# Colour Equivalence Patterns and Groups on Rubik's Cubes

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**Abstract**

We introduce the concept of colour equivalence patterns on Rubik's cubes. These are patterns where: (a) the six colours are first split into two sets of three, (b) within each set those three colours must be distributed equally, and (c) the distributions of the two sets must be mirror images of one another. The complete set of such patterns is first obtained by computer programs. The patterns are then analysed from both geometrical and group theoretical perspectives, which can differ significantly. The group theory results are presented at a level appropriate for students just learning the subject.

Key words and phrases:
Rubik's cube, three-dimensional patterns, group theory

2020 Mathematics Subject Classification: 97G20, 97H40

# 1 Introduction

It is 50 years now since the Rubik's cube was invented, with several 100 million cubes sold since then. Mathematicians immediately realised that it is far more than just a puzzle though, and has a deep underlying group theoretical structure [1, 2, 3, 4]. This reveals, for example, that it is not possible to exchange two side pieces while leaving everything else in place. This property is very familiar to anyone who has ever played around with cubes for any length of time, but to provide a rigorous proof that it could never be done requires some rather sophisticated group theory. Another interesting question is the maximum number of turns needed to solve an arbitrary position. Direct computer searches show this to be 11 for the $2 \times 2 \times 2$ cube, and 20 for the $3 \times 3 \times 3$ cube [8]. For the general $n \times n \times n$ cube this so-called group diameter scales as $n^2/\log n$ [5].

There are also enthusiasts who are simply interested in a variety of pretty patterns that can be constructed, without regard to any special mathematical properties they might have. An online search easily yields several dozen such patterns, many of them very intricate, but typically presented without any associated mathematical analysis. One recent attempt to bridge this gap between very abstract mathematics on the one hand versus pretty patterns on the other introduced the concept of *colour equality patterns* [6, 7], which were defined to be patterns where all six colours are distributed exactly equally. That is, if you had two identically arranged cubes, you could rotate them with respect to one another such that any given colour on the first cube and any other colour on the second cube would coincide in all of their positions.

Figure 1a shows an example of such a colour equality pattern. We begin by arbitrarily choosing the colours red/yellow/blue (r/y/b) to constitute the front, and orange/white/green (o/w/g) the back. Then take the three side pieces around the r/y/b front corner and the three side pieces around the o/w/g back corner – which will together be known as the '3+3' side pieces – and rotate each set clockwise as shown. All six colours are clearly distributed in exactly the same way. The study of colour equality patterns as a whole is then concerned with finding all such patterns, systematically classifying them according to their various symmetries, etc. One discovery is that there are patterns having different symmetries, so different that they are mutually incompatible. That is, two patterns might each separately satisfy colour equality, but if you try to combine them, colour equality is violated. This is certainly an interesting result, but it does also mean that the set of all colour equality patterns does not constitute a group.

Next, suppose we arrange those same '3+3' side pieces around the front and back corners as in Figure 1b. The r/y/b front pieces are still rotated clockwise, but now the o/w/g back pieces are rotated counter-clockwise. Strict colour equality, between all six colours, is then no longer satisfied. However, we can observe two things: First, within each {r,y,b} and {o,w,g} set separately, colour equality is still satisfied. Second, the patterns formed by red and orange, say, are not the same, but they are mirror images of each other. And since yellow and blue were already seen to be equal to red, and white and green equal to orange, we realise that we still have a nice, symmetrical equivalence between the two colour sets.

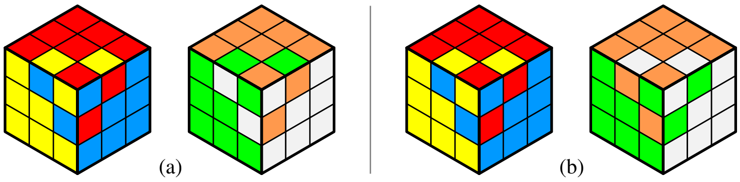


Figure 1: (a) A colour equality pattern. (b) A colour equivalence pattern. In each case the left panel shows the three faces on the front, and the right panel the three faces on the back, after the cube has been turned over. Supplementary Material at the end also includes templates to cut out and tape together to form actual three-dimensional cubes, which can then be rotated as desired to see the equality/equivalence relations between the different colours even more clearly.

Let us therefore define *colour equivalence patterns* to be patterns of this type, with {r,y,b} and {o,w,g} satisfying equality within each set, and being mirror images between the two sets. As before, we would then like to find all such patterns, systematically classify them in various ways, etc. One difference that will become apparent below is that the equivalence patterns are far more regular than the equality patterns were, with every pattern having the same symmetries, and thus being mutually compatible. From the point of view of studying different symmetries, this makes the equality patterns the more interesting set. On the other hand, precisely this mutual compatibility means that the set of all colour equivalence patterns does constitute a group.

The purpose of this article is then to derive the full set of equivalence patterns, and analyse them from both geometrical and group theoretical perspectives. At the end I also outline how the $2 \times 2 \times 2$ equality patterns can be split into two (partially overlapping) sets that do satisfy the requirements to be groups, with the details left as an exercise for readers. My hope is that both types of patterns might not only be interesting in their own right, in terms of symmetries of three-dimensional patterns, but that the resulting groups could serve as useful examples for students encountering group theory for the first time.

## 2 The Corners

We start with the $2 \times 2 \times 2$ cube consisting only of corners. The corners of all larger cubes follow exactly the same rules. The number of possible configurations, $7! \cdot 3^6 \approx 4 \cdot 10^6$, is sufficiently small that we can scan through them all with a computer and pick out the ones that satisfy the colour equivalence criterion. There turn out to be 18 such patterns, shown in Figure 2. The representation is as before, with three faces that constitute the front, then the cube is flipped over to show the three faces on the back. The mirror symmetry between red versus orange, yellow versus white, and blue versus green is again easily recognisable. The three-fold symmetry among {r,y,b} and {o,w,g} also shows the equality within the given set. Once again though, to fully appreciate the patterns it is best to construct actual 3D cubes, using the templates provided in the Supplementary Material at the end.

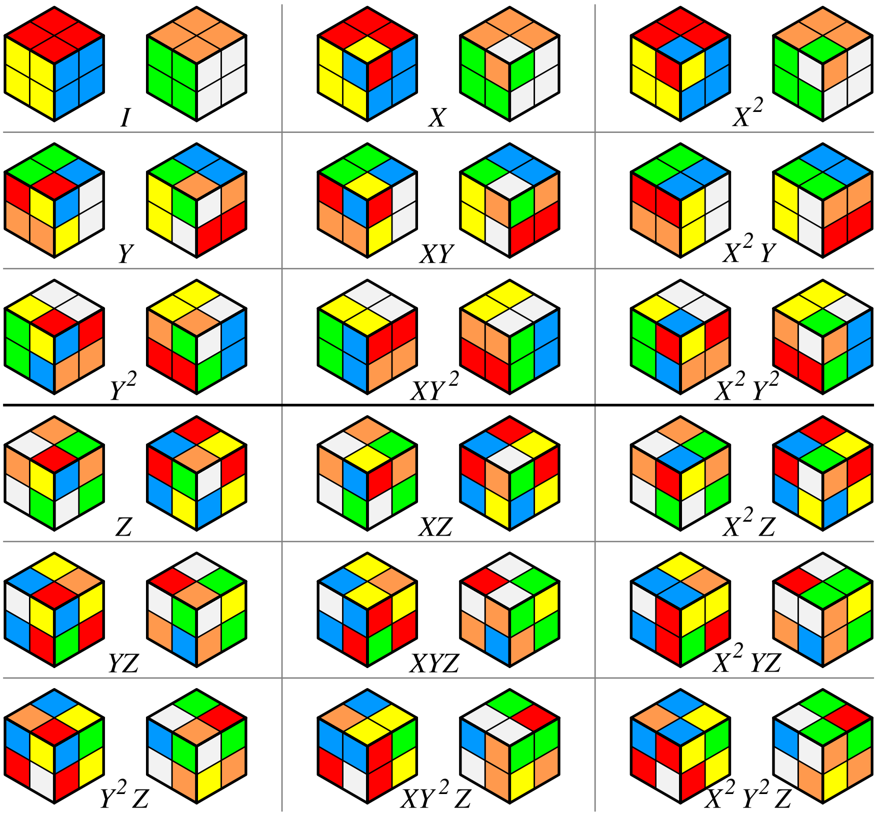


Figure 2: The 18 corner patterns, labeled according to how they are constructed from the basic actions $\{X, Y, Z\}$.

The labels in Figure 2 indicate the group structure. First, the standard solved configuration $I$ is the identity element. Let's focus next on the three patterns $X$, $Y$, and $Z$, since these allow us to generate the entire group. $X$ is easiest to describe; it simply rotates the r/y/b front corner one twist clockwise, and the o/w/g back corner one twist counter-clockwise. $Y$ is slightly more complicated, but just like $X$ it also leaves all eight corners in their original positions. The r/y/b front and o/w/g back pieces also retain their original orientation. The three corners around the r/y/b front piece each rotate one twist clockwise, the three corners around the o/w/g back piece one twist counter-clockwise. From these geometrical definitions we can immediately see that $X^3 = Y^3 = I$ and $XY = YX$, that is, $X$ and $Y$ commute.

The third pattern $Z$ has the most complicated geometrical action, but even it is relatively straightforward to understand. The front and back corners again remain in their original positions, and with their original orientation. The other six pieces switch places with their diagonally opposite partners, while maintaining the same relative orientation to one another. That is, they undergo a 180° flip as if they were a single connected object. From this definition we can again easily see that $Z^2 = I$ and also $XZ = ZX$. The commutation relation between $Y$ and $Z$ is less obvious, and indeed they don't commute. By directly carrying out the various steps involved, you should be able to convince yourself that $ZY = Y^2Z$ and similarly $ZY^2 = YZ$.

At this point, make sure also that you fully understand what combinations of this type mean. Think of $\{X, Y, Z\}$ not just as static patterns that are whatever they happen to be, but also think of them as *instructions* telling you what to do next. That is, a combination such as $XYZ$ tells you: First start with the instructions $Z$, which when applied to the original configuration gives you just the pattern called $Z$. Next apply the instructions $Y$ to this pattern $Z$, which gives you something else. Finally apply the instructions $X$ to whatever you just produced, and that is the final pattern called $XYZ$. Interpreting $\{X, Y, Z\}$ as instructions and not just as static patterns is precisely why it was so important to first clarify what exactly it is that they actually do, in terms of rotating and/or moving the various corners. You will probably find it useful also to construct actual cubes, including filling in some colours yourself on the blank templates provided, to really see how the different corners move around under combined actions like these.

Given the results $X^3 = Y^3 = Z^2 = I$ and the various commutation relations, it is straightforward to work out that the 18 elements $X^m Y^n$ and $X^m Y^n Z$, with $m, n = 0, 1, 2$, are the only possibilities. Any other combinations of these three elements could always be reduced to one of these 18 elements. And sure enough, Figure 2 consists of exactly these 18 elements. Furthermore, from the basic definitions we see that $X$ and $X^{-1} = X^2$ should be mirror images in terms of which corners rotate which way, and similarly $Y$ and $Y^{-1} = Y^2$ should be mirror images. In contrast, $Z$ is its own inverse, and correspondingly has no handedness. If you carefully compare various pairs of patterns, you find that this handedness of $X$ and $Y$ then also exactly carries over to all the combinations; for example $XY^2Z$ and $X^2YZ$ are mirror image partners, as we would expect if their $X$ versus $X^2$ and $Y^2$ versus $Y$ parts separately are mirror images.

There is one further twist to these patterns and how best to represent them: Suppose we define $W = XZ$. From a geometrical perspective this combination is needlessly complicated compared with the relative simplicity of $X$ and $Z$ separately. However, from the group theoretical perspective something rather interesting happens. Using our previous results that $XZ = ZX$ and $X^3 = Z^2 = I$, it is straightforward to obtain $W^2 = X^2$, $W^3 = Z$, $W^4 = X$, $W^5 = X^2Z$, and finally $W^6 = I$. The important ones here are $X = W^4$ and $Z = W^3$. That is, $X$ and $Z$ can both be reconstructed from the single element $W$. From an abstract group theoretical perspective, we recognise therefore that we don't need three generators $\{X, Y, Z\}$; we can also obtain the entire group from just the two generators $\{W, Y\}$.

It is important to emphasise also that this entire geometrical and group theoretical labeling and analysis was done only after the patterns had originally been obtained by the computer search. The computer program conducted its search based entirely on the original geometrical definition of colour equivalence, and simply produced the 18 patterns here. It was only afterwards that a careful examination revealed them to be a group, geometrically best described by the three elements $\{X, Y, Z\}$, but group theoretically most compactly defined by just the two elements $\{W, Y\}$.

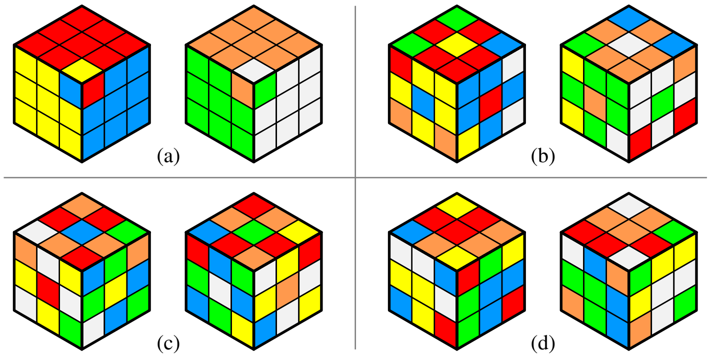


Figure 3: The centre and corner labelings for these patterns are: (a) $I_c X$, (b) $C Y$, (c) $C^2 Z$, (d) $I_c XYZ$. As noted also in the text, the labels ($I_c$, $C$, $C^2$, $I_c$) refer to the arrangements of the centres only, and constitute rotations about the r/y/b – o/w/g preferred axis. $X$ and $Y$ apply to the corners only, exactly as before. $Z$ applies to the corners as before, but additionally the arrangement of the side pieces in (c,d) is part of the new definition of $Z$.

# 3 Adding the Centres

Moving on to $3\times3\times3$ cubes, the first step is to consider how these corner patterns fit together with the centres. The centres form a rigid lattice, so there are only 24 ways it can be oriented with respect to the corner patterns. Three of these are compatible with colour equivalence, corresponding to successive rotations about the r/y/b – o/w/g preferred axis, as shown in Figure 3. In the language of group theory the Figure 3a orientation is the identity element $I_c$, then 3b is the generator $C$, 3c has the orientation $C^2$, and finally 3d is back to $C^3 = I_c$.

If we next examine the side pieces in Figure 3, in 3a and 3b they are all in their original positions. In 3c and 3d though, the same '3+3' side pieces as in Figure 1 have each been exchanged with their diagonally opposite partners. This exchange is *not* a side piece action though, which we will get to below. Instead, it is required by the corner element $Z$, which is present in both 3c and 3d. This requirement to diagonally exchange these '3+3' side pieces for the nine corner patterns containing $Z$ comes about because of the point noted in the introduction, that two side pieces cannot be exchanged while leaving everything else in place. In particular, if you try to construct any corner pattern containing $Z$ along with the side pieces as in 3a or 3b, the last two side pieces will be in the wrong positions. Adding these three diagonal exchanges gets you back to an even number of side piece switches, which can be done.

That is, the nine corner patterns that do not contain $Z$ *must* have their standard side piece arrangement as in Figures 3a and 3b, whereas the nine corner patterns that do contain $Z$ *must* have their standard side piece arrangement as in Figures 3c and 3d. The only remaining question is, how to describe this in the language of group theory? It's quite simple: we just extend the definition of the element $Z$ (or equivalently $W$, if we want to use the $\{W, Y\}$ representation). That is, $X$ and $Y$ remain as before, and act only on the corners, exactly as before. In contrast, $Z$ now acts on the corners exactly as before, but in addition also does this diagonal exchange of the '3+3' side pieces (which very conveniently is also still consistent with colour equivalence).

## 4 The Side Pieces

The number of possible configurations for the side pieces is $12! \cdot 2^{10} \approx 5 \cdot 10^{11}$, far greater than the $4 \cdot 10^6$ possibilities for the corner pieces. Nevertheless, even a number as large as this is still manageable by the direct computational approach, requiring only a few days on a reasonably powerful workstation. The results show that the 12 side pieces can be arranged in 144 different ways that are consistent with colour equivalence. The equivalent of Figure 2, that is, explicitly showing all 144 patterns, would then be rather unwieldy. However, just as the 18 patterns in Figure 2 could be constructed out of a much smaller number of generators, for the side pieces also the 144 patterns can be constructed out of a relatively small number of geometrically simple actions.

Indeed, we have already seen one of these actions before, namely the clockwise/counter-clockwise rotation of the '3+3' side pieces in Figure 1b. This clearly contributes a factor of three to our target of 144. Another action we can do is to flip these same '3+3' side pieces in place. Alternatively, we can take the other six side pieces, constituting a 'ring' that separates the front and back halves of the cube, and flip those in place. At this point we thus have $3 \cdot 2 \cdot 2 = 12$ possibilities. The remaining factor of 12 that we need to achieve our target of 144 comes from two further geometrical actions, one yielding a factor of two, and the other a factor of six. Can you think what they are? Solutions are provided in the Supplementary Material, but perhaps you can already discover these two geometrical actions yourself.

Another interesting point comes about when we consider the associated group structure. For the corners we already saw that geometrically the patterns are best described by the three actions $\{X, Y, Z\}$, but that group theoretically we can actually specify things more compactly in terms of just two generators $\{W, Y\}$. For the side pieces we will similarly find that geometrically these 144 patterns are best described by the three actions already mentioned here, plus the further two actions in the Supplementary Material. However, from the group theoretical perspective, we again do *not* need five generators. There is a certain redundancy among the geometrical actions, with the result that any one of them can be constructed from suitable combinations of the others. The entire group can ultimately be constructed from only three generators. Full details are again presented in the Supplementary Material. Any readers though who have deduced the missing two geometrical actions can further think about which actions might then be constructible from some of the others.

## 5 Conclusion

Starting from the previous concept of colour equality patterns on Rubik's cubes [6, 7], we constructed the new category of colour equivalence patterns. By direct computational searches, we found that there are 18 corner patterns and 144 side piece patterns. Together also with three possible arrangements of the centres, there are therefore $18 \cdot 144 \cdot 3 = 7776$ different equivalence patterns. For the 18 corner patterns we analysed the group structure in full detail, and showed that it can be obtained from just three generators $\{X, Y, Z\}$, corresponding to relatively simple geometrical actions, or even from just two generators $\{W, Y\}$, if we are willing to accept that $W = XZ$ is geometrically more complicated than either $X$ or $Z$ alone. For the 144 side piece patterns we left most of the details for the Supplementary Material, but here also there are relatively simple geometrical actions, and ultimately just three generators are needed to create the entire group.

Let us return also to our starting point, the colour equality patterns. As already noted in the introduction, equality patterns cannot constitute one single group, since there are mutually incompatible patterns. However, it turns out that the $2 \times 2 \times 2$ corner patterns can be suitably split up into two separate groups. The Supplementary Material includes two sets $\{x, y, z\}$ and $\{U, V\}$ that allow you to construct the entire groups. We then compare the $\{X, Y, Z\}$ equivalence group with the $\{x, y, z\}$ equality group, and show that the underlying group structure is the same, although curiously enough the way the elements have to be matched up is not quite as simple as $X \leftrightarrow x$, $Y \leftrightarrow y$, $Z \leftrightarrow z$.

Finally, it is interesting to note how colour equality is simpler than colour equivalence, in terms of the original geometrical definitions, but in terms of the group structures, equivalence is simpler than equality. For equivalence all of the patterns nicely constitute one single group, whereas for equality they require this somewhat awkward splitting into separate groups. Is there some underlying reason for this? I don't know, but if anyone can discover something here, I would certainly be very interested to hear about it!

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## Supplementary Material

As noted in the main article, there are several reasons for including supplementary material:

First, to properly understand the different patterns, and how they satisfy the colour equivalence condition, it is best to see and handle actual three-dimensional cubes. This is especially true if you want to think of the various results not just as static patterns, but as actions that tell you how to move the different pieces around. By carefully studying actual cubes it is possible to far better understand how pieces have moved around than if you were viewing only the two-dimensional figures in the article. For every figure in the article there are therefore templates provided below that can be cut out and folded together into three-dimensional cubes.

Second, to fully describe the side piece results in the article itself would probably have made it somewhat longer than appropriate. The article therefore just briefly mentioned the three simplest side piece actions, then encouraged readers to think about what other actions there might be, and promised the full details in this supplementary material section.

Third, as noted in the article, it is possible to apply a group theoretical analysis to the $2 \times 2 \times 2$ colour equality patterns as well, as long as they are split up into two different groups.

And finally, as also noted in the article, it is possible to make some comparisons between different groups and show that the underlying group structure is the same, even if the best geometrical descriptions of each group separately don't make that immediately obvious.

The following pages therefore consist of:

1. Eight pages (and two further figures) describing the colour equivalence side piece results in full detail, from both the geometrical and group theoretical perspectives.

2. Four pages (and two further figures) outlining how to do the group theoretical analysis of the $2 \times 2 \times 2$ colour equality patterns.

3. Four pages on comparisons between different groups.

4. Eighteen pages with templates for the three figures in the article plus the two additional figures in item Nr. 1.

5. One page with completely blank $2 \times 2 \times 2$ templates, to fill in yourself when constructing the various combinations such as $YZ$, $Y^2Z$, $ZY$, $ZY^2$, etc.

6. Nine pages with the 18 possible equivalence corner patterns already filled in on $3 \times 3 \times 3$ cubes, but the other slots left blank. The corner pattern $I$ template could be a useful starting point if you want to explore various combinations of side piece actions. In general all of them could be useful if you want to try out for yourself what particular arrangements of centres and sides look like together with a given corner pattern.

7. Finally, there are five pages with templates for the two figures in item Nr. 2.

## Geometrical and Group Theoretical Analysis of the Side Pieces

The main article already noted three geometrical actions that preserve colour equivalence. The first row in Figure 4 shows these three actions. The first one, $r$, is the same as Figure 1b in the article. It rotates the three side pieces around the r/y/b front corner in a clockwise direction, and the three side pieces around the o/w/g back corner in a counter-clockwise direction. These six pieces are again collectively referred to as the '3+3' pieces. The next action, $f$, takes these same '3+3' pieces and flips them in place. Finally, $F$ takes the other six side pieces, which constitute a 'ring of six' separating the front and back halves, and flips those in place. It is straightforward to verify that all three of $r$, $f$ and $F$ do indeed satisfy colour equivalence.

The article then asserted that there are two further geometrical actions, one yielding a factor of two and the other a factor of six to the total number of patterns, and challenged readers to try to come up with these actions themselves. The first of these, yielding the factor of two, is simply a

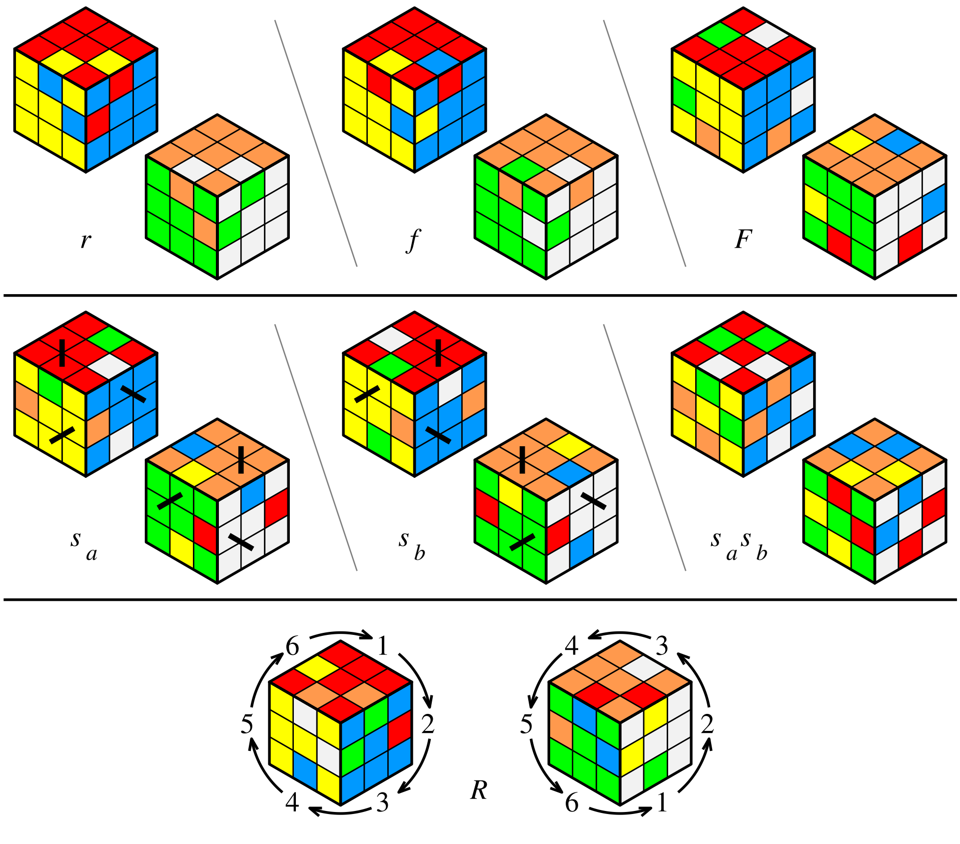


Figure 4: The top row shows the three actions $r$, $f$ and $F$. The second row shows the two possible ways of switching the '3+3' and 'ring of six' pieces, $s_a$ and $s_b$, as well as the combination $s_a s_b$. The thick lines superimposed on $s_a$ and $s_b$ indicate precisely which pieces are being switched. The final row shows the action $R$, with the circles indicating how the 'ring of six' rotates, clockwise as seen from the front and counter-clockwise as seen from the back.

switch between the '3+3' and the 'ring of six' pieces. There are actually several ways to accomplish this switch, none of them obviously preferable as a simplest, 'standard' action. The second row in Figure 4 shows two possible switches, the mirror image pair $s_a$ and $s_b$. The thick line segments superimposed on the cube faces indicate which pieces are being exchanged. The third entry in the second row also shows the combination $s_a s_b$, that is, the combined action of first switching according to $s_b$, then switching back again according to $s_a$. We will see below why it might be interesting to consider this combination.

The final geometrical action, $R$, is shown in the bottom row of Figure 4. It consists of a rotation of the 'ring of six' pieces, with the numbers beside the pieces indicating how the rotation proceeds, one position clockwise as seen from the front, and counter-clockwise as seen from the back. This action $R$ also has one additional complication, namely that the '3+3' pieces must be exchanged with their diagonally opposite partners, as indicated in the figure. This is exactly the same effect as for the generator $Z$ in the article, and for the same reason. Just the rotation of the 'ring of six' alone is readily seen to correspond to an odd number of pairwise exchanges of side pieces, namely 12345**6** $\to$ 1234**6**5 $\to$ 123**6**45 $\to$ 12**6**345 $\to$ 1**6**2345 $\to$ **6**12345. It is therefore not possible to accomplish this alone, but if we add the '3+3' pairwise exchanges we are back to an even number of exchanges, which can be done.

To summarise, we have the five geometrical actions:

**1.** $r$ rotates the '3+3' pieces, yielding a factor of three.

**2.** $f$ flips the '3+3' pieces in place, yielding a factor of two.

**3.** $F$ flips the 'ring of six' pieces in place, yielding a factor of two.

**4.** Either of $s_a$ or $s_b$ switches the '3+3' and 'ring of six' pieces, yielding a factor of two.

**5.** $R$ rotates the 'ring of six' pieces, and then necessarily also exchanges the '3+3' pieces with their diagonal opposites, yielding a factor of six.

The total number of colour equivalence side piece patterns is then indeed $3 \cdot 2 \cdot 2 \cdot 2 \cdot 6 = 144$, as noted in the article. Readers are further encouraged to use the templates below to construct actual cubes, and thereby convince themselves that all of these actions really do satisfy the colour equivalence criterion that red/yellow/blue and orange/white/green are distributed equally within each set, and that red versus orange, yellow versus white, and blue versus green have mirror image patterns.

In terms of combining these actions, the best way to think about how to systematically construct all 144 patterns is then as follows: First decide whether you want to leave the '3+3' and 'ring of six' pieces in their original positions, or exchange them. If you choose to switch, either $s_a$ or $s_b$ will accomplish that. That is, all 72 patterns where the '3+3' and 'ring of six' pieces have switched positions can be created by starting out with either $s_a$ or $s_b$. (We will see below also how the details after that differ slightly depending on whether you start with $s_a$ or $s_b$.) Next, use $r$ and $R$ to adjust the rotations of whichever pieces are now in the '3+3' and 'ring of six' positions. Finally, once both sets of pieces are in their desired positions, use $f$ and $F$ to fix whichever orientations you want.

Turning next to the group theoretical analysis, if we think of the set $\{r,\ f,\ F,\ s_a,\ s_b,\ R\}$ as elements of a group, it is immediately clear that we have $r^3 = f^2 = F^2 = s_a^2 = s_b^2 = R^6 = I_s$, the side piece identity element. Which elements do or don't commute with which other elements is usually also clear from the geometrical meanings. For example, $f$ commutes with $r$, since 'flip first, then rotate' and 'rotate first, then flip' clearly yield the same final result. Or, for an even simpler example, $F$ commutes with $r$, since they affect completely different pieces, so it makes no difference in which order the two actions are done. In contrast, the flips $f$ or $F$ do not commute with the switches $s_a$ or $s_b$.

Staying with the flips and switches, we recognise that they satisfy

$$f = s_a F s_a, \qquad f = s_b F s_b, \qquad F = s_a f s_a, \qquad F = s_b f s_b. \tag{1}$$

That is, whether we (a) flip either the '3+3' or 'ring of six' pieces directly ($f$ or $F$), or (b) first switch, then flip those same pieces, but now in the other positions, then switch back, clearly yields the same result. In terms of fundamental generators, we therefore do not need both of $f$ and $F$, but only one of them.

A similar result applies to $r$, which satisfies

$$r = s_a R^2 s_a, \qquad r = s_b R^2 s_b. \tag{2}$$

That is, whether we (a) rotate the '3+3' pieces directly, or (b) first switch, then use $R^2$ to rotate those same pieces two positions along the 'ring of six', then switch back, yields the same result. So again, in terms of fundamental generators, we do not need $r$.

The next result of this type we want to consider concerns the two switches $s_a$ and $s_b$. In terms of what they do, switching the '3+3' and 'ring of six' pieces, we suspect we don't really need both of them. Unlike the above results for $f$, $F$, and $r$ though, here it is not immediately clear how to express $s_a$ or $s_b$ in terms of each other combined with the other actions. To address this question, let's return to the $s_a s_b$ result in the middle row of Figure 4. If we examine this carefully, we see that its geometrical effect is the same as $FRr$ (or $rRF$, or any other permutation, since these three actions all commute). From $s_a s_b = FRr = rRF$, and using also $s_a^{-1} = s_a$ and $s_b^{-1} = s_b$, we can then obtain results such as

$$s_a = FRrs_b, \qquad s_b = s_a rRF, \tag{3}$$

or using the above results for $r$,

$$s_a = FRs_b R^2, \qquad s_b = R^2 s_a RF. \tag{4}$$

These are the desired relations then that express $s_a$ and $s_b$ in terms of each other and other actions, thereby verifying that we don't need both of them as fundamental generators.

There is another, perhaps somewhat unexpected result we can obtain at this point. We can further rearrange the expressions in Eq. (4) as

$$F = s_a R^{-2} s_b R^{-1}, \qquad F = R^{-1} s_a R^{-2} s_b. \tag{5}$$

Using the previous relationships between $F$ and $f$, these can be further converted to

$$f = R^{-2} s_b R^{-1} s_a, \qquad f = s_b R^{-1} s_a R^{-2}. \tag{6}$$

That is, we could potentially eliminate both of the flips $f$ and $F$ as fundamental generators, as long as we instead keep both of the switches $s_a$ and $s_b$. It may seem perverse to express actions as geometrically simple as the flips $f$ and $F$ by combinations of actions as complicated as $R$, $s_a$ and $s_b$, but it can be done.

And finally, from our previous results $s_a s_b = FRr = rRF$ we can even obtain

$$R = F\, s_a s_b\, r^{-1} = r^{-1}\, s_a s_b\, F, \tag{7}$$

or any other permutation of $F$, $s_a s_b$, and $r^{-1}$. That is, none of our six actions $f$, $F$, $s_a$, $s_b$, $r$ and $R$ seems to be truly fundamental! According to Eqs. (1)-(7), any one of them can be constructed from various combinations of the others. It is a simple matter also to directly verify the validity of (1)-(7), and indeed you are strongly encouraged to do so. Just evaluate the sequence of actions on the right-hand sides, and confirm that the final result is the same as the single action on the left. It requires a bit of time and effort filling in colours on some blank templates, but it is ultimately straightforward, provided you properly understand the six actions separately.

However, from the point of view of trying to construct a nice, simple set of fundamental generators for the entire 144-element group, this ability to eliminate any one, or even several of them in favour of the others is not particularly helpful, since it creates more choices than we can easily keep track of. Instead of addressing the 'fundamental generators' question directly, let's start with something much simpler, namely what subgroups — and specifically of what sizes — will we obtain if we gradually combine more and more of our six actions.

For example, if we consider the two actions $f$ and $F$, these two together will clearly generate a subgroup that consists of the four elements $\{I_s, f, F, fF\}$. Acting on any of these four elements with either $f$ or $F$ just generates a different member of the subgroup, but nothing outside it — this is after all the requirement to be a subgroup. So, for every combination of two of our six basic actions $f$, $F$, $s_a$, $s_b$, $r$, $R$, what is the size of the resulting subgroup? If you just keep applying the two chosen actions over and over again, in all possible combinations, how many elements will you obtain before you reach this point where you stop generating anything new?

Table 1 shows the results, for the 15 possible ways of choosing two of the six elements. Many of these results should be readily understandable in terms of the underlying geometrical definitions of the actions. For example, we see that either flip, $f$ or $F$, combined with either switch, $s_a$ or $s_b$, yields a subgroup containing eight elements. To see how this comes about, first note that according to Eq. (1) either flip combined with either switch will generate the other flip. So effectively you have both flips, together with a switch, and each of those three actions yields a factor of two to the total size of the subgroup.

There are also other results which are far less obvious geometrically, but are still straightforward to obtain. For example, Table 1 claims that combining the two switches $s_a$ and $s_b$ yields a subgroup

| $f, F$ | $f, r$ | $F, r$ | $f, s_a$ | $f, s_b$ | $F, s_a$ | $F, s_b$ |
|---|---|---|---|---|---|---|
| 4 | 6 | 6 | 8 | 8 | 8 | 8 |

| $f, R$ | $F, R$ | $s_a, s_b$ | $s_a, r$ | $s_b, r$ | $r, R$ | $s_a, R$ | $s_b, R$ |
|---|---|---|---|---|---|---|---|
| 12 | 12 | 12 | 18 | 18 | 18 | 72 | 72 |

Table 1: For every possible choice of two out of the six basic actions, the number of elements in the subgroup generated by those two actions.

containing twelve elements. To see how this comes about, recall our previous result $s_a s_b = FRr$, and also that $F$, $R$, $r$ all commute with one another. We then have that $(s_a s_b)^6 = F^6 R^6 r^6 = I_s$, the identity element. The sequence of elements $s_b$, $s_a s_b$, $s_b s_a s_b$, ... will therefore yield twelve patterns before it repeats. From $(s_a s_b)^6 = I_s$ we further have that $s_a = s_b(s_a s_b)^5$, $s_b s_a = (s_a s_b)^5$, $s_a s_b s_a = s_b(s_a s_b)^4$, etc. That is, the sequence of elements $s_a$, $s_b s_a$, $s_a s_b s_a$, ... yields these same twelve patterns, just in a different order. The only remaining item would be to verify that all twelve of these patterns really are distinct, which you are encouraged to do, by carrying out the actual geometrical actions involved.

The most complicated entries in Table 1 are for the pairs $\{s_a, R\}$ and $\{s_b, R\}$, which each yield subgroups having 72 elements. In principle even these could be worked out by hand, but it would be a *very* lengthy and tedious process. Instead, we can obtain results such as these computationally. We start by labeling the 24 little panels (2 for each of the 12 side pieces) in any order we like. The various geometrical actions can then be represented by permutations, indicating how all the panels move to new positions. Geometrical actions applied in sequence then become just permutations applied in sequence, which is obviously something a computer program can easily handle. It is in this way that all the entries in Table 1 were obtained, although again many of them are also intuitively clear from the underlying geometrical actions.

Now, what is the significance of Table 1 when applied to our original question of what to choose as our fundamental generators? Since none of the subgroups includes all 144 elements of the entire group, none of these 15 pairs here is sufficient to act as fundamental generators. We are tempted to conclude therefore that two fundamental generators are indeed not enough, and that we need (at least) three. However, we haven't actually proved this result just yet. All that Table 1 proves is that none of these particular pairs are sufficient to generate the entire group.

Suppose we defined two new actions though, for example $p = s_a f r$ and $q = FR^{-1}$. How do we know that the pair $\{p, q\}$ might not generate the entire group? That is, maybe we just weren't clever enough in choosing the right actions to try out in creating Table 1. Indeed, we have encountered exactly this situation before: if we describe the corner group as $\{X, Y, Z\}$, then no two of these alone will generate the entire group, but if we introduce $W = XZ$, then just $\{W, Y\}$ will generate it. So how do we know that the same thing might not happen here, for just the right choices of $p$ and $q$?

Fortunately, there is a simple way to address this question. If we have already represented our

| $f, F, s_a$ | $f, F, s_b$ | $f, F, r$ | $f, F, R$ | $f, r, R$ | $F, r, R$ | $s_a, s_b, r$ |
|---|---|---|---|---|---|---|
| 8 | 8 | 12 | 24 | 36 | 36 | 36 |

| $f, s_a, s_b$ | $F, s_a, s_b$ | $f, s_a, r$ | $f, s_b, r$ | $F, s_a, r$ | $F, s_b, r$ | $s_a, r, R$ | $s_b, r, R$ |
|---|---|---|---|---|---|---|---|
| 48 | 48 | 72 | 72 | 72 | 72 | 72 | 72 |

| $f, s_a, R$ | $f, s_b, R$ | $F, s_a, R$ | $F, s_b, R$ | $s_a, s_b, R$ |
|---|---|---|---|---|
| **144** | **144** | **144** | **144** | **144** |

Table 2: For every possible choice of three out of the six basic actions, the number of elements in the subgroup generated by those three actions.

six basic actions $f$, $F$, $s_a$, $s_b$, $r$, $R$ as permutations of the numbers 1–24, and then applied these permutations in sequence to obtain further permutations, we can easily combine all of them to obtain all 144 geometrical patterns represented as permutations. Then we just repeat the procedure in Table 1, but now with 'choose 2 out of 144' rather than 'choose 2 out of 6'. That is, the number of pairs we need to try out is now $144 \cdot 143/2 = 10,296$ rather than 15. This is still a sufficiently small number though that the calculations can easily be done (in about an hour on a workstation). And now the results really are conclusive: out of all of these 10,296 possible pairs, none of them generates the entire group.

We therefore move on to Table 2, which shows the sizes of the subgroups for the 20 possible ways of choosing three of our six elements $f$, $F$, $s_a$, $s_b$, $r$, $R$. Many of these results are again intuitively clear, especially if we also compare with Table 1, and imagine adding one extra action to some of the entries there. The most important entries in Table 2 are in the bottom row, where the numbers 144 are in boldface for emphasis. These entries indicate that the given choices of three elements work as fundamental generators, and will create the entire group. That is, any of the five sets

$$\{f, s_a, R\}, \qquad \{f, s_b, R\}, \qquad \{F, s_a, R\}, \qquad \{F, s_b, R\}, \qquad \{s_a, s_b, R\} \tag{8}$$

can be chosen as our fundamental generators.

Comparing these five possibilities, we notice that the first four all consist of one flip, one switch, and the rotation $R$. That any of these choices will indeed generate the entire group is also clear from our previous results: First, according to Eq. (1) either flip combined with a switch immediately gives us the other flip as well. Next, Eq. (2) gives us $r$, and finally Eq. (4) gives us whichever switch we didn't already have. That is, for any of the first four sets in (8), we can directly see how to generate the other three actions, and once we have all six we know from their geometrical definitions that they will indeed generate the entire group. Also, since neither of $f$ versus $F$ nor $s_a$ versus $s_b$ is in any way 'preferred' over the other choice, all four of these sets are on an equal footing as to which one we want to choose for our fundamental generators.

The fifth set in (8), $\{s_a, s_b, R\}$, is the one that is slightly anomalous, since it contains neither of the flips, but instead both of the switches. We can certainly recognise though that it does also work as a fundamental generators set, since Eqs. (5,6) create $F$ and $f$ from $s_a$, $s_b$, $R$, and Eq. (2)

still gives us $r$. The conclusion therefore is that any of the five sets in (8) will indeed work as fundamental generators. Four of these options are on an essentially equal footing, whereas the fifth one is a bit different. Whether 'a bit different' means better or worse is ultimately just a matter of personal preference though.

There is one final point to note regarding the five options in (8). All five of them include the 'ring of six' rotation $R$ as one of the three fundamental generators. So how do we reconcile this with Eq. (7), which showed that $R$ can be constructed from some of the other actions? There is no contradiction here though, since (7) merely states that $R$ can be constructed from four of the other actions. However, $R$ cannot be constructed from any combination of just three of the other actions, since otherwise there would be an option in (8) that didn't include $R$. Note in particular how Eqs. (1)-(6) indicate how each of $f$, $F$, $s_a$, $s_b$, $r$ can be created from at most three of the others, so no specific one of these five must be included among the fundamental generators. $R$ though is different: if you want to generate the entire 144-element group from just three of the six basic actions, then $R$ must be one of the three.

To summarise the side piece group then, the underlying geometry is best described by the six actions $\{f, F, r, R, s_a, s_b\}$, but only three of these are needed as fundamental generators for the entire 144-element group, with any of the five choices in (8) being equally valid. This distinction between the geometrical versus the 'fundamental generators' way of describing the side piece patterns is sufficiently important also that it is worth reviewing in some detail. We start by returning to our previous assertion that the best way to systematically describe all 144 patterns geometrically is (a) first decide whether or not to switch the '3+3' and 'ring of six' pieces, (b) next apply the rotations $r$ and $R$ to get all the pieces in their final positions, (c) finally apply $f$ and $F$ to fix the orientations. The 72 patterns where we decide not to switch can then be represented as

$$f^k\, F^l\, r^m\, R^n, \qquad k,l=0,1, \quad m=0,1,2, \quad n=0,1,2,3,4,5. \tag{9}$$

The other 72 patterns, where we do decide to switch, can be represented as either of the expressions

$$f^{k_a}\, F^{l_a}\, r^{m_a}\, R^{n_a}\, s_a, \qquad k_a, l_a=0,1, \quad m_a=0,1,2, \quad n_a=0,1,2,3,4,5, \tag{10$a$}$$

$$f^{k_b}\, F^{l_b}\, r^{m_b}\, R^{n_b}\, s_b, \qquad k_b, l_b=0,1, \quad m_b=0,1,2, \quad n_b=0,1,2,3,4,5. \tag{10$b$}$$

From Eq. (3) we also recall that $s_a = FRr\, s_b$. Inserting this into (10a), and using the fact that $F$, $R$, $r$ all commute, we see that the exponents in the two representations are related by

$$k_b = k_a, \quad l_b = (l_a+1) \bmod(2), \quad m_b = (m_a+1) \bmod(3), \quad n_b = (n_a+1) \bmod(6), \tag{11}$$

thereby confirming that $s_a$ and $s_b$ are equally valid ways of achieving the desired switching action, but ultimately both create the same 72 patterns when combined with $f$, $F$, $r$, $R$ as in (10a,b).

Now consider, for example, the pattern whose labels in this representation are $P = fr^2Rs_a = fFR^2s_b$. Geometrically these sequences of actions are both straightforward to understand, and you

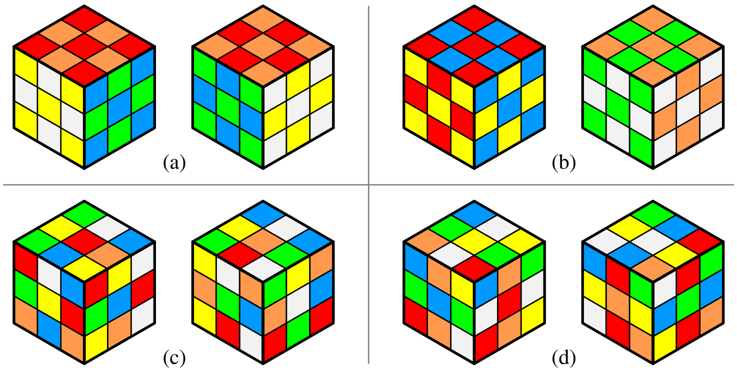


Figure 5: (a,b) Two different chessboard patterns that each satisfy the equivalence criterion. (c,d) Two equivalence patterns where every face contains all six colours.

should have no difficulty in constructing the actual pattern from either expression. Next, using the various relations (1)-(6), we could also convert $P$ into any one of the five expressions

$$P = fs_aR^{-2}s_aRs_a = fs_bfs_bR^2s_b = s_aFR^{-2}s_aRs_a = s_bFs_bFR^2s_b = R^{-2}s_bR^3s_aRs_a, \quad (12)$$

corresponding to the five choices of fundamental generator sets in (8). However, just because it is possible to convert $P$ into any of these five forms does not mean that it is desirable! All five are far more complicated to understand, and the actual pattern would be much harder to construct from these sequences of actions. Knowing that the group can be reduced to just three fundamental generators is important from the abstract group theoretical perspective, but in terms of understanding the geometry, and constructing actual patterns, the representations (9) and (10a) or (10b) are far superior.

As a final challenge then, consider the four colour equivalence patterns shown in Figure 5. For each of them, write down their correct labelings. For the corners and centres it is just a matter of comparing with Figures 2 and 3, and seeing which arrangements there match. For the side pieces, print out the templates below, construct actual cubes, and study them to see which side pieces have moved where, and what the correct descriptions therefore are, in terms of the representations (9,10).

These four patterns in Figure 5 were also chosen to have specific properties. The first two are familiar chessboards that can be found among the various patterns available online. In addition to being pretty, which is mathematically rather ill-defined, they also satisfy colour equivalence, which we have seen is very precisely defined. The second two patterns have been constructed to have a rather different property. If you examine them carefully, you'll notice that the colours have been spread out so uniformly that every face contains all six colours. This is relatively easy to accomplish for those corner patterns that already have four different colours on each face, but gets progressively harder the fewer colours you have on the corners. You can perhaps try to create a few more combinations with this property.

## Group Theory Applied to the $2 \times 2 \times 2$ Equality Patterns

Figure 6 presents two sets $\{x, y, z\}$ and $\{U, V\}$ which we will use to construct and analyse the group theoretical structure of the colour equality patterns on the $2 \times 2 \times 2$ cube. We begin by noting that $y$ and $V$ are actually the same pattern; that is, the two groups partially overlap. The pattern $z$ should also look familiar; it is the same as $Z$ in the main article. This pattern has no handedness, and thus satisfies both equivalence and equality.

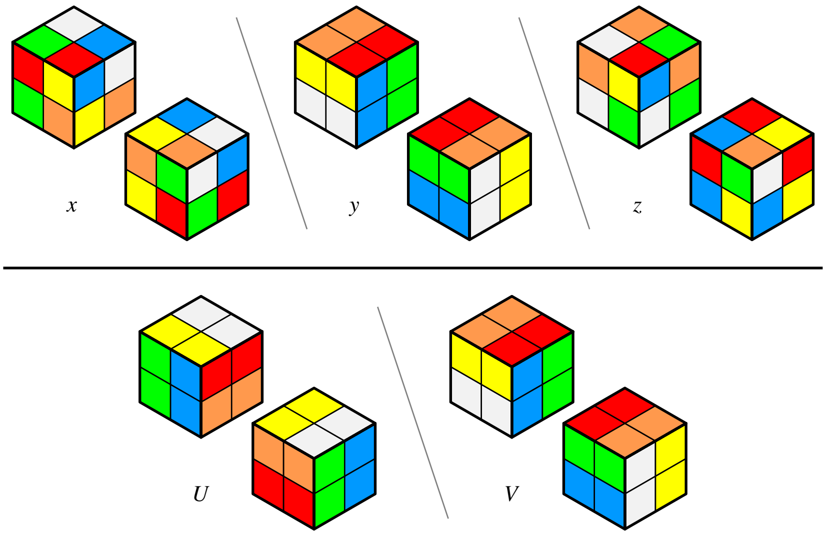


Figure 6: The two sets $\{x, y, z\}$ and $\{U, V\}$.

As before, the first step is to make sure you fully understand what these various generators do to the eight corners; otherwise it's impossible to work out any of the combinations that come later. We have:

**(a)** $x$ leaves all eight pieces in their original positions. The r/y/b front and o/w/g back corners also retain their original orientations. The other six corners all rotate one twist clockwise.

**(b)** $y = V$ also leaves the r/y/b front and o/w/g back corners untouched. The three corners adjacent to the r/y/b front, and similarly the three corners adjacent to the o/w/g back, then each move one slot over in a clockwise direction, as viewed from their orientation. That is, viewed from a 'neutral' perspective, the two sets of three corners rotate their positions in opposite directions. Perhaps not surprisingly, the action most similar to $y/V$ is the '3+3' clockwise/clockwise side piece equality pattern in Figure 1a of the article. This $y/V$ generator is probably the most complicated one we've encountered, so make sure you fully understand what it does, including the orientation of the pieces in their new positions. By using the templates to construct an actual 3D cube and looking at these two sets of three corners it should be possible to understand it though.

**(c)** Since $z = Z$, just remind yourself what $Z$ did before.

**(d)** $U$ is fortunately another relatively easy one. All eight corners remain in their original positions. They alternately rotate one twist clockwise for four corners including the r/y/b front, and counter-clockwise for the other four corners including the o/w/g back.

In the process of exploring and understanding what these actions do, make sure you also verify that all of them satisfy the colour equality criterion! Recall that unlike equivalence, where e.g. red and orange were mirror images, here all six colours must be distributed equally. There is no point trying to construct colour equality groups unless you have first convinced yourself that these starting points $\{x, y, z\}$ and $\{U, V\}$ do in fact satisfy the equality criterion.

Next, work out what $xU$, $Ux$, $zU$, $Uz$ look like. You should find that all four of them *violate* colour equality. This is precisely why we need two separate colour equality groups; trying to put them all into one group simply wouldn't work. Having even just one of these combinations violate equality would actually already be enough to establish that we need (at least) two separate groups, but it turns out that all four violate equality. The generator $U$ simply does not get along with either $x$ or $z$. (It had better get along with $y$ though, since $U$ and $V = y$ together are supposed to generate a group!)

**1.** Let's first construct the $\{x, y, z\}$ group. Directly from the geometrical actions, it should be clear that $x^3 = y^3 = z^2 = I$, where the identity element $I$ is again the standard solved configuration. ($I$ and $z = Z$, along with $U$ and $U^2$, are the only patterns that satisfy both equality and equivalence.) We note next that $x$ commutes with both $y$ and $z$; $y$ and $z$ both move all six corners except the r/y/b front and o/w/g back ones, but since $x$ rotates all of those six corners one twist clockwise, it doesn't matter whether they get rotated first and then moved, or moved first and then rotated. Given how complicated both $y$ and $z$ are, it is far less obvious whether they commute or not, so let's defer that question for now.

Instead, let's just start by constructing the same 18 combinations $x^m y^n$ and $x^m y^n z$, with $m, n = 0, 1, 2$, that we previously had for the $\{X, Y, Z\}$ equivalence group. Or rather, *you* should construct these 18 patterns! It will undoubtedly take a while, but by using the blank templates below, and suitably moving the pieces around according to the actions of $x$, $y$ and $z$ separately, you should manage to construct these 18 patterns.

At this point we don't know yet whether this is the complete group or not, since it is conceivable that $zy$ or $zy^2$ for example could still yield completely new patterns. That is, we really do need to know the commutation relation between $y$ and $z$. Well, go ahead and just construct $zy$ and $zy^2$. Then compare them with the 18 patterns you have already, and see whether they are completely new, or whether anything matches. You should find that $zy$ is the same as $x^2y^2z$ and $zy^2$ is the same as $xyz$, which we note are remarkably similar to our previous equivalence results $ZY = Y^2Z$ and $ZY^2 = YZ$, even though $y$ and $Y$ have completely different actions.

Having established these results for $zy$ and $zy^2$, we then also have the same conclusion as before for the equivalence group, that any arbitrary combination of $x$, $y$, $z$ can always be reduced to one of these 18 patterns $x^m y^n$ and $x^m y^n z$, with $m, n = 0, 1, 2$. That is, there really are exactly 18 elements in this group, and no others. Furthermore, even though the individual patterns are completely different from the previous equivalence group, they have the same general relationship among one another, as seen by these common forms of expressing them in terms of $\{x, y, z\}$.

There are still two further items to consider before we're done with this group. First, check that all 18 patterns you've constructed really do satisfy the equality criterion! That is, what you have so far is a group of some kind, just from the way it was constructed as a group, but if you want to call this an *equality* group you better check that every element really is an equality pattern.

Second, although the relations $ZY^2 = YZ$ and $zy^2 = xyz$ may seem very similar, there is also one significant difference between the two. Can you think what it is? Is there something you can do with $zy^2 = xyz$ that you cannot do with $ZY^2 = YZ$? The answer is given on the next page, so don't turn the page just yet, but instead try to come up with the answer yourself first.

**2.** We turn next to the $\{U, V\}$ group. We again easily see that $U^3 = V^3 = I$. A little thought further convinces us that $U$ and $V$ will commute. In particular, $U$ rotates corners clockwise and counter-clockwise in an alternating pattern, whereas $V$ moves them around. However, $V$ never moves any corners in a way that would change their sense of rotation by $U$. Whether we do $UV$ or $VU$ therefore does not matter; the result will be the same. Given $U^3 = V^3 = I$ and $UV = VU$, we immediately see then that any arbitrary combination of the two can always be reduced to one of the nine patterns $U^m V^n$, with $m, n = 0, 1, 2$. So, go ahead and construct these nine patterns, and as before check that all of them really do satisfy the equality criterion, and then you can legitimately call this a second equality group.

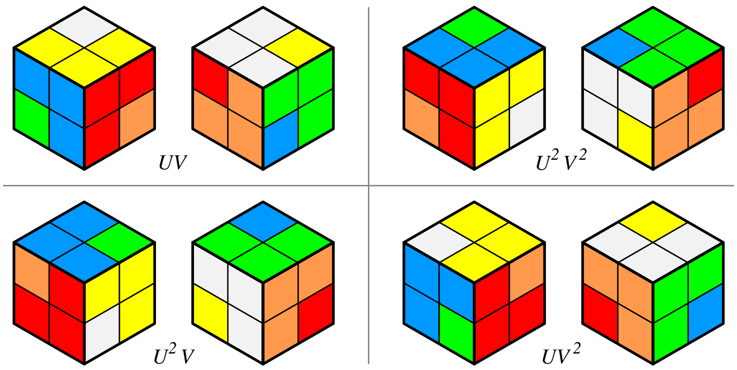


Figure 7: The four combinations $UV$, $U^2V^2$, $U^2V$, $UV^2$, as labeled.

Figure 7 shows the four members that combine $U$ and $V$. As we might expect, $\{UV, U^2V^2\}$ and $\{U^2V, UV^2\}$ are each mirror image pairs. We note next that all four patterns have a pronounced front-back asymmetry. If we instead pair them up as $\{UV, U^2V\}$ and $\{U^2V^2, UV^2\}$, the patterns are identical except with front and back reversed. That is, any one of these alone has a somewhat lopsided, asymmetrical appearance, but the set of all four together has a nice symmetry again.

Having obtained our two equality groups here, we can also return to Ref. [6] in the article, and compare with the patterns there. Every pattern there does indeed appear in one or both ($I$, $y$, $y^2$) of the groups derived here, and every pattern here also appears there (allowing for the fact that for all the various mirror image and front/back asymmetry pairs only one member was listed there).

So what exactly have we gained by this group theoretical analysis? We now have a proper understanding of the relationships between the different patterns. The computer search in [6] gives you a complete listing of the patterns, which is certainly an important first step, but it is ultimately just a random list of patterns, with no insight yet into the relationships between any of them.

For example, in [6] it was claimed that $U$ is the simplest pattern other than the solved state $I$ itself, based purely on the notion that this clockwise/counter-clockwise rotation of $U$ is so easy to understand. In contrast, the one member of the Figure 7 quadruplet that was included in [6] was presented as an interesting 'anomalous' pattern, but without any further comments. It is a testament to the elegance of group theory that we can now so clearly see where the whole quadruple set comes from, and how the seemingly simple element $U$ is in fact a crucial ingredient in creating these seemingly anomalous elements.

The last item then is to return to the question posed earlier, what is the significance of the result $zy^2 = xyz$? It can be rearranged to yield $x = zy^{-1}zy^{-1}$, and similarly $x^{-1} = zyzy$. (We also used $y^{-1} = y^2$ and $z^{-1} = z$ here.) In other words, $x$ isn't actually an independent generator at all, but can be created from $y$ and $z$. Try it out! Work through the four steps on the right-hand sides, and verify for yourself that the final results are precisely the actions of $x$ and $x^{-1}$.

This is very similar to our previous results (5,6), where we found that the geometrically very simple side piece flips $f$ and $F$ can be constructed from very complicated combinations of other actions. Here also we have complicated combinations of $y$ and $z$ that ultimately yield the very simple geometrical actions $x$ and $x^{-1}$.

This of course also means that, once again, in terms of fundamental generators we don't need the full set $\{x, y, z\}$, but instead just $\{y, z\}$ are already enough. Or alternatively, as before for the $\{X, Y, Z\}$ group, here also we can introduce $w = xz$, yielding $w^4 = x$ and $w^3 = z$, and thus generate the whole group with just $\{w, y\}$. This is analogous to (8), where we also had various choices of minimal generating sets.

Finally, if you're feeling especially heroic, you can also have a go yourself at trying to apply group theory to the side piece equality patterns on $3 \times 3 \times 3$ cubes. The basic patterns as such are listed in detail in Ref. [7], so it's 'just' a matter of figuring out the right group theoretical structure (which may possibly involve splitting things up even further).

## Comparing the Different Groups

Table 3 summarises results for the equivalence corner and side piece groups, and the two equality corner groups. The most striking aspect is probably how different the best geometrical and group theoretical descriptions are. Only one of the four groups, the $\{U, V\}$ equality group, has the geometry and the group theory agreeing on the best representation. For the other three the best way to describe the geometry involves more generators than the most compact group representation does.

| | Equivalence Corner Group | Equivalence Side Piece Group | Equality Corner Group 1 | Equality Corner Group 2 |
|---|---|---|---|---|
| Number of Elements | 18 | 144 | 18 | 9 |
| Best geometrical representation | $\{X, Y, Z\}$ | $\{f, F, r, R, s_a, s_b\}$ | $\{x, y, z\}$ | $\{U, V\}$ |
| Most compact group representation | $\{W, Y\}$ | Any of Eq. (8) | $\{y, z\}$ or $\{w, y\}$ | $\{U, V\}$ |

Table 3: A comparison of the key aspects of the four groups considered in this work.

Let's next consider the two 18-element groups in more detail. We recall that they were each described by

$$X^3 = Y^3 = Z^2 = I, \quad XY = YX, \quad XZ = ZX, \quad ZY = Y^2Z, \quad ZY^2 = YZ, \tag{13}$$

$$x^3 = y^3 = z^2 = I, \quad xy = yx, \quad xz = zx, \quad zy = x^2y^2z, \quad zy^2 = xyz, \tag{14}$$

yielding

$$\{X^mY^n\ ,\ X^mY^nZ\} \quad \text{and} \quad \{x^my^n\ ,\ x^my^nz\} \quad \text{with} \quad m, n = 0, 1, 2 \tag{15}$$

as the 18 elements in each group. And again, in each case we could describe the groups more compactly by introducing $W = XZ$ and $w = xz$, but the three-generator descriptions here are more convenient for what we want to do next.

One interesting question then is, do these ultimately describe the same group structure, or are the slight differences between the $\{Y, Z\}$ versus $\{y, z\}$ commutation relations enough to make them fundamentally different groups? To answer this question, let's start by constructing Table 4, showing the so-called *order* of each group element, where the order of an element $a$ is the smallest $k$ for which $a^k = I$.

Make sure also that you understand how these values are obtained. For example, the claim is that $YZ$ and $x^2yz$ each have order 2. To verify that this is correct, we evaluate

$$(YZ)(YZ) = Y(ZY)Z = Y(Y^2Z)Z = Y^3Z^2 = I, \tag{16$a$}$$

$$(x^2yz)(x^2yz) = x^4yzyz = xy(zy)z = xy(x^2y^2z)z = x^3y^3z^2 = I, \tag{16$b$}$$

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $I$ | 1 | $X$ | 3 | $X^2$ | 3 | $Z$ | 2 | $XZ$ | 6 | $X^2Z$ | 6 |
| $Y$ | 3 | $XY$ | 3 | $X^2Y$ | 3 | $YZ$ | 2 | $XYZ$ | 6 | $X^2YZ$ | 6 |
| $Y^2$ | 3 | $XY^2$ | 3 | $X^2Y^2$ | 3 | $Y^2Z$ | 2 | $XY^2Z$ | 6 | $X^2Y^2Z$ | 6 |
| $I$ | 1 | $x$ | 3 | $x^2$ | 3 | $z$ | 2 | $xz$ | 6 | $x^2z$ | 6 |
| $y$ | 3 | $xy$ | 3 | $x^2y$ | 3 | $yz$ | 6 | $xyz$ | 6 | $x^2yz$ | 2 |
| $y^2$ | 3 | $xy^2$ | 3 | $x^2y^2$ | 3 | $y^2z$ | 6 | $xy^2z$ | 2 | $x^2y^2z$ | 6 |

Table 4: The orders of the 18 elements in each of the two groups $\{X, Y, Z\}$ and $\{x, y, z\}$.

where you should be able to provide the detailed justifications for all the steps along the way. All entries in Table 4 are obtained in this way. For the ones where the order comes out to be 6 it can be a bit tedious to work out all the lower powers along the way, but it is good practice, so have a go at it and confirm at least some of these results yourself.

Comparing the various entries then, it is at least possible that the two groups might be the same. In particular, in addition to the identity $I$ itself, which obviously has order 1, in each upper/lower half of the table there are three 2s, eight 3s, and six 6s. If these numbers had not been the same, we would be done already, and the conclusion would be that the two groups would be fundamentally different. Since these overall totals are the same though, we can at least try to pair them up so that each element in one group has a partner in the other group with the same order.

Focusing on the three order-2 elements in each group, we have $\{Z, YZ, Y^2Z\}$ and $\{z, x^2yz, xy^2z\}$. Suppose we introduce $\tilde{y} = x^2y$. This relabels the second set as $\{z, \tilde{y}z, \tilde{y}^2z\}$, giving it the same appearance as the first set. And just as we left $z$ alone, we can also leave $x$ alone, since the conditions it satisfies in Eq. (14) are already the same as those satisfied by $X$ in Eq. (13).

That is, the suggestion anyway is that if we simply relabel the $\{x, y, z\}$ group in terms of $\{x, \tilde{y}, z\}$ instead, then $\{x, \tilde{y}, z\}$ could perhaps satisfy exactly the same commutation relations as $\{X, Y, Z\}$ in (13). Verifying the easy ones first, we certainly have

$$x^3 = \tilde{y}^3 = z^2 = I, \qquad x\tilde{y} = \tilde{y}x, \qquad xz = zx. \tag{17}$$

For the less obvious $\{y, z\}$ commutation relations that prompted this whole exercise in the first place, we first note that $\tilde{y} = x^2y$ can be rearranged as $y = x\tilde{y}$. Then just insert this into the previous $\{y, z\}$ commutation relations to obtain:

$$zy = x^2y^2z \quad \Longrightarrow \quad z(x\tilde{y}) = x^2(x\tilde{y})^2z \quad \Longrightarrow \quad xz\tilde{y} = x^4\tilde{y}^2z \quad \Longrightarrow \quad z\tilde{y} = \tilde{y}^2z, \tag{18}$$

$$zy^2 = xyz \quad \Longrightarrow \quad z(x\tilde{y})^2 = x(x\tilde{y})z \quad \Longrightarrow \quad x^2z\tilde{y}^2 = x^2\tilde{y}z \quad \Longrightarrow \quad z\tilde{y}^2 = \tilde{y}z. \tag{19}$$

If we therefore do nothing more than label the second group in terms of $\{x, \tilde{y}, z\}$ rather than $\{x, y, z\}$, then the various commutation relations are indeed seen to be exactly identical to those of the $\{X, Y, Z\}$ group, so clearly they are the same group.

We could of course have avoided all of this as far back as Figure 6, just by listing $\tilde{y}$ rather than $y$ as the relevant generator there. However, in terms of the geometry $y$ really is simpler and therefore better than $\tilde{y}$. Also, in terms of emphasising that the two equality groups share the common generator $y = V$, we really did need $y$ rather than $\tilde{y}$, since $x$ is not an element of the $\{U, V\}$ group, and therefore neither is $\tilde{y}$.

Since this worked so well, let's do one more easy comparison, and then at least set up a more challenging comparison for especially energetic readers to try. If we go all the way back to Table 1, the 144-element equivalence side piece group has the subgroups $\{r, R\}$, $\{s_a, r\}$ and $\{s_b, r\}$ that also each have 18 elements. So do any of these have the same underlying structure as our $\{X, Y, Z\}$ group?

The easy one is the $\{r, R\}$ group. Since $r$ and $R$ commute, the 18 elements are simply given by $r^m R^n$ with $m = 0, 1, 2$ and $n = 0, 1, 2, 3, 4, 5$. That is, $r$ and $R$ just have their own separate actions, with no coupling between them, which makes both the geometrical interpretation and the group structure very simple. Also, if the entire group is commutative, it cannot possibly be the same as the non-commutative $\{X, Y, Z\}$ group. We see therefore that there are at least two different 18-element groups, one commutative and one non-commutative.

The $\{s_a, r\}$ and $\{s_b, r\}$ groups are considerably trickier, especially since we don't even understand yet why they end up having 18 elements. That is actually quite straightforward, and yields a surprisingly simple geometrical interpretation as well. Let's first recall Eq. (2), stating that $r = sR^2s$, where we will generically use $s$ to denote either of the two switches $\{s_a, s_b\}$. At the time, we interpreted this result $r = sR^2s$ to mean that as long as we had either of the switches together with $R$, then we didn't need to include $r$ among our fundamental generators. However, we can also rearrange this result as $R^2 = srs$. That is, the groups generated by $\{s, r\}$ and $\{r, R^2, s\}$ are the same, since $R^2$ can be generated from the other two.

We therefore consider the group generated by $\{r, R^2, s\}$, together with the associated commutation relations

$$r^3 = (R^2)^3 = s^2 = I, \qquad rR^2 = R^2 r, \qquad sr = R^2 s, \qquad sR^2 = rs. \tag{20}$$

Any combination of these three elements can then always be converted into the form

$$r^m (R^2)^n s^k \qquad \text{with} \qquad m, n = 0, 1, 2, \quad k = 0, 1. \tag{21}$$

The geometrical interpretation is thus very simple: first decide whether you want to switch or not ($k = 1$ or 0), then the rotations $r$ and $R^2$ act completely separately, with three choices each, for a total of 18 elements. So yet again we see that the geometry of a particular group is best understood by including more generators than strictly needed in the most compact group representation.

So, the challenge for you is: is it possible to relabel this $\{r, R^2, s\}$ group so that its commutation relations (20) also end up looking exactly like the commutation relations (13) of the $\{X, Y, Z\}$ group? To discover the answer, again first work out the equivalent of Table 4. You should find that the overall numbers match, so there is at least the possibility of the individual elements correctly matching up as well. Then again focus on the three order-2 elements, and decide how you want to label them to match the previous result $\{Z, YZ, Y^2Z\}$. You should find that taking $s$ to be the equivalent of $Z$ works. You therefore 'just' need to decide what the second element, call it $\hat{y}$, should be.

Up to now everything is exactly the same as before in converting the $\{x, y, z\}$ group to $\{x, \tilde{y}, z\}$. For the final step there is an additional challenge though. For the $\{x, y, z\}$ group we could leave $x$ alone, since it already satisfied the desired conditions, most importantly $xz = zx$. For the $\{r, R^2, s\}$ group though, if we're taking $s$ to be the equivalent of $Z$, then the third element, call it $\hat{x}$, cannot be either of $r$ or $R^2$, since neither of these commutes with $s$. So how should we choose $\hat{x}$? Well, we want it to satisfy $\hat{x}^3 = I$, $\hat{x}\hat{y} = \hat{y}\hat{x}$ and $\hat{x}s = s\hat{x}$. So, just look at the various order-3 elements, and find the one that commutes with $s$, and that's the correct choice for $\hat{x}$.

Then, once you've chosen $\hat{x}$ and $\hat{y}$ as combinations of $r$ and $R^2$, it's again 'just' a matter of translating the original $\{r, R^2, s\}$ commutation relations (20) into new $\{\hat{x}, \hat{y}, s\}$ commutation relations, and checking whether they match the $\{X, Y, Z\}$ commutation relations (13). If you've correctly worked out the right choices for $\hat{x}$ and $\hat{y}$, you should find that they do match. That is, once the $\{r, R^2, s\}$ group has been suitably relabeled, it is also seen to have the same group structure as the previous $\{X, Y, Z\}$ and $\{x, y, z\}$ groups.

As a final comment on these specific three groups then, it is quite extraordinary that in terms of the geometry there is no question, they really are best described in terms of $\{X, Y, Z\}$, $\{x, y, z\}$, and $\{r, R^2, s\}$. To see that the underlying group structure is the same though, we have to do these complicated relabelings that would completely obscure the geometry, if we actually tried to describe it in terms of the new labelings. This demonstrates once again how different the geometrical and group theoretical perspectives can be.

In terms of just abstract group theory, these various comparisons certainly raise a great many further questions. For example, if the commutation relations (13), (14) and (20) ultimately all describe the same group structure, then which description is 'best' in some sense? Is there some standard form that we should be applying? Also, if we've seen that there are at least two different 18-element groups, one commutative and one non-commutative, then how many others might there be? And even if we succeeded in constructing one or more new 18-element groups that were clearly different from both of our groups here, at what point would we know that we've constructed all of them, and there simply aren't any further 18-element groups for us to discover?

Well, the answers to all these questions go far beyond the 'beginner level' knowledge we've assumed here! Once you've learned more group theory you can perhaps revisit these results here, and see how they fit into a more advanced understanding of the subject. I hope anyway that you've enjoyed seeing these examples, and learned something in the process. And finally, many thanks to the anonymous referee whose comments motivated this entire comparison section.

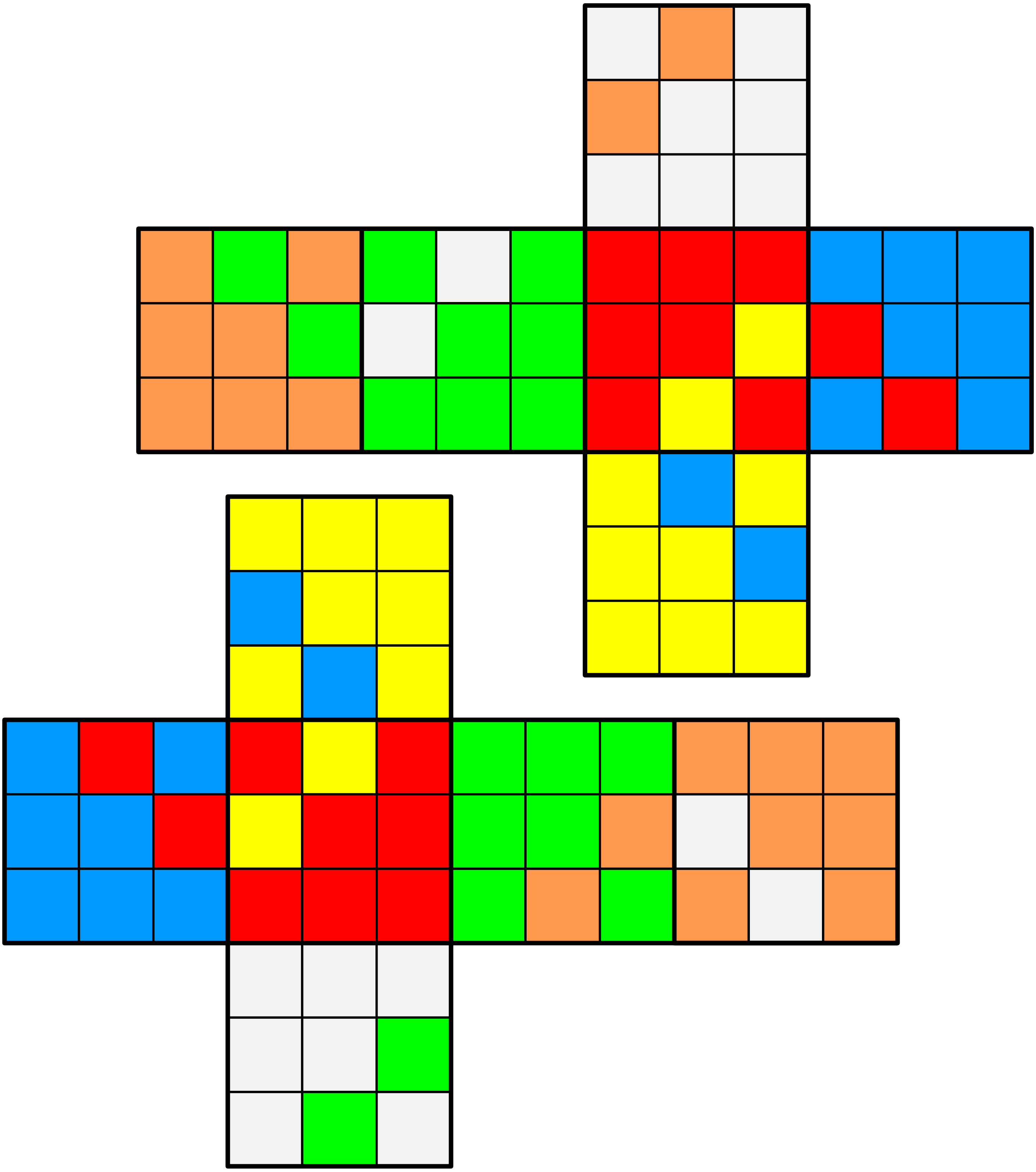

Figure 8: Templates for Figure 1 in the article. The second of these is also the side piece pattern $r$ in Figure 4.

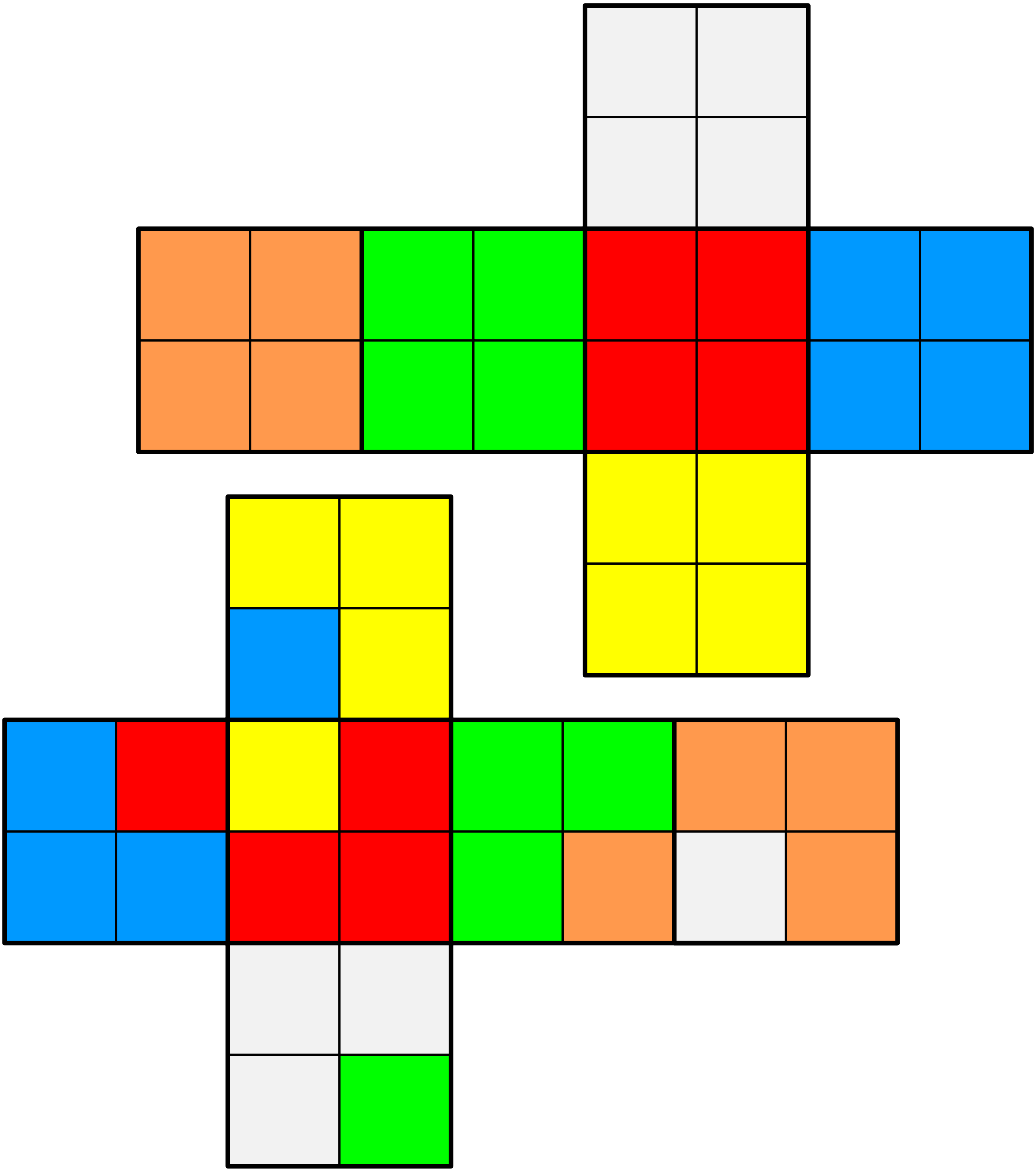

Figure 9: Templates for Figure 2, corner patterns $I$ and $X$.

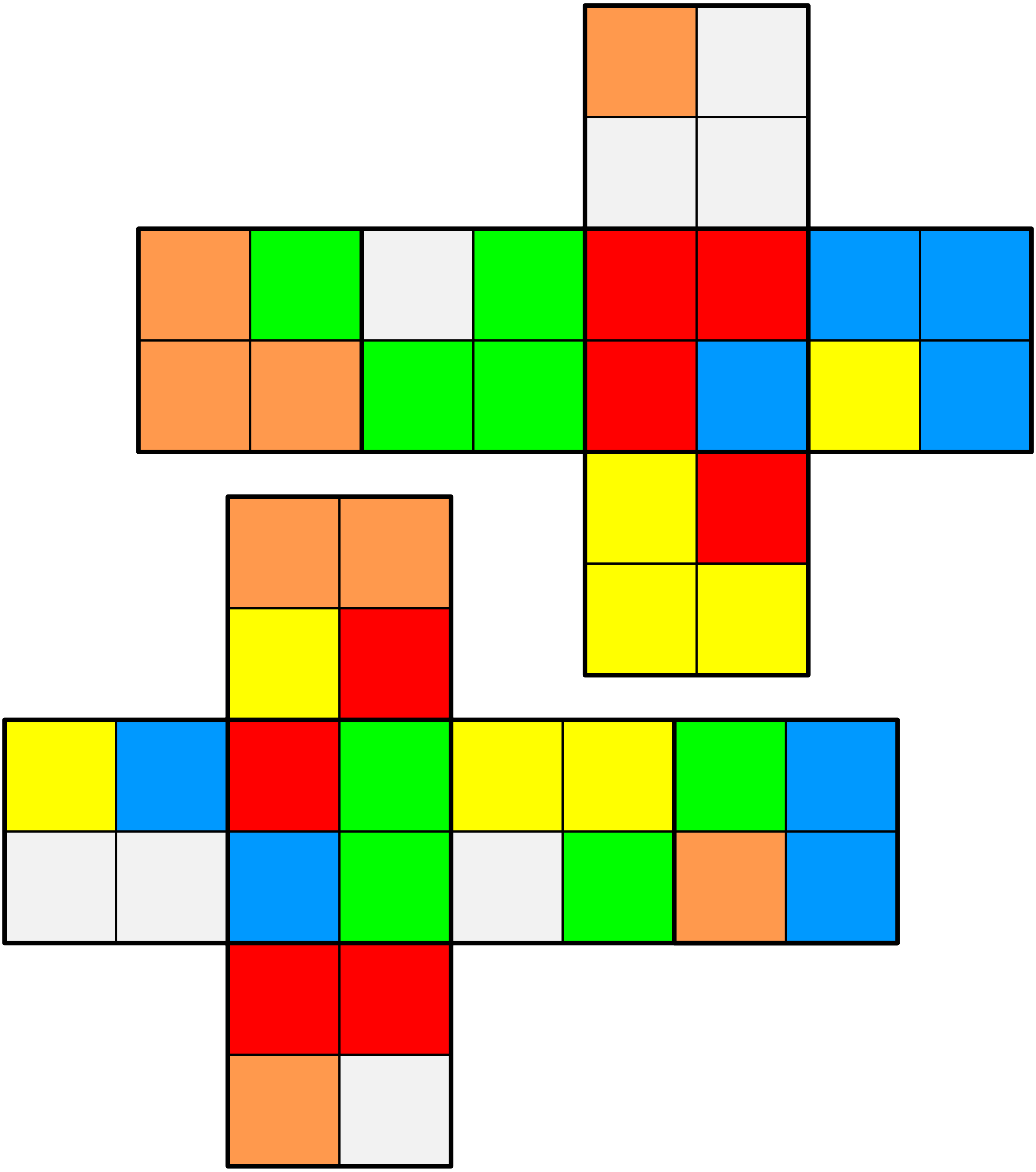

Figure 10: Templates for Figure 2, corner patterns $X^2$ and $Y$.

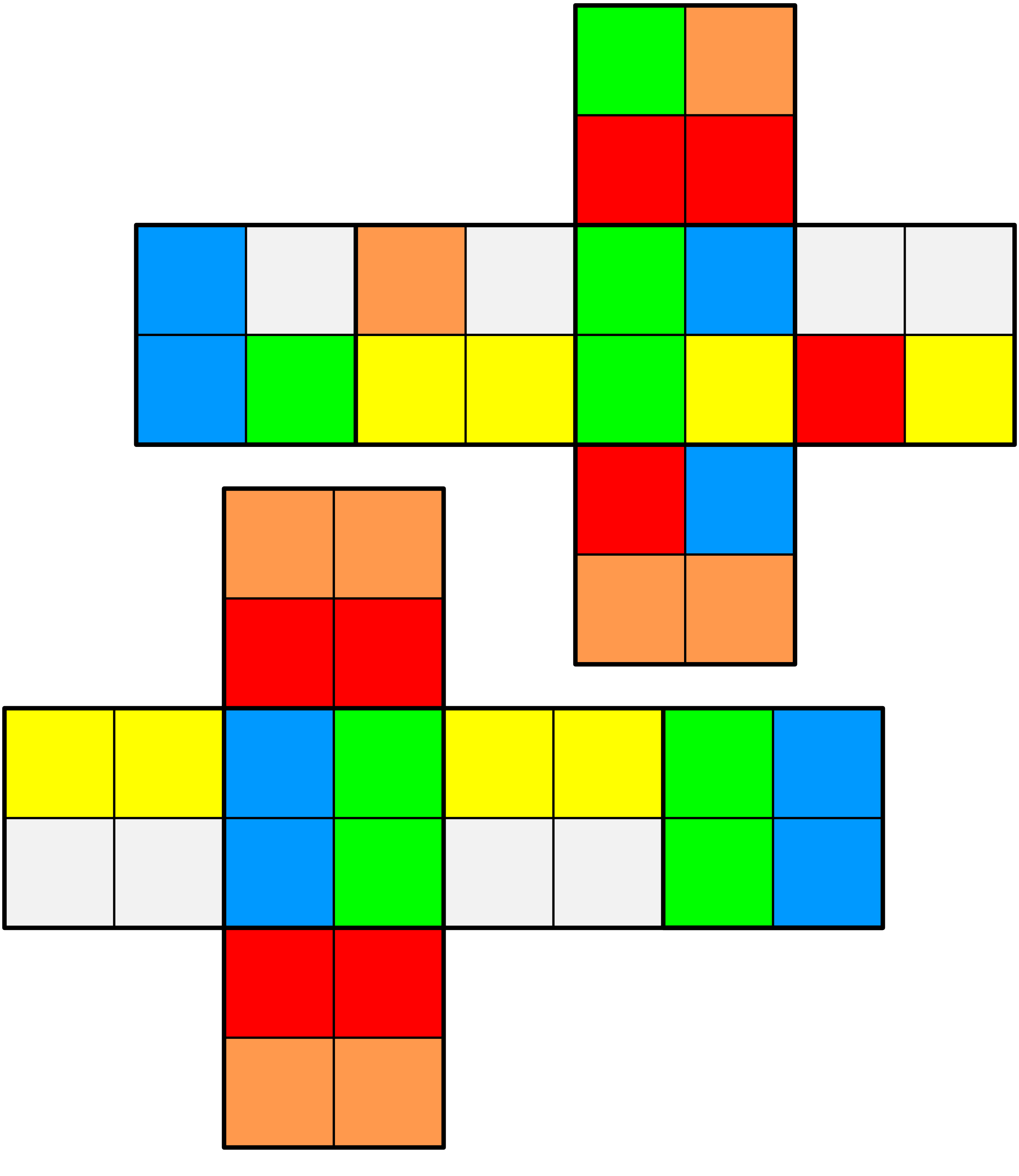

Figure 11: Templates for Figure 2, corner patterns $XY$ and $X^2Y$.

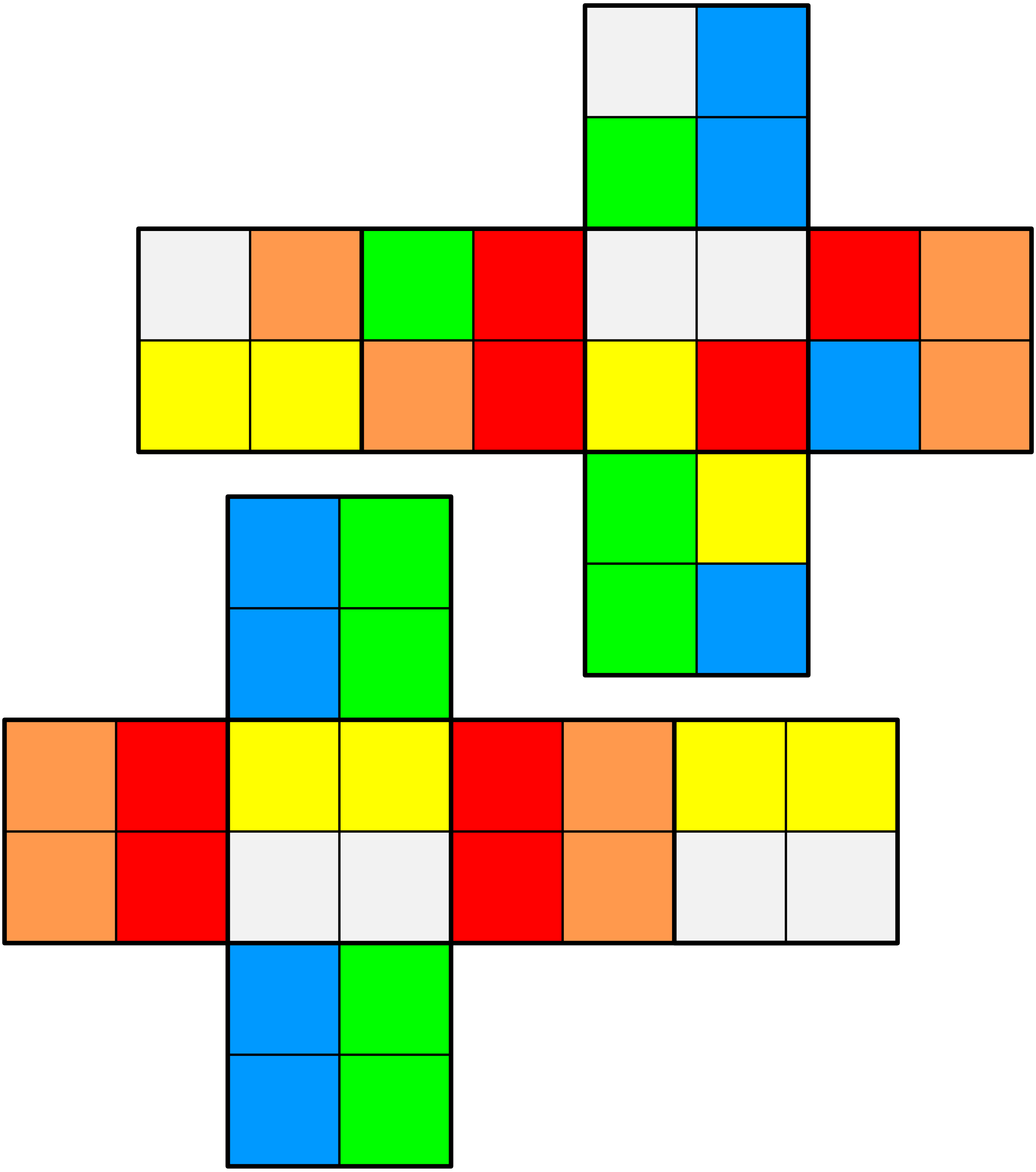

Figure 12: Templates for Figure 2, corner patterns $Y^2$ and $XY^2$.

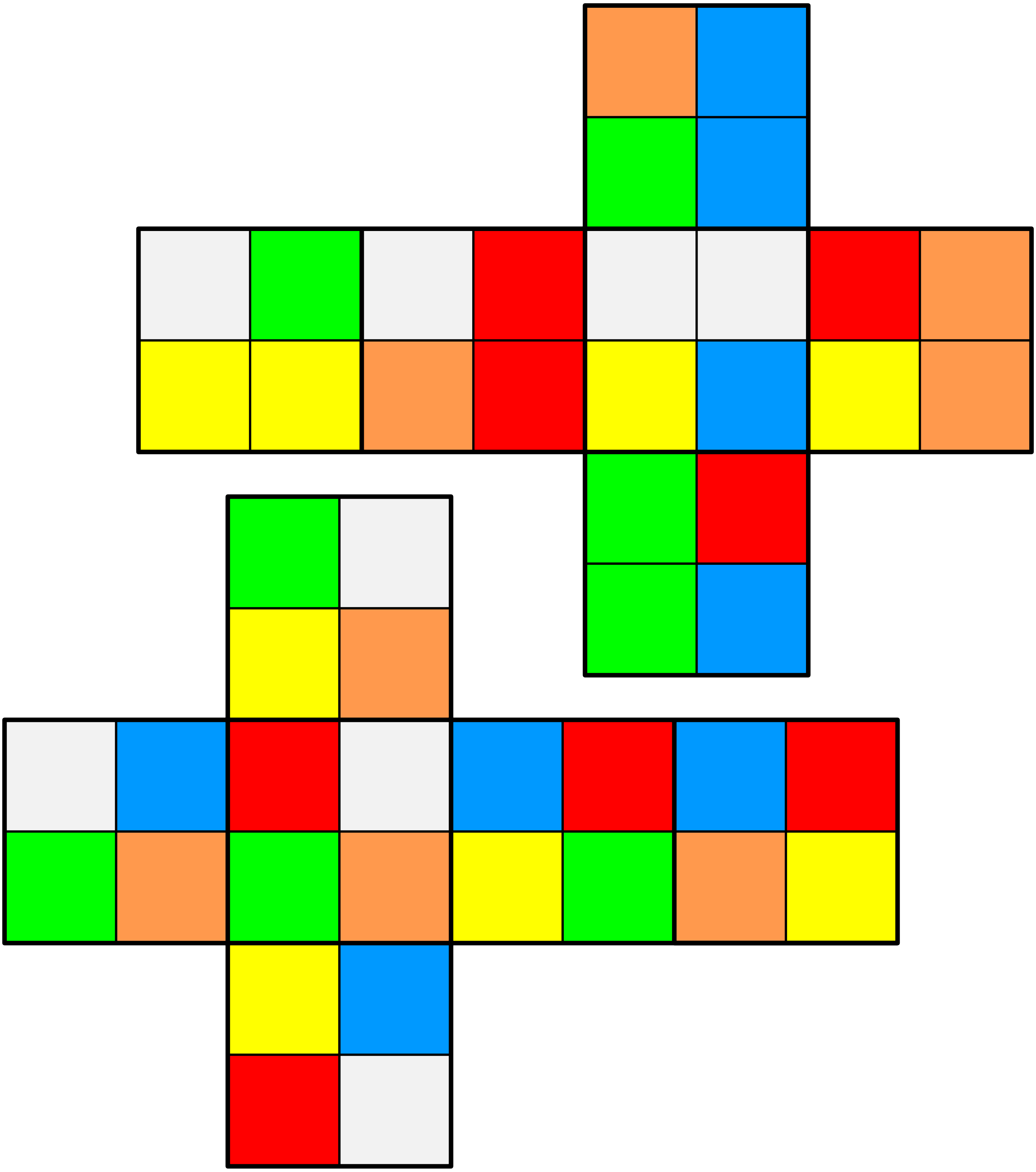

Figure 13: Templates for Figure 2, corner patterns $X^2Y^2$ and $Z$.

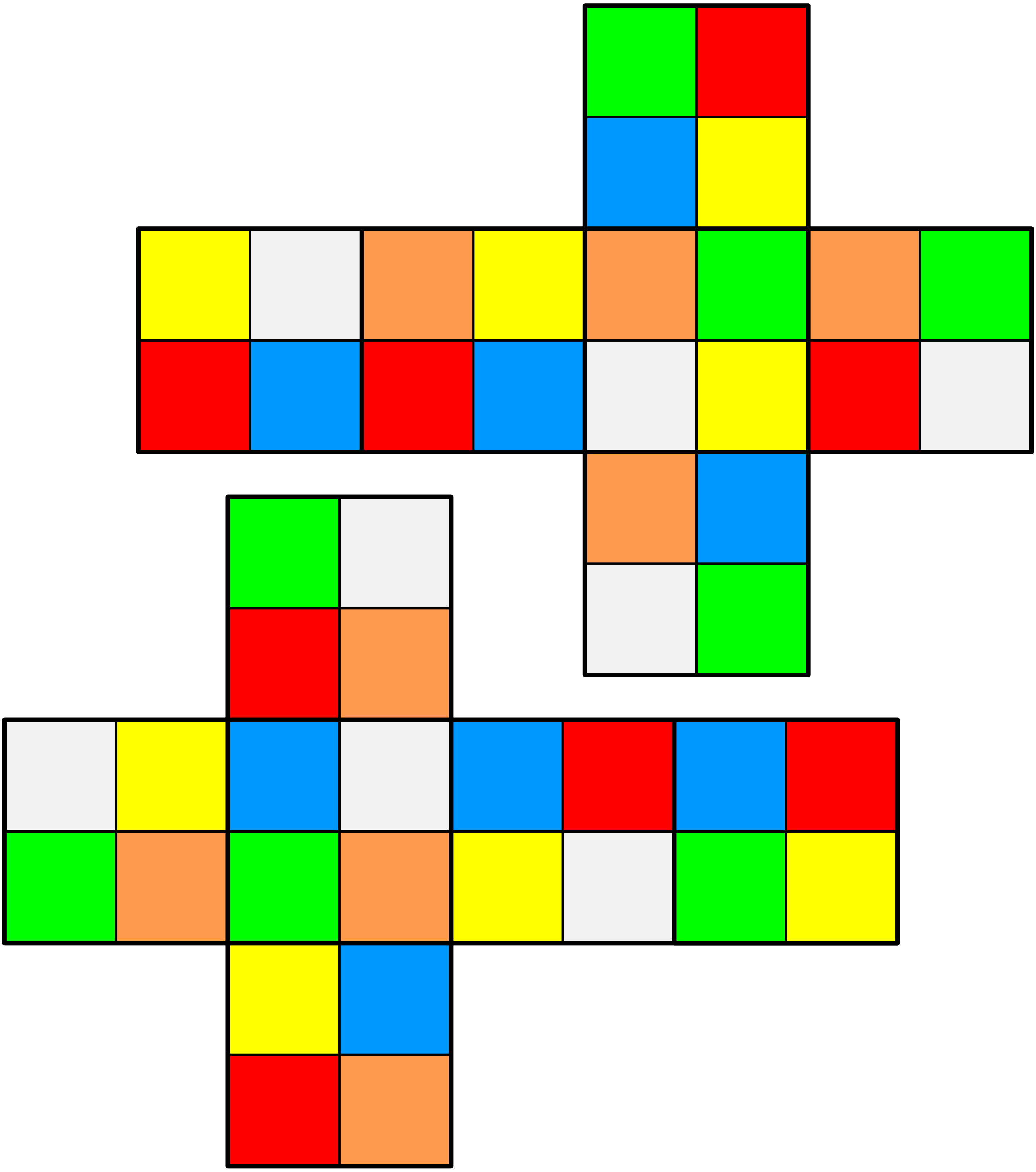

Figure 14: Templates for Figure 2, corner patterns $XZ$ and $X^2Z$.

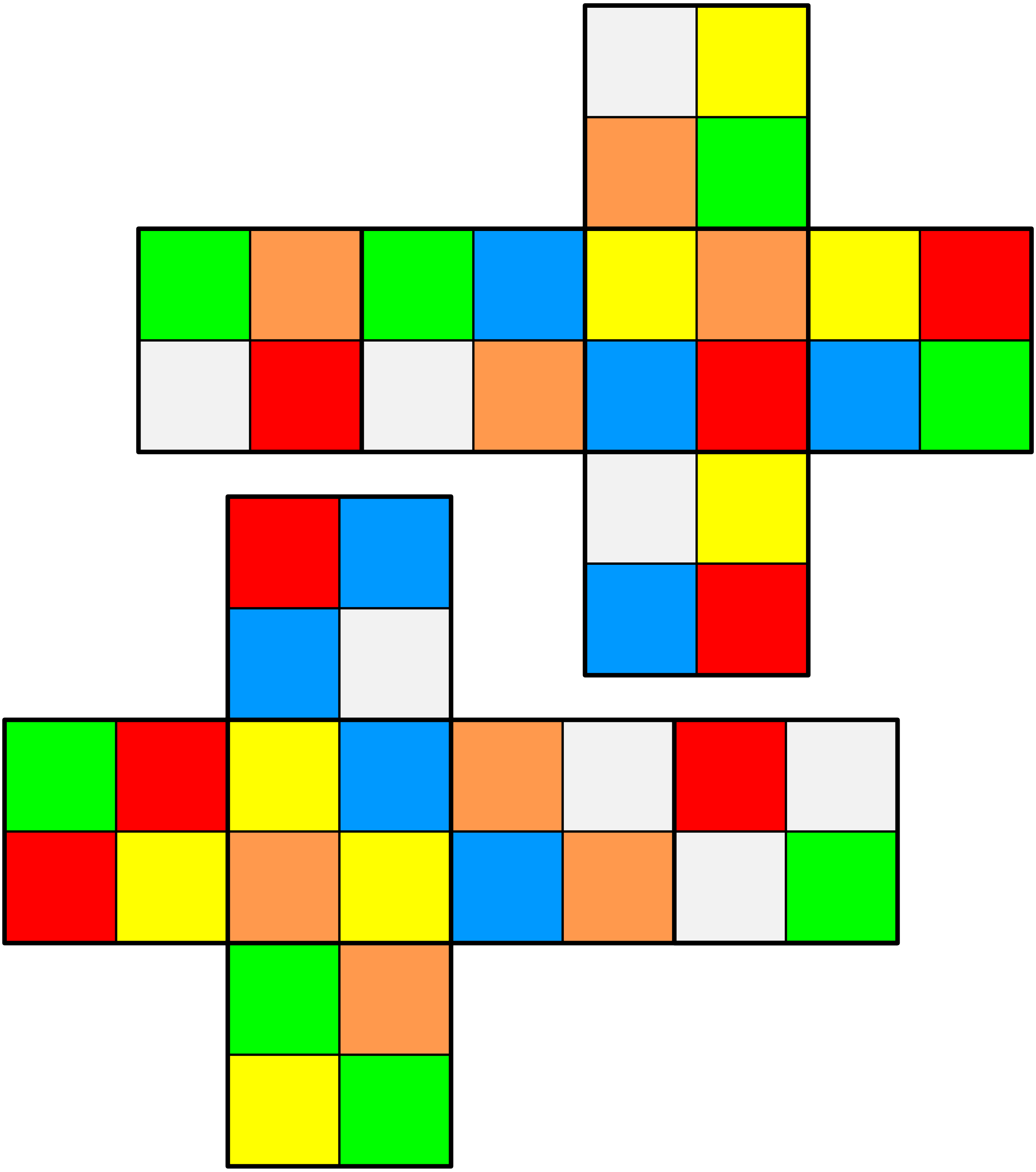

Figure 15: Templates for Figure 2, corner patterns $YZ$ and $XYZ$.

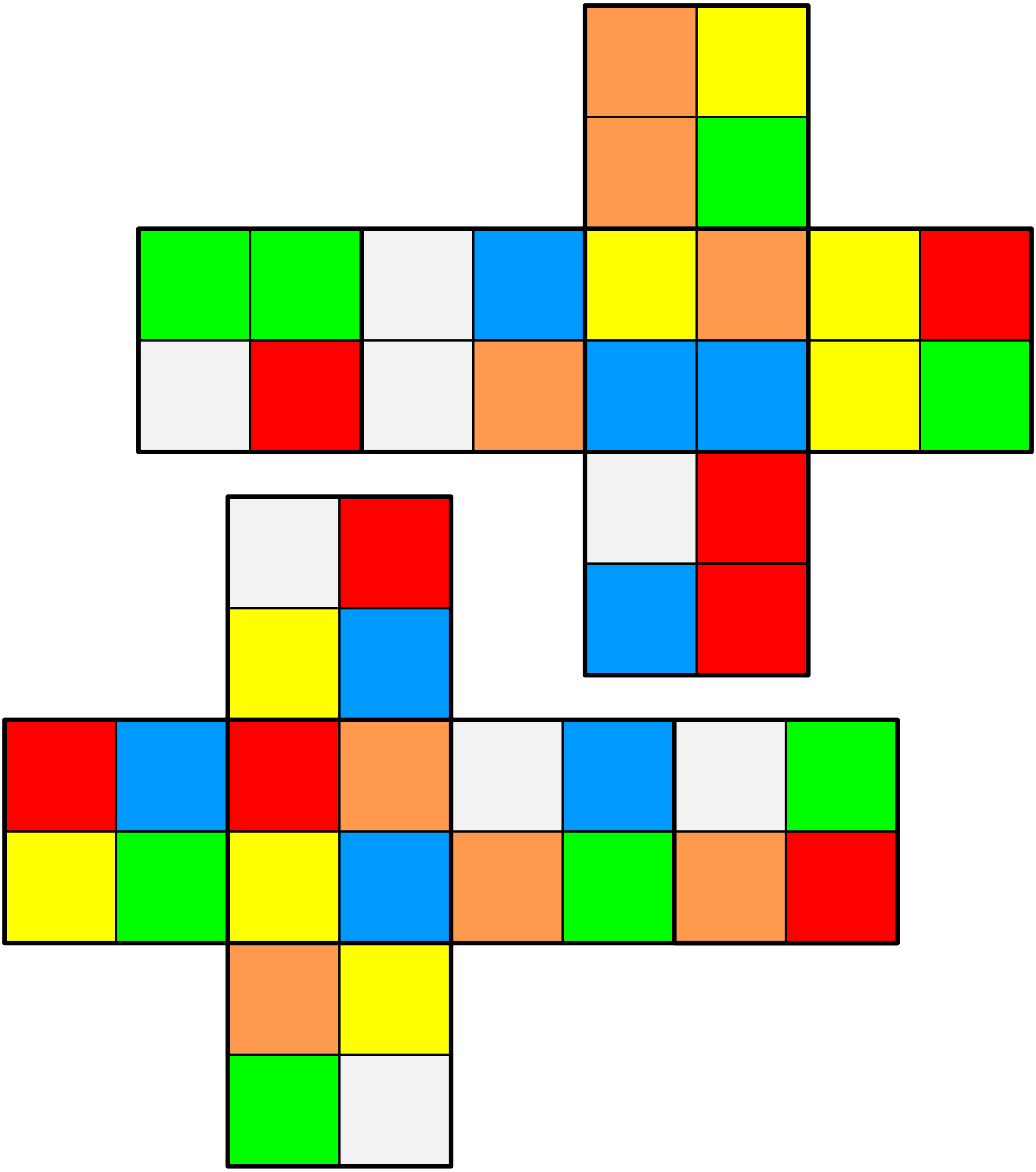

Figure 16: Templates for Figure 2, corner patterns $X^2YZ$ and $Y^2Z$.

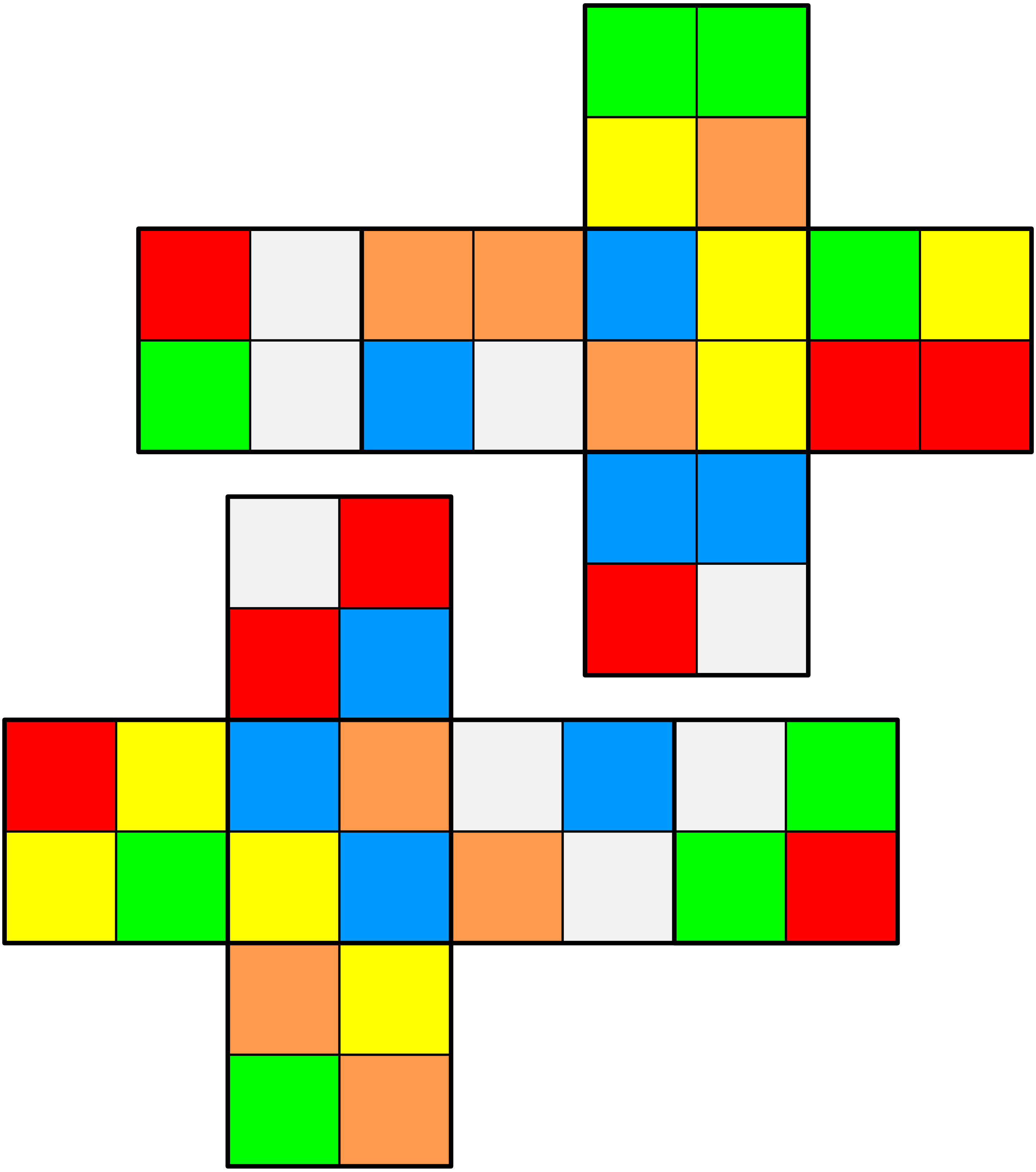

Figure 17: Templates for Figure 2, corner patterns $XY^2Z$ and $X^2Y^2Z$.

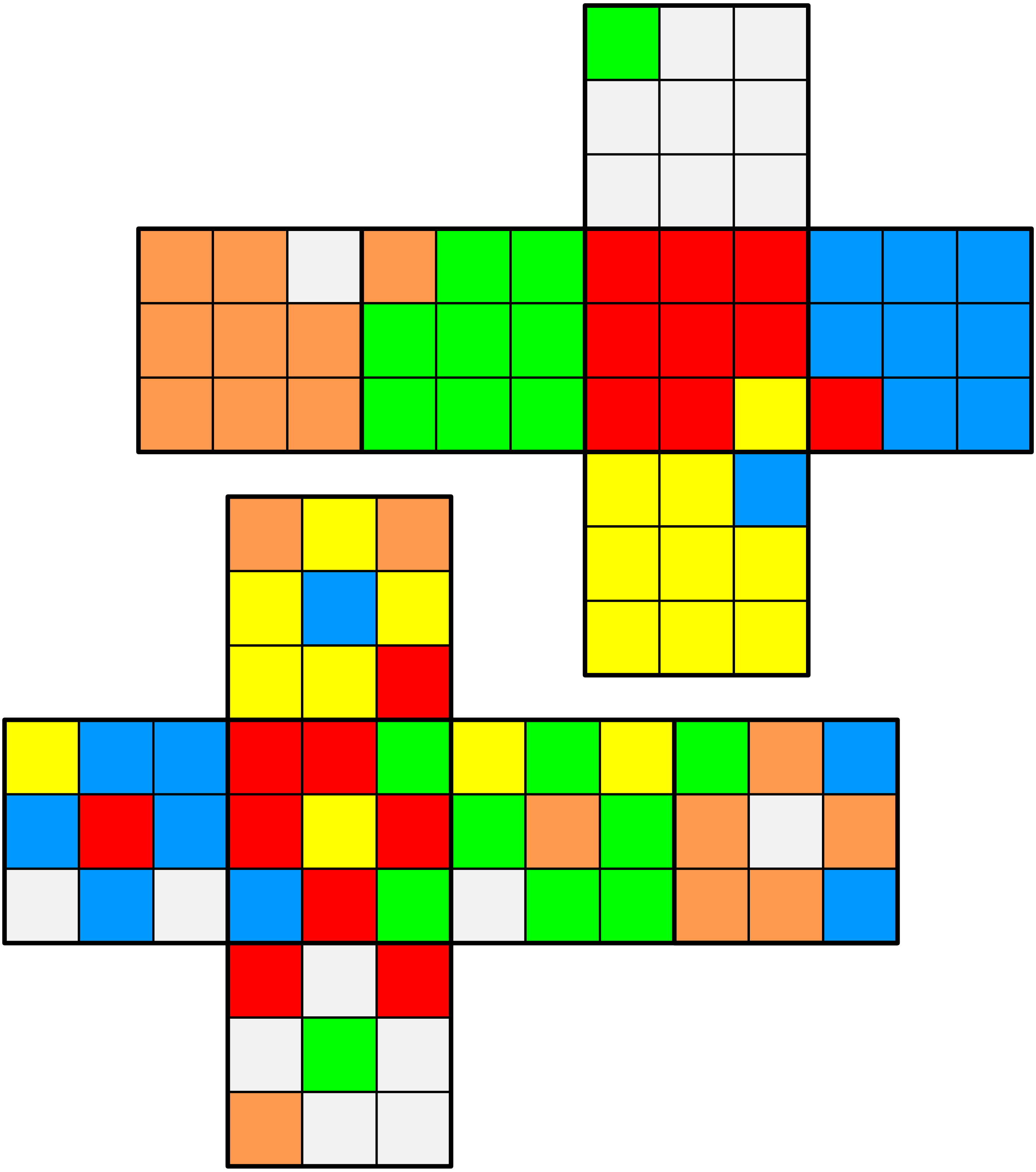

Figure 18: Templates for Figure 3, patterns $I_c X$ and $C Y$.

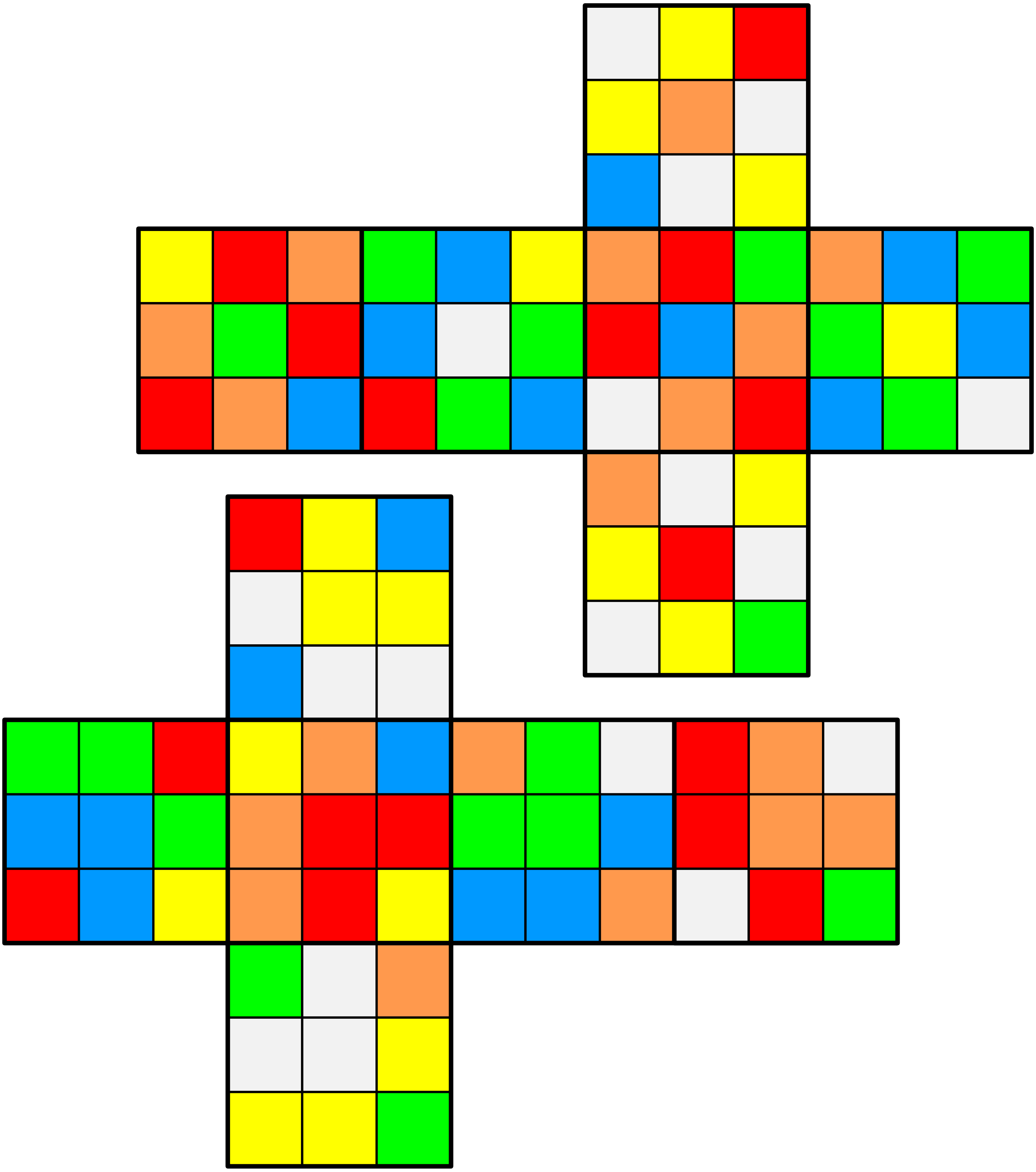

Figure 19: Templates for Figure 3, patterns $C^2\,Z$ and $I_c\,XYZ$.

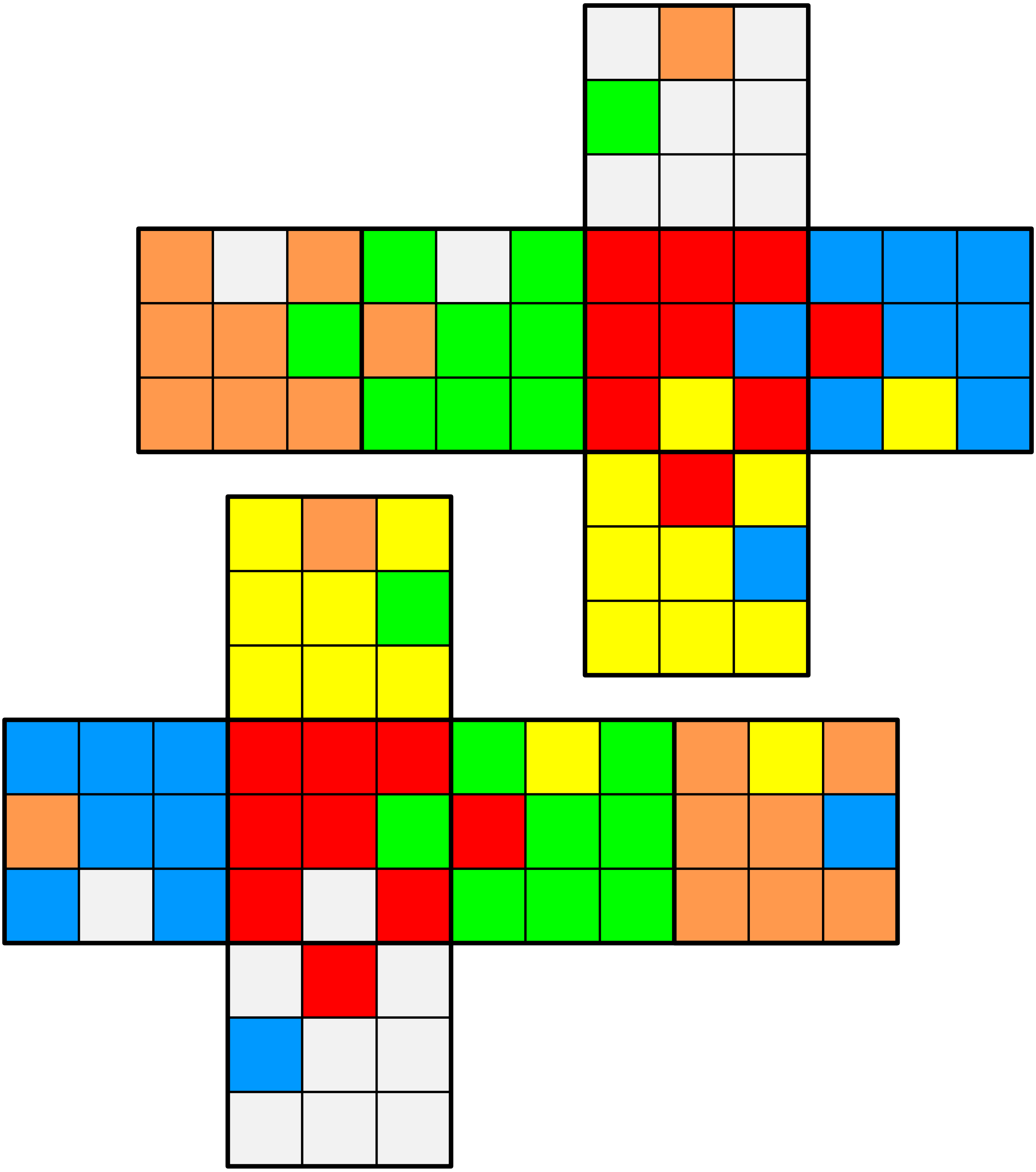

Figure 20: Templates for Figure 4, side piece actions $f$ and $F$.

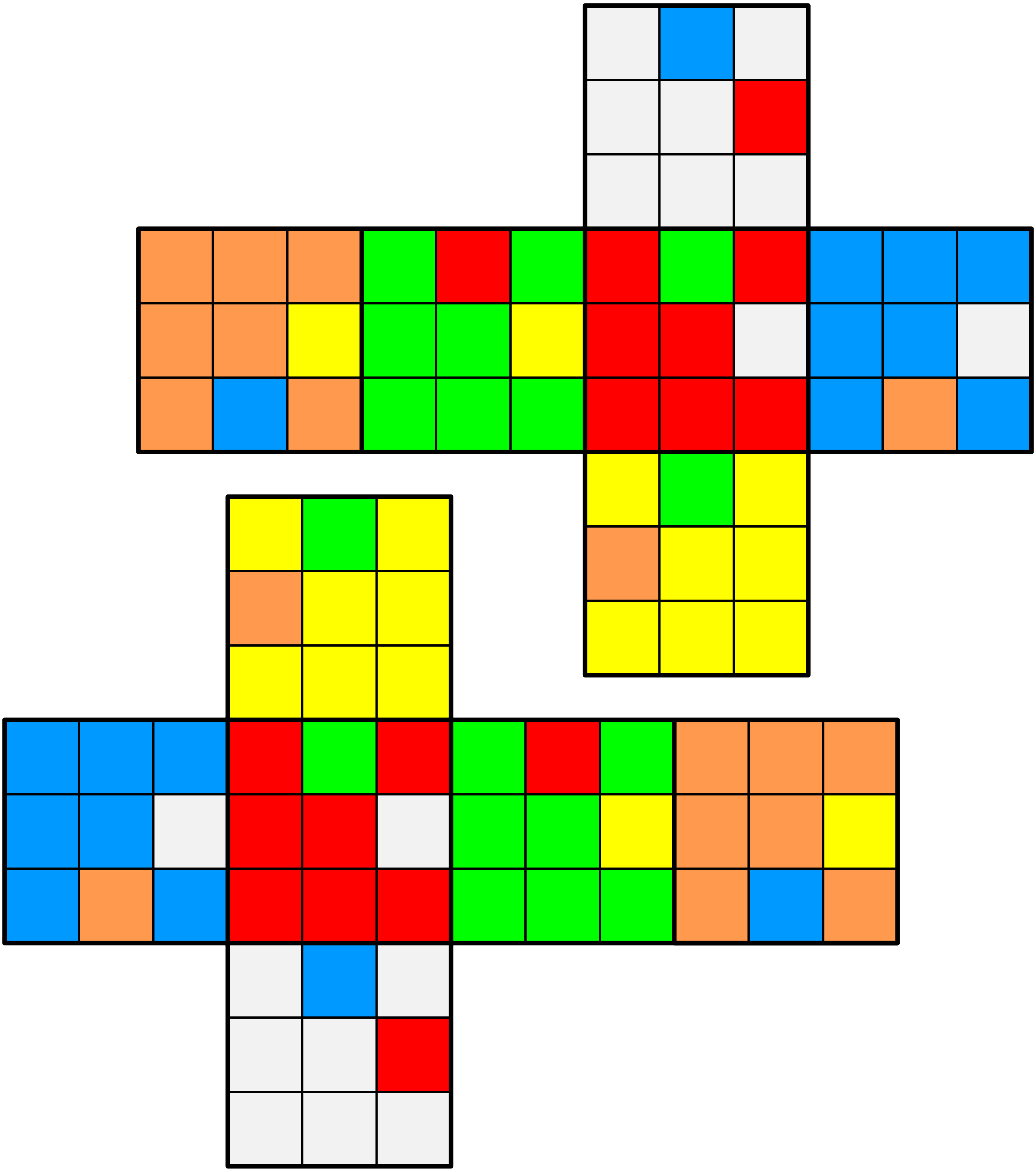

Figure 21: Templates for Figure 4, side piece actions $s_a$ and $s_b$.

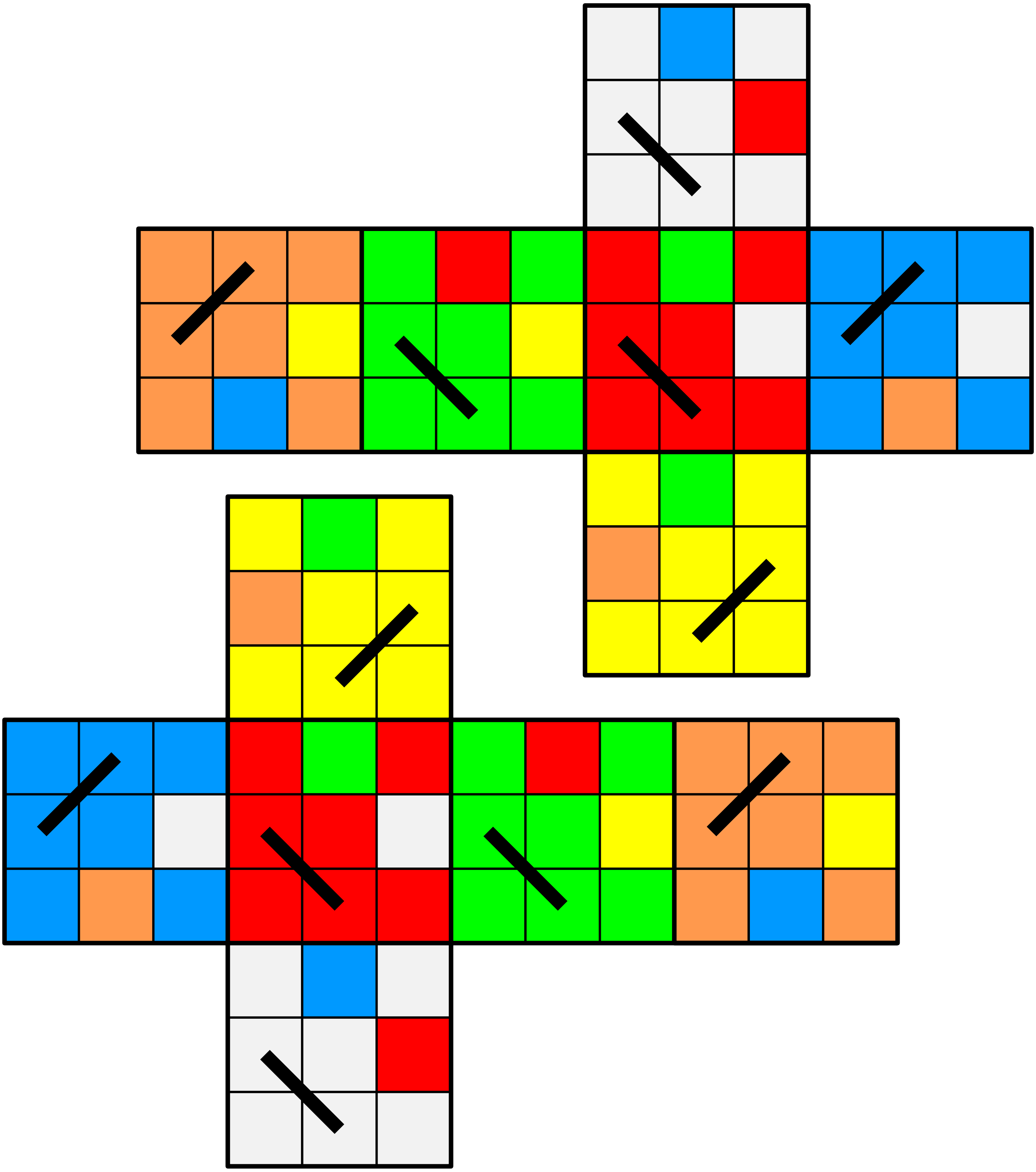

Figure 22: Templates for Figure 4, side piece actions $s_a$ and $s_b$, now with the lines added that explicitly show which side pieces are being exchanged.

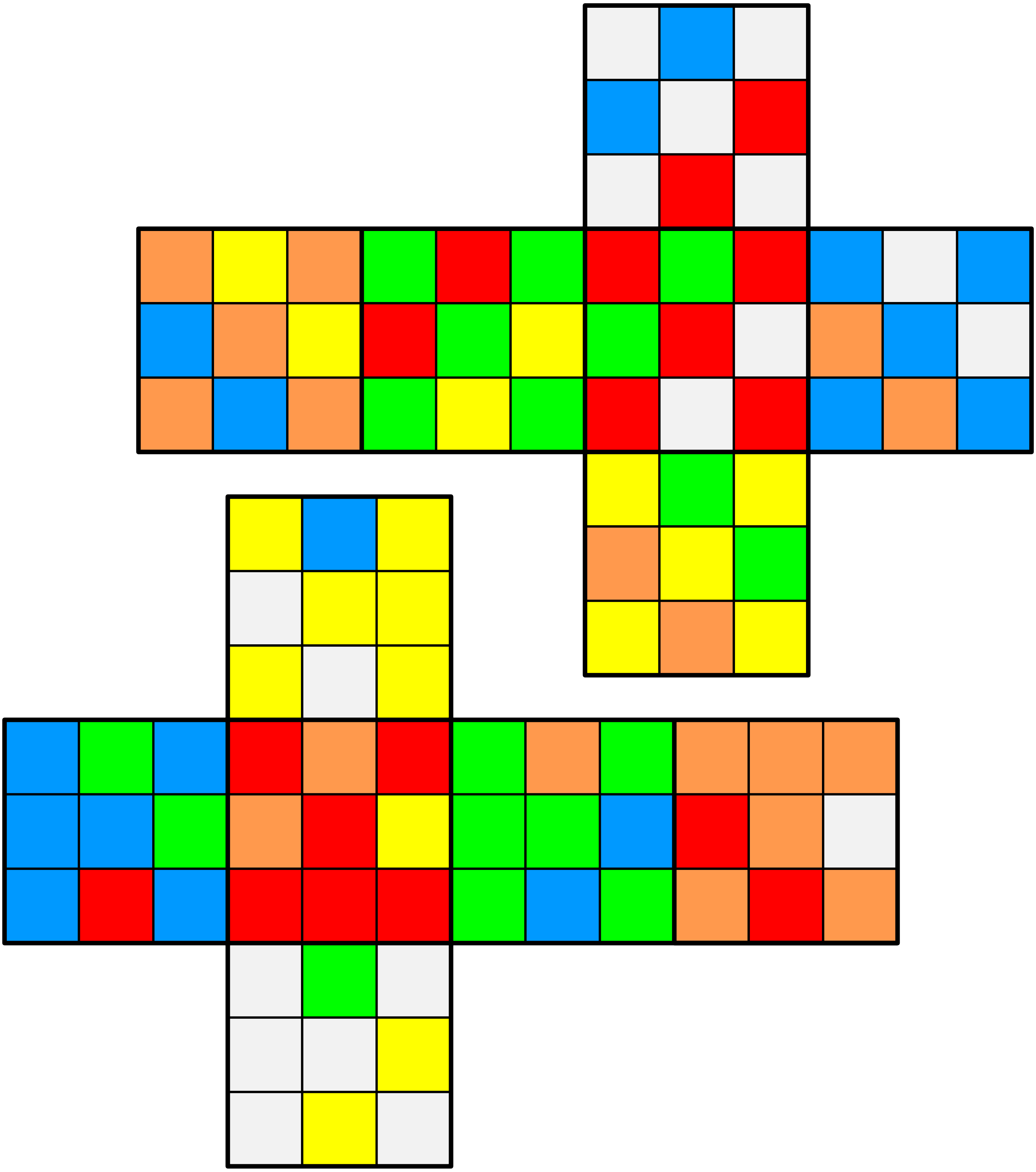

Figure 23: Templates for Figure 4, the combination $s_a s_b$ and the side piece action $R$.

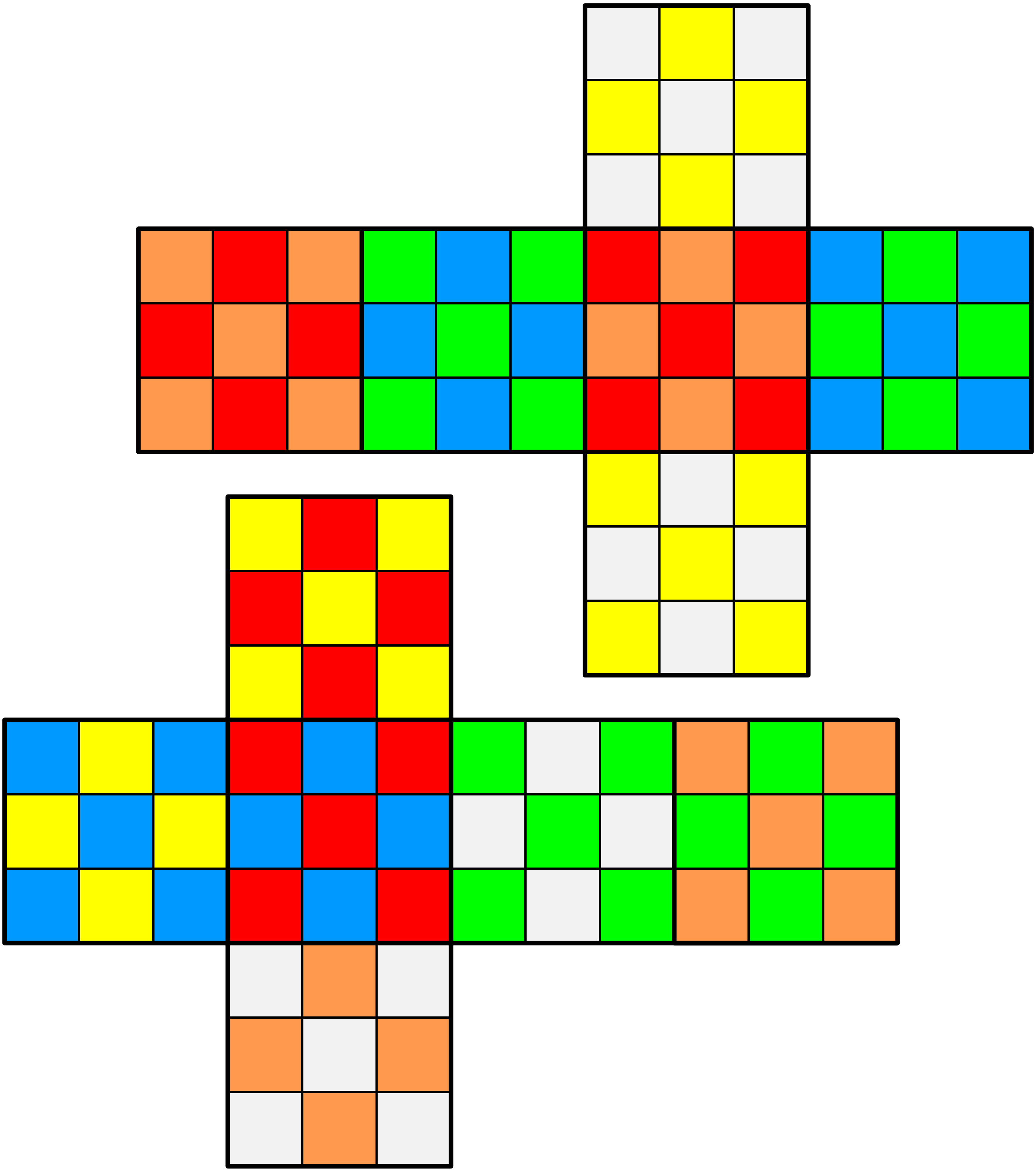

Figure 24: Templates for Figures 5a and 5b.

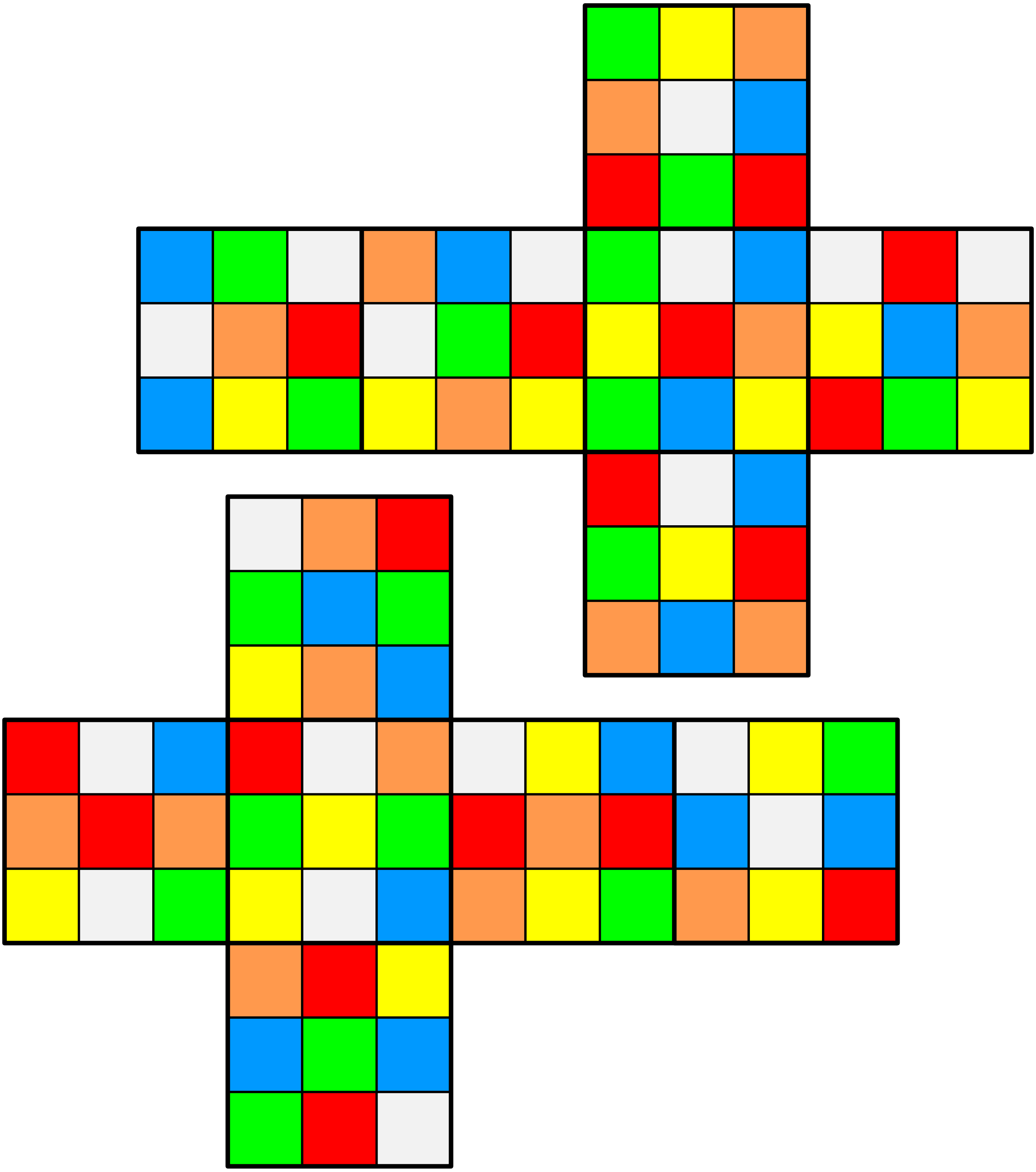

Figure 25: Templates for Figures 5c and 5d.

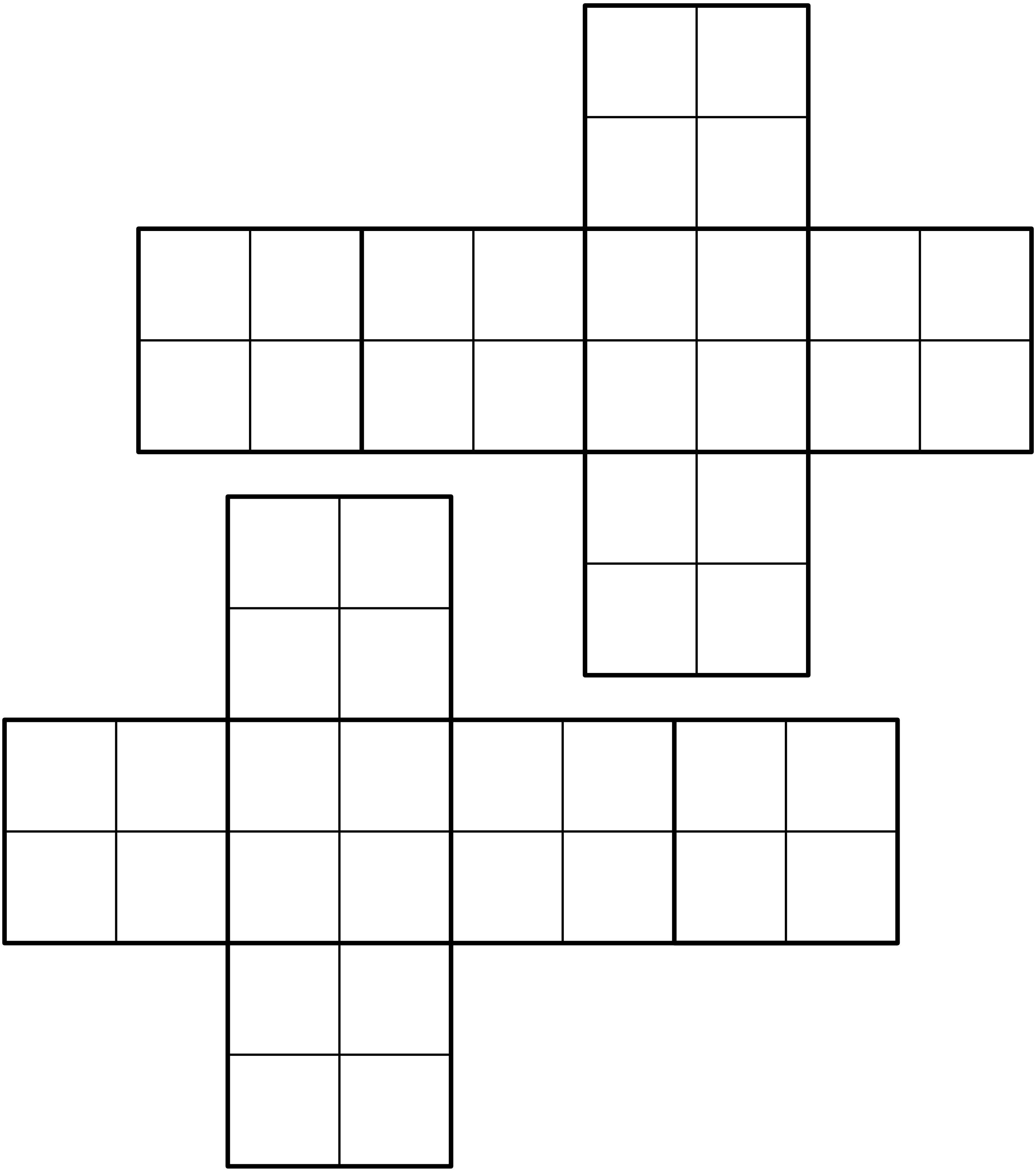

Figure 26: Some blank $2 \times 2 \times 2$ templates may also be useful.

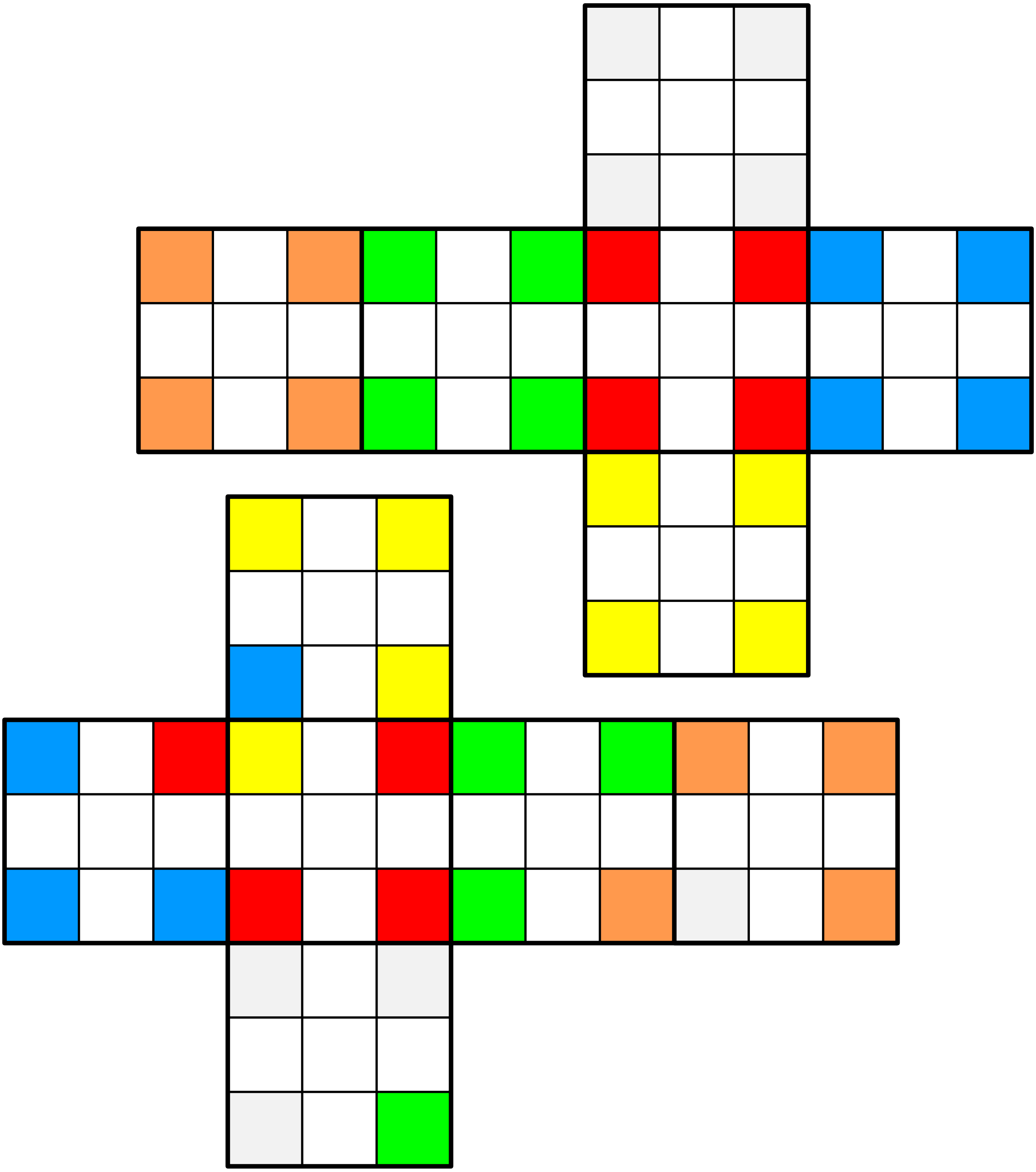

Figure 27: Corner patterns $I$ and $X$.

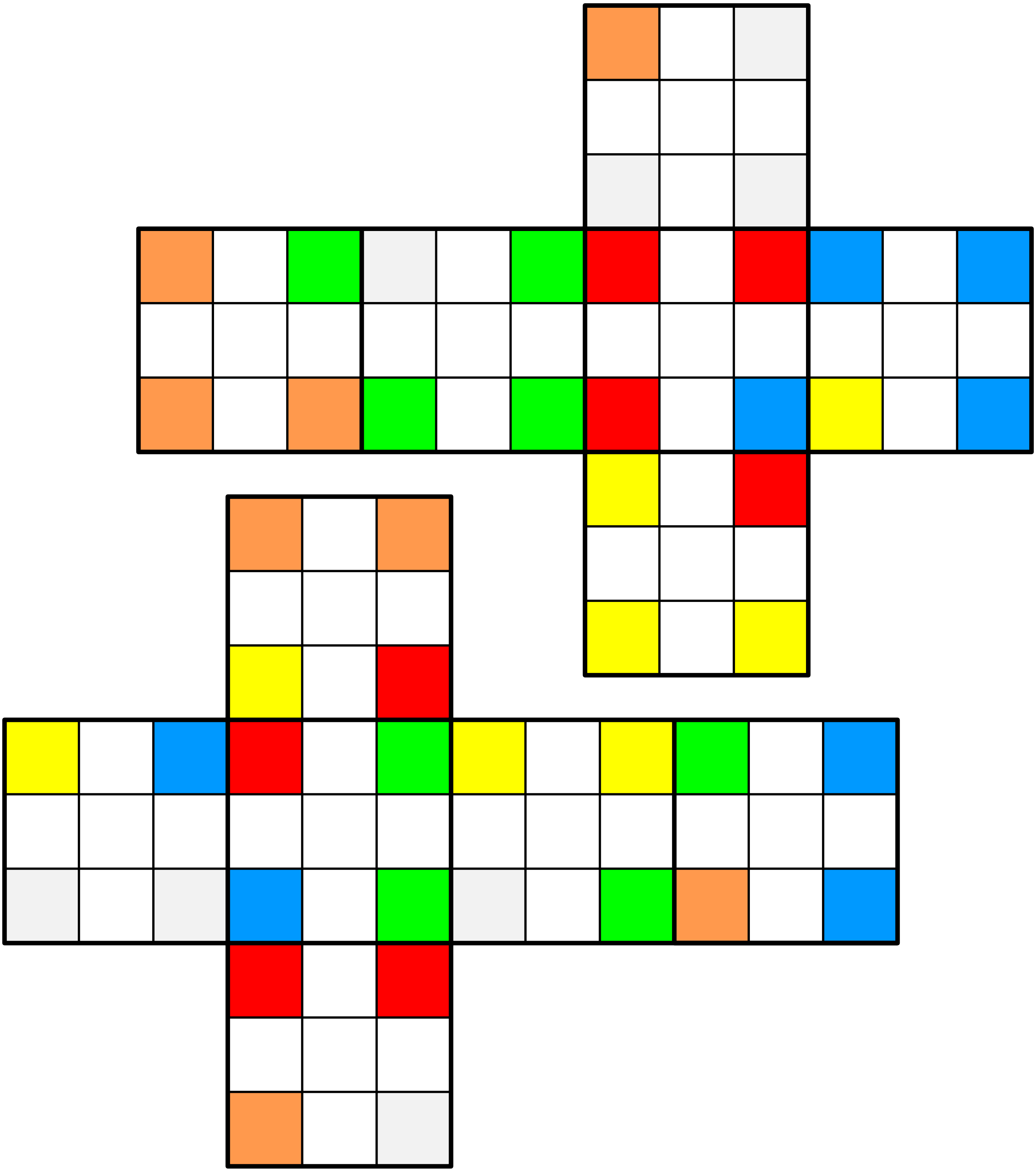

Figure 28: Corner patterns $X^2$ and $Y$.

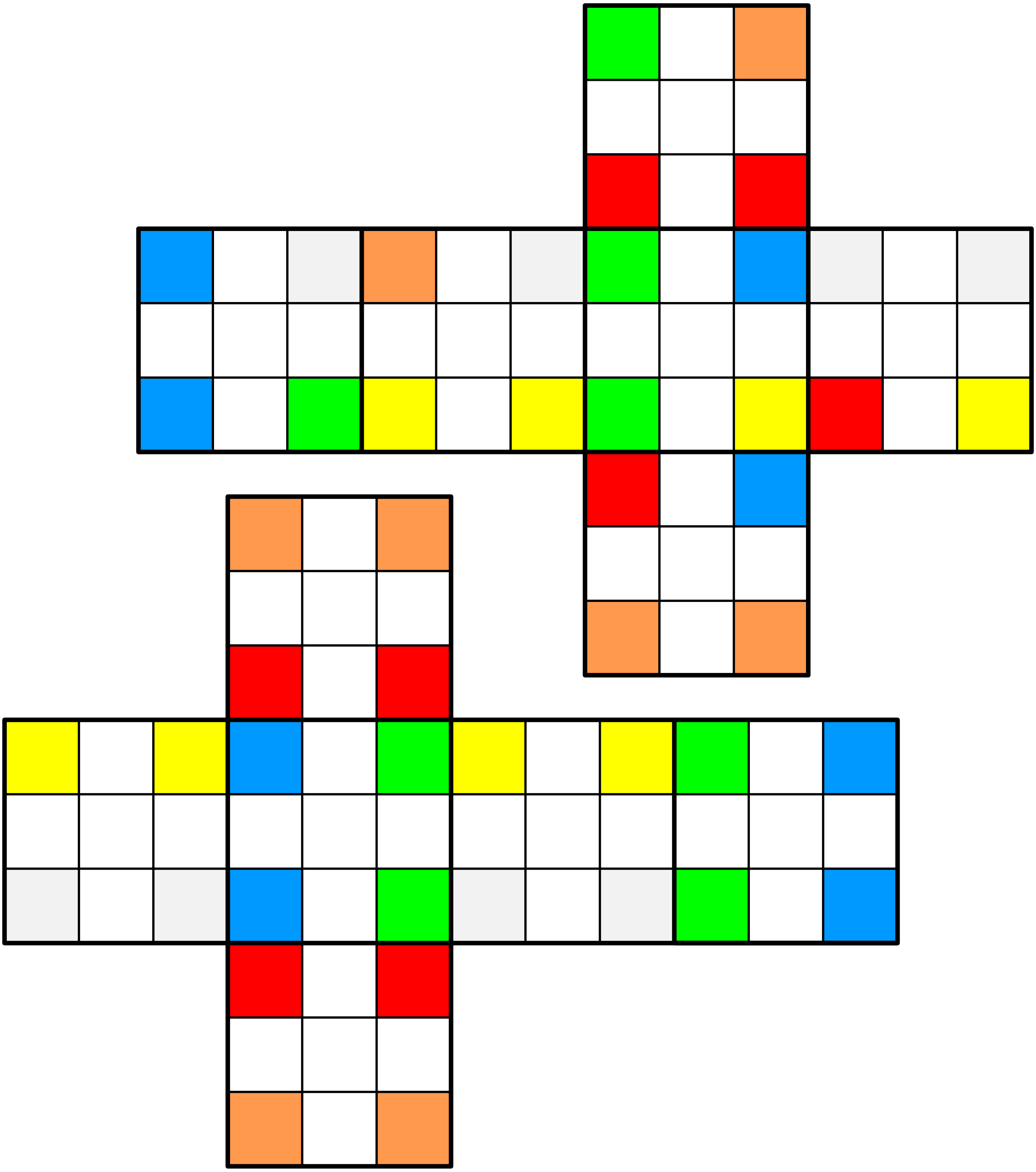

Figure 29: Corner patterns $XY$ and $X^2Y$.

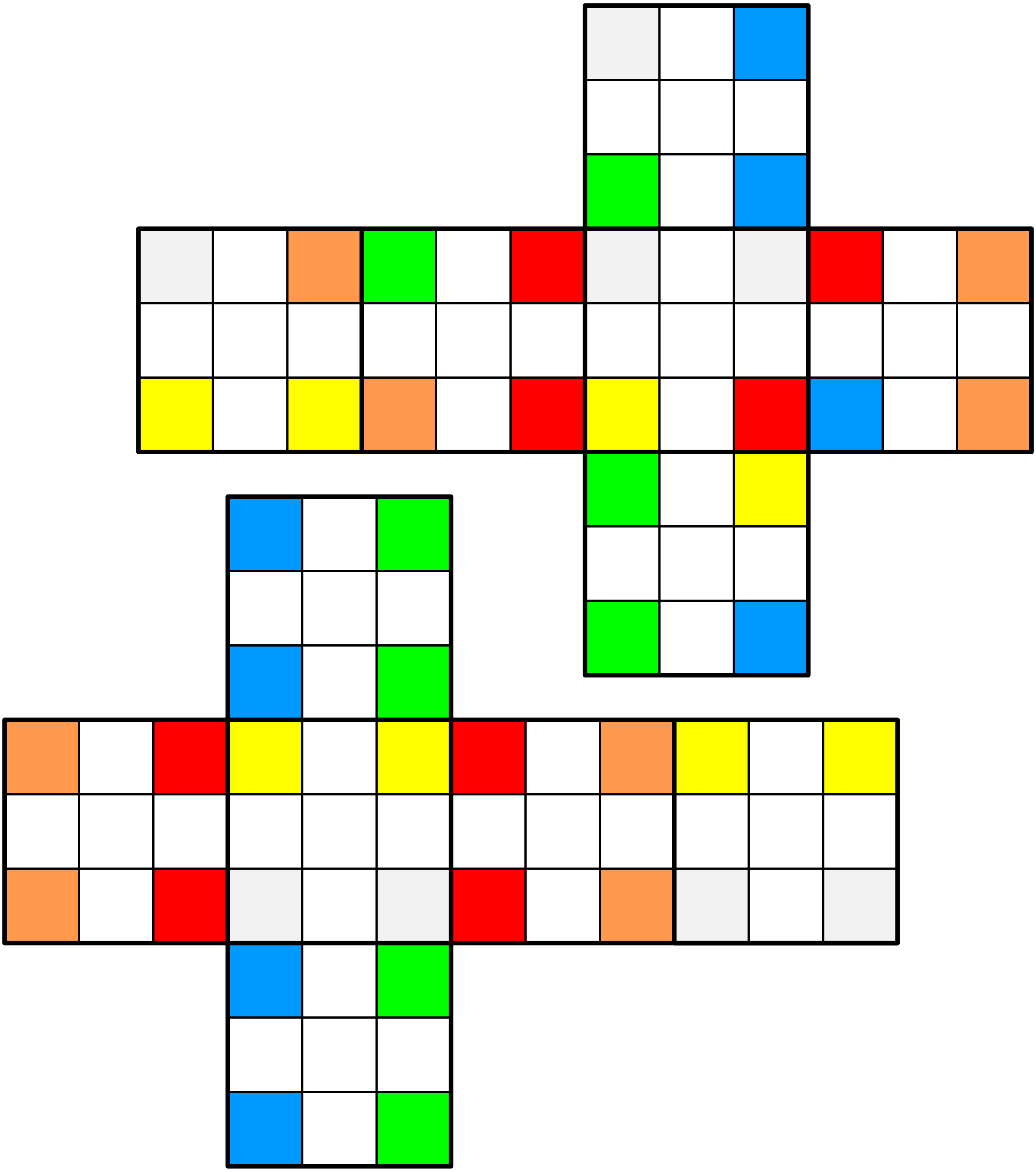

Figure 30: Corner patterns $Y^2$ and $XY^2$.

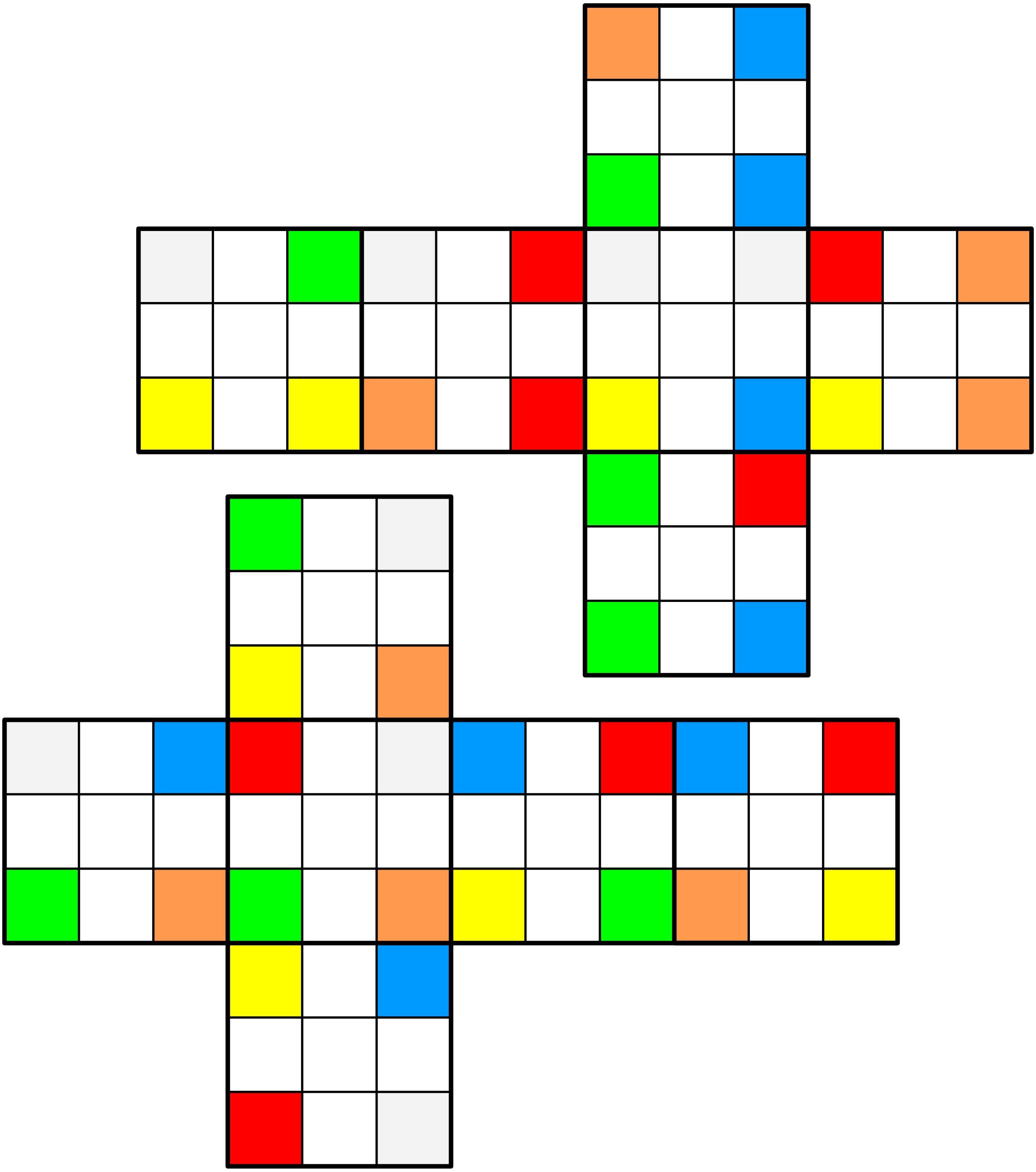

Figure 31: Corner patterns $X^2Y^2$ and $Z$.

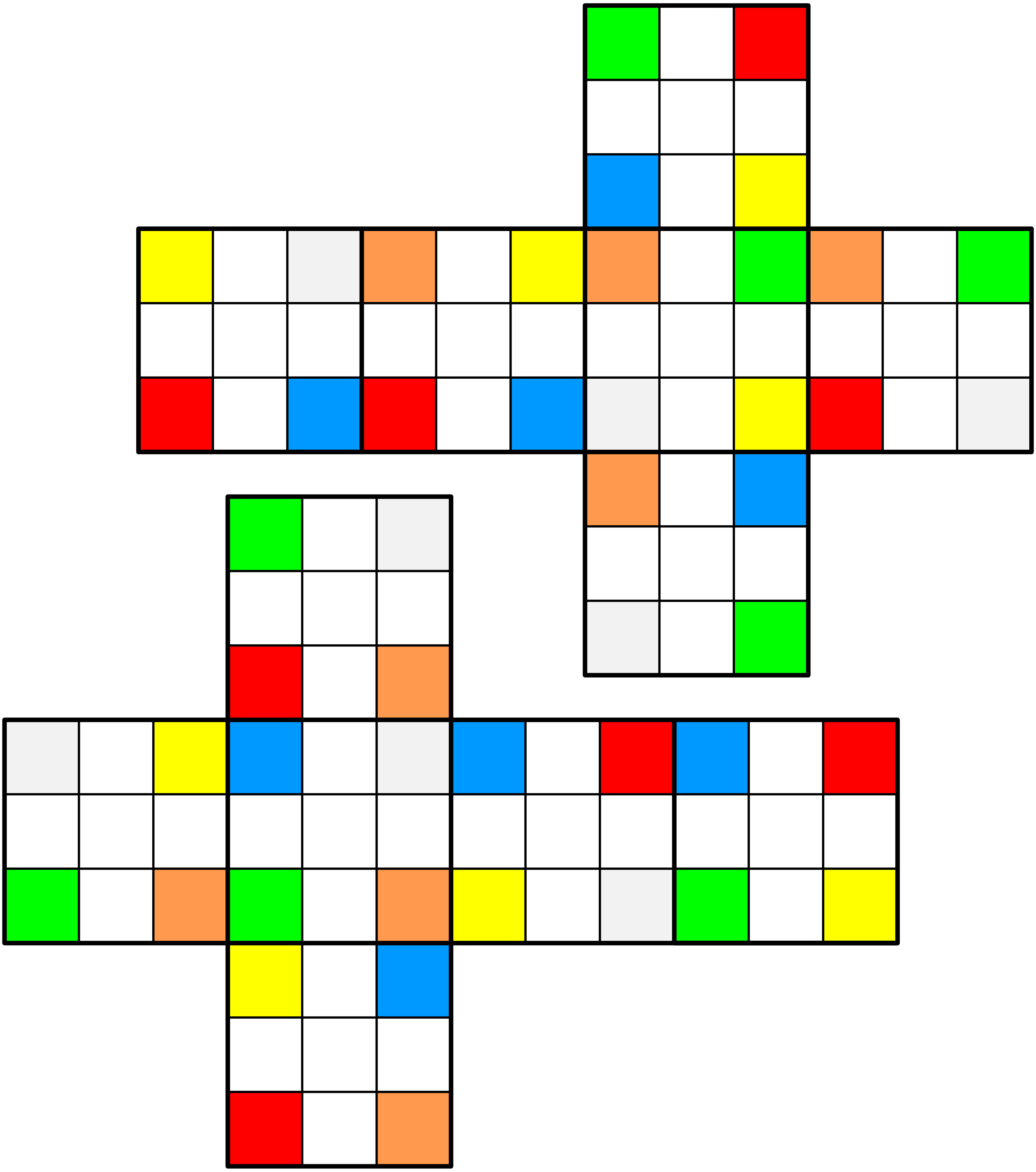

Figure 32: Corner patterns $XZ$ and $X^2Z$.

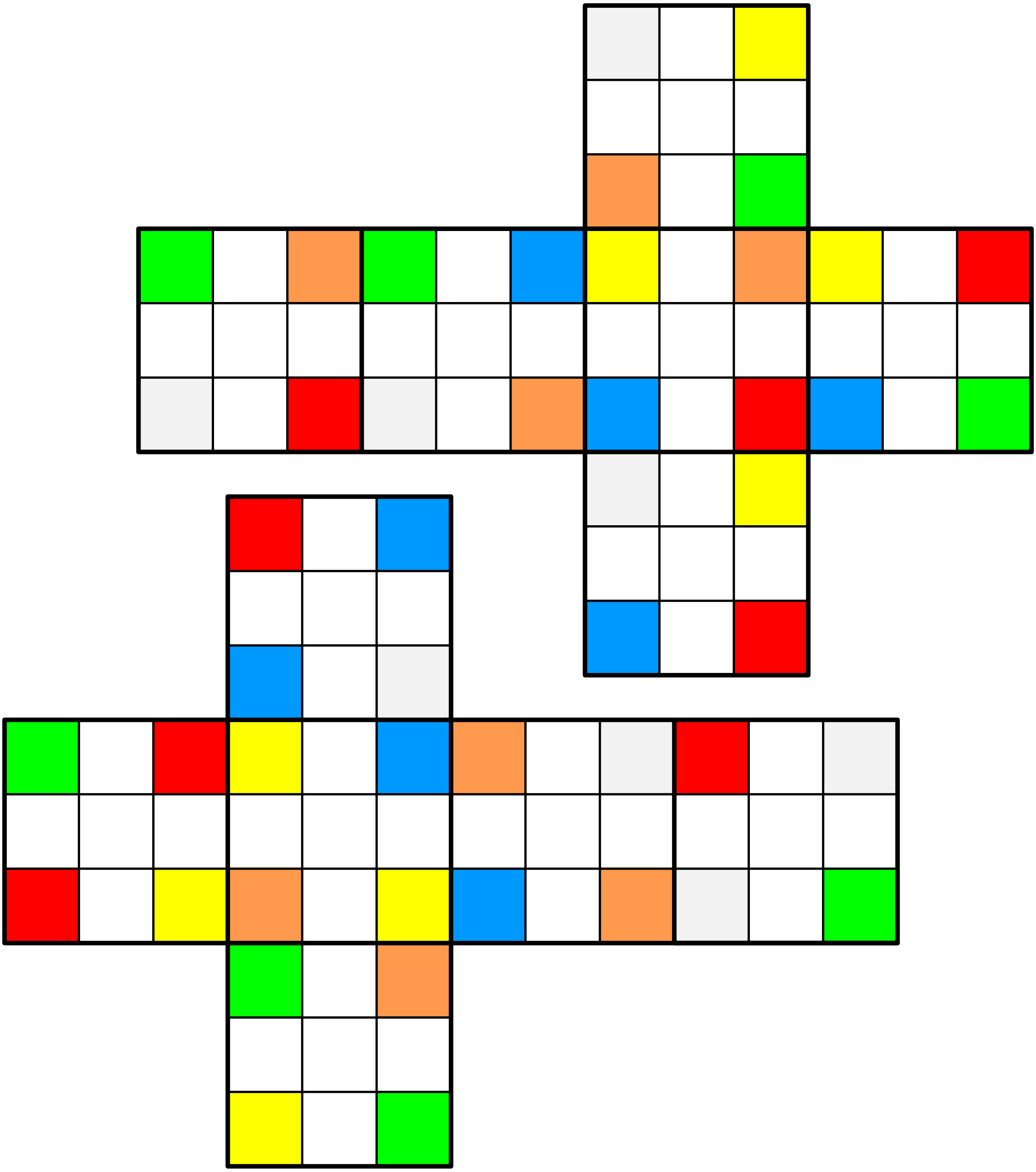

Figure 33: Corner patterns $YZ$ and $XYZ$.

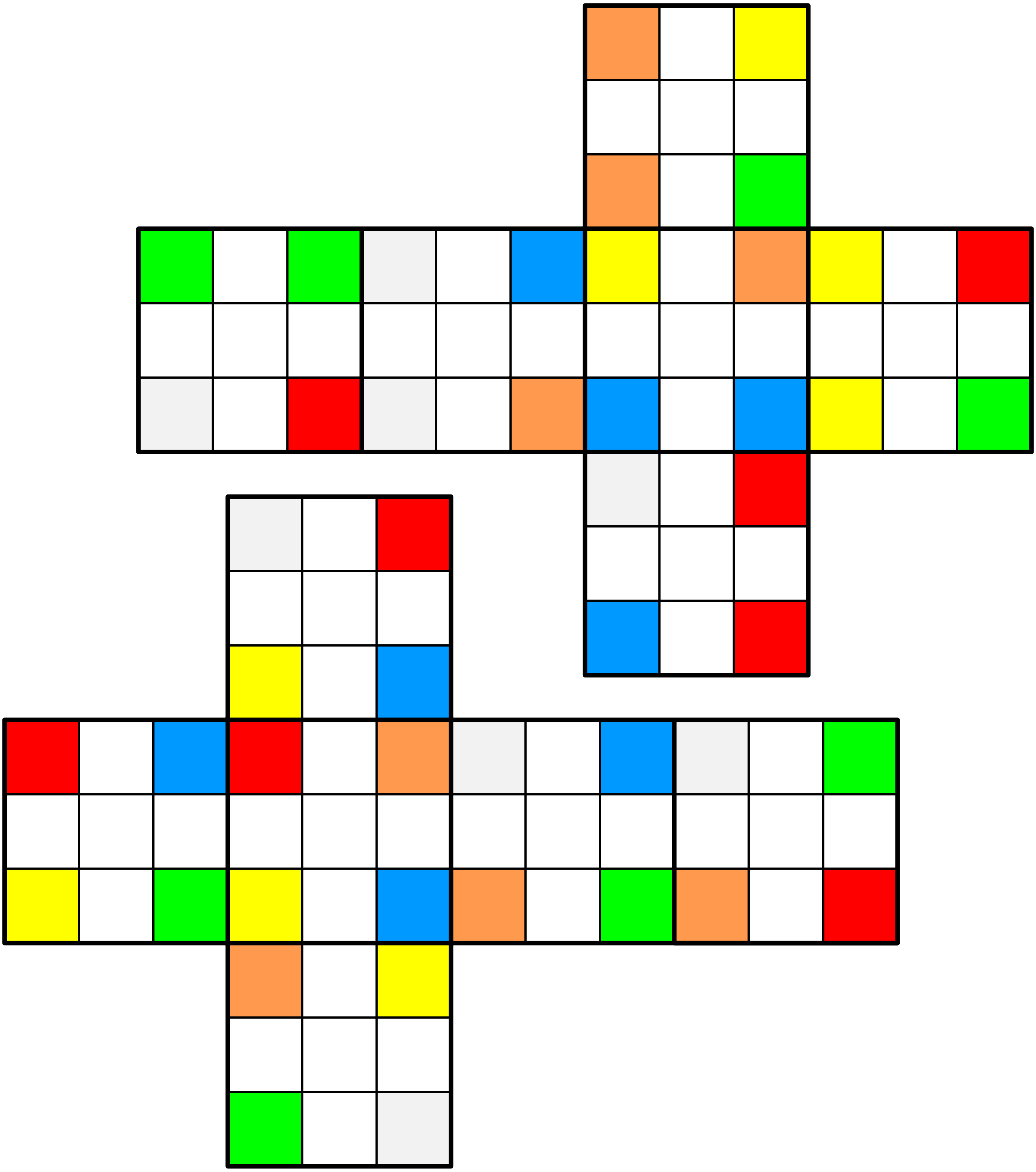

Figure 34: Corner patterns $X^2YZ$ and $Y^2Z$.

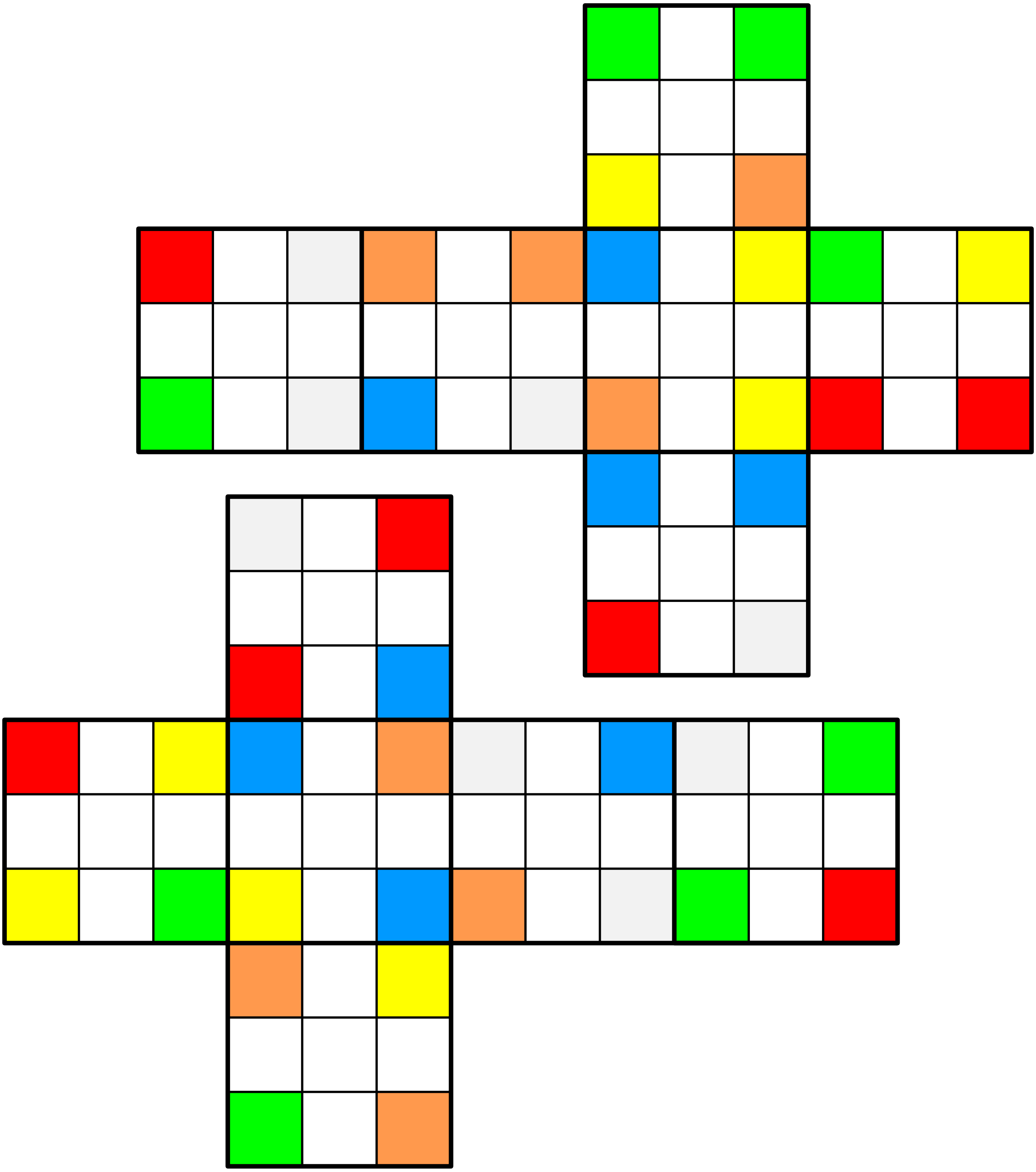

Figure 35: Corner patterns $XY^2Z$ and $X^2Y^2Z$.

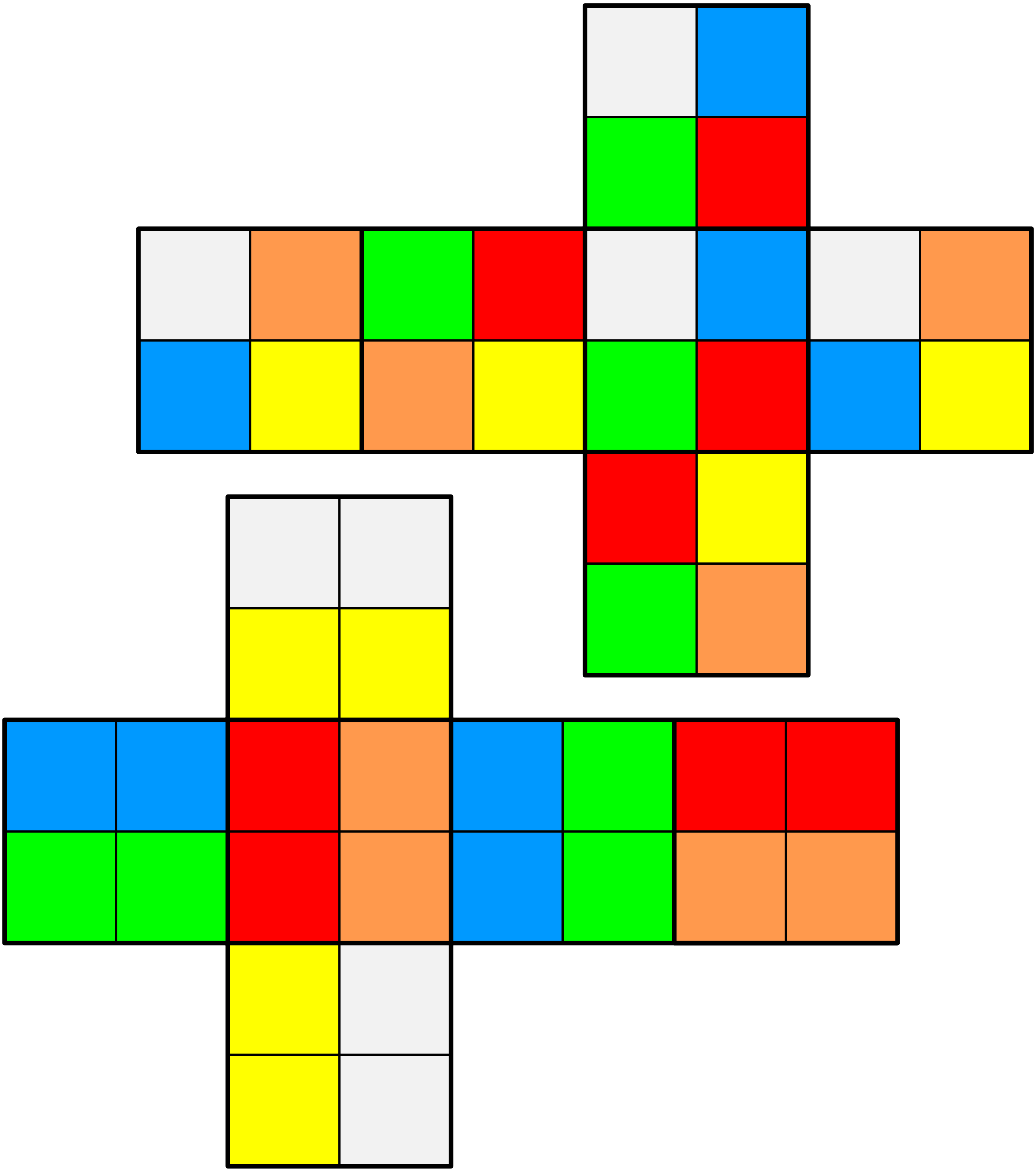

Figure 36: Templates for Figure 6, generators $x$ and $y$.

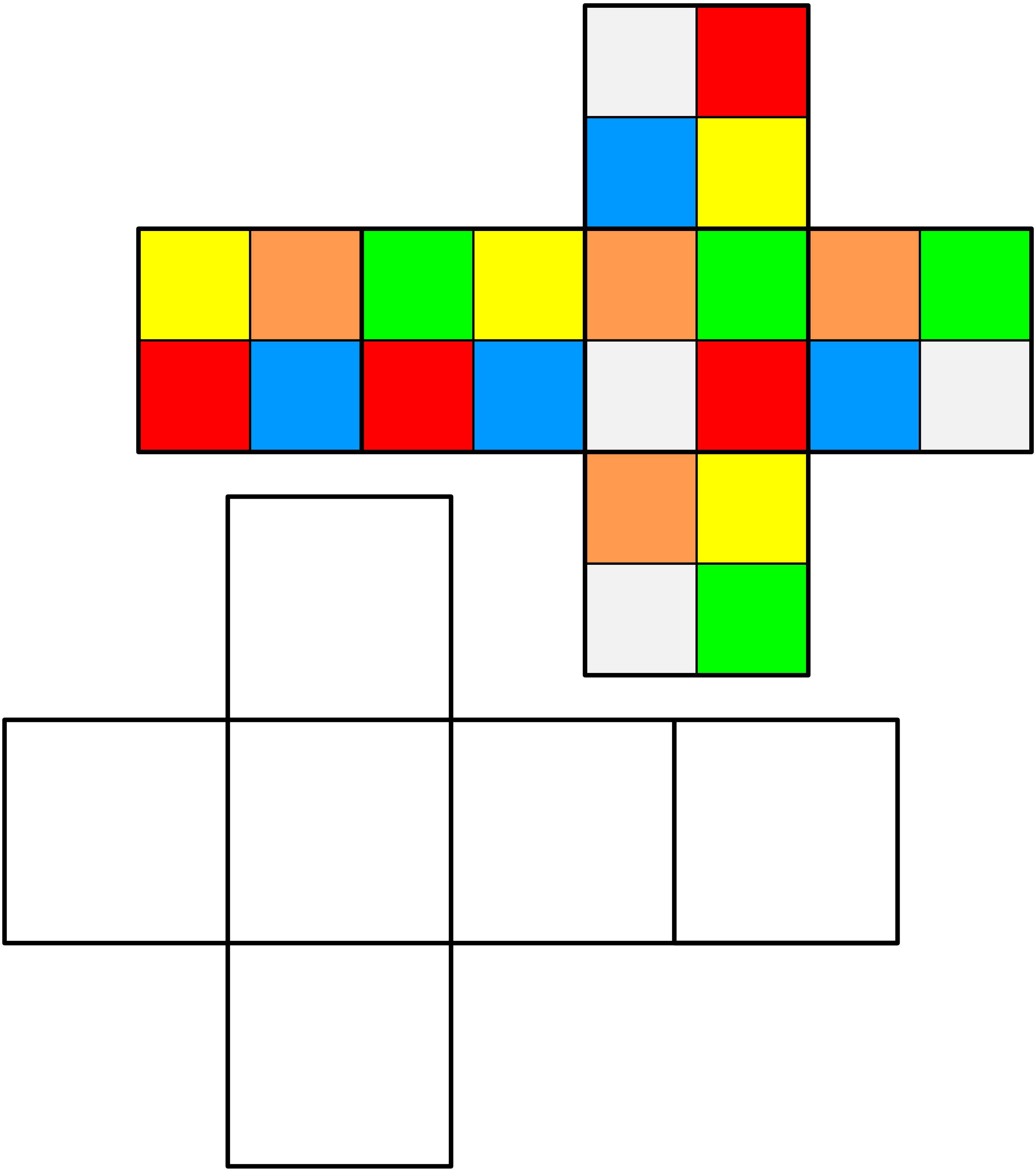

Figure 37: Template for Figure 6, generator $z$.

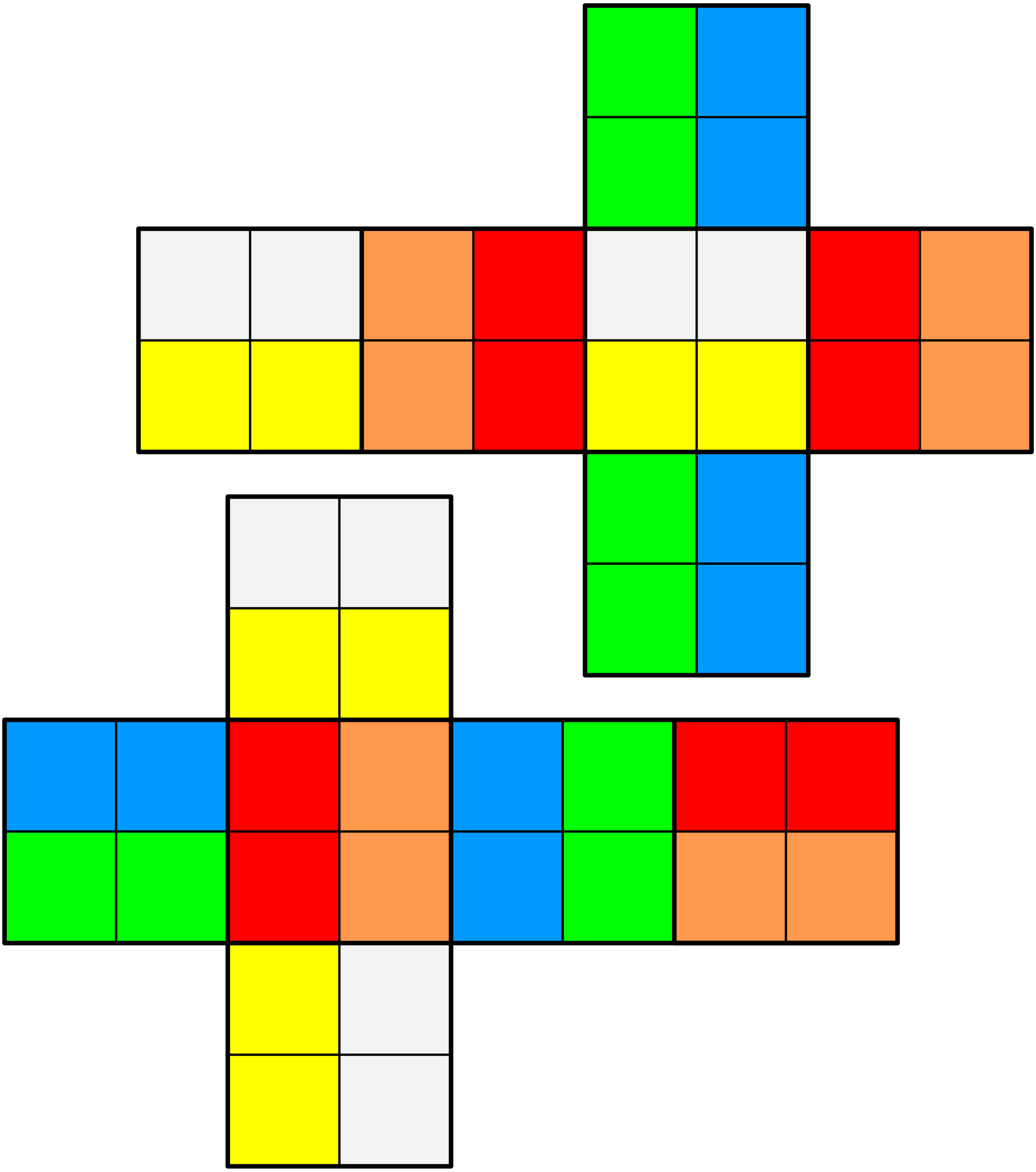

Figure 38: Templates for Figure 6, generators $U$ and $V$.

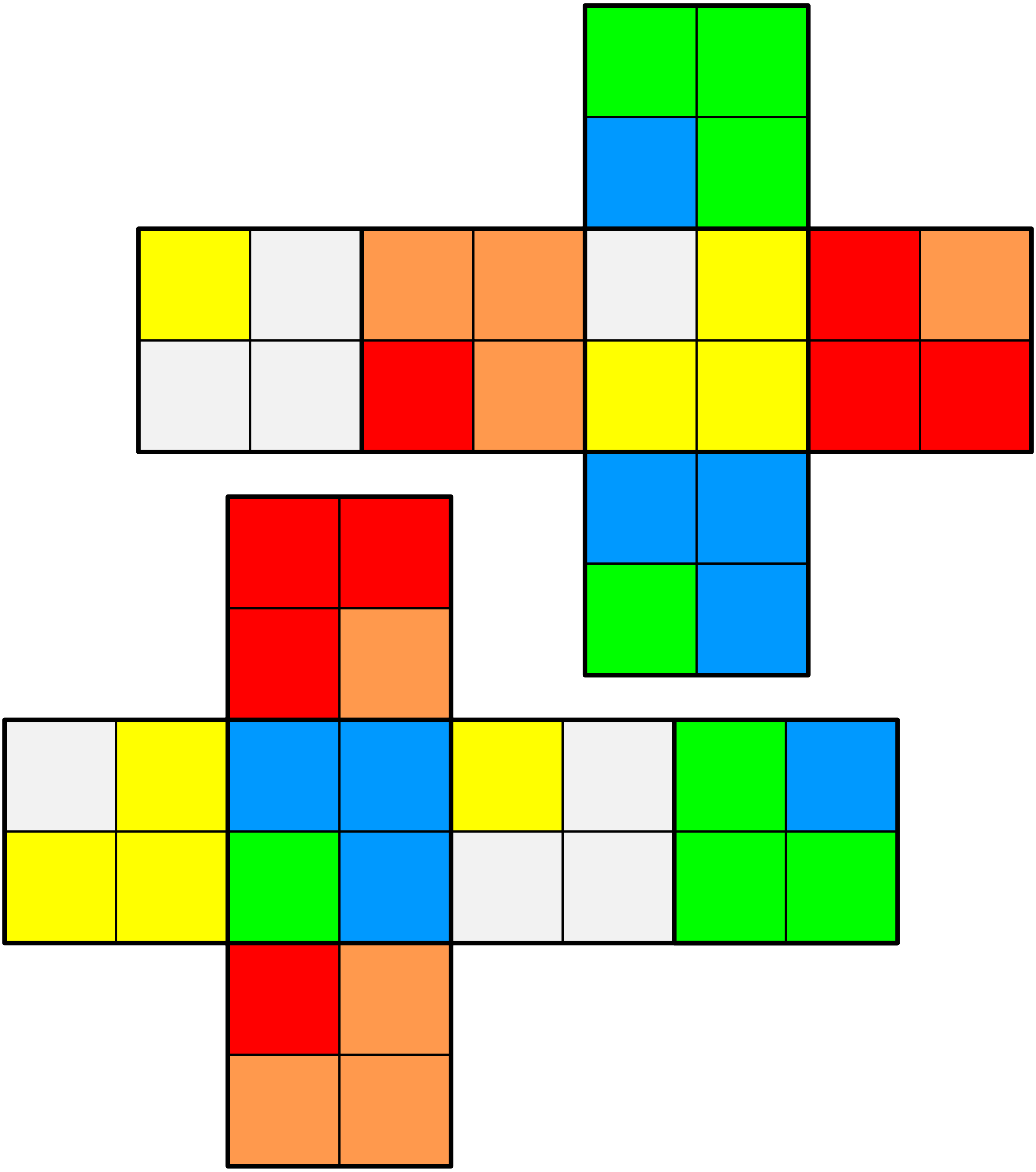

Figure 39: Templates for Figure 7, $UV$ and $U^2V$.

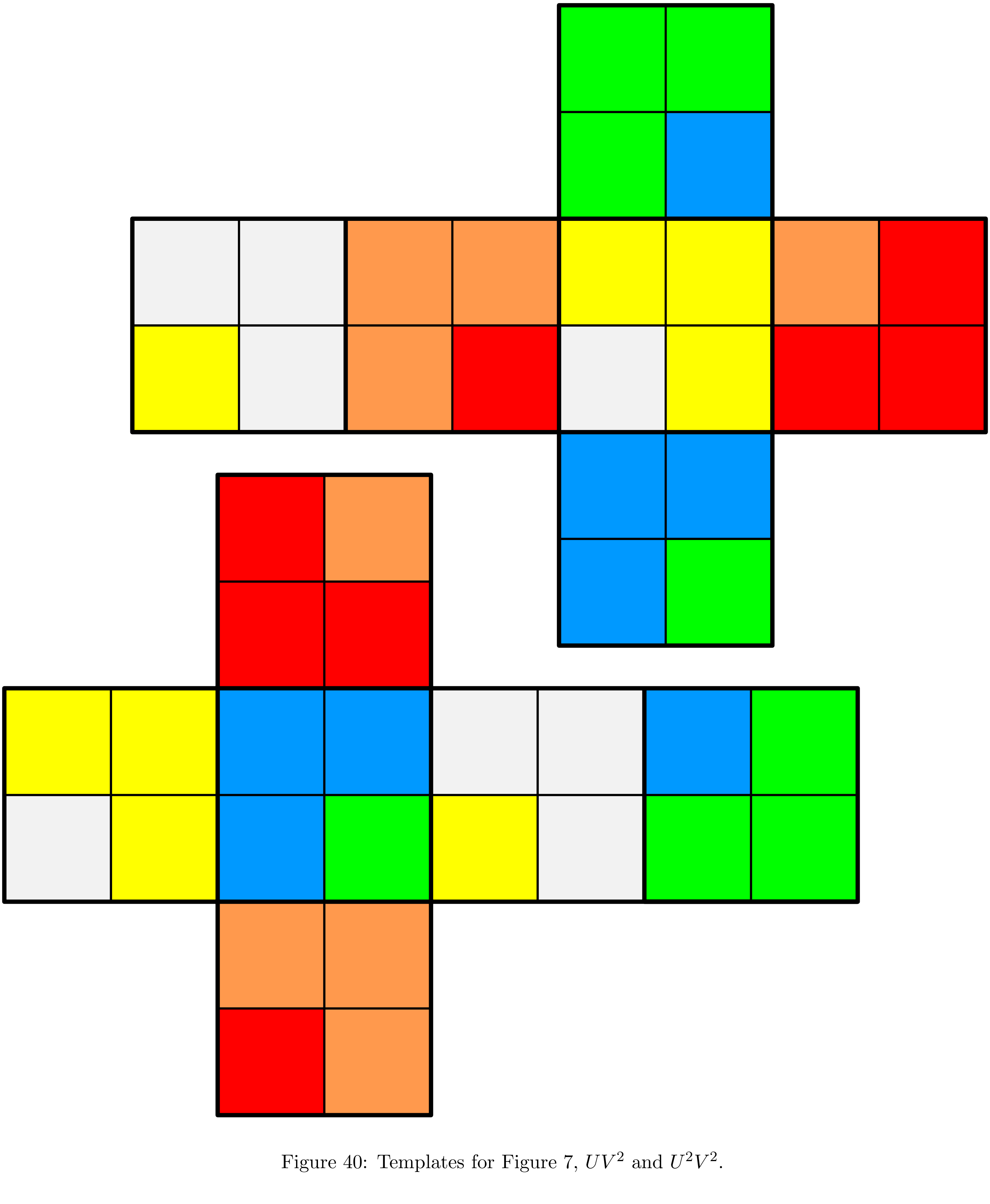

Figure 40: Templates for Figure 7, $UV^2$ and $U^2V^2$.