Theoretical Computer Science 1049 (2025) 115365

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

Additive approximation algorithm for geodesic centers in δ-hyperbolic graphs

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ARTICLE INFO

Section Editor: Pinyan Lu

Keywords:
Hyperbolicity
Approximation algorithms
Isometric paths
Minimum eccentricity shortest paths

ABSTRACT

For an integer $k \geq 1$, the objective of k-Geodesic Center is to find a set C of k isometric paths such that the maximum distance between any vertex v and C is minimised. Introduced by Gromov, δ-*hyperbolicity* measures how treelike a graph is from a metric point of view. Our main contribution in this paper is to provide an additive $O(\delta)$-approximation algorithm for k-Geodesic Center on δ-hyperbolic graphs. On the way, we define a coarse version of the pairing property introduced by Gerstel and Zaks (1994) [28] and show it holds for δ-hyperbolic graphs. This result allows to reduce the k-Geodesic Center problem to its rooted counterpart, a main idea behind our algorithm. We also adapt a technique of Dragan and Leitert (2017) [24] to show that for every $k \geq 1$, k-Geodesic Center is NP-hard even on partial grids.

1. Introduction

Given a graph G, the k-Geodesic Center problem asks to find a collection C of k isometric paths such that the maximum distance between any vertex and C is minimised. This problem may arise in determining a set of k "most accessible" speedy line routes in a network and can find applications in communication networks, transportation planning, water resource management and fluid transportation [24]. The decision version of this problem asks, given a graph G and two integers k and R, whether there exists a collection C of k isometric paths such that any vertex of G is at distance at most R from C.

k-Geodesic Center is related to several algorithmic problems studied in the literature. k-Geodesic Center is a generalisation of Minimum Eccentricity Shortest Path (MESP) where given an integer R, the objective is to decide if there exists an isometric path P such that the maximum distance between any vertex and P is at most R [24]. Clearly, 1-Geodesic Center is equivalent to MESP. If, instead of isometric paths, we ask whether there exists a subset of k vertices of eccentricity at most R, we obtain the decision version of k-center which is one of the most studied facility location problem in the literature [30,31,35,41,42]. The solution of a k-Geodesic Center can be thought of as a relaxation of k-Center. k-Geodesic Center is also related to Isometric Path Cover, where the objective is to find the minimum number of isometric paths that contains all vertices of the input graph. Study of the algorithmic aspects of Isometric Path Cover has garnered much attention recently [11,25,27,13].

All the three problems (i.e., IPC, MESP, and k-Center) are NP-hard for general graphs but are known to admit exact polynomial time algorithms when the given graph G is a tree [11,24,43]. This raises the question about the complexity of these problems when the input graph is *close to a tree*? In this paper, we consider the graph parameter δ-*hyperbolicity* [29], which measures how treelike a

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https://doi.org/10.1016/j.tcs.2025.115365
Received 4 April 2024; Received in revised form 1 March 2025; Accepted 27 May 2025

Available online 2 June 2025

graph is from a metric point of view. See Section 2 for a formal definition. Graphs with constant δ-hyperbolicity are called *hyperbolic* graphs. From a practical perspective, the study of δ-hyperbolicity of graphs is motivated by the fact that many real-world graphs are tree-like [1,2,32] or have small δ-hyperbolicity [8,26,40]. From a theoretical perspective, many popular graph classes like interval graphs, chordal graphs, α_i-metric graphs [22], graphs with bounded tree-length [21], link graphs of simple polygons [16] have constant δ-hyperbolicity.

When k is part of the input, k-Geodesic Center remains NP-hard even on chordal graphs. This follows from the fact that Isometric Path Cover remains NP-hard on chordal graphs [12]. Since chordal graphs have hyperbolicity at most 1, it follows that k-Geodesic Center remains NP-hard even on graphs with hyperbolicity at most 1.

Polynomial time approximation algorithms with an error (additive or multiplicative) depending only on the δ-hyperbolicity of G exist for MESP [39], k-Center [18,26], and Isometric Path Cover [12]. Motivated by the above results, in this paper, we provide an additive $O(\delta)$-approximation algorithm[1] for k-Geodesic Center on δ-hyperbolic graphs for arbitrary k. The same algorithmic approach leads to an exact polynomial time algorithm in case of trees.

Theorem 1. *Let G be a δ-hyperbolic graph and k be an integer. Then, there is a polynomial time $O(\delta)$-additive approximation algorithm for k-Geodesic Center on G.*

Our algorithm has mainly two stages. In the first stage, we solve the "rooted" version of $(2k-1)$-Geodesic Center, where we require that all isometric paths in the solution have a common end-vertex. Then to reduce the number of isometric paths, we use the *shallow pairing* property of δ-hyperbolic graphs. See Definition 6. Intuitively, this property ensures that the $2k$-many end-vertices of the $2k-1$ isometric paths obtained in the first stage can be "paired" to obtain k many isometric paths which together provide an additive $O(\delta)$-approximation algorithm for k-Geodesic Center. We think that the shallow pairing property could also be interesting in itself and for other algorithmic applications.

We also adapt a technique of Dragan & Leitert, (TCS '17) to show that for every $k \geq 1$, k-Geodesic Center is NP-hard even on *partial grids*. A graph is a *partial grid* if it is a subgraph of $(k \times k)$-grid for some positive integer k.

Theorem 2. *For every integer $k \geq 1$, k-Geodesic Centre is NP-hard even on partial grids.*

Related Works To the best of our knowledge, the computational complexity of k-Geodesic Center for arbitrary k have not been studied before. Therefore, we begin by surveying the relevant results on 1-Geodesic Center i.e., the MESP problem. Dragan & Leitart [24] gave several constant factor approximation algorithms for MESP with varying running times on general graphs. In another paper [23], the authors proposed polynomial time algorithms for MESP on graph classes like *chordal* graphs and *distance hereditary* graphs. In fact the authors proved that, MESP admits an $O(n^{\gamma+3})$-time algorithm on graphs with *projection gap* at most γ, and n vertices. The parameter *projection gap* generalizes the notion of δ-hyperbolicity. Their result implies that MESP admits an $O(n^{4\delta+4})$-time algorithm on graphs with δ-hyperbolicity at most δ, and n vertices. We do not know if MESP admits a fixed parameter algorithm with respect to δ-hyperbolicity. The same authors also proposed additive approximation algorithms for graphs with bounded *tree-length*. Fixed parameter tractability of MESP with respect to various graph parameters like modular width, distance to cluster graph, maximum leaf number, feedback edge set, etc. have been also studied recently [5,34]. As noted by Kučera and Suchý [34], the fixed parameter tractability of MESP with respect to *tree-width* is an interesting open problem. (Tree-width measures how far a graph is from a tree from a structural point of view.) Relation of MESP with other problems like the *minimum distortion embedding on a line* [24] and *k-laminar* problem [6] have been established.

Dragan & Leitart [23] observed that MESP admits an additive $O(\delta \log n)$-approximation algorithm on graphs with δ-hyperbolicity at most δ and n vertices. Their proof uses the fact that the tree-length of δ-hyperbolic graphs are at most $O(\delta \log n)$. As the best known bound for tree-length of δ-hyperbolic graphs is $O(\delta \log n)$, this method seems not directly provide constant error in case of hyperbolic graph. Then, in the PhD Thesis of A.O. Mohammed [39], an $O(\delta)$-approximation algorithm for MESP on δ-hyperbolic graphs has been proposed. Other examples include fast additive $O(\delta)$-approximation algorithms for finding the diameter, radius, and all eccentricities [16–18] as well as packing and covering for families of quasiconvex sets [15]. Theorem 1 adds k-Geodesic Center in the list of problems admitting an additive approximation algorithm depending only on the δ-hyperbolicity of the input graph. Recently, the computational complexity of maximum independent set of planar δ-hyperbolic graphs has been studied [33].

Organisation: In Section 2 we introduce some terminologies. In Section 3, we introduce the notion of shallow pairing and prove its existence in δ-hyperbolic graphs. In Sections 4 and 5, we prove Theorems 1 and 2, respectively.

2. Preliminaries

Basic notations: For two vertices $u, v \in V(G)$, $\sigma(u, v)$ shall denote an (u, v)-isometric path in G and the length (i.e., the number of edges) in $\sigma(u, v)$ is denoted as $d(u, v)$, the *distance* between u and v. If an isometric path P of G has a vertex r as end vertex, then P is called an *r-path*.

[1] A feasible solution for a minimization problem is said to be *additive α-approximate* if its objective value is at most the optimum value plus α. An *additive α-approximation algorithm* for a minimization problem is a polynomial time algorithm that produces an additive α-approximate solution for every instance of the input.

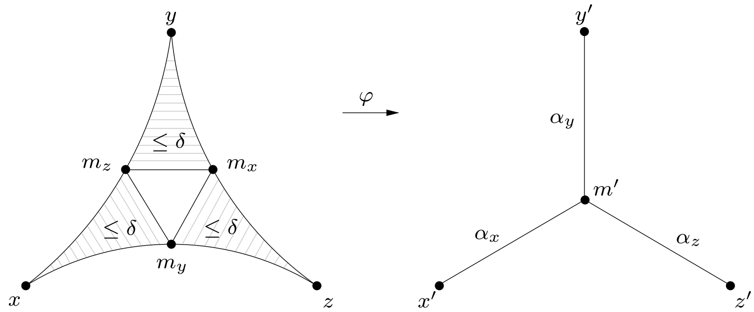


Fig. 1. A δ-thin geodesic triangle.

For two sets $S_1, S_2 \subseteq V(G)$ of vertices, $d(S_1, S_2) = \min\{d(u,v) : u \in S_1, v \in S_2\}$ is the distance between S_1 and S_2. For convenience, if one subset of vertices is a singleton, we abbreviate $d(\{v\}, S)$ by $d(v,S)$. For an integer k and a set of vertices S, the *k-neighbourhood (or k-ball) around S*, denoted as $B_k(S)$, is the set of all vertices v such that $d(v,S) \le k$. For an integer R, a collection C of isometric paths of G is an *R-cover* of G if

$$\bigcup_{P \in C} B_R(V(P)) = V(G)$$

For an integer k, the symbol R_k^* shall denote the minimum integer for which there is a R_k^*-cover C of G with $|C| = k$. If every path in C is an r-path, then C is an *(r,R)-cover* of G. For an integer R, and a vertex r, a subset $S \subseteq V(G)$ is a *(r,R)-packing* if the R-neighbourhood of any r-path P contains at most one vertex of S, i.e., $|B_R(V(P)) \cap S| \le 1$. Note that if S is an (r,R)-packing of G then any (r,R)-cover of G has size at least $|S|$. Indeed, the R-neighbourhood of any r-path covers at most one vertex in S. Hence, it is not possible to cover S with less than $|S|$ r-paths.

Definitions related to δ-hyperbolicity: Let (X,d) be a metric space. A geodesic segment joining two points x and y from X is a map ρ from the segments $\sigma(a,b)$ of length $|a-b| = d(x,y)$ to X such that $\rho(a) = x$, $\rho(b) = y$, and $d(\rho(s), \rho(t)) = |s-t|$ for all $s,t \in \sigma(a,b)$. A metric space (X,d) is geodesic if every pair of points in X can be joined by a geodesic. We will denote by $\sigma(x,y)$ any geodesic segment connecting the points x and y.

Introduced by Gromov [29], δ-hyperbolicity measures how treelike a graph is from a metric point of view. Recall that a metric space (X,d) embeds into a tree network (with positive real edge lengths), if and only if for any four points u,v,w,x the two larger of the distance sums $d(u,v)+d(w,x)$, $d(u,w)+d(v,x)$, and $d(u,x)+d(v,w)$ are equal. A metric space (X,d) is called δ-hyperbolic if the two larger distance sums differ by at most 2δ. For a metric space (X,d), the *Gromov product* of two points x,y with respect to a third point z is defined as

$$(x|y)_z = \frac{1}{2}\left(d(x,z) + d(z,y) - d(x,y)\right)$$

Equivalently, a metric space (X,d) is δ-hyperbolic if for any four points u,v,w,x,

$$(u|w)_x \ge \min\left\{(u|v)_x, (v|w)_x\right\} - \delta.$$

A connected graph $G = (V,E)$ equipped with standard graph metric d_G is δ-hyperbolic if (V, d_G) is a δ-hyperbolic metric space. The δ-hyperbolicity $\delta(G)$ of a graph G is the smallest δ such that G is δ-hyperbolic.

There exist several equivalent definitions of δ-hyperbolic metric spaces involving different but comparable values of δ. In the proof of Theorem 1, we will use the definition employing δ-thin geodesic triangles. A *geodesic triangle* $\Delta(x,y,z)$ is a union $\sigma(x,y) \cup \sigma(x,z) \cup \sigma(y,z)$ of three geodesic segments connecting these vertices. Let m_x be the point of the geodesic $\sigma(y,z)$ located at distance $\alpha_y := (x|z)_y$ from y. Then m_x is located at distance $\alpha_z := (x|y)_z$ from z because $\alpha_y + \alpha_z = d(y,z)$. Analogously, define the points $m_y \in \sigma(x,z)$ and $m_z \in \sigma(x,y)$ both located at distance $\alpha_x := (y|z)_x$ from x; see Fig. 1.

There exists a unique isometry φ which maps the geodesic triangle $\Delta(x,y,z)$ to a star $T(x,y,z)$ consisting of three solid segments $\sigma(x',m')$, $\sigma(y',m')$, and $\sigma(z',m')$ of length α_x, α_y, and α_z, respectively. This isometry maps the vertices x,y,z of $\Delta(x,y,z)$ to the respective leaves x',y',z' of $T(x',y',z')$ and the points m_x, m_y, and m_z to the center m of this tripod. Any other point of $T(x',y',z')$ is the image of exactly two points of $\Delta(x,y,z)$. A geodesic triangle $\Delta(x,y,z)$ is called δ-thin [3] if for all points $u,v \in \Delta(x,y,z)$, $\varphi(u) = \varphi(v)$ implies $d(u,v) \le \delta$.

The following result shows that the δ-hyperbolicity of geodesic space can be approximated by the maximum thinness of its geodesic triangles.

Proposition 3 *([3,9,29]). Geodesic triangles of any geodesic δ-hyperbolic space are 4δ-thin. Conversely, geodesic space with δ-thin triangles are δ-hyperbolic.*

Every graph $G=(V,E)$ equipped with its standard distance d_G can be transformed into a geodesic space (X,d) by replacing every edge $e=(u,v)$ by a solid segment $[u,v]$ of length 1. These segments may intersect only at their common ends. Then (V,d_G) isometrically embeds naturally in (X,d). The *thinness* $\tau(G)$ of a graph G is the smallest integer δ, such that all geodesic triangles of the geodesic space arising from G are δ-thin. When thinness of a graph is δ, then it is also called a δ-*thin* graph.

Subdivisions and partial grids: For a graph G, its ℓ-*subdivision*, denoted as G_ℓ, is obtained by replacement of all its edges by paths of a fixed length $\ell \geq 1$. An *equal subdivision* of G is an ℓ-subdivision for some $\ell \geq 1$. The vertices of G in G_ℓ are the *original vertices*. We shall use the following result.

Lemma 4 *([12]). Let G be a planar graph with maximum degree* 4. *Then there exists a partial grid graph H, which is an equal subdivision of G and contains at most* $O(|V(G)|^3)$ *vertices.*

3. Pairings and shallow pairings

In this section we first recall the definition of *the pairing property* following the terminology of [4]. Then, we introduce a coarse version of this property, called *shallow pairing property*,[2] and show that this relaxed property holds for δ-hyperbolic graphs.

Given a connected graph G, a *profile* of length n is any sequence $\pi=(x_1,\dots,x_n)$ of n vertices of G. The *total distance* of a vertex v of G is defined by $T_\pi(v)=\sum_{i=1}^n d(v,x_i)$. A *pairing* P is a partition of an even profile π of length $n=2k$, into k disjoint pairs. For a pairing P, define $D_\pi(P)=\sum_{\{a,b\}\in P} d(a,b)$. The notion of pairing was defined by Gerstel and Zaks [28]; they also proved the following weak duality between the functions T_π and D_π:

Lemma 5 *([28]). For any even profile π of length $n=2k$ of a connected graph G, for any pairing P of π and any vertex v of G,* $D_\pi(P)\leq T_\pi(v)$ *and the equality holds if and only if* $v\in\bigcap_{\{a,b\}\in P} I(a,b)$.

We say that a graph G satisfies the *pairing property* if for any even profile π there exists a pairing P of π and a vertex v of G such that $D_\pi(P)=T_\pi(v)$, i.e., the functions T_π and D_π satisfy the strong duality. Such a pairing is called a *perfect pairing*. By Lemma 5, the pairing property of [28] coincides with the *intersecting-intervals property* of [36]. It was shown in [28] that trees satisfy the pairing property. More generally, it was shown in [37] and independently in [19] that cube-free median graphs also satisfy the pairing property. It was proven in [36] that the complete bipartite graph $K_{2,n}$ satisfies the pairing property. As observed in [36], the 3-cube is a simple example of a graph not satisfying the pairing property.

In general, δ-hyperbolic graphs do not satisfy the pairing property, but, as shown below, they satisfy some coarse variant of the pairing property. Before defining this variant, let us first define the notion of γ-*shallow pairing*

Definition 6 *(γ-shallow pairing).* Let G be a graph and π be an even profile of length $2k$. A γ-*shallow pairing* of π is a pairing P such that, there exists a vertex v with $(x|y)_v\leq\gamma$ for every $\{x,y\}\in P$.

In the definition of a perfect pairing P, the vertex v belongs to every interval between a pair of vertices in P. As the following lemma shows, in the definition of a γ-shallow pairing P_γ, the vertex v is at distance at most $\gamma+\tau(G)$ from every isometric path between a pair of vertices in P_γ.

Lemma 7. *Let G be a δ-thin graph, π be an even profile of length $2k$ and P be a γ-shallow pairing of π. Then there exists a vertex v such that $d(v,\sigma(x,y))\leq\gamma+\delta$ for every geodesic $\sigma(x,y)$ with $\{x,y\}\in P$.*

Proof. By definition of a γ-shallow pairing, there exists a vertex v such that $(x|y)_v\leq\gamma$ for every $\{x,y\}\in P$. For any pair of vertices $\{x,y\}\in P$, consider the geodesic triangle $\Delta(x,y,v)$ and let $\sigma(v,x),\sigma(v,y)$, and Q be the sides of this triangle. Let x',y' be the points of $\sigma(v,x)$ and $\sigma(v,y)$, respectively, located at distance $(x|y)_v$ from v. Since G is δ-thin, $d(x',y')\leq\delta$, moreover $d(x',z')\leq\delta$ and $d(y',z')\leq\delta$, where z' is the point of Q at distance $(y|v)_x$ from v and at distance $(x|v)_y$ from y. Since, $(x|y)_v\leq\gamma$, we conclude that $d(v,Q)\leq d(v,z')\leq d(v,x')+d(x',z')\leq\gamma+\delta$. □

We say that a graph G satisfies the γ-*shallow pairing property* if for any even profile π there exists a γ-shallow pairing P of π.

The existence and the computation in polynomial time of a $(2\delta+\frac{1}{2})$-shallow pairing in a δ-thin graph G can be obtained using the concept of *fiber* that was introduced in [15]. For a vertex $u\in V(G)$ and a profile π, the *fiber* of x with respect to a vertex u is the set of vertices

$$F_u(x)=\{y\in\pi : (x|y)_u\geq 2\delta+1\}.$$

From Claim 1 and 2 of [15], the following lemma holds.

[2] The notion of approximate (shallow) pairing in hyperbolic graphs was defined by Victor Chepoi, who also asked the question about their existence.

Lemma 8. *For any graph G and any even profile π of length $2k$, there is a vertex $v \in V(G)$ such that $|F_v(x)| \leq k$ for any vertex $x \in \pi$.*

Lemma 8 is useful to prove the following result:

Proposition 9. *Any δ-thin graph G satisfies the $(2\delta + \frac{1}{2})$-shallow pairing property. Moreover, for any δ-thin graph G with n vertices and m edges, and any even profile π of length $2k$, a $(2\delta + \frac{1}{2})$-shallow pairing of π can be computed in $O(mn^2)$ time.*

Proof. First, we will prove that the vertex v whose existence is guaranteed by Lemma 8 can be calculated efficiently. Indeed, an $O(mn^2)$ time algorithm was given in [14] to compute the thinness $\delta = \tau(G)$ of a graph G with n vertices and m edges. The matrix of distances between every pair of vertices can be computed in $O(mn)$ time. Then, for every vertex $u \in V(G)$, it is possible to compute in $O(k^2)$ time the Gromov products $(x|y)_u$ for every pair of vertices $x, y \in \pi$. Within the same running time, it is possible to also compute the fibers $F_u(x)$ for every $x \in \pi$. Indeed, it suffices to add y to $F_u(x)$ and x to $F_u(y)$ when we compute a value $(x|y)_u$ exceeding $2\delta + \frac{1}{2}$. If the size of a fiber $F_u(x)$ becomes larger than k then we can abort and try the next vertex u. The procedure stops once we have found a vertex v such that $F_v(x) \leq k$ for every vertex $x \in \pi$. By Lemma 8, this happens for at least one vertex $v \in V(G)$. Hence, it is possible to find in $O(mn + nk^2)$ time a vertex v such that $|F_v(x)| \leq k$ for every $x \in \pi$. Let H be the graph defined on the vertices of π by adding an edge between two vertices x and y whenever $(x|y)_v \leq 2\delta + \frac{1}{2}$. By the choice of v, every vertex of H has degree at least k. By Dirac's theorem, H is Hamiltonian and thus has a perfect matching M. Such a perfect matching can be computed in $O(\sqrt{n}m)$ time [7,38]. The pairing defined by the end-vertices of edges in M is a $(2\delta + \frac{1}{2})$-shallow pairing of π that can be computed in $O(mn^2)$ time. Indeed, the running time is dominated by the algorithm that computes the thinness of G. □

In the next section, for any pairing P, we will denote by $\mathcal{S}(P)$ a collection of isometric paths, one for each pair of vertices in P, i.e., $\mathcal{S}(P) := \{\sigma(x, y) : \{x, y\} \in P\}$.

4. Additive approximation algorithm

In this section, we prove Theorem 10 which implies Theorem 1 by Proposition 3.

Theorem 10. *Let G be a δ-thin graph with m edges, n vertices and k be an integer. Then, Algorithm 3 is a $O(mn^2 \log n)$-time $(6\delta + 1)$-additive approximation algorithm for k-Geodesic Center on G.*

We provide a brief outline of the proof for the above theorem and organisation of this section. A collection C of isometric paths is "rooted" if all the paths in C have a common end-vertex. First we show in Section 4.1, that the rooted version of the k-Geodesic Center problem where we require that the collection of isometric paths is rooted can be solved in polynomial time up to an additive 2δ error in δ-thin graphs. For that, we use a primal dual algorithm and a dichotomy to find an integer R such that there is a collection of $2k - 1$ isometric rooted paths of eccentricity $R + 2\delta$ and no such collection has eccentricity smaller than R. Then in Section 4.2, we show that any collection C of k isometric paths can be transformed into a rooted collection C' of size $2k - 1$ such that the eccentricity of C' is at most the eccentricity of C plus δ. From this observation and the choice of R, we conclude that no collection of k isometric paths has eccentricity smaller than $R - \delta$. To transform the rooted collection returned by the primal-dual algorithm into a non rooted collection of size k, we also need a converse result. For that, using the $(2\delta + \frac{1}{2})$-shallow pairing property of δ-thin graphs, in Section 4.3, we show that any rooted collection C' of $2k - 1$ isometric paths can be transformed into a non rooted collection C of k isometric paths such that the eccentricity of C is at most the eccentricity of C' plus $3\delta + 1$. We complete the proof in Section 4.4 as follows: the rooted collection of eccentricity $R + 2\delta$ computed by our primal-dual algorithm can be transformed into a collection of size k and eccentricity $R + 5\delta + 1$. Since there is no such collection with an eccentricity less than $R - \delta$, the collection of eccentricity $R + 5\delta + 1$ is optimal up to a $6\delta + 1$ error.

4.1. An algorithm for the Rooted k-Geodesic Center problem

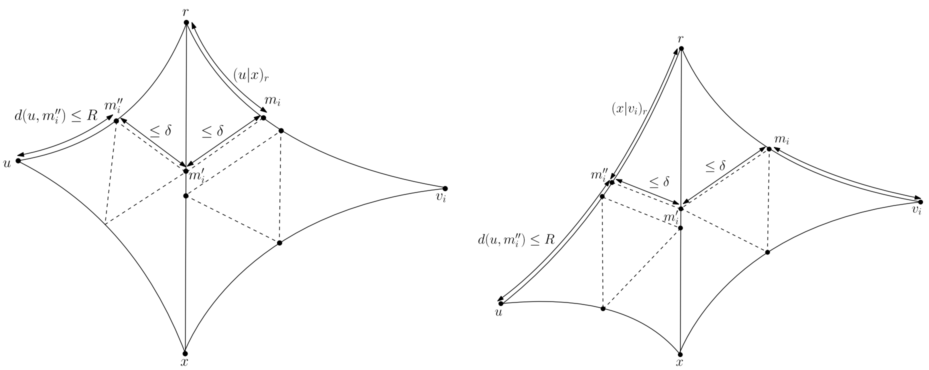
In this Section, we present an algorithm that, given a δ-thin graph G, computes an integer R such that no collection of $2k - 1$ rooted isometric paths has eccentricity smaller than R and there is a collection of $2k-1$ isometric rooted paths of eccentricity $R + 2\delta$. Our description of this algorithm proceeds in two steps. First, we describe Algorithm 1. Given a graph G, a root $r \in V(G)$ and integer R, Algorithm 1 outputs either an $(r, R+2\delta)$-cover of G of size $2k-1$ or a (r, R)-packing of size $2k$. Then, Algorithm 2 uses Algorithm 1 to perform a dichotomy. For every vertex $u \in V(G)$, Algorithm 2 computes the smallest value R_u for which Algorithm 1 outputs a cover. We show that $R := \min\{R_u : u \in V(G)\}$ is an integer such that no collection of $2k - 1$ rooted isometric paths has eccentricity smaller than R and there is a collection of $2k - 1$ isometric rooted paths of eccentricity $R + 2\delta$. We start with the following technical lemma.

Lemma 11. *Let $\sigma(u, v) \cup \sigma(v, w) \cup \sigma(u, w)$ be a geodesic triangle with $d(w, \sigma(u, v)) \leq R$. Then, $(u|v)_w \leq R$.*

Algorithm 1: Algorithm for the ROOTED k-GEODESIC CENTER problem for fixed root.

Input : A δ-thin graph G, $r \in V(G)$, $R \in \mathbb{N}$, an integer $k \leq |V(G)|$
Output: $(r, R + 2\delta)$-cover of G of size $2k - 1$ or an (r, R)-packing of size $2k$.

$X = V(G)$; $i = 0$;
while $i \leq 2k - 1$ *and* $X \neq \emptyset$ **do**
  Let $v_i \in X$ be a vertex with $d(r, v_i) \geq d(r, z)$ for all $z \in X$;
  Let σ_i be any (r, v_i)-isometric path;
  $X_i = \{u \in X :\ \exists$ an r-path P such that $d(u, P) \leq R$ and $d(v_i, P) \leq R\}$
  $X = X \setminus X_i$;
  $\mathcal{P} = \mathcal{P} \cup \{v_i\}$;
  $\mathcal{C} = \mathcal{C} \cup \{\sigma_i\}$.
  $i = i + 1$;
if $X = \emptyset$ **then**
  return $\mathcal{C}$;
else
  return $\mathcal{P}$



Case 1. $(u|x)_r \leq (v_i|x)_r$ **Case 2.** $(u|x)_r > (v_i|x)_r$

Fig. 2. Illustrations for the two cases in the proof of Lemma 12.

Proof. Let y be a vertex of $\sigma(u, v)$ at distance at most R from w. Let y' be the vertex of $\Delta(u, v, w)$ such that $\varphi(y) = \varphi(y')$. Without loss of generality, we can assume that $y' \in \sigma(v, w)$. By triangle inequality, $d(w, v) = d(w, y') + d(y', v) \leq d(w, y) + d(y, v)$. Since $d(v, y') = d(v, y)$, we get $d(w, y') \leq d(w, y)$. Hence, $(u|v)_w \leq d(w, y') \leq d(w, y) \leq R$. □

Lemma 12. *Algorithm 1 either returns an (r, R)-packing of size $2k$ or an $(r, R + 2\delta)$-cover of G of size at most $2k - 1$.*

Proof. First assume that Algorithm 1 returns a subset of vertices $\mathcal{P}$. Suppose there exists two vertices $\{v_i, v_j\} \subseteq \mathcal{P}$ with $i < j$ such that **(a)** v_i, v_j were included in $\mathcal{P}$ at the i^{th} and j^{th} iteration of Algorithm 1, and **(b)** there exists an r-path P such that $\{v_i, v_j\} \subseteq B_R(V(P)) \cap \mathcal{P}$. But then $v_j \in X_i$ and therefore was removed from X in the i^{th} iteration, a contradiction.

Now assume that Algorithm 1 returns a collection of r-paths $\mathcal{C}$. For a vertex $u \in V(G)$, let $v_i \in \mathcal{P}$ be the vertex such that $u \in X_i$ when u was removed from X. By definition of X_i, there exists an r-path P such that $d(u, P) \leq R$ and $d(v_i, P) \leq R$. Let x be the end-vertex of P distinct from r. Let σ_i be the r-path added to $\mathcal{C}$ by Algorithm 1 during the i^{th} iteration and $\sigma(r, u)$ be any isometric path between r and u. We distinguish two cases (see Fig. 2).

- **Case 1.** First suppose that $(u|x)_r \leq (v_i|x)_r$. Let m_i, m_i', m_i'' be the points of σ_i, P and $\sigma(r, u)$ at distance $(u|x)_r$ from r. Since $d(u, \sigma(r, x)) \leq R$, Lemma 11 implies $d(u, m_i'') = (r|x)_u \leq R$. Hence, the δ-thinness of the geodesic triangles $\sigma(r, u) \cup \sigma(u, x) \cup P$ and $\sigma_i \cup \sigma(v_i, x) \cup P$ implies $d(u, m_i) \leq d(u, m_i'') + d(m_i'', m_i') + d(m_i', m_i) \leq R + 2\delta$.
- **Case 2.** Now, assume that $(v_i|x)_r \leq (u|x)_r$. Let m_i, m_i', m_i'' be the vertices of σ_i, P and $\sigma(r, u)$ at distance $(v_i|x)_r$ from r. By the choice of v_i, $d(r, m_i) + d(m_i, v_i) = d(r, v_i) \geq d(r, u) = d(r, m_i'') + d(m_i'', u)$. Since $d(r, m_i) = d(r, m_i'')$, we deduce that $d(m_i'', u) \leq d(m_i, v_i)$. Since $d(v_i, \sigma(r, x)) \leq R$, Lemma 11 implies $d(m_i'', u) \leq d(m_i, v_i) = (r|x)_{v_i} \leq R$. Using the thinness of geodesic triangles $\sigma(r, u) \cup \sigma(u, x) \cup P$ and $\sigma_i \cup \sigma(v_i, x) \cup P$, we derive that $d(u, m_i) \leq d(u, m_i'') + d(m_i'', m_i') + d(m_i', m_i) \leq R + 2\delta$.

We conclude that any vertex $u \in V(G)$ is at distance at most $R + 2\delta$ from some path in $\mathcal{C}$. □

Algorithm 2: Algorithm for the ROOTED k-GEODESIC CENTER problem.

Input : A δ-thin graph G, an integer $k \leq |V(G)|$.
Output: An integer R and an $(u, R+2\delta)$-cover of size $2k-1$ such that there is no (v, R')-cover with $R' < R$.

for $v \in V(G)$ **do**
  Let R_v be the smallest R integer for which Algorithm 1 returns a $(v, R+2\delta)$-cover C_v of size $2k-1$.
  /* The above steps can be implemented by using Algorithm 1 in combination with a binary search on $R_v \in \{0, 1, \dots, |V(G)|\}$. */
Let $u \in V(G)$ such that $R_u = \min\{R_v : v \in V(G)\}$.
return (C_u, R_u).

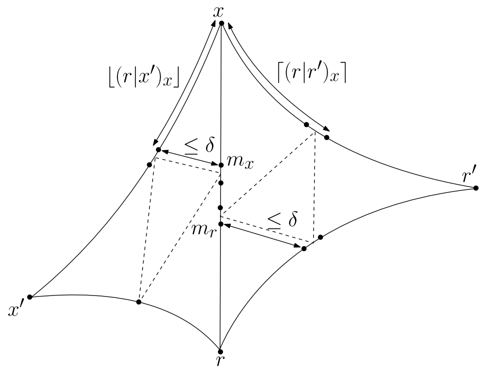


Fig. 3. Illustration of the notations used in Proof of Lemma 15.

Lemma 13. *Algorithm 2 returns an integer R and a $(u, R+2\delta)$-cover C_u of G of size $2k-1$ such that, there is no (v, R')-cover of size $2k-1$ with $R' < R$.*

Proof. Let (C_u, R_u) be the output of Algorithm 2. Since $R_u = \min\{R_v : v \in V(G)\}$ is the minimum integer R for which Algorithm 1 returns $(u, R+2\delta)$-cover C_u, Algorithm 1 returns a (v, R')-packing of size $2k$ for any $R' < R_u$ and any $v \in V(G)$. Hence, there is no (v, R')-cover of size $2k-1$ with $R' < R$. □

4.2. From non rooted to rooted collection of paths

In the following lemma, we show that if there is an R-cover of size k of a δ-thin graph then there is a rooted $(r, R+\delta)$-cover of size $2k-1$ of G, for some $r \in V(G)$.

Lemma 14. *Let C be an R-cover of a δ-thin graph G with $|C| = k$, X_C the set of end-vertices of paths in C and $r \in X_C$. Then, any collection of isometric paths $C_r = \{\sigma(r, x) : x \in X_C \setminus \{r\}\}$ is an $(r, R+\delta)$-cover of G.*

Proof. For a vertex $u \in V(G)$, let $P = \sigma(v_1, v_2) \in C$ be a path such that $d(u, P) \leq R$ and $w \in V(P)$ be a vertex with $d(u, w) \leq R$. Let $P_1, P_2 \in C_r$ where $P_i = \sigma(r, v_i)$. Since the geodesic triangle $P \cup P_1 \cup P_2$ of G is δ-thin, either $d(w, P_1) \leq \delta$ or $d(w, P_2) \leq \delta$. Therefore, either $d(u, P_1) \leq R+\delta$ or $d(u, P_2) \leq R+\delta$. Hence, C_r is a $(r, R+\delta)$-cover of G. □

4.3. From rooted to non rooted collection of paths

Conversely, the next lemma shows that, from a set of rooted isometric paths of a δ-thin graph of size $2k-1$ and eccentricity R, it is possible to construct a $(R+3\delta+1)$-cover of G of size k.

Lemma 15. *Let r be a vertex of a δ-thin graph G. For integers R and k, let C_r be an (r, R)-cover of G with $|C_r| = 2k-1$. Let π_r be a profile of length $2k$ containing all end vertices of the paths in C_r and P_r be a $(2\delta + \frac{1}{2})$-shallow pairing of π_r. Then, $S(P_r)$ is a $(R+3\delta+1)$-cover of G of size k.*

Proof. Let $u \in V(G)$ and $P = \sigma(r, x) \in C_r$ be an r-path with $d(u, P) \leq R$. Let $\{x', r'\} \subseteq \pi_r$ be the vertices such that $\{x, x'\} \in P_r$ and $\{r, r'\} \in P_r$. See Fig. 3 for notations used in this proof. By definition of a $(2\delta + \frac{1}{2})$-shallow pairing, there is a vertex m such that $(x|x')_m \leq 2\delta + \frac{1}{2}$ and $(r|r')_m \leq 2\delta + \frac{1}{2}$ which imply the following inequalities:

$$d(x, m) + d(m, x') - d(x, x') \leq 4\delta + 1 \Rightarrow d(x, x') \geq d(x, m) + d(m, x') - (4\delta + 1)$$

Algorithm 3: Algorithm for k-GEODESIC CENTER.

Input : A δ-thin graph G, an integer $k \le |V(G)|$.
Output: A collection of k isometric paths with eccentricity at most $R_k^* + (6\delta + 1)$

1 Let C_u be the (u, R_u)-cover of G returned by Algorithm 2 with G and k as input.
2 Let π be the even profile consisting of u and every other end vertex of the u-paths in C_u.
3 Compute a $(2\delta + \frac{1}{2})$-shallow pairing P of π.
4 **return** $\mathcal{S}(P)$.

$$d(r,m) + d(m,r') - d(r,r') \le 4\delta + 1 \Rightarrow d(r,r') \ge d(r,m) + d(m,r') - (4\delta + 1)$$

Combining the above inequalities we derive:

$$d(x,x') + d(r,r') \ge (d(x,m) + d(m,r')) + (d(x',m) + d(m,r)) - (8\delta + 2) \ge d(x,r') + d(x',r) - (8\delta + 2) \quad (1)$$

Adding $d(r,x)$ to both sides of (1) we get:

$$d(r,x) + d(x,x') - d(r,x') \ge d(r,x) + d(x,r') - d(r,r') - (8\delta + 2)$$

which further implies:

$$\left(r|x'\right)_x \ge \left(r|r'\right)_x - (4\delta + 1) \quad (2)$$

Recall that $P = \sigma(r,x) \in C_r$. Let m_x be the point of the geodesic P such that $d(x,m_x) = \lfloor \left(r|x'\right)_x \rfloor$ and m_r be the point of the geodesic P such that $d(x,m_r) = \lceil \left(r|r'\right)_x \rceil$. Let z be the vertex of P with $d(u,z) \le R$. Consider the following cases.

- If z lies in the (x,m_x)-subpath of P, consider any isometric path $\sigma(r,x')$. Since the geodesic triangle $P \cup \sigma(r,x') \cup \sigma(x,x')$ is δ-thin, we have that $d(z,\sigma(x,x')) \le \delta$. Hence, $d(u,\sigma(x,x')) \le d(u,z) + d(z,\sigma(x,x')) \le R + \delta$.
- If z lies in the (r,m_r)-subpath of P, consider any isometric path $\sigma(r',x)$. Since the geodesic triangle $P \cup \sigma(r,r') \cup \sigma(r',x)$ is δ-thin, we have that $d(z,\sigma(r,r')) \le \delta$. Hence, $d(u,\sigma(r,r')) \le d(u,z) + d(z,\sigma(r,r')) \le R + \delta$.
- Otherwise, z must lie in the (m_r,m_x)-subpath of P. Due to inequality (2) we have that $d(m_r,m_x) \le 4\delta + 2$. This implies either $d(z,m_x) \le 2\delta + 1$ or $d(z,m_r) \le 2\delta + 1$. Since the geodesic triangles $P \cup \sigma(r,x') \cup \sigma(x,x')$ and $P \cup \sigma(r,r') \cup \sigma(x,r')$ are δ-thin, we have $d(m_x,\sigma(x,x')) \le \delta$ and $d(m_r,\sigma(r,r')) \le \delta$. If $d(z,m_x) \le 2\delta + 1$ then $d(u,\sigma(x,x')) \le d(u,z) + d(z,m_x) + d(m_x,\sigma(x,x')) \le R + 3\delta + 1$. Otherwise, $d(z,m_r) \le 2\delta + 1$ and $d(u,\sigma(r,r')) \le d(u,z) + d(z,m_r) + d(m_r,\sigma(r,r')) \le R + 3\delta + 1$.

In all three above cases, either $d(u,\sigma(x,x')) \le R + 3\delta + 1$ or $d(u,\sigma(r,r')) \le R + 3\delta + 1$. We conclude that, for any $u \in V(G)$, the collection $\mathcal{S}(P_r)$ contains an isometric path Q such that $d(u,Q) \le R + 3\delta + 1$, i.e., $\mathcal{S}(P_r)$ is a $(R + 3\delta + 1)$-cover. □

4.4. Proof of Theorem 10

Let G be a connected δ-thin graph. Let C^* be a R_k^*-cover of G, π_{C^*} the profile containing the end-vertices of paths in C^* and $r \in \pi_{C^*}$. Consider the set $C_r^* = \left\{\sigma(r,x) : x \in \pi_{C^*} \setminus \{r\}\right\}$. Due to Lemma 14, we have

$$C_r^* \text{ is a } \left(r, R_k^* + \delta\right)\text{-cover of } G. \quad (3)$$

Now let $\left(C_u, R_u\right)$ be the output of Algorithm 2. By Lemma 13, there is no (r,R')-cover of size $2k-1$ with $R' < R_u$. Due to (3), there exists a $(r, R_k^* + \delta)$-cover of G of size $2k-1$. Hence, $R_u \le R_k^* + \delta$ and C_u has eccentricity at most $R_u + 2\delta \le R_k^* + 3\delta$. Let π_u be the profile consisting of u and every other end vertex of the u-paths in C_u. Observe that π_u is an even profile of length of $2k$. By Proposition 9, it is possible to compute in $O(mn^2)$-time a $(2\delta + \frac{1}{2})$-shallow pairing P' of π_u. Due to Lemma 15, $\mathcal{S}(P')$ is an $(R_k^* + 6\delta + 1)$-cover of size k. This completes the proof of Theorem 10. Algorithm 3 describes our complete algorithm for k-GEODESIC CENTER on δ-thin graphs.

Clearly, the running times of Algorithm 1 and Algorithm 2 are $O(k(n+m))$ and $O(nk(n+m)\log n)$, respectively. Due to Proposition 9, computing a shallow pairing takes $O(mn^2)$-time. Therefore, the total running time of Algorithm 3 is $O(mn^2 \log n)$.

4.5. The special case of trees

In case of trees, the same algorithmic approach leads to an exact polynomial time algorithm. Indeed, since trees are 0-hyperbolic, Lemma 13 implies that Algorithm 2 computes a rooted (r,R)-cover C of size $2k-1$ such that there is no (r',R')-cover with $R' < R$. By Lemma 12, an optimal R^*-cover of size k can be transformed into a rooted (r',R^*)-cover of size $2k-1$. Hence, $R^* \ge R$. Since trees satisfy the pairing property, any (r,R)-cover of size $2k-1$ can be transformed into an R-cover of size k. This implies $R \ge R^*$. Hence, in case of trees, $R = R^*$ and the solution returned by Algorithm 3 is an optimal R^*-cover of size k.

5. NP-hardness for partial grids

In this section we prove Theorem 2. Our proof is an adaptation of the NP-hardness of 1-Geodesic Center on planar bipartite graphs proved by Dragan & Leitert (Corollary 8, [24]). First we prove the following lemmas.

Lemma 16. *Let G be a graph and $H = G_\ell$ for some $\ell \geq 1$. Then for integers m, k, if G has an m-cover of size k then H has an $(m\ell + \lfloor \ell/2 \rfloor)$-cover of size k.*

Proof. Let C be an m-cover of G of size k and C' be a set of paths in H which are ℓ-subdivision of the paths in C. Clearly, all paths in C' consist of isometric paths in H. Let u be an original vertex of H and $P \in C$ be a path such that $d(u, V(P)) \leq m$ in G. Let $P' \in C'$ be the ℓ-subdivision of P. Clearly, $d(u, V(P')) \leq m\ell$. Now consider a vertex $u \in V(H)$ which is not a vertex of G. Hence there exists an original vertex $u' \in V(H)$ with $d(u, u') \leq \lfloor \ell/2 \rfloor$. Let $P \in C$ be a path such that $d(u', V(P)) \leq m$ in G and $P' \in C'$ be the ℓ-subdivision of P. Then $d(u, V(P')) \leq d(u, u') + d(u', V(P')) \leq m\ell + \lfloor \ell/2 \rfloor$. □

Let G be a graph and $H = G_\ell$ for some $\ell \geq 1$. For an isometric path P of H between two original vertices u, v, let $G(P)$ denote the (u, v)-isometric path in G such that P is an ℓ-subdivision of $G(P)$. Intuitively, $G(P)$ is the original isometric path whose ℓ-subdivision created P in H.

Lemma 17. *Let G be a graph and let $H = G_\ell$ for some $\ell \geq 1$. Let P be an isometric path of H between two original vertices u, v. Let w be an original vertex of G such that $d(w, V(P)) < (r+1)\ell$ for some positive integer r. Then, for $Q = G(P)$ we have $d(w, V(Q)) \leq r$.*

Proof. Let $w' \in V(P)$ be a vertex which is closest to w in H and P' be an (w, w')-isometric path in H. Clearly, w' is an original vertex and therefore $G(P')$ exists. Observe that, the number of original vertices in P' is at most $r + 1$. (Otherwise length of P' is at least $(r + 1)\ell$ in H which is a contradiction.) Hence length of $G(P)$ is at most r in G. □

Dragan & Leitert [24] reduced the NP-complete Planar Monotone 3-SAT [20] to show that 1-Geodesic Center is NP-hard on bipartite planar subcubic graphs. Given an Planar Monotone 3-SAT instance I, the authors constructed a planar bipartite subcubic graph $B(I)$ and an integer m' with the following properties.

- $B(I)$ has an isometric path with eccentricity at most m' if and only if I is satisfiable;
- there are two special cut vertices v_0, v_n of $B(I)$ such that any isometric path with eccentricity at most m' will contain v_0 and v_n.

To prove our result, we modify the graph $B(I)$ slightly. First construct a gadget as follows. Take a path P of length $2k$ and let the vertices of P be $u_1, u_2, \ldots, u_{2k+1}$. For each $j \in [2, 2k]$, take a new path Q_j of length m' and make one of the end-vertex of Q_j adjacent to u_j. Let T be the union of P and $Q_j, j \in [2, 2k-1]$. Now make the vertex v_0 adjacent to u_1.

We call the modified graph $B'_k(I)$. It is easy to verify that a set $\mathcal{P}$ of isometric paths in $B'_k(I)$ is an m'-cover if and only if the following holds:

- There are $k - 1$ isometric paths in $B'_k(I)$ whose vertices completely lie in T.
- There is a special isometric path P in $B'_k(I)$ containing v_0, v_n such that P has eccentricity m' in $B(I)$.

The above discussion implies that $B'_k(I)$ has an m'-cover of size k if and only if $B(I)$ has an m'-cover of cardinality one. Moreover, $B'_k(I)$ is planar and has maximum degree at most 3. Now we construct a $H = P_k(I)$ by applying Lemma 4 on $G = B'_k(I)$ and ℓ be the integer such that $H = G_\ell$. We prove that G has an m'-cover of cardinality k if and only if H has an $\left(m'\ell + \lceil \ell/2 \rceil\right)$-cover of cardinality k.

If G has an m'-cover of cardinality k, then Lemma 16 implies that H has an $\left(m'\ell + \lceil \ell/2 \rceil\right)$-cover of cardinality k. Now assume that H has an $\left(m'\ell + \lceil \ell/2 \rceil\right)$-cover C of cardinality k. Let T_ℓ be the induced subgraph of $P_k(I)$ isomorphic to the ℓ-subdivision of T. The structure of T_ℓ implies that there will be $k - 1$ isometric paths whose vertices lie completely in T_ℓ. Let B_ℓ denote the subgraph of $P_k(I)$ isomorphic to the ℓ-subdivision of $B(I)$. Since the original vertices v_0 and v_n are still cut vertices in $P_k(I)$, there is a special path $P \in C$ such that all vertices in B_ℓ are at a distance at most $m'\ell + \lceil \ell/2 \rceil$ from P. Now apply Lemma 17 to conclude that all vertices of $B(I)$ which also belongs to $B'_k(I)$ are at distance m' from $G(P)$. Therefore, the set $C' = \{G(P) : P \in C\}$ is an m'-cover of G.

The above discussion implies that $P_k(I)$ has an $\left(m'\ell + \lceil \ell/2 \rceil\right)$-cover of cardinality k if and only if I is satisfiable. This completes the proof.

6. Conclusion

Constant factor approximability of k-Geodesic Center in general graphs for $k \geq 2$ remains open. From both application and theoretical point of views, approximability of k-Geodesic Center on planar graphs is an important open question. An approximation scheme even for MESP on planar graphs is open. Studying whether k-Geodesic Center (for arbitrary k) is fixed-parameter tractable with respect to tree-width or hyperbolicity are interesting research questions. Polynomial solvability of k-GEODESIC CENTER in superclasses of trees (e.g. outerplanar graphs), solid grids (i.e., partial grids where internal faces have unit area), are open as well. For graphs with bounded isometric path complexity (including hyperbolic graphs), IPC admits a constant factor approximation algorithm [10]. It would be interesting to explore if k-GEODESIC CENTER admits an additive approximation algorithm on graphs with bounded isometric path complexity? Finally, investigating the approximability or fixed-parameter tractability of k-GEODESIC CENTER on weighted graphs are interesting directions.

CRediT authorship contribution statement

Dibyayan Chakraborty: Writing – original draft, Methodology, Investigation. **Yann Vaxès:** Writing – original draft, Methodology, Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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