CORRIGENDUM TO "A ROCKAFELLAR-TYPE THEOREM FOR NON-TRADITIONAL COSTS"

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ABSTRACT. This note describes corrections to an error in the published version of the paper "A Rockafellar-type theorem for non-traditional costs" regarding the solvability of an uncountable family of inequalities. In this note, we describe the mathematical error and show that one must add an extra assumption - either countability of the family or an assumption on the coefficients not allowing the existence of what we call an infinite "black hole".

1. The mistake in the published version, and the corrected statements

In Theorem 3.2 from the paper [1] we used Zorn's lemma to show that there always exists a solution to the system of inequalities

$$a_{i,j} \le x_i - x_j,\ i,j \in I$$

where I is some index set, $a_{i,j} \in [-\infty, \infty)$ satisfy for all $i_1, \dots, i_m$ that

$$a_{i_1,i_m} \ge \sum_{k=1}^{m-1} a_{i_k,i_{k+1}}. \tag{1}$$

Within the proof a mistake was made, when we assumed that, fixing $J_0 \subset I$ and some solution f of the system for $i,j \in J_0$, we have for some $i_0 \notin J_0$

$$\sup_{j\in J_0}(a_{i_0,j} + f(j)) < \infty \quad\text{and}\quad \inf_{j\in J_0}(f(j) - a_{j,i_0}) > -\infty.$$

This was needed to *extend* a solution found on the index set $J_0 \subset I$ to some $i_0 \in I \setminus J_0$ (and conclude that we have a solution on the whole index set I). We showed that this supremum is bounded from above by $f(j) - a_{j,i_0}$ for any $j \in J_0$, however, if for all $j \in J_0$ and all $i_0 \notin J_0$ we have $a_{j,i_0} = -\infty$, it could still be the case that the supremum is infinite.

In the same paper we prove Theorem 3.2 by other means in the case where I is countable. That proof is valid (and does not rely on the extendability of a solution). For the uncountable case, not only is the proof using Zorn's lemma not valid, but we can find a counterexample for the statement of [1, Theorem 3.2]. We present it in Section 2. Nevertheless, if we add an extra condition on the system that guarantees existence of some $j \in J_0$ with $a_{j,i_0} \neq -\infty$, the proof from [1] carries over. The corrected version of [1, Theorem 2] is as follows

Theorem 2 (Corrected)**.** *Let $\{\alpha_{i,j}\}_{i,j\in I} \in [-\infty, \infty)$, where I is some arbitrary index set, and with $\alpha_{i,i} = 0$. The system of inequalities*

$$\alpha_{i,j} \le x_i - x_j, \quad i,j \in I \tag{2}$$

has a solution if (a) for any $i, j \in I$ there exists some constant $M(i,j)$ such that for any m and any $i_2, \cdots, i_{m-1}$, letting $i = i_1$ and $j = i_m$ one has that $\sum_{k=1}^{m-1} \alpha_{i_k,i_{k+1}} \leq M(i,j)$, and (b) either I is at most countable, or, if I is uncountable then for every infinite subset $J \subset I$ there exist some $j \in J, i \notin J$ with $\alpha_{j,i} > -\infty$.

For the corrected version of [1, Theorem 1] we need the following.

Definition 1.1. *Let $c : X \times Y \to (-\infty, \infty]$ be a cost function. We say that a set $G \subset X \times Y$ does not have an infinite black hole if for every infinite subset $G_0 \subset G$ there exists $y \in P_Y G_0$ and $z \in P_X(G \setminus G_0)$ such that $c(z, y) < \infty$.*

Theorem 1 (Corrected)**.** *Let X, Y be two arbitrary sets and $c : X \times Y \to (-\infty, \infty]$ an arbitrary cost function. Assume that $G \subset X \times Y$ is a c-path-bounded subset that is countable, or if it is uncountable, it does not have an infinite black hole. Then there exists a c-class function $\varphi : X \to [-\infty, \infty]$ such that $G \subset \partial^c \varphi$.*

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2. An uncountable Counterexample

In this section, we present a counterexample to the statement of [1, Theorem 3.2]. To this end, we choose the uncountable index set I to be $I = \mathbb{N} \cup S$ where S denotes the subset of all non-decreasing real-valued sequences, $S \subset \mathbb{R}^{\mathbb{N}}$. Next, for $i, j \in \mathbb{N}$ we define $a_{i,j} = i - j$ when $i \leq j$ and $-\infty$ when $i > j$. For $s \neq t \in S$ we let $a_{s,t} = -\infty$ (and, as usual, $a_{s,s} = 0$). For $s \in S$ and $j \in \mathbb{N}$ we let $a_{j,s} = -\infty$ and $a_{s,j} = s(j)$, the j^{th} element in the sequence s.

Let us check that the system satisfies the condition (1).

The system of inequalities restricted to $\mathbb{N}$ becomes $i - j \leq x_i - x_j$ for $i \leq j$, that is, $x_i - i$ should be a non-increasing sequence. In particular, condition (1) is satisfied when all indices belong to $\mathbb{N}$, since the restricted system admits a solution $x_i = i$.

When at least one of the indices appearing in (1) belongs to S, then we need to distinguish two cases: $i_1 \in S$ or $i_k \in S$ with $k \geq 2$. Let us start with the latter, since then it is easy to see that then at least one of the $a_{i_k,i_{k+1}}$ on the right-hand side is $-\infty$ and (1) holds. Thus, we only need to be concerned with the case where $i_1 \in S$ and $i_2, \dots, i_m \in \mathbb{N}$. If for some index we have $i_k > i_{k+1}$ then again the right-hand side is $-\infty$ and (1) holds. The remaining case is where $i_1 \in S$ and all other elements are in $\mathbb{N}$ and satisfy $i_k < i_{k+1}$ (if they are equal, we can omit one of them as $a_{i,i} = 0$). Then, the condition amounts to

$$s(i_m) = a_{s,i_m} \geq a_{s,i_2} + \sum_{k=2}^{m-1} a_{i_k,i_{k+1}} = s(i_2) + \sum_{k=2}^{m-1} (i_k - i_{k+1}) = s(i_2) + i_2 - i_m.$$

We see that this inequality is satisfied, precisely if

$$s(i_m) + i_m \geq s(i_2) + i_2.$$

Recalling that we consider the case when $i_m > i_2$, the above condition is precisely that the sequence $s(j) + j$ is non-decreasing. In particular, if $s \in S$ is a non-decreasing sequence, this condition is satisfied.

Now, assume towards a contradiction that our system does have a solution. In particular, this solution restricted to $\mathbb{N}$, denoted, say, by $(x_j)_{j \in \mathbb{N}}$, must satisfy $x_j = j + b_j$ with b_j non-increasing. Assigning a value $x(s)$ for any sequence in S, it must satisfy

$$s(j) = a_{s,j} \leq x(s) - x_j = x(s) - j - b_j.$$

This means that

$$\sup_j \big(s(j) + j + b_j\big) < \infty$$

for any $s \in S$.

However, one of the sequences $s \in S$ which we can consider is the sequence $s(j) = -b_j$, which is non-decreasing. For this specific sequence, the condition above reads

$$\sup_j j < \infty,$$

which is clearly false. This implies that we do not have a solution $x : \mathbb{N} \cup S \to \mathbb{R}$ for the original system, although it does satisfy the condition (1).

3. The corrected statement of [1, Theorem 3.2], and its proof

As mentioned above, in the case of a countable index set we gave an alternative proof (see Appendix A in [1]) that does not rely on extendability of a solution of a subsystem. We have proved the following theorem.

Theorem 3.1. *Let $\{a_{i,j}\}_{i,j \in I} \in [-\infty, \infty)$, where I is a countable index set. Assume that for any $m \geq 1$ and any $i_1, i_2, \cdots, i_m$ it holds that $a_{i_1,i_m} \geq \sum_{k=1}^{m-1} a_{i_k,i_{k+1}}$. Then the system of inequalities*

$$a_{i,j} \leq x_i - x_j, \quad i, j \in I$$

has a solution.

In the case of an uncountable index set, the counterexample shows that we need to add some additional condition on the set $\{a_{i,j}\}$ to guarantee the existence of a solution. As we mentioned above, the issue arises when we attempt to extend the maximal solution f_{J_0} indexed by J_0 given by Zorn's lemma. For the (real-valued) extension to exist we need $\sup_{j \in I_0}(a_{i_0,j} + f(j))$ to not be $+\infty$ and $\inf_{j \in I_0}(f(j) - a_{j,i_0})$ to not be $-\infty$ (the latter is always more than the former, but they might both be $+\infty$ or $-\infty$ if no extra assumption is made).

Note that in general, we cannot expect to be able to extend a solution. It may be that, after adding the additional index i_* and all the $a_{i_*,j}, a_{j,i_*}$ for $j \in J_0$ one needs an entirely different solution. Nevertheless, we pursue the path of extending the given solution and show that the proof of solvability of a system of linear inequalities can be made valid under additional assumptions on $a_{i,j}$'s that guarantee that the supremum is not $\pm\infty$. Therefore, the condition we add is sufficient not only for the existence of a solution but also for the extendability of given sub-solution.

Definition 3.2. *Consider a collection of numbers $\{a_{i,j}\}_{i,j\in I} \in [-\infty,\infty)$, where I is an index set. We say that the collection $\{a_{i,j}\}_{i,j\in I}$ has a* black hole *in the index set $J_0 \subset I$ if for all $j \in J_0$ and all $i \in I \setminus J_0$ we have that $a_{j,i} = -\infty$. We say that it has a black hole of infinite cardinality if such J_0 exists and is of infinite cardinality.*

Remark 3.3. By definition, if $J_\alpha \subset I$ are black holes for the system $\{a_{i,j}\}_{i,j\in I} \in [-\infty,\infty)$ for any $\alpha \in A$ then so is $J = \cup_{\alpha\in A} J_\alpha$. This means that one can take a maximal black hole $J \subseteq I$ by taking the union over all black holes, and this J includes, as a subset, any black hole of any cardinality.

We see that if the collection $\{a_{i,j}\}_{i,j\in I}$ does not have a black hole then for every $J \subset I$ there exists $i \notin J_0$ and $j \in J_0$ such that $a_{j,i_0} \neq -\infty$. In fact, black holes of finite cardinality are not of worry to us as we shall readily see.

Theorem 3.4. *Let I be an uncountable index set, and let $\{a_{i,j}\}_{i,j\in I} \in [-\infty,\infty)$ be a collection that does not have a black hole of infinite cardinality. Assume that for any $m \geq 1$ and any $i_1, i_2, \cdots, i_m$ it holds that $a_{i_1,i_m} \geq \sum_{k=1}^{m-1} a_{i_k,i_{k+1}}$. Then the system of inequalities*

$$a_{i,j} \leq x_i - x_j, \quad i,j \in I$$

has a solution.

Proof of Theorem 3.4. We start by letting J_1 be the union of all black holes in I. By Remark 3.3, the set J_1 is a black hole and by the *added* assumption J_1 is finite. We take any countably infinite set $J_2 \subseteq I$ which includes it. Then the system of inequalities indexed by J_2 has a solution due to Theorem 3.1. Denote this solution by f_2.

We shall now use Zorn's Lemma. Consider the partially ordered set of pairs (J, f_J) where $J_2 \subset J \subset I$ and $f_J : J \to \mathbb{R}$ satisfies $f_J|_{J_2} = f_2$, and such that for any $i,j \in J$ we have $f_J(i) - f_J(j) \geq a_{i,j}$. We know the set is non-empty because it contains the pair (J_2, f_2). The partial order we consider is $(J, f_J) \leq (K, f_K)$ if $J \subset K$ and $f_K|_J = f_J$.

First, let us notice that every chain has an upper bound. Assume $(J_\alpha, f_{J_\alpha})_{\alpha\in A}$ is a chain (namely any two elements are comparable). Consider $J = \cup_\alpha J_\alpha$ and $f_J = \cup_\alpha f_{J_\alpha}$. This function is well defined because of the chain properties (at a point $i \in J$ it is defined as $f_{J_\alpha}(i)$ for any α with $i \in J_\alpha$). The pair $(J, f|_J)$ is in our set because if $i,j \in J$ then for some α we have $i,j \in J_\alpha$, so $f|_{J_\alpha}$ satisfies the inequality on $f_J(i) - f_J(j) \geq a_{i,j}$ and so does f_J. Finally, (J, f_J) is clearly an upper bound for the chain. So, we have shown that every chain has an upper bound, and we may use Zorn's lemma to find a maximal element. Denote the maximal element by (J_0, f_{J_0}).

Assume towards a contradiction that $J_0 \neq I$. Note that the non-empty set $I \setminus J_0$ has no black holes since we assumed that $J_2 \subset J_0$ contains all the black holes in I. Therefore, there is some $i_0 \in I \setminus J_0$ and some $j_{i_0} \in J_0$ with $a_{i_0,j_{i_0}} \neq -\infty$.

If we are able to extend f_{J_0} to be defined on $\{i_0\}$ in such a way that all inequalities with indices of the form (i_0, j) and (j, i_0) with $j \in J_0$ still hold, we will contradict maximality and complete the proof.

First, recall that under our assumptions $a_{k,j} \geq a_{k,i_0} + a_{i_0,j}$. Moreover, since f_{J_0} already satisfies the inequality $a_{k,j} \leq f_{J_0}(k) - f_{J_0}(j)$ for all $j, k \in J_0$, we get that

$$a_{k,i_0} + a_{i_0,j} \leq a_{k,j} \leq f_{J_0}(k) - f_{J_0}(j)$$

holds for all $j, k \in J_0$. In particular, this gives us

$$a_{i_0,j} + f_{J_0}(j) \leq f_{J_0}(k) - a_{k,i_0}$$

for any $j, k \in J_0$. This means that f_{J_0} must satisfy that

$$\sup_{j \in J_0} \left(a_{i_0,j} + f_{J_0}(j)\right) \leq \inf_{j \in J_0} \left(f_{J_0}(j) - a_{j,i_0}\right). \tag{3}$$

In particular, since we chose i_0 so that there exists $j_{i_0} \in J_0$ with $a_{i_0,j_{i_0}} \neq -\infty$ we know that the supremum is not $-\infty$, and therefore, the infimum is not $-\infty$. We will now show that $\inf_{j \in J_0} \left(f_{J_0}(j) - a_{j,i_0}\right)$ is not $+\infty$, from which we will conclude that both the infimum and supremum are finite.

To this end, we will show that for all $i \in I \setminus J_0$ there is some $j \in J_0$ such that $a_{j,i} \neq -\infty$. Let $J_3 \subseteq I \setminus J_0$ denote all those $i \in I \setminus J_0$ for which there is some $j_i \in J_0$ with $a_{j_i,i} \neq -\infty$. We claim that $J_3 = I \setminus J_0$. Towards a contradiction, assume that $I \setminus (J_0 \cup J_3) \neq \emptyset$. Then, as $J_0 \cup J_3$ is not a black hole (since it has infinite cardinality), there is some $k \in J_0 \cup J_3$ such that $a_{k,l} \neq -\infty$ for some $l \in I \setminus (J_0 \cup J_3)$. The fact that $l \notin J_3$ means $k \in J_3$ (and not in J_0). However, since $k \in J_3$ there is some $j_k \in J_0$ with $a_{j_k,k} \neq -\infty$. Together with our assumption that $a_{i_1,i_3} \geq a_{i_1,i_2} + a_{i_2,i_3}$ for any indexes $i_1, i_2, i_3 \in I$, this means

$$a_{j_k,l} \geq a_{j_k,k} + a_{k,l} > -\infty$$

in contradiction to the fact that $l \notin J_3$. Hence, as claimed, for all $i \in I \setminus J_0$ there is some $j \in J_0$ such that $a_{j,i} \neq -\infty$. In particular, this is true for $i = i_0$ which we chose before.

We conclude that both sides of the inequality (3) are finite, and hence we may take $f(i_0) \in \mathbb{R}$ such that

$$\sup_{j \in J_0} \left(a_{i_0,j} + f_{J_0}(j)\right) \leq f(i_0) \leq \inf_{j \in J_0} \left(f_{J_0}(j) - a_{j,i_0}\right).$$

This means that we may extend the function f_{J_0} to i_0, which is a contradiction to maximality, and we conclude that $J_0 = I$. This finished the proof, as we have found a solution to the full system of inequalities. □

Remark 3.5. Note that the additional condition of ‘not having an infinite black hole’ is not a necessary one. Indeed, one can come up with a system where, for example, all the $a_{i,j}$ are equal $-\infty$, and this system admits a solution, e.g. we can take a constant solution.

4. Completing the proof for the corrected theorems 1 and 2

In [1] we used Theorem 3.2 to prove Theorem 2, which is again a statement on the solvability of a system of linear inequalities. We then used the latter to prove Theorem 1 regarding the existence of a potential for a given set G. In this section, we provide the corrected version for these two theorems.

First, we examine the relation of the collection of $a_{i,j}$'s from [1, Theorem 3.2] and $\alpha_{i,j}$'s [1, Theorem 2], where we defined

$$a_{i,j} = \sup\{\sum_{k=1}^{m-1} \alpha_{i_k,i_{k+1}} : m \in \mathbb{N}, m \geq 2,\, i_2, \ldots, i_{m-1} \in I\}.$$

We see that the collection $\{a_{i,j}\}_{i,j\in I}$ does not have an infinite black hole, if for every $J_0 \subset I$ of infinite cardinality, there exists a "path" of finite valued coefficients. More precisely, we need that for every $J_0 \subset I$ of infinite cardinality, there exists $i_0 \notin J_0$, $j \in J_0$ and $m \in \mathbb{N}$ indexes $i_1, \ldots, i_m \in I$ such that $a_{j,i_1}, a_{i_1,i_2}, \ldots, a_{i_m,i_0} \neq -\infty$.

This in turn means the same as the system $\{\alpha_{i,j}\}$ not having a black hole. Indeed, the existence of such a path implies there is some first index $i_k \in J_0$, so that $i_{k+1} \notin J_0$ and we get a finite $\alpha_{i_k,i_{k+1}}$ escaping J_0. Therefore, the proof of the implication in the paper remains valid. The reasoning in [1] thus carries through and completes the proof of Theorem 2.

Next we trace back the correspondence of the above result to [1, Theorem 1], in which we seek a condition on a set $G \subset X \times X$ to lie on a c-subgradient of a c-class function. As explained in [1], we consider $I = G \subset X \times X$, and for any two elements $(x, y), (z, w) \in G$, we define $\alpha_{(x,y),(z,w)} := c(x, y) - c(z, y)$. We observe that the condition of having no black holes can be rewritten in the following way: For any infinite subset $G_0 \subset G$ there exists $(x, y) \in G_0$ and $(z, w) \in G \setminus G_0$ such that

$$c(x, y) - c(z, y) \neq -\infty.$$

Since the cost c takes values in $(-\infty, \infty]$ the above condition is equivalent to $c(z, y) < \infty$. This is precisely Definition 1.1.

With this definition, joined with Theorem 2 and [1, Theorem 3.1], we have proved the above corrected version of Theorem 1.

Remark 4.1. In [1], we presented two results for special cases of a set G and continuous cost functions, Corollary 4.1 and Proposition 4.2. It is easy to check that in these special instances the sets do not have black holes.

References

[1] Shiri Artstein-Avidan, Shay Sadovsky, and Katarzyna Wyczesany. A Rockafellar-type theorem for non-traditional costs. *Advances in Mathematics*, 395, 2022