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Abstraction and Semantic Presuppositions

Abstract

According to the neo-Fregean abstractionism, numerical expressions of the form 'the number of Fs', introduced by Hume's Principle, should be read as purportedly referential singular terms. I will explore the prospects of a version of abstractionism in which such expressions have presuppositional content, as in Strawson's account. I will argue that the thesis that 'the number of Fs' semantically presupposes the existence of a number is inconsistent with the required 'modest' stipulative character of the truth of Hume's Principle: since Hume's Principle is true and provably presupposes that numbers exist, what it presupposes is also be true; and so, numbers exist. This, however, means that numbers are conjured into existence by a direct stipulation.

1 *Opening*

Abstractionism in the philosophy of mathematics is the thesis that Frege-style abstraction principles underwrite our knowledge of mathematical truths, of the existence of mathematical objects, and our capacity to have singular thoughts about these objects. Abstraction principles are sentences of the following form:

(AP) $$\S\alpha = \S\beta \leftrightarrow \alpha \sim \beta$$

where α and β are variables of some type, § is a term-forming operator that applies to such variables, and $\sim$ stands for an equivalence relation on the kinds of items over which the variables range. Any admissible instance of AP implicitly defines § in accordance with the rule that the abstract of α is identical to the abstract of β if and only if α and β are related by $\sim$. An important instance of an abstraction principle is known as Hume's Principle, which defines the cardinality operator # by specifying that the cardinal number F is identical to the cardinal number of G precisely when there is a one-to-one correspondence between F and G:

(Hume's Principle) $\#F = \#G \leftrightarrow F \approx G$

where '$F \approx G$' is the second-order formalization of the claim that there is a relation that one-to-one correlates the Fs and the Gs.

Most abstractionists, notably Hale and Wright (2001), Heck (2011), and Linnebo (2018), have followed Frege who both in his *Die Grundlagen* and *Grundgesetze*, takes the category of singular terms to include both semantically simple expressions, such as proper names and demonstrative pronouns, and semantically complex expressions including functional terms and definite descriptions. Since the publication of Russell's (1905) 'On Denoting', however, there have been serious doubts as to whether the class of singular terms is as inclusive as Frege had proposed. According to the Russellian analysis, part of what is asserted when we assert 'The present King of France is bald' is that there is one and only one King of France. In the context of abstractionism, MacFarlane (2009: §1) asks whether it is essential for the abstractionist programme to regard abstract expressions of the form 'the number of Fs' as singular terms, as opposed to the Russellian quantified expressions. In this paper, I will articulate a similar question for the abstractionists: what would be lost, if anything, if such numerical expressions were treated along the presuppositional view à la Strawson's 'On Referring' (1950)? According to the Strawsonian semantics, that there is one and only one King of France is not part of what is asserted by 'The present King of France is bald'; it is, rather, a presupposition of that assertion.[1]

I will argue that the presuppositional treatment of abstract expressions is incompatible with a crucial condition that admissible abstraction principles must respect. According to this condition, often called *non-arrogance* or *modesty*, an abstraction principle must

[1]According to Russell (1905), a definite description is any expression of the form 'the so-and-so'. In this syntactic sense, 'the number of Fs' is a definite description. Hale and Wright, however, take 'the number of Fs' as a functional term, and argue (2009b: 461–2) against the Russellian analysis of functional terms in terms of definite descriptions. Their central claim is that the Russellian analysis flies in the face of our ability to engage in singular, object-involving, thoughts about numbers. This, however, would not convince the Russellian: in her view, precisely because we do not have this capacity, numerical expressions introduced by Hume's Principle cannot qualify as genuine singular terms.

do no more than define the expression it purports to define. It must not, in particular, tell us that the expression refers. Its only function is to 'modestly' stipulate what it takes for a statement of identity, occurring on its left-hand side, to be true. In this paper, I will be concerned with a particular abstraction principle, namely Hume's Principle, but it will be clear that my conclusion holds of all admissible abstraction principles. Thus, the generality of the paper's title is justified.

I thereby explore the prospects for a version of abstractionism which employs the resources of a presuppositional language. While I put forward an analysis of why the modesty constraint must be given up if numerical expressions are interpreted in the presuppositional framework, I do not offer any argument for the claim that they should be so interpreted. Thus, the incompatibility of the presuppositional view with modesty is no threat to those abstractionists whose interpretation of numerical expressions does not leave room for the presuppositional view. (Hale (1996), for example, gives extensive Fregean arguments for treating numerical expressions as singular terms.)

Nevertheless, there are strong motivations for adopting a presuppositional construal of definite descriptions. Semantic, pragmatic, and inferential considerations have led many philosophers and linguists to conclude with Strawson that definite descriptions trigger presuppositions. (See Soames (1989) for a review of the linguistic evidence for presuppositions.) Furthermore, to the extent to which it is a desideratum of an abstractionist philosophy of mathematics to receive support from natural language, the presuppositional treatment of numerical expressions cannot be neglected. It is intriguing to discover that despite plenty of syntactic, inferential, and semantic considerations in favour of the presuppositional view, the literature contains no discussion of this issue.

2 *Arrogance and existence*

What is it for an abstraction principle to be arrogant? It does not seem that there is a uniform notion in play here. In Hale and Wright's terminology, an arrogant stipulation 'carries', 'asserts', 'calls for', 'requires', and 'presupposes' the existence of a range of objects.[2] A common example in the literature is the definition of 'Jack the Ripper' through the following stipulation:

(J) Jack the Ripper is the perpetrator of these killings.

As Hale and Wright point out, we cannot, just by laying down (J), be warranted that a unique murderer was responsible for the 1890s murders in London. The success of (J) depends on the truth of the assumption that there was indeed a unique perpetrator of the murders. However, to ascertain whether there was a unique perpetrator of the Whitechapel murders, additional epistemic work is required. In this sense, it would be 'presumptuous' to lay down (J) as the definition of 'Jack the Ripper'. In Hale and Wright's view, what can safely be stipulated is the following conditional:

(CJ) If there exists a unique perpetrator of the killings, it is Jack the Ripper.

This is a properly modest stipulation: it does not do anything more than introduce 'Jack the Ripper'.

For a mathematical example, Hale and Wright focus on the implicit definition of *natural number* that would be given by the stipulation of the Dedekind-Peano axioms. In their view, these axioms arrogantly stipulate the existence of the natural numbers. Hume's Principle, by contrast, is only the stipulation of the truth of a biconditional, doing nothing more than stipulating that the truth-conditions of identities involving numerical terms are the same as those of the corresponding statements of one-to-one correspondence. Hume's Principle, together with the conceptual resources of second-order logic, entails existential claims, but that does not mean that Hume's Principle, by itself, entails those claims. The existence of numbers follows from the stipulation together with the truth of the right-hand side.[3]

[2]See, for example, Hale and Wright (2000: 127, 145–7, 2003: 261–3, 2009b: §§2–4) and Wright (2016: 173).

[3]See Hale (2001: 343), Hale and Wright (2000: 146–50, 2003: 262, 2008: 16–19, and 2009a: 209).

The proof of the existence of numbers thus rests on the very manner in which the truth-conditions of self-identities featuring numerical expressions are tied to the reflexivity of the relevant equivalence relation:

(1) $\#A = \#A \leftrightarrow A \approx A$

(2) $A \approx A$

(3) $\#A = \#A$

(4) $\exists x(x = \#A)$

(1) is an instance of Hume's Principle; (2) is a second-order logical truth; (3) follows, in propositional logic, from (1) and (2); and (4) follows from (3) by Existential Generalization. The background logic must be a version of free logic. Since in non-free classical logic, every term is non-empty, (4) would directly follow from (3) without the use of (1) and (2), and so, Hume's Principle would be left totally idle in the proof of the existence of numbers. Which free logic is needed though? In positive free logic, (3) can be true even if the ingredient terms are empty. Thus, (4) would not follow from (3): in order to apply Existential Generalization in (3), one must already have (4) as an additional premise. In negative free logic, however, this issue does not arise: (4) follows from (3), since in this version of free logic, an atomic formula can be true only if all of the component terms refer.[4]

The argument (1)-(4) manifests the modesty of Hume's Principle, in that the existence of numbers could be inferred from Hume's Principle, only given the truth of the instances of its right-hand side. That is the crucial role played by the additional premise (2), which, by itself, is independent from the stipulation of the truth of Hume's Principle. However, as we will see in the next section, the non-arrogance constraint fails in the presuppositional

[4]For more on this proof, see Hale and Wright (2000: 146 n.48, 2001: 309–310) and MacFarlane (2009: 448–449). The use of (negative) free logic has been endorsed in Hale and Wright (2003: 260, 2008: §5) and Linnebo (2018: 48–9). In the next section, I will assume that the background logic of the presuppositional reading of Hume's Principle is also negative free logic. Since in non-free classical logic, every singular term has a referent, the presupposition requirement is trivially satisfied.

setting: we won't need (2) anymore. The existence of numbers will be a direct consequence of Hume's Principle.

3 *The presuppositions of Hume's Principle*

How does Hume's Principle presuppose the existence of numbers? (The presupposition of uniqueness is not the point of our focus.) Following Strawson (1950), when we say that a sentence ϕ presupposes the sentence ψ, we mean that a particular semantic relation obtains between ϕ and ψ. For example, 'Alma drinks, too' presupposes that someone besides Alma drinks: it would be absurd to assert 'Alma drinks, too' and yet deny that anyone besides Alma drinks. 'Alma drinks, too' seems to suggest or imply that someone besides Alma drinks without asserting it. Expressions like 'too' giving rise to presupposition are known as *presupposition triggers*, and an important feature of this class of expressions is that any presupposition that they trigger survives embedding under negation, conjunction, conditionals, and other operators. For instance:

(M) My neighbour is reading Hegel.

presupposes that I have a neighbour; and so does its negation. The phenomenon whereby a presupposition survives an operation is called *presupposition projection*. The projection rules that linguists have formulated for determining the presuppositions of complex sentences on the basis of the presuppositions of their parts are largely based on an empirical basis and a *pragmatic* approach to presuppositions.[5] However, the projection rules that I will use here concern the logico-semantic approach to presuppositions. As it will be clear, though, considerations concerning the pragmatic approach are compatible with our projection rules.

We would like to see whether Hume's Principle 'inherits' the existential presupposition of 'The number of Fs = the number of Gs'; so, we need to explore projection rules in

[5]For an overview of the projection problem, see Heim (1991). The seminal paper for the pragmatic approach to presupposition projection is Karttunen (1973).

conditional contexts. Let us first introduce the two-place connective $\rightsquigarrow$ standing for the relation of presupposition. For example, where 'K' stands for 'present King of France' and 'B' for 'bald', we can formalize Strawson's analysis of the presuppositions of 'The present King of France is bald' as

(F) The K is $B \rightsquigarrow \exists x \forall y(Ky \leftrightarrow x = y)$

This is merely a device for indicating that the right-hand side of $\rightsquigarrow$ is the part of the overall content of 'The present King of France is bald' which is presupposed. If we think of a theory of presupposition as giving a semantics for a fragment of our language, this notation is just a way of translating our base sentences into that fragment, and then the relevant interpretation rules will recursively specify how to interpret presuppositional sentences. According to the standard trivalent semantics, 'ϕ, presupposing that ψ' is true iff ϕ is true and ψ is true; it is false iff ϕ is false and ψ is true; and otherwise it is undefined. Thus, given (F), in a situation where there is no unique present King of France, 'The present King of France is bald' will be undefined; whereas in a situation where there is one king, it will be true or false if that individual is, or is not, bald.

It is well established in work on the projection of presuppositions that where ϕ is a sentence with a presupposition π, and ψ is a sentence with no presupposition, then $\phi \rightarrow \psi$ presupposes π:

(I) If $\phi \rightsquigarrow \pi$, then $(\phi \rightarrow \psi) \rightsquigarrow \pi$.

For example, suppose that 'Alma reads Hegel, too' presupposes 'Somebody else, other than Alma, reads Hegel'. Then in an utterance of

(S) If Alma reads Hegel too, I am baffled.

what is presupposed by the antecedent is presupposed by the whole conditional. However,

(P) If Paul reads Hegel, Alma reads Hegel, too.

does not inherit the presupposition of 'Alma reads Hegel, too'. For the antecedent of (P) already entails that somebody else, other than Alma, reads Hegel. The presupposition of 'Alma reads Hegel, too' has been 'cancelled' in (P); and, as is well known, one way to cancel a sentence's presuppositions is to embed it as the consequent of a conditional whose antecedent already entails its presuppositions.

In view of our discussion in §2, an interesting example is the case of stipulative definitions. The stipulation of 'Jack the Ripper' involves the presupposition that there exists a unique perpetrator of the relevant killings, and so, in any situation where there is no such perpetrator, (J) will be undefined. However, (CJ) does not presuppose the existence of the perpetrator. Since the antecedent of (CJ) entails the presupposition of (J), the relevant presupposition is cancelled. The semantic treatment of presupposition projection correctly anticipates the abstractionists' distinction between the arrogance of (J) and the modesty of (CJ).

Another dominant hypothesis on presupposition projection in conditional contexts is that if the antecedent ϕ is a sentence with no presupposition and the consequent ψ is a sentence with a presupposition π, then the presupposition of the conditional is the conditional presupposition that if ϕ, then π:

(II) If $\psi \rightsquigarrow \pi$, then $(\phi \rightarrow \psi) \rightsquigarrow (\phi \rightarrow \pi)$.

(See, for example, Rothschild (2008: §2) for a justification of this principle.) What about biconditionals? The trivalent semantics we are working with would predict that $\phi \leftrightarrow \psi$ presupposes whatever ϕ presupposes and whatever ψ presupposes. (This is also true of Schlenker's (2010) pragmatic local contexts.) Here is an outline of a general justification. The main idea is that the undefinedness of arguments that occurs 'later' projects only if the truth-value of the sentence cannot already be determined on the basis of the truth-value of the arguments that occur earlier. For example, $\phi \vee \psi$ is undefined if ϕ is undefined.

But if ψ is undefined, the disjunction is not necessarily undefined. In particular, if ϕ is true and ψ is undefined, then the sentence is true; not undefined. In the case of $\phi \leftrightarrow \psi$, the presupposition of ϕ will project, which is inevitable in this semantics. But since we can never determine the truth-value of the biconditional merely on the basis of the truth-value of ϕ, then the presupposition of ψ projects, too. Thus:

(III) $\phi \leftrightarrow \psi$ presupposes whatever ϕ and ψ presuppose.

In light of the above projection rules, let us now see how Hume's Principle presupposes the existence of numbers. My argument proceeds from the assumption that in '$\#F = \#G$', the terms flanking the identity sign trigger the presupposition that numbers exist. Then in accordance with our projection rules, it can be proved that the presupposition of '$\#F = \#G$' projects on Hume's Principle. But since Hume's Principle is true, what it presupposes is also true. Hence, numbers exist. There is thus no need, in establishing the existence of numbers, to ensure whether the condition given on the right-hand side of Hume's Principle is satisfied. However, precisely because the satisfaction of this condition is bypassed, the stipulation of Hume's Principle cannot be modest. Let us summarize the argument as follows:

(i) $(\#F = \#G \leftrightarrow F \approx G) \rightsquigarrow \exists x(x = \#F)$

(ii) $\#F = \#G \leftrightarrow F \approx G$

(C) $\exists x(x = \#F)$

(i) is the formulation of the claim that Hume's Principle presupposes that numbers exist. (ii) is Hume's Principle in which the expressions formed on its left-hand side trigger the presupposition of numerical existence; and the step from (ii) to (C) is based on an analogue of *modus ponens* in presuppositional environments: from $\phi \rightsquigarrow \pi$ and ϕ, infer π. This immediately follows from the above characterization of presupposition. So, the crucial claim is (i). How can we defend it?

Let us start with the assumption that '$\#F = \#G$' presupposes the existence of the number of Fs. The assumption is not that '$\#F = \#G$, presupposing that numbers exist' is true. For in that case, the truth of the presuppositional identity statement would require the satisfaction of its presupposition and also a one-to-one correspondence between F and G, as Hume's Principle demands. So, we cannot assume '$\#F = \#G$, presupposing that numbers exist' unless we antecedently make sure that numbers exist. What is assumed, rather, is the weaker claim that the numerical identity statement presupposes the existence of numbers – in the same way in which 'My neighbour is reading Hegel' presupposes that I have a neighbour.

The justification of this assumption lies in the Frege-Strawson's widely endorsed thesis that uses of singular terms such as 'The number of Fs' trigger the presupposition that they stand for some objects. A singular term is, after all, an expression whose function is to pick out the object that satisfies the presupposition. So, that '$\#F = \#G$' presupposes the existence of numbers is a general assumption, resting merely on the status of singular terms as expressions triggering existential presuppositions.

Furthermore, the assumption runs into no conflict with the requirements of free logic, endorsed by the abstractionists: one may hold that the numerical identity statement presupposes the existence of numbers without assuming that the numerical terms are referential. For example, in either foundation for free logic (positive or negative), when it is enriched with presuppositional resources, 'Vulcan is burning' will have no truth-value, because it presupposes the false statement that 'Vulcan exists'. Similarly, '$\#F = \#G$' presupposes '$\exists x(x = \#F)$', and where one or more of the terms is non-referential, the statement will have no truth-value.[6]

So, when we say that '$\#F = \#G$' presupposes the existence of numbers, we are neither stipulating that numbers exist, nor that '$\#F = \#G$, presupposing that numbers exist' is true. The existential presupposition of the numerical identity statement is, this sense,

[6] For an application of free logic to Strawson-style presuppositions, see van Fraassen (1969: 96–91).

compatible with abstractionism. The neo-Fregean abstractionists have always emphasized that the truth-value of the left-hand side of Hume's Principle is no part of the stipulation. What is stipulated to be true, rather, is that the two sides of Hume's Principle have the same truth-conditions. This will leave entirely open the question whether the numerical terms have reference or not.

Let us turn to the premise (i) of the main argument. First, note that Hume's Principle is a biconditional, consisting of the following two conditionals:

(HP1) $$\#F = \#G \rightarrow F \approx G$$

(HP2) $$F \approx G \rightarrow \#F = \#G$$

As above, assume that '$\#F = \#G$' presupposes that the number of Fs exists:

(iii) $(\#F = \#G) \rightsquigarrow \exists x(x = \#F)$

HP1 and (iii), by (I), entail:

(iv) $(\#F = \#G \rightarrow F \approx G) \rightsquigarrow \exists x(x = \#F)$

which states that HP1 presupposes the existence of the number of Fs. However, in order to establish (i), we need to show that Hume's Principle, in its both directions, presupposes that numbers exist. According to (II), HP2 has the following conditional presupposition:

(v) $(F \approx G \rightarrow \#F = \#G\ ) \rightsquigarrow (F \approx G \rightarrow \exists x(x = \#F))$.

However, this does not affect the presupposition of Hume's Principle as a whole. For according to (III), a biconditional presupposes whatever each of its constituent conditionals does. In particular, given (iii), we can infer:

(i) $(\#F = \#G \leftrightarrow F \approx G) \rightsquigarrow \exists x(x = \#F)$

which states that Hume's Principle presupposes that numbers exist.

The moral should be clear. The mere fact that Hume's Principle, in its presuppositional reading, is true suffices to entail the existence of numbers. There is thus no need to ensure whether the condition given on its right-hand side is satisfied: facts about one-one correspondence are bypassed in the proof of the existence of numbers. In this sense, Hume's Principle is arrogant: it is not merely the truth of a biconditional purporting to explain what it is for an object to be the cardinal number of a given concept; it also postulates that such objects exist.[7]

References

Hale, B. 1996. Singular Terms (1). In B. Hale and C. Wright, 2001, *The Reason's Proper Study: Essays towards a Neo-Fregean Philosophy of Mathematics*, 31–47. Oxford: Clarendon Press.

Hale, B. 2001. A response to Potter and Smiley: abstraction by recarving. *Proceedings of the Aristotelian Society* 101: 339–358.

Hale, B. and Wright, C. 2000. In B. Hale and C. Wright, 2001, *The Reason's Proper Study: Essays towards a Neo-Fregean Philosophy of Mathematics*, 117–150. Oxford: Clarendon Press.

Hale, B. and Wright, C. 2001. *The Reason's Proper Study: Essays towards a Neo-Fregean Philosophy of Mathematics*. Oxford: Clarendon Press.

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Hale, B. and Wright, C. 2003. Responses to commentators. *Philosophical Books* 44: 245–263.

Hale, B. and Wright, C. 2008. Abstraction and additional nature. *Philosophia Mathematica* 16: 182–208.

Hale, B. and Wright, C. 2009a. The metaontology of abstraction. In *Metametaphysics: New Essays on the Foundations of Ontology*, eds. D. Chalmers, D. Manley, and R. Wasserman, 178–212. Oxford: Oxford University Press.

Hale, B. and Wright, C. 2009b. Focus restored: comments on John MacFarlane. *Synthese* 170: 457–482.

Heck, R. K. 2011. The existence (and non-existence) of abstract objects. In Heck, *Frege's Theorem*, 200–226. Oxford: Clarendon Press.

Heim, I. 1991. On the projection problem for presuppositions. In *Pragmatics*, ed. S. Davis, 397–405. New York: Oxford University Press.

Karttunen, L. 1973. Presuppositions of compound sentences. *Linguistic Inquiry* 4: 169–93.

Linnebo, Ø. 2018. *Thin Objects: An Abstractionist Account*. Oxford: Oxford University Press.

MacFarlane, J. 2009. Double vision: two questions about the neo-Fregean program. *Synthese* 170: 443–456.

Rothschild, D. 2008. Presupposition projection and logical equivalence. *Philosophical Perspectives* 22: 473–497.

Russell, B. 1905. On denoting. *Mind* 14: 479–493.

Schlenker, P. 2010. Presuppositions and local contexts. *Mind* 119: 377–391.

Soames, S. 1989. Presuppositions. In *Handbook of Philosophical Logic,* eds. D. Gabbay and F. Guenther, 553–616. Dordrecht: Reidel.

Strawson, P. F. 1950. On referring. *Mind* 59: 320–44.

van Fraassen, B. 1969. Presuppositions, supervaluations, and free logic. In *The Logical Way of Doing Things*, ed. K. Lambert, 67–91. New Haven: Yale University Press.

Wright. C. 2016. Abstraction and epistemic entitlement: on the epistemological status of Hume's Principle. In *Abstractionism: Essays in Philosophy of Mathematics*, eds. P. Ebert and M. Rossberg, 161–85. Oxford: Oxford University Press.