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Higher Catoids, Higher Quantales and their Correspondences

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Abstract
We introduce ω-catoids as generalisations of (strict) ω-categories and in particular the higher path categories generated by computads or polygraphs in higher-dimensional rewriting. We also introduce ω-quantales that generalise the ω-Kleene algebras recently proposed for algebraic coherence proofs in higher-dimensional rewriting. We then establish correspondences between ω-catoids and convolution ω-quantales. These are related to Jónsson-Tarski-style dualisms between relational structures and lattices with operators. We extend these correspondences to (ω, p)-catoids, catoids with a groupoid structure above some dimension, and convolution (ω, p)-quantales, using Dedekind quantales above some dimension to capture homotopic constructions and proofs in higher-dimensional rewriting. We also specialise them to finitely decomposable (ω, p)-catoids, an appropriate setting for defining (ω, p)-semirings and (ω, p)-Kleene algebras. These constructions support the systematic development and justification of ω-Kleene algebra and ω-quantale axioms, improving on the recent approach mentioned, where axioms for ω-Kleene algebras have been introduced in an ad hoc fashion.

Keywords Higher catoids · Higher quantales · Multisemigroups · Convolution algebras · Categorification · Higher rewriting

Mathematics Subject Classification 18A05 · 18F75 · 06F07 · 18N30 · 68Q42 · 68V15

1 Introduction

Rewriting systems are fundamental models of computation. Their rules generate computations as sequences or paths of rewriting steps. Computational structure, such as confluence or Church–Rosser properties, is modelled geometrically in terms of rewriting diagrams and algebraically via inclusions between rewriting relations, which form abstractions of sets

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Extended author information available on the last page of the article of rewriting paths [1]. Coherence properties of rewriting systems, like the Church–Rosser theorem, Newman's lemma or normal form theorems, can be derived in algebras that support reasoning with binary relations: Kleene algebras, quantales or relation algebras [2–5].

Higher-dimensional rewriting generalises and categorifies this approach, using computads or polygraphs as higher-dimensional rewriting systems and the higher-dimensional paths they generate instead of rewriting relations, relations between rewriting relations, and so forth [6, 7], see also [8] for a recent textbook. This approach has been developed mainly in the context of (strict) ω-categories, where higher-dimensional rewriting diagrams have globular shape. They are filled with higher-dimensional cells instead of relational inclusions (two-cells in Rel) as witnesses for the $\forall\exists$-relationships between the higher-dimensional paths that form their faces. In fact, strict (ω, p)-categories, where cells of dimension greater than p are invertible, are often used in practice, for instance for showing that all parallel reduction cells of a higher-dimensional rewriting system are contractible. Applications of higher-dimensional rewriting range from string rewriting [9, 10] to the computational analysis of coherence properties and cofibrant approximations in categorical algebra [11, 12].

It has recently been argued that higher Kleene algebras, which support algebraic reasoning about sets of higher-dimensional rewriting paths, can be used for calculating categorical coherence proofs in higher-dimensional rewriting [13]—just as Kleene algebras in the classical case. To capture such properties, their axioms must reflect the shapes of globular cells of (ω, p)-categories and their pasting schemes: the relationships between their face maps and the interchange laws that relate the cell compositions in different dimensions and directions. Yet a systematic construction of these algebras and a systematic justification of their axioms relative to the underlying (ω, p)-categories and polygraphs has so far been missing.

Drawing from a seemingly unrelated field, we use the correspondences, in the sense of modal logic, associated with the Jónsson-Tarski duality between $(n + 1)$-ary relational structures and boolean algebras with n-ary operators—a Stone-type dual equivalence—as a guide. In light of this duality, we might consider the multiplication of a Kleene algebra, for instance, as a binary modal operator and the fact that an arrow in a category is a composition of two others as a ternary relation, and look at correspondences between identities that hold in these structures, for instance, whether an associativity law on the relational structure makes the multiplication of the Kleene algebra associative, and vice versa. Balancing such correspondences leads us from ω-categories to ω-catoids, which are isomorphic to ternary-relational structures with suitable relational laws, and from higher Kleene algebras to ω-quantales. Imposing axioms on ω-catoids then allows us to derive axioms on ω-quantales and vice versa, until balance is achieved. While this could be considered for ω-catoids on a set X and ω-quantales on the powerset $\mathcal{P}X$, we generalise this construction to correspondence triangles for ω-catoids C, ω-quantales Q and convolution ω-quantales on function spaces Q^C, and further to (ω, p)-structures.

We briefly outline the simplest case to supply some basic intuition. A powerset quantale on a set X is a quantale on $\mathcal{P}X$, that is, the complete lattice $(\mathcal{P}X, \subseteq)$ and at the same time a monoid $(\mathcal{P}X, \cdot, 1)$ such that the binary operator preserves arbitrary sups in both arguments. In every powerset quantale, the multiplication on singleton sets, the atoms in this structure, can be arranged into a ternary relation $\{x\} \subseteq \{y\} \cdot \{z\}$, which satisfies a certain relational associativity law and has the elements of the set 1 as relational units (the laws of a monoid object in Rel). As each element $x \in X$ has exactly one left and one right relational unit, they can be assigned to x by a source map $s : X \to X$ and a target map $t : X \to X$. Using a multioperation $\odot : X \times X \to \mathcal{P}X$ instead of the ternary relation $X \times X \times X \to 2$, so that $x \in y \odot z \Leftrightarrow \{x\} \subseteq \{y\} \cdot \{z\}$, and emphasising source and target maps instead of unit elements, leads to the definition of a catoid [14, 15] as a structure $(X, \odot, s, t)$ where, for all $x, y, z \in C$,

$$\bigcup\{x \odot v \mid v \in y \odot z\} = \bigcup\{u \odot z \mid u \in x \odot y\},$$
$$x \odot y \neq \emptyset \Rightarrow t(x) = s(y), \qquad s(x) \odot x = \{x\}, \qquad x \odot t(x) = \{x\}.$$

Powerset quantales on $\mathcal{P}X$ thus give rise to catoids: associativity of $\odot$ is derived using associativity of . Conversely, starting from a catoid $(X, \odot, s, t)$, one can construct a quantale on $\mathcal{P}X$ with composition $A \cdot B = \{c \in a \odot b \mid a \in A, b \in B\}$. Tying these constructions together yields a dual equivalence between the category of catoids (with suitable morphisms) and the category of powerset quantales (with suitable homomorphisms), an instance of the Jónsson-Tarski duality mentioned. The multioperation of the catoid is thus simply an alternative encoding of a ternary relation, while the multiplication of the powerset quantale is seen as binary operator on a boolean algebra. We are, however, not interested in such a duality itself, but in the correspondences between equations that are typical for the modal logics and algebras associated with it, as illustrated in the example of associativity above.

In the above construction, the powerset on X corresponds to a map $X \to 2$ into the quantale 2 of booleans, and 2 can be replaced by an arbitrary quantale Q. This leads to correspondence triangles between catoids C, value quantales Q and convolution quantales Q^C on function spaces, as depicted in Fig. 1. In the simplest case, if C is a catoid and Q a quantale, then Q^C is a quantale; if Q^C and Q are quantales, then C is a catoid; and if Q^C is a quantale and C a catoid, then Q is a quantale [16, 17]. The last two of these two-out-of-three properties require mild conditions on C or Q explained in Sect. 8. In the construction of the convolution quantale Q^C, if $\odot : C \times C \to \mathcal{P}C$ is the multioperation on the catoid C, if $\cdot : Q \times Q \to Q$ is the composition in the value quantale Q and if f, g are maps in $C \to Q$, then the quantalic composition $* : Q^C \times Q^C \to Q^C$ is the convolution

$$(f * g)(x) = \bigvee_{x \in y \odot z} f(y) \cdot g(z),$$

where the sup is taken with respect to y and z. Functions $\delta_x^\alpha : C \to Q$, which map $y \in C$ to $\alpha \in Q$ if $y = x$ and to the minimal element $\perp$ of the quantale Q otherwise, now replace singleton sets as atoms. They allow us to obtain equations in C from those in Q^C and Q, as well as Q from Q^C and C. Units in two of these structures then give rise to a corresponding unit in the third, see [17] and Sect. 8 for details. To construct a convolution or power

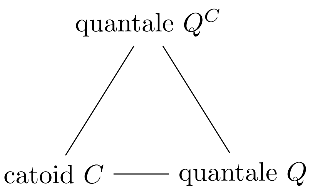


Fig. 1 Basic two-out-of-three correspondence between catoid C, quantale Q and convolution quantale Q^C

set quantale, it is thus necessary and sufficient to understand the structure of the underlying catoid (if the value quantale is fixed).

Our correspondence results for ω-catoids and ω-quantales are based on two extensions of this basic triangle. The first is a correspondence triangle between interchange catoids C, interchange quantales Q and convolution interchange quantales Q^C [17]. Catoids are then equipped with two multioperations and quantales with two monoidal structures that interact via interchange laws. This first extension helps us to deal with the interchange laws in ω-catoids which we wish to reflect within ω-quantales.

The second extension is a correspondence triangle between local catoids C, which exhibit the typical composition pattern of categories, modal value quantales Q and modal convolution quantales Q^C [15]. Here, correspondences between properties of the source and target maps of catoids and the domain and codomain maps of modal quantales extend the basic ones. This second extension justifies the laws of modal Kleene algebras [4] and modal quantales [18] relative to the catoid and category axioms. In powerset quantales, for instance, the domain and codomain operations in a powerset quantale arise as the direct images of the source and target maps of the corresponding catoid, and source and target maps in a catoid are obtained by restricting the application of domain and codomain operations of a powerset quantale to singleton sets. The second extension thus sets up the correspondence between the source and target structure in ω-catoids and the domain and codomain structure on ω-catoids.

In combination, the correspondence for interchange laws and that for source/target and domain/codomain therefore help us to reflect the full structure of strict ω-categories in powerset or convolution algebras, using ω-catoids to balance the equational axioms in the two kinds of structures in correspondence proofs.

The correspondence triangles between ω-catoids and ω-quantales, shown in Fig. 2, the main technical results in this article, require first of all definitions of these two structures. ω-Catoids are introduced in Sect. 6. They are obtained by adding globular shape axioms to those of local and interchange catoids and then generalising beyond two dimensions in light of previous axiomatisations of (single-set) ω-categories [19–22]. Based on the ω-catoid axioms and on previous axioms for globular n-Kleene algebras [13], we then introduce ω-quantales in Sect. 7 as our main conceptual contribution, and justify their axioms through the correspondence proofs in Sect. 8. Results for n-structures can be obtained in the standard way by truncation; correspondence results for powerset ω-quantales are discussed in Sect. 11. In this article, ω-catoids, as simple generalisations of strict ω-categories, are therefore merely tools for deriving the ω-quantale axioms and vice versa. They do not constitute any attempt towards infinity categories.

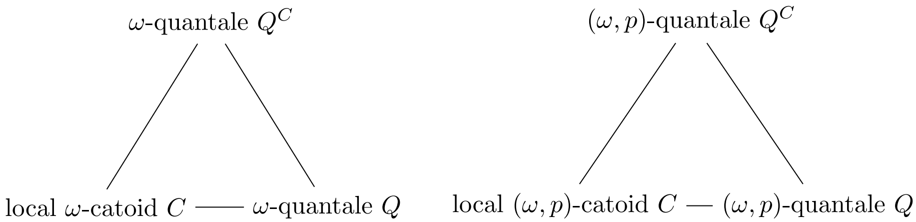


Fig. 2 Correspondences between local ω-catoid C, ω-quantale Q and ω-quantale Q^C on the left, and local (ω, p)-catoid C, (ω, p)-quantale Q and (ω, p)-quantale Q^C on the right

The ω-catoid axioms depend on rather delicate definedness conditions for compositions, captured multioperationally by mapping to non-empty sets. These are sensitive to Eckmann-Hilton-style collapses, as discussed in Appendix A. They satisfy natural functorial properties, specialise to ω-category axioms and yield the globular cell structure expected, see Sect. 6. The Isabelle/HOL proof assistant with its automated theorem provers and counterexample generators has allowed us to simplify these structural axioms quite significantly, which in turn simplified the development of ω-quantale axioms in Sect. 7 and the proofs of correspondence triangles in Sect. 8.

The extended correspondence triangles for (ω, p)-catoids and (ω, p)-quantales in Sect. 10, shown in Fig. 2, are compositional with respect to the ω-correspondences and those for groupoids and quantales with a suitable notion of involution or converse. These are established in Sect. 9, see Fig. 3. They adapt Jónsson and Tarski's classical duality between groupoids and relation algebras [23] and extend it to convolution algebras. Catoids specialise automatically to groupoids when the two obvious axioms for inverses are added, see Sect. 3. Yet we use Dedekind quantales instead of relation algebras here. These are involutive quantales [24] equipped with a variant of the Dedekind law from relation algebra, see Sect. 5. Apart from the interaction between the converse and the modal structure needed for (ω, p)-quantales, we also discuss weaker variants of converse useful for semirings or Kleene algebras in this section.

The content of the remaining sections of this paper is summarised as follows. In Sect. 2 and 4 we recall the basic properties of catoids and modal quantales. In Sect. 12 we develop the basic laws for modal box and diamond operators in ω-quantales in preparation for more advanced future coherence proofs in higher-dimensional rewriting. In Sect. 13 and 14 we specialise the ω- and (ω, p)-quantale axioms to those of ω-semirings and ω-Kleene algebras and their (ω, p)-variants. For general convolution algebras, this requires a finite decomposition property on ω- and (ω, p)-catoids. Sections 13 and 14 also contain a detailed comparison of the ω-Kleene algebras and (ω, p)-Kleene algebras introduced in this paper with previous globular n-Kleene algebras [13] and their slightly different axioms. Overall, ω-quantales offer greater flexibility when reasoning about higher rewriting diagrams than ω-Kleene algebras. They admit arbitrary suprema and additional operations such as residuals, and they support reasoning with least and greatest fixpoints beyond the Kleene star. Already the proof of the classical Newman's lemma in modal Kleene algebras [5] assumes certain suprema that are not present in all Kleene algebras. Nevertheless, convolution semiring and Kleene algebras have been widely studied in computer science [25] and their higher variants therefore certainly deserve an exploration.

Most results in this paper have been verified with the Isabelle/HOL proof assistant, but the development of interactive proof support for higher categories, higher-dimensional rewriting and categorical algebra remains part of a larger research programme, which requires substantial additions. Our Isabelle components can be found in the Archive of For-

$$\begin{tikzcd}
 & \text{Dedekind quantale } Q^C \arrow[dl, dash] \arrow[dr, dash] & \\
\text{groupoid } C \arrow[rr, dash] & & \text{Dedekind quantale } Q
\end{tikzcd}$$

Fig. 3 Correspondence between groupoid C, Dedekind quantale Q and convolution Dedekind quantale Q^C, where both quantales are assumed to carry a complete Heyting algebra structure mal Proofs [26–28]. The components contain specifications and basic libraries for 2-catoids, groupoids, 2-quantales and their ω-variants, as well as for Dedekind quantales, and they cover the basic properties of these structures that feature in this paper. All extensions to powersets have been formalised with Isabelle/HOL, but neither the constructions of convolution algebras nor the full correspondence triangles. In addition to the Isabelle proofs, we present the most important proofs for this article in the relevant sections. All other proofs can be found in Appendix B or the references given, unless they are trivial.

Finally, Appendix C provides diagrams for the most important structures used in this articles and their relationships.

2 Catoids

In preparation for the ω-catoids in Sect. 6 we start with recalling the definitions and basic properties of catoids and related structures [15]; see also the Isabelle theories [26] for details. General background on multisemigroups can be found in [29] and the references given there. All proofs in this section have been checked with Isabelle.

A *catoid* $(C, \odot, s, t)$ consists of a set C, a multioperation $\odot : C \times C \to \mathcal{P}C$ and *source* and *target* maps $s, t : C \to C$. These satisfy, for all $x, y, z \in C$,

$$\bigcup\{x \odot v \mid v \in y \odot z\} = \bigcup\{u \odot z \mid u \in x \odot y\},$$
$$x \odot y \neq \emptyset \Rightarrow t(x) = s(y), \qquad s(x) \odot x = \{x\}, \qquad x \odot t(x) = \{x\}.$$

The first catoid axiom expresses multirelational *associativity*. If we extend $\odot$ from $C \times C \to \mathcal{P}C$ to $\mathcal{P}C \times \mathcal{P}C \to \mathcal{P}C$ such that, for all $X, Y \subseteq C$,

$$X \odot Y = \bigcup_{x \in X, y \in Y} x \odot y,$$

and write $x \odot X$ for $\{x\} \odot X$ and likewise, then the associativity axiom simplifies to $x \odot (y \odot z) = (x \odot y) \odot z$. A *multisemigroup* can thus be defined as a set equipped with an associative multioperation [29]. From now on, we often write xy instead of $x \odot y$. We also write $s(X)$ and $t(X)$ for the direct images of X under s and t, respectively, for instance, $s(x \odot y)$ or $s(xy)$ as well as $t(x \odot y)$ or $t(xy)$.

The second catoid axiom, the *weak locality* axiom, states that the target $t(x)$ of x and the source $s(y)$ of y are equal whenever the composite of x and y is defined, that is, $x \odot y \neq \emptyset$. We write $\Delta(x, y)$ for $x \odot y \neq \emptyset$ and call Δ the *domain of definition* of $\odot$.

The third and fourth catoid axioms are *left* and *right unit* axioms and we refer to $s(x)$ and $t(x)$ as the *left unit* and *right unit* of x, respectively.

A catoid C is *functional* if $x, x' \in yz$ imply $x = x'$ for all $y, z \in C$, and *local* if $t(x) = s(y) \Rightarrow xy \neq \emptyset$ for all $x, y \in C$. A *category* is then a local functional catoid.

Catoids thus generalise categories beyond locality and functionality. Local functional catoids formalise categories in single-set style [30], see also [21, Chapter XII], and with arrow composition in diagrammatic order. In functional catoids, composition is a partial operation that maps either to singleton sets or to the empty set. Locality imposes the standard composition pattern $\Delta(x, y) \Leftrightarrow t(x) = s(y)$ of arrows in categories.

Composition in a catoid is *total* if $\Delta = C \times C$. In every total catoid C, there is precisely one element which is the source and target element of every element in C. Total multioperations are known as *hyperoperations*. Total operations are therefore total functional multioperations.

Remark 2.1 In the definition of catoids and throughout this text, we are using "set" indifferently for small sets and classes and ignore the well known foundational issues, which do not arise in our setting. Distinctions can be made as for standard categories. See, for instance, Mac Lane's book [21] and Example 2.8 for a discussion.

Catoids form a category with respect to several notions of morphism. A *catoid morphism* $f : C \to D$ between catoids C and D must preserve compositions, sources and targets:

$$f(x \odot_C y) \subseteq f(x) \odot_D f(y), \qquad f \circ s_C = s_D \circ f, \qquad f \circ t_C = t_D \circ f,$$

where, on the left-hand side of the left identity the image of the set $x \odot_C y$ with respect to f is taken. A morphism $f : C \to D$ is *bounded* if $f(x) \in u \odot_D v$ implies that $x \in y \odot_C z$, $u = f(y)$ and $v = f(z)$ for some $y, z \in C$.

Morphisms of categories, as local functional catoids, are functors. The inclusion in the definition of morphisms reflects that $x \odot_C y = \emptyset$ whenever the composition is undefined. Bounded morphisms are widely used in modal and substructural logics.

Example 2.2 Bounded morphisms between catoids need not satisfy $f(x \odot_C y) = f(x) \odot_D f(y)$. Consider the discrete catoid on $C = \{a, b\}$ with $s = id_C = t$ and

$$xy = \begin{cases} \{x\} & \text{if } x = y, \\ \emptyset & \text{otherwise.} \end{cases}$$

The constant map $f_b : x \mapsto b$ on C is clearly a catoid endomorphism. It is bounded because every $x \in C$ satisfies $f_b(x) \in bb$, $x \in xx$ and $b = f_b(x)$. Nevertheless we have $f_b(ab) = \emptyset \neq \{b\} = f_b(a)f_b(b)$.

The *opposite* of a catoid is defined as for categories. It is a structure in which s and t are exchanged and so are the arguments in $\odot$. The opposite of a (local, functional, total) catoid is again a (local, functional, total) catoid. Properties of catoids translate through this duality.

Properties of catoids and related structures have been collected in [15] and our Isabelle theories [26]. Here we list only some that are structurally interesting or needed in proofs below.

Lemma 2.3 *In every catoid,*

1. $s \circ s = s$, $t \circ t = t$, $s \circ t = t$ and $t \circ s = s$,
2. $s(x) = x \Leftrightarrow x = t(x)$,
3. $s(x)s(x) = \{s(x)\}$ and $t(x)t(x) = \{t(x)\}$,

4. $s(x)t(y) = t(y)s(x)$,
5. $s(s(x)y) = s(x)s(y)$ and $t(xt(y)) = t(x)t(y)$.

Note that direct images with respect to s and t are taken in the left-hand sides of the identities in (5). According to (2), the set of fixpoints of s equals the set of fixpoints of t. We henceforth write C_0 for this set.

Lemma 2.4 *Let C be a catoid. Then $C_0 = s(C) = t(C)$.*

The proof is immediate from Lemma 2.3(1). Hence C_0 consists of the units in C which we also call 0-*cells*, by analogy with categories. Similarly, the elements of C can be viewed as 1-*cells*, the elements of C_0 as degenerate 1-cells ($s(x) = t(x)$ holds for all $x \in C_0$ by Lemma 2.3(2)) and those of $C - C_0$ as *proper* or *non-degenerate* 1-cells. Further, units of catoids can be seen as orthogonal idempotents.

Lemma 2.5 *Let C be a catoid. For all $x, y \in C_0$, $\Delta(x, y) \Leftrightarrow x = y$ and*

$$xy = \begin{cases} \{x\} & if x = y, \\ \emptyset & otherwise. \end{cases}$$

The next two lemmas recall an alternative equational characterisation of locality, which is important for the correspondence with modal quantales in Sect. 8.

Lemma 2.6 *In every catoid,*

1. $s(xy) \subseteq s(xs(y))$ and $t(xy) \subseteq t(t(x)y)$,
2. $\Delta(x, y)$ implies $s(xy) = \{s(x)\}$ and $t(xy) = \{t(y)\}$.

Item (2) of the following lemma features the equational characterisation of locality mentioned.

Lemma 2.7 *In every catoid C, the following statements are then equivalent:*

1. C is local,
2. $s(xy) = s(xs(y))$ and $t(xy) = t(t(x)y)$, for all $x, y \in C$,
3. $\Delta(x, y) \Leftrightarrow t(x)s(y) \neq \emptyset$, for all $x, y \in C$.

Many examples of catoids and related structures are listed in [15]. Here we mention only a few. First we summarise the relationship with categories.

Example 2.8 The category of local functional catoids with (bounded) morphisms is isomorphic to the category of single-set categories á la MacLane [21, Chapter XII] with (bounded) morphisms [15, Proposition 3.10]. The elements of single-set categories are arrows or 1-cells of traditional categories. The objects of 0-cells of traditional categories correspond bijectively to identity 1-cells and thus to units of catoids.

Example 2.9 The free category or *path category* generated by a given digraph $s, t : E \to V$, for a set E of edges or 1-generators and a set V of vertices or 0-generators, is a fortiori a catoid. Recall that morphisms of path categories are finite paths between vertices, represented as alternating sequences of vertices and edges. Source and target maps extend from edges to paths, and we write $\pi : v \to w$ for a path π with source v and target w. Paths $\pi_1 : u \to v$ and $\pi_2 : v \to w$ can be composed to the path $vw : u \to w$ by gluing their ends. Identities are constant paths of length zero such as $v : v \to v$. See [21] for details.

Digraphs themselves can be modelled as single-set structures (X, s, t) satisfying $s \circ s = s$, $t \circ t = t$, $s \circ t = t$ and $t \circ s = s$. These conditions make $X_0 = s(X) = t(X)$ the set of vertices.

In higher-dimensional rewriting, digraphs are referred to as 1-computads or 1-polygraphs; in traditional rewriting, they correspond to abstract rewriting systems. The free category generated by a digraph supplies rewriting paths, the main object of study in rewriting systems. The recursive construction of higher-dimensional computads or polygraphs has two-steps: for any dimension $n \geq 0$, assuming that k-generators have been supplied for all $0 \leq k \leq n$, form the free n-category on n-generators, then add $(n + 1)$-generators over this free category.

The next example presents a catoid which is not a category and which will recur across this text.

Example 2.10 Let Σ^* be the free monoid generated by the finite set Σ. The *shuffle catoid*$(\Sigma^*, \|, s, t)$ on Σ^* has the total commutative multioperation $\| : \Sigma^* \times \Sigma^* \to \mathcal{P}\Sigma^*$ as its composition. For all $a, b \in \Sigma$ and $v, w \in \Sigma^*$, it is defined recursively as

$$\varepsilon \| v = \{v\} = v \| \varepsilon \qquad \text{and} \qquad (av)\|(bw) = a(v\|(bw)) \cup b((av)\|w),$$

where ε denotes the empty word. The source and target structure of the shuffle catoid is trivial: $s(w) = \varepsilon = t(w)$ for all $w \in \Sigma^*$. It is local because of totality and triviality of s and t. It is obviously not functional and hence not a category.

Remark 2.11 Catoids can be seen as multimonoids, that is, multisemigroups with multiple units, which generalise single-set categories with multiple units [21, Chapter I] to multi-operations. Multimonoid morphisms are then unit-preserving multisemigroup morphisms, which ignore source- and target-preservation. Categories of catoids and multimonoids, both with the obvious morphisms, are isomorphic; the functional relationship between elements of multimonoids and their left and right units determines source and target maps [14, 15, Proposition 3.10]. Categories of local partial multimonoids are therefore isomorphic to categories. See [14] for the appropriate notion of locality.

Remark 2.12 Multioperations $X \times X \to \mathcal{P}X$ are isomorphic to ternary relations $X \to X \to X \to 2$. Writing R^x_{yz} for $x \in y \odot z$ allows us to axiomatise catoids alternatively as relational structures (C, R, s, t) that satisfy the relational associativity axiom $\exists v.\ R^w_{xv} \wedge R^v_{yz} \Leftrightarrow \exists u.\ R^u_{xy} \wedge R^w_{uz}$, the weak locality axiom $\exists z.\ R^z_{xy} \Rightarrow t(x) = s(z)$ and the relational unit axioms $R^x_{s(x)x}$ and $R^x_{xt(x)}$. Such *relational monoids* are, in fact, monoids in the monoidal category Rel with the standard tensor [31, 32]. Functionality and locality translate readily to relations. Relational variants of catoids have been studied in [14]. Morphisms and bounded morphisms of ternary relations are standard for modal and substructural logics [33, 34]; categories of catoids and relational monoids are once again isomorphic [15].

Ternary and more generally $(n+1)$-ary relations appear as duals of binary and more generally n-ary modal operators on boolean algebras in Jónsson and Tarski's duality theory for boolean algebras with operators [23, 33–35].

Remark 2.13 A partial operation $\hat{\odot} : \Delta \to C$, where $y \,\hat{\odot}\, z$ denotes the unique element that satisfies $y \,\hat{\odot}\, z \in y \odot z$ whenever $\Delta(y,z)$, can be defined in any functional catoid. Using this partial operation, $x \in y \odot z$ if and only if $\Delta(y,z)$ and $x = y \,\hat{\odot}\, z$.

3 Groupoids

Higher-dimensional rewriting usually requires rewriting steps to be invertible above a certain dimension. This amounts to using groupoids, see [36] for a survey. Particularly relevant to us is the work of Jónsson and Tarski [23, Sect. 5] on the correspondence between groupoids and relation algebras, which we revisit in the slightly different setting of Dedekind quantales in Sect. 9. Several properties that feature in this section are theirs. In Sect. 10, catoids are combined with groupoids to (n, p)-catoids and (ω, p)-catoids, which specialise to the (n, p)-categories and (ω, p)-categories used in higher-dimensional rewriting. Interestingly, catoids become groupoids and hence categories when the natural axioms for inverses are added. All proofs in this section have been checked with Isabelle [26].

A *groupoid* is a catoid C with an *inversion* operation $(-)^- : C \to C$ such that, for all $x \in C$,

$$xx^- = \{s(x)\} \qquad \text{and} \qquad x^-x = \{t(x)\}.$$

To justify this definition, we proceed in two steps to derive locality and functionality, showing selected proofs only. The remaining ones can be found in Appendix B.

Lemma 3.1 *Let X be a catoid with operation $(-)^- : C \to C$ that satisfies $s(x) \in xx^-$ and $t(x) \in x^-x$ for all $x \in C$. Then*

1. C is local,
2. $s(x^-) = t(x)$ and $t(x^-) = s(x)$,
3. $xy = \{s(x)\}$ implies $x^- = y$ and $yx = \{t(x)\}$ implies $x^- = y$,
4. $s(x)^- = s(x)$ and $t(x)^- = t(x)$.

Proof For (1), suppose $t(x) = s(y)$. Then $\{x\} = xt(x) = xs(y) = x(yy^-) = (xy)y^-$. Hence $x = uy^-$ and $u \in xy$ hold for some $u \in C$. Thus $\Delta(x,y)$ and C is local. $\square$

For proofs of (2)-(4) see Appendix B.

Lemma 3.2 *In every groupoid,*

1. $(x^-)^- = x$,
2. $x \in yz \Leftrightarrow y \in xz^- \Leftrightarrow z \in y^- x$.

See Appendix B for proofs.

Proposition 3.3 *Every groupoid (as defined above) is a category.*

Proof In light of Lemma 3.1(1) it remains to check functionality. Suppose $x, x' \in yz$. Then $z \in y^- x$ by Lemma 3.2(2) and $x' \in yy^- x = s(y)x = s(x)x = \{x\}$ using the first assumption and $v \in xy \Rightarrow s(v) = s(x)$, which holds in any catoid by Lemma 2.7, in the second step. Thus $x' = x$. □

Lemma 3.4 *The following cancellation properties hold in every groupoid:*

1. $s(x) = t(z) = s(y)$ and $zx = zy$ imply $x = y$,
2. $t(x) = s(z) = t(y)$ and $xz = yz$ imply $x = y$.

See Appendix B for proofs. The cancellation properties correspond to the fact that every morphism in a groupoid is both an epi and a mono.

Lemma 3.5 *In every groupoid,*

1. $x \mathbin{\hat{\odot}} x^- \mathbin{\hat{\odot}} x = x$,
2. $t(x) = s(y)$ implies $x^- \mathbin{\hat{\odot}} x = y^- \mathbin{\hat{\odot}} y$,
3. $(x \mathbin{\hat{\odot}} y)^- = y^- \mathbin{\hat{\odot}} x^-$.

The last identity follows from Lemma 3.2(2), the others are straightforward.

The following example is particularly interesting in the context of Dedekind quantales in Sect. 5.

Example 3.6 The *pair groupoid*$(X \times X, \odot, s, t, (-)^-)$ on the set X is defined, for all $a, b, c, d \in X$, by

$$(a, b) \odot (c, d) = \begin{cases} \{(a, d)\} & \text{if } b = c, \\ \emptyset & \text{otherwise,} \end{cases}$$

$$s((a, b)) = (a, a), \qquad t((a, b)) = (b, b), \qquad (a, b)^- = (b, a).$$

Pair groupoids give rise to algebras of binary relations, see Example 5.5 below.

The final example in this section builds on the path categories from Example 2.9. It is relevant for modelling higher homotopies in the context of (n, p)-catoids and (ω, p)-catoids and categories.

Example 3.7 The free groupoid generated by a digraph is obtained by adding formal inverses to each edge and considering paths up to simplification. For instance, if a, b, c, d are appropriate edges, then $abc^{-1}cb^{-1}d = ad$, see also [37, Chapter 4].

4 Modal Quantales

The second ingredient of the convolution algebras in this article are quantales, more specifically quantales that extend not only the composition and unit structure of catoids, categories and groupoids, but also their source and target structure. It has been shown in [15] that source and target maps extend to domain and codomain maps on quantales, which further allow defining modal diamond and box operators on them. The resulting quantales with axioms for domain and codomain operators are therefore known as modal quantales [18]. In this section we recall their definition and basic properties. Most of this development has been formalised with Isabelle [27].

A *quantale*$(Q, \leq, \cdot, 1)$ is a complete lattice $(Q, \leq)$ with an associative composition , which preserves all sups in both arguments and has a unit 1. See [38] for an introduction. We write $\bigvee$, $\vee$, $\bigwedge$ and $\wedge$ for sups, binary sups, infs and binary infs in a quantale, and $\perp$ and $\top$ for the smallest and greatest element, respectively. A *subidentity* of a quantale Q is an element $\alpha \in Q$ such that $\alpha \leq 1$.

A quantale is *distributive* if its underlying lattice is distributive and *boolean* if its underlying lattice is a boolean algebra, in which case we write − for boolean complementation.

We henceforth use greek letters for elements of quantales to distinguish them from elements of catoids, and we often write $\alpha\beta$ for $\alpha \cdot \beta$.

Example 4.1 We need the *quantale of booleans* 2, which is a boolean quantale with carrier set $\{0, 1\}$, order $0 < 1$, $\max$ as binary sup, $\min$ as binary inf and composition, 1 as its unit and $\lambda x.\ 1 - x$ as complementation. It allows constructing powerset quantales over catoids and categories, using 2 as a value algebra.

A *Kleene star*$(-)^* : Q \to Q$ can be defined on any quantale Q, for $\alpha^0 = 1$ and $\alpha^{i+1} = \alpha\alpha^i$, as

$$\alpha^* = \bigvee_{i \geq 0} \alpha^i.$$

A *domain quantale* [18] is a quantale Q with an operation $dom : Q \to Q$ such that, for all $\alpha, \beta \in Q$,

$$\alpha \leq dom(\alpha)\alpha, \qquad dom(\alpha dom(\beta)) = dom(\alpha\beta), \qquad dom(\alpha) \leq 1,$$
$$dom(\perp) = \perp, \qquad dom(\alpha \vee \beta) = dom(\alpha) \vee dom(\beta).$$

These domain axioms are known from domain semirings [4], see also Sect. 13. We call the first axiom the *absorption axiom*. The second expresses *locality* of domain. The third is the *subidentity* axiom, the fourth the *bottom* axiom and the final the *(binary) sup* axiom. Most properties of interest translate from domain semirings to domain quantales. An equational absorption law $dom(\alpha)\alpha = \alpha$ is derivable.

We define $Q_dom = \{\alpha \in Q \mid dom(\alpha) = \alpha\}$. As for catoids, $Q_dom = dom(Q)$ holds because $dom \circ dom = dom$ is derivable from the domain quantale axioms. Moreover, $(Q_dom, \leq)$ forms a bounded distributive lattice with $\vee$ as binary sup, as binary inf, $\perp$ as least element and 1 as greatest element. We call Q_dom the lattice of *domain elements* or simply *domain algebra*. In a boolean quantale, Q_dom is the set of all subidentities and hence a complete boolean algebra.

Quantales are closed under opposition, which exchanges the arguments in compositions. A *codomain quantale*(Q, cod) is then a domain quantale (Q^{op}, dom). Further, a *modal quantale* [18] is a domain and codomain quantale $(Q, \leq, \cdot, 1, dom, cod)$ that satisfies the *compatibility* axioms

$$dom \circ cod = cod \qquad \text{and} \qquad cod \circ dom = dom.$$

These guarantee that $Q_dom = Q_cod$, a set which we denote by Q_0 by analogy to catoids.

Example 4.2 Let $(C, \odot, s, t)$ be a local catoid. Then $\mathcal{P}C$ can be equipped with a modal quantale structure. The monoidal structure is given by the extended composition $\odot : \mathcal{P}C \times \mathcal{P}C \to \mathcal{P}C$ of the catoid and the set C_0. Its lattice structure is given by $\subseteq$, $\bigcup$ and $\bigcap$. Its domain and codomain structure is given by $dom(X) = s(X)$ and $cod(X) = t(X)$, the images of any set $X \subseteq C$ with respect to *s* and *t* [18]. As a powerset quantale, that is, a quantale on a power set, $\mathcal{P}C$ is in fact an atomic boolean quantale. Note that locality of *C* is needed for locality of $\mathcal{P}C$. This result has the following instances, among others.

1. Every category *C* extends to a modal quantale on $\mathcal{P}C$.
2. The path category over any digraph, more specifically, extends to a modal quantale at powerset level. Domain and codomain elements are sets of vertices, 1 is the set *V* of all vertices of the digraph. In the context of rewriting, such quantales allow reasoning about sets of rewriting paths and in particular shapes of rewriting diagrams.
3. Every pair groupoid on the set *X* extends to a modal quantale of binary relations on *X* with the standard relational domain and codomain maps $dom(R) = \{(x, x) \mid \exists y.\ (x, y) \in R\}$ and $cod(R) = \{(y, y) \mid \exists x.\ (x, y) \in R\}$, and with $\odot$ extended to the relational composition $R \cdot S = \{(x, y) \mid \exists z.(x, z) \in R \wedge (z, y) \in S\}$. The quantalic unit is the identity relation $\{(x, x) \mid x \in X\}$. Quantales of binary relations and similar algebras can be used to reason about rewrite relations and once again about shapes of rewrite diagrams [2, 3].
4. The shuffle catoid on Σ^*, considered in Example 2.10 extends to the commutative quantale of shuffle languages on Σ, a standard model of interleaving concurrency in computing. The quantalic composition is the shuffle product $X \| Y = \bigcup\{x \| y \mid x \in X \wedge y \in Y\}$ of languages, which are subsets of Σ^*. Its monoidal unit is $\{\varepsilon\}$, the domain/codomain structure is therefore trivial.

For further examples of modal quantales and their underlying catoids see Sect. 8 and [18].

Remark 4.3 Locality of catoids in the form $x \odot y \neq \emptyset \Leftrightarrow t(x)s(y) \neq \emptyset$, as in Lemma 2.7, corresponds to

$$\alpha \cdot \beta \neq \bot \Leftrightarrow \mathit{cod}(\alpha) \cdot \mathit{dom}(\beta) \neq \bot$$

in modal quantales. This is a consequence of locality of dom and cod in modal semirings [4] and modal quantales. In modal quantales, it is even equivalent to locality of dom and cod. Yet the more precise locality property $t(x)s(y) \neq \emptyset \Leftrightarrow t(x) = s(y)$ does not hold in all modal quantales.

Consider for instance the path category over the digraph $v_1 \xleftarrow{e_1} v_2 \xrightarrow{e_2} v_3 \xrightarrow{e_3} v_4$ and the sets of paths $X = \{(v_2, e_1, v_1), (v_2, e_2, v_3)\}$ and $Y = \{(v_3, e_3, v_4)\}$. Then $\mathit{cod}(X) = \{v_1, v_3\} \neq \{v_3\} = \mathit{dom}(Y)$ whereas $X \cdot Y = \{(v_2, e_2, v_3, e_3, v_4)\} \neq \emptyset$.

Nevertheless, for any local catoid *C* with elements *a* and *b*,

$$\begin{aligned} \mathit{cod}(\{a\}) \cap \mathit{dom}(\{b\}) \neq \emptyset &\Leftrightarrow \{t(a)\} \cap \{s(b)\} \neq \emptyset \\ &\Leftrightarrow \{t(a)\} = \{s(b)\} \\ &\Leftrightarrow \mathit{cod}(\{a\}) = \mathit{dom}(\{b\}) \end{aligned}$$

holds at least for the atoms $\{a\}$ and $\{b\}$ in the quantale $\mathcal{P}C$.

5 Dedekind Quantales

In the previous section we have seen how catoids and categories give rise to modal quantales at powerset level. One may therefore wonder how the inverse structure of groupoids is reflected in quantales. In a slightly different setting, Jónsson and Tarski have already given an answer, extending groupoids to (powerset) relation algebras [23] along the lines outlined in the previous section, so that the inverse of the groupoid corresponds to the converse of the relation algebra. Yet this correspondence ignores the source/target and domain/codomain structures. To translate their results to quantales, we consider Dedekind quantales: quantales with an involution that satisfies the Dedekind law from relation algebra. Interestingly, domain and codomain operations can be defined explicitly in Dedekind quantales, whereas they need to be axiomatised in weaker kinds of quantales. For applications in higher-dimensional rewriting along the lines of [13], quantales are combined with Dedekind quantales in Sect. 10. This yields (n, p)-quantales and (ω, p)-quantales, which are related to (n, p)- and (n, ω)-catoids via correspondence proofs.

Dedekind quantales are single-object versions of the modular quantaloids studied by Rosenthal [39], but much of the material introduced in this section is new. All proofs in this section (except for Lemma 5.14) can be found in Appendix B and our Isabelle theories [27] (including the proofs for this Lemma). Isabelle has also been instrumental in finding the counterexamples in this section.

An *involutive quantale* [24] is a quantale Q with an operation $(-)^\circ : Q \to Q$ that satisfies

$$\alpha^{\circ\circ} = \alpha, \qquad (\bigvee A)^\circ = \bigvee\{\alpha^\circ \mid \alpha \in A\}, \qquad (\alpha\beta)^\circ = \beta^\circ\alpha^\circ.$$

Involution thus formalises opposition within the language of quantales.

Remark 5.1 Replacing the first two axioms by $\alpha^\circ \leq \beta \Leftrightarrow \alpha \leq \beta^\circ$ yields an equivalent axiomatisation. Involution is therefore self-adjoint.

Lemma 5.2 *In every involutive quantale, the following properties hold:*

1. $\alpha \leq \beta \Rightarrow \alpha^\circ \leq \beta^\circ$,
2. $(\alpha \vee \beta)^\circ = \alpha^\circ \vee \beta^\circ$,
3. $(\bigwedge A)^\circ = \bigwedge\{\alpha^\circ \mid \alpha \in A\}$ and $(\alpha \wedge \beta)^\circ = \alpha^\circ \wedge \beta^\circ$,
4. $\bot^\circ = \bot$, $1^\circ = 1$ and $\top^\circ = \top$,
5. $\alpha^\circ \wedge \beta = \bot \Leftrightarrow \alpha \wedge \beta^\circ = \bot$,
6. $\alpha^{*\circ} = \alpha^{\circ *}$.

A *Dedekind quantale* is an involutive quantale in which the *Dedekind law*

$$\alpha\beta \wedge \gamma \leq (\alpha \wedge \gamma\beta^\circ)(\beta \wedge \alpha^\circ\gamma).$$

holds. It is standard in relation algebra. Next we present an alternative definition.

Lemma 5.3 *An involutive quantale is a Dedekind quantale if and only if the following modular law holds:*

$$\alpha\beta \wedge \gamma \leq (\alpha \wedge \gamma\beta^\circ)\beta.$$

The modular law is standard in relation algebra as well.

An extensive list of properties of Dedekind quantales can be found in our Isabelle theories. Here we only list some structurally important ones.

Lemma 5.4 *The following properties hold in every Dedekind quantale:*

1. the strong Gelfand property $\alpha \leq \alpha\alpha^\circ\alpha$,
2. *Peirce's law*$\alpha\beta \wedge \gamma^\circ = \bot \Leftrightarrow \beta\gamma \wedge \alpha^\circ = \bot$,
3. the *Schröder laws*$\alpha\beta \wedge \gamma = \bot \Leftrightarrow \beta \wedge \alpha^\circ\gamma = \bot \Leftrightarrow \alpha \wedge \gamma\beta^\circ = \bot$.

The strong Gelfand property (the name has been borrowed from [40]) has been used previously by Ésik and co-workers to axiomatise relational converse in semigroups and Kleene algebras, where infs are not available [41, 42]. Similarly, and for the same reason, it appears in globular n-Kleene algebras and their applications in higher-dimensional rewriting [13]. This is our main reason for including it here and revisiting it in Sect. 14. Peirce's law and the Schröder laws are standard in relation algebra. Indeed, Dedekind quantales bring us close to relation algebras [43–45]; but see our more precise comparison below.

Example 5.5 Let G be a groupoid. Then $\mathcal{P}G$ forms a relation algebra over G [23]. As a powerset algebra, the underlying lattice of $\mathcal{P}G$ is complete (even boolean and atomic). Hence every groupoid extends to a Dedekind quantale in which $dom(X) = G_0 \cap XX^\circ$ and $cod(X) = G_0 \cap X^\circ X$ for every $X \subseteq G$. The derivation of the Dedekind law follows more or less that of Jónsson and Tarski for relation algebras [23]. It needs neither the boolean algebra structure present in relation algebras nor the completeness of the lattice of the Dedekind quantale. We present a more general derivation of this law in Theorem 9.2 and revisit this example more formally in Corollary 11.4.

Again there are interesting instances.

1. Each free groupoid generated by some digraph extends to a Dedekind quantale on sets of paths. The converses in this quantale are sets of formal inverses in the groupoid.
2. The pair groupoid on a set X extends to the Dedekind quantale of binary relations on $\mathcal{P}(X \times X)$ with standard relational converse $R^\circ = \{(y, x) \mid (x, y) \in R\}$ extended from the inverse operation on the pair groupoid.

Next we turn to the domain and codomain structure on Dedekind quantales. By contrast to general quantales, it can be defined explicitly in involutive quantales using

$$dom(\alpha) = 1 \wedge \alpha\alpha^\circ \qquad \text{and} \qquad cod(\alpha) = 1 \wedge \alpha^\circ\alpha,$$

or alternatively $dom(\alpha) = 1 \wedge \alpha\top$ and $cod(\alpha) = 1 \wedge \top\alpha$, as in relation algebra.

But only Dedekind quantales are expressive enough to make these two definitions coincide, derive the natural domain and codomain laws needed for defining suitable modal operators (as in Sect. 12 below) and establish the correspondence with respect to groupoids in Sects. 9 and 11.

Proposition 5.6 *Every Dedekind quantale is a modal quantale.*

A proof using the explicit definitions of dom and cod can be found in Appendix B. In addition, we list some properties that are not available in modal quantales.

Lemma 5.7 *In every Dedekind quantale, the following properties hold:*

1. $dom(\alpha) = 1 \wedge \alpha\top$,
2. $dom(\alpha)\top = \alpha\top$,
3. $(dom(\alpha))^\circ = dom(\alpha)$,
4. $dom(\alpha^\circ) = cod(\alpha)$.

Next we present a natural example of a modal quantale that is not Dedekind.

Example 5.8 The law (1) fails in the modal path quantales from Example 4.2, where formal inverses are not assumed in the underlying catoid. Recall that, in this model, 1 is the set V of all vertices of the digraph. Thus $V \cap P\top = \emptyset$ unless the set P of paths contains a path of length one and $dom(P) = \emptyset$ if and only if $P = \emptyset$.

The next lemma shows more directly that Dedekind quantales are more expressive than involutive quantales

Lemma 5.9 *Neither the strong Gelfand property nor the modular law holds in all involutive quantales.*

Proof In the involutive quantale $\bot < a < \top = 1$ with multiplication $aa = \bot($ the rest being fixed) and $(-)^\circ = id$, the strong Gelfand property fails because $aa^\circ a = \bot < a$. The modular and (Dedekind law) fail in this involutive quantale because

$$(1 \wedge aa)a(1 \wedge aa)(a \wedge 1a) = 0 < 1a \wedge a = a.$$

□

The following example refines this result.

Example 5.10 Adding the strong Gelfand property to the involutive quantale axioms does not imply the modular law. The involutive quantale defined by

$$\begin{tikzcd} & \top & \\ 1 \arrow[ur, dash] & & a \arrow[ul, dash] \\ & \bot \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

$\cdot$	$\bot$	1	a	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
1	$\bot$	1	a	$\top$
a	$\bot$	a	a	a
$\top$	$\bot$	$\top$	a	$\top$

and $(-)^\circ = id$ satisfies the strong Gelfand property, but

$$1a \wedge \top = a > \bot = (1 \wedge \top a)a.$$

The domain and codomain operations of modal semirings, even finite ones, need not be uniquely determined [4, Lemma 6.4]. A modal semiring can therefore carry several domain/codomain structures. Yet they are uniquely determined in modal semirings over boolean algebras. Finite modal semirings and finite modal quantales are the same. One may therefore ask whether there can be other domain/codomain structures on Dedekind quantales than that given by the explicit definitions above. The answer is the same as for modal semirings.

For the sake of this argument, we call *modal Dedekind quantale* a Dedekind quantale that is also a modal quantale, that is, it is equipped with a map δ^-(domain) and a map δ^+(codomain) that satisfy axioms from Sect. 4. We start with a technical lemma.

Lemma 5.11 *In every modal Dedekind quantale,*

$$\delta^-(\alpha)^\circ = \delta^-(\alpha) \qquad \textit{and} \qquad \delta^+(\alpha)^\circ = \delta^+(\alpha).$$

Lemma 5.12 *There is a modal distributive Dedekind quantale in which $\delta^- \neq dom$ and $\delta^+ \neq cod$.*

Proof In the modal distributive Dedekind quantale with $\bot < a < \top = 1$, multiplication $aa = a$, $\delta^-(a) = 1 = \delta^+(a)$ and $(-)^\circ = id$, we have $\delta^-(a) = \delta^+(a) = \top \neq a = 1 \wedge aa^\circ = 1 \wedge a^\circ a$. □

Remark 5.13 In any boolean modal quantale Q, the set Q_0 equals the boolean subalgebra of all subidentities, and domain and codomain are uniquely defined. The proof for modal semirings [4, proof of Theorem 6.12] translates directly. Thus, in any boolean modal Dedekind quantale, $\delta^-(\alpha) = 1 \wedge \alpha\alpha^\circ = dom(x)$ and $\delta^+(\alpha) = 1 \wedge \alpha^\circ\alpha = cod(x)$.

Boolean Dedekind quantales are strongly related to relation algebras.

Lemma 5.14 *In any boolean Dedekind quantale,* $(-\alpha)^\circ = -(\alpha^\circ)$ *and the residual law* $\alpha^\circ - (\alpha\beta) \leq -\beta$ *is derivable.*

A residual law appears in Tarski's original axiomatisation of relation algebra [43]. A boolean Dedekind quantale is thus a relation algebra over a complete lattice, and a relation algebra a boolean Dedekind quantale in which only finitary sups and infs are required to exist. Relation algebras are formed over boolean algebras that need not be complete.

Finally we relate the explicit definition of dom and cod in Dedekind quantales with a definition previously used in higher-dimensional rewriting in a Kleene-algebraic structure where infs are not available.

Remark 5.15 The conditions $dom(\alpha) \leq \alpha\alpha^\circ$ and $cod(\alpha) \leq \alpha^\circ\alpha$ have been used for higher-dimensional rewriting with globular 2-Kleene algebras [46], see also Sect. 14. They are consequences of the explicit definition of domain and codomain in Dedekind quantales. In involutive modal quantales, each of them implies the strong Gelfand property, $dom(\alpha) \leq \alpha\alpha^\circ$ is equivalent to $dom(\alpha) = 1 \wedge \alpha\alpha^\circ$ and $cod(\alpha) = 1 \wedge \alpha^\circ\alpha$ is equivalent to $cod(\alpha) \leq \alpha^\circ\alpha$. Yet none of these laws need to hold in such quantales: in the modal distributive Dedekind quantale used in the proof of Lemma 5.12,

$$dom(a) = cod(a) = \top > a = aa^\circ = a^\circ a.$$

6 Higher Catoids

In this section we present our first conceptual contribution: axioms for n-catoids and ω-catoids that generalise definitions of strict n-categories and ω-categories. These are the structures from which we develop axioms for higher quantales in Sect. 7, using the proofs in Sect. 8. Mac Lane has outlined axiomatisations of single-set 2-categories, n-categories and ω-categories, imposing a 2-category structure on each pair of single-set categories C_i and C_j for $0 \leq i < j < \omega$ [21, Chapter XII]. Here, ω-category means strict globular ∞-category. Similar single-set approaches appear, for instance, in [19, 20, 22]. We adapt MacLane's axioms to catoids. We start from a uniform axiomatisation that includes the case of n or ω, but then focus mainly on ω-catoids. As previously, most of the material in this section has been formalised with Isabelle [28], and Isabelle has been instrumental in analysing and reducing this axiomatisation.

For an ordinal $\alpha \in \{0, 1, \ldots, \omega\}$, an α-*catoid* is a structure $(C, \odot_i, s_i, t_i)_{0 \leq i < \alpha}$ such that each $(C, \odot_i, s_i, t_i)$ is a catoid and these structures interact as follows:

- for all $i \neq j$,

$$s_i \circ s_j = s_j \circ s_i, \qquad s_i \circ t_j = t_j \circ s_i, \qquad t_i \circ t_j = t_j \circ t_i,$$
$$s_i(x \odot_j y) \subseteq s_i(x) \odot_j s_i(y), \qquad t_i(x \odot_j y) \subseteq t_i(x) \odot_j t_i(y),$$

- and for all $i < j$,

$$(w \odot_j x) \odot_i (y \odot_j z) \subseteq (w \odot_i y) \odot_j (x \odot_i z),$$
$$s_j \circ s_i = s_i, \qquad s_j \circ t_i = t_i, \qquad t_j \circ s_i = s_i, \qquad t_j \circ t_i = t_i.$$

An α-*category* is a local functional α-catoid, that is, each $(C, \odot_i, s_i, t_i)$ is local and functional.

As the $(C, \odot_i, s_i, t_i)$ are catoids, $s_i(C) = t_i(C)$ for each $i < \alpha$ by Lemma 2.4. We write C_i for this set of *i-cells* of *C*. We also write Δ_i for the domain of definition of $\odot_i$ and refer to the source and target cells of cells as (*lower* and *upper*) *faces*.

The axioms after the first bullet point impose that source and target maps at each dimension *i* are catoid (endo)morphisms of the catoid $(C, \odot_j, s_j, t_j)$ at each dimension $j \neq i$. In the local functional case of α-categories, these morphisms become functors, as expected. (In the ω-category of globular sets, the axioms in the first line are known as *globular laws*.)

The *interchange* axioms in the first line of the second bullet point ensure that $\odot_i$ is a catoid bi-morphism with respect to $\odot_j$, for all $i < j$.

The *whisker* axioms in the second line of this bullet point, together with the catoid laws in Lemma 2.3(1), imply that $s_j(x) = t_j(x) = x$ for all $x \in C_i$, and thus $x \in C_j$ for all $j \geq i$. Lower dimensional cells thus remain units in higher dimensions and

$$C_0 \subseteq C_1 \subseteq C_2 \subseteq \cdots \subseteq C.$$

We may thus regard *i*-cells as degenerate cells or whiskers in which sources and targets at each dimension greater than $i - 1$ coincide. Further, Lemma 2.5 implies that all lower dimensional cells are orthogonal idempotents with respect to higher compositions. For all $i, j \leq k$,

$$s_i(x) \odot_k s_j(y) = \begin{cases} \{s_i(x)\} & \text{if } s_i(x) = s_j(y), \\ \emptyset & \text{otherwise.} \end{cases}$$

All higher-dimensional compositions of lower dimensional cells are therefore trivial. The structure of units across dimensions can be seen in Example 6.6 below.

We now turn to ω-catoids and mention *n*-catoids only occasionally. In this context, one often adds an axiom guaranteeing that for all $x \in C$ there exists and i, ω such that $s_i(x) = x = t_i(x)$, that is all cells in *C* have finite dimension. In the special case of strict ω-categories, for instance, Steiner [22], adds such an axiom, whereas Street [20] and Mac Lane [21] do not feature this condition and Brown and Higgins mention both variants [19]. Our ω-categories, as local functional ω-catoids, are therefore the same as ω-categories in the sense of Street and Mac Lane. We need the above condition in the following construction, but not beyond this paragraph. Because of the whisker axioms and the catoid axioms, the chain $C_0 \hookrightarrow C_1 \hookrightarrow C_2 \hookrightarrow \cdots \hookrightarrow C$ is a filtration of ω-catoids (when $\alpha = \omega$). Catoid C is the (co)limit of an increasing chain of sub-ω-catoids C_n, which themselves are n-catoids. See [22, Proposition 2.3] for a related discussion on ω-categories.

ω-Catoids, form a category in several ways. Their morphisms are catoid morphisms that preserve all source and target maps s_i and t_i and all compositions $\odot_i$ at each dimension $i < \omega$. These kind of categories are also categories with respect to bounded morphisms.

As in the one-dimensional case, ω-catoids and n-catoids generalise ω-categories and n-categories to multioperational compositions, and beyond functionality and locality. Yet the underlying ω-graphs or n-graphs, where compositions are forgotten, are the same for both. The globular cell shape of strict higher categories is therefore present in the corresponding catoids, too.

Lemma 6.1 *In every ω-catoid, the globular laws hold. For all $0 \leq i < j < \omega$,*

$$s_i \circ s_j = s_i, \qquad s_i \circ t_j = s_i, \qquad t_i \circ t_j = t_i, \qquad t_i \circ s_j = t_i.$$

The proofs are immediate from the globular and whisker axioms.

Example 6.2 The globular cell shape of ω-catoids can be visualised, in two dimensions, as

$$\begin{tikzcd} s_0(x) \arrow[r, bend left, "s_1(x)"] \arrow[r, bend right, "t_1(x)"'] \arrow[r, phantom, "\Downarrow x"] & t_0(x) \end{tikzcd}$$

The relationships between 0-cells and 1-cells in this diagram can be calculated using Lemma 6.1.

In light of Example 2.2, the morphism laws of ω-categories are rather strong.

Lemma 6.3 *In every ω-category, for $0 \leq i < j < \omega$, the following strong morphism laws hold:*

$$s_j(x \odot_i y) = s_j(x) \odot_i s_j(y) \qquad \textit{and} \qquad t_j(x \odot_i y) = t_j(x) \odot_i t_j(y).$$

Proof First we derive the law for s_j. Suppose the right-hand side is equal to $\emptyset$. Then $s_j(x \odot_i y) = s_j(x) \odot_i s_j(y)$, using the morphism axiom $s_j(x \odot_i y) \subseteq s_j(x) \odot_i s_i(y)$. Otherwise, if $\Delta_i(s_j(x), s_j(y))$, then $t_i(s_j(x)) = s_i(s_j(x))$ and therefore $t_i(x) = s_i(y)$ by Lemma 6.1. Locality then implies that $\Delta_i(x, y)$, and $s_j(x \odot_i y) = s_j(x) \odot_i s_j(y)$ follows from the morphism axiom for s_j and functionality. The proof for t_j follows by opposition. $\square$

By contrast to the other morphism axioms, which require that if left-hand sides are defined, then so are right-hand sides, the strong morphism laws state that one side is defined if and only if the other is. Accordingly, an ω-catoid is *strong* if it satisfies the strong morphism laws.

Corollary 6.4 *In every ω-category, $\Delta_i(x, y) \Leftrightarrow \Delta_i(s_j(x), s_j(y)) \Leftrightarrow \Delta_i(t_j(x), t_j(y))$ for all $0 \leq i < j < \omega$.*

Example 6.5 The strong morphism laws of ω-categories can be explained diagrammatically for 2-categories and in particular in Cat, the category of all small categories, where $\odot_1$ is the vertical composition of 2-cells or natural transformations, and $\odot_0$ the horizontal one. Lemma 6.3, Corollary 6.4 and locality imply that the horizontal composition $x \odot_0 y$ is defined if and only if the horizontal compositions $s_1(x) \odot_0 s_1(y)$ and $t_1(x) \odot_0 t_1(y)$ are defined. In all cases, this means that $t_0(x) = s_0(y)$.

$$\begin{tikzcd}
s_0(x) \arrow[r, bend left, "s_1(x)"] \arrow[r, bend right, "t_1(x)"'] \arrow[r, phantom, "\Downarrow x"] & t_0(x) \arrow[r, bend left, "s_1(y)"] \arrow[r, bend right, "t_1(y)"'] \arrow[r, phantom, "\Downarrow y"] & t_0(y)
\end{tikzcd}$$

The morphism axioms $s_1(x \odot_0 y) \subseteq s_1(x) \odot_0 s_1(y)$ and $t_1(x \odot_0 y) \subseteq t_1(x) \odot_0 t_1(y)$ then become strong because the compositions $x \odot_0 y$, $s_1(x) \odot_0 s_1(y)$ and $t_1(x) \odot_0 t_1(y)$ are functional, each yields at most a single cell. The upper and lower faces $s_1(x \odot_0 y)$ and $t_1(x \odot_0 y)$ of $x \odot_0 y$ must thus be equal to $s_1(x) \odot_0 s_1(y)$ and $t_1(x) \odot_0 t_1(y)$, respectively.

The next two examples explain the weak morphism laws in the absence of locality or functionality.

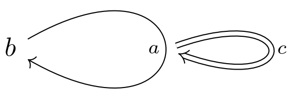
Example 6.6 In the local 2-catoid on the set $\{a, b, c\}$ with

	s_0	t_0	s_1	t_1
a	b	b	a	a
b	b	b	b	b
c	b	b	a	a

$\odot_0$	a	b	c
a	$\{a, b\}$	$\{a\}$	$\{c\}$
b	$\{a\}$	$\{b\}$	$\{c\}$
c	$\{c\}$	$\{c\}$	$\{c\}$

$\odot_1$	a	b	c
a	$\{a\}$	$\emptyset$	$\{c\}$
b	$\emptyset$	$\{b\}$	$\emptyset$
c	$\{c\}$	$\emptyset$	$\{c\}$

we have $s_1(a \odot_0 c) = s_1(\{c\}) = \{a\} \subset \{a, b\} = \{a\} \odot_0 \{a\} = s_1(a) \odot_0 s_1(a)$ and the same inequality holds for t_1 because $s_1 = t_1$. In this example, c is a non-degenerate 2-cell with 1-faces a and 0-faces b, while a is a non-degenerate 1-cell with 0-faces b and b is a 0-cell. So the strong morphism laws fail because $\odot_0$ may map to more than one cell. This degenerate situation can be depicted as



This 2-catoid is functional with respect to $\odot_1$, but not with respect to $\odot_0$.

It is also worth considering the unit structure given by the source and target maps and the whisker axioms, and its effect on $\odot_0$ and $\odot_1$. They determine all compositions except $a \odot_0 a$, $c \odot_0 c$ and $c \odot_1 c$. The composition $\odot_1$ is trivial because of the whisker axioms.

Example 6.7 In the functional 2-catoid on the set $\{a, b\}$ with

	s_0	t_0	s_1	t_1
a	b	b	b	b
b	b	b	b	b

$\odot_0$	a	b
a	$\emptyset$	$\{a\}$
b	$\{a\}$	$\{b\}$

$\odot_1$	a	b
a	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{b\}$

we have $s_1(a \odot_0 a) = s_1(\emptyset) = \emptyset \subset \{b\} = \{b\} \odot_0 \{b\} = s_1(a) \odot s_1(a)$. The corresponding inequality holds for t_1 because $s_1 = t_1$. This example is a monoid as a category with respect to $\odot_1$, and a "broken monoid", hence simply a graph, with respect to $\odot_0$, as $a \odot_0 a$ is undefined. Now, the strong morphism laws fail because the broken monoid is not local: $s_0(a) = b = t_0(a)$, but $a \odot_0 a = \emptyset$. So $x \odot_0 y$ and therefore $s_1(x \odot_0 y)$ may be $\emptyset$, whereas $s_1(x) \odot_0 s_1(y)$ or $t_1(x) \odot_0 t_1(y)$ are not.

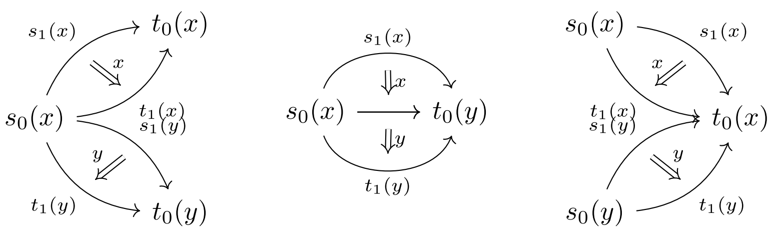
Next we explain the weakness of the remaining morphism axioms. First, we need a lemma.

Lemma 6.8 *In every ω-catoid, for all $0 \leq i < j < \omega$, if $\Delta_j(x, y)$, then*

$$s_i(x \odot_j y) = \{s_i(x)\} = \{s_i(y)\} \qquad \textit{and} \qquad t_i(x \odot_j y) = \{t_i(x)\} = \{t_i(y)\}.$$

A proof can be found in Appendix B.

Example 6.9 In 2-catoids, for instance, weakness of $s_0(x \odot_1 y) \subseteq s_0(x) \odot_1 s_0(y)$ and $t_0(x \odot_1 y) \subseteq t_0(x) \odot_1 t_0(y)$ allows $x \odot_1 y$ to be undefined while both $s_0(x) \odot_1 s_0(y)$ or $t_0(x) \odot_1 t_0(y)$ are defined. The last two compositions are defined if $s_0(x)$ is equal to $s_0(y)$, and $t_0(x)$ is equal to $t_0(y)$, respectively. For the first one, $t_1(x)$ must be equal to $s_1(y)$, from which $s_0(x) = s_0(t_1(x)) = s_0(s_1(y)) = s_0(y)$ and $t_0(x) = t_0(t_1(x)) = t_0(s_1(y)) = t_0(y)$ follow using the globular laws in Lemma 6.1. The left diagram below shows a situation where $s_0(x \odot_1 y) \subset s_0(x) \odot_1 s_0(y)$ because $s_0(x) = s_0(y)$ whereas $t_1(x) \neq s_1(y)$. The right diagram shows the opposite situation were $t_0(x \odot_1 y) \subset t_0(x) \odot_1 t_0(y)$ because $t_0(x) = t_0(y)$ whereas $t_1(x) \neq s_1(y)$. The middle diagram shows a situation where both sides are defined because $t_1(x) = s_1(y)$. The globular structure is imposed by Lemma 6.8.



The weak morphism axioms are thus consistent with vertical compositions of globes.

Example 6.10 In 2-catoids, weakness of $(w \odot_1 x) \odot_0 (y \odot_1 z) \subseteq (w \odot_0 y) \odot_1 (x \odot_0 z)$ allows the right-hand side of the interchange axiom to be defined while the left-hand side is undefined. The right-hand side is defined if $t_1(w \odot_0 y)$ is equal to $s_1(x \odot_0 z)$ and both sets are nonempty. The globular laws in Lemma 6.1 and Lemma 2.6 then imply that $s_0(w) = s_0(w \odot_0 y) = s_0(t_1(w \odot_0 y)) = s_0(s_1(x \odot_0 z)) = s_0(x \odot_0 z) = s_0(x)$, and $t_0(y) = t_0(z)$ holds for similar reasons. The left-hand side is defined if $t_0(w \odot_1 x)$ is equal to $s_0(y \odot_1 z)$ and both sets are nonempty. Therefore $s_0(w) = s_0(w \odot_1 x) = s_0(x)$

and $t_0(y) = t_0(y \odot_1 z) = t_0(z)$, but also $t_0(w) = t_0(w \odot_1 x) = t_0(x)$ and $s_0(y) = s_0(y \odot_1 z) = s_0(y)$ by Lemma 6.8. Thus in particular $t_0(w) = t_0(x) = s_0(y) = s_0(z)$. The right diagram below shows a situation where $(w \odot_1 x) \odot_0 (y \odot_1 z) \subset (w \odot_0 y) \odot_1 (x \odot_0 z)$, because all compositions on the right are defined, but $t_0(w) = s_0(y) \neq t_0(x) = s_0(z)$. The left diagram below shows a situation where both sides are defined.

$$\begin{tikzcd}
s_0(x) \arrow[r] \arrow[r, bend left=60, "s_1(w)"] \arrow[r, bend right=60, "t_1(x)"'] \arrow[r, phantom, shift left=4, "\Downarrow w"] \arrow[r, phantom, shift right=4, "\Downarrow x"] & t_0(x) \arrow[r] \arrow[r, bend left=60, "s_1(y)"] \arrow[r, bend right=60, "t_1(z)"'] \arrow[r, phantom, shift left=4, "\Downarrow y"] \arrow[r, phantom, shift right=4, "\Downarrow z"] & t_0(y)
\end{tikzcd}
\qquad
\begin{tikzcd}
s_0(x) \arrow[r] \arrow[rr, bend left=40, "s_1(w)"] \arrow[r, bend right=60, "t_1(x)"'] \arrow[r, phantom, shift left=5, "\Downarrow w"] \arrow[r, phantom, shift right=4, "\Downarrow x"] & t_0(x) \arrow[r] \arrow[rr, bend right=40, "t_1(z)"'] & s_0(y) \arrow[r] \arrow[r, bend left=60, "s_1(y)"] \arrow[r, phantom, shift left=4, "\Downarrow y"] \arrow[phantom, shift right=5, "\Downarrow z"] & t_0(y)
\end{tikzcd}$$

The interchange axiom is thus consistent with horizontal and vertical compositions of globes. The difference to the standard equational interchange laws of category theory is that, using multioperations, we express partiality by mapping to the empty set.

The following example confirms that the interchange axiom and the morphism axioms for s_0 and t_0 remain inclusions in ω-categories.

Example 6.11 Consider the 2-category with $X = \{a, b\}$ and

	s_0	t_0	s_1	t_1
a	b	b	a	a
b	b	b	b	b

$\odot_0$	a	b
a	$\{b\}$	$\{a\}$
b	$\{a\}$	$\{b\}$

$\odot_1$	a	b
a	$\{a\}$	$\emptyset$
b	$\emptyset$	$\{b\}$

It is actually a monoid as a category with 1-cell a, 0-cell b, and composition $a \odot_0 a = b$, where b is seen as a unit arrow. Further, $\odot_1$ is trivial because of the whisker axioms. Because of this, $(b \odot_1 a) \odot_0 (b \odot_1 a) = \emptyset \subset \{b\} = (b \odot_0 b) \odot_1 (a \odot_0 a)$ as well as $s_0(a \odot_1 b) = \emptyset \subset \{b\} = s_0(a) \odot_1 s_0(b)$ and $t_0(a \odot_1 b) = \emptyset \subset \{b\} = t_0(a) \odot_1 t_0(b)$.

Remark 6.12 The inclusions in the morphism axioms for s_i and t_i cannot be strengthened to equations. We show in Appendix A that this would collapse the entire structure. A similar collapse happens with an equational interchange law.

The α-catoid axioms contain redundancy. We have used Isabelle's SAT-solvers and automated theorem provers to analyse them. For irredundancy of a formula φ with respect to a set X of formulas, we ask the SAT-solvers for a model of $X \cup \{\neg\varphi\}$. For redundancy, we ask the automated theorem provers for a proof of $X \vdash \varphi$. This proofs-and-refutations game often succeeds in practice. Because of the set-up of ω-catoids as pairs of 2-catoids, an analysis of 2-catoids suffices.

Proposition 6.13 *The following α-catoid axioms are irredundant and imply the other α-catoid axioms from the beginning of this section. For all $0 \leq i < j \leq \alpha$,*

$$s_j(x \odot_i y) \subseteq s_j(x) \odot_i s_j(y), \qquad t_j(x \odot_i y) \subseteq t_j(x) \odot_i t_j(y),$$
$$(w \odot_j x) \odot_i (y \odot_j z) \subseteq (w \odot_i y) \odot_j (x \odot_i z).$$

A proof can be found in Appendix B. This reduction is convenient for relating structures, and it streamlines our correspondence proofs below. More generally, the single-set approach makes n-categories accessible to SMT-solvers and first-order automated theorem provers, using Δ and $\hat{\odot}$ in specifications like in Sect. 2.

Remark 6.14 We cannot replace the morphism laws for s_j and t_j by those for s_i and t_i in the reduced axiomatisation for α-catoids. Otherwise the first morphism laws would no longer be derivable: Isabelle produces counterexamples. Likewise, if we use only the whisker axioms and the morphism axioms in the first line of the non-reduced axiomatisation that commute source and target maps, we can neither derive the interchange axiom nor the morphism axioms for s_1 and t_1 in the special case of 2-categories. Isabelle produces once again counterexamples. Coincidentally, the morphism laws for s_0 and t_0 are derivable; proofs can be found in our Isabelle theories.

Example 6.15 Let $(\Sigma^*, \odot_0, \varepsilon)$ denote the free monoid generated by the finite alphabet Σ, with word concatenation $\odot_0$ modelled as a multirelation and the empty word ε. It can be viewed as a category with $s_0(w) = \varepsilon = t_0(w)$ for all $w \in \Sigma^*$. Further, let $(\Sigma^* \odot_1, \varepsilon)$ be the shuffle multimonoid on Σ^* from Example 2.10 with $\odot_1 = \|$. Then $(\Sigma^*, \odot_0, \odot_1, \{\varepsilon\})$ forms a 2-catoid. It has one single 0-cell, $\{\varepsilon\}$, which is also the only 1-cell. The source/target structure is therefore trivial, but an interchange law between word concatenation and word shuffle holds.

7 Higher Quantales

As our main conceptual contribution, we now define the quantalic structures that match higher catoids in Jónsson-Tarski-type correspondences. We start with an axiomatisation of α-quantales, which correspond to α-catoids, but then turn our attention mainly to ω-quantales. Once again we have checked all proofs in this section with Isabelle [28].

An *α-quantale* is a structure $(Q, \leq, \cdot_i, 1_i, dom_i, cod_i)_{0 \leq i < \alpha}$, for an ordinal $\alpha \in \{0, 1, \dots, \omega\}$, such that each $(Q, \leq, \cdot_i, 1_i, dom_i, cod_i)$ is a modal quantale and the structures interact as follows:

- for all $i \neq j$,

$$dom_i(\alpha \cdot_j \beta) \leq dom_i(\alpha) \cdot_j dom_i(\beta) \qquad \text{and} \qquad cod_i(\alpha \cdot_j \beta) \leq cod_i(\alpha) \cdot_j cod_i(\beta),$$

- and for all $i < j$

$$(\alpha \cdot_j \beta) \cdot_i (\gamma \cdot_j \delta) \leq (\alpha \cdot_i \gamma) \cdot_j (\beta \cdot_i \delta) \qquad \text{and} \qquad dom_j(dom_i(\alpha)) = dom_i(\alpha).$$

An α-quantale is *strong* if for all $i < j$,

$$dom_j(\alpha \cdot_i \beta) = dom_j(\alpha) \cdot_i dom_j(\beta) \qquad \text{and} \qquad cod_j(\alpha \cdot_i \beta) = cod_j(\alpha) \cdot_i cod_j(\beta).$$

These axiom systems are already reduced and irredundant in the sense of Sect. 6.

Example 7.1 In the double modal quantale on $\bot < a < \top$ with $\cdot_0 = \cdot_1 = \wedge$, $1_0 = 1_1 = \top$, $dom_0 = id$ and $dom_1(a) = \top = cod_1(a)$(the rest being fixed), the first five globular 2-quantale axioms hold, but the last one does not: $dom_1(dom_0(a)) = \top \neq a = dom_0(a)$.

Irredundancy of the two weak morphism laws is established using similar counterexamples with 5-elements found by Isabelle. Their particular form is of little interest.

Remark 7.2 The strong resemblance of the α-quantale axioms and the α-catoid ones is caused by correspondences that are developed in the following sections. Nevertheless, there is a mismatch: $dom_j \circ dom_i = dom_i$ is an α-quantale axiom while $s_j \circ s_i = s_i$ is derivable in α-catoids and the same holds for the two morphism axioms of α-quantales. For $dom_j \circ dom_i = dom_i$ with $i < j$, this can be explained as follows. Our proof of $s_j \circ s_i = s_i$ relies on $\Delta_k(x, y) \Rightarrow t_k(x) = s_k(y)$, but Remark 4.3 shows that the corresponding property is not available for quantales. The related properties $\alpha \cdot_k \beta \neq \bot \Rightarrow cod_k(\alpha) \wedge dom_k(\beta) \neq \bot$ *are* available, but too weak to translate the proof of $s_j \circ s_i = s_i$ to quantales. Any formal proof is of course ruled out by Example 7.1.

We now turn to ω-quantales.

Lemma 7.3 *In every ω-quantale Q, for $0 \leq i < j < \omega$,*

1. $dom_j \circ cod_i = cod_i$, $cod_j \circ dom_i = dom_i$ and $cod_j \circ cod_i = cod_i$,
2. $1_j \leq 1_j \cdot_i 1_j$, $1_j \cdot_i 1_j = 1_j$ if Q is strong, and $1_i \cdot_j 1_i = 1_i$,
3. $1_i \leq 1_j$,
4. $dom_j(1_i) = 1_i$, $dom_i(1_j) = 1_j$, $cod_j(1_i) = 1_i$ and $cod_i(1_j) = 1_j$,
5. $dom_i \circ dom_j = dom_j \circ dom_i$, $dom_i \circ cod_j = cod_j \circ dom_i$, $cod_i \circ dom_j = dom_j \circ cod_i$ and $cod_i \circ cod_j = cod_j \circ cod_i$,
6. $dom_i(\alpha \cdot_j \beta) = dom_i(\alpha \cdot_j dom_j(\beta))$ and $cod_i(\alpha \cdot_j \beta) = cod_i(cod_j(\alpha) \cdot_j \beta)$.

A proof can be found in Appendix B. By (1), the sets $Q_i = Q_{dom_i}$ form a chain: $Q_0 \subseteq Q_1 \subseteq Q_2 \subseteq \cdots \subseteq Q$. Each Q_i is a complete distributive lattice with $+$ as binary sup and $\odot_i$ as binary inf according to the properties of domain quantales recalled in Sect. 4. Similarly to the situation for ω-catoids in Sect. 6, the elements of Q_i remain domain elements in all higher dimensions, hence each Q_i is a distributive sublattice of Q_j for all $j \geq i$ and of Q. We have

$$dom_i(x) \odot_k dom_j(y) = dom_i(x) \wedge dom_j(y),$$
$$cod_i(x) \odot_k cod_j(y) = cod_i(x) \wedge cod_j(y)$$

for all $i, j \leq k$. At the same time, all truncations Q_i of Q are i-quantales, so that the chain of the Q_i is a filtration of ω-quantales. The ω-quantale Q is the union of the quantales Q_i if we add an axiom guaranteeing that for all $x \in Q$ there exists and $i \leq \omega$ such that

$dom_i(x) = x = cod_i(x)$. As for ω-catoids, we keep this optional and do not require this property in the considerations that follow. The same results hold for strong ω-quantales.

Remark 7.4 An interchange law $(\alpha \cdot_1 \beta) \cdot_0 (\gamma \cdot_1 \delta) \leq (\alpha \cdot_0 \gamma) \cdot_1 (\beta \cdot_0 \delta)$ features in concurrent semirings [47] and concurrent Kleene algebras and quantales [48]. It has often been contrasted with the seemingly equational interchange laws of category theory. Yet this ignores the weak nature of equality in categories, which may depend on definedness conditions of terms, and which is captured explicitly and precisely by the multioperational language. Example 6.11 shows that the interchange laws of ω-categories are as weak as those of ω-quantales and ω-Kleene algebras (defined below), of which concurrent quantales and Kleene algebras are special cases. See Appendix A for pitfalls of strong interchange laws and related morphisms.

Example 7.5 The identity $1_i = 1_j$ need not hold for $i < j$ in strong ω-quantales. There is a strong 2-quantale on $0 < 1_0 < 1_1$ in which $1_1 \cdot_0 1_1 = 1_1$ and $1_0 \cdot_1 1_1 = 1_1$(the rest is fixed), and $dom_0(1_1) = 1_0 = cod_0(1_1)$ and $dom_1(1_0) = 1_0 = cod_1(1_0)$(the rest is again fixed). This makes strong 2-quantales different from the original concurrent quantales mentioned and prevents smaller interchange laws with two or three variables. See [17] for a discussion of how the condition $1_0 = 1_1$ leads to a partial Eckmann-Hilton-style collapse.

Next we consider the interactions of the Kleene stars with the ω-structure.

Lemma 7.6 *In every ω-quantale Q, for $0 \leq i < j < \omega$,*

1. $dom_i(\alpha) \cdot_i \beta^{*_j} \leq (dom_i(\alpha) \cdot_i \beta)^{*_j}$ and $\alpha^{*_j} \cdot_i cod_i(\beta) \leq (\alpha \cdot_i cod_i(\beta))^{*_j}$,
2. $dom_j(\alpha) \cdot_i \beta^{*_j} \leq (dom_j(\alpha) \cdot_i \beta)^{*_j}$ and $\alpha^{*_j} \cdot_i cod_j(\beta) \leq (\alpha \cdot_i cod_j(\beta))^{*_j}$ if Q is strong,
3. $(\alpha \cdot_j \beta)^{*_i} \leq \alpha^{*_i} \cdot_j \beta^{*_i}$.

See Appendix B for proofs. The properties in (1) and (2) feature as axioms of globular n-Kleene algebras in [13]. In sum, all axioms of these n-Kleene algebras have now been derived from our smaller, but slightly different set of axioms for ω-quantales and n-quantales. However, these quantales presuppose that arbitrary joins and meets exist, while globular n-Kleene algebras are based on globular n-semirings, where only finite sups are assumed to exist and meets are not part of the language. See Sects. 13 and 14 for a detailed discussion. We summarise this discussion as follows.

Proposition 7.7 *Every strong ω-quantale is a globular ω-Kleene algebra á la [13, Definition 3.2.7] with Kleene stars $\alpha^{*_j} = \bigvee_{i \geq 0} \alpha^{i_j}$.*

Strictly speaking, only globular n-Kleene algebras are considered in [13], but the axioms for ω are the same.

Finally we list further properties of domains and codomains that are useful below.

Lemma 7.8 *In every ω-quantale, for $0 \leq i < j < \omega$, the following properties hold:*

1. $dom_i(\alpha) \cdot_j dom_i(\alpha) = dom_i(\alpha)$ and $cod_i(\alpha) \cdot_j cod_i(\alpha) = cod_i(\alpha)$,
2. $dom_i(\alpha \cdot_j \beta) = dom_i(\alpha \cdot_j dom_j(\beta))$ and $cod_i(\alpha \cdot_j \beta) = cod_i(cod_j(\alpha) \cdot_j \beta)$,
3. $dom_i(\alpha \cdot_j \beta) = dom_i(cod_j(\alpha) \cdot_j \beta)$ and $cod_i(\alpha \cdot_j \beta) = cod_i(\alpha \cdot_j dom_j(\beta))$,
4. $dom_i(\alpha \cdot_i \beta) = dom_i(\alpha \cdot_i dom_j(\beta))$ and $cod_i(\alpha \cdot_i \beta) = cod_i(cod_j(\alpha) \cdot_i \beta)$,
5. $dom_i(\alpha \cdot_i \beta) \leq dom_i(cod_j(\alpha) \cdot_i \beta)$ and $cod_i(\alpha \cdot_i \beta) \leq cod_i(\alpha \cdot_i dom_j(\beta))$, and equalities hold if Q is strong,

6.
$$dom_i(\alpha) \cdot_i (\beta \cdot_j \gamma) \leq (dom_i(\alpha) \cdot_i \beta) \cdot_j (dom_i\alpha) \cdot_i \gamma),$$
$$(\alpha \cdot_j \beta) \cdot_i dom_i(\gamma) \leq (\alpha \cdot_i dom_i(\gamma)) \cdot_j (\beta \cdot_i dom_i(\gamma)),$$

7. $dom_i(dom_j(\alpha) \cdot_j \beta) \leq dom_i(\alpha) \cdot_j dom_i(\beta)$ and $cod_i(\alpha \cdot_j cod_j(\beta)) \leq cod_i(\alpha) \cdot_j cod_i(\beta)$,
8. $dom_j(dom_i(\alpha) \cdot_j \beta) = dom_i(\alpha) \cdot_j dom_j(\beta)$ and $cod_j(\alpha \cdot_j cod_i(\beta)) = cod_i(\alpha) \cdot_i cod_j(\beta)$,
9. $dom_i(\alpha) \cdot_j dom_i(\beta) = dom_i(\alpha) \cdot_i dom_i(\beta)$ and $cod_i(\alpha) \cdot_j cod_i(\beta) = cod_i(\alpha) \cdot_i cod_i(\beta)$,

10.
$$\begin{aligned}&(dom_i(\alpha) \cdot_j dom_i(\beta)) \cdot_i (dom_i(\gamma) \cdot_j dom_i(\delta))\\ &= (dom_i(\alpha) \cdot_i dom_i(\gamma)) \cdot_j (dom_i(\beta) \cdot_i dom_i(\delta)),\end{aligned}$$
$$(cod_i(\alpha) \cdot_j cod_i(\beta)) \cdot_i (cod_i(\gamma) \cdot_j cod_i(\delta)) = (cod_i(\alpha) \cdot_i cod_i(\gamma)) \cdot_j (cod_i(\beta) \cdot_i cod_i(\delta)).$$

See again Appendix B for proofs. The laws in (2)-(5) are extended locality laws, those in (6) are weak distributivity laws for compositions, those in (7) and (8) extended export laws. Note that export laws $dom(dom(\alpha)\beta) = dom(\alpha)dom(\beta)$ and $cod(\alpha cod(\beta)) = cod(\alpha)cod(\beta)$ hold in any modal semiring and quantale. Finally, the laws in (9) are useful for proving the strong interchange laws in (10).

Example 7.9 The shuffle 2-catoid on Σ in Example 6.15 extends to the shuffle language 2-quantale on Σ under the standard language product

$$XY = \{vw \mid v \in X, w \in Y\}$$

and the shuffle product of languages discussed in Example 4.2. The domain/codomain structures are trivial, as the empty word language $\{\varepsilon\}$ is the joint identity of the two underlying quantales, but an interchange law $(W\|X) \cdot (Y\|Z) \subseteq (W \cdot Y)\|(X \cdot Z)$ at language level can be derived. 2-Quantales satisfying such more restrictive conditions are known as *interchange quantales* [17].

8 Higher Convolution Quantales and their Correspondences

We can now extend the correspondence triangles between local catoids C, modal quantales Q and modal convolution quantales Q^C [15] as well as those for interchange multimonoids, interchange quantales and interchange convolution quantales [17], which we mentioned briefly in the introduction, to local ω-catoids, ω-quantales and convolution ω-quantales, as well as their truncations at dimension *n*. These constitute the main technical contribution in this article. In Sect. 11 we specialise these results to modal powerset ω-quantales, which, at dimension *n*, have applications in higher-dimensional rewriting [13]. We start with a formal summary of the 1-dimensional and 2-dimensional cases considered in [15, 17].

Let *C* be a catoid and *Q* a quantale, henceforth called *value* or *weight quantale*. We write Q^C for the set of functions from *C* to *Q*. We define, for $f, g : C \to Q$, the *convolution operation* $* : Q^C \times Q^C \to Q^C$ as

$$(f * g)(x) = \bigvee_{x \in y \odot z} f(y) \cdot g(z)$$

and the *unit function* $id_0 : C \to Q$ as

$$id_0(x) = \begin{cases} 1 & \text{if } x \in Q_0, \\ \perp & \text{otherwise.} \end{cases}$$

We define $\bigvee F : C \to Q$ by pointwise extension, $(\bigvee F)(x) = \bigvee \{f(x) \mid f \in F\}$ for $F \subseteq Q^C$, and in particular $\perp : C \to Q$ by $\perp(x) = \perp$ for all $x \in C$, overloading notation. We also extend the order on *Q* pointwise to a relation on Q^C. It is consistent with the standard order on the lattice $C \to Q$.

Following [15, 17] we introduce further notation. For any predicate *P* we define

$$[P] = \begin{cases} 1 \text{ if } P, \\ \perp \text{ otherwise,} \end{cases}$$

and then $\delta_x(y) = [x = y]$. Any $f : C \to Q$ can now be written as

$$f = \bigvee_{x \in C} f(x) \cdot \delta_x,$$

where is a module-style action between the "scalars" $f(x) \in Q$ and functions $\delta_x \in Q^C$. More generally, we often write δ_x^α for $\alpha \cdot \delta_x$. Then

$$id_{Q_0} = [e \in Q_0] = \bigvee_{e \in Q_0} \delta_e,$$

$$\bigvee F = \bigvee_{x \in C} \bigvee \{f(x) \mid f \in F\} \cdot \delta_x,$$

$$f \vee g = \bigvee_{x \in C} (f(x) \vee g(x)) \cdot \delta_x.$$

Also, for convolution,

$$(f * g)(x) = \bigvee_{y,z \in C} f(y) \cdot g(z) \cdot [x \in y \odot z],$$
$$f * g = \bigvee_{x,y,z \in C} f(y) \cdot g(z) \cdot [x \in y \odot z] \cdot \delta_x.$$

Finally, whenever Q is a modal quantale, we define $Dom, Cod : Q^C \to Q^C$ as

$$Dom(f) = \bigvee_{x \in C} dom(f(x)) \cdot \delta_{s(x)},$$
$$Cod(f) = \bigvee_{x \in C} cod(f(x)) \cdot \delta_{t(x)}.$$

Correspondence triangles for relational monoids and quantales as well as relational interchange monoids and interchange quantales are already known [17]. They have been extended to local catoids and modal quantales [15]. The resulting 2-out-of-3 laws between catoids C, value algebras Q and convolution algebras Q^C require mild non-degeneracy conditions on C or Q. They have previously been given at a fine level of granularity to explain correspondences between individual laws in C, Q and Q^C. Here we only summarise those results relevant to higher-dimensional extensions.

The following fact translates results for relational monoids and related structures to the setting of catoids.

Theorem 8.1 *[[16, Theorem 4.8], [17, Proposition 16, Corollary 21]]*

1. Let C be a catoid and Q a quantale. Then Q^C is a quantale with the convolution and unit structure defined above.
2. Let X be a set, let Q^X and Q be quantales such that $\bot \neq 1$ and $\alpha \cdot (\beta \cdot \gamma) \neq \bot$ for some $\alpha, \beta, \gamma \in Q$. Then X can be equipped with a catoid structure.
3. Let Q be a complete lattice equipped with a multiplication that preserves arbitrary sups and has a unit. Let Q^C be a quantale and C a catoid such that $C_0 \neq \emptyset$ and $w \in (x \odot y) \odot z$ for some $w, x, y, z \in C$. Then Q is a quantale.

We henceforth refer to quantales on function spaces, such as Q^C, as *convolution quantales*. The two-out-of-three correspondence in Theorem 8.1 is illustrated in Fig. 1 in the introduction.

In the following lemma, *double catoid* refers to a set equipped with two catoid structures that do not interact. Likewise, *double quantale* refers to a complete lattice equipped with two monoidal structures that do not interact. In particular, there are no interchange laws.

Theorem 8.2 *[[17, p, 934]] Let $(C, \cdot_0, s_0, t_0, \cdot_1, s_1, t_1)$ be a double catoid, and further let $(Q, \leq, \cdot_0, 1_0, \cdot_1, 1_1)$ be a double quantale. Let $(Q^C, \leq, *_0, id_0, *_1, id_1)$ be the associated double convolution quantale.*

1. The interchange law holds in Q^C if it holds in C and Q.
2. The interchange law holds in C if it holds in Q and Q^C, and if $(\alpha \cdot_1 \beta) \cdot_0 (\gamma \cdot_1 \delta) \neq \bot$ for some $\alpha, \beta, \gamma, \delta \in Q$.

Fig. 4 Correspondence between interchange catoid C, interchange quantale Q and interchange quantale Q^C from [17]

$$\begin{tikzcd}[arrows=dash] & \text{interchange quantale } Q^C \arrow[dl] \arrow[dr] & \\ \text{interchange catoid } C \arrow[rr] & & \text{interchange quantale } Q \end{tikzcd}$$

Fig. 5 Correspondence between local catoid C, modal quantale Q and modal convolution quantale Q^C

$$\begin{tikzcd}[arrows=dash] & \text{modal quantale } Q^C \arrow[dl] \arrow[dr] & \\ \text{local catoid } C \arrow[rr] & & \text{modal quantale } Q \end{tikzcd}$$

3. The interchange law holds in Q if it holds in C and Q^C, and if $y \in u \odot_1 v$, $z \in w \odot_1 x$ and $\Delta(y, z)$ hold for some $u, v, w, x, y, z \in C$.

Once again we have translated the original statement in [17] from relational monoids to catoids along the isomorphism between these structures. This two-out-of-three correspondence is illustrated in Fig. 4. Double quantales in which the interchange law holds are known as *interchange quantales*. The results in [17] establish in fact correspondences between 2-catoids and interchange quantales, but only the interchange law of the double catoid contributes to the interchange law in the double quantale and vice versa.

Theorem 8.3 *[[15, Theorems 7.1, 8.4, 8.5]] Let $(C, \odot, s, t)$ be a catoid and $(Q, \leq, \cdot, 1)$ be a quantale. Let $(Q^C, \leq, *, id_0)$ be the associated convolution quantale.*

1. Q^C is modal if C is local and Q modal.
2. C is local if Q and Q^C are modal, and if $1 \neq \bot$ in Q.
3. Q is modal if C is local and Q^C modal, and if $\Delta(\ell(x), r(y))$ and $\Delta(z, w)$ for some $w, x, y, z \in C$.

This two-out-of-three correspondence is illustrated in Fig. 5.

Example 8.4 ([15, 17]) Theorems 8.2 and 8.3 have the following instances that generalise Example 4.2. Let Q be a modal value quantale.

1. Every category C extends to a modal convolution quantale Q^C. It is similar to a category algebra, using a quantale as value algebra instead of a ring or field and extending the source and target structure of the category, which category algebras ignore.
2. The modal powerset quantales from Example 4.2 are convolution quantales with $Q = 2$.
3. Every path category over a digraph extends to a modal convolution quantale of Q-valued paths.
4. Every pair groupoid extends to a modal convolution quantale of Q-valued relation.

5. Every shuffle 2-catoid extends to a convolution 2-quantale of Q-weighted shuffle languages if Q is a 2-quantale.

We now extend the constructions in the proofs of Theorems 8.2 and 8.3 to proofs of 2-out-of-3 correspondence triangles between ω-catoids and ω-quantales, as shown in the diagram on the left of Fig. 2. Given the proofs in these theorems it remains to consider the globular structure. Theorem 8.2 guarantees a 2-out-of-3 correspondence between the interchange laws in ω-catoids and ω-quantales, while Theorem 8.3 supplies such a correspondence between the source and target structure in ω-catoids and the domain and codomain structure in ω-quantales in each dimension. Locality of ω-catoids, in particular, is needed to reflect locality of the domain and codomain structure. This, in turn, is needed for defining modal operators as actions in Sect. 12.

In the ω-setting, our notation for $[P]$ or δ-functions requires dimension indices, strictly speaking. But in practice, we can always pick and swap such indices. Hence we usually suppress them, as will become clear in the proofs below.

Theorem 8.5 *Let C be a local ω-catoid and Q an ω-quantale. Then Q^C is an ω-quantale.*

Proof In light of the proof of Theorems 8.2 and 8.3 it remains to extend the morphism axioms as well as the axiom $Dom_j \circ Dom_i = Dom_i$ for $0 \leq i < j < \omega$. Although the proof of the interchange law is covered, for 2-catoids and 2-quantales, by Theorem 8.2, we state it explicitly for the case of ω. We omit indices related to square brackets as mentioned. For $0 \leq i < j < \omega$,

$$
\begin{aligned}
&((f *_j g) *_i (h *_j k))(x)\\
&= \bigvee_{y,z} (f *_j g)(y) \cdot_i (h *_j k)(z) \cdot [x \in y \odot_i z]\\
&= \bigvee_{y,z} \left(\bigvee_{t,u} f(t) \cdot_j g(u) \cdot [y \in t \odot_j u] \right) \cdot_i \left(\bigvee_{v,w} h(v) \cdot_j k(w) \cdot [z \in v \odot_j w] \right) \cdot [x \in y \odot_i z]\\
&= \bigvee_{t,u,v,w} (f(t) \cdot_j g(u)) \cdot_i (h(v) \cdot_j k(w)) \cdot [x \in (t \odot_j u) \odot_i (v \odot_j w)]\\
&\leq \bigvee_{t,u,v,w} (f(t) \cdot_i h(v)) \cdot_j (g(u) \cdot_i k(w)) \cdot [x \in (t \odot_i v) \odot_j (u \odot_i w)]\\
&= \bigvee_{y,z} \left(\bigvee_{t,v} f(t) \cdot_i h(v) \cdot [y \in t \odot_i v] \right) \cdot_j \left(\bigvee_{u,w} g(u) \cdot_i k(w) \cdot [z \in u \odot_i w] \right) \cdot [x \in y \odot_j z]\\
&= ((f *_i h) *_j (g *_i k))(x).
\end{aligned}
$$

For the morphism axiom $Dom_j(f *_i g) \leq Dom_j(f) *_i Dom_j(g)$,

$$\begin{aligned}
&Dom_j(f *_i g)(x)\\
&= \bigvee_u dom_j\left(\bigvee_{v,w} f(v) \cdot_i g(w)) \cdot [u \in v \odot_i w]\right) \cdot \delta_{s_j(u)}(x)\\
&= dom_j\left(\bigvee_{v,w} f(v) \cdot_i g(w)\right) \cdot [x \in s_j(v \odot_i w)]\\
&= \bigvee_{v,w} dom_j(f(v) \cdot_i g(w)) \cdot [x \in s_j(v \odot_i w)]\\
&\leq \bigvee_{v,w} dom_j(f(v)) \cdot_i dom_j(g(w)) \cdot [x \in s_j(v) \odot_i s_j(w)]\\
&= \bigvee_{t,u}\left(\bigvee_v dom_j(f(v)) \cdot_j \delta_{s_j(v)}(t)\right) \cdot_i \left(\bigvee_w dom_j(g(w)) \cdot_j \delta_{s_j(w)}(u)\right) \cdot [x \in t \odot_i u]\\
&= (Dom_j(f) *_i Dom_j(g))(x).
\end{aligned}$$

The proof of $Cod_j(f *_i g) \leq Cod_j(f) *_i Cod_j(g)$ follows by opposition.

The proofs of

$$Dom_i(f *_j g) \leq Dom_i(f) *_j Dom_i(g),$$
$$Cod_i(f *_j g) \leq Cod_i(f) *_j Cod_i(g)$$

are obtained by re-indexing these proofs.

Finally, for $Dom_j \circ Dom_i = Dom_i$,

$$\begin{aligned}
Dom_j(Dom_i(f)) &= \bigvee_u dom_j\left(\bigvee_v dom_i(f(v)) \cdot \delta_{s_i(v)}(u)\right) \cdot \delta_{s_j(u)}\\
&= \bigvee_v dom_j(dom_i(f(v))) \cdot \delta_{s_j(s_i(v))}\\
&= \bigvee_v dom_i(f(v)) \cdot \delta_{s_i(v)}\\
&= Dom_i(f).
\end{aligned}$$

□

For strong ω-catoids, and hence ω-categories, we obtain a stronger result.

Corollary 8.6 *Let C be a strong local ω-catoid and Q a strong ω-quantale. Then Q^C is a strong ω-quantale.*

Proof It suffices to replay the proofs for the two strong morphism laws for Dom_j and Cod_j with an equality step in the fourth proof step for Dom_j above. The proof for Cod_j follows by opposition. □

For the following two theorems we tacitly assume the non-degeneracies needed for Theorem 8.2 and 8.3, calling the respective structures *sufficiently supported*, and mention those on C and Q explicitly in the proof. See [17] for the general construction and more detailed explanations of the mechanics of proofs. First we recall three properties from [15, Lemma 8.2] that generalise readily beyond one dimension.

By analogy to the double catoids and quantales considered in Theorem 8.2, we define an ω-fold catoid as a set equipped with ω catoid structures which do not interact. Likewise, an *ω-fold quantale* is a complete lattice equipped with ω sup-preserving monoidal structures which do not interact.

Lemma 8.7 *Let C be an ω-fold catoid and Q be an ω-fold modal quantale, all without globular structure. Let Q^C be the associated ω-fold modal quantale. Then, for $i \neq j$,*

1. $Dom_i(\delta_x^\alpha) = \bigvee_y dom_i(\delta_x^\alpha(y))\delta_{s_i(y)} = dom_i(\alpha) \cdot \delta_{s_i(x)}$,
2. $Dom_i(\delta_x^\alpha *_j \delta_y^\beta)(z) = dom_i(\alpha \cdot_j \beta) \cdot_i [z \in s(x \odot_j y)]$,
3. $(Dom_i(\delta_x^\alpha) *_j Dom_i(\delta_y^\beta))(z) = dom_i(\alpha) \cdot_j dom_i(\beta) \cdot_j [z \in s_i(x) \odot_j s_i(y)]$,
4. $(\delta_x^\alpha * \delta_y^\beta)(z) = \alpha \cdot \beta \cdot [z \in x \odot y]$.

Theorem 8.8 *Let X be a set, let Q^X and Q be ω-quantales with Q sufficiently supported. Then X can be equipped with a local ω-catoid structure.*

Proof Given Theorems 8.2 and 8.3 we need to check the homomorphism axioms. As in Theorem 8.5, all proofs are similar and we show only one. Suppose

$$Dom_i(\delta_x^\alpha *_j \delta_y^\beta)(z) \leq (Dom_i((\delta_x^\alpha) *_j Dom_i(\delta_y^\beta))(z)$$

holds in Q^C and $dom_i(\alpha *_j \beta) \leq dom_i(\alpha) *_j dom_i(\beta)$ in Q. Suppose also, for non-degeneracy, that $dom_i(\alpha \cdot_j \beta) \neq \bot$. Then $s_i(x \odot_j y) \leq s_i(x) \odot_j s_i(y)$ using Lemma 8.7(2) and (3). $\square$

Once again we get stronger results for strong quantales. The proofs are obvious.

Corollary 8.9 *Let Q^C and Q be strong ω-quantales with Q sufficiently supported. Then C is a strong local ω-catoid.*

Additional assumptions are needed to obtain ω-categories. We do not explain them in this article.

Theorem 8.10 *Let Q be a complete lattice equipped with a multiplication that preserves arbitrary sups and has a unit. Let Q^C be an ω-quantale and C a local ω-catoid that is sufficiently supported. Then Q is an ω-quantale.*

Proof Given Theorems 8.2 and 8.3 we need to check the homomorphism axioms and $dom_j \circ dom_i = dom_i$. As in Theorem 8.5, all proofs of homomorphism axioms are similar and we show only one. Suppose $Dom_i(\delta_x^\alpha *_j \delta_y^\beta)(z) \leq (Dom_i(\delta_x^\alpha) *_j Dom_i(\delta_y^\beta))(z)$ holds in Q^C and $s_i(x \odot_j y) \leq s_i(x) \odot_j s_i(y)$ in C. Suppose also, for non-degeneracy, that $z \in s_i(x) \odot_j s_i(y)$. Then, using Lemma 8.7(2) and (3),

$$
\begin{aligned}
dom_i(\alpha \cdot_j \beta) &= Dom_i(\delta_x^\alpha *_j \delta_y^\beta)(z) \\
&\leq (Dom_i(\delta_x^\alpha) *_j Dom_i(\delta_y^\beta))(z) \\
&= dom_i(\alpha) \cdot_j dom_i(\beta).
\end{aligned}
$$

Finally, using Lemma 8.7(1),

$$
dom_j(dom_i(\alpha)) = Dom_j(Dom_i(\delta_x^\alpha))(s_j(s_i(x))) = Dom_i(\delta_x^\alpha)(s_i(x)) = dom_i(\alpha).
$$

□

The two-out-of-three correspondence captured by Theorems 8.5, 8.8 and 8.8 is depicted in the left diagram of Fig. 2 in the introduction.

Corollary 8.11 *Let Q^C be a strong ω-quantale and C a strong local ω-catoid that is sufficiently supported. Then Q is a strong ω-quantale.*

Example 8.12

1. The category Cat extends to a convolution 2-quantale Q^{Cat} for every value 2-quantale Q.
2. The ω-category of globular sets with the standard globular compositions extends to a convolution ω-quantale for every value ω-quantale Q.

Applications of the powerset case $Q = 2$ relevant to higher-dimensional rewriting are discussed in Sects. 11, 14 and 10. A convolution 2-quantale relevant to concurrency theory based on weighted languages of isomorphism classes of labelled posets, equipped with a so-called serial and a parallel composition is discussed in [17]. In this case, the only unit is the empty poset, and the domain/codomain structure is trivial, as for weighted languages with shuffle.

9 Dedekind Convolution Quantales and their Correspondences

In this section we study correspondence triangles between groupoids, Dedekind value quantales and Dedekind convolution quantales, adapting the correspondence triangles between groupoids and relation algebras established by Jónsson and Tarski [23, Sect. 5].

We define, for every $f : C \to Q$ from a groupoid C into an involutive quantale Q,

$$
f^\circ(x) = (f(x^-))^\circ.
$$

Alternatively, we write $f^\circ = \bigvee_{x \in C} (f(x))^\circ \cdot \delta_{x^-}$.

Proposition 9.1 *Let C be a groupoid and Q an involutive quantale. Then Q^C is an involutive quantale.*

Proof Given Theorem 8.1 it remains to check the three involution axioms.

For sup-preservation,

$$
\begin{aligned}
(\bigvee F)^\circ &= \bigvee_x ((\bigvee F)(x))^\circ \cdot \delta_{x^-} \\
&= \bigvee_x ((\bigvee \{f(x) \mid f \in F\})^\circ \cdot \delta_{x^-} \\
&= \bigvee \{\bigvee_x f(x)^\circ \cdot \delta_{x^-} \mid f \in F\} \\
&= \bigvee \{f^\circ \mid f \in F\}.
\end{aligned}
$$

For involution proper,

$$
\begin{aligned}
f^{\circ\circ} &= \bigvee_x \left(\bigvee_y (f(y))^\circ \cdot \delta_{y^-}(x)\right)^\circ \cdot \delta_{x^-} \\
&= \left(\bigvee_y (f(y))^\circ \cdot \delta_{y^{--}}\right)^\circ \\
&= \bigvee_y (f(y))^{\circ\circ} \cdot \delta_{y^{--}} \\
&= \bigvee_y (f(y)) \cdot \delta_y \\
&= f.
\end{aligned}
$$

For contravariance,

$$
\begin{aligned}
(f * g)^\circ &= \bigvee_x \left(\bigvee_{y,z} f(y) \cdot g(z)\right)^\circ \cdot [x \in y \odot z] \cdot \delta_{x^-} \\
&= \bigvee_{x,y,z} (f(y) \cdot g(z))^\circ \cdot [x \in (y \odot z)^-] \cdot \delta_x \\
&= \bigvee_{x,y,z} g(z)^\circ \cdot f(y)^\circ \cdot [x \in z^- \odot y^-] \cdot \delta_x \\
&= \bigvee_{x,y,z} g(z^-)^\circ \cdot f(y^-)^\circ \cdot [x \in z \odot y] \cdot \delta_x \\
&= \bigvee_x \left(\bigvee_{z,y} g^\circ(z) \cdot f^\circ(y) \cdot [x \in z \odot y]\right) \cdot \delta_x \\
&= \bigvee_x (g^\circ * f^\circ)(x) \cdot \delta_x \\
&= (g^\circ * f^\circ).
\end{aligned}
$$

□

Theorem 9.2 *Let C be a groupoid and Q a Dedekind quantale in which binary inf distributes over all sups. Then Q^C is a Dedekind quantale (in which binary inf distributes over all sups).*

Proof

$$\begin{aligned}
f * g \wedge h &= \left(\left(\bigvee_{y,z} f(y) \cdot g(z) \cdot [x \in y \odot z]\right) \wedge h(x)\right) \cdot \delta_x \\
&= \bigvee_{x,y,z} (f(y) \cdot g(z) \wedge h(x)) \cdot [x \in y \odot z] \cdot \delta_x \\
&\leq \bigvee_{y,z} (f(y) \wedge h(x) \cdot g(z)^\circ) \cdot g(z) \cdot [x \in y \odot z] \\
&= \bigvee_{x,y,z} (f(y) \wedge h(x) \cdot g^\circ(z^-)) \cdot g(z) \cdot [x \in y \odot z] \cdot \delta_x \\
&\leq \bigvee_{x,y,z} \left(f(y) \wedge \bigvee_{v,w} h(v) \cdot g^\circ(w) \cdot [y \in v \odot w]\right) \cdot g(z) \cdot [x \in y \odot z] \cdot \delta_x \\
&= \bigvee_{x,y,z} (f(y) \wedge (h * g^\circ)(y)) \cdot g(z) \cdot [x \in y \odot z] \cdot \delta_x \\
&= \bigvee_{x,y,z} (f \wedge h * g^\circ)(y) \cdot g(z) \cdot [x \in y \odot z] \cdot \delta_x \\
&= (f \wedge h * g^\circ) * g.
\end{aligned}$$

The distributivity law is used in the first step of the proof. The sixth step works because $x \in y \odot z$ if and only if $y \in x \odot z^-$ by Lemma 3.2(2), so that the pair (x, z^-) is considered in the sup introduced. □

The condition that binary infs distribute over all sups is well known from frames or locales, that is, complete Heyting algebras. It holds in every power set Dedekind quantale.

Theorem 9.3 *Let X be a set, let Q^X and Q be Dedekind quantales such that $\perp \neq 1$ in Q. Then X can be equipped with a groupoid structure.*

Proof We show that $x \odot x^- = \{s(x)\}$. Suppose $Dom(\delta_x^\alpha) \leq \delta_x^\alpha * (\delta_x^\alpha)^\circ$ holds in Q^C and $dom(\alpha) \leq \alpha \cdot \alpha^\circ$ holds in Q. Suppose also for non-degeneracy that $dom(\alpha) \neq \perp$; we can pick $\alpha = 1$ and simply require $\perp \neq 1$. By Lemma 8.7(1) and (4),

$$\delta_{s(x)}(y) = Dom(\delta_x)(y) \leq (\delta_x * (\delta_x)^\circ)(y) = [y \in x \odot x^-].$$

Thus $s(x) \in x \odot x^-$. Moreover, $\{s(x)\} = x \odot x^-$ because, in fact, $Dom(\delta_x) = \delta_x * (\delta_x)^\circ$, that is, δ_x is functional. □

Theorem 9.4 *Let Q be a complete lattice with a binary multiplication that preserves sups in both arguments and has a unit. Let Q^C be a Dedekind quantale and C a groupoid. Then Q is a Dedekind quantale.*

Proof We start with the involutive quantale axioms. First, suppose $f^{\circ\circ} = f$ in Q^C and $x^{--} = x$ in X. Then $\alpha^{\circ\circ} = (\delta_x^\alpha)^{\circ\circ}(x^{--}) = \delta_x^\alpha(x) = \alpha$.

Second, suppose $(\bigvee\{\delta_x^\alpha \mid \alpha \in A)^\circ = \bigvee\{(\delta_x^\alpha)^\circ \mid \alpha \in A\}$ in Q^X. Then

$$(\bigvee A)^\circ = (\bigvee\{\delta_x^\alpha(x^-) \mid \alpha \in A\})^\circ = \bigvee\{(\delta_x^\alpha)^\circ(x^-) \mid \alpha \in A\} = \bigvee\{\alpha^\circ \mid \alpha \in A\}.$$

Third, suppose $(\delta_y^\alpha * \delta_z^\beta)^\circ = (\delta_z^\beta)^\circ * (\delta_y^\alpha)^\circ$ in Q^C and $(y \odot z)^- = z^- \odot y^-$ in C. Also assume $z \in x \odot y$ for non-degeneracy. Then

$$(\alpha \cdot \beta)^\circ = (\delta_y^\alpha * \delta_z^\beta)^\circ(x^-) = ((\delta_y^\beta)^\circ * (\delta_z^\alpha)^\circ))(x^-) = \beta^\circ \cdot \alpha^\circ.$$

Finally, for the modular law, suppose $\delta_u^\alpha * \delta_v^\beta \wedge \delta_w^\gamma \leq (\delta_u^\alpha \wedge \delta_w^\gamma * (\delta_v^\beta)^\circ) * \delta_v^\beta$. Assume, for non-degeneracy, that $x \in u \odot v$. Then

$$\begin{aligned}
\alpha \cdot \beta \wedge \gamma &= \alpha \cdot \beta \cdot [x \in u \odot v] \wedge \gamma \cdot \delta_x(x) \\
&= (\delta_u^\alpha * \delta_v^\beta \wedge \delta_x^\gamma)(x) \\
&\leq ((\delta_u^\alpha \wedge \delta_x^\gamma * (\delta_v^\beta)^\circ) * \delta_v^\beta)(x) \\
&= (\alpha \wedge \gamma \cdot \beta^\circ \cdot [u \in x \odot v^-]) \cdot \beta \cdot [x \in u \odot v] \\
&= (\alpha \wedge \gamma \cdot \beta^-) \cdot \beta,
\end{aligned}$$

where $x \in u \odot v \Leftrightarrow u \in x \odot v^-$, which holds by Lemma 3.2(2), is used in the fourth step. □

It is straightforward to check that Q carries a complete Heyting structure if Q^C does. Theorem 9.2 and 9.4 clearly depend on the quantales Q or Q^C in their hypotheses being Dedekind, while Theorem 9.3 uses slightly more general properties of the shape $dom(x) \leq xx^\circ$ and $cod(x) \leq x^\circ x$, which are further discussed in the context of ω-semirings and ω-Kleene algebras in Sect. 14.

The two-out-of-three correspondence captured by Theorems 9.2, 9.3 and 9.4 is illustrated by the left diagram of Fig. 3 in the introduction.

Example 9.5 Example 5.5 generalises from powerset quantales to convolution quantales. Let Q be a Dedekind value quantale.

1. Every free groupoid extends to a modal Q-valued path Dedekind quantale in which formal inverses extend to converses.
2. Every pair groupoid extends to a modal Dedekind quantale of Q-valued relations.

10 (ω, p)-Catoids and (ω, p)-Quantales

In this section we combine the structures of ω-catoids and ω-quantales from Sects. 6, 7 and 8 with the structures of groupoids and Dedekind quantales from Sects. 3, 5 and 9 to define the corresponding (ω, p)-structures in higher-dimensional rewriting theory. We also show how 2-out-of-3 correspondence triangles for the (ω, p)-structures can be obtained from those of their component structures, see also Fig. 2 in the introduction. We start from ω-catoids which have a groupoid structure above some dimension $p < \omega$. These (ω, p)-catoids and the corresponding (ω, p)-quantales, which are Dedekind quantales above dimension p, are suitable for defining higher homotopies.

An (ω, p)-*catoid* is an ω-catoid C with operations $(-)^{-i} : C \to C$ for all $p < i \leq \omega$ such that $(x)^{-i} \in C_i$ for all $x \in C_i$ and the groupoid axioms hold, for all $p < i \leq \omega$ and $x \in C$:

$$x \odot_{i-1} x^{-i} = \{s_{i-1}(x)\} \quad \text{and} \quad x^{-i} \odot_{i-1} x = \{t_{i-1}(x)\}.$$

An (ω, p)-*category* is a local functional (ω, p)-catoid.

Likewise, an (ω, p)-*quantale* is an ω-quantale Q with operations $(-)^{\circ_i} : Q \to Q$ for all $p < i \leq \omega$ such that $(\alpha)^{\circ_j} \in Q_i$ for all $\alpha \in Q_i$ and that satisfy the involution axioms and the modular law with respect to the composition $\cdot_{i-1}$. Strong (ω, p)-quantales are defined as for higher quantales in Sect. 7.

All inverses in (ω, p)-catoids and all converses in (ω, p)-quantales are trivial on elements of lower dimensions. It follows from Lemma 3.1 that $x^{-j} = x$ for all $x \in C_i$ with $i < j$. Similarly, it follows from Lemma 5.11 that $\alpha^{\circ_j} = \alpha$ for all $\alpha \in Q_i$ with $i < j$.

The correspondence results from Sects. 8 and 9 can then be combined. We use the notion of sufficient support as for ω-catoids, ω-groupoids and ω-quantales defined in these sections.

Theorem 10.1

1. Let C be a local (ω, p)-catoid and Q an (ω, p)-quantale in which binary inf distributes over all sups. Then Q^C is an (ω, p)-quantale.
2. Let X be a set, let Q^X be an (ω, p)-quantale and C a local (ω, p)-catoid that is sufficiently supported. Then X can be equipped with an (ω, p)-catoid structure.
3. Let Q be a complete lattice equipped with a multiplication that preserves arbitrary sups and has a unit. Let Q^C and Q be (ω, p)-quantales with Q sufficiently supported. Then C is a local (ω, p)-quantale.

These results specialise to correspondence triangles for strong (ω, p)-catoids and strong (ω, p)-quantales as usual. The complete Heyting structure can be used as part of the definition of (ω, p)-quantale. Further, we obtain correspondences for powerset quantales as instances.

Corollary 10.2

1. Let C be a local (ω, p)-catoid. Then $\mathcal{P}C$ is an (ω, p)-quantale.
2. Let X be a set and $\mathcal{P}C$ an (ω, p)-quantale in which $id_i \neq \emptyset$ and the atoms are functional. Then X can be equipped with a local (ω, p)-catoid structure.

Again there are special cases for strong structures, and Proposition 7.7 can be extended to show that every strong (ω, p)-quantale is a globular (ω, p)-Kleene algebra á la [13, Definition 4.4.2]. The laws of converse used in the definition of globular (ω, p)-Kleene algebras are subsumed by the laws of of Dedekind quantales, as shown in Sect. 5; see Sects. 13 and 14 for further discussion of globular Kleene algebras.

11 Correspondences for Powerset Quantales

The correspondence triangles of Sect. 8 and 9 specialise to powerset algebras for the value quantale $Q = 2$ of booleans (Example 4.1), using the isomorphism between the map $C \to 2$ and $\mathcal{P}C$. We have already seen several examples of powerset extensions in previous sections. Here we give standalone correspondence proofs for the globular and the converse structure, because they are interesting for higher-dimensional rewriting. Most of the results in this sections have also been checked with Isabelle [28].

The first corollary is an immediate instance of Theorems 8.5 and 8.8.

Corollary 11.1

1. Let C be a local ω-catoid. Then $(\mathcal{P}C, \subseteq, \odot_i, C_i, s_i, t_i)_{0 \leq i < \omega}$ is an ω-quantale.
2. Let X be a set and $\mathcal{P}X$ be an ω-quantale in which $id_i \neq \emptyset$. Then X can be equipped with a local ω-catoid structure.
3. If C is strong, then $\mathcal{P}C$ is strong and vice versa.

We present a set-theoretic proof in Appendix B because it might provide additional intuition. We emphasise the one-to-one relationship between laws of ω-catoids and those of ω-quantales in the proofs in Appendix B to highlight the underlying correspondence, including the proof of this corollary. The extension from C to a modal quantale $\mathcal{P}C$ has been described in Example 4.2, see also [15, Theorem 6.5]. In the converse direction, one can recover a catoid from the atom structure of the powerset structure; its singleton sets, as discussed in the introduction: $x \in y \odot z \Leftrightarrow \{x\} \subseteq \{y\} * \{z\}$, while source and target maps correspond to domain and codomain maps on singleton sets. Note that the relation between $\odot$ and $*$ fixes also $\subseteq$ as the order on the powerset quantale, and it is compatible with the standard definition of a lattice order in terms of sups and infs.

Remark 11.2 A classical result by Gautam [49] shows that equations extend to the powerset level if each variable in an equation occurs precisely once in each side. These results have later been generalised by Grätzer and Whitney [50] (see also [51] for an overview). It is therefore no surprise that all the (unreduced) axioms of ω-catoids extend directly to corresponding properties, which we have already derived from the ω-quantale axioms in Lemma 7.3. Nevertheless Gautam's result does not prima facie cover multioperations, let alone constructions of convolution algebras.

Next we show how groupoids extend to Dedekind quantales and relation algebras. This mainly reproduces Jónsson and Tarski's results [23] with groupoids based on catoids. In addition, we provide explicit extensions for *dom* and *cod*, which can otherwise be derived in powerset quantales or relation algebras. In any groupoid C, we write X^- for $X \subseteq C$. First we list some properties that are not directly related to the extension.

Lemma 11.3 *Let C be a groupoid. For all $X, Y \in \mathcal{P}(C)$ and all $\mathcal{X} \in \mathcal{P}(\mathcal{P}(C))$,*

1. $(\bigcup \mathcal{X})^- = \bigcup\{X^- \mid X \in \mathcal{X}\}$ and $(X \cup Y)^- = X^- \cup X^-$,
2. $s(X) \subseteq XX^-$ and $t(X) \subseteq X^-X$,
3. $X \subseteq XX^-X$,
4. $s(X) = C_0 \cap XX^-$ and $t(X) = C_0 \cap X^-X$,
5. $s(X) = C_0 \cap X\top$ and $r(X) = C_0 \cap \top X$,
6. $X\top = s(X)\top$ and $\top X = \top t(X)$.

A proof can be found in Appendix B. Properties (2) and (3) are of course the conditions for involutive quantales from Sect. 5. Next we specialise the results of Sect. 9, revisiting and Example 5.5 more formally and extending it to a correspondence.

Corollary 11.4

1. Let C be a groupoid. Then $(\mathcal{P}C, \odot, s, t, (-)^-)$ is a Dedekind quantale.
2. Let X be a set and $\mathcal{P}X$ be a Dedekind quantale in which $id_0 \neq \emptyset$ and the atoms are functional. Then X can be equipped with a groupoid structure.

As mentioned in Example 5.5, the proofs are due to Jónsson and Tarski; we have also checked (1) with Isabelle. The functionality requirement on atoms in (2), in particular, has been noticed and used by Jónsson and Tarski.

Remark 11.5 Jónsson and Tarski have considered relation algebras based on boolean algebras instead of complete lattices. They also use the residual law mentioned in Sect. 5 instead of the modular or Dedekind law. This makes no difference. See [44, 45] for details.

Examples of powerset extensions for specific groupoids have been given in Example 5.5. This section provides a formal development that subsumes these results.

12 Modal Operators and their Laws

We have already mentioned that modal semirings and modal quantales carry their name because modal operators, akin to those of modal logics, can be defined and related using the domain and codomain structure. As already mentioned, the main application of the higher quantales introduced in Sect. 7 and the Dedekind quantales studied in Sect. 5 are proofs in higher-dimensional rewriting [13]. These require the modal operators that can be defined on these quantales. In this section, we briefly recall these modal structures and present some new modal laws that hold in higher quantales. The entire content of this section has been formalised with Isabelle [27, 28].

Modal diamond operators can be defined in modal semirings or quantales [4, 18] as

$$|\alpha\rangle\beta = dom(\alpha\beta) \qquad \text{and} \qquad \langle\alpha|\beta = cod(\beta\alpha).$$

By domain and codomain locality, $|\alpha\rangle\beta = |\alpha\rangle dom(\beta)$ and $\langle\alpha|\beta = \langle\alpha|dom(\beta)$. Accordingly, we mostly use those modal operators with β a domain element. See [15, 18] for additional properties for modal operators on quantales. In particular, we can "demodalise" diamonds at left-hand sides of inequalities: for $p, q \in Q_0$,

$$|\alpha\rangle p \leq q \Leftrightarrow \alpha p \leq q\alpha \qquad \text{and} \qquad \langle\alpha|p \leq q \Leftrightarrow p\alpha \leq \alpha q.$$

In boolean modal quantales, and hence in modal powerset quantales, we can define forward and backward modal box operators

$$[\alpha|p = \bigvee\{q \mid |\alpha\rangle q \leq p\} \qquad \text{and} \qquad |\alpha]p = \bigvee\{q \mid \langle\alpha|q \leq p\},$$

for all $\alpha \in Q$ and $p \in Q_0$. (In arbitrary modal quantales, we cannot guarantee that the sups, which feature in the definienda, are again elements of Q_0.)

The modal box and diamond operators are then adjoints in the Galois connections

$$|\alpha\rangle p \leq q \Leftrightarrow p \leq |\alpha]q \qquad \text{and} \qquad \langle\alpha|p \leq q \Leftrightarrow p \leq [\alpha|q,$$

for all $\alpha \in Q$ and $p, q \in Q_0$. The additional demodalisation laws $p \leq [\alpha|q \Leftrightarrow \alpha p \leq q\alpha$ and $p \leq |\alpha]q \Leftrightarrow p\alpha \leq \alpha q$ are helpful in deriving them. Further, in boolean modal quantales, boxes and diamonds are related by De Morgan duality,

$$|\alpha]p = -|\alpha\rangle - p, \qquad |\alpha\rangle p = -|\alpha] - p, \qquad [\alpha|p = -\langle\alpha| - p, \qquad \langle\alpha|p = -[\alpha| - p.$$

Finally, in any modal Dedekind quantale, $\langle\alpha| = |\alpha^\circ\rangle$, $|\alpha\rangle = \langle\alpha^\circ|$, $[\alpha| = |\alpha^\circ]$ and $|\alpha] = [\alpha^\circ|$.

Modal diamond operators satisfy the following module-style laws.

Lemma 12.1 *In every modal quantale,*

1. $|\alpha\beta\rangle p = |\alpha\rangle|\beta\rangle p$,
2. $|\alpha \vee \beta\rangle p = |\alpha\rangle p \vee |\beta\rangle p$,
3. $|\alpha\rangle(p \vee q) = |\alpha\rangle p \vee |\alpha\rangle q$ and $|\alpha\rangle\bot = \bot$,
4. $|\bot\rangle p = \bot$, $|1\rangle p = p$ and $|\alpha\rangle 1 = dom(\alpha)$.

These laws hold already in domain semirings [4], hence we do not present proofs. Dual properties hold for backward diamonds, and for boxes, if the modal quantale is boolean.

Beyond these one-dimensional properties, we list some new modal laws for the ω-structure. We write $\langle\alpha\rangle_i$ for either $|\alpha\rangle_i$ or $\langle\alpha|_i$ in the following lemma.

Lemma 12.2 *In every modal ω-quantale Q, for all $0 \leq i < j < \omega$,*

1. $\langle\alpha\rangle_i|\beta\rangle_j\gamma = \langle\alpha\rangle_i(\beta \cdot_j \gamma)$ and $\langle\alpha\rangle_i\langle\beta|_j\gamma = \langle\alpha\rangle_i(\gamma \cdot_j \beta)$,
2. $|\alpha\rangle_i\langle\beta\rangle_j\gamma \leq |\alpha\rangle_i(dom_i(\beta) \cdot_j dom_i(\gamma))$ and $\langle\alpha|_i\langle\beta|_j\gamma \leq \langle\alpha|_i(cod_i(\gamma) \cdot_j cod_i(\beta))$,
3. If $\alpha = dom_k(\alpha)$ for some $k \leq j$, then

$$|\alpha\rangle_i|\beta\rangle_j\gamma \leq |\alpha\rangle_i\beta \cdot_j |\alpha\rangle_i\gamma \qquad \text{and} \qquad \langle\alpha|_i\langle\beta|_j\gamma \leq \langle\alpha|\beta \cdot_j \langle\alpha|\gamma,$$

4. $|\alpha\rangle_j\langle\beta\rangle_i\gamma \leq |\alpha\rangle_j\langle dom_j(\beta)\rangle_i\gamma$ and $\langle\alpha|_j\langle\beta\rangle_i\gamma \leq \langle\alpha|_j\langle cod_j(\beta)\rangle_i\gamma$, and equality holds if Q is strong,
5. $dom_i(\alpha) \cdot_i \langle\beta\rangle_j\gamma \leq \langle dom_i(\alpha) \cdot_i \beta\rangle_j(dom_i(\alpha) \cdot_i \gamma)$ and, whenever Q is strong, then $\langle\alpha\rangle_j\beta \cdot_i dom_i(\gamma) \leq \langle\alpha \cdot_i dom_i(\gamma)\rangle_j(\beta \cdot_i dom_i(\gamma))$.

Proofs can be found in Appendix B.

13 Higher Semirings

In this section we weaken our value algebras from quantales to dioids, which are additively idempotent semirings. Convolution semirings have been studied widely in mathematics and computer science; the higher globular semirings in [13] form a starting point for the investigations in this article. Here we generalise from the powerset structures in [13] to convolution algebras. As usual in the construction of formal power series or convolution algebras, some restriction on the domain algebra or the function space need to be imposed. Beyond the relationship with [13], the consideration of ω-semirings and ω-Kleene algebras in this and the following section also sheds some light on the design space of algebras for higher-dimensional rewriting. While ω-quantales are certainly more expressive, ω-semirings and ω-Kleene algebras might lead to stronger computational properties such as decision procedures. But an exploration of this design space in rewriting applications remains beyond the scope of this article.

In convolution algebras on S^C, where S is an additively idempotent semiring, the complete lattice structure of the value quantale is replaced by a semilattice. To compensate for the lack of infinite sups in convolutions

$$(f * g)(x) = \bigvee_{x \in y \odot z} f(y) \cdot g(z),$$

when infinitely many pairs (y, z) satisfy the ternary relation $x \in y \odot z$, we require *finitely decomposable* catoids C. That is, for each $x \in C$ the fibre $\odot^{-1}(x) = \{(y, z) \mid x \in y \odot z\}$ must be finite. Now algebras with finite sups such as additively idempotent semirings suffice. This is standard, for instance, for the incidence algebras in combinatorics [52], where such a finiteness condition is imposed on intervals on the real line or a poset. In our context this means that modal value quantales can be replaced by modal value semirings [4] if catoids are finitely decomposable. Similar adaptations have been made for concurrent quantales and concurrent semirings [17].

Example 13.1 ([17]) In the Q-valued shuffle language 2-quantale from Example 8.4, the fibres $\{(u, v) \mid w = uv\}$ and $\{(u, v) \mid w = u\|v\}$ are finite for every word w. It therefore suffices that Q is a 2-semiring, as defined below.

The main results in this section show how finitely decomposable ω-catoids extend to convolution ω-semirings, where the ω-structure is defined precisely as for ω-quantales, and to (modal) semirings with a converse structure. These results prepare for the study of ω -Kleene algebras and (modal) Kleene algebras with converse in the following section, and in particular for a comparison with the higher globular Kleene algebras previously introduced [13].

We start with a summary of the structure of α-dioids. In a nutshell, their overall structure is the same as for α-quantales, except that the complete lattice in the latter is replaced by a semilattice in the former.

- A *dioid* is an additively idempotent semiring $(S, +, \cdot, 0, 1)$. Thus $(S, +, 0)$ is a semilattice with least element 0. We write $\leq$ for its order.
- A *modal semiring* [4] is an additively idempotent semiring (or *dioid*) equipped with operations $dom, cod : S \to S$ that satisfy precisely the modal quantale axioms for domain and codomain up to notational differences: we write $dom(0) = 0$, replacing $\perp$ by 0, $dom(\alpha + \beta) = dom(\alpha) + dom(\beta)$, replacing $\vee$ by $+$, and likewise for *cod*.
- An *α-semiring* is a structure $(S, +, 0, \cdot_i, 1_i, dom_i, cod_i)_{0 \leq i < \alpha}$, for an ordinal $\alpha \in \{0, 1, \ldots, \omega\}$, such that the $(S, +, 0, \cdot_i, 1_i, dom_i, cod_i)$ are modal semirings and we impose the same interchange and domain/codomain axioms as for α-quantales.
- An α-semiring is *strong* if it satisfies the same domain/codomain axioms as for strong α-quantales.

A formal list of axioms can be found in our Isabelle theories [28] and their proof document. Most properties derived for ω-quantales in previous sections are already available in ω -semirings, in particular Lemmata 7.3, 7.8, 12.1 and 12.1, the results on chains of C_i cells and ω-quantales as filtrations, and the construction of *n*-quantales by truncation.

Along the lines of Sect. 5 we can also define a converse structure $(-)^\circ : S \to S$ on a dioid *S*.

- In an *involutive dioidS* we impose the axioms $\alpha^{\circ\circ} = \alpha$, $(\alpha + \beta)^\circ = \alpha^\circ + \beta^\circ$ and $(\alpha\beta)^\circ = \beta^\circ\alpha^\circ$.
- In a *dioid with converse* we also require the strong Gelfand law $\alpha \leq \alpha\alpha^\circ\alpha$.
- *Modal involutive semirings* and *modal semirings with converse* are then defined in the obvious way.
- A *modal semiring with strong converse* is a modal involutive semiring in which $dom(\alpha) \leq \alpha\alpha^\circ$ and $cod(\alpha) \leq \alpha^\circ\alpha$.

The involutive dioid axioms imply $0^\circ = 0$ and $1^\circ = 1$, but are too weak to relate *dom* and *cod* in the modal case. Dioids with converse have been introduced in [41]. These provide the domain/codomain interaction expected.

Lemma 13.2 *In every modal semiring with converse,*

1. $dom(\alpha)^\circ = dom(\alpha)$ and $cod(\alpha)^\circ = cod(\alpha)$.
2. $dom(\alpha^\circ) = cod(\alpha)$ and $cod(\alpha^\circ) = dom(\alpha)$.

See Appendix B for proofs. Modal semirings with strong converse have been proposed in [46]. The strong Gelfand property holds in this setting: $\alpha = dom(\alpha)\alpha \leq \alpha\alpha^\circ\alpha$. The strong converse axioms can be seen as shadows of the explicit definitions of dom and cod in Dedekind quantales in the absence of $\wedge$.

For finitely decomposable ω-catoids C, where each of the ω underlying catoids has this property, our main correspondence triangles from Sects. 8 and 9 transfer to ω-semirings and semirings with converse. Here we consider only the extensions from groupoids to convolution algebras. First we present a corollary to Theorem 8.5, but to deal with the sups in the definition of Dom and Cod, we need to impose another restriction: a catoid C has *finite valency* if for each $x \in C_0$ the sets $\{y \in C \mid s(y) = x\}$ and $\{y \in C \mid t(y) = x\}$ are finite. An ω-catoid has finite valency if each of the underlying catoids has this property. we present a corollary to Theorem 8.5.

Corollary 13.3 *Let C be a finitely decomposable local ω-catoid of finite valency and S a (strong) ω-semiring. Then S^C is a (strong) ω-semiring.*

Proof In the construction of the convolution algebra in the proof of Theorem 8.5, it is routine to check that all sups remain finite when C is finitely decomposable. In particular, all sups in the definitions of Dom and Cod are finite because of finite valency. $\square$

Next we consider the converse structure.

Proposition 13.4 *Let C be a finitely decomposable groupoid.*

1. If S is a dioid with converse, then so is S^C.
2. If S is a modal semiring with strong converse, then so is S^C, whenever \$C\$ has finite valency.

Proof First note that the infinite sups in the proof of Proposition 9.1 can be replaced by finite sups when C is finitely decomposable and has finite valency. It remains to extend the strong Gelfand property for (1) and the axioms $dom(\alpha) \leq \alpha\alpha^\circ$ and $cod(\alpha) \leq \alpha^\circ\alpha$ for (2). In the proofs we still write $\bigvee$, but tacitly assume that all sups used are finite and can therefore be represented using $+$.

For dioids with converse,

$$\begin{aligned}
f * (f^\circ * f) &= \bigvee_{w,x,y,z \in C} f(x) f(y^-)^\circ g(z) [w \in x \odot y \odot z] \delta_w \\
&= \bigvee_{w,x,y,z \in C} f(x) f(y)^\circ g(z) [w \in x \odot y^- \odot z] \delta_w \\
&\geq \bigvee_{w \in C} f(w) f(w)^\circ g(w) [w \in w \odot w^- \odot w] \delta_w \\
&\geq \bigvee_{w \in C} f(w) [w \in s(w) w] \delta_w \\
&= \bigvee_{w \in C} f(w) \delta_w \\
&= f.
\end{aligned}$$

For semirings with strong converse,

$$\begin{aligned}
Dom(f) &= \bigvee_{x\in C} dom(f(x))\delta_{s(x)} \\
&\leq \bigvee_{x\in C} f(x)f(x)^{\circ}[s(x) \in x \odot x^{-}]\delta_x \\
&\leq \bigvee_{x,y,z\in C} f(y)f(z^{-})^{\circ}[x \in y \odot z^{-}]\delta_x \\
&= \bigvee_{x\in C} (f * f^{\circ})(x)\delta_x \\
&= f * f^{\circ}
\end{aligned}$$

and the proof for Cod follows by opposition. □

Next we present two examples that separate the three converse structures introduced above in the modal case.

Example 13.5 In the involutive modal semiring given by

$$\begin{array}{ccccc}
 & & 1 & & \\
 & / & & \backslash & \\
a & & & & b \\
 & \backslash & & / & \\
 & & 0 & &
\end{array}$$

with as inf, $dom = id = cod$ and $a^{\circ} = b$, $b^{\circ} = a$ we have

$$dom(a^{\circ}) = dom(b) = b \neq a = cod(a)$$

and likewise $cod(a^{\circ}) \neq dom(a)$. Hence it is not a modal semiring with converse.

Similar examples show that $dom(\alpha)^{\circ}$ and $cod(\alpha)^{\circ}$ need not be equal to $dom(\alpha)$ and $cod(\alpha)$ in involutive modal semirings, respectively.

Example 13.6 In the modal semiring with converse given by $0 < a < 1$, $aa = a$, $dom(a) = cod(a) = 1$ and $(-)^{\circ} = id$, we have $dom(a) = cod(a)1 > a = aa^{\circ} = a^{\circ}a$. It is therefore not a modal semiring with strong converse.

Remark 13.7 In an involutive domain semiring or domain semiring with converse S, one can of course define codomain explicitly as $cod(\alpha) = dom(\alpha^{\circ})$, from which $dom(\alpha) = cod(\alpha^{\circ})$ follows, but then S_dom need not be equal to S_cod: $dom \circ cod = cod$ may hold, but $cod \circ dom = dom$ may fail. We do not consider such alternative definitions of modal semirings with converse any further.

Remark 13.8 As in Sect. 10, we can consider (ω, p)-semirings and (ω, p)-Kleene algebras. We start with the former and outline the latter in Sect. 14.

An (ω, p)-*dioid* is an ω-dioid S equipped with operations $(-)^{\circ_j} : S \to S$ for all $p < j \leq n$ such that $(\alpha)^{\circ_j} \in S_j$ for all $\alpha \in S_j$ and such that the involution axioms and $\alpha \leq \alpha\alpha^{\circ_j}\alpha$ hold for $p < j \leq n$.

Corollary 10.2 then specialises further to (ω, p)-semirings owing to Corollary 14.3. We can still extend to (ω, p)-semirings, that is, whenever C is a finitely decomposable local (ω, p)-catoid of finite valency and S an (ω, p)-semiring, then S^C is an (ω, p)-semiring. Truncations at dimension n work as expected and the globular (n, p)-semirings introduced in [13] arise as special cases.

Finally, we compare ω-semirings with the previous slightly different axiomatisation of higher globular semirings [13, Definition 3.2.6]. These have been proved sound with respect to a powerset model of path n-categories [13, Sect. 3.3], which is the basis of higher-dimensional rewriting, but axioms have been introduced in an ad hoc fashion with a view on higher-dimensional rewriting proofs. Our correspondence triangles with respect to ω-catoids and ω-categories yield a more systematic structural justification of the ω-quantale and ω-semiring axioms.

One difference between the globular variant and ours is that the morphism axioms $dom_i(\alpha \cdot_j \beta) \leq dom_i(\alpha) \cdot_j dom_i(\beta)$ and $cod_i(\alpha \cdot_j \beta) \leq cod_i(\alpha) \cdot_j cod_i(\beta)$ are missing in the globular axiomatisation, while they are irredundant among the ω-semiring and ω-quantale axioms (Example 7.1). Yet they can be derived in our construction of convolution ω-quantales over any ω-catoid and hence hold in the powerset model of path n-categories. A second difference is that the previous axiomatisation contains redundant assumptions and axioms, for instance on the bounded distributive lattice structure of S_i or the relationships between quantalic units at different dimensions, which are now derivable. This discussion can be summarised as follows.

Proposition 13.9 *Every strong ω-semiring is a globular ω-semiring. Every globular ω-semiring, which satisfies $dom_i(\alpha \cdot_j \beta) \leq dom_i(\alpha) \cdot_j dom_i(\beta)$ and $cod_i(\alpha \cdot_j \beta) \leq cod_i(\alpha) \cdot_j cod_i(\beta)$, is a strong ω-semiring.*

The result generalises to (ω, p)-semirings with or without strong converses.

14 Higher Kleene Algebras

In this section we extend the results in the previous one from semirings to Kleene algebras K, adding a Kleene star $(-)^* : K \to K$. By contrast to the quantalic Kleene star $\alpha^* = \bigvee_{i \geq 0} \alpha^i$ in Sect. 4, it is modelled in terms of least fixpoints. Intuitively, the Kleene star models an unbounded finite iteration or repetition of a computation or a sequence of rewrite steps. In the quantale of binary relations in Example 4.2, for instance, the Kleene star of a binary relation models its reflexive-transitive closure, in particular the iterated execution of a rewrite relation. In the same example, the Kleene star on the quantale of paths models the iterative gluing of paths in a given set, thus in particular of rewriting sequences. The same observations can be made about Kleene algebras of relations or sets of paths. For evidence that Kleene stars are crucial for reasoning algebraically about higher-dimensional rewriting properties see [13].

The axioms of α-Kleene algebras are not as straightforward as those for α-dioids; additional identities capturing the interaction between higher stars, domain and codomain are needed. For convolution algebra constructions, a star on convolution Kleene algebras K^C has previously been defined only for finitely decomposable catoids *C* with a single unit that satisfy a certain grading condition [17]. This does not cover the α-semirings with multiple units in this paper. Instead of general convolution algebras, we therefore only consider powerset extensions in this section, where the finite decomposibility of catoids is not needed.

A *Kleene algebra* is a dioid *K* equipped with a star operation $(-)^* : K \to K$ that satisfies the star unfold and star induction axioms

$$1 + \alpha \cdot \alpha^* \leq \alpha^*, \qquad \gamma + \alpha \cdot \beta \leq \beta \Rightarrow \alpha^* \cdot \gamma \leq \beta$$

and their opposites, where the arguments in compositions have been swapped. A *modal Kleene algebra* is simply a Kleene algebra which is also a modal semiring.

The star unfold and induction axioms of Kleene algebras thus model α^* in terms of the least pre-fixpoints of the maps $x \mapsto 1 + \alpha \cdot x$ and $x \mapsto 1 + x \cdot \alpha$, and hence their least fixpoints. Every quantale is a Kleene algebra: the quantalic Kleene star defined in Sect. 4 satisfies the Kleene algebra axioms.

An adaptation of the definition of ω-quantales to Kleene algebras is straightforward—except for the last two axioms.

For an ordinal $\alpha \in \{0, 1, \ldots, \omega\}$, an α-*Kleene algebra* is an α-semiring *K* equipped with Kleene stars $(-)^{*_i} : K \to K$ that satisfy the usual star unfold and star induction axioms, for all $0 \leq i < j < \alpha$,

$$\begin{aligned}
1_i + \alpha \cdot_i \alpha^{*_i} &\leq \alpha^{*_i}, &\qquad \gamma + \alpha \cdot_i \beta \leq \beta &\Rightarrow \alpha^{*_i} \cdot_i \gamma \leq \beta,\\
1_i + \alpha^{*_i} \cdot_i \alpha &\leq \alpha^{*_i}, &\qquad \gamma + \beta \cdot_i \alpha \leq \beta &\Rightarrow \gamma \cdot_i \alpha^{*_i} \leq \beta,\\
dom_i(\alpha) \cdot_i \beta^{*_j} &\leq (dom_i(\alpha) \cdot_i \beta)^{*_j}, &\qquad \alpha^{*_j} \cdot_i cod_i(\beta) &\leq (\alpha \cdot_i cod_i(\beta))^{*_j}.
\end{aligned}$$

An α-Kleene algebra is *strong* if the underlying α-semiring is, and for all $0 \leq i < j < \alpha$,

$$dom_j(\alpha) \cdot_i \beta^{*_j} \leq (dom_j(\alpha) \cdot_i \beta)^{*_j}, \qquad \alpha^{*_j} \cdot_i cod_j(\beta) \leq (\alpha \cdot_i cod_j(\beta))^{*_j}.$$

Remark 14.1 The axioms mentioning domain and codomain are derivable in (strong) α -quantales. In α-Kleene algebras we have neither proofs nor counterexamples to show redundancy or irredundancy of these axioms with respect to the remaining ones, yet these laws are needed for coherence proofs in higher-dimensional rewriting [13].

The converse structure on Kleene algebras is inherited from dioids. This leads immediately to *involutive Kleene algebras*, *Kleene algebras with converse*, their modal variants and *modal Kleene algebras with strong converse*.

Lemma 14.2 *In every involutive Kleene algebra,* $\alpha^{*\circ} = \alpha^{\circ *}$.

A proof can be found in Appendix B, as usual.

We now present a correspondence result for higher Kleene algebras– in the special case of powerset extensions. As usual, we consider ω-structures. As every quantale is a Kleene algebra and the Kleene star on a set is simply a union of powers, it is an immediate consequence of Corollary 11.1.

Corollary 14.3

1. Let C be a local ω-catoid. Then $\mathcal{P}C$ is an ω-Kleene algebra.
2. Let X be a set and $\mathcal{P}X$ an ω-Kleene algebra in which $id_i \neq 0$. Then X can be equipped with a local ω-catoid structure.
3. If C is strong, then $\mathcal{P}C$ is strong and vice versa.

Similarly, the following correspondence result is a specialisation of Corollary 11.4.

Corollary 14.4

1. Let C be a groupoid. Then $\mathcal{P}C$ is a modal Kleene algebra with strong converse.
2. Let X be a set and $\mathcal{P}X$ a modal Kleene algebra with strong converse in which $id_0 \neq \emptyset$ and all atoms are functional. Then X can be equipped with a groupoid structure.

Proof For (1), it remains to extend the strong converse axioms, as the modular law is not available. This has been done in Lemma 11.3.

For (2), note that the strong converse axioms are used in the proof of Theorem 9.3 instead of the Dedekind or modular law, but the strong Gelfand law would be too weak. For a direct proof, $x \odot x^- = \{x\} \cdot \{x\}^\circ = dom(\{x\}) = \{s(x)\}$ and likewise for the target axiom, using the strong converse axiom for domain in the second step (as an equation because atoms are functional). □

Typical examples of powerset ω-Kleene algebras are subalgebras of powerset ω-quantales generated by some finite set and closed under the operations of ω-Kleene algebras. Languages over a finite alphabet Σ, for instance, form a quantale, whereas the regular languages generated by Σ form a Kleene algebra, but not a quantale.

Remark 14.5 The definition of (ω, p)-*Kleene algebra* is a straightforward extension of that of (ω, p)-semiring. The fact that $\alpha^{\circ_j} = \alpha$ for all $\alpha \in S_{dom_i}$(or K_j) with $i < j$ now follows from Lemma 13.2.

Corollary 10.2 now specialises to (ω, p)-Kleene algebras owing to Corollary 14.3. In particular, an (ω, p)-Kleene algebra with strong converse is needed to recover the (ω, p)-catoid among the atoms. For the more general convolution algebra constructions, the limitations mentioned in Sect. 14 remain; an extension to Kleene algebras requires further thought. Truncations at dimension *n* work as expected.

Finally, we briefly compare ω-Kleene algebras with globular Kleene algebras [13, Definitions 3.2.7]. As both are based on the higher semirings discussed in the previous section, we focus on the star axioms. On one hand, as discussed in Remark 14.1, we need a strong ω - Kleene algebra to derive two of the globular ω-Kleene algebra axioms. On the other hand, two globular ω-Kleene algebra axioms are derivable in the context of ω-Kleene algebras, as the following lemma shows.

Lemma 14.6 *In every ω-Kleene algebra, $(\alpha \cdot_j \beta)^{*_i} \leq \alpha^{*_i} \cdot_j \beta^{*_i}$ for all $0 \leq i < j < \omega$.*

See Appendix B for a proof. The following result summarises this discussion.

Proposition 14.7 *Every strong ω-Kleene algebra is a globular ω-Kleene algebra. Every globular ω-Kleene algebra, which satisfies $dom_i(\alpha \cdot_j \beta) \leq dom_i(\alpha) \cdot_j dom_i(\beta)$ and $cod_i(\alpha \cdot_j \beta) \leq cod_i(\alpha) \cdot_j cod_i(\beta)$, is a strong ω-Kleene algebra.*

This proposition extends to strong (ω, p)-Kleene algebras and globular (ω, p)-Kleene algebras, as introduced in [13, Definition 4.4.2] for the case (n, p), using either the notion of converses or strong converses discussed in the previous section.

15 Conclusion

This paper combines two lines of research on higher globular algebras for higher-dimensional rewriting and on the construction of convolution algebras on catoids and categories. More specifically, we have introduced ω-catoids and ω-quantales, and established correspondence triangles between them. These extend and justify the axioms of higher globular Kleene algebras [13], which have previously been used for coherence proofs in higher-dimensional rewriting, up to some modifications. We have also introduced several extensions and specialisations of these constructions, in particular to (ω, p)-catoids and (ω, p)-quantales and to variants of such quantales based on semirings or Kleene algebras. While the technical focus has been on convolution algebras, which often lead to interesting applications in quantitative program semantics and verification in computer science, we currently have no use for convolution ω-quantales or (ω, p)-quantales with value quantales different from the quantale of booleans. In the latter case, however, we can use power set ω-quantales instead of globular Kleene algebras to reason about higher-dimensional rewriting systems with greater flexibility and expressive power. A detailed introduction to the relationship with higher-dimensional rewriting, to globular Kleene algebras and to their use in coherence proofs such as higher Church–Rosser theorems and higher Newman's lemmas can be found in [13].

Our results add a new perspective to higher-dimensional rewriting and they yield new tools for reasoning with higher categories. But further work is needed for exploring them in practice. A first line of research could consider categorical variations of the strict globular ω-catoids and ω-quantales introduced in this work to provide a categorical framework for the construction of coherence proofs and more generally of polygraphic resolutions, which are cofibrant replacements in higher categories:

- It is worth considering cubical versions of ω-catoids and ω-quantales, as confluence diagrams in higher-dimensional rewriting have a cubical shape [53, 54] and proofs of higher confluence properties can be developed more naturally in a cubical setting. A single-set axiomatisation of cubical categories has recently been developed [55]. It remains

to generalise it to cubical catoids, to introduce cubical quantales and to study their correspondences along the lines of the globular case. Cubical catoids and quantales may also be beneficial for explicit constructions of resolutions. Beyond that, we are interested in applications to precubical sets and higher-dimensional automata, where Kleene algebras or quantales describing their languages remain to be defined [56, 57], and to cubical ω-categories with connections [58], where single-set formalisations might be of interest.
- To expand the constructions in this article to algebraic rewriting such as string, term, linear or diagrammatic rewriting, the development of catoids and quantales internal to categories of algebras over an operad would be needed.
- The homotopic properties of algebraic rewriting require weakening the exchange law of higher categories [59, 60] and, in our context, Gray-variants of the ω-catoids and ω-quantales, where interchange law holds only up to coherent isomorphism [61].

A second line of research could investigate the link of the correspondence triangles between catoids, value quantales and convolution quantales with duality results for convolution algebras beyond the Jónsson-Tarski case and similar structural results, and the categorification of such an approach. Catoids, for instance, are equivalent to monoids in the monoidal category Rel with the standard tensor and unit. Related to this are questions about free powerset or convolution ω-quantales generated by polygraphs, or computads, and about coherent rewriting properties of these internal monoids.

A third line of research could consider the formalisation of higher category theory, and higher-dimensional rewriting support for these, in the single-set framework of catoids developed in [26–28] or by other means with proof assistants such as Coq, Lean or Isabelle. The formalisation of the theorems in [13] could serve as stepping stones towards coherence theorems like Squier's theorem in higher dimensions [9, 62] or the computation of resolutions by rewriting in categorical and homological algebra [10–12].

Finally, the study of domain, codomain and converse in the setting of quantales leads to interesting questions about Dedekind quantales and the interplay of these operations in variants of allegories [63]. These will be addressed in successor articles.

Eckmann–Hilton-Style Collapses

Strengthening the weak homomorphism axioms

$$s_i(x \odot_j y) \subseteq s_i(x) \odot_j s_i(y) \qquad \text{and} \qquad t_i(x \odot_j y) \subseteq t_i(x) \odot_j t_i(y),$$

for $0 \leq i < j < \omega$ to equations in the ω-catoid axioms collapses the structure. The equational homomorphism laws for s_j and t_j and $\odot_i$ for $i < j$ in ω-categories are therefore rather exceptional. In this Appendix, following [15], we call *st-multimagma* a catoid in which associativity of $\odot$ has been forgotten. We extend this notion to ω-st-magmas in the obvious way.

Lemma 1 *If the inclusions*

$$s_i(x \odot_j y) \subseteq s_i(x) \odot_j s_i(y) \qquad \textit{and} \qquad t_i(x \odot_j y) \subseteq t_i(x) \odot_j t_i(y),$$

for $0 \leq i < j < \omega$ are replaced by equations in the axiomatisation of ω-st-multimagmas, then

1. $s_i = s_j$ and $t_i = t_j$,
2. $s_i = t_i$ and $s_j = t_j$.

Proof For $s_i = s_j$,

$$\begin{aligned} s_i(s_j(x) \odot_j s_i(s_j(x))) &= s_i(s_j(x)) \odot_j s_i(s_i(s_j(x))) \\ &= s_j(s_i(s_j(x))) \odot_j s_i(s_j(x)) \\ &= \{s_i(s_j(x))\}, \end{aligned}$$

thus $s_i(s_j(x) \odot_j s_i(s_j(x))) \neq \emptyset$ and therefore $\Delta_j(s_j(x), s_i(s_j(x)))$. It follows that

$$s_j(x) = t_j(s_j(x)) = s_j(s_i(s_j(x))) = s_j(s_j(s_i(x))) = s_j(s_i(x)) = s_i(x).$$

By opposition, therefore, $t_i = t_j$. Also, $s_i(x) = t_j(s_i(x)) = s_i(t_j(x)) = s_i(t_i(x)) = t_i(x)$, and $s_j = t_j$ then follows from the previous properties. □

Lemma 2 *If the same replacement is made for ω-categories, then $\odot_i = \odot_j$ for $0 \leq i < j \leq \omega$ and both operations commute.*

Proof If $x \odot_i y = \emptyset$, then $x \odot_i y \subseteq x \odot_j y$. Otherwise, if $x \odot_i y \neq \emptyset$, then $t_i(x) = s_i(y)$ and

$$\begin{aligned} \{x \odot_i y\} &= (x \odot_j t_j(x)) \odot_i (s_j(y) \odot_j y) \\ &\subseteq (x \odot_i s_j(y)) \odot_j (t_j(x) \odot_j y) \\ &= (x \odot_i s_i(y)) \odot_j (t_i(x) \odot_j y) \\ &= (x \odot_i t_i(x)) \odot_j (s_i(y) \odot_j y) \\ &= \{x \odot_j y\} \end{aligned}$$

and therefore $x \odot_i y = x \odot_j y$ by functionality.

Likewise, if $x \odot_i y = \emptyset$, then $x \odot_i y \subseteq y \odot_i x$. If $x \odot_i y \neq \emptyset$, then $t_i(x) = s_i(y)$ and

$$\begin{aligned} \{x \odot_i y\} &= (s_j(x) \odot_j x) \odot_i (y \odot_j t_j(x)) \\ &\subseteq (s_j(x) \odot_i y) \odot_j (x \odot_i t_j(y)) \\ &= (t_i(x) \odot_i y) \odot_j (x \odot_i s_i(y)) \\ &= (s_i(y) \odot_i y) \odot_j (x \odot_i t_i(x)) \\ &= \{y \odot_j x\}. \end{aligned}$$

Then $x \odot_i y = y \odot_i x$ and $x \odot_j y = y \odot_j x$ by functionality and the previous result. □

Unlike the classical Eckmann-Hilton collapse, the resulting structure is *not* an abelian monoid: different elements can still have different units and $\odot_i$(and therefore $\odot_j$) need not be total. There are 2-element counterexamples for $\odot_0$ and $\odot_1$ in the case of 2-categories.

Finally we obtain a stronger collapse in the presence of an equational interchange law in the more general setting of multioperations and several units.

Lemma 3 *If the inclusions* $(w \odot_j x) \odot_i (y \odot_j z) \subseteq (w \odot_i y) \odot_j (x \odot_i z)$, *for* $0 \leq i < j < \omega$, *are replaced by equations in the axiomatisation of* ω*-st-multimagmas, then* $\odot_i$ *and* $\odot_j$ *coincide,* $s_i = s_j = t_i = t_j$ *and* $\odot_i$*(as well as* $\odot_j$*) is associative and commutative.*

We have verified this result with Isabelle in two dimensions, but leave a proof on paper to the reader. Once again, the resulting structure is not automatically an abelian monoid: different elements can have different units and $\odot_i$(and therefore $\odot_j$) need neither be total nor functional. There are again 2-element counterexamples.

Proofs

Proof of Lemma 3.1(2)-(4) Item (2) is immediate from the axioms. For (3), suppose $xy = \{s(x)\}$. Then $t(x) = s(y)$ and hence $\{y\} = t(x)y$. Thus $y \in x^- xy = x^- s(x) = x^- t(x^-) = x^-$ and therefore $x^- = y$. Finally, (4) is immediate from (3) because $s(x)t(x) = \{s(x)\}$ and $t(x)t(x) = t(x)$ by Lemma 2.3(3). □

Proof of Lemma 3.2 For (1), $x^- x = \{t(x)\} = \{s(x^-)\}$ implies $(x^-)^- = x$ by Lemma 3.1(2).

For (2), suppose $x \in yz$. Then $t(y) = s(z)$ and $xz^- \subseteq yzz^- = ys(z) = yt(y) = \{y\}$. Moreover, by assumption, $t(x) = t(z)$ and therefore $\Delta(x, z^-)$ by locality. It then follows that $xz^- = \{y\}$ and hence $y \in xz^-$. The converse implication follows from (1). The remaining equivalence follows by opposition. □

Proof of Lemma 3.4 For (1), suppose $s(x) = t(z) = s(y)$ and $zx = zy$. Then $z^- zx = z^- zy$, therefore $t(z)x = t(z)y$, $s(x)x = s(y)y$ and finally $x = y$. (2) follows by opposition. □

Proof of Lemma 5.2 (1) and (2) are immediate consequences of sup-preservation.

For (3),

$$\gamma \leq (\alpha \wedge \beta)^\circ \Leftrightarrow \gamma^\circ \leq \alpha \wedge \beta \Leftrightarrow \gamma^\circ \leq \alpha \wedge \gamma^\circ \leq \beta \Leftrightarrow \gamma \leq \alpha^\circ \wedge \gamma \leq \beta^\circ \Leftrightarrow \gamma \leq \alpha^\circ \wedge \beta^\circ$$

implies the claim for $\wedge$, and the proof for $\bigwedge$ is similar.

For (4), $\top^\circ \leq \top$ and hence $\top = \top^{\circ\circ} \leq \top^\circ$, and the other proofs are equally simple.

For (5), $\alpha^\circ \wedge \beta = \bot \Leftrightarrow (\alpha^\circ \wedge \beta)^\circ = \bot \Leftrightarrow \alpha \wedge \beta^\circ = \bot$.

For (6), we know that every quantale is a Kleene algebra an can use their induction axioms (see Sect. 14). First, $\alpha^{\circ*\circ} = (1 + \alpha^\circ \alpha^{\circ*})^\circ = 1 + \alpha\alpha^{\circ*\circ}$ and hence $\alpha^* \leq \alpha^{\circ*\circ}$ by star induction. Thus $\alpha^{*\circ} \leq \alpha^{\circ*}$. For the converse direction, $\alpha^{*\circ} = (1 + \alpha^* \alpha)^\circ = 1 + \alpha^\circ \alpha^{*\circ}$ and therefore $\alpha^{\circ*} \leq \alpha^{*\circ}$ by star induction. An alternative inductive proof uses the definition of the star in quantales. □

Proof of Lemma 5.3 Let Q be an involutive quantale. Suppose the modular law holds. We derive two auxiliary properties before deriving the Dedekind law. First, the modular law can be strengthened to

$$\alpha\beta \wedge \gamma = (\alpha \wedge \gamma\beta^\circ)\beta \wedge \gamma,$$

because $\alpha\beta \wedge \gamma = \alpha\beta \wedge \gamma \wedge \gamma \leq (\alpha \wedge \gamma\beta^\circ)\beta \wedge \gamma$ using the modular law and $(\alpha \wedge \gamma\beta^\circ)\beta \wedge \gamma \leq \alpha\beta \wedge \gamma$ by properties of infs and order preservation. Second, a dual equational modular law

$$\alpha\beta \wedge \gamma = \alpha(\beta \wedge \alpha^\circ\gamma) \wedge \gamma$$

then follows because $\alpha\beta \wedge \gamma = (\beta^\circ\alpha^\circ \wedge \gamma^\circ)^\circ = ((\beta^\circ \wedge \gamma^\circ\alpha^{\circ\circ})\alpha^\circ \wedge \gamma^\circ)^\circ = \alpha(\beta \wedge \alpha^\circ\gamma) \wedge \gamma$ using properties of converse and the first equational modular law in the second step. Therefore,

$$\begin{aligned}\alpha\beta \wedge \gamma &= \alpha(\beta \wedge \alpha^\circ\gamma) \wedge \gamma\\ &\leq (\alpha \wedge \gamma(\beta \wedge \alpha^\circ\gamma)^\circ)(\beta \wedge \alpha^\circ\gamma)\\ &= (\alpha \wedge \gamma(\beta^\circ \wedge \gamma^\circ\alpha))(\beta \wedge \alpha^\circ\gamma)\\ &\leq (\alpha \wedge \gamma\beta^\circ)(\beta \wedge \alpha^\circ\gamma),\end{aligned}$$

using properties of converse and order preservation as well as the dual equational modular in the first and the modular law in the second step. This proves the Dedekind law.

Finally, $\alpha\beta \wedge \gamma \leq (\alpha \wedge \gamma\beta^\circ)(\beta \wedge \alpha^\circ\gamma) \leq (\alpha \wedge \gamma\beta^\circ)\beta$ yields the modular law from the Dedekind law using properties of inf and order preservation.

These proofs are standard in relation algebra. □

Proof of Lemma 5.4 For the strong Gelfand property, $\alpha = 1\alpha \wedge \alpha \leq \top\alpha \wedge \alpha \leq (\top \wedge \alpha\alpha^\circ)\alpha = \alpha\alpha^\circ\alpha$ using the modular law in the third step.

For Peirce's law, suppose $\alpha\beta \wedge \gamma^\circ = \bot$. Then $(\alpha\beta \wedge \gamma^\circ)\beta^\circ \wedge \alpha = \bot$ and therefore $\beta\gamma \wedge \alpha^\circ = \bot$ using the first equational modular law from the proof of Lemma 5.3 and properties of convolution. The converse implication is similar, using the dual equational modular law from Lemma 5.3.

For the first Schröder law, $\alpha\beta \wedge \gamma = \bot \Leftrightarrow \beta \wedge \alpha^\circ\gamma = \bot$, $\alpha\beta \wedge \gamma = \bot$ implies $\beta^\circ\alpha^\circ \wedge \gamma^\circ = \bot$ by properties of converse and therefore $\beta \wedge \alpha^\circ\gamma = \bot$ by Peirce's law. The proof of the converse direction is similar.

The second Schröder law, $\alpha\beta \wedge \gamma = \bot \Leftrightarrow \alpha \wedge \gamma\beta^\circ = \bot$ follows in a similar way from Peirce's law, the first Schröder law and properties of converse.

These proofs are once again standard in relation algebra. □

Proof of Proposition 5.6 The proofs of $dom(\alpha) \leq 1$ and $dom(\bot) = \bot$ are trivial.

For $\alpha \leq dom(\alpha)\alpha$, $\alpha = \alpha \wedge \alpha 1 \leq (1 \wedge \alpha\alpha^\circ)\alpha = dom(\alpha)\alpha$ by the modular law, and the converse inequality follows from the second axiom.

For $dom(\alpha dom(\beta)) = dom(\alpha\beta)$, we have $dom(\alpha dom(\beta)) = 1 \wedge \alpha dom(\beta)\top = 1 \wedge \alpha\beta\top = dom(\alpha\beta)$.

For $dom(\alpha \vee \beta) = dom(\alpha) \vee dom(\beta)$, note that $dom(dom(\alpha)) = dom(\alpha)$ is immediate from the previous axiom. We first show that $dom(dom(\alpha) \vee dom(\beta)) = dom(\alpha) \vee dom(\beta)$. Indeed,

$$\begin{aligned} dom(dom(\alpha) \vee dom(\beta)) &= 1 \wedge (dom(\alpha) \vee dom(\beta))(dom(\alpha) \vee dom(\beta))^\circ \\ &= 1 \wedge (dom(\alpha) \vee dom(\beta))(dom(\alpha) \vee dom(\beta)) \\ &= 1 \wedge (dom(\alpha) \vee dom(\beta)) \wedge (dom(\alpha) \vee dom(\beta)) \\ &= 1 \wedge (dom(\alpha) \vee dom(\beta)) \\ &= (1 \wedge dom(\alpha)) \vee (1 \wedge dom(\beta)) \\ &= dom(\alpha) \vee dom(\beta), \end{aligned}$$

where the third step uses the fact that multiplication of domain elements is commutative and idempotent. Proofs can be found in our Isabelle theories. Using this property with the alternative definition of domain,

$$\begin{aligned} dom(\alpha \vee \beta) &= 1 \wedge (\alpha \vee \beta)\top \\ &= 1 \wedge (\alpha\top \vee \beta\top) \\ &= 1 \wedge (dom(\alpha)\top \vee dom(\beta)\top)) \\ &= 1 \wedge (dom(\alpha) \vee dom(\beta))\top \\ &= dom(dom(\alpha) \vee dom(\beta)) \\ &= dom(\alpha) \vee dom(\beta). \end{aligned}$$

For the compatibility property note that, by the strong Gelfand property and previous domain axioms $dom(\alpha)dom(\alpha)dom(\alpha) = dom(\alpha)$, therefore $dom(\alpha)dom(\alpha) = dom(\alpha)$ and likewise for $cod(\alpha)$. The proof of $cod(dom(\alpha)) = dom(\alpha)$ is similar. □

Proof of Lemma 5.7 For (1), $1 \wedge \alpha\top = 1 \wedge \alpha(\top \wedge \alpha^\circ 1) = 1 \wedge \alpha\alpha^\circ = dom(\alpha)$ using the modular law.

For (2), $\alpha\top = dom(\alpha)\alpha\top \leq dom(\alpha)\top\top = dom(\alpha)\top$ and $dom(\alpha)\top = (1 \wedge \alpha\top)\top \leq \top \wedge \alpha\top\top = \alpha\top$.

Item (3) and (4) are trivial.

Finally, (4) is immediate from the definition of dom. □

Proof of Lemma 6.8 By the morphism axiom for s_i, $s_i(x \odot_j y) \subseteq s_i(x) \odot_j s_i(y)$. The right-hand side must not be empty whenever the left-hand side is defined, which is assumed. Hence it must be equal to $\{s_i(x)\}$ because compositions of lower cells in higher dimensions are trivial. □

Proof of Lemma 5.11 By the strong Gelfand property,

$$dom(\alpha) \leq dom(\alpha)dom(\alpha)^\circ dom(\alpha) \leq 1 dom(\alpha)^\circ 1 = dom(\alpha)^\circ,$$

from which $dom(\alpha)^\circ \leq dom(\alpha)$ follows using the adjunction in Remark 5.1. The proof for cod follows by duality. $\square$

Proof of Proposition 6.13 Before the proof proper, we derive $s_i \circ t_j \circ s_i = s_i$ and $t_i \circ s_j \circ s_i = s_i$. We have

$$\begin{aligned}
\{s_i(x)\} &= s_i(x) \odot_i s_i(x)\\
&= (s_j(s_i(x)) \odot_j s_i(x)) \odot_i (s_i(x) \odot_j t_j(s_i(x)))\\
&\subseteq (s_j(s_i(x)) \odot_i s_i(x)) \odot_j (s_i(x) \odot_i t_j(s_i(x))),
\end{aligned}$$

hence $\Delta_j(s_j(s_i(x)) \odot_i s_i(x), s_i(x) \odot_i t_j(s_i(x)))$. Thus $\Delta_i(s_j(s_i(x)), s_i(x))$ and $\Delta_i(s_i(x), t_j(s_i(x)))$, and therefore $t_i(s_j(s_i(x))) = s_i(s_i(x)) = s_i(x)$ as well as $s_i(t_j(s_i(x))) = t_i(s_i(x)) = s_i(x)$.

Next we derive the missing *n*-catoid axioms. First we consider $s_j \circ s_i = s_i$, $s_j \circ t_i = t_i$, $t_j \circ s_i = s_i$, and $t_j \circ t_i = t_i$. For the first one,

$$\begin{aligned}
\{s_i(x)\} &= s_i(x) \odot_i s_i(x)\\
&= (s_j(s_i(x)) \odot_j s_i(x)) \odot_i (s_i(x) \odot_j t_j(s_i(x)))\\
&\subseteq (s_j(s_i(x)) \odot_i s_i(x)) \odot_j (s_i(x) \odot_i t_j(s_i(x)))\\
&= (s_j(s_i(x)) \odot_i t_i(s_j(s_i(x)))) \odot_j (s_i(t_j(s_i(x))) \odot_i t_i(t_j(s_i(x))))\\
&= s_j(s_i(x)) \odot_j t_j(s_i(x))
\end{aligned}$$

and therefore $s_i(x) = s_j(s_i(x))$ as well as $s_i(x) = t_j(s_i(x))$. The remaining identities hold by opposition.

Second, we derive the identities $s_i \circ s_j = s_j \circ s_i$, $s_i \circ t_j = t_j \circ s_i$, $t_i \circ s_i = s_j \circ t_i$ and $t_i \circ t_j = t_j \circ t_i$. For the first, $\{s_j(x)\} = s_j(s_i(x)) \odot_i s_j(x) = s_i(x) \odot_i s_j(x)$ and therefore $\Delta_i(s_i(x), s_j(x))$. It follows that $s_j(s_i(x)) = s_i(x) = t_i(s_i(x)) = s_i(s_j(x))$. The remaining proofs are similar.

Note that none of the proofs so far requires an associativity law.

It remains to derive $s_i(x \odot_j y) \subseteq s_i(x) \odot_j s_i(y)$ and $t_i(x \odot_j y) \subseteq t_i(x) \odot_j t_i(y)$. We prove the first inclusion by cases. If $s_i(x \odot_j y) = \emptyset$, then the claim is trivial. Otherwise, if $s_i(x \odot_j y) \neq \emptyset$, then $t_j(x) = s_i(y)$ and thus

$$\begin{aligned}
s_i(x \odot_j y) &= s_i(s_j(x \odot_j y))\\
&= s_i(s_j(x \odot_j s_j(y)))\\
&= s_i(s_j(x \odot_j t_j(y)))\\
&= s_i(s_j(x))\\
&= \{s_i(x)\}\\
&= s_i(x) \odot_j t_j(s_i(x))\\
&= s_i(x) \odot_j s_i(t_j(x))\\
&= s_i(x) \odot_j s_i(s_j(x))\\
&= s_i(x) \odot_j s_i(x).
\end{aligned}$$

This uses weak locality laws that are available in all catoids if the composition under the source operation is defined. See Lemma 2.6 and [15] for details. The second inclusion follows by opposition. □

Proof of Lemma 7.3 For (1) $cod_j \circ dom_i = cod_j \circ dom_j \circ dom_i = dom_j \circ dom_i = dom_i$, and the remaining identities in (1) follow by opposition.

For (2), $1_j \cdot_i 1_j = d_j(1_j) \cdot_i d_j(1_j) = d_j(1_j \cdot_i 1_j) \le 1_j$ if Q is strong and

$$1_j = 1_j \cdot_i 1_i = (1_j \cdot_j 1_j) \cdot_i (1_i \cdot_j 1_j) \le (1_j \cdot_i 1_i) \cdot_j (1_j \cdot_i 1_j) = 1_j \cdot_j (1_j \cdot_i 1_j) = 1_j \cdot_i 1_j.$$

Further, $1_i \cdot_j 1_i = d_i(1_i) \cdot_j d_i(1_i) = d_j(d_i(1_i)) \cdot_j d_i(1_i) = d_i(1_i) = 1_i$.

For (3), $1_i = 1_i \cdot_i 1_i = (1_j \cdot_j 1_i) \cdot_i (1_i \cdot_j 1_i) \le (1_j \cdot_i 1_i) \cdot_j (1_i \cdot_i 1_j) = 1_j \cdot_j 1_j = 1_j$.

For (4), $d_j(1_i) = d_j(d_i(1_i)) = d_i(1_i) = 1_i$, and $d_i(1_j) \le 1_i$ as well as $1_i = d_i(1_i) \le d_i(1_j)$. The other identities in (4) then follow by opposition.

For (5), $dom_i \circ dom_j = dom_j \circ dom_i$ follows from

$$\begin{aligned}
dom_i(dom_j(\alpha)) &= dom_i(dom_j(dom_i(\alpha) \cdot_i \alpha)) \\
&= dom_i(dom_j(dom_i(\alpha)) \cdot_i dom_j(\alpha)) \\
&= dom_i(dom_i(\alpha) \cdot_i dom_j(\alpha)) \\
&= dom_i(\alpha) \cdot_i dom_i(dom_j(\alpha)) \\
&= dom_j(dom_i(\alpha)) \cdot_i dom_i(dom_j(\alpha)) \\
&\le dom_j(dom_i(\alpha)) \cdot_i 1_i \\
&= dom_j(dom_i(\alpha))
\end{aligned}$$

and

$$\begin{aligned}
dom_j(dom_i(\alpha)) &= dom_i(\alpha) \\
&= dom_i(dom_j(\alpha) \cdot_j \alpha) \\
&\le dom_i(dom_j(\alpha)) \cdot_j dom_i(\alpha) \\
&\le dom_i(dom_j(\alpha)) \cdot_j 1_i \\
&\le dom_i(dom_j(\alpha)) \cdot_j 1_j \\
&= dom_i(dom_j(\alpha)).
\end{aligned}$$

Moreover, $dom_i \circ cod_j = cod_j \circ dom_i$ follows from

$$\begin{aligned}
dom_i(cod_j(\alpha)) &= dom_i(cod_j(dom_i(\alpha) \cdot_i \alpha)) \\
&= dom_i(cod_j(dom_i(\alpha)) \cdot_i cod_i(\alpha)) \\
&= dom_i(dom_i(\alpha) \cdot_i cod_i(\alpha)) \\
&= dom_i(\alpha) \cdot_i dom_i(cod_i(\alpha)) \\
&= cod_j(dom_i(\alpha)) \cdot_i dom_i(cod_i(\alpha)) \\
&\le cod_j(dom_i(\alpha)) \cdot_i 1_i \\
&= cod_j(dom_i(\alpha))
\end{aligned}$$

and

$$\begin{aligned} cod_j(dom_i(\alpha)) &= dom_i(\alpha) \\ &= dom_i(\alpha \cdot_j cod_j(\alpha)) \\ &\leq dom_i(\alpha) \cdot_j dom_i(cod_j(\alpha)) \\ &\leq 1_i \cdot_j dom_i(cod_j(\alpha)) \\ &\leq 1_j \cdot_j dom_i(cod_j(\alpha)) \\ &= dom_i(cod_j(\alpha)). \end{aligned}$$

The remaining two identities in (6) then follow by opposition.

Finally, for (6),

$$\begin{aligned} dom_i(\alpha \cdot_j \beta) &= dom_i(dom_j(\alpha \cdot_j \beta)) \\ &= dom_i(dom_j(\alpha \cdot_j dom_j(\beta))) \\ &= dom_i(\alpha \cdot_j dom_j(\beta)) \end{aligned}$$

and the second identity in (7) then follows by opposition. □

Proof of Lemma 7.6 For the first property in (1) and $k = i$, we first use induction on k to prove $dom_i(\alpha) \cdot_i \beta^{k_1} \leq (dom_i(\alpha) \cdot_i \beta)^{k_j}$, where $(-)^{k_j}$ indicates that powers are taken with respect to $\cdot_j$. The base case follows from $dom_i(\alpha) \cdot_i 1_j \leq 1_i \cdot_i 1_j = 1_j$. For the induction step, suppose $dom_i(\alpha) \cdot_i \beta^{k_j} \leq (dom_i(\alpha) \cdot_i \beta)^{k_j}$. Then

$$\begin{aligned} dom_i(\alpha) \cdot_i \beta^{(k+1)_j} &= dom_j(dom_i(\alpha)) \cdot_i (\beta \cdot_j \beta^{k_j}) \\ &= (dom_j(dom_i(\alpha)) \cdot_j dom_j(dom_i(\alpha))) \cdot_i (\beta \cdot_j \beta^{k_j}) \\ &= (dom_i(\alpha) \cdot_j dom_i(\alpha)) \cdot_i (\beta \cdot_j \beta^{k_j}) \\ &\leq (dom_i(\alpha) \cdot_i \beta) \cdot_j (dom_i(\alpha) \cdot_i \beta^{k_j}) \\ &\leq (dom_i(\alpha) \cdot_i \beta) \cdot_j (dom_i(\alpha) \cdot_i \beta)^{k_j} \\ &= (dom_i(\alpha) \cdot_i \beta)^{(k+1)_j}. \end{aligned}$$

Using this property yields

$$\begin{aligned} dom_i(\alpha) \cdot_i \beta^{*_j} &= dom_i(\alpha) \cdot_i \bigvee_{k \geq 0} \beta^{k_j} \\ &= \bigvee \{dom_i(\alpha) \cdot_i \beta^{k_j} \mid k \geq 0\} \\ &\leq \bigvee_{k \geq 0} (dom_i(\alpha) \cdot_i \beta)^{k_j} \\ &= (dom_i(\alpha) \cdot_i \beta)^{*_j}. \end{aligned}$$

The proof of the second property in (1) for $k = i$ is dual.

The proofs for (2) are very similar, but in the base case, $dom_j(x) \cdot_i 1_j \leq 1_j \cdot_i 1_j$ needs to be shown, which follows from Lemma 7.3(2) and needs a strong ω-quantale.

For (3), we first use induction on *k* to show that $(\alpha \cdot_j \beta)^{k_i} \leq \alpha^{k_i} \cdot_j \beta^{k_i}$. In the base case, $1_i = 1_i \cdot_j 1_i$. For the induction step, suppose $(\alpha \cdot_j \beta)^{k_i} \leq \alpha^{k_i} \cdot_j \beta^{k_i}$. Then

$$\begin{aligned}(\alpha \cdot_j \beta)^{(k+1)_i} &= (\alpha \cdot_j \beta) \cdot_i (\alpha \cdot_j \beta)^{k_i} \\ &\leq (\alpha \cdot_j \beta) \cdot_i \alpha^{k_i} \cdot_j \beta^{k_i} \\ &\leq (\alpha \cdot_i \alpha^{k_i}) \cdot_j (\beta \cdot_i \beta^{k_i}) \\ &= \alpha^{k+1_i} \cdot_j \beta^{(k+1)_i}.\end{aligned}$$

Using this property, we get $(\alpha \cdot_j \beta)^{k_i} \leq \alpha^{*_i} \cdot_j \beta^{*_i}$ for all $k \geq 0$ and thus $(\alpha \cdot_j \beta)^{*_i} \leq \alpha^{*_i} \cdot_j \beta^{*_i}$ by properties of sup. □

Proof of Lemma 7.8 For (1),

$$\begin{aligned}dom_0(\alpha) \cdot_1 dom_0(\alpha) &= dom_1(dom_0(\alpha)) \cdot_1 dom_1(dom_0(\alpha)) \\ &= dom_1(dom_0(\alpha) \\ &= dom_0(\alpha)\end{aligned}$$

and the proof for codomain follows by opposition.

For (2),

$$\begin{aligned}dom_i(\alpha \cdot_j \beta) &= dom_i(dom_j(\alpha \cdot_j \beta)) \\ &= dom_i(dom_j(\alpha \cdot_j dom_j(\beta))) \\ &= dom_j(\alpha \cdot_j dom_j(\beta)),\end{aligned}$$

and the proof for codomain follows by opposition.

The proofs of (3) are similar to those of (2), inserting cod_j instead of dom_j in the first step.

For (4),

$$\begin{aligned}dom_i(\alpha \cdot_i \beta) &= dom_i(\alpha \cdot_i dom_i(\beta)) \\ &= dom_i(\alpha \cdot_i dom_j(dom_i(\beta))) \\ &= dom_i(\alpha \cdot_i dom_i(dom_j(\beta))) \\ &= dom_i(\alpha \cdot_i dom_j(\beta))\end{aligned}$$

and the proof for codomain follows by opposition.

For (5),

$$\begin{aligned}dom_i(x \cdot_i y) &= dom_i(cod_j(x \cdot_i y)) \\ &\leq dom_i(cod_j(x) \cdot_i cod_j(y)) \\ &= dom_i(cod_j(x) \cdot_i dom_i(cod_j(y))) \\ &= dom_i(cod_j(x) \cdot_i dom_i(y)) \\ &= dom_i(cod_j(x) \cdot_i y).\end{aligned}$$

The second step uses the morphism law for cod_j. If the quantale is strong, it can be replaced by an equational step. The proofs for cod_i follow by opposition.

Item (6) is immediate from (1) and interchange.

Items (7) and (8) are immediate consequences of the weak homomorphism laws and the homomorphism laws, respectively.

For (9),

$$\begin{aligned} dom_i(\alpha) \cdot_j dom_i(\beta) &= dom_i(dom_i(\alpha) \cdot_j dom_i(\beta)) \cdot_0 (dom_i(\alpha) \cdot_j dom_i(\beta)) \\ &\leq (dom_i(dom_i(\alpha)) \cdot_j dom_i(dom_i(\beta)))) \cdot_0 (dom_i(\alpha) \cdot_j dom_i(\beta)) \\ &\leq (dom_i(\alpha) \cdot_j 1_i) \cdot_0 (1_i \cdot_j dom_i(\beta)) \\ &\leq (dom_i(\alpha) \cdot_j 1_j) \cdot_0 (1_i \cdot_j dom_j(\beta)) \\ &\leq dom_i(\alpha) \cdot_i dom_i(\beta), \end{aligned}$$

where the first step uses domain absorption, the second a weak homomorphism law, and the remaining steps are straightforward approximations. For the converse direction,

$$\begin{aligned} dom_i\alpha) \cdot_i dom_i(\beta) &= (dom_i(\alpha) \cdot_j dom_i(\alpha)) \cdot_i (dom_i(\beta) \cdot_j dom_i(\beta)) \\ &\leq (dom_i(\alpha) \cdot_i dom_i(\beta)) \cdot_j (dom_i(\alpha) \cdot_i dom_i(\beta)) \\ &\leq (dom_i(\alpha) \cdot_i 1_i) \cdot_j (1_i \cdot_i dom_i(\beta)) \\ &= dom_i(\alpha) \cdot_j dom_i(\beta), \end{aligned}$$

where the first step uses (1), the second step the interchange law, and the remaining steps are obvious.

For (10), using in particular (9),

$$\begin{aligned} &(dom_i(\alpha) \cdot_j dom_i(\beta)) \cdot_i (dom_i(\gamma) \cdot_j dom_i(\delta)) \\ &= (dom_i(\alpha) \cdot_i dom_i(\beta)) \cdot_i (dom_i(\gamma) \cdot_i dom_i(\delta)) \\ &= (dom_i(\alpha) \cdot_i dom_i(\gamma)) \cdot_i (dom_i(\beta) \cdot_i dom_i(\delta)) \\ &= dom_i(dom_i(\alpha) \cdot_i dom_i(\gamma)) \cdot_i dom_i(dom_i(\beta) \cdot_i dom_i(\delta)) \\ &= dom_i(dom_i(\alpha) \cdot_i dom_i(\gamma)) \cdot_j dom_i(dom_i(\beta) \cdot_i dom_i(\delta)) \\ &= (dom_i(\alpha) \cdot_i dom_i(\gamma)) \cdot_j (dom_i(\beta) \cdot_i dom_i(\delta)). \end{aligned}$$

□

Proof of Corollary 11.1 For (1), we show explicit proofs for the globular structure, starting with the interchange laws. For $W, X, Y, Z \subseteq C$,

$$\begin{aligned} a \in (W \odot_j X) \odot_i (Y \odot_j Z) &\Leftrightarrow \exists w \in W, x \in X, y \in Y, z \in Z.\ a \in (w \odot_j x) \odot_i (y \odot_j z) \\ &\Rightarrow \exists w \in W, x \in X, y \in Y, z \in Z.\ a \in (w \odot_i y) \odot_j (x \odot_i z) \\ &\Leftrightarrow a \in (W \odot_i X) \odot_j (Y \odot_i Z). \end{aligned}$$

This requires only the interchange law in *C*. It remains to extend the globular laws.

For $dom_j(x \cdot_i y) \leq dom_j(x) \cdot_i dom_j(y)$ and $X, Y \subseteq X$,

$$\begin{aligned}
a \in s_j(X \odot_i Y) &\Leftrightarrow \exists b \in X \odot_i Y.\ a = s_j(b)\\
&\Leftrightarrow \exists b, c \in X, d \in Y.\ a = s_j(b) \wedge b \in c \odot_i d\\
&\Leftrightarrow \exists c \in X, d \in Y.\ a \in s_j(c \odot_i d)\\
&\Rightarrow \exists c \in X, d \in Y.\ a \in s_j(c) \odot_i s_j(d)\\
&\Leftrightarrow \exists c \in s_j(X), d \in s_j(Y).\ a \in c \odot_i d\\
&\Leftrightarrow a \in s_j(X), \cdot_i s_j(Y).
\end{aligned}$$

The implication in the fourth step can be replaced by $\Leftrightarrow$ if C is an ω-category. The proofs of $cod_j(x \cdot_i y) \leq cod_j(x) \cdot_i cod_j(y)$ and its strong variant follows by opposition. These proofs require only the respective morphism laws in C.

For $dom_i(x \cdot_j y) \leq dom_i(x) \cdot_j dom_i(y)$, note that $s_i(x \odot_j y) \subseteq s_i(x) \odot_j s_i(y)$ is derivable even in the reduced axiomatisation of n-catoids (Proposition 6.13). The proof is then similar to the previous one.

Finally, for $dom_j \circ dom_i = dom_i$, note that $s_j \circ s_i = s_i$ is derivable in the reduced axiomatisation of n-catoids (Proposition 6.13). Hence let $X \subseteq C$. Then

$$a \in s_j(s_i(X)) \Leftrightarrow \exists b \in X.\ a = s_j(s_i(b)) \Leftrightarrow \exists b \in X.\ a = s_i(b) \Leftrightarrow a \in s_i(X).$$

For (2), we consider the morphism laws for s_i and t_i. We consider these correspondences one by one between quantales and catoids and thus mention the laws for s_i and t_i for the context without locality. As for Theorem 8.5, we start with an explicit proof of interchange for multioperations and powersets.

$$\begin{aligned}
(w \odot_j x) \odot_i (y \odot_j z) &= (\{w\} \cdot_j \{x\}) \cdot_i (\{y\} \cdot_j \{z\})\\
&\subseteq (\{w\} \cdot_i \{y\}) \cdot_j (\{x\} \cdot_i \{z\})\\
&= (w \odot_i y) \odot_j (x \odot_i z).
\end{aligned}$$

For the morphism law for s_j,

$$s_j(x \odot_i y) = dom_j(\{x\} \cdot_i \{y\}) \subseteq dom_j(\{x\}) \cdot_i dom_j(\{y\}) = s_j(x) \odot_i s_j(y).$$

The inclusion in the second step becomes an equality if Q is strong. The proofs for the morphism laws for t_j are dual. The proofs of the laws for s_i and t_i are very similar. $\square$

Proof of Lemma 11.3 The proofs of (1) and (2) are obvious; (3) is immediate from (2): $X = s(X)X \subseteq XX^{-}X$. For (4),

$$\begin{aligned}
a \in C_0 \cap XX^{-} &\Leftrightarrow s(a) = a \wedge \exists b \in X, c \in X^{-}\ s(a) \in b \odot c\\
&\Leftrightarrow s(a) = a \wedge \exists b \in X.\ s(a) = s(b)\\
&\Leftrightarrow a \in s(X)
\end{aligned}$$

and the proof for t is dual. The proofs for (5) are very similar:

$$\begin{aligned}a \in C_0 \cap X\top &\Leftrightarrow s(a) = a \wedge \exists b \in X, c \in C.\ s(a) \in b \odot c\\&\Leftrightarrow s(a) = a \wedge \exists b \in X.\ s(a) = s(b)\\&\Leftrightarrow a \in s(X)\end{aligned}$$

and likewise for t. Finally, for (6), $X\top = s(X)X\top \subseteq s(X)\top\top \subseteq s(X)\top$ and $s(X)\top \subseteq XX^{-}\top \subseteq X\top\top \subseteq X\top$. □

Proof of Lemma 12.2 The equations in (1) follow immediately from Lemma 7.8(2)-(5).

For (2), $dom_i(\alpha \cdot_i dom_j(\beta \cdot_j \gamma)) \leq dom_i(\alpha \cdot_i (dom_i(\beta) \cdot_j dom_i(\gamma)))$ follows from the above using the weak morphism axiom, and a dual property holds for cod_i and cod_j.

For (3),

$$\begin{aligned}&dom_i(dom_k(\alpha) \cdot_i dom_j(\beta \cdot_j \gamma))\\&\leq dom_i(dom_k(\alpha) \cdot_i (dom_i(\beta) \cdot_j dom_i(\gamma)))\\&\leq dom_i((dom_k(\alpha) \cdot_i dom_i(\beta)) \cdot_j (dom_k(\alpha) \cdot_i dom_i(\gamma)))\\&\leq dom_i(dom_k(\alpha) \cdot_i dom_i(\beta)) \cdot_j dom_i(dom_k(\alpha) \cdot_i dom_i(\gamma))\\&= dom_i(dom_k(\alpha) \cdot_i \beta) \cdot_j dom_i(dom_k(\alpha) \cdot_i \gamma).\end{aligned}$$

The first step uses an approximation from (2), the second Lemma 7.8(6), the third a weak morphism axiom and the last domain locality. The proof for codomains is similar.

For (4),

$$\begin{aligned}dom_j(\alpha \cdot_j dom_j(dom_i(\beta \cdot_i \gamma))) &= dom_j(\alpha \cdot_j dom_i(dom_j(\beta \cdot_i \gamma)))\\&\leq dom_j(\alpha \cdot_j dom_i(dom_j(\beta) \cdot_i dom_j(\gamma)))\\&= dom_j(\alpha \cdot_j dom_i(dom_j(\beta) \cdot_i \gamma)).\end{aligned}$$

The first step uses Lemma 7.3(5), the second applies a morphism axiom, the third 7.8(4). The second step becomes an equality if Q is strong. The remaining proofs are similar.

For (5),

$$\begin{aligned}dom_0(\gamma) \cdot_0 dom_1(\alpha \cdot_1 \beta) &= dom_1(dom_0(\gamma)) \cdot_0 dom_1(\alpha \cdot_1 \beta)\\&= dom_1(dom_0(\gamma) \cdot_0 (\alpha \cdot_1 \beta))\\&\leq dom_1((dom_0(\gamma) \cdot_0 \alpha) \cdot_1 (dom_0(\gamma) \cdot_0 \beta))\end{aligned}$$

The first step uses an axiom, the second a strong morphism axiom and the third Lemma 7.8(6). The second proof is similar. □

Proof of Lemma 13.2 For (1), $dom(\alpha) \leq dom(\alpha)dom(\alpha)^{\circ}dom(\alpha) \leq dom(\alpha)^{\circ}$, and thus $dom(\alpha)^{\circ} \leq dom(\alpha)^{\circ\circ} = dom(\alpha)$ as converse is order preserving. The property for cod then follows by opposition.

For (2),

$$
\begin{aligned}
dom(\alpha^\circ) &= dom((\alpha cod(\alpha))^\circ) \\
&= dom(cod(\alpha)^\circ x^\circ) \\
&= dom(cod(\alpha)\alpha^\circ) \\
&= cod(\alpha)dom(\alpha^\circ) \\
&= cod(\alpha dom(\alpha^\circ)) \\
&= cod(\alpha^{\circ\circ}dom(\alpha^\circ)^\circ) \\
&= cod((dom(\alpha^\circ)\alpha^\circ)^\circ) \\
&= cod(\alpha^{\circ\circ}) \\
&= cod(\alpha).
\end{aligned}
$$

The export steps in the fourth and fifth lines of the proof work because of the compatibility laws $dom \circ cod = cod$ and $cod \circ dom = dom$ of modal semirings. The second property then follows by opposition. □

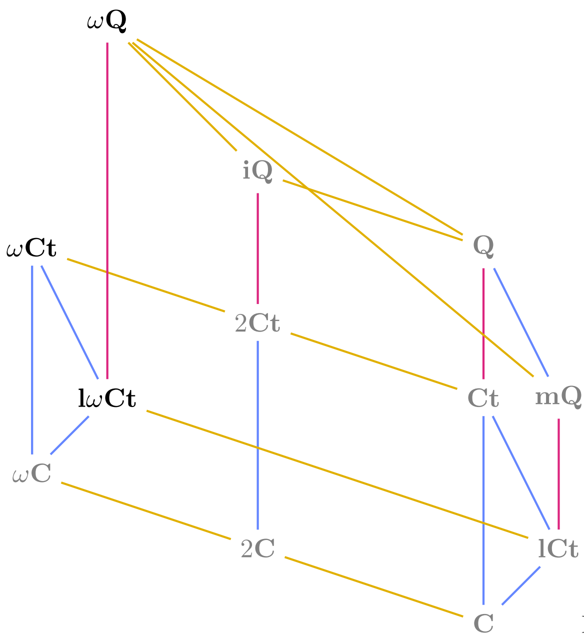
Proof of Lemma 14.2 First, $\alpha^{\circ*\circ} = (1 + \alpha^{\circ*}\alpha^\circ)^\circ = 1 + \alpha \cdot \alpha^{\circ*\circ}$ by star unfold and properties of involution. Thus $\alpha^* \le \alpha^{\circ*\circ}$ by star induction and therefore $\alpha^{*\circ} \le \alpha^{\circ*}$.

Second, $\alpha^{*\circ} = (1 + \alpha\alpha^*)^\circ = 1 + \alpha^\circ\alpha^{*\circ}$ by star unfold and properties of involution. Thus $\alpha^{\circ*} \le \alpha^{*\circ}$ by star induction. □

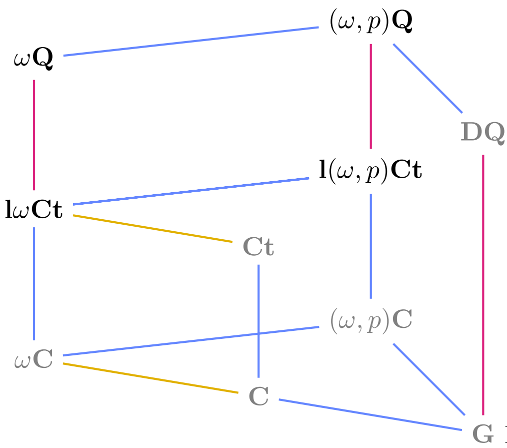
Proof of Lemma 14.6 It suffices to show that $1_i + (\alpha \cdot_j \beta) \cdot_i (\alpha^{*_i} \cdot_j \beta^{*_i}) \le (\alpha^{*_i} \cdot_j \beta^{*_i})$ by star induction. Then $1_i \le (\alpha^{*_i} \cdot_j \beta^{*_i})$ holds because $1_i = 1_i \cdot_j 1_i$ and $1_i \le \alpha^{*_i}$, $1_i \le \beta^{*_i}$ by standard Kleene algebra. Also, we have $(\alpha \cdot_j \beta) \cdot_i (\alpha^{*_i} \cdot_j \beta^{*_i}) \le (\alpha^{*_i} \cdot_j \beta^{*_i})$ by interchange and $\alpha \cdot_i \alpha^{*_i} \le \alpha^{*_i}$ and $\beta \cdot_i \beta^{*_i} \le \beta^{*_i}$, again by standard Kleene algebra. □

Diagrams for Main Structures

The main structures used in this article are related in the following two diagrams. The diagrams are drawn by analogy to the Hasse diagrams of order theory. An increasing ***blue*** line means that the class at the lower node is a subclass of the class at the higher node. For instance, every (single-set) category is a local catoid. An increasing orange line means that every element in the lower class can be obtained by truncation from an element of the higher one. For instance, every 2-catoid can be obtained by truncation from an ω-catoid. An increasing *red* line means that the upper and lower class are related by correspondence in the sense of Jónsson-Tarski duality. This the case, for instance, for local ω-catoids and ω -quantales.



In the diagram above we write C, 2C and ωC for the classes of (single-set) categories, strict 2-categories and strict ω-categories, respectively. Further Ct, lCt, 2Ct, ωCt lωCt stand for the classes of catoids, local catoids, 2-catoids, ω-catoids and local ω-catoids, respectively. Finally, Q, iQ, mQ, ωQ, indicate the classes of quantales, interchange quantales, modal quantales and ω-quantales, respectively. Structures introduced in this articles are shown in black, all others are shown in **gray**.



In the second diagram we further write G for the class of groupoids, (ω, p)C for that of $(\omega, p$-categories, l(ω, p)Ct for the class of (ω, p)-catoids, DQ for that of Dedekind quantales and (ω, p)Q for that of (ω, p)-quantales.

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Data Availability The Isabelle/HOL components supporting this work can be found in the Archive of Formal Proofs (see references in article).

Code Availability The Isabelle/HOL components supporting this work can be found in the Archive of Formal Proofs [26–28].

Declarations

Competing interest The authors have no competing interest to declare.

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