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A look back at the core of games in characteristic function form: some new axiomatization results *

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Abstract

The main contribution of this paper is to provide three new results axiomatizing the core of games in characteristic function form (not necessarily with transferable utility) obeying an innocuous condition (that the set of individually rational pay-off vectors is bounded). One novelty of this exercise is that our domain is the *entire* class of such games: i.e., restrictions like "non-levelness" (a restriction not very appealing in several real-life situations) or "balancedness", usually imposed in the related literature, are not required.

Keywords:

NTU games, core, axioms, consistency, weak continuity

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1. Introduction

For analyzing coalitional behaviour, (cooperative) games in characteristic function form with not necessarily transferable utility (i.e., with "non-transferable utility" or "NTU" games) constitute a canonical framework. Naturally, the core, as a natural set-valued prediction or "solution" for such games, received quite an amount of justifiable attention.[1] Also, quite naturally, like several other solutions within such frameworks, the core has been analyzed from axiomatic standpoint (Peleg (1985), Keiding (1986), Nagahisa and Yamato (1992), Tadenuma (1992), Hwang and Sudholter (2001), Hwang (2006) and most recently, Arribillaga (2016)).

The main motivation for this exercise: However, apart from Keiding (1986), most other authors have axiomatized the core as a solution concept *only within the class of NTU games which satisfy the "non-levelness" condition.* We have recalled this condition formally in the following section. Somewhat informally stated, non-levelness means that if a pay-off vector lies on the boundary of a coalition's set of feasible pay-offs, then lowering any coordinate pushes it strictly inside. This ensures that every weakly dominated outcome is also strictly dominated–implicitly assuming some form of utility transfer. Frequent inclusion of this assumption for axiomatizing the core is possibly owing to the fact that for NTU games that do not satisfy this condition, the core may not obey many well-known and useful consistency properties such as Davis–Maschler consistency and Moulin consistency (formally stated

[1] See, e.g., among others, Kannai (1992) or part A of Mertens and Sorin (1994).

in Section 3) etc. However, the non-levelness condition may not be very appealing *especially in several contexts of games without transferable utility* because, as we have mentioned above, by this condition, some transferability of pay-off is implicitly smuggled in even in set-ups of non-transferable utility which may not be intuitively meaningful in several real-life situations. In particular, at least an important class of games–the hedonic games (defined precisely in Section 3 below), which have been studied extensively over the last decade or so especially in contexts of matching models–do not necessarily satisfy the non-levelness condition. Another important environment is that of voting, which, when represented in characteristic function form, may not satisfy the non-levelness condition.[2] While we discuss this motivational issue further with further formal details in Section 3 below, in the penultimate paragraph of this introductory section we give a real-life example illustrating this issue.

To address such situations and the corresponding conceptual categories, terms like "imperfectly transferable" utility, in contrast to just "non-transferable" utility, have been put forward (Galichon et al., 2019). We quote from Galichon et al. (2019) (pp. 2876-77) for a possibly helpful explanation of such distinctions within the specific environment of matching:

> . . . Becker (1973) and Shapley and Shubik (1972)—mainly focuses on matching patterns and the sharing of the surplus in a transferable utility (TU) setting. . . . However, the assumption that the bargaining frontier has this particular [linear] shape may be in-

[2]We provide details on this environment of voting in Section 3.

> appropriate; one can think of many cases in which there are nonlinearities that partially impede the transfer of utility between matched partners. Such nonlinearities arise naturally in marriage markets, where the transfers between partners might take any form (e.g., cash, favor exchanges and change in time use or consumption patterns) and the utility cost of a concession to one partner may not exactly equal the benefit to the other. An extreme case is the nontransferable utility (NTU) framework (Gale and Shapley 1962), in which there is no possibility of compensating transfer between partners . . . well suited to settings like school choice (where transfers are often explicitly ruled out) . . .

A summary of the main contribution of this paper: Given this context, this paper provides mainly three new axiomatic characterizations of the core as a solution on the *general* class of games in characteristic function form obeying only an innocuous condition: that the set of individually rational pay-off vectors is bounded for each coalition. We call this class Γ.

The first of these results (Theorem 1) is a tight complete characterization mainly using an axiom of consistency and the corresponding converse consistency property. The consistency property used, named *Strong Secession Consistency*, similar to some such properties already known in the literature, is, however, new. Theorem 1 shows that there is a unique solution on Γ that satisfies all of Pareto Optimality, Non-emptiness for Single Player Games, Strong Secession Consistency and Converse Strong Secession Consistency and it is the core. Further, these four axioms are independent on Γ : i.e., for each of these axioms there exists a solution which, on Γ, violates this axiom but satisfies the other three. In the next characterization result we retain Pareto Optimality and Strong Secession Consistency, replace the axiom Non-emptiness for Single Player Games by another weak non-emptiness property and substitute the converse consistency axiom by two other axioms including a weak continuity-like property. These other two axioms are Antimonotonicity, already quite well-known in the literature, and *Weak Continuity*. Theorem 2 shows that the core is the unique solution on Γ that satisfies these five axioms together. Further, these five axioms are independent on Γ : i.e., for each of these axioms there exists a solution which, on Γ, violates this axiom but satisfies the other four. Finally, we look into a variant of the axiom Strong Secession Consistency which we call *Weak Secession Consistency*. In Theorem 3 we replace Strong Secession Consistency by Weak Secession Consistency, retain the other four axioms as in Theorem 2 and add a sixth axiom called *Weak Internal Stability for Proximal Coalitions*. With these axioms we find a weaker result: Theorem 3 shows that the core is the *minimal* among the solutions which satisfy all of these six axioms on Γ.

We would like to mention another possibly interesting feature of our characterization results. While, especially Peleg (1985, 1992) characterized the core by consistency and converse-consistency-like axioms, the novelty of Keiding's approach was to provide an axiomatization result without invoking consistency at all. Bhattacharya (2004) and then Llerena and Rafels (2007) took a mixed approach (in a transferable utility set-up): they used both consistency-like axioms as well as axioms like Antimonotonicity used by Keiding (1986). Theorems 2 and 3 of this paper adopt a similar mixed approach.

One motivating example: Consider a set of two persons, $N = \{1, 2\}$. Each of them, to begin with, remains single and can allocate their time into maintaining their respective household and into income generating activities. By doing so, each $i \in N$ can get some maximum income ω^i. Then they can become partners and can decide to live together in a single household and reallocate their time into maintaining that household and into income generating activities. Thus, together they can generate optimally a total income ω^N and distribute that to themselves. Now consider some income pair (y_1, y_2) such that $y_1 + y_2 = \omega^N$. Take another worse income pair $(y_1 - \varepsilon, y_2)$ in which the income of person 1 falls by $\varepsilon > 0$ but that of person 2 remains intact. Then these persons can still generate the total income ω^N together and *because of the transferability of income*, redistribute that total income as $(y_1 - \varepsilon/2, y_2 + \varepsilon/2)$ and by doing so, both can become better off from that worse income-pair. If this situation is represented as a game in characteristic function form then the assumption of non-levelness is intuitively acceptable. Now take a contrasting scenario in which persons 1 and 2, instead, have started to date. While alone, each $i \in N$ has some reservation level of pleasure/satisfaction (i.e., "utility") π^i. But when together on a date, each of them can take some actions from some well-specified set of actions and each of them gets some pleasure/satisfaction (i.e., "utility") from the profile of actions chosen. For each $i \in N$, the maximum utility they can get from being together on a date is π_i^N. Take a utility-pair $(\pi_1^N - \varepsilon, \pi_2^N)$ in which the utility of person 1 falls from its potentially maximal value by $\varepsilon > 0$ but that of person 2 remains at its maximal level. In this scenario, however, interpersonal transfer of utility seems highly unrealistic: in particular, one does not typically use money or similar instruments to compensate for a loss of pleasure when on a date. Thus, if this situation is represented as a game in characteristic function form then the assumption of non-levelness is intuitively unappealing.

The structure of the remainder of this paper: The following section gives the preliminary definitions and notation. Section 3 expands on the issue with non-levelness further and provides a motivating example of a hedonic game for which the core does not satisfy either Davis-Maschler consistency or Moulin consistency; thus demonstrating that most of the existing literature cannot provide axiomatic justification of the core for such classes of games. The main axioms and some discussions of these are given in Section 4. Section 5 discusses two of the main axiomatization results. The implications of replacing Strong Secession Consistency by its variant, Weak Secession Consistency, have been explored in Section 6. We provide a couple of concluding remarks in Section 7. Finally, some results supplementary to the central ones in the main body of this paper are given in an Appendix at the very end.

2. Preliminary definitions and notation

Let U be a set of potential players that may be finite or countably infinite. For a set A we shall denote the cardinality of A by $|A|$ and the proper subset relation is denoted by $\subset$. For any finite subset S of U, by $\mathbb{R}^S$ we denote the set of all functions from S to $\mathbb{R}$, the set of real numbers. We would think of elements of $\mathbb{R}^S$ as $|S|-$dimensional vectors whose coordinates are indexed by the members of S and we shall use the Euclidean distance as the metric on $\mathbb{R}^S$.

Definition 1. A **(cooperative) game in characteristic function form (with non-transferable utility)** or an NTU game is a pair (N, V) where N is a finite subset of U and V is the (characteristic) set function which assigns to every $S \subseteq N$ a set $V(S)$ such that:
(1) $V(\emptyset) = \emptyset$;
(2) For each coalition[3] $S \subseteq N$, $V(S)$ is a non-empty proper subset of $\mathbb{R}^S$;
(3) For each coalition $S \subseteq N$, $V(S)$ is closed in $\mathbb{R}^S$;
(4) For each coalition $S \subseteq N$, $V(S)$ is comprehensive, i.e., if $x \in V(S)$, and $y \in \mathbb{R}^S$ is such that $y \leq x$ then $y \in V(S)$.

For analyzing coalitional behaviours a more general framework is that of *partition functions.* To recall that first we set up the following preliminary ideas and notation. For any finite $N \subset U$, denote by $\Pi(N)$ be the set of all *partitions* of N. An *embedded coalition* is a pair (S, π) such that $S \subseteq N$, $\pi \in \Pi(N)$ and $S \in \pi$. Then (see also Bimonte et al. (2024)), a partition function (not necessarily with transferable utility) can be defined, in general, as follows.

Definition 2. A **(cooperative) game (with a finite set of players $N \subseteq U$) in partition function form (with non-transferable utility)** or an NTU game in partition function form is a function $\mathcal{V}$ which assigns to every

[3]In what follows, often, by a *coalition* S we shall mean a *non-empty* $S \subseteq N$ with no possibility of confusion.

embedded coalition (S, π); $S \subseteq N; \pi \in \Pi(N)$ a set $\mathcal{V}(S, \pi)$ such that:

(1) For every $\pi \in \Pi(N)$, $\mathcal{V}(\emptyset, \pi) = \emptyset$;

(2) For all non-empty $S \in \pi$, $\mathcal{V}(S, \pi)$ is a non-empty proper subset of $\mathbb{R}^S$;

(3) For all non-empty $S \in \pi$, $\mathcal{V}(S, \pi)$ is closed in $\mathbb{R}^S$;

(4) For all non-empty $S \in \pi$, $\mathcal{V}(S, \pi)$ is comprehensive, i.e., if $x \in \mathcal{V}(S, \pi)$, and $y \in \mathbb{R}^S$ is such that $y \leq x$ then $y \in \mathcal{V}(S, \pi)$.

Recall that the idea underlying partition functions is that the set of pay-offs attainable by a coalition depends on which other coalitions have formed. A characteristic function is a more restrictive framework where, the set of pay-offs attainable by a coalition remains fixed irrespective of which other coalitions form. The analyses in this paper will remain within the framework of characteristic functions; only at Section 7 we make a remark with respect to partition functions.

For the rest of this paper we shall often refer to an NTU game (in characteristic function form) simply as a game with no possibility of confusion.

For any finite $N \subseteq U$ and any vector $x \in \mathbb{R}^N$ we shall denote the i-th component of it by x_i and the S coordinates of it (where $S \subseteq N$) by x_S. For any two vectors $a, b \in \mathbb{R}^S$ for some $S \subseteq N$, if $a_i > b_i$ for all $i \in S$ then we shall denote that as $a \gg b$. Given a set $A \subseteq \mathbb{R}^S$; (with $S \subseteq N$) the boundary of A is denoted by $bd(A)$ and the interior of A by $int(A)$.

Definition 3. Recall that a game (N, V) is said to be of *transferable utility* or a TU game if for every non-empty $S \subseteq N$ there exists a real number $v(S) \in \mathbb{R}$ such that $V(S) = \{x \in \mathbb{R}^S | \sum_{i \in S} x_i \leq v(S)\}$.

We assume henceforth that every game (N, V) we consider satisfies the following regularity condition (see, e.g., Scarf (1967)).

C1 (Boundedness of Individually Rational Pay-off Vectors): For any $j \in N$, let $b_j := max\{x|x \in V(\{j\})\}$.[4] For all $S \subseteq N$, the set $\{x \in V(S)|x_j \geq b_j$ for all $j \in S\}$ is bounded.

Another regularity condition, as we mentioned in the previous section, imposed frequently while analyzing the class of NTU games, is the following (see, e.g., Aumann (1985), Peleg (1985), Peleg (1992) etc).

C2 (Non-levelness): For each $S \subseteq N$, if $x \in bd(V(S))$ then $y \in \mathbb{R}^S$ and $[y \leq x \; ; y \neq x]$ imply that $y \in int(V(S))$.

But *we shall not impose this condition* (apart from for illustrating a minor contrasting result in the Appendix). Recall that every TU game, in particular, satisfies C2 and we repeat: if a general NTU game satisfies C2 then some (possibly imperfect) transferability of pay-offs is implicitly assumed even in the set-up of non-transferable utility.

By Γ we denote the class of games which satisfy C1.

Given any game $(N, V) \in \Gamma$, a vector $x \in \mathbb{R}^T$, $(T \subseteq N, \; T \neq \emptyset)$ is said to be blocked or dominated by a (finite dimensional) real vector y if there is a coalition $S \subseteq T$ such that $y_i > x_i$ for all $i \in S$ and $y_S \in V(S)$. We indicate this domination relation as $y \succ_S x$, i.e., y dominates x via coalition S. If a

[4] Note that for any $j \in N$, $V(\{j\})$ is comprehensive as well as a closed proper subset of $\mathbb{R}$. Therefore, for each $j \in N$, a unique b_j exists.

vector y dominates a vector x via some coalition S then we shall denote that as $y \succ x$.

Definition 4. The **core** of the game (N, V), denoted by $C(N, V) = \{x \in V(N) |$ there is no y such that $y \succ x\}$.

The set of Pareto-efficient pay-off vectors of (N, V), denoted by $X(N, V) = \{x \in V(N) |$ there is no y such that y dominates x via $N\}$. The set of individually rational pay-off vectors of (N, V), denoted by $I(N, V) = \{x \in V(N) | x_i \geq b_i$ for all $i \in N\}$ (where the piece of notation b_i has been defined in the statement of C1 above).

Definition 5. Given some $\Gamma_0 \subseteq \Gamma$, a **solution** on Γ_0 is a mapping $\sigma(.)$ which associates with each game $(N, V) \in \Gamma_0$ a subset $\sigma(N, V)$ of $V(N)$.

We repeat that almost all of our exercises are on the entire Γ which is a central feature of this paper.

Definition 6. For a game (N, V), a **subgame** of (N, V) on $T \subseteq N$, denoted by (T, V_T), is defined as

$$\text{for all } S \subseteq T,\ V_T(S) = V(S).$$

3. Further motivating our exercise: a couple of examples

Recall that the primitive of a hedonic game, mentioned in Section 1, is a coalition structure: i.e., a partition of N. For each partition P of N, for each $i \in N$, let $S_i(P)$ be the coalition that contains player i. Player i has a complete, reflexive and transitive preference ordering $\succeq_i$ over $\{S_i(P) \mid P$

is a partition of $N\}$. Generally, a hedonic game is defined as the collection $\langle N, (\succeq_i)_{i \in N} \rangle$ (see, e.g., Section 5.1.2 of Chalkiadakis et al. (2012)). In words, each $i \in N$ has a preference ordering over the possible coalitions to which they may belong.

If we further assume that for each $i \in N$, $\succeq_i$ has a cardinal representation–i.e., each $i \in N$ gets some cardinal pay-off from the possible coalitions to which they may belong–then a hedonic game can be written as an NTU game as follows (see, again, e.g., Section 5.1.2 of Chalkiadakis et al. (2012)). Whereas, in general, such hedonic games can be written in partition function form as well, a characteristic function suffices for such games as the cardinal pay-off of every player in any coalition remains invariant irrespective of what other coalitions are present.

Definition 7. A game (N, V) is a hedonic game if for every non-empty $S \subseteq N$, there exists $\pi^S \in \mathbb{R}^S$ such that $V(S) = \{x \in \mathbb{R}^S \mid \text{for each } i \in S, x_i \leq \pi_i^S\}$.

Denote by Γ_H the class of such hedonic games. Note that each game in Γ_H satisfies C1, but does not satisfy C2.

As another instance of a real-life game-scenario for which C2 may not be quite acceptable, consider the following example of voting by committees. Let N be a finite set of judges who are to decide on selecting some winner in a gymnastics competition. Let C be a finite set of competitors. A coalition $S \subseteq N$ can decide to select any $c \in C$ rather then any $c' \in C$ if and only if $S \subseteq N$ is a majority coalition (i.e., if and only if $|S| > |N|/2$). For each competitor $c \in C$, given their performance in the competition, each

judge $i \in N$ gets some pleasure/satisfaction (“utility") if c wins. For each $i \in N$, this utility is measured by a function $u_i : C \mapsto \mathbb{R}$ and without loss of generality, let us assume that for each $i \in N$, and each $c \in C$, $0 \leq u_i(c) < \infty$. This situation can be represented as a game in characteristic function form (N, V^{gym}) in the following way (see also Chapter 7 of Ordeshook (1986)):
if $|S| > |N|/2$, then $V^{gym}(S) = \{x \in \mathbb{R}^S \mid \text{for each } i \in S, \text{ there exists } c \in C \text{ such that } x_i \leq u_i(c)\}$;
if $0 < |S| < |N|/2$, then $V^{gym}(S) = \{x \in \mathbb{R}^S \mid \text{for each } i \in S,\ x_i \leq 0\}$; and
$V^{gym}(\emptyset) = \emptyset$.
Note that $(N, V^{gym}) \in \Gamma_H$.

Next we recall some consistency axioms that have been utilized for axiomatizing the core on a restricted class of NTU games (those that satisfy non-levelness and have a non-empty core; call that class Γ_C). We start with Davis-Maschler consistency, used by Peleg (1985) for axiomatizing the core on Γ_C.

Definition 8. Consider any game (N, V) in some subclass $\Gamma_0 \subseteq \Gamma$. Let $x \in V(N)$. The **Davis-Maschler reduced game** on $S \subset N$, $(S \neq \emptyset)$ with respect to x, $(S, V_S^{\delta x})$, is given by:

$$V_S^{\delta x}(S) = \{y \in \mathbb{R}^S \mid (y, x_{N \setminus S}) \in V(N)\};$$
$$V_S^{\delta x}(T) = \bigcup_{Q \subseteq N \setminus S} \{y \in \mathbb{R}^T \mid (y, x_Q) \in V(T \cup Q)\};\ T \subset S.$$

For the intuitive idea underlying Davis-Maschler reduced games, see, if necessary, Section 3 of Peleg (1985). Then Davis-Maschler consistency (Peleg

(1985)) is defined as follows.

Davis-Maschler Consistency (DMC):
Consider any game (N, V) in some subclass $\Gamma_0 \subseteq \Gamma$. Then, a solution $\sigma(.)$ satisfies DMC on Γ_0 if the following holds: if $x \in \sigma(N, V)$ then for any coalition S, $(S, V_S^{\delta x}) \in \Gamma_0$ and $x_S \in \sigma(S, V_S^{\delta x})$.

Next we recall Moulin consistency, used by Tadenuma (1992) for axiomatizing the core on Γ_C.

Definition 9. Consider any game (N, V) in some subclass $\Gamma_0 \subseteq \Gamma$. Let $x \in V(N)$. The **Moulin reduced game** on $S \subset N$, $(S \neq \emptyset)$ with respect to x, $(S, V_S^{\mu x})$, is given as follows: for every non-empty $T \subseteq S$,

$$V_S^{\mu x}(T) = \{y \in \mathbb{R}^T \mid (y, x_{N \setminus S}) \in V(T \cup N \setminus S)\}.$$

Then Moulin consistency (Tadenuma, 1992) is defined as follows.

Moulin Consistency (MC):
Consider any game (N, V) in some subclass $\Gamma_0 \subseteq \Gamma$. Then, a solution $\sigma(.)$ satisfies MC on Γ_0 if the following holds: if $x \in \sigma(N, V)$ then for any coalition S, $(S, V_S^{\mu x}) \in \Gamma_0$ and $x_S \in \sigma(S, V_S^{\mu x})$.

Next we get on to the following Observation.

Observation 1. *The core does not satisfy either DMC or MC on Γ_H.*

PROOF. The proof is in the form of the following simple example of a hedonic game (a more general example proving this Observation can be generated easily using this very example).

$N = \{1, 2\}$; $\pi^N = (2, 2)$; $\pi^{\{1\}} = \pi^{\{2\}} = 0$.

Note that this game satisfies C1 but not C2.

Note that the vector $x = (1, 2)$ is in the core of this game. Then,

$V^{\delta x}_{\{1\}}(\{1\}) = V^{\mu x}_{\{1\}}(\{1\}) = \{y \in \mathbb{R} \mid y \leq 2\}$.

But then $x_{\{1\}}$ is not in the core of either of these reduced games. ∎

Observation 1 indicates that much of the existing results axiomatizing the core are inapplicable for such (and similar) classes of games. Our work tackles this issue.

4. Our main axioms

Some of the axioms used in this paper are quite well-known in the literature and hardly need much discussion. Below we discuss in some detail the axioms we consider relatively less familiar.

We state the axioms by invoking an arbitrary solution $\sigma(.)$ on the general domain of our study–the entire Γ; but restating these on some subsets of Γ, if required, is unproblematic. Take $(N, V) \in \Gamma$.

1. **Pareto Optimality** (PO): $\sigma(N, V) \subseteq X(N, V)$.

2. **Non-emptiness for Single Player Games** (NESPG): If $|N| = 1$, $\sigma(N, V) \neq \emptyset$.

3. **Irrelevance of σ-empty Coalitions** (IREC):

Suppose that for every non-singleton and non-empty $S \subset N$, $\sigma(S, V_S) = \emptyset$. In that case, if there exists $x \in I(N, V)$, then $\sigma(N, V) \neq \emptyset$.

Operationally this is a technical but weak "non-emptiness" criterion. However, an intuitive normative idea behind IREC is as follows. If a coalition is to affect the solution set for the whole game involving the grand coalition N, then it must have a non-empty solution set for its own sub-situation (i.e., the respective subgame). It is straightforward that if a game satisfies IREC then that satisfies NESPG as well.

Next we provide a consistency-like axiom and toward that goal first we define a reduced game.

Definition 10. Let $x \in V(N)$. The **strong secession reduced game** (SS reduced game hereafter) on $S \subset N$, $(S \neq \emptyset)$ with respect to x, (S, V_S^x), is given by:

$$V_S^x(S) = \begin{cases} V(S) & \text{if } x_S \in V(S), \\ \{y \in \mathbb{R}^S |\ y \leq x_S\} & \text{if } x_S \notin V(S), \end{cases}$$

$$V_S^x(T) = V(T) \text{ for } T \subset S.$$

This reduced game reflects the following situation. Suppose, a pay-off vector x is agreed upon by N. Then, the players in $N \setminus S$ leave, no cooperation with them is possible any more. But the grand coalition S in the "reduced" situation can still renegotiate on the pay-off distribution. If it finds that it cannot possibly improve upon this agreed pay-off x then this distribution is maintained in the reduced situation as well. Otherwise, the members of S oppose the pay-off distribution according to x and completely secede from the original game making a coalition for themselves. Moreover, since no co-operation with the players in $N \setminus S$ is possible, the worth of each $T \subset S$ in the reduced game remains what it was in the original game. This is similar in spirit to the reduced game introduced by Nagahisa and Yamato (1992). Bhattacharya (2004) used such a reduced game in transferable utility set-up. Llerena and Rafels (2007) also used this idea (but they called it "projection" reduced game).

4. **Strong Secession Consistency** (SSC):
If $x \in \sigma(N, V)$ then for any coalition S, $(S, V_S^x) \in \Gamma$ and $x_S \in \sigma(S, V_S^x)$.

Note further that the idea underlying SS reduced games is quite opposite to that for Davis-Maschler reduced games. For the latter, in the reduced game on any proper coalition S, for any coalition $T \subset S$, any other coalition $Q \subseteq (N \setminus S)$ is available for joint play in the underlying strategic sub-situation whereas for the former, no cooperation with $N \setminus S$ is possible at all. It might be intersting to note that on Γ_C the core satisfies consistency with respect to both these kinds of behaviorally quite opposite reduced sub-situations.

The corresponding "converse" property is:

5. **Converse Strong Secession Consistency** (CSSC):
Suppose $x \in X(N, V)$ and for every coalition S, $S \neq N$, $x_S \in \sigma(S, V_S^x)$. Then

$x \in \sigma(N, V)$.

A tiny point to note is that unlike, say, in Peleg (1985), in the statement of CSSC we require the vector x to be in $X(N, V)$, not merely in $V(N)$. It is easy to see that if we replace $X(N, V)$ by $V(N)$ in the statement of CSSC then the core may not satisfy the resulting variant of the axiom on Γ.

The next axiom is akin to continuity (but much weaker than continuity) which is a desirable feature for a solution. For stating this axiom, to represent distance between two subsets of a finite-dimensional Euclidean space we have used below the Hausdorff distance[5] as that is possibly the most widely used in such contexts.

6. **Weak continuity** (WC):
Let $\{(N, V^k)\}$ be a sequence of games belonging to Γ such that $\forall k$, $V^k(S) = V(S)$ for $S \subset N$ and $V^k(N)$ converges to $V(N)$ (in the Hausdorff distance). Let $\{x^k\}$ be a sequence such that $x^k \in \sigma(N, V^k)$ for all k and x^k converges to x. Then $x \in \sigma(N, V)$.

In the Appendix we introduce a variant of this continuity-like axiom (called "Modified weak continuity") and Supplementary Result 2 in the Appendix shows that this axiom also works in axiomatizing the core on Γ.

7. **Antimonotonicity** (AM):

[5] For a definition and some properties of this, see, if necessary, e.g., Hildenbrand (1974).

Let $(N, V') \in \Gamma$ be such that $V'(S) \subseteq V(S)$ for all $S \subset N$ and $V'(N)=V(N)$. Then $\sigma(N, V) \subseteq \sigma(N, V')$.

The intuition is that if the coalitions get impoverished then the pay-off vectors in the solution of the original game remain in the solution of the new game and additionally some more pay-off vectors feasible for the grand coalition may qualify as solution vectors. Keiding (1986) introduced this axiom in the literature.

It should be straightforward to see that the core satisfies PO, NESPG and AM on Γ. In the next section we demonstrate that the core satisfies the other four axioms on Γ as well.

5. The main characterization results

Theorem 1. *There is a unique solution on Γ that satisfies PO, NESPG, SSC and CSSC and it is the core. Further, these four axioms are independent on Γ : i.e., for each of these axioms there exists a solution which, on Γ, violates this axiom but satisfies the other three.*

For proving the characterization part of this Theorem we use three lemmas given below. The idea behind this result–especially that of invoking NESPG for a single-player game and then using converse consistency and the method of induction–has been adopted from Nagahisa and Yamato (1992).

Lemma 1.1. *For any $(N, V) \in \Gamma$, $C(N, V)$ satisfies SSC and CSSC*

PROOF. Take any $(N, V) \in \Gamma$.

SSC: Take any $x \in C(N, V)$ and for any $S \subset N$, consider the SS reduced

game (S, V_S^x). If $x_S \in V(S)$ then $V_S^x(S) = V(S)$ and since $(N, V) \in \Gamma$, by the definition of SS reduced games, (S, V_S^x) also belongs to Γ. If $x_S \notin V(S)$ then for each $y \in V_S^x(S)$, $y \leq x_S$ and so, again by the definition of SS reduced games, (S, V_S^x) belongs to Γ. Further, since $x \in C(N, V)$, for both these cases, by the definition of SS reduced games, there cannot exist $y \in V_S^x(S)$ such that $y \succ_S x_S$. Next, consider, if possible, a vector y and a coalition $T \subset S$ such that $y \succ_T x_S$. But then $y \succ_T x$ as well which contradicts the supposition that $x \in C(N, V)$. Therefore, $x_S \in C(S, V_S^x)$.
CSSC: Conversely, suppose $x \in X(N, V)$ and for every coalition S, $S \neq N$, $x_S \in C(S, V_S^x)$. First, from the definition of SS reduced games it is straightforward that there cannot exist any vector y and coalition T such that $|T| < |N| - 1$ and $y \succ_T x$. Next, consider any $S \subset N$ such that $|S| = |N| - 1$. If there exists any $y \in \mathbb{R}^S$ for which $y \succ_S x$ then $x_S \in int(V(S))$. But then $x_S \notin C(S, V_S^x)$ leading to a contradiction. Therefore, since $x \in X(N, V)$ as well, $x \in C(N, V)$. ■

Lemma 1.2 *If a solution $\sigma(.)$ satisfies PO and SSC on Γ then for any $(N, V) \in \Gamma$, $\sigma(N, V) \subseteq C(N, V)$.*

Proof. Take $(N, V) \in \Gamma$ and suppose that $x \in \sigma(N, V) \backslash C(N, V)$. Then, for some $S \subset N$, $x_S \in int(V(S))$. Therefore, by the definition of a SS reduced game, $V_S^x(S) = V(S)$. Moreover, by SSC, $x_S \in \sigma(S, V_S^x)$. But since $x_S \in int(V(S))$, then $\sigma(.)$ violates PO. ■

Lemma 1.3 *If a solution $\sigma(.)$ satisfies PO, NESPG and CSSC on Γ then for any $(N, V) \in \Gamma$, $C(N, V) \subseteq \sigma(N, V)$.*

PROOF. We shall prove this by induction on the number of players. Take $(N, V) \in \Gamma$. If $|N| = 1$, then by NESPG, $\sigma(N, V) \neq \emptyset$ and by PO, $C(N, V) \subseteq \sigma(N, V)$ (in fact, these two sets are equal). Assume that the result is true whenever $|N|$ is less than or equal to some positive integer $k - 1$. Consider $(N, V) \in \Gamma$ such that $|N| = k$. Let $x \in C(N, V)$. Then, for every $S \subset N$, $x_S \in C(S, V_S^x)$ and therefore, by the induction hypothesis, $x_S \in \sigma(S, V_S^x)$. Then, by CSSC, $x \in \sigma(N, V)$. ■

This completes the proof of the characterization part.

PROOF OF THE REMAINDER OF THEOREM 1.

Next we show that each of the axioms is independent of the other three.

PO: Consider a solution $\sigma(.)$ on Γ as follows: for every $(N, V) \in \Gamma$, $\sigma(N, V) = I(N, V)$, the set of individually rational pay-off vectors. Then $\sigma(.)$ does not satisfy PO but obviously satisfies NESPG. Next, if $x \in I(N, V)$ then for every $S \subset N$, irrespective of whether $x_S \in V(S)$ or not, for every $i \in S$, $x_i \geq b_i$. Therefore, for each $S \subset N$, $x_S \in \sigma(S, V_S^x)$ and thus, $\sigma(.)$ satisfies SSC. Finally, suppose $x \in X(N, V)$ and for every coalition S, $S \neq N$, $x_S \in \sigma(S, V_S^x)$. Then, for each $i \in N$, $x_i \geq b_i$ and so, $x \in \sigma(N, V)$.

NESPG: Consider a solution $\sigma(.)$ on Γ as follows: for every $(N, V) \in \Gamma$, $\sigma(N, V) = \emptyset$. Then $\sigma(.)$ violates NESPG but satisfies each of the other three axioms for this Theorem.

SSC: Consider a solution $\sigma(.)$ on Γ as follows: for every $(N,V) \in \Gamma$, $\sigma(N,V) = X(N,V)$. Then it is straightforward to see that $\sigma(.)$ satisfies PO, NESPG and CSSC but violates SSC.

CSSC: Consider a solution $\sigma(.)$ on Γ as follows: for every (N,V) for which $|N| = 1$, $\sigma(N,V) = C(N,V)$ and if $|N| > 1$ then $\sigma(N,V) = \emptyset$. Then $\sigma(.)$ violates CSSC (which is easy to see considering examples of games with $|N| = 2$) but satisfies the other three axioms. ■

The proof of the next characterization result–Theorem 2–uses some ideas from Bhattacharya (2004).

Theorem 2. *There is a unique solution on Γ that satisfies PO, IREC, SSC, WC and AM and it is the core. Further, these five axioms are independent on Γ : i.e., for each of these axioms there exists a solution which, on Γ, violates this axiom but satisfies the other four.*[6]

PROOF. We prove this result along the following three steps.

Step 1: First we show that for each $(N,V) \in \Gamma$, $C(N,V)$ satisfies IREC and WC.

IREC: Take $(N,V) \in \Gamma$ such that for every non-singleton and non-empty

[6]In the Appendix we prove a minor contrasting result: Supplementary Result 1. Let $\Gamma_{L2} \subset \Gamma$ be the subclass of 2-player games which satisfy C2 (i.e., non-levelness) as well. Then there is a unique solution on Γ_{L2} that satisfies PO, IREC, SSC and AM and it is the core (i.e., for characterization of the core for this sub-class, WC is not required).

$S \subset N$, $C(S, V_S) = \emptyset$. First note that since each $(N, V) \in \Gamma$ satisfies C1, for each such (N, V) with a non-empty $I(N, V)$, every $y \in I(N, V)$ cannot be in the interior of $V(N)$ and thus, $X(N, V) \cap I(N, V) \neq \emptyset$. Pick $x \in X(N, V) \cap I(N, V)$. We claim that $x \in C(N, V)$. Suppose not. Then, arguing as in Ray (1989), since $C(S, V_S) = \emptyset$ for every non-singleton and non-empty $S \subset N$, there must exist $i \in N$ for which $b_i > x_i$. But this contradicts the supposition that $x \in I(N, V)$ and thus, $C(N, V)$ satisfies IREC.

WC: Let $\{(N, V^k)\}$ be a sequence of games belonging to Γ such that $\forall k$, $V^k(S) = V(S)$ for $S \subset N$ and $V^k(N)$ converges to $V(N)$ (in the Hausdorff distance). Let $\{x^k\}$ be a sequence such that $x^k \in C(N, V^k)$ for all k and x^k converges to x. Suppose $x \notin V(N)$: i.e., $x \in \mathbb{R}^N \setminus V(N)$. Since x does not belong to the closure of $V(N)$, the distance between the point x and the set $V(N)$ is some finite number greater than 0. But then, since x^k converges to x and for each k, $x^k \in V^k(N)$, $V^k(N)$ cannot converge to $V(N)$ in the Hausdorff distance. This leads to a contradiction. Therefore, $x \in V(N)$. Next we show that $x \in C(N, V)$. Suppose not. Since the sequence $\{x^k\}$ converges to x, for each k, $x_k \in C(N, V^k)$ and for each $S \subset N$, $V^k(S) = V(S)$ it cannot be the case that $x_S \in int(V(S))$ for some coalition $S \subset N$. Then it must be that $x \in int((V(N))$. Then there is some open ball $B(x)$ centred on x such that $B(x) \subset V(N)$. But then, since for the sequence $\{(N, V^k)\}$, $V^k(N)$ converges to $V(N)$ in the Hausdorff distance, there is a positive integer $\bar{k}$ such that for each $k \geq \bar{k}$, the ball $B(x) \subset V^k(N)$ as well. But then, since the sequence $\{x^k\}$ converges to x, it cannot be the case that for each k, $x^k \in C(N, V^k)$ which leads to a contradiction.

Step 2: Next we prove the main characterization result.

Suppose a solution $\sigma(.)$ satisfies PO, IREC, SSC, WC and AM on Γ. Take $(N, V) \in \Gamma$. By Lemma 1.2 above, $\sigma(N, V) \subseteq C(N, V)$.
Take $x \in C(N, V)$. Fix a real number $\varepsilon > 0$ and construct the game (N, V^{ε}) as follows:

$$V^{\varepsilon}(N) = V(N) \cup \{y \in \mathbb{R}^N | \text{ for each } i \in N,\ y_i \leq x_i + \varepsilon/|N|\},$$

and for $S \subset N$,

$$V^{\varepsilon}(S) = V(S).$$

Construct the vector x^{ε}, given by $x_i^{\varepsilon} = x_i + \varepsilon/|N|$ for all $i \in N$.
Now, further construct the game $(N, V^{\varepsilon,x})$ for which $V^{\varepsilon,x}(S) = V^{\varepsilon}(S)$ for every non-singleton coalition $S \subseteq N$ and for every $i \in N$, $V^{\varepsilon,x}(\{i\}) = \{y \in \mathbb{R} | y \leq x_i + \varepsilon/|N|\}$.
We claim that for any proper coalition $S \subset N$ such that $|S| > 1$, $\sigma(S, V_S^{\varepsilon,x})$, i.e., the solution for the subgame of $(N, V^{\varepsilon,x})$ on the coalition S, is empty. Suppose otherwise. Fix $S \subset N$, $|S| > 1$, such that $\sigma(S, V_S^{\varepsilon,x}) \neq \emptyset$ and let $y \in \sigma(S, V_S^{\varepsilon,x})$. By Lemma 1.2, $y_i \geq x_i + \varepsilon/|N|$ for every $i \in S$. But then $y \succ_S x$ which contradicts the supposition that $x \in C(N, V)$. Hence, the claim is proved.

Then, by IREC, $\sigma(N, V^{\varepsilon,x}) \neq \emptyset$ and by PO and Lemma 1.2, $\sigma(N, V^{\varepsilon,x}) = \{x^{\varepsilon}\}$. Therefore, by AM, $x^{\varepsilon} \in \sigma(N, V^{\varepsilon})$. Now, take a decreasing sequence of positive numbers $\{\varepsilon^k\}$ such that $\varepsilon^1 = \varepsilon$ and $\varepsilon^k \longrightarrow 0$. For each k, construct a game (N, V^{ε^k}) such that:

$$V^{\varepsilon^k}(N) = V(N) \cup \{y \in \mathbb{R}^N | y_i \leq x_i + \varepsilon^k/|N| \text{ for each } i \in N\},$$

and for $S \subset N$,

$$V^{\varepsilon^k}(S) = V(S).$$

Let x^{ε^k} be the vector given by $x_i^{\varepsilon^k} = x_i + \varepsilon^k/|N|\ \forall i \in N$. By our argument above, for each k, x^{ε^k} is in $\sigma(N, V^{\varepsilon^k})$. Then for the sequence $\{(N, V^{\varepsilon^k})\}$, $V^{\varepsilon^k}(N)$ converges to $V(N)$ (in the Hausdorff distance) and x^{ε^k} converges to x. Then, by WC, $x \in \sigma(N, V)$.

Step 3: Next we show that each of the axioms is independent of the other four.

PO: Consider a solution $\sigma(.)$ on Γ as follows: for every $(N, V) \in \Gamma$, $\sigma(N, V) = I(N, V)$, the set of individually rational pay-off vectors for (N, V). Then it is straightforward to see that $\sigma(.)$ violates PO but satisfies the other four axioms.

IREC: Consider a solution $\sigma(.)$ on Γ as follows: for every $(N, V) \in \Gamma$, $\sigma(N, V) = \emptyset$. Then it is straightforward to see that $\sigma(.)$ violates IREC but satisfies the other four axioms.

SSC: Consider a solution $\sigma(.)$ on Γ as follows: for every $(N, V) \in \Gamma$, $\sigma(N, V) = X(N, V)$. Then it is straightforward to see that $\sigma(.)$ satisfies PO, WC and AM but violates SSC.
To demonstrate that this $\sigma(.)$ satisfies IREC, note that since each $(N, V) \in \Gamma$ satisfies C1, for each such (N, V) with a non-empty $I(N, V)$, every $y \in$

$I(N,V)$ cannot be in the interior of $V(N)$ and thus, $X(N,V) \cap I(N,V) \neq \emptyset$.

WC: Fix a set of players $\hat{N} = \{1,2,3\}$. Fix a game $(\hat{N}, V^{\varepsilon^k}) \in \Gamma_H$ as follows:
$\hat{V}(\hat{N}) = \{x \in \mathbb{R}^{\hat{N}} | x_1 \leq 4; x_2 \leq 4; x_3 \leq 4\}$;
for $S \subset \hat{N}$ such that $|S| = 2$; $\hat{V}(S) = \{x \in \mathbb{R}^S | x_i \leq 2$ for each $i \in S\}$;
for each $i \in \hat{N}$, $\hat{V}(\{i\}) = \{x \in \mathbb{R} | x \leq 1\}$.
Let a solution $\sigma(.)$ be as follows. For any $(N,V) \in \Gamma$ such that $(\hat{N}, \hat{V})$ is a subgame of (N,V), $\sigma(N,V) = \emptyset$. For $(\hat{N}, \hat{V})$, $\sigma(\hat{N}, \hat{V}) = (4,4,4)$. For any other $(N,V) \in \Gamma$, $\sigma(N,V)$ is the core.
Then it is easy to see that $\sigma(.)$ satisfies the four axioms other than WC.
To see that $\sigma(.)$ violates WC, take a decreasing sequence of positive numbers $\{\varepsilon^k\}$ such that $\varepsilon^k \longrightarrow 0$. For each k, construct a game $(\hat{N}, V^{\varepsilon^k}) \in \Gamma_H$ such that:
$V^{\varepsilon^k}(\hat{N}) = \{x \in \mathbb{R}^{\hat{N}} | x_1 \leq 4 + \varepsilon^k; x_2 \leq 4 + \varepsilon^k; x_3 \leq 4 + \varepsilon^k\}$;
and for $S \subset \hat{N}$,
$V^{\varepsilon^k}(S) = V(S)$.
Consider the sequence $\{x^k\}$ such that for each k, $x^k = (3+\varepsilon^k, 4+\varepsilon^k, 4+\varepsilon^k)$. Then, for each k, $x^k \in \sigma(\hat{N}, V^{\varepsilon^k})$. The sequence $\{V^{\varepsilon^k}(\hat{N})\}$ converges to $\hat{V}(\hat{N})$ and the sequence $\{x^k\}$ converges to $(3,4,4)$ which, however, does not belong to $\sigma(\hat{N}, \hat{V})$.

AM: Fix a set of players $\hat{N} = \{1,2\}$. Consider the subset of games $\Gamma' \subset \Gamma_H$ such that for each $(N,V) \in \Gamma'$, $N = \hat{N}$ and
$V(N) = \{x \in \mathbb{R}^N | x_1 \leq 1; x_2 \leq 1\}$;
$b_1 = 1$; $0 \leq b_2 \leq 1$.

Let a solution $\sigma(.)$ be as follows. For any $(N,V) \in \Gamma$ such that $N = \hat{N}$, $\sigma(N,V) = x \in C(N,V)$ such that $x_2 = b_2$. For any $(N,V) \in \Gamma$ such that $\hat{N}$ is a proper subset of N and for which $C(\hat{N}, V_{\hat{N}}) \neq \emptyset$, $\sigma(N,V) = \emptyset$. For any other $(N,V) \in \Gamma$, $\sigma(N,V)$ is the core.
Then it is easy to see that $\sigma(.)$ violates AM but satisfies the other four axioms.

Remark 1. Note that for proving the independence of the four axioms in Theorem 1, we did not have to impose any restriction on Γ. With respect to Theorem 2, while proving independence of the five axioms, for two of the contrasting example solutions we used games belonging to Γ_H and the rest of the example solutions did not require any restriction at all. This shows that these (tight) axiomatizations are valid for characterizing the core on Γ_H which is in sharp contrast to what we saw in Section 3.

6. A related consistency property and its implication

Note that following the idea of secession consistency, a similar consistency axiom can be introduced as follows. First we define a corresponding reduced game. As before, take some $(N,V) \in \Gamma$.

Definition 11. Let $x \in V(N)$. The **weak secession reduced game** on $S \subset N$, $(S \neq \emptyset)$ with respect to x, $(S, \tilde{V}_S^x)$, is given by:

$$\tilde{V}_S^x(S) = \{y \in \mathbb{R}^S | y \leq x\};$$

$$\tilde{V}_S^x(T) = V(T) \text{ for } T \subset S.$$

This, obviously, is similar in spirit to the strong secession reduced game. But here, the coalition S *must* maintain the pay-off agreed upon by N and so, the power of seceding is weaker. The corresponding consistency property is:

8. **Weak Secession Consistency** (WSC):
If $x \in \sigma(N, V)$ then for any coalition S, $(S, \tilde{V}_S^x) \in \Gamma$ and $x_S \in \sigma(S, \tilde{V}_S^x)$.

The corresponding "converse consistency" condition is:

8′. **Converse Weak Secession Consistency** (CWSC):
Suppose $x \in X(N, V)$ and for every coalition S, $S \neq N$, $x_S \in \sigma(S, \tilde{V}_S^x)$. Then $x \in \sigma(N, V)$.

In this section we explore the possibility of axiomatizing the core using this related but distinct consistency condition instead of SSC. But first we make a couple of preliminary observations.

Observation 2. *(i) The core does not satisfy CWSC on* Γ;
(ii) There exists a solution $\sigma(.)$ *on* Γ *which satisfies WSC but not SSC.*

Proof. (i): Consider the following game. $N = \{1, 2, 3\}$ and
$V(N) = \{x \in \mathbb{R}^N | \sum_{i \in N} x_i \leq 6\}$;
$V(\{1, 2\}) = \{x \in \mathbb{R}^{\{1,2\}} | x_1 + x_2 \leq 4\}$;
for every other $S \subset N$, $V(S) = \{x \in \mathbb{R}^S | \sum_{i \in S} x_i \leq 0\}$.
Consider the vector $x = (2, 1, 3)$. Then $x \in X(N, V)$ and for every coalition S, $S \neq N$, $x_S \in C(S, \tilde{V}_S^x)$. But $x \notin C(N, V)$ as, say, the vector $(2.1, 1.9)$

dominates x via the coalition $\{1, 2\}$.

(ii): Consider the following solution: for each $(N, V) \in \Gamma$, the solution is $X(N, V)$. Consider the game used in proving part (i) of this Observation. For that game, $X(N, V)$ satisfies WSC but not SSC. To see this, consider again the vector $x = (2, 1, 3) \in X(N, V)$ and the coalition $S = \{1, 2\}$. Then, $\tilde{V}_S^x(S) = \{y \in \mathbb{R}^S | \ y \leq (2, 1)\}$ whereas $V_S^x(S) = \{y \in \mathbb{R}^S | y_1 + y_2 \leq 4\}$. Therefore, while $x \in X(S, \tilde{V}_S^x)$, $x \notin X(S, V_S^x)$. ■

Next we introduce our final axiom.

9. **Weak Internal Stability for Proximal Coalitions** (WISPC):
Let $x \in \sigma(N, V)$. Consider any $S \subset N$ such that $|S| = |N| - 1$. Then for all $y \in \sigma(S, V_S)$,

$$max_{j \in S} \ x_j \geq min_{j \in S} \ y_j.$$

This axiom is somewhat egalitarian in spirit. Suppose for a coalition S proximal to N (obtained by dropping only one player) even the worst-paid player in a pay-off vector y in the *solution* of the subgame on S gets more than that is given to *any* player of S in an allocation x for the grand coalition. Then, this axiom specifies that if x is *so bad* for possibly such a *large fraction of the players* then x should not be in the solution of the whole game. Bhattacharya (2004) introduced this axiom in context of transferable utility environments.

Our final characterization result is as follows.

Theorem 3. *The core is the minimal among the solutions which satisfy PO,* *IREC, WSC, WC, WISPC and AM on* Γ.[7]

We prove this theorem via the following three lemmas.

Lemma 3.1. *The core satisfies WSC and WISPC on* Γ.

PROOF. Take any $(N, V) \in \Gamma$ and any $x \in C(N, V)$.
WSC: Now, for any $S \subset N$, consider the weak secession reduced game $(S, \tilde{V}_S^x)$. By the definition of this reduced game, for each $y \in \tilde{V}_S^x(S)$, $y \leq x_S$. Further, for each $T \subset S$, $\tilde{V}_S^x(T) = V(T)$. Therefore, since $(N, V) \in \Gamma$, $(S, \tilde{V}_S^x) \in \Gamma$ as well. Next, by the definition of $\tilde{V}_S^x(S)$, there cannot exist $y \in \tilde{V}_S^x(S)$ such that $y \succ_S x_S$. Next, consider, if possible, a vector y and a coalition $T \subset S$ such that $y \succ_T x_S$. But then $y \succ_T x$ as well which contradicts the supposition that $x \in C(N, V)$. Therefore, $x_S \in C(S, \tilde{V}_S^x)$.
WISPC: Consider any $S \subset N$ such that $|S| = |N| - 1$. Suppose there exists $y \in \sigma(S, V_S)$, such that $max_{j \in S}\ x_j < min_{j \in S}\ y_j$. Then $y \succ_S x$ leading to a contradiction. ■

Lemma 3.2. *If a solution* $\sigma(.)$ *satisfies PO, IREC, WSC and WISPC on* Γ *then for any* $(N, V) \in \Gamma$, $\sigma(N, V) \subseteq I(N, V)$.

PROOF. Take $x \in \sigma(N, V)$. Note that by IREC, for any single-player game (N, V), $\sigma(N, V) \neq \emptyset$. Then, by PO, for every $i \in N$, $\sigma(\{i\}, V_{\{i\}}) = \{b_i\}$.

[7]Supplementary Result 3 in the Appendix shows that the minimality in the statement of this theorem is non-trivial: i.e., there exists a solution $\sigma(.)$ satisfying all these six axioms on Γ such that for every $(N, V) \in \Gamma$, $C(N, V) \subseteq \sigma(N, V)$ and for some $(N, V) \in \Gamma$, $C(N, V)$ is a proper subset of $\sigma(N, V)$.

Now, let $|N| > 1$ and suppose $x_i < b_i$ for some $i \in N$. If $|N| = 2$, then $\sigma(.)$ violates WISPC as $b_i > x_i$. If $|N| > 2$, then pick $j \in N \setminus \{i\}$ and construct $(\{i,j\}, \tilde{V}^x_{\{i,j\}})$, the weak secession reduced game on $\{i,j\}$ with respect to x. Then by WSC, the vector $(x_i, x_j) \in \sigma(\{i,j\}, \tilde{V}^x_{\{i,j\}})$. Note that the subgame of the game $(\{i,j\}, \tilde{V}^x_{\{i,j\}})$ on the singleton coalition $\{i\}$ is, by the definition of weak secession reduced games, precisely $(\{i\}, V_{\{i\}})$. Therefore, since $\sigma(\{i\}, V_{\{i\}}) = \{b_i\}$ and $b_i > x_i$, $\sigma(.)$ violates WISPC. ■

Lemma 3.3 *If a solution $\sigma(.)$ satisfies PO, IREC, WSC, WC, AM and WISPC on Γ then for any $(N,V) \in \Gamma$, $C(N,V) \subseteq \sigma(N,V)$.*

PROOF. The proof is exactly similar to the Step 2 of the proof of Theorem 2 above, but for completeness we (essentially) reproduce the first part of the proof.

Take $x \in C(N,V)$. Fix $\varepsilon > 0$ and construct the game (N, V^ε) as follows:

$$V^\varepsilon(N) = V(N) \cup \{y \in \mathbb{R}^N | \text{ for each } i \in N,\ y_i \leq x_i + \varepsilon/|N|\},$$

and for $S \subset N$,

$$V^\varepsilon(S) = V(S).$$

Construct the vector x^ε, given by $x^\varepsilon_i = x_i + \varepsilon/|N|$ for each $i \in N$.

Now, further construct the game $(N, V^{\varepsilon,x})$ for which $V^{\varepsilon,x}(S) = V^\varepsilon(S)$ for every non-singleton coalition $S \subseteq N$ and for every $i \in N$, $V^{\varepsilon,x}(\{i\}) = \{y \in \mathbb{R} | y \leq x_i + \varepsilon/|N|\}$.

We claim that for any proper coalition $S \subset N$ such that $|S| > 1$, $\sigma(S, V^{\varepsilon,x}_S)$, i.e., the solution for the subgame of $(N, V^{\varepsilon,x})$ on the coalition S, is empty. Suppose otherwise. Fix $S \subset N$, $|S| > 1$, such that $\sigma(S, V^{\varepsilon,x}_S) \neq \emptyset$ and let

$y \in \sigma(S, V_S^{\varepsilon,x})$. By Lemma 3.2, $y_i \geq x_i + \varepsilon/|N|$ for every $i \in S$. But then $y \succ_S x$ which contradicts the supposition that $x \in C(N, V)$. Hence, the claim is proved.

The remainder of the proof is exactly identical to the Step 2 of the proof of Theorem 2 above. ■

7. Some concluding remarks

Note that while the core has an immediate intuitive explanation, some other solution concepts (like the kernel) are intuitively (apparently) less straightforward and the acceptability of these depends more strongly on axiomatic justification (e.g., Inarra et al. (2020)). But much of such exercises have been under the restriction of "non-levelness". Perhaps some axioms used in this paper may be fruitfully used to explore other such solutions axiomatically on richer classes of games (but, of course, at the moment this remark is entirely speculative).

Further, the core of games (not necessarily with transferable utility) in partition (rather than characteristic) function form (see, e.g., Bimonte et al. (2024)) has not been well-explored and to our knowledge axiomatic analyses of such cores are entirely absent. It might be interesting to study further whether the axiomatic analysis similar to this paper can be made for the core in the environment of games in partition function form.

Appendix

In this Appendix we present three supplementary results.

Supplementary Result 1. *Let* $\Gamma_{L2} \subset \Gamma$ *be the subclass of games such that each* (N, V) *in* Γ_{L2} *satisfies C2 (i.e., non-levelness) and for each* $(N, V) \in \Gamma_{L2}$, $|N| = 2$. *Then there is a unique solution on* Γ_{L2} *that satisfies PO, IREC, SSC and AM and it is the core. Further, these four axioms are independent on* Γ_{L2} *: i.e., for each of these axioms there exists a solution which, on* Γ, *violates this axiom but satisfies the other three.*

PROOF. Take any 2-player game $(N, V) \in \Gamma_{L2}$. First, by by Lemma 1.2, $\sigma(N, V^x) \subseteq C(N, V^x)$.

Take $x \in C(N, V)$. Construct $(N, V^x) \in \Gamma_{L2}$ for which $V^x(N) = V(N)$ and for every $i \in N$, $V^x(\{i\}) = \{y \in \mathbb{R} | y \leq x_i\}$.

Then, by IREC, $\sigma(N, V^x) \neq \emptyset$. Note that $C(N, V^x) = \{x\}$. To see this consider some $y \neq x$ such that $y \in C(N, V^x)$. Then $y_i \geq x_i$ for each $i \in N$ and $y_i > x_i$ for at least one $i \in N$. But then, since (N, V^x) satisfies C2 (non-levelness), x is in the interior of $V(N)$ which leads to a contradiction. Since, by Lemma 1.2, $\sigma(N, V^x) \subseteq C(N, V^x)$, $\sigma(N, V^x) = C(N, V^x) = \{x\}$. Then, by AM $x \in \sigma(N, V)$: i.e., $\sigma(N, V) = C(N, V)$.

Next we demonstrate the independence of AM from the other three axioms. Recall that $N = \{1, 2\}$. Consider the subset of games $\Gamma' \subset \Gamma_{L2}$ such that for each $(N, V) \in \Gamma'$,

$V(N) = \{x \in \mathbb{R}^N | \sum_{i \in N} x_i \leq 2\}$;

$b_1 = 1$; $0 \leq b_2 \leq 1$.

Let a solution $\sigma(.)$ be as follows. For any $(N, V) \in \Gamma'$, $\sigma(N, V) = \{x \in C(N, V)$ such that $x_2 = b_2\}$. For any other $(N, V) \in \Gamma_{L2}$, $\sigma(N, V)$ is the core.

Then it is easy to see that $\sigma(.)$ violates AM but satisfies the other three axioms.

For each of the other three axioms, the example-solution we used to demonstrate its independence in Step 3 of the proof of Theorem 2 works for this result too. ∎

Next we obtain a variant of Theorem 2 by replacing the axiom WC by a modified continuity-like axiom (and retaining the other four axioms). The new axiom is given below.

6′. **Modified weak continuity** (MWC):
Take $(N, V) \in \Gamma$ and let $x \in V(N)$. Let $\{(N, V^k)\}$ be a sequence of games belonging to Γ such that $\forall k$, $V^k(S) = V(S)$ for $S \subset N$ and $V(N) \supseteq V^{k+1}(N) \supseteq V^k(N)$. Let $\{x^k\}$ be a sequence such that $x^k \in \sigma(N, V^k)$ for all k and x^k converges to x. Then $x \in \sigma(N, V)$.

Then the analogue of Theorem 2 is as below.

Supplementary Result 2. *There is a unique solution on* Γ *that satisfies PO, IREC, SSC, MWC and AM and it is the core. Further, these five axioms are independent on* Γ *: i.e., for each of these axioms there exists a solution which, on* Γ*, violates this axiom but satisfies the other four.*

Proof. To demonstrate that the core satisfies MWC on Γ, consider a sequence of games belonging to Γ such that $\forall k$, $V^k(S) = V(S)$ for $S \subset N$ and $V(N) \supset V^{k+1}(N) \supset V^k(N)$. Let $\{x^k\}$ be a sequence such that $x^k \in \sigma(N, V^k)$ for all k and x^k converges to x. Suppose, if possible, $x \in V(N) \setminus C(N, V)$. Then there exists a real vector y such that for some $S \subseteq N$, $y \succ_S x$. But, since, for each k, $V^k(S) = V(S)$ for $S \subset N$ and $V(N) \subseteq V^k(N)$, there is a positive integer $\bar{k}$ such that for each $k \geq \bar{k}$, $y \succ_S x^k$ (since the sequence $\{x^k\}$ converges to x). This leads to a contradiction to the supposition that for each k, $x^k \in C(N, V^k)$.

The proof of rest of this result, including the demonstration of independence of the axioms, is exactly similar to that for Theorem 2. ■

Finally, we demonstrate that the minimality in the statement of Theorem 3 is non-trivial.

Supplementary Result 3. *There exists a solution $\sigma(.)$ which satisfy PO, IREC, WSC, WC, WISPC and AM on Γ such that for every $(N, V) \in \Gamma$, $C(N, V) \subseteq \sigma(N, V)$ and for some $(N, V) \in \Gamma$, $C(N, V)$ is a proper subset of $\sigma(N, V)$.*

Proof. Consider the following solution $\sigma(.)$ on Γ :

If (N, V) is a TU game then $\sigma(N, V) = \{x \in X(N, V)|$ for no $S \subset N$ is it the case that $(v(S)/|S|) > x_i$ for each $i \in S\}$ (where, recall that $v(S)$ is as in Definition 3 above); and

$\sigma(N, V) = C(N, V)$ otherwise.

It is straightforward to see that $\sigma(.)$ satisfies PO, WSC, WC, AM and WISPC.

To show that $\sigma(.)$ satisfies IREC we proceed as follows. Naturally, it suffices to confine attention to $\Gamma_{TU} \subset \Gamma$, the sub-class of TU games. First it is straightforward that if $|N| = 2$ then $\sigma(.)$ satisfies IREC. Now suppose that for some game (N, V), with $|N| > 2$, IREC is violated. If for some proper coalition $S \subset N$ such that $|S| > 1$, $\sigma(S, V_S) \neq \emptyset$, then IREC is (vacuously) satisfied. Therefore, for every proper coalition $S \subset N$ such that

$|S| > 1$, $\sigma(S, V_S) = \emptyset$. This implies that if $|S| = 2$, then for some $i \in S$, $b_i > (v(S)/|S|)$. Suppose such an inequality holds if $|N| - 1 > |S| = k \geq 2$: i.e., for every S with $|S| = k$, for some $i \in S$, $b_i > (v(S)/|S|)$. But suppose that for some proper coalition $S \subset N$ such that $|S| = k + 1$, $(v(S)/|S|) \geq b_i$ for every $i \in S$. But then $(v(S)/|S|) \geq (v(T)/|T|)$ for every proper subcoalition T of S. But this implies that $\sigma(S, V_S) \neq \emptyset$ which leads to a contradiction. Now take $x \in X(N, V) \cap I(N, V)$. Then it must be that $x_S \geq (v(S)/|S|)$ for every proper coalition S of N. This is because, otherwise, for some $i \in N$, $b_i > (v(S)/|S|) > x_i$ which leads to a contradiction to the supposition that $x \in I(N, V)$. Then $x \in \sigma(N, V)$. ■

Declaration of generative AI and AI-assisted technologies in the writing process

I have not used any generative AI and AI-assisted technologies at all.

Data availability

No data was used for research described in this article.

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