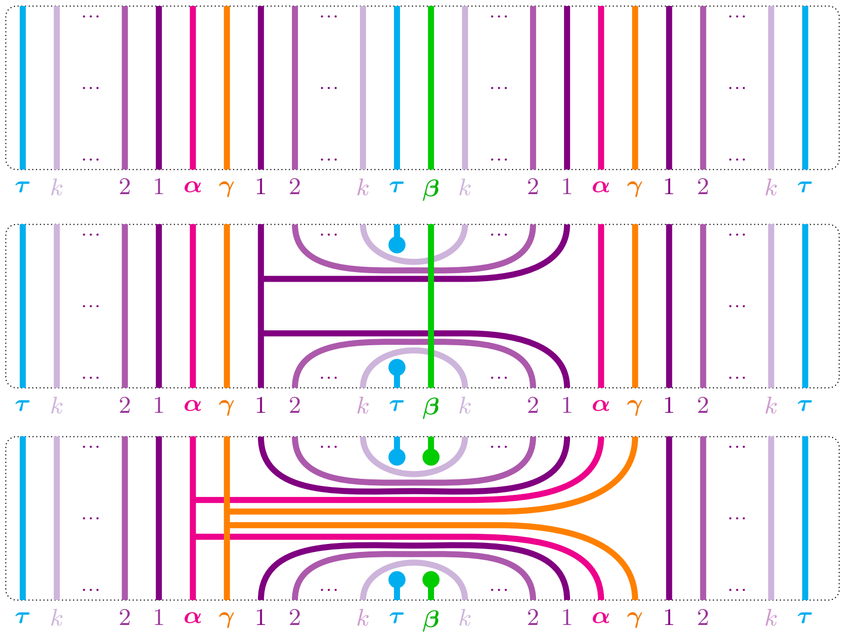



FIGURE 63. The three steps in proving equation (6.8).

$(-1)^k$ times the third diagram of Figure 63) and hence obtain $-\mathsf{dil}_{\tau}(\mathsf{spot}^{\sigma\beta\sigma}_{\sigma\emptyset\sigma})\mathsf{dil}_{\tau}(\mathsf{dork}^{\sigma\sigma}_{\sigma\sigma})\mathsf{dil}_{\tau}(\mathsf{spot}^{\sigma\emptyset\sigma}_{\sigma\beta\sigma})$ as required.

6.5. **The dilated barbell relations.** We now consider the one and two colour barbell relations.

**Lemma 6.1.** *For $\alpha, \gamma, \tau \in S^3_W$ with $m(\alpha,\tau) = 3 = m(\gamma,\tau)$ and $m(\alpha,\gamma) = 2$ we have that*

$$\begin{aligned}\mathsf{fork}^{\tau}_{\tau\tau}(1_{\tau} \otimes \mathsf{bar}(\alpha) \otimes \mathsf{bar}(\gamma) \otimes 1_{\tau})\mathsf{fork}^{\tau\tau}_{\tau} &= -\left(\mathsf{bar}(\alpha) + \mathsf{bar}(\tau) + \mathsf{bar}(\gamma)\right) \otimes 1_{\tau}\\ &= -1_{\tau} \otimes \left(\mathsf{bar}(\alpha) + \mathsf{bar}(\tau) + \mathsf{bar}(\gamma)\right).\end{aligned}$$

*Proof.* We prove the first equality, the second is given by equation (4.4) and recorded here only for reference. We first move the $\alpha$ barbell to the left through the $\tau$ strand using 4.3 (and hence obtain 3 terms); for the first two of these terms (in which the $\tau$-strand remains in tact) we then again use 4.3 to move the $\gamma$ barbell to the left through the $\tau$-strand. We hence obtain a sum involving 7 terms, 4 of which are zero by the $\tau$-circle-annihilation relation; this leaves us with the required 3 terms. □

We now "augment" the previous lemma so that it applies to augmented tricornes.

**Lemma 6.2.** *Let $\sigma$ be an augmented tricorne, $\sigma = \underline{x}\alpha\gamma\underline{x}^{-1}\tau$ and $\underline{x} = \sigma_k \dots \sigma_1$. We have that*

$$\begin{aligned}&\mathsf{fork}^{\tau}_{\tau\tau}(1_{\tau} \otimes \mathsf{cap}^{\emptyset}_{\underline{x}\underline{x}^{-1}} \otimes 1_{\tau})(1_{\tau\underline{x}} \otimes \mathsf{bar}(\alpha) \otimes \mathsf{bar}(\gamma) \otimes 1_{\underline{x}^{-1}\tau})(1_{\tau} \otimes \mathsf{cup}^{\underline{x}\underline{x}^{-1}}_{\emptyset} \otimes 1_{\tau})\mathsf{fork}^{\tau\tau}_{\tau}\\ &\quad= (-1)^{k+1}1_{\tau} \otimes \left(\textstyle\sum_{i=1}^{k} 2\mathsf{bar}(\sigma_i) + \mathsf{bar}(\alpha) + \mathsf{bar}(\gamma) + \mathsf{bar}(\tau)\right) \qquad (6.9)\\ &\quad= (-1)^{k+1}\left(\textstyle\sum_{i=1}^{k} 2\mathsf{bar}(\sigma_i) + \mathsf{bar}(\alpha) + \mathsf{bar}(\gamma) + \mathsf{bar}(\tau)\right) \otimes 1_{\tau} \qquad (6.10)\end{aligned}$$

*Proof.* We proceed by induction on $k \geqslant 0$, with the $k = 0$ base case taken care of in Lemma 6.1. By induction, we can rewrite the left-hand side of 6.9 as follows

$$\mathsf{fork}^{\tau}_{\tau\tau}\mathsf{spot}^{\tau\emptyset\tau}_{\tau\sigma_k\tau}(1_{\tau} \otimes (-1)^k 1_{\sigma_k} \otimes (\textstyle\sum_{i=1}^{k-1} 2\mathsf{bar}(\sigma_i) + \mathsf{bar}(\alpha) + \mathsf{bar}(\gamma) + \mathsf{bar}(\sigma_k)) \otimes 1_{\tau})\mathsf{spot}^{\tau\sigma_k\tau}_{\tau\emptyset\tau}\mathsf{fork}^{\tau\tau}_{\tau}.$$

which is equal to

$$(-1)^k\mathsf{fork}^{\tau}_{\tau\tau}(1_{\tau} \otimes \mathsf{bar}(\sigma_k) \otimes (\textstyle\sum_{i=1}^{k} 2\mathsf{bar}(\sigma_i) + \mathsf{bar}(\alpha) + \mathsf{bar}(\gamma) + \mathsf{bar}(\sigma_k)) \otimes 1_{\tau})\mathsf{fork}^{\tau\tau}_{\tau}.$$

The term involving a tensor product $\mathsf{bar}(\sigma_k) \otimes \mathsf{bar}(\sigma_k)$ can be rewritten using equation (4.14). The remaining terms involve a tensor product of two distinctly coloured barbells, one of which commutes with the $\tau$-strand; thus we can apply equation (4.12) to these terms. Rewriting all the terms in the above manner and summing over the resulting elements, we obtain 6.9. Equation (6.10) follows by equation (4.4). □

We are now ready to construct the dilated barbell diagrams.

**Lemma 6.3.** *Let $\sigma$ be an augmented tricorne, $\sigma = \underline{x}\boldsymbol{\alpha}\boldsymbol{\gamma}\underline{x}^{-1}\boldsymbol{\tau}$ and $\underline{x} = \boldsymbol{\sigma}_k \dots \boldsymbol{\sigma}_1$. We have that*

$$\mathsf{dil}_{\boldsymbol{\tau}}(\mathsf{bar}(\sigma)) = 1_{\boldsymbol{\tau}} \otimes (\textstyle\sum_{i=1}^{k} 2\mathsf{bar}(\boldsymbol{\sigma}_i) + \mathsf{bar}(\boldsymbol{\alpha}) + \mathsf{bar}(\boldsymbol{\gamma}) + \mathsf{bar}(\boldsymbol{\tau})) \tag{6.11}$$

$$= (\textstyle\sum_{i=1}^{k} 2\mathsf{bar}(\boldsymbol{\sigma}_i) + \mathsf{bar}(\boldsymbol{\alpha}) + \mathsf{bar}(\boldsymbol{\gamma}) + \mathsf{bar}(\boldsymbol{\tau})) \otimes 1_{\boldsymbol{\tau}} \tag{6.12}$$

$$\mathsf{dil}_{\boldsymbol{\tau}}(1_\sigma) \circledast \mathsf{dil}_{\boldsymbol{\tau}}(\mathsf{bar}(\sigma)) = 1_{\boldsymbol{\tau}\underline{x}\boldsymbol{\alpha}} \otimes \mathsf{bar}(\boldsymbol{\alpha}) \otimes 1_{\boldsymbol{\gamma}\underline{x}^{-1}\boldsymbol{\tau}} \tag{6.13}$$

$$\mathsf{dil}_{\boldsymbol{\tau}}(\mathsf{bar}(\sigma)) \circledast \mathsf{dil}_{\boldsymbol{\tau}}(1_\sigma) = 1_{\boldsymbol{\tau}\underline{x}} \otimes \mathsf{bar}(\boldsymbol{\alpha}) \otimes 1_{\boldsymbol{\alpha}\boldsymbol{\gamma}\underline{x}^{-1}\boldsymbol{\tau}} \tag{6.14}$$

$$\mathsf{dil}_{\boldsymbol{\tau}}(\mathsf{gap}(\sigma)) = 1_{\boldsymbol{\tau}\underline{x}} \otimes \mathsf{gap}(\boldsymbol{\alpha}) \otimes 1_{\boldsymbol{\gamma}\underline{x}^{-1}\boldsymbol{\tau}} \tag{6.15}$$

*Proof.* Equation (6.11) follows directly from Lemma 6.2. We now consider equation (6.13) and equation (6.14). We have that

$$\begin{aligned}
\mathsf{dil}_{\boldsymbol{\tau}}(1_\sigma) \circledast \mathsf{dil}_{\boldsymbol{\tau}}(\mathsf{bar}(\sigma)) &= 1_{\boldsymbol{\tau}\underline{x}\boldsymbol{\alpha}\boldsymbol{\gamma}\underline{x}^{-1}\boldsymbol{\tau}} \otimes (\textstyle\sum_{i=1}^{k} 2\mathsf{bar}(\boldsymbol{\sigma}_i) + \mathsf{bar}(\boldsymbol{\alpha}) + \mathsf{bar}(\boldsymbol{\gamma}) + \mathsf{bar}(\boldsymbol{\tau}))\\
&= 1_{\boldsymbol{\tau}\underline{x}\boldsymbol{\alpha}\boldsymbol{\gamma}\underline{x}^{-1}} \otimes (\textstyle\sum_{i=1}^{k} 2\mathsf{bar}(\boldsymbol{\sigma}_i) + \mathsf{bar}(\boldsymbol{\alpha}) + \mathsf{bar}(\boldsymbol{\gamma}) + \mathsf{bar}(\boldsymbol{\tau})) \otimes 1_{\boldsymbol{\tau}}\\
&= 1_{\boldsymbol{\tau}\underline{x}\boldsymbol{\alpha}\boldsymbol{\gamma}\underline{x}^{-1}} \otimes (\textstyle\sum_{i=1}^{k-1} 2\mathsf{bar}(\boldsymbol{\sigma}_i) + \mathsf{bar}(\boldsymbol{\alpha}) + \mathsf{bar}(\boldsymbol{\gamma}) + \mathsf{bar}(\boldsymbol{\sigma}_k)) \otimes 1_{\boldsymbol{\tau}}
\end{aligned}$$

where the first equality follows equation (6.11); the second from summing over relations R4 and R5; the third from Lemma 4.5. We repeat the final two steps above a further $k-2$ times and hence obtain

$$\begin{aligned}
\mathsf{dil}_{\boldsymbol{\tau}}(1_\sigma) \circledast \mathsf{dil}_{\boldsymbol{\tau}}(\mathsf{bar}(\sigma)) &= 1_{\boldsymbol{\tau}\underline{x}\boldsymbol{\alpha}\boldsymbol{\gamma}\boldsymbol{\sigma}_1} \otimes (2\mathsf{bar}(\boldsymbol{\sigma}_1) + \mathsf{bar}(\boldsymbol{\alpha}) + \mathsf{bar}(\boldsymbol{\gamma}) + \mathsf{bar}(\boldsymbol{\sigma}_2)) \otimes 1_{\boldsymbol{\sigma}_2\dots\boldsymbol{\sigma}_k\boldsymbol{\tau}}\\
&= 1_{\boldsymbol{\tau}\underline{x}\boldsymbol{\alpha}\boldsymbol{\gamma}\boldsymbol{\sigma}_1} \otimes (\mathsf{bar}(\boldsymbol{\alpha}) + \mathsf{bar}(\boldsymbol{\gamma}) + \mathsf{bar}(\boldsymbol{\sigma}_1)) \otimes 1_{\boldsymbol{\sigma}_2\dots\boldsymbol{\sigma}_k\boldsymbol{\tau}}\\
&= 1_{\boldsymbol{\tau}\underline{x}\boldsymbol{\alpha}\boldsymbol{\gamma}} \otimes (\mathsf{bar}(\boldsymbol{\alpha}) + \mathsf{bar}(\boldsymbol{\gamma}) + \mathsf{bar}(\boldsymbol{\sigma}_1)) \otimes 1_{\underline{x}^{-1}\boldsymbol{\tau}}\\
&= 1_{\boldsymbol{\tau}\underline{x}\boldsymbol{\alpha}\boldsymbol{\gamma}} \otimes \mathsf{bar}(\boldsymbol{\alpha}) \otimes 1_{\underline{x}^{-1}\boldsymbol{\tau}}\\
&= 1_{\boldsymbol{\tau}\underline{x}\boldsymbol{\alpha}} \otimes \mathsf{bar}(\boldsymbol{\alpha}) \otimes 1_{\boldsymbol{\gamma}\underline{x}^{-1}\boldsymbol{\tau}}
\end{aligned}$$

where the second and fourth equalities follow from Lemma 4.5; the third from equation (4.4); the fifth from the $\boldsymbol{\gamma}\boldsymbol{\alpha}$-commutativity relations. We now consider equation (6.14). We have that

$$\begin{aligned}
\mathsf{dil}_{\boldsymbol{\tau}}(\mathsf{bar}(\sigma)) \circledast \mathsf{dil}_{\boldsymbol{\tau}}(1_\sigma) &= 1_{\boldsymbol{\tau}} \otimes (\textstyle\sum_{i=1}^{k} 2\mathsf{bar}(\boldsymbol{\sigma}_i) + \mathsf{bar}(\boldsymbol{\alpha}) + \mathsf{bar}(\boldsymbol{\gamma}) + \mathsf{bar}(\boldsymbol{\tau})) \otimes 1_{\underline{x}\boldsymbol{\alpha}\boldsymbol{\gamma}\underline{x}^{-1}\boldsymbol{\tau}}\\
&= 1_{\boldsymbol{\tau}\underline{x}} \otimes \mathsf{bar}(\boldsymbol{\alpha}) \otimes 1_{\boldsymbol{\alpha}\boldsymbol{\gamma}\underline{x}^{-1}\boldsymbol{\tau}}
\end{aligned}$$

where the first equality follows from equation (6.11) and the second follows by the exact same argument as for the case of equation (6.13). Finally, we consider equation (6.15). We have that

$$\begin{aligned}
\mathsf{dil}_{\boldsymbol{\tau}}(\mathsf{gap}(\sigma)) &= \mathsf{dil}_{\boldsymbol{\tau}}(\mathsf{spot}^{\sigma}_{\emptyset})\mathsf{dil}_{\boldsymbol{\tau}}(\mathsf{spot}^{\emptyset}_{\sigma})\\
&= (-1)^{k+1}\mathsf{spot}^{\boldsymbol{\tau}\underline{x}\boldsymbol{\alpha}\boldsymbol{\gamma}\underline{x}^{-1}\boldsymbol{\tau}}_{\boldsymbol{\tau}\underline{x}\emptyset\emptyset\underline{x}^{-1}\boldsymbol{\tau}}(1_{\boldsymbol{\tau}} \otimes \mathsf{cup}^{\underline{x}\underline{x}^{-1}}_{\emptyset} \otimes 1_{\boldsymbol{\tau}})\mathsf{dork}^{\boldsymbol{\tau}\boldsymbol{\tau}}_{\boldsymbol{\tau}\boldsymbol{\tau}}(1_{\boldsymbol{\tau}} \otimes \mathsf{cap}^{\emptyset}_{\underline{x}\underline{x}^{-1}} \otimes 1_{\boldsymbol{\tau}})\mathsf{spot}^{\boldsymbol{\tau}\underline{x}\emptyset\emptyset\underline{x}^{-1}\boldsymbol{\tau}}_{\boldsymbol{\tau}\underline{x}\boldsymbol{\alpha}\boldsymbol{\gamma}\underline{x}^{-1}\boldsymbol{\tau}}\\
&= 1_{\boldsymbol{\tau}\underline{x}} \otimes \mathsf{gap}(\boldsymbol{\alpha}) \otimes 1_{\boldsymbol{\gamma}\underline{x}^{-1}\boldsymbol{\tau}}
\end{aligned}$$

where the first and second equalities are by definition; the third follows by applying the $\boldsymbol{\tau}\boldsymbol{\sigma}_k$-null-braid relation followed by the $\boldsymbol{\sigma}_i\boldsymbol{\sigma}_{i+1}$-null-braid relations for $k > i \geqslant 1$ followed by the $\boldsymbol{\gamma}\boldsymbol{\sigma}_1$-null-braid relation. □

6.5.1. *The dilated one colour barbell relation.* Let $\sigma$ be a reading word of some $\boldsymbol{\tau}$-contraction tile. We now verify the rightmost relation in (R4). We have that

$$\begin{aligned}
&\mathsf{dil}_{\boldsymbol{\tau}}(\mathsf{bar}(\sigma)) \circledast \mathsf{dil}_{\boldsymbol{\tau}}(1_\sigma) + \mathsf{dil}_{\boldsymbol{\tau}}(1_\sigma) \circledast \mathsf{dil}_{\boldsymbol{\tau}}(\mathsf{bar}(\sigma))\\
&\qquad = 1_{\boldsymbol{\tau}\underline{x}} \otimes \mathsf{bar}(\boldsymbol{\alpha}) \otimes 1_{\boldsymbol{\alpha}\boldsymbol{\gamma}\underline{x}^{-1}\boldsymbol{\tau}} + 1_{\boldsymbol{\tau}\underline{x}\boldsymbol{\alpha}} \otimes \mathsf{bar}(\boldsymbol{\alpha}) \otimes 1_{\boldsymbol{\gamma}\underline{x}^{-1}\boldsymbol{\tau}}\\
&\qquad = 2 \cdot 1_{\boldsymbol{\tau}\underline{x}} \otimes \mathsf{gap}(\boldsymbol{\alpha}) \otimes 1_{\boldsymbol{\gamma}\underline{x}^{-1}\boldsymbol{\tau}}\\
&\qquad = 2 \cdot \mathsf{dil}_{\boldsymbol{\tau}}(\mathsf{gap}(\sigma))
\end{aligned}$$

as required. Here the first equality follows from equation (6.13) and (6.14); the second follows from the one-colour-barbell relation; and the third from equation (6.15).

6.5.2. *The dilated two colour barbell relations.* Let $\beta \in S_{W^\tau}$, as noted in Subsection 6.4, we can assume that $\beta$ is a singleton which commutes every label in $\sigma$ except $\alpha$. We have that

$$\begin{aligned}
&\mathsf{dil}_\tau(\mathsf{bar}(\sigma)) \circledast \mathsf{dil}_\tau(1_\beta) - \mathsf{dil}_\tau(1_\beta) \circledast \mathsf{dil}_\tau(\mathsf{bar}(\sigma)) \\
&\qquad = 1_\tau \otimes (\textstyle\sum_{i=1}^k 2\mathsf{bar}(\sigma_i) + \mathsf{bar}(\alpha) + \mathsf{bar}(\gamma) + \mathsf{bar}(\tau)) \otimes 1_\beta \\
&\qquad\quad - 1_{\tau\beta} \otimes (\textstyle\sum_{i=1}^k 2\mathsf{bar}(\sigma_i) + \mathsf{bar}(\alpha) + \mathsf{bar}(\gamma) + \mathsf{bar}(\tau)) \\
&\qquad = 1_\tau \otimes \mathsf{bar}(\alpha) \otimes 1_\beta - 1_{\tau\beta} \otimes \mathsf{bar}(\alpha) \\
&\qquad = 1_{\tau\beta} \otimes \mathsf{bar}(\beta) - 1_\tau \otimes \mathsf{gap}(\beta) \\
&\qquad = \mathsf{dil}_\tau(1_\beta) \circledast \mathsf{dil}_\tau(\mathsf{bar}(\beta)) - \mathsf{dil}_\tau(\mathsf{gap}(\beta))
\end{aligned}$$

as required. Here, the first equality follows from equation (6.11); the second from the commutativity relations; the third from the $\alpha\beta$-barbell relation; the fourth follows by definition.

We now turn to the other two-colour barbell relation (in which the roles of $\beta$ and $\sigma$ are swapped). We have that

$$\begin{aligned}
&\mathsf{dil}_\tau(\mathsf{bar}(\beta)) \circledast \mathsf{dil}_\tau(1_\sigma) - \mathsf{dil}_\tau(1_\sigma) \circledast \mathsf{dil}_\tau(\mathsf{bar}(\beta)) \\
&\qquad = 1_\tau \otimes \mathsf{bar}(\beta) \otimes 1_{\underline{x}\alpha\gamma\underline{x}^{-1}\tau} - 1_{\tau\underline{x}\alpha\gamma\underline{x}^{-1}\tau} \otimes \mathsf{bar}(\beta) \\
&\qquad = 1_{\tau\underline{x}} \otimes \mathsf{bar}(\beta) \otimes 1_{\alpha\gamma\underline{x}^{-1}\tau} - 1_{\tau\underline{x}\alpha} \otimes \mathsf{bar}(\beta) \otimes 1_{\gamma\underline{x}^{-1}\tau} \\
&\qquad = 1_{\tau\underline{x}} \otimes \mathsf{bar}(\alpha) \otimes 1_{\alpha\gamma\underline{x}^{-1}\tau} - 1_{\tau\underline{x}\alpha} \otimes \mathsf{bar}(\alpha) \otimes 1_{\gamma\underline{x}^{-1}\tau} \\
&\qquad = \mathsf{dil}_\tau(1_\sigma) \circledast \mathsf{dil}_\tau(\mathsf{bar}(\sigma)) - \mathsf{dil}_\tau(\mathsf{gap}(\sigma))
\end{aligned}$$

as required. Here the first equality follows by definition; the second by the commutativity relations; the third by the $\alpha\beta$-barbell; the fourth by equation (6.13) and (6.15).

6.6. **The dilated $m = 2$ relations.** For $\sigma, \beta, \pi \in S_{W^\tau}$ with $m(\sigma, \beta) = m(\pi, \beta) = m(\sigma, \pi) = 2$ we need to check the dilated versions of the relations

$$\mathsf{braid}^{\beta\sigma}_{\sigma\beta}\mathsf{braid}^{\sigma\beta}_{\beta\sigma} = 1_{\beta\sigma} \qquad \mathsf{braid}^{\beta\pi\sigma}_{\pi\beta\sigma}\mathsf{braid}^{\pi\beta\sigma}_{\sigma\pi\beta} = \mathsf{braid}^{\beta\pi\sigma}_{\sigma\beta\pi}\mathsf{braid}^{\sigma\beta\pi}_{\sigma\pi\beta}$$

$$\mathsf{braid}^{\beta\sigma}_{\sigma\beta}\mathsf{fork}^{\sigma\beta}_{\sigma\sigma\beta} = \mathsf{fork}^{\beta\sigma}_{\beta\sigma\sigma}\mathsf{braid}^{\beta\sigma\sigma}_{\sigma\sigma\beta} \qquad (1_{\sigma\beta} \otimes \mathsf{cap}^{\emptyset}_{\sigma\sigma})\mathsf{braid}^{\sigma\beta\sigma\sigma}_{\sigma\sigma\beta\sigma}(\mathsf{cup}^{\sigma\sigma}_{\emptyset} \otimes 1_{\beta\sigma}) = \mathsf{braid}^{\sigma\beta}_{\beta\sigma}$$

and their horizontal and vertical flips, along with the diagrams obtained by swapping the roles of $\beta$ and $\sigma$. Note that by Proposition 4.8, both sides of all of these equations vanish when $(W, P)^\tau = (A_n, A_{n-1})$, or $(W, P)^\tau = (D_n, D_{n-1})$, or $(W, P)^\tau = (D_n, A_{n-1})$ with $\{\beta, \sigma\} = \{s_0, s_1\}$. In all other cases, we have that $\sigma$ is a (possibly) composite label and $\beta$ (and $\pi$) are singleton labels which commute with every constituent label of $\sigma$. Thus all these relations are trivially satisfied.

6.7. **The cyclotomic relations.** We finish by showing that the dilations of the non-local relation R15 and R16 are also preserved by $\varphi_\tau$. It is easy to see that

$$\varphi_\tau(1_\sigma \otimes 1_{\underline{w}}) = 1_{T_{0\to 1}} \circledast \mathsf{dil}_\tau(1_\sigma) \circledast \mathsf{dil}_\tau(1_{\underline{w}}) = 0$$

whenever $\sigma \in S_{P^\tau}$ using (possibly) the null-braid relations, the commutativity relations, and the cyclotomic relation in $\mathcal{H}_{(W,P)}$. It remains to show that

$$\varphi_\tau(\mathsf{bar}(\sigma) \otimes 1_{\underline{w}}) = 0$$

for $\sigma$ the unique element of $S_{W^\tau} \setminus S_{P^\tau}$. We will show that

$$1_{T_{0\to 1}} \circledast \mathsf{dil}_\tau(\mathsf{bar}(\sigma)) = 0$$

for such $\sigma$ and hence deduce the result. For the remainder of this section, we set $T_{0\to 1} = \rho_1\rho_2\dots\rho_r$ and we note that $\rho_r = \tau$.

**Case 1.** Suppose that $\sigma$ is a singleton. Then there exists $1 \leqslant j \leqslant r$ such that $m(\sigma, \rho_i) = 2$ for all $i \neq j$ and $m(\sigma, \rho_j) = 3$; this can be seen by inspection of Figures 43 to 48. We have that

$$\begin{aligned}
1_{T_{0\to 1}} \circledast \mathsf{dil}_\tau(\mathsf{bar}(\sigma)) &= 1_{\rho_1\rho_2\dots\rho_r} \otimes \mathsf{bar}(\sigma) \\
&= 1_{\rho_1\rho_2\dots\rho_j} \otimes \mathsf{bar}(\sigma) \otimes 1_{\rho_{j+1}\dots\rho_r} \\
&= 1_{\rho_1\rho_2\dots\rho_{j-1}} \otimes (\mathsf{bar}(\sigma) + \mathsf{bar}(\rho_j)) \otimes 1_{\rho_j\rho_{j+1}\dots\rho_r} \\
&\quad - 1_{\rho_1\rho_2\dots\rho_{j-1}} \otimes \mathsf{gap}(\rho_j) \otimes 1_{\rho_{j+1}\dots\rho_r}
\end{aligned}$$

$$= 1_{\rho_1\rho_2\dots\rho_{j-1}} \otimes \mathsf{bar}(\rho_j) \otimes 1_{\rho_j\rho_{j+1}\dots\rho_r}$$
$$= 0$$

as required. Here the first equality is the definition; the second follows by the commuting relations; the third by the two-colour barbell relation; the fourth by the commuting and cyclotomic relations; the fifth follows by repeating the arguments above.

**Case 2.** We now suppose that $\sigma = \alpha\gamma\tau$, a tricorne. By inspecting Figures 43 to 48, we deduce that $m(\alpha, \rho_i) = 2 = m(\gamma, \rho_i)$ for all $1 \leqslant i \leqslant r$. We have that

$$\begin{aligned}
1_{T_{0\to1}} \circledast \mathsf{dil}_{\tau}(\mathsf{bar}(\sigma)) &= 1_{\rho_1\rho_2\dots\rho_{r-1}} \otimes 1_{\tau} \otimes (\mathsf{bar}(\alpha) + \mathsf{bar}(\gamma) + \mathsf{bar}(\tau)) \\
&= 1_{\rho_1\rho_2\dots\rho_{r-1}} \otimes (\mathsf{bar}(\alpha) + \mathsf{bar}(\gamma) + \mathsf{bar}(\tau)) \otimes 1_{\tau} \\
&= 1_{\rho_1\rho_2\dots\rho_{r-1}} \otimes \mathsf{bar}(\tau) \otimes 1_{\tau} \\
&= 0
\end{aligned}$$

as required. Here the first equation follows from equation (6.11); the second by equation (4.4); the third by the commuting and cyclotomic relations; the fourth follows as in Case 1.

**Case 3.** We now suppose that $\sigma = \underline{x}\alpha\gamma\underline{x}^{-1}\tau$, an augmented tricorne. By inspecting Figures 44 to 48, we deduce that $m(\alpha, \rho_i) = m(\gamma, \rho_i) = m(\sigma_j, \rho_i) = 2$ for $j \neq k$, $1 \leqslant i \leqslant r$; and $m(\sigma_k, \rho_r) = 3$ (recall $\tau = \rho_r$). We have that

$$\begin{aligned}
1_{T_{0\to1}} \circledast \mathsf{dil}_{\tau}(\mathsf{bar}(\sigma)) &= 1_{\rho_1\rho_2\dots\rho_{r-1}} \otimes 1_{\tau} \otimes (\textstyle\sum_{i=1}^{k} 2\mathsf{bar}(\sigma_i) + \mathsf{bar}(\alpha) + \mathsf{bar}(\gamma) + \mathsf{bar}(\tau)) \\
&= 1_{\rho_1\rho_2\dots\rho_{r-1}} \otimes 1_{\tau} \otimes (2\mathsf{bar}(\sigma_k) + \mathsf{bar}(\tau)) \\
&= 1_{\rho_1\rho_2\dots\rho_{r-1}} \otimes (2\mathsf{bar}(\sigma_k) + \mathsf{bar}(\tau)) \otimes 1_{\tau} \\
&= 1_{\rho_1\rho_2\dots\rho_{r-1}} \otimes \mathsf{bar}(\tau) \otimes 1_{\tau} \\
&= 0
\end{aligned}$$

as required. The first equality follows from equation (6.11); the second from the commutativity relations; the third from equation (4.4); the fourth by commutativity relations; the fifth equality follows as in Case 1.

## 7. Graded decomposition numbers and Koszul resolutions

We are now ready to determine the main structural results concerning the Hecke categories of Hermitian symmetric pairs. Specifically, we will calculate the graded composition multiplicities and radical filtrations of standard modules in Theorem 7.2 and Corollary 7.11. In order to prove that the grading and radical layers coincide, we will prove that the algebra $h_{(W,P)}$ satisfies the strong cohomological property of standard Koszulity (see [BGS96, ADL03] for the definition of standard Koszul); this amounts to constructing linear projective resolutions of standard modules as in Theorem 7.9. Our treatment of this material is inspired by similar ideas in [BS10].

**Proposition 7.1** ([BDF+25, Corollary 6.2])**.** *Let $(W, P)$ be a simply laced Hermitian symmetric pair. For any $\lambda \neq \mu$, we have that*

$$\sum_{\mathsf{S}\in\mathrm{Path}(\lambda,\mathsf{t}_\mu)} q^{\deg(\mathsf{S})} \in q\mathbb{Z}_{\geqslant 0}[q].$$

*In particular, all the non-zero terms occur in strictly-positive degree.*

**Theorem 7.2.** *Let $(W, P)$ be an arbitrary Hermitian symmetric pair and $\Bbbk$ be a field of characteristic $p \geqslant 0$. The p-Kazhdan–Lusztig polynoials*

$${}^{p}n_{\lambda,\mu}(q) = \sum_{k\in\mathbb{Z}} [\Delta(\lambda) : L(\mu)\langle k\rangle] q^k$$

*of $h_{(W,P)}$ are independent of the prime $p \geqslant 0$. For $(W, P)$ of simply laced type, the algebra $h_{(W,P)}$ is basic and the modules $1_{\mathsf{t}_\lambda} h_{(W,P)}$ for $\lambda \in \mathscr{P}_{(W,P)}$ provide a complete set of non-isomorphic projective indecomposable right $h_{(W,P)}$-modules.*

*Proof.* By Theorem 4.24, it is enough to restrict our attention to simply laced type. By Proposition 7.1, we have that $h_{(W,P)}$ is a positively $\mathbb{Z}$-graded $\Bbbk$-algebra with

$$\dim_q(h_{(W,P)})|_{q=0} = (\textstyle\sum_{\lambda\in\mathscr{P}_{(W,P)}}(\sum_{\mu\in\mathscr{P}_{(W,P)}}\dim_q(\Delta(\lambda)1_\mu))^2)|_{q=0} = |\{\lambda \mid \lambda \in \mathscr{P}_{(W,P)}\}| \quad (7.1)$$

where the latter equality follows again from Proposition 7.1. Now, we have that $\dim_q(L(\lambda)) \in \mathbb{Z}_{\geqslant 0}[q+q^{-1}]$ (by [HM10, Proposition 2.18]) and so by (7.1) we deduce that $\dim_q(L(\lambda)) = 1$ and that the degree zero subalgebra of $h_{(W,P)}$ is isomorphic to $\oplus_{\lambda\in\mathscr{P}_{(W,P)}}L(\lambda)$ (regardless of the characteristic of $\Bbbk$). Thus the algebra is basic (as all the simple modules are 1-dimensional) and we have that

$${}^p n_{\lambda,\mu}(q) = \sum_{k\in\mathbb{Z}}[\Delta(\lambda) : L(\mu)\langle k\rangle]q^k = \dim_q(\Delta(\lambda)1_\mu) = \sum_{\mathsf{S}\in \mathrm{Path}(\lambda,\mathsf{t}_\mu)} q^{\deg(\mathsf{S})} \in q\mathbb{Z}_{\geqslant 0}[q]$$

again by Theorem 7.2 (again, regardless of the characteristic $p$ of the field $\Bbbk$). ☐

In the remainder of this section, we will prove the Koszulity of the Hecke categories for Hermitian symmetric pairs. Using Section 4, we can reduce to the simply-laced cases where we will use the Coxeter truncation to work by induction on the rank.

7.1. **Induction.** Assume that $(W,P)$ is simply laced and let $\tau \in S_W$. Define

$$e_\tau = \sum_{\substack{\mu\in\mathscr{P}_{(W,P)}\\ \mu<\mu\tau}} 1_{\mathsf{t}_\mu}\otimes 1_\tau$$

We have that $e_\tau h_{(W,P)}$ carries the structure of a $(h^\tau_{(W,P)}, h_{(W,P)})$-bimodule. The action on the right is by concatenation of diagrams. The action on the left is given by first conjugating $h^\tau_{(W,P)}$ by a (commuting) braid so that the colour sequences match-up, and then concatenating diagrams. (Recall from Remark 3.1 that this simply amounts to changing our choice of tableaux.) With this isomorphism in place (and the isomorphism of Theorem 5.9) we are now able to define an induction functor

$$G^\tau : h_{(W,P)^\tau}\text{-mod} \longrightarrow h_{(W,P)}\text{-mod}$$
$$M \mapsto M \otimes_{h_{(W,P)^\tau}} e_\tau h_{(W,P)}\langle -1\rangle$$

using the identification $h_{(W,P)^\tau} \cong h^\tau_{(W,P)} \subseteq h_{(W,P)}$. The degree shift in this definition ensures that the functor $G^\tau$ commutes with duality (see Theorem 7.5 below). We have that

$$\mathscr{P}^\tau_{(W,P)} := \{\lambda \in \mathscr{P}_{(W,P)} \mid \tau \in \mathrm{Rem}(\lambda)\} \leftrightarrow \mathscr{P}_{(W,P)^\tau}$$

and for $\lambda \in \mathscr{P}^\tau_{(W,P)}$, we write $\lambda{\downarrow_\tau}$ for the image on the righthand-side (so that $\varphi_\tau(\lambda{\downarrow_\tau}) = \lambda$). We say that $\lambda{\downarrow_\tau}$ is the contraction of $\lambda$ at $\tau$. In what follows, we will write $1_\mu$ instead of $1_{\mathsf{t}_\mu}$ to simplify notations.

**Theorem 7.3.** *The functor $G^\tau$ is exact.*

*Proof.* We need to show that $e_\tau h_{(W,P)}$ is projective as both a right $h_{(W,P)}$-module and as a left $h_{(W,P)^\tau}$-module. As a right $h_{(W,P)}$-module, $e_\tau h_{(W,P)}$ is a direct summand of $h_{(W,P)}$ (as $e_\tau$ is an idempotent) and so it is clearly projective. It remains to show that $e_\tau h_{(W,P)}$ is projective as a left $h_{(W,P)^\tau}$-module. We can decompose this module as follows

$$e_\tau h_{(W,P)} = \oplus_\mu e_\tau h_{(W,P)}1_\mu.$$

We will show that each of these summands is projective as a left $h_{(W,P)^\tau}$-module. For the remainder of the proof, all statements concerning modules or homomorphisms will be taken implicitly to be of left $h_{(W,P)^\tau}$-modules. In all of the following cases, we will use the fact that $c^\lambda_{\mathsf{ST}} \in e_\tau h_{(W,P)}$ implies $\mathsf{S} \in \mathrm{Path}(\lambda,\mathsf{t}_\nu)$ such that $\tau \in \mathrm{Rem}(\nu)$. This, in turn, implies that $\tau \in \mathrm{Rem}(\lambda)$ or in $\mathrm{Add}(\lambda)$.

**Case 1**. We first assume that $\tau \in \mathrm{Rem}(\mu)$. We claim that in this case

$$e_\tau h_{(W,P)}1_\mu \cong h_{(W,P)^\tau}1_{\mu\downarrow_\tau} \oplus h_{(W,P)^\tau}1_{\mu\downarrow_\tau}\langle 2\rangle.$$

The module $e_\tau h_{(W,P)}1_\mu$ has a basis

$$B = \{c^\lambda_{\mathsf{ST}} \mid \mathsf{S} \in \mathrm{Path}(\lambda,\mathsf{t}_\nu), \mathsf{T} \in \mathrm{Path}(\lambda,\mathsf{t}_\mu), \text{ with } \lambda \in \mathscr{P}_{(W,P)} \text{ and } \nu \in \mathscr{P}^\tau_{(W,P)}\}$$

which decomposes as a disjoint union $\{c^\lambda_{\mathsf{ST}} \in B \mid \tau \in \mathrm{Rem}(\lambda)\} \sqcup \{c^\lambda_{\mathsf{ST}} \in B \mid \tau \in \mathrm{Add}(\lambda)\}$. Now we have

$$\langle c^\lambda_{\mathsf{ST}} \in B \mid \tau \in \mathrm{Rem}(\lambda)\rangle = h^{\tau}_{(W,P)} 1_\mu \cong h_{(W,P)^\tau} 1_{\mu\downarrow_\tau}.$$

Now consider the quotient $e_\tau h_{(W,P)} 1_\mu / h^\tau_{(W,P)} 1_\mu$. It has a basis given by the elements $c^\lambda_{\mathsf{ST}} + h^\tau_{(W,P)} 1_\mu$ with $\tau \in \mathrm{Add}(\lambda)$. These satisfy $\mathsf{S} = X^-_\tau(\mathsf{S}')$ and $\mathsf{T} = X^-_\tau(\mathsf{T}')$ for a (possibly different) choice of $X = A$ or $R$ for each one. If we take $\mathsf{U} = X^+_\tau(\mathsf{S}')$ and $\mathsf{V} = X^+_\tau(\mathsf{T}')$ then we can write

$$c^\lambda_{\mathsf{ST}} = c^*_{\mathsf{S}} c_{\mathsf{T}} = c^*_{\mathsf{U}}(1_\lambda \otimes \mathsf{gap}(\tau)) c_{\mathsf{V}}$$

If $\mathsf{T} = A^-_\tau(\mathsf{T}')$ and so $\mathsf{V} = A^+_\tau(\mathsf{T}')$ then it becomes

$$c^\lambda_{\mathsf{ST}} = c^{\lambda\tau}_{\mathsf{UV}}(1_{\mu-\tau} \otimes \mathsf{gap}(\tau)).$$

If $\mathsf{T} = R^-_\tau(\mathsf{T}')$ and so $\mathsf{V} = R^+_\tau(\mathsf{T}')$ then we can factorise $c^\lambda_{\mathsf{ST}}$ as

$$c^\lambda_{\mathsf{ST}} = c^*_{\mathsf{U}}(1_\lambda \otimes (\mathsf{spot}^\tau_\emptyset \mathsf{cap}^\emptyset_{\tau\tau}))(c_{\mathsf{T}'} \otimes 1_\tau). \tag{7.2}$$

Now applying $1_\tau \otimes \mathsf{spot}^\emptyset_\tau$ to equation Figure 9 we get

$$1_\tau \otimes \mathsf{spot}^\emptyset_\tau = \mathsf{spot}^\emptyset_\tau \otimes 1_\tau + \mathsf{spot}^\tau_\emptyset \mathsf{cap}^\emptyset_{\tau\tau} - \mathsf{bar}(\tau) \otimes \mathsf{fork}^\tau_{\tau\tau}.$$

Thus we can rewrite equation (7.2) as

$$c^\lambda_{\mathsf{ST}} = c^*_{\mathsf{U}}(1_\lambda \otimes (1_\tau \otimes \mathsf{spot}^\emptyset_\tau))(c_{\mathsf{T}'} \otimes 1_\tau) - c^*_{\mathsf{U}}(1_\lambda \otimes (\mathsf{spot}^\emptyset_\tau \otimes 1_\tau))(c_{\mathsf{T}'} \otimes 1_\tau) + c^*_{\mathsf{U}}(1_\lambda \otimes (\mathsf{bar}(\tau) \otimes \mathsf{fork}^\tau_{\tau\tau}))(c_{\mathsf{T}'} \otimes 1_\tau)$$

Now note that the last two terms belong to $h^\tau_{(W,P)} 1_\mu$ and the first one can be rewritten as

$$c^{\lambda\tau}_{\mathsf{UV}}(1_{\mu-\tau} \otimes \mathsf{gap}(\tau)).$$

where $c^{\lambda\tau}_{\mathsf{UV}} \in h^\tau_{(W,P)} 1_\mu$. This shows that the quotient is isomorphic to $h^\tau_{(W,P)} 1_\mu (1_{\mu-\tau} \otimes \mathsf{gap}(\tau))$. As it is projective, it splits and we have

$$e_\tau h_{(W,P)} 1_\mu \cong h^\tau_{(W,P)} 1_\mu \oplus h^\tau_{(W,P)} 1_\mu (1_{\mu-\tau} \otimes \mathsf{gap}(\tau)),$$

thus proving the claim.

**Case 2**. We now assume that $\tau \in \mathrm{Add}(\mu)$. We claim that in this case

$$e_\tau h_{(W,P)} 1_\mu \cong h_{(W,P)^\tau} 1_{\mu\tau\downarrow_\tau} \langle 1 \rangle.$$

To see this, we will show that

$$e_\tau h_{(W,P)} 1_\mu \cong h^\tau_{(W,P)} 1_{\mu\tau}(1_\mu \otimes \mathsf{spot}^\tau_\emptyset).$$

Indeed for any $c^\lambda_{\mathsf{ST}} \in e_\tau h_{(W,P)} 1_\mu$ we have that $\mathsf{S} = X^\pm_\tau(\mathsf{S}')$. If $\mathsf{S} = X^-_\tau(\mathsf{S}')$ then we must have $\tau \in \mathrm{Add}(\lambda)$ and we define $\mathsf{U} = X^+_\tau(\mathsf{S}')$ and $\mathsf{V} = A^+_\tau(\mathsf{T})$. If $\mathsf{S} = X^+_\tau(\mathsf{S}')$ then we must have $\tau \in \mathrm{Rem}(\lambda)$ and we define $\mathsf{U} = \mathsf{S}$ and $\mathsf{V} = R^+_\tau(\mathsf{T})$. Then in both cases we can write

$$c^\lambda_{\mathsf{ST}} = c_{\mathsf{UV}}(1_\mu \otimes \mathsf{spot}^\tau_\emptyset).$$

Note that $c_{\mathsf{UV}} \in h^\tau_{(W,P)} 1_{\mu\tau}$ so we're done.

**Case 3**. It remains to consider the case that $\tau \notin \mathrm{Rem}(\mu)$ or $\mathrm{Add}(\mu)$. We now consider the case that $\tau \notin \mathrm{Rem}(\mu)$ or $\mathrm{Add}(\mu)$, but there exists $\sigma \in \mathrm{Rem}(\mu)$ with $m(\sigma, \tau) = 3$. Note that we can assume that $\tau \in \mathrm{Rem}(\mu - \sigma)$ as otherwise we would be in Case 2. This will serve as the base case for the inductive step in Case 4. We claim that in this case

$$e_\tau h_{(W,P)} 1_\mu \cong h_{(W,P)^\tau} 1_{(\mu-\sigma)\downarrow_\tau} \langle 1 \rangle.$$

To see this, we will show that

$$e_\tau h_{(W,P)} 1_\mu = h^\tau_{(W,P)} 1_{\mu-\sigma}(1_{\mu-\sigma} \otimes \mathsf{spot}^\emptyset_\sigma)$$

Our assumptions that $\sigma \in \mathrm{Rem}(\mu)$ and $\tau \in \mathrm{Rem}(\nu)$ imply that there are two cases to consider: $\sigma \in \mathrm{Rem}(\lambda)$ and $\tau \in \mathrm{Add}(\lambda)$ versus $\sigma \in \mathrm{Add}(\lambda)$ and $\tau \in \mathrm{Rem}(\lambda)$. In the latter case, we have that $\mathsf{T} = A^-_\sigma(\mathsf{T}')$ and so

$$c_{\mathsf{ST}} = c^*_{\mathsf{S}}(c_{\mathsf{T}'} \otimes \mathsf{spot}^\emptyset_\sigma) = c_{\mathsf{ST}'} \otimes \mathsf{spot}^\emptyset_\sigma$$

with $c_{\mathsf{ST}'} \in h^{\tau}_{(W,P)}1_{\mu-\sigma}$ as required. In the first case, we have $\mathsf{T} = R^+_{\sigma} X^-_{\tau}(\mathsf{T}')$ and $\mathsf{S} = X^-_{\tau}(\mathsf{S}')$. Setting $\mathsf{U} = X^+_{\tau}(\mathsf{S}')$ and $\mathsf{V} = X^+_{\tau}(\mathsf{T}')$, we can write

$$c_{\mathsf{ST}} = c^*_{\mathsf{U}}(1_{\lambda-\sigma} \otimes \mathsf{trid}^{\sigma}_{\sigma\tau\sigma} \otimes \mathsf{spot}^{\tau}_{\emptyset})(c_{\mathsf{V}} \otimes 1_{\sigma}) = -c_{\mathsf{UV}} \otimes \mathsf{spot}^{\emptyset}_{\sigma}$$

where the last equality follows by applying $1_{\sigma\tau} \otimes \mathsf{spot}^{\emptyset}_{\sigma}$ to the $\sigma\tau$-nullbraid relations. Again we have that $c_{\mathsf{UV}} \in h^{\tau}_{(W,P)}1_{\mu-\sigma}$ so we are done.

**Case 4.** If $\mu$ is not as in cases 1 to 3, then we must have $\sigma \in \mathrm{Rem}(\mu)$ with $\sigma$ and $\tau$ commuting. We will show that $e_{\tau}h_{W,P)}1_{\mu}$ is either 0 or projective-indecomposable as a left $h_{(W,P)^{\tau}}$-module. We proceed by induction on the rank of $W$. Note that as $\sigma$ and $\tau$ commute, $\sigma$ labels a node in the Dynkin diagram for $(W,P)^{\tau}$ and so it makes sense to consider $e^{(W,P)^{\tau}}_{\sigma} \in h_{(W,P)^{\tau}}$ and $(W,P)^{\tau\sigma}$. We claim that

$$e_{\tau}h_{(W,P)}1_{\mu} \cong h_{(W,P)^{\tau}}e^{(W,P)^{\tau}}_{\sigma} \otimes_{h_{(W,P)^{\sigma\tau}}} e^{(W,P)^{\sigma}}_{\tau} h_{(W,P)^{\sigma}}1_{\mu\downarrow_{\sigma}} \tag{7.3}$$

as a left $h_{(W,P)^{\tau}}$-module. Note that any basis element in $e_{\tau}h_{(W,P)}1_{\mu}$ has the form $c^{\lambda}_{\mathsf{ST}}$ for $\mathsf{S} \in \mathrm{Path}(\lambda, \mathsf{t}_{\nu})$, $\mathsf{T} \in \mathrm{Path}(\lambda, \mathsf{t}_{\mu})$ with $\tau \in \mathrm{Rem}(\nu)$ and $\sigma \in \mathrm{Rem}(\mu)$. So either $\sigma \in \mathrm{Rem}(\lambda)$ or $\sigma \in \mathrm{Add}(\lambda)$ and similarly either $\tau \in \mathrm{Rem}(\lambda)$ or $\tau \in \mathrm{Add}(\lambda)$. To prove the claim, it is enough to show that any such $c^{\lambda}_{\mathsf{ST}}$ can be written as a product

$$c^{\lambda}_{\mathsf{ST}} = c^{\alpha}_{\mathsf{PQ}}c^{\beta}_{\mathsf{UV}}$$

where $c^{\alpha}_{\mathsf{PQ}} \in h^{\tau}_{(W,P)}$ and $c^{\beta}_{\mathsf{UV}} \in h^{\sigma}_{(W,P)}$. There are four distinct cases to consider. If $\sigma, \tau \in \mathrm{Rem}(\lambda)$ then $\mathsf{S} = X^+_{\tau}(\mathsf{S}'), \mathsf{T} = X^+_{\sigma}(\mathsf{T}')$ and we pick $\mathsf{P} = \mathsf{S}$, $\mathsf{Q} = \mathsf{U} = \mathsf{t}_{\lambda}$ and $\mathsf{V} = \mathsf{T}$. If $\sigma \in \mathrm{Rem}(\lambda)$ and $\tau \in \mathrm{Add}(\lambda)$ then $\mathsf{S} = X^-_{\tau}(S'), \mathsf{T} = X^+_{\sigma}(\mathsf{T}')$ and we pick $\mathsf{P} = X^+_{\tau}(\mathsf{S}')$, $\mathsf{Q} = A^+_{\tau}(\mathsf{t}_{\lambda})$, $\mathsf{U} = A^-_{\tau}(\mathsf{t}_{\lambda})$ and $\mathsf{V} = \mathsf{T}$. If $\sigma \in \mathrm{Add}(\lambda)$ and $\tau \in \mathrm{Rem}(\lambda)$ then $\mathsf{S} = X^+_{\tau}(S'), \mathsf{T} = X^-_{\sigma}(\mathsf{T}')$ and we pick $\mathsf{P} = \mathsf{S}$, $\mathsf{Q} = A^-_{\sigma}(\mathsf{t}_{\lambda})$, $\mathsf{U} = A^+_{\sigma}(\mathsf{t}_{\lambda})$ and $\mathsf{V} = X^+_{\sigma}(\mathsf{T}')$. If $\sigma, \tau \in \mathrm{Add}(\lambda)$ then $\mathsf{S} = X^-_{\tau}(S'), \mathsf{T} = X^-_{\sigma}(\mathsf{T}')$ and we pick $\mathsf{P} = X^+_{\tau}(\mathsf{S}')$, $\mathsf{Q} = A^+_{\tau}A^-_{\sigma}(\mathsf{t}_{\lambda})$, $\mathsf{U} = A^-_{\tau}A^+_{\sigma}(\mathsf{t}_{\lambda})$ and $\mathsf{V} = X^+_{\sigma}(\mathsf{T}')$. Hence we have proven equation (7.3).

By induction, $e^{(W,P)^{\sigma}}_{\tau} h_{(W,P)^{\sigma}}1_{\mu\downarrow_{\sigma}}$ is either 0, or it is a projective indecomposable $h_{(W,P)^{\sigma\tau}}$-module, say $h_{(W,P)^{\sigma\tau}}1_{\eta}$. Substituting into equation (7.3), we obtain that $e_{\tau}h_{(W,P)}1_{\mu}$ is either 0, or

$$\begin{aligned} e_{\tau}h_{(W,P)}1_{\mu} &\cong h_{(W,P)^{\tau}}e^{(W,P)^{\tau}}_{\sigma} \otimes_{h_{(W,P)^{\sigma\tau}}} h_{(W,P)^{\sigma\tau}}1_{\eta} \\ &\cong h_{(W,P)^{\tau}}1_{\varphi_{\sigma}(\eta)} \end{aligned}$$

which is projective indecomposable. □

**Lemma 7.4.** *There is a graded* $(h_{(W,P)^{\tau}}, h_{(W,P)^{\tau}})$*-bimodule homomorphism*

$$\psi : e_{\tau}h_{(W,P)}e_{\tau} \to h_{(W,P)^{\tau}}\langle 2\rangle.$$

*Proof.* The module $e_{\tau}h_{(W,P)}e_{\tau}$ has basis given by

$$B = \{c^{\lambda}_{\mathsf{ST}} \mid \mathsf{S} \in \mathrm{Path}(\lambda, \mathsf{t}_{\mu}), \mathsf{T} \in \mathrm{Path}(\lambda, \mathsf{t}_{\nu}), \text{ with } \lambda \in \mathscr{P}_{(W,P)} \text{ and } \mu, \nu \in \mathscr{P}^{\tau}_{(W,P)}\}$$

which decomposes as a disjoint union $\{c^{\lambda}_{\mathsf{ST}} \in B \mid \tau \in \mathrm{Rem}(\lambda)\} \sqcup \{c^{\lambda}_{\mathsf{ST}} \in B \mid \tau \in \mathrm{Add}(\lambda)\}$. By Theorem 5.9, we have a $(h_{(W,P)^{\tau}}, h_{(W,P)^{\tau}})$-bimodule isomorphism

$$h_{(W,P)^{\tau}} \cong h^{\tau}_{(W,P)} = \langle c^{\lambda}_{\mathsf{ST}} \in B \mid \tau \in \mathrm{Rem}(\lambda)\rangle \subseteq e_{\tau}h_{(W,P)}e_{\tau}.$$

Following the proof of case 1 of Theorem 7.3, we see that

$$e_{\tau}h_{(W,P)}e_{\tau}/\Bbbk\{c^{\lambda}_{\mathsf{ST}} \in B \mid \tau \in \mathrm{Rem}(\lambda)\} \cong h^{\tau}_{(W,P)}(\textstyle\sum_{\mu \in \mathscr{P}^{\tau}_{(W,P)}} 1_{\mathsf{t}_{\mu-\tau}} \otimes \mathsf{gap}(\tau))$$

as left $h_{(W,P)^{\tau}}$-modules and similarly, flipping diagrams across the horizontal axis we get that

$$e_{\tau}h_{(W,P)}e_{\tau}/\Bbbk\{c^{\lambda}_{\mathsf{ST}} \in B \mid \tau \in \mathrm{Rem}(\lambda)\} \cong (\textstyle\sum_{\mu \in \mathscr{P}^{\tau}_{(W,P)}} 1_{\mathsf{t}_{\mu-\tau}} \otimes \mathsf{gap}(\tau))h^{\tau}_{(W,P)}$$

as right $h_{(W,P)^{\tau}}$-modules. This shows that

$$e_{\tau}h_{(W,P)}e_{\tau}/\Bbbk\{c^{\lambda}_{\mathsf{ST}} \in B \mid \tau \in \mathrm{Rem}(\lambda)\} \cong h^{\tau}_{(W,P)}\langle 2\rangle \cong h_{(W,P)^{\tau}}\langle 2\rangle$$

as $(h_{(W,P)^{\tau}}, h_{(W,P)^{\tau}})$-bimodules as required. □

Let $M$ be a right $h_{(W,P)}$-module. Define the right $h_{(W,P)}$-module $M^*$ by $M^* = \mathrm{Hom}_{\Bbbk}(M, \Bbbk)$ as a vector space and for $f \in M^*$, $a \in h_{(W,P)}$ we define $fa \in M^*$ by $(fa)(m) = f(ma^*)$ where $a^*$ is the dual element in $h_{(W,P)}$ (given by flipping a diagram across the horizontal axis).

**Theorem 7.5.** *For $M$ an $h_{(W,P)}$-module we have that $G^{\tau}(M^*) \cong (G^{\tau}(M))^*$.*

*Proof.* We have that

$$G^{\tau}(M^*) = M^* \otimes_{h_{(W,P)^{\tau}}} e_{\tau} h_{(W,P)}\langle -1\rangle \qquad (G^{\tau}(M))^* = \mathrm{Hom}_{\Bbbk}(M \otimes_{h_{(W,P)^{\tau}}} e_{\tau} h_{(W,P)}\langle -1\rangle, \Bbbk)$$

We define $\vartheta : G^{\tau}(M^*) \to (G^{\tau}(M))^*$ by setting $f \otimes a \mapsto \vartheta_{f\otimes a}$ for $f \in M^*$ and $a \in e_{\tau} h_{(W,P)}\langle -1\rangle$ where

$$\vartheta_{f\otimes a}(m \otimes b) = f(m\psi(ba^*))$$

for $m \in M$ and $b \in e_{\tau} h_{(W,P)}\langle -1\rangle$. Note that this makes sense because $ba^* \in e_{\tau} h_{(W,P)} e_{\tau}\langle -2\rangle$ and so $\psi(ba^*) \in h_{(W,P)^{\tau}}$. Also $\vartheta$ is well-defined as $\psi$ is a bimodule homomorphism.

We now show that $\vartheta$ is a $h_{(W,P)}$-homomorphism. On one hand, we have

$$\vartheta_{(f\otimes a)x}(m \otimes b) = \vartheta_{f\otimes ax}(m \otimes b) = f(m\psi(bx^*a^*)).$$

On the other hand, we have

$$(\vartheta_{(f\otimes a)}x)(m \otimes b) = \vartheta_{f\otimes a}((m \otimes b)x^*) = \vartheta_{f\otimes a}(m \otimes bx^*) = f(m\psi(bx^*a^*))$$

as required. We now show that $\vartheta$ is a vector space isomorphism. It is enough to check that $\vartheta : G^{\tau}(M^*)1_{\mathsf{t}_\mu} \to (G^{\tau}(M))^* 1_{\mathsf{t}_\mu}$ is a vector space isomorphism for each $\mu \in \mathscr{P}_{(W,P)}$. We have

$$\begin{aligned} G^{\tau}(M^*)1_{\mathsf{t}_\mu} &= M^* \otimes_{h_{(W,P)^{\tau}}} e_{\tau} h_{(W,P)}\langle -1\rangle 1_{\mathsf{t}_\mu} \\ (G^{\tau}(M))^* 1_{\mathsf{t}_\mu} &= \mathrm{Hom}_{\Bbbk}(M \otimes_{h_{(W,P)^{\tau}}} e_{\tau} h_{(W,P)}\langle -1\rangle, \Bbbk) 1_{\mathsf{t}_\mu} \\ &= \mathrm{Hom}_{\Bbbk}(M \otimes_{h_{(W,P)^{\tau}}} e_{\tau} h_{(W,P)}\langle -1\rangle 1_{\mathsf{t}_\mu}, \Bbbk) \\ &= (M \otimes_{h_{(W,P)^{\tau}}} e_{\tau} h_{(W,P)}\langle -1\rangle 1_{\mathsf{t}_\mu})^* \end{aligned}$$

We have seen in the proof of Theorem 7.3 that $e_{\tau} h_{(W,P)} 1_{\mathsf{t}_\mu}$ is either zero or isomorphic to (possibly two shifted copies of) $h_{(W,P)^{\tau}} 1_{\mathsf{t}_\nu}$ for some $\nu \in \mathscr{P}_{(W,P)^{\tau}}$. So it is enough to note that

$$M^* 1_{\mathsf{t}_\nu} = M^* \otimes_{h_{(W,P)^{\tau}}} h_{(W,P)^{\tau}} 1_{\mathsf{t}_\nu} \cong (M \otimes_{h_{(W,P)^{\tau}}} h_{(W,P)^{\tau}} 1_{\mathsf{t}_\nu})^* = (M 1_{\mathsf{t}_\nu})^*$$

as required. □

Using our induction functor, we will relate (sequences of) $h_{(W,P)^{\tau}}$-modules labelled by $\lambda \in \mathscr{P}_{(W,P)^{\tau}}$ with (sequences of) $h_{(W,P)}$-modules labelled by

$$\lambda^+ := \varphi_{\tau}(\lambda) \quad \text{ and } \quad \lambda^- := \varphi_{\tau}(\lambda) - \tau.$$

We note that this is the typical Kazhdan–Lusztig "doubling-up" that we expect.

**Proposition 7.6.** *For each $\lambda \in \mathscr{P}_{(W,P)^{\tau}}$, we have $G^{\tau}(P(\lambda)) = P(\lambda^+)\langle -1\rangle$.*

*Proof.* Recall that $(W,P)$ is a simply laced Hermitian symmetric pair. By Theorem 7.2, the projective indecomposable modules are $P(\lambda) = 1_{\mathsf{t}_\lambda} h_{(W,P)}$ for $\lambda \in \mathscr{P}_{(W,P)}$. Therefore

$$G^{\tau}(P(\lambda)) = 1_{\mathsf{t}_\lambda} h_{(W,P)^{\tau}} \otimes_{h_{(W,P)^{\tau}}} e_{\tau} h_{(W,P)}\langle -1\rangle = 1_{\varphi_{\tau}(\mathsf{t}_\lambda)} h_{(W,P)}\langle -1\rangle = P(\lambda^+)\langle -1\rangle$$

as required. □

**Proposition 7.7.** *For each $\mu \in \mathscr{P}_{(W,P)^{\tau}}$, we have*

$$0 \to \Delta(\mu^-) \to G^{\tau}(\Delta(\mu)) \to \Delta(\mu^+)\langle -1\rangle \to 0$$

*Proof.* We have an exact sequence

$$0 \to h^{<\mu}_{(W,P)} \to P(\mu) \to \Delta(\mu) \to 0$$

where $h^{<\mu}_{(W,P)} = \sum_{\nu<\mu} 1_\mu h_{(W,P)} 1_\nu h_{(W,P)}$. The modules $P(\mu)$ and $h^{<\mu}_{(W,P)}$ have bases

$$\{1_\mu c^\nu_{\mathsf{ST}} \mid \mathsf{S},\mathsf{T} \in \mathrm{Path}(\nu,-), \nu \leqslant \mu\} \qquad \{1_\mu c^\nu_{\mathsf{ST}} \mid \mathsf{S},\mathsf{T} \in \mathrm{Path}(\nu,-), \nu < \mu\}$$

respectively. Since $G^{\tau}$ is exact, we obtain an exact sequence

$$0 \to G^{\tau}(h^{<\mu}_{(W,P)}) \to G^{\tau}(P(\mu)) \to G^{\tau}(\Delta(\mu)) \to 0$$

where $G^{\tau}(P(\mu)) \cong P(\mu^{+})\langle -1\rangle$. Therefore $G^{\tau}(\Delta(\mu)) = P(\mu^{+})\langle -1\rangle / G^{\tau}(h^{<\mu}_{(W,P)})$ has basis given by

$$\{c_{\mathsf{U}}\langle -1\rangle, c_{\mathsf{V}} \otimes \mathsf{spot}^{\tau}_{\emptyset}\langle -1\rangle \mid \mathsf{U} \in \mathrm{Path}(\mu^{+}, -), \mathsf{V} \in \mathrm{Path}(\mu^{-}, -)\}.$$

It is clear that, as a right $h_{(W,P)}$-module

$$\{c_{\mathsf{V}} \otimes \mathsf{spot}^{\tau}_{\emptyset}\langle -1\rangle \mid \mathsf{V} \in \mathrm{Path}(\mu^{-}, -)\}$$

is a submodule of $G^{\tau}(\Delta(\mu))$ isomorphic to $\Delta(\mu^{-})$ and the quotient is isomorphic to $\Delta(\mu^{+})\langle -1\rangle$. □

7.2. **Koszulity.** We are now able to use the ideas of the previous section in order to prove that $h_{(W,P)}$ is standard Koszul. First, we continue to assume that $(W, P)$ is simply laced.

**Definition 7.8.** *For $\lambda, \mu \in \mathscr{P}_{(W,P)}$, we define polynomials $p_{\lambda,\mu}(q)$ inductively on the rank and Bruhat order as follows. We set $p_{\lambda,\lambda}(q) = 1$ and for $\lambda \not\subseteq \mu$ we set $p_{\lambda,\mu}(q) = 0$. If $\lambda \subset \mu$, pick $\tau$ such that $\tau \in \mathrm{Rem}(\lambda)$. We set*

$$p_{\lambda,\mu}(q) = \begin{cases} p_{\lambda\downarrow_{\tau},\mu\downarrow_{\tau}}(q) + q \times p_{\lambda-\tau,\mu}(q) & \text{if } \tau \in \mathrm{Rem}(\mu); \\ q \times p_{\lambda-\tau,\mu}(q) & \text{if } \tau \notin \mathrm{Rem}(\mu). \end{cases}$$

*We write $p_{\lambda,\mu}(q) = \sum_{n \geqslant 0} p^{(n)}_{\lambda,\mu} q^{n}$.*

**Theorem 7.9.** *For $\lambda \in \mathscr{P}_{(W,P)}$, we have an exact sequence*

$$\cdots \to P_2(\lambda) \to P_1(\lambda) \to P_0(\lambda) \to \Delta(\lambda) \to 0$$

*where $P_0(\lambda) = P(\lambda)$ and for $n \geqslant 1$ we have $P_n(\lambda) = \oplus_{\mu \in \mathcal{P}_{(W,P)}} p^{(n)}_{\lambda,\mu} P(\mu)\langle n\rangle$.*

*Proof.* We proceed by induction on the rank of $W$ and the Bruhat order on $\mathscr{P}_{(W,P)}$. If $\lambda = \varnothing$ is the minimal element in the Bruhat order, then $\Delta(\varnothing) = P(\varnothing)$ and we are done. Assume $\varnothing \neq \lambda \in \mathscr{P}_{(W,P)}$, then there exists some $\tau \in \mathrm{Rem}(\lambda)$ and we have that $\lambda - \tau \in \mathscr{P}_{(W,P)}$ and $\lambda\downarrow_{\tau} \in \mathscr{P}_{(W,P)^{\tau}}$. By induction we have exact sequences,

$$\cdots \to P_2(\lambda-\tau) \to P_1(\lambda-\tau) \to P_0(\lambda-\tau) \to \Delta(\lambda-\tau) \to 0$$
$$\cdots \to P_2(\lambda\downarrow_{\tau}) \to P_1(\lambda\downarrow_{\tau}) \to P_0(\lambda\downarrow_{\tau}) \to \Delta(\lambda\downarrow_{\tau}) \to 0$$

in $h_{(W,P)}$−mod and $h_{(W,P)^{\tau}}$−mod respectively. Applying the induction functor $G^{\tau}$ to the latter sequence, and lifting the injective homomorphism from Proposition 7.7 we obtain a commutative diagram with exact rows.

$$\begin{array}{ccccccccc} \cdots \longrightarrow & P_2(\lambda-\tau) & \longrightarrow & P_1(\lambda-\tau) & \longrightarrow & P_0(\lambda-\tau) & \longrightarrow & \Delta(\lambda-\tau) & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & G^{\tau}(P_2(\lambda\downarrow_{\tau})) & \longrightarrow & G^{\tau}(P_1(\lambda\downarrow_{\tau})) & \longrightarrow & G^{\tau}(P_0(\lambda\downarrow_{\tau})) & \longrightarrow & G^{\tau}(\Delta(\lambda\downarrow_{\tau})) & \longrightarrow 0 \end{array}$$

Taking the total complex of this double complex (that is, summing over the dotted lines) and then taking the quotient by the complex

$$\ldots \to 0 \to 0 \to \Delta(\lambda-\tau) \to \Delta(\lambda-\tau) \to 0$$

we obtain

$$\cdots \longrightarrow G^{\tau}(P_2(\lambda\downarrow_{\tau})) \oplus P_1(\lambda-\tau) \longrightarrow G^{\tau}(P_1(\lambda\downarrow_{\tau})) \oplus P_0(\lambda-\tau) \longrightarrow G^{\tau}(P_0(\lambda\downarrow_{\tau})) \longrightarrow \Delta(\lambda)\langle -1\rangle \to 0.$$

We have $G^{\tau}(P_0(\lambda\downarrow_{\tau})) = P(\lambda)\langle -1\rangle$. By induction, for $n \geqslant 1$ we have that

$$\begin{aligned} & G^{\tau}(P_n(\lambda\downarrow_{\tau})) \oplus P_{n-1}(\lambda-\tau) \\ = & \bigoplus_{\mu\downarrow_{\tau} \in \mathscr{P}_{(W,P)^{\tau}}} p^{(n)}_{\lambda\downarrow_{\tau},\mu\downarrow_{\tau}} G^{\tau}(P(\mu\downarrow_{\tau})\langle n\rangle) \bigoplus_{\mu \in \mathscr{P}_{(W,P)}} p^{(n-1)}_{\lambda-\tau,\mu} P(\mu)\langle n-1\rangle \end{aligned}$$

$$= \bigoplus_{\mu\downarrow_{\tau} \in \mathscr{P}_{(W,P)^{\tau}}} p^{(n)}_{\lambda\downarrow_{\tau},\mu\downarrow_{\tau}} P(\mu)\langle n-1\rangle \bigoplus_{\mu \in \mathscr{P}_{(W,P)}} p^{(n-1)}_{\lambda-\tau,\mu} P(\mu)\langle n-1\rangle$$

$$= \bigoplus_{\mu \in \mathscr{P}^{\tau}_{(W,P)}} (p^{(n)}_{\lambda\downarrow_{\tau},\mu\downarrow_{\tau}} + p^{(n-1)}_{\lambda-\tau,\mu}) P(\mu)\langle n-1\rangle \bigoplus_{\mu \in \mathscr{P}_{(W,P)} \setminus \mathscr{P}^{\tau}_{(W,P)}} p^{(n-1)}_{\lambda-\tau,\mu} P(\mu)\langle n-1\rangle$$

$$= \bigoplus_{\mu \in \mathscr{P}_{(W,P)}} p^{(n)}_{\lambda,\mu} P(\mu)\langle n-1\rangle$$

where the last equality follows by the definition of $p_{\lambda,\mu}(q)$. Thus we obtain an exact sequence

$$\cdots \to P_2(\lambda)\langle -1\rangle \to P_1(\lambda)\langle -1\rangle \to P_0(\lambda)\langle -1\rangle \to \Delta(\lambda)\langle -1\rangle \to 0.$$

Applying a degree shift $\langle 1\rangle$ gives the required linear projective resolution for $\Delta(\lambda)$. □

**Corollary 7.10.** *Let $(W,P)$ be any Hermitian symmetric pair. The algebra $h_{(W,P)}$ is standard Koszul.*

*Proof.* Using Theorem 4.24, it is enough to consider the simply laced types. The algebra $h_{(W,P)}$ is graded quasi-hereditary algebra with (right) standard modules $\Delta(\lambda)$; the linear projective resolutions of these modules are given in Theorem 7.9. Twisting with the anti-automorphism $*$ we also get that its left standard modules have linear projective resolutions. Therefore $h_{(W,P)}$ is Koszul by [ADL03, Theorem 1]. □

**Corollary 7.11.** *Let $(W,P)$ be any Hermitian symmetric pair. For $\mu \in \mathscr{P}_{(W,P)}$, we have that the radical filtration of $\Delta(\mu)$ coincides with the grading filtration*

$$\Delta(\mu) = \Delta_{\geqslant 0}(\mu) \supset \Delta_{\geqslant 1}(\mu) \supset \Delta_{\geqslant 2}(\mu) \supset \dots$$

*where we define $\Delta_{\geqslant k}(\mu) = \{c_{\mathsf{S}} \mid \mathsf{S} \in \mathrm{Path}(\lambda, \mathsf{t}_{\mu}), \deg(\mathsf{S}) \geqslant k\}$.*

*Proof.* We have that $h_{(W,P)}$ is standard Koszul by Corollary 7.10. That the radical and grading series coincide follows from [BGS96, Proposition 2.4.1]. □

**Acknowledgements.** *The first and third authors are grateful for funding from EPSRC grant EP/V00090X/1 and the Royal Commission for the Exhibition of 1851, respectively. The authors are very thankful for the referee's careful reading of an earlier version of this manuscript.*

## References

[ADL03] I. Ágoston, V. Dlab, and E. Lukács, *Quasi-hereditary extension algebras*, Algebr. Represent. Theory **6** (2003), no. 1, 97–117.

[Bai] J. Baine, *(Co)minuscule Hecke categories*, arXiv:2409.16131, preprint.

[BCH23] C. Bowman, A. Cox, and A. Hazi, *Path isomorphisms between quiver Hecke and diagrammatic Bott-Samelson endomorphism algebras*, Adv. Math. **429** (2023), Paper No. 109185, 106.

[BCHM22] C. Bowman, A. Cox, A. Hazi, and D. Michailidis, *Path combinatorics and light leaves for quiver Hecke algebras*, Math. Z. **300** (2022), no. 3, 2167–2203.

[BDD+a] C. Bowman, A. Dell'Arciprete, M. De Visscher, A. Hazi, R. Muth, and C. Stroppel, *Faithful covers of Khovanov arc algebras*, arXiv:2411.15788, preprint.

[BDD+b] C. Bowman, A. Dell'Arciprete, M. De Visscher, A. Hazi, R. Muth, and C. Stroppel, *Quiver presentations and Schur–Weyl duality for Khovanov arc algebras*, arXiv:2411.15520, preprint.

[BDF+25] C. Bowman, M. De Visscher, N. Farrell, A. Hazi, and E. Norton, *Oriented Temperley–Lieb algebras and combinatorial Kazhdan–Lusztig theory*, Canadian Journal of Mathematics (2025), 1–43.

[BDHS] C. Bowman, M. De Visscher, A. Hazi, and C. Stroppel, *Quiver presentations and isomorphisms of Hecke categories and Khovanov arc algebras*, arXiv:2309.13695, preprint.

[BDHS24] C. Bowman, M. De Visscher, A. Hazi, and C. Stroppel, *Dyck combinatorics in p-Kazhdan-Lusztig theory*, Sém. Lothar. Combin. **91B** (2024), Art. 17, 12.

[Ben98] D. J. Benson, *Representations and cohomology. I*, second ed., Cambridge Studies in Advanced Mathematics, vol. 30, Cambridge University Press, Cambridge, 1998, Basic representation theory of finite groups and associative algebras.

[BGS96] A. Beilinson, V. Ginzburg, and W. Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9** (1996), no. 2, 473–527.

[BHN22] C. Bowman, A. Hazi, and Emily Norton, *The modular Weyl-Kac character formula*, Math. Z. **302** (2022), no. 4, 2207–2232.

[Boe88] B. Boe, *Kazhdan-Lusztig polynomials for Hermitian symmetric spaces*, Trans. Amer. Math. Soc. **309** (1988), no. 1, 279–294.
[Bow25] C. Bowman, *Diagrammatic algebra*, Universitext, Springer, Cham, 2025.
[Bre07] F. Brenti, *Parabolic Kazhdan-Lusztig R-polynomials for Hermitian symmetric pairs*, J. Algebra **318** (2007), no. 1, 412–429.
[Bre09] ———, *Parabolic Kazhdan-Lusztig polynomials for Hermitian symmetric pairs*, Trans. Amer. Math. Soc. **361** (2009), no. 4, 1703–1729.
[Bru03] J. Brundan, *Kazhdan-Lusztig polynomials and character formulae for the Lie superalgebra $gl(m|n)$*, J. Amer. Math. Soc. **16** (2003), no. 1, 185–231.
[BS10] J. Brundan and C. Stroppel, *Highest weight categories arising from Khovanov's diagram algebra. II. Koszulity*, Transform. Groups **15** (2010), no. 1, 1–45.
[BS11a] ———, *Highest weight categories arising from Khovanov's diagram algebra I: cellularity*, Mosc. Math. J. **11** (2011), no. 4, 685–722, 821–822.
[BS11b] ———, *Highest weight categories arising from Khovanov's diagram algebra III: category $\mathcal{O}$*, Represent. Theory **15** (2011), 170–243.
[BS12a] ———, *Gradings on walled Brauer algebras and Khovanov's arc algebra*, Adv. Math. **231** (2012), no. 2, 709–773.
[BS12b] ———, *Highest weight categories arising from Khovanov's diagram algebra IV: the general linear supergroup*, J. Eur. Math. Soc. (JEMS) **14** (2012), no. 2, 373–419.
[BS24] ———, *Semi-infinite highest weight categories*, Mem. Amer. Math. Soc. **293** (2024), no. 1459, vii+152.
[BW] S. Barmeier and Z. Wang, *$A_\infty$ deformations of extended Khovanov arc algebras and Stroppel's conjecture*, arXiv:2211.03354, preprint.
[Car35] E. Cartan, *Sur les domaines bornés homogènes de l'espace den variables complexes*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg **11** (1935), 116–162.
[CD11] A. Cox and M. De Visscher, *Diagrammatic Kazhdan-Lusztig theory for the (walled) Brauer algebra*, J. Algebra **340** (2011), 151–181.
[CIS88] D. Collingwood, R. Irving, and B. Shelton, *Filtrations on generalized Verma modules for Hermitian symmetric pairs*, J. Reine Angew. Math. **383** (1988), 54–86.
[CPS88] E. Cline, B. Parshall, and L. Scott, *Finite-dimensional algebras and highest weight categories*, J. Reine Angew. Math. **391** (1988), 85–99.
[Deo87] V. V. Deodhar, *On some geometric aspects of Bruhat orderings II. The parabolic analogue of Kazhdan–Lusztig polynomials*, J. Algebra **111** (1987), no. 2, 483–506.
[EHP14] T. Enright, M. Hunziker, and A. Pruett, *Diagrams of Hermitian type, highest weight modules, and syzygies of determinantal varieties*, Symmetry: representation theory and its applications, Progr. Math., vol. 257, Birkhäuser/Springer, New York, 2014, pp. 121–184.
[EK23] B. Elias and H. Ko, *A singular Coxeter presentation*, Proc. Lond. Math. Soc. (3) **126** (2023), no. 3, 923–996.
[EKLP] B. Elias, H. Ko, N. Libedinsky, and L. Patimo, *Singular light leaves*, arXiv:2401.03053, preprint.
[EKLP24] ———, *On reduced expressions for core double cosets*, arXiv:2402.08673, preprint.
[Eli16] B. Elias, *Thicker Soergel calculus in type A*, Proc. Lond. Math. Soc. (3) **112** (2016), no. 5, 924–978.
[EMTW20] B. Elias, S. Makisumi, U. Thiel, and G. Williamson, *Introduction to Soergel bimodules*, RSME Springer Series, vol. 5, Springer, Cham, [2020] ©2020.
[ES87] T. Enright and B. Shelton, *Categories of highest weight modules: applications to classical Hermitian symmetric pairs*, Mem. Amer. Math. Soc. **67** (1987), no. 367, iv+94.
[ES16a] M. Ehrig and C. Stroppel, *2-row Springer fibres and Khovanov diagram algebras for type D*, Canad. J. Math. **68** (2016), no. 6, 1285–1333.
[ES16b] ———, *Koszul gradings on Brauer algebras*, Int. Math. Res. Not. IMRN (2016), no. 13, 3970–4011.
[ES17] ———, *On the category of finite-dimensional representations of $\mathrm{OSp}(r|2n)$: Part I*, Representation theory—current trends and perspectives, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2017, pp. 109–170.
[EW14] B. Elias and G. Williamson, *The Hodge theory of Soergel bimodules*, Ann. of Math. (2) **180** (2014), no. 3, 1089–1136.
[EW16] ———, *Soergel calculus*, Represent. Theory **20** (2016), 295–374.
[Fie10] P. Fiebig, *The multiplicity one case of Lusztig's conjecture*, Duke Math. J. **153** (2010), no. 3, 551–571.
[Fie12] ———, *An upper bound on the exceptional characteristics for Lusztig's character formula*, J. Reine Angew. Math. **673** (2012), 1–31.
[GJW23] J. Gibson, L. T. Jensen, and G. Williamson, *Calculating the p-canonical basis of Hecke algebras*, Transform. Groups **28** (2023), no. 3, 1121–1148.
[HM10] J. Hu and A. Mathas, *Graded cellular bases for the cyclotomic Khovanov–Lauda–Rouquier algebras of type A*, Adv. Math. **225** (2010), no. 2, 598–642.
[JW17] L. Jensen and G. Williamson, *The p-canonical basis for Hecke algebras*, Categorification and higher representation theory, Contemp. Math., vol. 683, Amer. Math. Soc., Providence, RI, 2017, pp. 333–361.
[KX98] S. König and C. Xi, *On the structure of cellular algebras*, Algebras and modules, II (Geiranger, 1996), CMS Conf. Proc., vol. 24, Amer. Math. Soc., Providence, RI, 1998, pp. 365–386.

[LM21] M. Lanini and P. McNamara, *Singularities of Schubert varieties within a right cell*, SIGMA Symmetry Integrability Geom. Methods Appl. **17** (2021), Paper No. 070, 9.

[Los16] I. Losev, *Proof of Varagnolo–Vasserot conjecture on cyclotomic categories $\mathcal{O}$*, Selecta Math. **22** (2016), no. 2, 631–668.

[LS81] A. Lascoux and M. Schützenberger, *Polynômes de Kazhdan & Lusztig pour les grassmanniennes*, Young tableaux and Schur functors in algebra and geometry (Toruń, 1980), Astérisque, vol. 87, Soc. Math. France, Paris, 1981, pp. 249–266.

[LW17] N. Libedinsky and G. Williamson, *A non-perverse Soergel bimodule in type A*, C. R. Math. Acad. Sci. Paris **355** (2017), no. 8, 853–858.

[LW18] G. Lusztig and G. Williamson, *Billiards and tilting characters for* $\mathrm{SL}_3$, SIGMA Symmetry Integrability Geom. Methods Appl. **14** (2018), 015, 22 pages.

[LW22] N. Libedinsky and G. Williamson, *The anti-spherical category*, Adv. Math. **405** (2022), Paper No. 108509, 34.

[Mar15] P. Martin, *The decomposition matrices of the Brauer algebra over the complex field*, Trans. Amer. Math. Soc. **367** (2015), no. 3, 1797–1825.

[Pat22] L. Patimo, *Bases of the intersection cohomology of Grassmannian Schubert varieties*, J. Algebra **589** (2022), 345–400.

[Per07] N. Perrin, *Small resolutions of minuscule Schubert varieties*, Compos. Math. **143** (2007), no. 5, 1255–1312.

[Pla17] D. Plaza, *Graded cellularity and the monotonicity conjecture*, J. Algebra **473** (2017), 324–351.

[RSVV16] R. Rouquier, P. Shan, M. Varagnolo, and E. Vasserot, *Categorifications and cyclotomic rational double affine Hecke algebras*, Invent. Math. **204** (2016), no. 3, 671–786.

[RW18] S. Riche and G. Williamson, *Tilting modules and the p-canonical basis*, Astérisque (2018), no. 397, ix+184.

[Sha12] P. Shan, *Graded decomposition matrices of v-Schur algebras via Jantzen filtration*, Represent. Theory **16** (2012), 212–269.

[Soe97] W. Soergel, *Kazhdan-Lusztig polynomials and a combinatoric[s] for tilting modules*, Represent. Theory **1** (1997), 83–114.

[Ste96] J. Stembridge, *On the fully commutative elements of Coxeter groups*, J. Algebraic Combin. **5** (1996), no. 4, 353–385.

[Web17] B. Webster, *Rouquier's conjecture and diagrammatic algebra*, Forum Math. Sigma **5** (2017), e27, 71.

[Wil11] G. Williamson, *Singular Soergel bimodules*, Int. Math. Res. Not. IMRN (2011), no. 20, 4555–4632.

[Wil17] ———, *Schubert calculus and torsion explosion*, J. Amer. Math. Soc. **30** (2017), no. 4, 1023–1046, With a joint appendix with A. Kontorovich and P. J. McNamara.

[Zel83] A. Zelevinski, *Small resolutions of singularities of Schubert varieties*, Funktsional. Anal. i Prilozhen. **17** (1983), no. 2, 75–77.

Department of Mathematics, University of York, Heslington, York, UK
*Email address*: Chris.Bowman-Scargill@york.ac.uk

Department of Mathematics, City, University of London, London, UK
*Email address*: Maud.DeVisscher@city.ac.uk

School of Mathematics, University of Leeds, Leeds, UK
*Email address*: A.Hazi@leeds.ac.uk

School of Mathematics, Statistics and Actuarial Science, University of Kent, Canterbury, UK
*Email address*: E.Norton@kent.ac.uk