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INTERMEDIATE MODELS AND KINNA–WAGNER PRINCIPLES

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Abstract. Kinna–Wagner Principles state that every set can be mapped into some fixed iterated power set of an ordinal, and we write KWP to denote that there is some α for which this holds. The Kinna–Wagner Conjecture, formulated by the first author in [9], states that if V is a model of $\mathsf{ZF}+\mathsf{KWP}$ and G is a V-generic filter, then whenever W is an intermediate model of ZF, that is $V \subseteq W \subseteq V[G]$, then $W = V(x)$ for some x if and only if W satisfies KWP. In this work we prove the conjecture and generalise it even further. We include a brief historical overview of Kinna–Wagner Principles and new results about Kinna–Wagner Principles in the multiverse of sets.

1. Introduction

The Axiom of Choice was formulated by Zermelo in order to prove that every set can be well-ordered, and we know that the converse statement holds as well. Namely, if we assume that every set can be well-ordered, then the Axiom of Choice holds. Since every well-ordered set is isomorphic to a unique ordinal, we can phrase the Axiom of Choice as "Every set injects into an ordinal".

Kinna and Wagner formulated a weakening of the Axiom of Choice in [11]. Formulated as a selection principle, they prove that it is equivalent to "Every set injects into the power set of an ordinal". So, an immediate consequence of this principle is that every set can be linearly ordered. The Kinna–Wagner selection principle was studied extensively, both by looking at various weaker versions (mostly derived from the formulation of the selection principle, see [4] for example), as well as studying its independence and consequences in the broader context of choice principles which imply linear ordering, see [15] for example).

In this paper we look at a different weakening of the Kinna–Wagner Principle, derived from the concept of "Every set injects into the nth-power set of an ordinal". This generalisation is originally due to Monro [12], later extended to the transfinite hierarchy by the first author in [7, 8]. Kinna–Wagner Principles are fairly stable under generic extensions, but can be violated using symmetric extensions to a certain degree. The *Kinna–Wagner Conjecture* states, informally, that if V satisfied any Kinna–Wagner Principle, then the intermediate models between V and $V[G]$, a set-generic extension of V, which satisfy some kind of Kinna–Wagner Principle are exactly of the form $V(x)$.

The Kinna–Wagner Conjecture can be understood as a generalisation of the intermediate model theorem, which states that if $V \subseteq M \subseteq V[G]$ are models of ZFC, where G is a V-generic filter, then M is a generic extension of V and $V[G]$ is a generic extension of M. This is not true if M does not satisfy the Axiom of Choice, as generic extensions preserve the Axiom of Choice. But a natural question

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is whether M must be a symmetric extension of V, at least in the presence of the Axiom of Choice, or some Kinna–Wagner Principle.

Grigorieff [2] studied intermediate models of ZF, mainly in the form of symmetric extensions, even without the assumption of AC in V or $V[G]$. The work on the Bristol model in [7, 9] showed that if c is a Cohen real over L, then the structure of intermediate models is far more complex than previously imagined. Not every intermediate model has the form $L(x)$ for any set x, and even those that do, are not necessarily symmetric extension given by the Cohen real itself. These results were recently extended by Hayut and Shani [3], where they show that given any model of ZF, V, if c is a Cohen real over V, then there is an intermediate $V \subseteq M \subseteq V[c]$ such that $M \neq V(x)$ for all x. Hayut and Shani show that in these intermediate models all Kinna–Wagner Principles fail.

We show in the final section of this work that more can be said. Theorem 4.6 shows that if $V \subseteq M \subseteq V[G]$ are models of ZF, with G a V-generic filter, and some Kinna–Wagner Principle holds in V, then M has the form $V(x)$ for some set x if and only if it satisfies some Kinna–Wagner Principle, and that the set x is uniformly defined from the Kinna–Wagner Principle. The notion of a Kinna–Wagner Principle is generalised to allow different “starting classes” than the ordinals, and consequently we obtain Theorem 5.5.

2. Preliminaries

We will use a superscript with M to denote the relativisation of classes to M, e.g. $\mathcal{P}(x)^M$ would be the power set of x as computed in M. The exception would be the von Neumann hierarchy of M, which we denote by M_α rather than V_α^M. All our “models of ZF” are class models, they are all transitive classes in the context of any larger model. We will also write $\mathcal{P}(X)$, even if X is a proper class, to mean $\{x \mid x \subseteq X\}$, but in the case where X is a transitive class[1] this is tantamount to $\bigcup\{\mathcal{P}(x) \mid x \in X\}$.

Suppose that x is a set, we define $L(x)$ to be the smallest transitive model of ZF containing the ordinals and x itself. This model has a constructible hierarchy, much like L does, with the modification that $L_0(x) = \operatorname{tcl}(\{x\})$, rather that $\varnothing$. We write $L(x, y)$ to mean $L(\{x, y\})$.

We can further extend this notation and define for an inner model V and a set x, the class $V(x)$ to be the smallest transitive model of ZF containing V and having x as an element. The following equalities hold:

$$V(x) = \bigcup\{L(x, y) \mid y \in V\} = \bigcup\{L(V_\alpha, x) \mid \alpha \in \mathrm{Ord}\} = \bigcup\{L_\alpha(V_\alpha, x) \mid \alpha \in \mathrm{Ord}\}.$$

Often, especially in the context of ZFC, we take V to be some definable inner model, however, this can be defined in any second-order set theory such as Gödel–Bernays set theory, for any inner model, or by adding V as a predicate when possible (e.g., when V is a ground of the universe).

The definition of $V(x)$ can be also extended to proper classes, although this requires a bit more care. Given $V \subseteq W$ and $X \subseteq W$, we define

$$V(X) = \bigcup\{V(X \cap W_\alpha) \mid \alpha \in \mathrm{Ord}\}.$$

This definition makes the most sense when W is a model of a second-order set theory and both V and X are proper classes of W, but this can be easily understood in general.

[1] Which will be the typical case.

Finally, we will use the shorthand notation $x \leq y$ to mean that there exists an injection from the set x into the set y. We use the $x \leq^* y$ notation to denote that either x is empty or that there is a surjection from y onto x, or alternatively, that there is a partial function from y onto x.

2.1. Generic and symmetric extensions.

Generic extensions are given by forcing. We will hardly use the mechanics of forcing and we refer the reader to standard sources, e.g. Jech [5], for a good review of the technique. While generic extensions are incredibly useful in the study of ZFC, they preserve the Axiom of Choice from the ground model, and as such are not quite the needed tool for proving independence results related to the Axiom of Choice.

Symmetric extensions are formed by identifying an intermediate model between V and $V[G]$. We will not be using the technique directly, but we refer the readers to [10] for an overview of the technique, along with several interesting[2] results on the technique, as well as the limitations of forcing in ZF.

One of the most important choice principles related to symmetric extensions was given by Blass [1], where he suggests that the failure of choice is, in a way, due to "one bad apple", or "small".

Definition 2.1 (Small Violations of Choice)**.** We say that $V \models \mathsf{SVC}(X)$ if for every set A there is an ordinal η and a surjection $f\colon X \times \eta \to A$. We write $V \models \mathsf{SVC}$ to mean that there exists some X such that $\mathsf{SVC}(X)$ holds.

We can define an injective version of SVC, namely, requiring that A injects into $X \times \eta$. However, the two notions are equivalent when working in ZF, as the surjection from $X \times \eta$ onto A translates to an injection from A into $\mathcal{P}(X \times \eta)$, which can be then modified to an injection into $\mathcal{P}(A) \times \eta$. For a brief discussion on this subject, see [16].

Small Violation of Choice turned out to be an incredibly useful choice principle. Through a combination of results due to Blass [1] and Usuba [18] the following theorem holds.

Theorem 2.2. *The following are equivalent for $W \models \mathsf{ZF}$.*

1. $W \models \mathsf{SVC}$.
2. *The Axiom of Choice can be forced over W.*
3. *W is a symmetric extension of a model of ZFC.*
4. *$W = V(x)$ where $V \models \mathsf{ZFC}$ and x is a set.* □

3. Kinna–Wagner Principles

3.1. Historical background.

The Axiom of Choice can be phrased as "For every set M, there is a function $f\colon \mathcal{P}(M) \to \mathcal{P}(M)$ such that $f(A) \subseteq A$ for all $A \in \mathcal{P}(M)$, and if A is non-empty, then $f(A)$ is a singleton". This lends itself to a natural weakening. Instead of selecting a singleton, we merely select a non-empty proper subset when possible. This is the original formulation of the Kinna–Wagner selection principle in [11].

Formally, the *Kinna–Wagner Selection Principle* states that for every set M there is a function $f\colon \mathcal{P}(M) \to \mathcal{P}(M)$ such that $f(A) \subseteq A$ for all $A \in \mathcal{P}(M)$, and if $|A| \geq 2$, then $\varnothing \subsetneq f(A) \subsetneq A$.

[2] In the humble view of the authors, at least.

In their original paper, Kinna and Wagner prove that $\mathcal{P}(M)$ admits such a function if and only if M itself can be injected into the power set of some well-ordered set. Let us sketch a brief proof of this equivalence.

If $M \subseteq \mathcal{P}(\eta)$ and $|A| \geq 2$, we define $\alpha_A = \min(\bigcup A \setminus \bigcap A)$, note that if $|A| \geq 2$ then this is well-defined. Next define $f(A) = \{a \in A \mid \alpha_A \notin a\}$, or $f(A) = \varnothing$ if $|A| \leq 1$. Then $f(A)$ produces a non-empty, proper subset of A as wanted.

In the other direction, starting from $f\colon \mathcal{P}(M) \to \mathcal{P}(M)$ we can build a tree by recursion, starting from the root as M and at each node, A, we have two successors: $f(A)$ and $A \setminus f(A)$. At limit points we take intersections along the branches (this is a copy of $2^{<\alpha}$ for some α, so branches do exist), and if they have more than two points, we continue. This recursion must end at some stage η, which can only happen once all the terminal nodes are singletons and every singleton has been reached. For every $m \in M$ we consider the "trace" of m in this tree: the set of all $\alpha < \eta$ such that for some A in the αth level of the tree, $m \in f(A)$. Easily this provides us an injection from M into $\mathcal{P}(\eta)$.

Many selection principles were derived from the one above, often by restricting the size of M or requiring that the size of the selected set will have some properties (e.g., [4]). Monro, however, generalised these principles based on the embeddibility into iterated power sets, starting from an ordinal in [12], extended further by the first author in [8], as well as studied by Shani in [17].

3.2. **Higher-order generalisation.** We define the iterated power set of a set x, $\mathcal{P}^\alpha(x)$, by recursion:

(1) $\mathcal{P}^0(x) = x$,
(2) $\mathcal{P}^{\alpha+1}(x) = \mathcal{P}(\mathcal{P}^\alpha(x))$, and
(3) $\mathcal{P}^\alpha(x) = \bigcup\{\mathcal{P}^\beta(x) \mid \beta < \alpha\}$ when α is a limit ordinal.

In the case where x is an infinite set and $x^2 \leq x$, we immediately get that $(\mathcal{P}^\alpha(x))^2 \leq \mathcal{P}^\alpha(x)$ for all α. In the case where x is an infinite ordinal (or Ord itself) this allows us to encode any binary relation on $\mathcal{P}^\alpha(\mathrm{Ord})$ as a subset of $\mathcal{P}^\alpha(\mathrm{Ord})$.

Definition 3.1. The **Kinna–Wagner $\boldsymbol{\alpha}$** principle, denoted by KWP_α, states that every set x injects into $\mathcal{P}^\alpha(\mathrm{Ord})$. That is, there is an ordinal η such that x injects into $\mathcal{P}^\alpha(\eta)$. We omit the index to write KWP as a shorthand for $\exists\alpha\, \mathsf{KWP}_\alpha$.

We define also the surjective version, **Kinna–Wagner* $\boldsymbol{\alpha}$** principle, denoted by KWP^*_α, which asserts that for every set x, there is a partial function from $\mathcal{P}^\alpha(\mathrm{Ord})$ onto x. Namely, there is some ordinal η and a partial function from $\mathcal{P}^\alpha(\eta)$ onto x. We use KWP^* to denote $\exists\alpha\, \mathsf{KWP}^*_\alpha$ as well.

The next proposition is an immediate corollary, following from the fact that for every x and y, $x \leq y \to x \leq^* y \to x \leq \mathcal{P}(y)$.

Proposition 3.2. *For all α, $\mathsf{KWP}_\alpha \to \mathsf{KWP}^*_\alpha \to \mathsf{KWP}_{\alpha+1}$.* □

Consequently, KWP is equivalent to KWP^*, but we can say even more.

Proposition 3.3. *If α is not a successor ordinal, then $\mathsf{KWP}_\alpha = \mathsf{KWP}^*_\alpha$.*

In the case of $\alpha = 0$ this is immediate, as both KWP^*_0 and KWP_0 are equivalent to the Axiom of Choice. In the case of limit ordinals this is a corollary of a much more general lemma.

Lemma 3.4. *Suppose that α is a limit ordinal, then $x \leq \mathcal{P}^\alpha(y) \iff x \leq^* \mathcal{P}^\alpha(y)$.*

Proof. Since ZF proves that if $x \leq \mathcal{P}^\alpha(y)$, then $x \leq^* \mathcal{P}^\alpha(y)$, we only need to verify the other implication. Suppose that $x \leq^* \mathcal{P}^\alpha(y)$. If x is empty, then $x \subseteq \mathcal{P}^\alpha(y)$, so we may assume that it is not empty. Therefore, there is a surjection $F\colon \mathcal{P}^\alpha(y) \to x$. For each $u \in x$, let $\alpha_u = \min\{\beta \mid F^{-1}(u) \cap \mathcal{P}^\beta(y) \neq \varnothing\}$, this is a well-defined ordinal since F is surjective and $\mathcal{P}^\alpha(y) = \bigcup\{\mathcal{P}^\beta(y) \mid \beta < \alpha\}$.

Define $f\colon x \to \mathcal{P}^\alpha(y)$ given by $f(u) = \{v \in \mathcal{P}^{\alpha_u}(y) \mid F(v) = u\}$. Since α is a limit ordinal, $f(u) \in \mathcal{P}^{\alpha_u+1}(y) \subseteq \mathcal{P}^\alpha(y)$, so it is well-defined, and since F is a function, f must be injective. □

Theorem 3.5 (Balcar–Vopěnka–Monro). *Suppose that M and N are two models of* ZF *with $\mathcal{P}^{\alpha+1}(\mathrm{Ord})^M = \mathcal{P}^{\alpha+1}(\mathrm{Ord})^N$, if $M \models \mathsf{KWP}^*_\alpha$, then $M = N$.*

Proof. Since M satisfies KWP^*_α, every set in M is the image of $\mathcal{P}^\alpha(\eta)$ of some ordinal η. Therefore, in M we can code $\mathrm{tcl}(\{x\})$ as the extensional quotient of a relation on $\mathcal{P}^\alpha(\eta)$ for some η, and by iterating Gödel's pairing, we have some $X \in \mathcal{P}^{\alpha+1}(\eta)$ from which we can recover the extensional and well-founded $\in$-relation on $\mathrm{tcl}(\{x\})$. By the assumption, $X \in N$, and it is still a well-founded relation (otherwise there would be a subset of X witnessing that in both M and N), so we can take the extensional quotient and use the Mostowski collapse lemma in N and get $x \in N$ as wanted, so $M \subseteq N$.

Suppose that $N \neq M$, then let δ be the least ordinal such that $M_\delta \neq N_\delta$. Note that both models have the same ordinals, so it is certainly the case that M_δ exists. Easily, δ must be a successor ordinal of the form $\gamma+1$, so in N there is some $x \subseteq M_\gamma$ such that $x \notin M$. However, since $M_\gamma = N_\gamma$ can be coded as an element of $\mathcal{P}^{\alpha+1}(\eta)$ for some ordinal η, and therefore x can be coded as a subset of that, in N, but the assumption guarantee that this code is in M, and so $x \in M$ as well. □

Corollary 3.6. *Suppose that $V \models \mathsf{KWP}^*_\alpha$, then $V = L(\mathcal{P}^{\alpha+1}(\mathrm{Ord}))$.* □

Proposition 3.7. *Suppose that V satisfies* KWP *and x is a set, then $V(x) \models \mathsf{KWP}$. Moreover, if $V \models \mathsf{KWP}^*_\beta$ and $x \in \mathcal{P}^{\alpha+1}(\mathrm{Ord})$, then $V(x) \models \mathsf{KWP}^*_{\max\{\alpha,\beta\}}$.*

Proof. Let $\gamma = \max\{\alpha, \beta\}$. Recall that $V(x) = \bigcup\{L(x, y) \mid y \in V\}$, and more specifically, $V(x) = \bigcup\{L(V_\delta, x) \mid \delta \in \mathrm{Ord}\}$. So, if $z \in V(x)$, then there is some δ such that $z \in L(V_\delta, x)$. It is enough to show that $L(V_\delta, x) \models \mathsf{KWP}^*_\gamma$ for all δ.

Recall that in general $L(a)$ has a recursive construction given by $L_0(a) = \mathrm{tcl}(\{a\})$, $L_{\xi+1}(a) = \mathrm{Def}(L_\alpha(a), \in)$, where Def denotes all the definable (with parameters) subsets of a given structure, and $L_\xi(a) = \bigcup\{L_\zeta(a) \mid \zeta < \xi\}$ for a limit ordinal ξ.

We will construct, in $V(x)$, a surjection from $\mathcal{P}^\gamma(\mathrm{Ord})$ onto $L(V_\delta, x)$, for any fixed δ, given by recursion, where we keep track of the surjections constructed in previous steps. Since $x \subseteq \mathcal{P}^\alpha(\mathrm{Ord})$ and V_δ is the image of $\mathcal{P}^\beta(\eta)$ for some η, there is a surjection, f_0, from $\mathcal{P}^\gamma(\mathrm{Ord})$ onto $L_0(V_\delta, x)$. If ξ is a limit ordinal and $\langle f_\zeta \mid \zeta < \xi\rangle$ is known, then by shifting the domains of the f_ζ—if necessary—we get an obvious surjection from $\mathcal{P}^\gamma(\mathrm{Ord})$ onto $L_\xi(V_\delta, x)$. Finally, for a successor step, there is a surjection, the interpretation map, from $\omega \times L_\xi(V_\delta, x)^{<\omega}$ onto $L_{\xi+1}(V_\delta, x)$ given by mapping $\langle n, \vec{a}\rangle$ to the set defined by $\varphi_n(x, \vec{a})$, where φ_n is some enumeration of all formulas with the understanding that we map the pair to $\varnothing$ if there is an arity mismatch.[3] Since $\omega \times L_\xi(V_\delta, x)$ is the image of $\mathcal{P}^\gamma(\mathrm{Ord})$, given by f_ξ, we can define $f_{\xi+1}$ as the composition of f_ξ with the interpretation map. □

Remark 3.8. The converse statement, $V(x) \models \mathsf{KWP} \to V \models \mathsf{KWP}$ is false. If $M \subseteq L[c]$ is the Bristol model, then $M(c) = L[c] \models \mathsf{KWP}_0$, but $M \models \neg\mathsf{KWP}$.

[3]We can also agree that any unassigned parameters are given the value 0, etc.

In Theorem 4.2 we will show that the statement does hold for the case of generic extensions (and consequently, symmetric extensions).

Proposition 3.9. *$L(\mathcal{P}^\alpha(\mathrm{Ord})) \models \mathsf{KWP}^*_\alpha$.*

Proof. Note that $L(\mathcal{P}^\alpha(\mathrm{Ord})) = \bigcup\{L(\mathcal{P}^\alpha(\delta)) \mid \delta \in \mathrm{Ord}\}$. By the Proposition 3.7, $L(\mathcal{P}^\alpha(\delta)) \models \mathsf{KWP}^*_\alpha$ for all δ, so for every $x \in L(\mathcal{P}^\alpha(\mathrm{Ord}))$ there is some δ such that $x \in L(\mathcal{P}^\alpha(\delta))$ and therefore there is some η such that $\mathcal{P}^\alpha(\eta)$ maps onto x. □

The following proposition follows directly from the definition of SVC and the fact that $\mathcal{P}^\alpha(\mathrm{Ord})$ definably maps onto $\mathcal{P}^\alpha(\mathrm{Ord}) \times \mathrm{Ord}$.

Proposition 3.10. *If $\mathsf{SVC}(x)$ holds and $x \in \mathcal{P}^{\alpha+1}(\mathrm{Ord})$, then KWP^*_α holds. In particular $\mathsf{SVC} \to \mathsf{KWP}$.* □

Of course, if the injective version of SVC holds, we get KWP_α. This is the core of the proof that in Cohen's first model every set can be linearly ordered: there is a set of reals, A, such that $\mathsf{SVC}([A]^{<\omega})$ holds in its injective form (see an example of such proof in [6, §5.5]). It should be noted that ZF does not prove that $\mathsf{KWP} \to \mathsf{SVC}$. Monro [13] constructed a model in which there is a proper class of Dedekind-finite sets using a class forcing, where SVC fails, but it can be shown that KWP_1 holds in that model.

Lemma 3.11. *Suppose $M = V(x)$ for some $V \models \mathsf{ZF}$ and a set x and $M \models \mathsf{KWP}^*_\alpha$. Then $M = V(y)$ for some $y \in \mathcal{P}^{\alpha+1}(\mathrm{Ord})$.*

Proof. Since $M \models \mathsf{KWP}^*_\alpha$, there is some $\eta \in \mathrm{Ord}$ and a surjection from $\mathcal{P}^\alpha(\eta)$ to $\mathrm{tcl}(\{x\})$. Exactly as in the proof of Theorem 3.5, we find $y \subseteq \mathcal{P}^\alpha(\eta)$ in M from which we can recover x. □

4. The Kinna–Wagner Conjecture and its proof

4.1. **Kinna–Wagner in the forcing multiverses.**

Proposition 4.1. *If $V \models \mathsf{KWP}^*_\alpha$ and G is a V-generic filter, then $V[G] \models \mathsf{KWP}^*_\alpha$.*

Proof. Let $x \in V[G]$ be any non-empty set, then there is some $\dot{x} \in V$ such that $\dot{x}^G = x$. Fixing some $x_0 \in x$, let $F\colon \dot{x} \to x$ be the function $F(p, \dot{y}) = \dot{y}^G$ when $p \in G$, otherwise $F(p, \dot{y}) = x_0$. This function is surjective, since $\dot{x} \in V$, there is an ordinal η such that $\dot{x}$ injects into $\mathcal{P}^\alpha(\eta)$, and so F can be extended to $\mathcal{P}^\alpha(\eta)$ via precomposition. □

As an immediate corollary, if $V \models \mathsf{KWP}_\alpha$, then $V[G] \models \mathsf{KWP}^*_\alpha$ and therefore $\mathsf{KWP}_{\alpha+1}$, for any generic filter G. On the other hand, Monro showed that there is a generic extension of Cohen's first model in which there is an amorphous set [14]. Since Cohen's first model satisfies KWP_1, and amorphous sets cannot be linearly ordered, it must be that Monro's generic extension satisfies $\mathsf{KWP}^*_1 + \neg\mathsf{KWP}_1$. However, by the above proposition, any further extension must satisfy KWP^*_1, since generic extensions of generic extensions are themselves generic extensions.

In the modal logic of forcing, KWP^*_α is a button. Once it is true, it must remain true in any further generic extension (it may be true that for some $\beta < \alpha$, KWP^*_β holds in a generic extension, of course). The converse statement is not necessarily true, of course, since if c is a Cohen real over L, then $L[c] \models \mathsf{KWP}^*_0$, but it has grounds where KWP^*_α fails for arbitrarily high α, as shown in [9]. On the other hand, as the following theorem shows, the failure cannot be complete.

Theorem 4.2. *If $V[G] \models \mathsf{KWP}^*_\alpha$, where $G \subseteq \mathbb{P} \subseteq \mathcal{P}^\beta(\mathrm{Ord})$ is V-generic, then $V \models \mathsf{KWP}^*_{\beta+\alpha}$.*

Proof. We define a sequence of classes of $\mathbb{P}$-names, S_δ, by recursion:

(1) $S_0 = \{\check{\xi} \mid \xi \in \mathrm{Ord}\}$,
(2) $S_{\delta+1} = \mathcal{P}(\mathbb{P} \times S_\delta)$,
(3) $S_\delta = \bigcup_{\gamma<\delta} S_\gamma$ for a limit ordinal δ.

If $a \in \mathcal{P}^\delta(\mathrm{Ord})^{V[G]}$, then there is some $\dot{a} \in S_\delta$ such that $\dot{a}^G = a$. Moreover, in V there is a definable surjection $\tau_\delta \colon \mathcal{P}^{\beta+\delta}(\mathrm{Ord}) \to S_\delta$, since $\mathbb{P} \subseteq \mathcal{P}^\beta(\mathrm{Ord})$.

Now letting $x \in V$ be an arbitrary set, there are $\mathbb{P}$-names $\dot{a} \in S_{\alpha+1}$ and $\dot{f}$, and a condition $p \in G$ such that $p \Vdash$ "$\dot{f} \colon \dot{a} \to \check{x}$ is a surjection". Let g be the function $g(q,s) = y$ if and only if $q \Vdash \dot{f}(\tau_\alpha(s)) = \check{y}$, then g is a surjection from a subset of $\mathbb{P} \times \mathcal{P}^{\beta+\alpha}(\mathrm{Ord})$ onto x. So $V \models \mathsf{KWP}^*_{\beta+\alpha}$, as wanted. □

Corollary 4.3. *Suppose that $V \models \neg\mathsf{KWP}_{\beta+\alpha}$ and $G \subseteq \mathbb{P} \subseteq \mathcal{P}^\beta(\mathrm{Ord})$ is a V-generic filter, then $V[G] \models \neg\mathsf{KWP}_\alpha$. In particular, you cannot force the Axiom of Choice with a well-orderable forcing from a model of* $\mathsf{ZF} + \neg\mathsf{AC}$.[4]

This means that KWP is an absolute truth not only in the generic multiverse, but indeed in the symmetric multiverse. Grigorieff showed in [2] that if $V \subseteq W \subseteq V[G]$, where W is a symmetric extension of V, then $V[G]$ is a generic extension of W. Therefore, if $V \models \mathsf{KWP}$, so must $V[G]$, and therefore W must satisfy this as well. The Bristol model is an example of an intermediate model between L and $L[c]$, where c is a Cohen real, where KWP fails. Of course, the Bristol model is not of the form $L(x)$ for any set x, which leads to the very natural conjecture made by the first author in [9, Conjecture 8.9].

Conjecture 4.4 (The Kinna–Wagner Conjecture)**.** Suppose that $V \models \mathsf{KWP}$ and G is a V-generic filter. If M is an intermediate model between V and $V[G]$ and $M \models \mathsf{KWP}$, then $M = V(x)$ for some set x.

4.2. Proof of the Kinna–Wagner Conjecture.

We begin with a lemma, before proving the Kinna–Wagner Conjecture.

Lemma 4.5. *Let $W \models \mathsf{ZF}$ and let $\mathbb{Q} \in W$ be a forcing notion, $f \colon X \to \mathbb{Q}$ a function whose image is dense, and $H \subseteq \mathbb{Q}$ is a W-generic filter. For every $x \in W[H]$ such that $x \subseteq W$, there is $y \subseteq X$ such that $W(x) = W(y)$.*

Proof. It is enough to prove this in the case where $\mathbb{Q}$ is a complete Boolean algebra, since every forcing notion embeds densely into its Boolean completion. Let $\dot{x}$ be a $\mathbb{Q}$-name such that $\dot{x}^H = x$, and let $\mathbb{Q}_{\dot{x}}$ be the complete subalgebra generated by the conditions of the form $[\![\check{z} \in \dot{x}]\!]$ for $z \in V$. Then $W[H \cap \mathbb{Q}_{\dot{x}}] = W[x]$, see the proof of [5, Corollary 15.42].[5]

Let $\pi \colon \mathbb{Q} \to \mathbb{Q}_{\dot{x}}$ be the projection map given by $\pi(q) = \inf\{p \in \mathbb{Q}_{\dot{x}} \mid q \leq p\}$. This is well-defined since $\mathbb{Q}_{\dot{x}}$ is a complete subalgebra, so the computation of the infimum is the same in both $\mathbb{Q}$ and $\mathbb{Q}_{\dot{x}}$. Since $\pi``f``X$ is dense in $\mathbb{Q}_{\dot{x}}$, by [5, Lemma 15.40],

$$W[x] = W[H \cap \mathbb{Q}_{\dot{x}}] = W[H \cap \pi``f``X],$$

[4] Note that the latter is also a consequence of the fact that a well-orderable forcing must preserve empty products. That is, if $\prod\{A_i \mid i \in I\} = \varnothing$, then adding a generic for a well-orderable forcing must preserve that.

[5] The proof in Jech is in the context of ZFC, but the Axiom of Choice is not used in the proof beyond the fact that KWP_0 and the Balcar–Vopěnka theorem reduce all cases to sets of ordinals and that sets of ordinals are subsets of V.

and therefore taking $y = f^{-1}(\pi^{-1}(H \cap \mathbb{Q}_{\dot{x}})) \subseteq X$ and we get that $W[x] = W[y]$. $\square$

One can view this lemma from a different direction: we use f to define a quasi-order on X which forcing equivalent to $\mathbb{Q}$, and then apply the restriction lemmas.

We can now proceed with our proof of the Conjecture. We will, in fact, prove a stronger form of it.

Theorem 4.6. *There is a definable sequence $\langle \eta_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ in $V[G]$, using V as a predicate, such that given any intermediate model $V \subseteq M \subseteq V[G]$ of KWP^*_α, then $M = V(x)$, for some $x \subseteq \mathcal{P}^\alpha(\eta_\alpha)$.*

As the definability of the ground model is still an open problem in ZF at the time of writing this paper, it is not even clear if V itself is definable, which is why we must add it as a predicate. Note that even in this context, we do not require (a priori) that M is definable in $V[G]$ (even with V as a predicate), although that is a consequence of the theorem.

Proof of Theorem 4.6. We begin by defining a different sequence, $\langle \eta'_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ in $V[G]$, using V as a predicate. Define $\eta'_0 = 0$, for a limit ordinal α we let $\eta'_\alpha = \sup\{\eta'_\beta \mid \beta < \alpha\}$. Suppose the η'_α was defined, let us define $\eta'_{\alpha+1}$.

First let β be the least such that $V \models \mathsf{KWP}^*_\beta$. Next, for each $z \subseteq \mathcal{P}^{\beta+\alpha}(\eta'_\alpha)$, there is a complete Boolean algebra $\mathbb{Q} \in V(z)$ and a $V(z)$-generic filter, H, such that $V(z)[H] = V[G]$. By Proposition 3.7, $V(z) \models \mathsf{KWP}^*_{\beta+\alpha}$, and therefore there is an ordinal η and a surjection from $\mathcal{P}^{\beta+\alpha}(\eta)^{V(z)}$ onto $\mathbb{Q}$ in $V(z)$, we let η^z be the least such ordinal. Then, let $\eta'_{\alpha+1} = \sup\{\eta^z \mid z \subseteq \mathcal{P}^{\beta+\alpha}(\eta'_\alpha)\}$.

Let M be an intermediate model of KWP^*_α. We claim that $M = V(z)$ for some $z \subseteq \mathcal{P}^{\beta+\alpha}(\eta'_{\alpha+1})^M$. Let X_δ denote $\mathcal{P}^\delta(\mathrm{Ord})^M$. First, note that $V(X_\delta) \models \mathsf{KWP}_{\beta+\delta}$, and since $M \models \mathsf{KWP}^*_\alpha$, $M = L(X_{\alpha+1}) = V(X_{\alpha+1})$. We claim that $V(X_\delta) = V(z)$ for some $z \subseteq \mathcal{P}^{\beta+\delta}(\eta'_\delta)$. We prove this claim by induction on δ. For $\delta = 0$, this is trivial, $V(X_0) = V$ so we can take $z = \varnothing$. If δ is a limit, by the continuity of X_δ, it follows that for $\gamma < \delta$, since $V(X_\gamma) = V(z_\gamma)$ for some z_γ, then $V(X_\gamma) = V(\mathcal{P}^{\beta+\gamma+1}(\eta'_\gamma)^M)$. Therefore, $V(X_\delta) = V(\mathcal{P}^{\beta+\delta}(\eta'_\delta)^M)$ as wanted.

It remains to prove this in the successor case for $\delta+1$. Note that $V(X_\delta) = V(z)$ for some $z \subseteq \mathcal{P}^{\beta+\delta}(\eta'_\delta)^M$, therefore we can fix some $\mathbb{Q} \in V(z)$ such that for some $V(z)$-generic filter, H, $V(z)[H] = V[G]$, by the choice of $\eta'_{\delta+1}$ there is a surjection from $\mathcal{P}^{\beta+\delta}(\eta'_{\delta+1})^{V(z)}$ onto $\mathbb{Q}$ in $V(z)$. By Lemma 4.5, for every $x \in X_{\delta+1}$, since $x \subseteq V(z)$, there is some $y \subseteq \mathcal{P}^{\beta+\delta}(\eta'_{\delta+1})$ such that $V(z,x) = V(z,y)$. Therefore,

$$V(X_{\delta+1}) \subseteq V(\mathcal{P}^{\beta+\delta+1}(\eta'_{\delta+1})^M).$$

Therefore, $V(X_{\delta+1}) = V(\{y \subseteq \mathcal{P}^{\beta+\delta}(\eta'_{\delta+1})^M \mid V(z') \subseteq V(X_{\delta+1})\})$.

Finally, we define the sequence $\langle \eta_\alpha \mid \alpha \in \mathrm{Ord}\rangle$. If $M \models \mathsf{KWP}^*_\alpha$ and $M = V(z)$, by Lemma 3.11 there is a minimal η^z and $y \subseteq \mathcal{P}^\alpha(\eta^z)$ such that $M = V(y)$. We let

$$\eta_\alpha = \sup\{\eta^z \mid z \subseteq \mathcal{P}^{\beta+\alpha}(\eta'_{\alpha+1}), V(z) \models \mathsf{KWP}^*_\alpha\}.$$

Given any intermediate $M \models \mathsf{KWP}^*_\alpha$, we have by the first part that $M = V(z)$ for some $z \subseteq \mathcal{P}^{\beta+\alpha}(\eta'_{\alpha+1})$, and by the definition of η_α, there is some $y \subseteq \mathcal{P}^\alpha(\eta_\alpha)$ such that $M = V(y)$. $\square$

5. Generalisation and applications

Theorem 5.1. *Suppose that $V \subseteq M \subseteq V[G]$ are models of ZF, where $G \subseteq \mathbb{P}$ is a V-generic filter, then if $M \models \mathsf{KWP}^*_\alpha$, then $M = V(\mathcal{P}^{\alpha+1}(\mathbb{P}^{<\omega})^M)$.*

Proof. Let $X_\delta = \mathcal{P}^\delta(\mathrm{Ord})^M$, we claim that for all δ, $V(X_\delta) = V(\mathcal{P}^\delta(\mathbb{P}^{<\omega})^M)$. Since $M = V(X_{\alpha+1})$, by Corollary 3.6, this will complete the proof. To simplify the notation let us denote by $x_\delta = \mathcal{P}^\delta(\mathbb{P}^{<\omega})^M$. We use $\dot{x}_\delta \in V$ to denote the $\mathbb{P}$-name for x_δ, given by the construction in the proof of Theorem 4.2.

We prove the claim by induction on δ. For $\delta = 0$, this is trivial, since $X_0 = \mathrm{Ord}$ and $x_0 \in V$. If δ is a limit ordinal, this follows by the continuity of the sequences X_δ and x_δ. It remains to prove the successor case.

Let $x \in X_{\delta+1}$, then $x \subseteq V(X_\delta) = V(x_\delta)$. Let $H \subseteq x_\delta^{<\omega}$ be a $V[G]$-generic filter. We make the following observations.

(1) $G * H$ is a V-generic filter for $\mathbb{P} * \dot{x}_\delta^{<\omega}$.
(2) $V(x_\delta)[H]$ is a generic extension of $V(x_\delta)$ and it is an intermediate model between V and $V[G*H]$. Thus, $V[G*H]$ is a generic extension of $V(x_\delta)[H]$ by a quotient of $B(\mathbb{P} * \dot{x}_\delta^{<\omega})^V$, where $B(\cdot)$ is the Boolean completion of the forcing. In particular, there is a $x_\delta^{<\omega}$-name, $\dot{I} \in V(x_\delta)$, for an ideal defining this quotient of the Boolean algebra.
(3) $V[G * H]$ is a generic extension of $V(x_\delta)$ by $\mathbb{Q} = x_\delta^{<\omega} * B(\mathbb{P} * \dot{x}_\delta^{<\omega})^V/\dot{I}$.

We define a map in $V(x_\delta)$ from $x_\delta^{<\omega} \times \mathbb{P} \times \mathcal{P}^\delta(\mathbb{P}^{<\omega})^V$ to $\mathbb{Q}$, defined in $V(x_\delta)$, given by $\langle c, p, y\rangle \mapsto \langle c, \langle \check{p}, \tau(y)\rangle_I\rangle$, where $\tau(y)$ is the name for an element of x_δ decoded from y via the same τ function as in the proof of Theorem 4.2, and the notation $\langle p, \tau(y)\rangle_I$ denotes a name for the equivalence class of the condition in the quotient by I. It is easy to check that the image of this map is dense in $\mathbb{Q}$.

Moreover, it is to verify that $x_\delta \times \mathbb{P} \times \mathcal{P}^\delta(\mathbb{P}^{<\omega})^V$ is in bijection with x_δ, so we get a function from x_δ onto $\mathbb{Q}$ whose image is dense. By Lemma 4.5 we have that $V(x_\delta, x) = V(x_\delta, y)$ for some $y \subseteq x_\delta$. Therefore, $V(x_{\delta+1}) = V(X_{\delta+1})$. □

The extension of the Kinna–Wagner Conjecture given in Theorem 4.6 gives us an interesting immediate corollary.

Corollary 5.2. *Suppose that $V \models \mathsf{KWP}$ and $V[G]$ is a generic extension of V, then there are only set-many intermediate models of KWP^*_α for any given α.* □

As not every model of ZF satisfies KWP, one has to wonder about the necessity of the KWP assumption in Theorem 4.6. It turns out that we can relativise the notion of KWP principles to any class (not necessarily a definable class, as working in ZF the problem of ground model definability remains wide open at this time).

Definition 5.3. Suppose that $V \subseteq W$ is a class. We write $W \models \mathsf{KWP}_\alpha(V)$ to mean that for every $x \in W$ there is some α such that x injects into $\mathcal{P}^\alpha(V)$. Namely, there is some $y \in V$ such that $x \leq \mathcal{P}^\alpha(y)$. We define $\mathsf{KWP}^*_\alpha(V)$ using surjections, and we omit the subscripts using the same convention as before.

In this generalised notion, KWP is simply $\mathsf{KWP}(\mathrm{Ord})$. It is also not hard to see that if G is a V-generic filter, then $V[G] \models \mathsf{KWP}^*_0(V)$, and much of the basic properties of KWP transfer almost directly to this relativised case. Note that $V[G] \models \mathsf{KWP}_0(V)$ can very well be false, since that would imply that if $V \models \mathsf{KWP}_\alpha$, then $V[G] \models \mathsf{KWP}_\alpha$, but as we mentioned, Monro proved in [14] that KWP_1 can be violated generically, so $\mathsf{KWP}_0(V)$ must fail in that generic extension.

Definition 5.4. Suppose that $V \subseteq W$ is a class. We say that $W \models \mathsf{SVC}_V(X)$ if for every set A there is some $e \in V$ and a surjection $f\colon X \times e \to A$. We write $W \models \mathsf{SVC}_V$ to mean that there exists some X such that $\mathsf{SVC}_V(X)$ holds.

Much like in the case of SVC, $W \models \mathsf{SVC}_V$ if and only if $\mathsf{KWP}^*_0(V)$ can be forced over W. We can also define the injective version of SVC_V as well, however, it is not at all true that the two are equivalent for an arbitrary V.

Theorem 5.5. *Suppose that $V \subseteq M \subseteq V[G]$ are models of* ZF *where G is a V-generic filter. Then the following are equivalent:*

(1) *$M = V(x)$ for some $x \in V[G]$.*
(2) *M is a symmetric extension of V.*
(3) *$M \models \mathsf{KWP}^*(V)$.*
(4) *$M \models \mathsf{SVC}_V$.*

Proof. The equivalence between (1) and (2) had been established by Grigorieff [2]. The implication from (1) to (4) follows directly from the definitions of $V(x)$ and SVC_V. The proof that (4) implies (3) has a similar proof to Proposition 3.10. Finally, the proof of Theorem 4.6 translates directly[6] to this case, establishing the implication from (3) to (1). □

Theorem 1.3 in [3] argues that in their Bristol-like models, M, KWP and SVC are both false. As a consequence of Theorem 5.5 we actually have that the two failures go hand-in-hand when $V \models \mathsf{ZFC}$. And indeed, $\mathsf{KWP}^*(V)$ fails if and only if SVC_V fails.

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