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On tori periods of Weil representations of unitary groups

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Abstract

We determine the restriction of Weil representations of unitary groups to maximal tori. In the local case, we show that the Weil representation contains a pair of compatible characters if and only if a root number condition holds. In the global case, we show that a torus period corresponding to a maximal anisotropic torus of the global theta lift of a character does not vanish if and only if the local condition is satisfied everywhere and a central value of an L -function does not vanish. Our proof makes use of the seesaw argument and of the well-known theta lifting results from $U(1)$ to $U(1)$. Our results are used in [1, 2] to construct Arthur packets for G_2 .

Keywords Theta correspondence · Periods · Special values of L -functions · Gan–Gross–Prasad conjectures

Mathematics Subject Classification 22E50 · 11F27; 11F67

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1 Introduction

Branching problems are a fascinating topic in representation theory and in the theory of automorphic representations. The most famous examples are the Gan–Gross–Prasad conjectures [9–11] and their refinements [16, 18, 33], extending the original conjectures of Gross–Prasad [13, 14].

In this paper, we study the restriction of Weil representations of unitary groups to maximal tori. Our results are both local and global, and they are similar in nature to the Gan–Gross–Prasad conjectures. Let us describe the problems we concern.

Let F be a field with characteristic different than 2 and let K/F be a quadratic étale algebra with involution $x \mapsto x^c$, whose set of fixed points is F . Let V be a non-degenerate n -dimensional hermitian space over K , and let W be a non-degenerate one-dimensional skew-hermitian space over K .

When F is a local field, we consider the following branching problem: given a maximal torus T of $U(V)$ and characters $\alpha: T \rightarrow \mathbb{C}^\times$ and $\beta: U(W) \rightarrow \mathbb{C}^\times$, we would like to investigate whether the restriction of the Weil representation of the metaplectic group $\text{Mp}_\psi(V, W)$ to $T \times U(W)$ contains the representation $\alpha \boxtimes \beta$ as a sub-quotient. Reformulating this using the theta correspondence, this is equivalent to asking whether the space $\text{Hom}_T(\Theta(\beta), \alpha)$ is non-zero, where $\Theta(\beta)$ is the big theta lift of β from $U(W)$ to $U(V)$.

Suppose that $F = \mathbf{F}$ is a number field and that \mathbf{K}/\mathbf{F} is a quadratic field extension. For an algebraic group G , we write $[G] = G(\mathbf{F}) \backslash G(\mathbb{A}_{\mathbf{F}})$. We consider the following branching problem: given a maximal torus T of $U(V)$ and automorphic characters $\alpha: [T] \rightarrow \mathbb{C}^\times$ and $\beta: [U(W)] \rightarrow \mathbb{C}^\times$, we would like to investigate whether the α -period of the global theta lift $\Theta(\beta)$ of β from $U(W)(\mathbb{A}_{\mathbf{F}})$ to $U(V)(\mathbb{A}_{\mathbf{F}})$ is non-zero.

That is, we are asking whether the assignment

$$\mathcal{P}_{T,\alpha}(f) = \int_{[T]} f(t) \overline{\alpha(t)} dt$$

is identically zero on the space $\Theta(\beta)$. In order to avoid convergence issues, we will assume that the torus T is anisotropic in the global setting, so that the integrals in question converge absolutely.

Notice that when V is one-dimensional, the theta lift of β is either zero or a character, and our problems reduce to determining whether $\Theta(\beta)$ equals α or not. This problem, of determining the theta lift from $U(1)$ to $U(1)$, is well understood. It dates back to Moen [24], Rogawski [28], and Harris–Kudla–Sweet [15] in the non-archimedean local field case, to Paul [25] in the archimedean local field case, to Minguez [22], Fang–Sun–Xue [5] and Gan [7] in the split local case, and to Rogawski [28], Yang [36] and Yamana [35] in the global case. See also Section 9 of [8] and the last paragraph of Section 7 of [8].

Our technique for solving these problems in the general case, where $\dim V$ is arbitrary, involves a seesaw identity that reduces the problems to the well-known case discussed above. This idea has been used before by Gan and his collaborators, see for example [6, Sections 9 and 10], [7, Section 5] and [8, Section 10].

In order to state our results, we need a classification of maximal tori in $U(V)$. In Section 2, we recall the classification given in [27]. Each maximal torus $T \subset U(V)$ corresponds to an étale algebra E of degree n over F and an element $\lambda \in E^\times$, such that the space $(V_{E,\lambda}, \langle \cdot, \cdot \rangle_\lambda)$ is isomorphic to V as hermitian spaces, where $V_{E,\lambda} = \text{Res}_{L_E/K} L_E$, equipped with the hermitian product $\langle x, y \rangle_\lambda = \text{tr}_{L_E/K}(\lambda xy^c)$, where $L_E = K \otimes_F E$. In this case, the maximal torus $T_{E,\lambda}$ is isomorphic to the norm one torus of L_E , that is,

$$L_E^1 = \{x \in \text{Res}_{E/F} L_E^\times \mid x \cdot x^c = 1\}.$$

For some of our results, we would like to iterate over the different embeddings $L_E^1 \hookrightarrow U(V)$, modulo $U(V)$ -conjugation. However, there are too many of these. To tackle this obstacle, we follow an idea presented in [8, Section 3] and define the notion of an *admissible embedding* $i: L_E^1 \rightarrow U(V)$ (Section 2.4). We show that the set of admissible embeddings of L_E^1 forms a stable conjugacy class in $U(V)$. Moreover, we construct a natural bijection between certain classes in $E^\times / N_{L_E/E}(L_E^\times)$ and admissible embeddings $i: L_E^1 \rightarrow U(V)$, up to $U(V)$ -conjugation (Theorem 2.8):

Theorem 1.1 *There exists a natural bijection between the sets*

$$\{\lambda \in E^\times / N_{L_E/E}(L_E^\times) \mid V_{E,\lambda} \text{ is isomorphic to } V \text{ as hermitian spaces}\}$$

and

$$\Sigma_{E,V} = \{i: L_E^1 \rightarrow U(V) \mid i \text{ is admissible}\} / U(V)\text{-conjugation}.$$

We now move to describe our main results. In order to make the results look cleaner, we will no longer mention the one-dimensional skew-hermitian space W in the introduction, but instead use a trace zero element δ that encodes the discriminant of such space.

Suppose that F is a local field. By choosing a trace zero element $\delta \in K^\times$, a character $\mu: K^\times \rightarrow \mathbb{C}^\times$ such that $\mu|_{F^\times} = \omega_{K/F}$ is the quadratic character given by local class field theory, and a non-trivial character $\psi: F \rightarrow \mathbb{C}^\times$, we can lift characters of L_F^1 to representations of $U(V)$. Suppose that E is an étale algebra of degree n over F . Let $\beta: L_F^1 \rightarrow \mathbb{C}^\times$, and $\alpha: L_E^1 \rightarrow \mathbb{C}^\times$ be characters. We give the following answer (Theorem 5.1) to the local problem discussed above.

Theorem 1.2 *Let $i: L_E^1 \rightarrow U(V)$ be an admissible embedding that corresponds to the element $\lambda \in E^\times$ under Theorem 1.1. Then the space*

$$\mathrm{Hom}_{i(L_E^1)} \left(\Theta_{\delta, V, \mu, \psi}(\beta), \alpha \circ i^{-1} \right) \quad (1)$$

is non-zero if and only if the following conditions hold:

- (1) *Character compatibility:* $\beta = \alpha|_{L_F^1}$.
- (2) *Root number condition:* $\omega_{L_E/E}(\lambda) = \varepsilon_{L_E/E}(\alpha_{L_E} \cdot \mu^{-1} \circ N_{L_E/K}, \psi, \delta)$.

Moreover, in this case, this Hom-space is one-dimensional.

Here, $\Theta_{\delta, V, \mu, \psi}(\beta)$ is the big theta lift of β to $U(V)$ with respect to the data (δ, μ, ψ) , see Section 3.1.2. It is either zero or irreducible in our case (since β is supercuspidal), and therefore equals the small theta lift $\theta_{\delta, V, \mu, \psi}(\beta)$. We refer the reader to Sections 2.2.2 and 3.3 for the definition of the vector of quadratic characters $\omega_{L_E/E}(\lambda)$, the vector of root numbers $\varepsilon_{L_E/E}(\alpha_{L_E} \cdot \mu^{-1} \circ N_{L_E/K}, \psi, \delta)$, and other notation appearing in the theorem.

We also show that for characters α and β satisfying the compatibility condition, there exists a unique non-degenerate hermitian space H of dimension n , up to isomorphism, and a unique admissible embedding $i: L_E^1 \rightarrow U(H)$, up to conjugation, such that the space (1) attached to i is non-zero. More precisely, we show the following theorem (Corollary 5.4).

Theorem 1.3 *For every choice of E , α , and β as above, we have*

$$\sum_{H \in \mathrm{Her}_n} \sum_{i \in \Sigma_{E, H}} \dim_{\mathbb{C}} \mathrm{Hom}_{i(L_E^1)} \left(\Theta_{\delta, H, \mu, \psi}(\beta), \alpha \circ i^{-1} \right) = \begin{cases} 1 & \beta = \alpha|_{L_F^1}, \\ 0 & \text{otherwise.} \end{cases}$$

Here, H runs over representatives of classes of

$$\mathrm{Her}_n = \{H \text{ is a non-degenerate hermitian space} \mid \dim H = n\} / \text{isomorphism},$$

and i runs over representatives of classes of $\Sigma_{E, H}$.

These two local theorems are similar to the local Gan–Gross–Prasad conjectures, in the sense that for compatible α and β , there exists a unique non-degenerate hermitian space H of dimension n (up to isomorphism), a unique admissible embedding (up to $U(H)$ -conjugation) $i: L_E^1 \rightarrow U(H)$, such that the space attached to i is not zero, and we pinpoint the tuple (H, i) in terms of the vector of root numbers attached to the data defining i . In our case, the set

$$\mathcal{V}_{\delta, E}(\beta) = \bigcup_{H \in \text{Her}_n} \left\{ \left(\Theta_{\delta, H, \mu, \psi}(\beta), \alpha \circ i^{-1} \right) \mid i \in \Sigma_{E, H} \right\},$$

is analogous to the local Vogan L -packet appearing in the Gan–Gross–Prasad conjectures, consisting of irreducible representations of the group and of its pure inner forms. Here, $\mathcal{V}_{\delta, E}$ consists of tuples whose first component is an irreducible representation of a pure inner form of $U(V)$, and whose second component is a character of a torus (of the aforementioned pure inner form) isomorphic to L_E^1 .

We move to explain our global result. Let $F = \mathbf{F}$ be a number field and let $K = \mathbf{K}$ be a quadratic field extension of \mathbf{F} .

By choosing a trace zero element $\delta \in \mathbf{K}^\times$, an automorphic character μ of $\mathbb{A}_{\mathbf{K}}^\times$, such that $\mu|_{\mathbb{A}_{\mathbf{F}}^\times} = \omega_{\mathbf{K}/\mathbf{F}}$ is the quadratic character given by global class field theory, and a non-trivial character $\psi: \mathbf{F} \backslash \mathbb{A} \rightarrow \mathbb{C}^\times$, we can lift automorphic characters of $L_{\mathbf{F}}^1(\mathbb{A}_{\mathbf{F}})$ to automorphic representations of $U(V)(\mathbb{A}_{\mathbf{F}})$. Suppose that \mathbf{E} is an n -dimensional étale algebra over \mathbf{F} , such that there exists $\lambda \in \mathbf{E}^\times$ satisfying that $V_{E, \lambda}$ is isomorphic to V as hermitian spaces, and such that $L_{\mathbf{E}}^1$ is anisotropic (equivalently, there is no embedding of \mathbf{F} -algebras $\mathbf{K} \hookrightarrow \mathbf{E}$). Let $\beta: [L_{\mathbf{F}}^1] \rightarrow \mathbb{C}^\times$ and $\alpha: [L_{\mathbf{E}}^1] \rightarrow \mathbb{C}^\times$ be automorphic characters. Our global result (Theorem 5.6) classifies when the α -period is identically zero on the space of the global theta lift $\Theta_{\delta, V, \mu, \psi}(\beta)$ of β to $U(V)(\mathbb{A}_{\mathbf{F}})$. As before, and as in the global Gan–Gross–Prasad conjectures, this classification is expressed in terms of root numbers and central values of L -functions.

Theorem 1.4 *Let $i: L_{\mathbf{E}}^1 \rightarrow U(V)$ be an admissible embedding corresponding to $\lambda \in \mathbf{E}^\times$. The $\alpha \circ i^{-1}$ -period $\mathcal{P}_{i(L_{\mathbf{E}}^1), \alpha \circ i^{-1}}$ is non-zero on the global theta lift $\Theta_{\delta, V, \mu, \psi}(\beta)$, that is,*

$$\int_{[L_{\mathbf{E}}^1]} f(i(t)) \overline{\alpha(t)} dt \neq 0 \text{ for some } f \in \Theta_{\delta, V, \mu, \psi}(\beta),$$

if and only if the following conditions hold:

- (1) *Character compatibility:* $\beta = \alpha|_{L_{\mathbf{F}}^1(\mathbb{A}_{\mathbf{F}})}$.
- (2) *Root number condition:* for every place v ,

$$\omega_{L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbf{F}_v / \mathbf{E} \otimes_{\mathbf{F}} \mathbf{F}_v}(\lambda) = \varepsilon_{L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbf{F}_v / \mathbf{E} \otimes_{\mathbf{F}} \mathbf{F}_v} \left(\alpha_{v, L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbf{F}_v} \cdot \mu_v^{-1} \circ N_{L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbf{F}_v / \mathbf{K} \otimes_{\mathbf{F}} \mathbf{F}_v}, \psi_v, \delta \right).$$

- (3) *Central L -function value condition:* the following value is non-zero

$$\mathcal{L} \left(\alpha_{L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbb{A}_{\mathbf{F}}} \cdot \mu^{-1} \circ N_{L_{\mathbf{E}}/\mathbf{K}} \right) \neq 0.$$

We refer the reader to Section 4.3 for the definition of the central L -function value $\mathcal{L}(\alpha_{L_E \otimes_F \mathbb{A}_F} \cdot \mu^{-1} \circ N_{L_E/\mathbf{K}})$ and other notation appearing in the theorem.

Notice that the first two conditions in Theorem 1.4 are equivalent to requiring that for every place v , the conditions in Theorem 1.2 hold. It is clear that the condition of the Hom-space not vanishing for every v is a necessary condition for the global period to not vanish. Moreover, the root number condition implies that the global root number equals one, i.e.,

$$\prod_v \varepsilon_{L_E \otimes_F F_v / E \otimes_F F_v} \left(\alpha_{v, L_E \otimes_F F_v} \cdot \mu_v^{-1} \circ N_{L_E \otimes_F F_v / \mathbf{K} \otimes_F F_v}, \psi_v, \delta \right) = 1,$$

and therefore the third condition regarding the central L -value $\mathcal{L}(\alpha_{L_E \otimes_F \mathbb{A}_F} \cdot \mu^{-1} \circ N_{L_E/\mathbf{K}})$ can be satisfied.

Similarly to the local case, we show in Corollary 5.8 that for characters α and β satisfying the compatibility condition and such that the central L -function value in question does not vanish, there exists a unique non-degenerate hermitian space H of dimension n and a unique class $i \in \Sigma_{E,H}$, such that the period $\mathcal{P}_{i(L_E^1), \alpha \circ i^{-1}}$ does not identically vanish on $\Theta_{\delta, H, \mu, \psi}(\beta)$.

Theorem 1.5 *For every choice of E , α , and β , as above, there exists a non-degenerate hermitian space H of dimension n and an admissible embedding $i: L_E^1 \rightarrow U(H)$, such that $\mathcal{P}_{i(L_E^1), \alpha \circ i^{-1}}$ is not identically zero on $\Theta_{\delta, H, \mu, \psi}(\beta)$, if and only if the following conditions hold:*

- (1) *Character compatibility: $\beta = \alpha|_{L_F^1(\mathbb{A}_F)}$.*
- (2) *Central L -function value condition: the following value is non-zero*

$$\mathcal{L}(\alpha_{L_E \otimes_F \mathbb{A}_F} \cdot \mu^{-1} \circ N_{L_E/\mathbf{K}}) \neq 0.$$

Moreover, in this case, the class of such H and the class of $i: L_E^1 \rightarrow U(H)$ in $\Sigma_{E,H}$ is unique.

Once again, these results are similar to the global Gan–Gross–Prasad conjectures, in the sense that for α and β satisfying the compatibility condition, the non-vanishing of the period $\mathcal{P}_{i(L_E^1), \alpha \circ i^{-1}}$ is determined by a central L -function value, and in this case there exists a unique non-degenerate hermitian space H of dimension n (up to isomorphism) and a unique (up to $U(H)$ -conjugacy) admissible embedding, for which this period does not vanish. As in the local case, the set

$$\mathcal{V}_{\delta, E}(\beta) = \bigcup_{H \in \text{Her}_n} \left\{ \left(\Theta_{\delta, H, \mu, \psi}(\beta), \alpha \circ i^{-1} \right) \mid i \in \Sigma_{E, H} \right\},$$

serves as a substitute for the global Vogan packet appearing in the global Gan–Gross–Prasad conjectures, consisting of automorphic representations of the group and its pure inner forms.

The results of this paper, combined with the exceptional theta correspondence of Bakić and Savin [3] for $(\mathrm{PU}_3 \rtimes \mathbb{Z}/2\mathbb{Z}) \times G_2$, are used in order to construct local and global Arthur packets for the exceptional group G_2 , see [1] and [2].

The paper is organized as follows. In Section 2, we recall the notion of ϵ -hermitian spaces and the invariants attached to them. Then we discuss the classification of maximal tori in unitary groups associated to ϵ -hermitian spaces, and discuss the notion of an admissible embedding. In Section 3, we recall the theta correspondence for unitary groups over local fields. We use it to define the notion of the big theta lift for characters of unitary groups of one-dimensional ϵ -hermitian spaces over an étale algebra. We then explain a seesaw identity for this theta lift. This is a key ingredient needed for our main results. We then discuss the well-known results of theta lifting from $\mathrm{U}(1)$ to $\mathrm{U}(1)$, and use them to obtain similar results for theta lifting from $\mathrm{U}(1)$ to $\mathrm{U}(1)$ for a one-dimensional ϵ -hermitian space over an étale algebra. In Section 4 we discuss the global analogs of the statements in Section 3. In Section 5, we state and prove our main theorems regarding toric periods of Weil representations of unitary groups. In the appendix, we prove statements regarding two embeddings of a maximal torus that are conjugate.

2 Maximal tori in unitary groups

In this section we describe how one can classify maximal tori in unitary groups, following the results of [27]. Moreover, we introduce the notion of an admissible embedding, which will be of importance in Section 5.

2.1 ϵ -hermitian spaces and their corresponding unitary groups

Let F be a field. We will always assume that F has characteristic $\neq 2$. Let K/F be a quadratic étale algebra with an involution, denoted $x \mapsto x^c$, whose fixed points are F . That is, K is either a quadratic field extension of F , in which case the involution is the nontrivial Galois action on K , or $K = F \times F$, in which case the involution is given by $(x, y) \mapsto (y, x)$ for $x, y \in F$. The latter case is known as the *split* case, and the unitary groups for hermitian spaces over these two quadratic étale algebras will be treated separately.

For $\epsilon \in \{\pm 1\}$, a *finite dimensional ϵ -hermitian space* is a finite dimensional vector space V over K , equipped with an ϵ -hermitian form, i.e., there exists a function $\langle \cdot, \cdot \rangle : V \times V \rightarrow K$, such that for all $w, x, y \in V$, and $\alpha \in K$:

- (1) $\langle x, y \rangle = \epsilon \langle y, x \rangle^c$,
- (2) $\langle w + x, y \rangle = \langle w, y \rangle + \langle x, y \rangle$,
- (3) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$.

If $\epsilon = 1$, we call V *hermitian*, and if $\epsilon = -1$, we call V *skew-hermitian*. If $\delta \in K^\times$ is a trace zero element, that is, $\mathrm{tr}_{K/F}(\delta) = 0$, consider the space $(W^\delta, \langle \cdot, \cdot \rangle_{W^\delta}) = (V, \langle \cdot, \cdot \rangle^\delta)$ equipped with the form defined by $\langle x, y \rangle^\delta = \delta \langle x, y \rangle$. We have that the space W^δ is a $-\epsilon$ -hermitian space. Throughout the text, hermitian spaces will always

be denoted using the symbol V or H , and skew-hermitian spaces will always be denoted using the symbol W .

We say that the space V is *non-degenerate* if for every $0 \neq x \in V$, there exists $y \in V$, such that $\langle x, y \rangle \neq 0$. In this case, if $T: V \rightarrow V$ is a linear map, then there exists a unique linear map $T^*: V \rightarrow V$, such that

$$\langle Tx, y \rangle = \langle x, T^*y \rangle,$$

for every $x, y \in V$. We call T^* the *adjoint* of T . The assignment $\text{End } V \rightarrow \text{End } V$ mapping $T \mapsto T^*$ is an involution, such that for any $T, S \in \text{End } (V)$ and any $\alpha \in K$,

$$\begin{aligned} (\alpha T)^* &= \alpha^c T^*, \\ (T \circ S)^* &= S^* \circ T^*. \end{aligned}$$

If V is a non-degenerate finite dimensional ϵ -hermitian space, we define its *unitary group* to be

$$U(V) = \{g \in \text{Res}_{K/F} \text{GL}_K(V) \mid \langle gx, gy \rangle = \langle x, y \rangle, \forall x, y \in V\}.$$

Note that if W^δ is given as above, then $U(V) = U(W^\delta)$.

If V is a non-degenerate one-dimensional ϵ -hermitian space over K , we have that $U(V)$ is isomorphic to the group

$$K^1 = \{x \in K^\times \mid xx^c = 1\}$$

by the map sending $x \in K^1$ to the multiplication by x map $m_x: V \rightarrow V$. We refer to the inverse of this map as the *obvious isomorphism* $U(V) \rightarrow K^1$.

2.1.1 Invariants of ϵ -hermitian spaces

Let V be a non-degenerate ϵ -hermitian space such that $\dim_K V = n$. The space V has a natural invariant known as the *discriminant*, which we explain below.

Let $\mathbf{e} = \{e_i\}_{i=1}^n$ be a basis of V . Then the *determinant* of the ϵ -hermitian form $\langle \cdot, \cdot \rangle$ with respect to the basis $\{e_i\}_{i=1}^n$ is defined to be

$$\det(\langle \cdot, \cdot \rangle)_{\mathbf{e}} := \det((e_i, e_j))_{ij}.$$

If V is hermitian, then $\det(\langle \cdot, \cdot \rangle)_{\mathbf{e}} \in F^\times$ and it is well-known that the class of $\det(\langle \cdot, \cdot \rangle)_{\mathbf{e}}$ in $F^\times / N_{K/F}(K^\times)$ is independent of the chosen basis \mathbf{e} . Hence, we may omit \mathbf{e} from the notation and denote $\det V = \det(\langle \cdot, \cdot \rangle)_{\mathbf{e}} \in F^\times / N_{K/F}(K^\times)$. Note that in the split case where $K = F \times F$, we have that $F^\times = N_{K/F}(K^\times)$, and therefore the invariant $\det V$ is always trivial.

We define the *discriminant* of V as

$$\text{disc } V := (-1)^{n(n-1)/2} \det V.$$

For a non-degenerate finite dimensional skew-hermitian space W and a trace zero element $\delta \in K^\times$, we have that the space $(V^\delta, \langle \cdot, \cdot \rangle_{V^\delta}) = (W, \langle \cdot, \cdot \rangle^\delta)$ equipped with the form

$$\langle x, y \rangle^\delta = \delta \langle x, y \rangle$$

is a hermitian space and we define

$$\text{disc } W = \delta^{-\dim W} \text{disc } (V^\delta) \in \delta^{-\dim W} \cdot F^\times / N_{K/F}(K^\times).$$

Suppose now that F is a local field (we allow both archimedean and non-archimedean local fields) and that K/F is a quadratic field extension. Using the non-trivial quadratic character $\omega_{K/F}$ of $F^\times / N_{K/F}(K^\times)$ associated to the quadratic extension K/F by local class field theory, we can encode the discriminant as a sign. For a hermitian space V as above we define

$$\epsilon(V) = \omega_{K/F}(\text{disc } V).$$

Similarly, for a skew-hermitian space W as above we define

$$\epsilon_\delta(W) := \epsilon(V^\delta) = \omega_{K/F}(\delta^{\dim W} \text{disc } W).$$

Note that the invariant $\epsilon_\delta(W)$ depends on the choice of δ .

2.2 One-dimensional hermitian spaces with respect to an étale algebra

Let L be an étale algebra of degree n over K . We say that (L, σ) is an étale *algebra with involution* if $\sigma : L \rightarrow L$ is an involution, such that for any $a \in K$, $\sigma(a) = a^c$. Given such an étale algebra with involution, we may define the notion of an L -hermitian (or L -skew-hermitian) space as in Section 2.1 by replacing K with L and the involution $x \mapsto x^c$ with the involution σ in the definitions.

It is well-known that any étale algebra with involution is of the form $L = L_E = K \otimes_F E$, where E is an étale algebra of degree n over F and $\sigma : L_E \rightarrow L_E$ is defined on pure tensors by

$$\sigma(k \otimes h) = k^c \otimes h,$$

for $k \in K$ and $h \in E$. Henceforth we will often write $\sigma(x) = x^c$ for $x \in L_E$.

Let E be an étale algebra as above. For any $\lambda \in E^\times$, let $L_{E,\lambda} = (L_E, \langle \cdot, \cdot \rangle_{L_{E,\lambda}})$ be the one-dimensional L_E -hermitian space equipped with the following form:

$$\langle x, y \rangle_{L_{E,\lambda}} = \lambda x y^c.$$

As before, given a trace zero element $\delta \in L_E^\times$, we may define a one-dimensional L_E -skew-hermitian space $W_{E,\lambda}^\delta = (L_E, \langle \cdot, \cdot \rangle_{W_{E,\lambda}^\delta})$ by setting $\langle x, y \rangle_{W_{E,\lambda}^\delta} = \delta \langle x, y \rangle_{L_{E,\lambda}}$.

Consider the unitary group $U(L_{E,\lambda})$, consisting of all L_E -linear maps that preserve the L_E -hermitian structure on $L_{E,\lambda}$. It is naturally isomorphic to the group of norm one elements of L_E :

$$L_E^1 = \{x \in \text{Res}_{E/F} L_E^\times \mid xx^c = 1\}.$$

The isomorphism is given by the map sending $x \in L_E^1$ to the multiplication by x map $m_x: L_{E,\lambda} \rightarrow L_{E,\lambda}$. We refer to the inverse of this isomorphism as the *obvious isomorphism* $U(L_{E,\lambda}) \rightarrow L_E^1$. Similarly, the unitary group $U(W_{E,\lambda}^\delta)$ is also naturally isomorphic to L_E^1 , and we define the notion of the *obvious isomorphism* $U(W_{E,\lambda}^\delta) \rightarrow L_E^1$. The group L_E^1 will be fundamental for defining maximal tori in unitary groups in the next sections.

If $E = \prod_{j=1}^m F_j$, where for every j , F_j/F is a field extension, and $\lambda = (\lambda_1, \dots, \lambda_m) \in \prod_{j=1}^m F_j^\times$, then we have that $U(L_{E,\lambda}) = \prod_{j=1}^m U(L_{F_j,\lambda_j})$, which is naturally isomorphic to $\prod_{j=1}^m L_{F_j}^1$. We also have a similar statement for the unitary group $U(W_{E,\lambda}^\delta)$ of the skew-hermitian L_E -space $W_{E,\lambda}^\delta$.

2.2.1 The discriminant of an étale algebra

Let E be an étale algebra of degree n over F . Let $\mathbf{e} = \{e_i\}_{i=1}^n$ be a basis of E over F . The *determinant* of the étale algebra E/F with respect to the basis \mathbf{e} is defined to be

$$\det(\text{tr}_{E/F}(\cdot))_{\mathbf{e}} := \det(\text{tr}_{E/F}(e_i e_j))_{i,j}$$

It turns out that the class of $\det(\text{tr}_{E/F}(\cdot))_{\mathbf{e}}$ in $F^\times / (F^\times)^2$ does not depend on the choice of the basis \mathbf{e} , and hence we can define

$$\det_F(E) = \det(\text{tr}_{E/F}(e_i e_j))_{i,j} \in F^\times / (F^\times)^2.$$

The *discriminant* of the étale algebra E/F is defined as

$$\text{disc}_F(E) = (-1)^{n(n-1)/2} \det_F(E) \in F^\times / (F^\times)^2.$$

Although these elements are classes of $F^\times / (F^\times)^2$, we will always regard them as classes of $F^\times / N_{K/F}(K^\times)$ using the quotient map $F^\times / (F^\times)^2 \rightarrow F^\times / N_{K/F}(K^\times)$ (recall that $(F^\times)^2 = N_{K/F}(F^\times) \subset N_{K/F}(K^\times)$).

2.2.2 Invariants of one-dimensional hermitian spaces over an étale algebra

Let $\lambda \in E^\times$. We define the discriminant of the space $L_{E,\lambda}$ as

$$\text{disc } L_{E,\lambda} = \lambda \in E^\times / N_{L_E/E}(L_E^\times).$$

We define the discriminant of the space $W_{E,\lambda}^\delta$ as

$$\text{disc } W_{E,\lambda}^\delta = \delta\lambda \in \delta E^\times / N_{L_E/E} (L_E^\times).$$

Suppose that F is a local field. We may encode each of these invariants as a tuple of signs. If $E = \prod_{j=1}^m F_j$ as before, we define a homomorphism

$$\omega_{L_E/E} : E^\times / N_{L_E/E} (L_E^\times) \rightarrow \{\pm 1\}^m$$

as follows. For $\lambda = (\lambda_1, \dots, \lambda_m) \in \prod_{j=1}^m F_j^\times$ let

$$\omega_{L_E/E} (\lambda) = \left(\omega_{L_{F_1}/F_1} (\lambda_1), \dots, \omega_{L_{F_m}/F_m} (\lambda_m) \right),$$

where $\omega_{L_{F_j}/F_j}$ is the non-trivial quadratic character if L_{F_j} is a field, and $\omega_{L_{F_j}/F_j}$ is the trivial character otherwise. We define

$$\epsilon (L_{E,\lambda}) = \omega_{L_E/E} (\lambda) \text{ and } \epsilon_\delta (W_{E,\lambda}^\delta) = \omega_{L_E/E} (\lambda).$$

2.3 Restriction of scalars of one-dimensional hermitian spaces over L_E

Let F be a field and let K/F be a quadratic field extension equipped with an involution $x \mapsto x^c$. Let E be an étale algebra of degree n over F . For any $\lambda \in E^\times$, consider the following hermitian space over K

$$V_{E,\lambda} = \left(\text{Res}_{L_E/K} L_E, \langle \cdot, \cdot \rangle_\lambda \right),$$

equipped with the following hermitian form

$$\langle x, y \rangle_\lambda = \text{tr}_{L_E/K} \langle x, y \rangle_{L_E,\lambda} = \text{tr}_{L_E/K} (\lambda x y^c),$$

where $x, y \in L_E$.

Proposition 2.1 *We have the equalities*

$$\det V_{E,1} = \det_F (E)$$

and

$$\text{disc } V_{E,1} = \text{disc}_F (E)$$

in $F^\times / N_{K/F} (K^\times)$.

Proof Choose an F -basis $\mathbf{e} = \{e_i\}_{i=1}^n$ for E . Then $1 \otimes_F \mathbf{e} = \{1 \otimes e_i\}_{i=1}^n$ is a K -basis for $V_{E,1}$. We have that $\sigma(1 \otimes e_j) = 1 \otimes e_j$, and therefore

$$(\langle e_i, e_j \rangle_1)_{ij} = (\mathrm{tr}_{L_E/K}(1 \otimes e_i e_j))_{ij} = (\mathrm{tr}_{E/F}(e_i e_j))_{ij}.$$

Taking the determinant, we get that

$$\det(\langle e_i, e_j \rangle_1)_{i,j} = \det(\mathrm{tr}_{E/F}(e_i e_j))_{i,j}, \quad (2)$$

which implies that the classes of these elements in $F^\times / N_{K/F}(K^\times)$ are the same. The class of the left hand side of (2) is $\det V_{E,1}$, while the class of the right hand side of (2) is the image of $\det_F(E)$ under the quotient map $(F^\times) / (F^\times)^2 \rightarrow F^\times / N_{K/F}(K^\times)$. \square

Combining Proposition 2.1 with [4, Corollary 1.2.2], we have the following relation between the determinants and discriminants of $V_{E,\lambda}$ and E .

Lemma 2.2 *Let $\lambda \in E^\times$. Then we have*

$$\det V_{E,\lambda} = N_{E/F}(\lambda) \cdot \det_F(E)$$

and

$$\mathrm{disc} V_{E,\lambda} = N_{E/F}(\lambda) \cdot \mathrm{disc}_F(E).$$

2.3.1 Classification of maximal tori of unitary groups

In this section, we utilize the space $V_{E,\lambda}$ and the unitary group $U(L_{E,\lambda})$ in order to classify the maximal tori of $U(V)$ where V is a non-degenerate n -dimensional hermitian space.

For any $x \in L_E$, we may consider the K -linear map $m_x: L_E \rightarrow L_E$ defined by $m_x(y) = xy$. For any $\lambda \in E^\times$ and any $x \in L_E^1$, we have that the map m_x preserves the hermitian form $\langle \cdot, \cdot \rangle_\lambda$. Let

$$T = \{m_x \mid x \in L_E^1\} \subset U(V_{E,\lambda}).$$

Consider the central simple algebra $\mathrm{End}(V_{E,\lambda})$ with involution $S \mapsto S^*$. It contains the n -dimensional étale K -subalgebra L_E realized by the embedding $L_E \rightarrow \mathrm{End}(V_{E,\lambda})$ given by $x \mapsto m_x$. Notice that

$$T = \mathrm{Res}_{L_E/F}(\mathrm{GL}_1) \cap U(V_{E,\lambda}).$$

By [27, Proposition 2.3] we have that T is a maximal torus in $U(V_{E,\lambda})$.

If V is a non-degenerate n -dimensional hermitian space, all maximal tori of $U(V)$ can be described in a similar way. First, if $r: V_{E,\lambda} \rightarrow V$ is an isomorphism of hermitian

spaces, then the map $U(V_{E,\lambda}) \rightarrow U(V)$ given by $g \mapsto r \circ g \circ r^{-1}$ is an isomorphism, and therefore

$$T_{E,r} = \left\{ r \circ m_x \circ r^{-1} \mid x \in L_E^1 \right\}$$

is a maximal torus in $U(V)$.

For the other direction, by [27, Proposition 2.3] if $T \subset U(V)$ is a maximal torus, then there exists an n -dimensional étale algebra E and an embedding $i': L_E \rightarrow \text{End}(V)$, such that $T = i'(\text{Res}_{L_E/F}(\text{GL}_1)) \cap U(V)$, or equivalently $T = i'(L_E^1)$. Finally, by [4, Proposition 1.4.1], if E is an n -dimensional étale algebra over F , then there exists an embedding $i': L_E \rightarrow \text{End}(V)$ if and only if there exists $\lambda \in E^\times$ and an isomorphism of hermitian spaces $r: V_{E,\lambda} \rightarrow V$. By the proof of [4, Proposition 1.4.1], in this case we have $i'(x) = r \circ m_x \circ r^{-1}$ for any $x \in L_E$. We summarize these results in the following theorem.

Theorem 2.3 *For any étale algebra E of degree n over F , an element $\lambda \in E^\times$ and an isomorphism of hermitian spaces $r: V_{E,\lambda} \rightarrow V$, we have that*

$$T_{E,r} = \left\{ r \circ m_x \circ r^{-1} \mid x \in L_E^1 \right\}$$

is a maximal torus in $U(V)$. Conversely, any maximal torus T in $U(V)$ can be realized in this form.

We remark that this theorem is also true in the split case $K = F \times F$ from easier considerations. In this case, the unitary group $U(V)$ is isomorphic to $\text{GL}_n(F)$. We also have in this case that $L_E = E \times E$ and that

$$L_E^1 = \left\{ (x, x^{-1}) \mid x \in E^\times \right\} \cong E^\times.$$

Hence, the statement in the split case is equivalent to the well-known fact that maximal tori in $\text{GL}_n(F)$ are in bijection with subgroups of the form $\{[m_x]_{\mathcal{B}_E} \mid x \in E^\times\}$ for some étale algebra E of degree n over F and some F -basis \mathcal{B}_E of E , where $[m_x]_{\mathcal{B}_E}$ is the matrix representing m_x with respect to the basis \mathcal{B}_E . See for example [29, Proposition 3.2.29] or [31, Section 6.1].

2.4 Admissible embeddings

Let V be a non-degenerate n -dimensional hermitian space over K . Let E be an étale algebra of rank n over F . In this section, we define the notion of an *admissible* embedding $i: L_E^1 \rightarrow U(V)$. We classify the set of admissible embeddings modulo conjugation by elements of $U(V)$. These results are needed for Section 5.

We say that an embedding $i: L_E^1 \rightarrow U(V)$ is *admissible* if there exists $\lambda \in E^\times$ and an isomorphism of hermitian spaces $r: V_{E,\lambda} \rightarrow V$, such that for any $x \in L_E^1$,

$$i(x) \circ r = r \circ m_x.$$

This definition is inspired by [8, Section 3].

Remark 2.4 [Uniqueness of λ and r] Suppose that $r_1: V_{E,\lambda} \rightarrow V$ and $r_2: V_{E,\lambda'} \rightarrow V$ are isomorphisms of hermitian spaces, where $\lambda, \lambda' \in E^\times$. Then we have that

$$r_1 \circ m_x \circ r_1^{-1} = r_2 \circ m_x \circ r_2^{-1} \quad (3)$$

for every $x \in L_E^1$, if and only if there exists $y \in L_E^\times$, such that $\lambda = N_{L_E/E}(y) \lambda'$ and $r_1 = r_2 \circ m_y$.

Indeed, if $\lambda = \lambda' \cdot N_{L_E/E}(y)$ for $y \in L_E^\times$ and $r_1 = r_2 \circ m_y$, it is easy to check that (3) holds. On the other hand if (3) holds, we have from Proposition 2.5 below that $r_2^{-1} \circ r_1: L_E \rightarrow L_E$ is an isomorphism of L_E -algebras, and therefore there exists $y \in L_E$ such that $r_2^{-1} \circ r_1 = m_y$. Since $r_2^{-1} \circ r_1: V_{E,\lambda} \rightarrow V_{E,\lambda'}$ is an isomorphism of hermitian spaces, we must have that for every $x_1, x_2 \in L_E$,

$$\langle x_1, x_2 \rangle_\lambda = \langle m_y x_1, m_y x_2 \rangle_{\lambda'} = \langle x_1, x_2 \rangle_{N_{L_E/E}(y) \lambda'},$$

which implies that $\lambda = N_{L_E/E}(y) \lambda'$.

We say that two embeddings $i_1, i_2: L_E^1 \rightarrow U(V)$ are *conjugate* if there exists $h \in U(V)$, such that for any $x \in L_E^1$,

$$h \circ i_1(x) \circ h^{-1} = i_2(x).$$

The proof of the following property is technical and will be postponed to the appendix (Proposition A.1).

Proposition 2.5 For $j = 1, 2$, let $i_j: L_E^1 \rightarrow U(V)$ be an admissible embedding corresponding to the data $r_j: V_{E,\lambda_j} \rightarrow V$, where $\lambda_j \in E^\times$. Let $i'_j: L_E \rightarrow \text{End}(V)$ be the map

$$i'_j(x) = r_j \circ m_x \circ r_j^{-1}.$$

Suppose that there exists $g \in \text{GL}(V)$, such that for any $x \in L_E^1$,

$$i_1(x) = g \circ i_2(x) \circ g^{-1}.$$

Then for any $x \in L_E$,

$$i'_1(x) = g \circ i'_2(x) \circ g^{-1}.$$

The following lemma serves as a key for our classification of admissible embeddings modulo conjugacy.

Lemma 2.6 Let $\lambda, \lambda' \in E^\times$, such that $V_{E,\lambda}$ and $V_{E,\lambda'}$ are isomorphic to V as hermitian spaces. Let $i: L_E^1 \rightarrow U(V)$ be an admissible embedding corresponding to the data

$r: \mathbb{V}_{E,\lambda} \rightarrow \mathbb{V}$. Let $r': \mathbb{V}_{E,\lambda'} \rightarrow \mathbb{V}$ be an isometry. Denote by $i_{r'}: L_E^1 \rightarrow \mathbb{U}(\mathbb{V})$ the admissible embedding

$$i_{r'}(x) = \left(r \circ r'^{-1} \right)^{-1} \circ i(x) \circ \left(r \circ r'^{-1} \right),$$

with respect to the data $r': \mathbb{V}_{E,\lambda'} \rightarrow \mathbb{V}$. Then $i_{r'}$ is conjugate to i if and only if there exists $y \in L_E^\times$, such that

$$\lambda^{-1}\lambda' = N_{L_E/E}(y).$$

Proof By Proposition 2.5, the embeddings i and $i_{r'}$ are conjugate if and only if the maps $i', i'_{r'}: L_E \rightarrow \text{End}(\mathbb{V})$, given by $i'(x) = r \circ m_x \circ r^{-1}$ and $i'_{r'}(x) = r' \circ m_x \circ r'^{-1}$ are conjugate. This happens if and only if there exists $h \in \mathbb{U}(\mathbb{V})$, such that

$$r^{-1} \circ h \circ r': L_E \rightarrow L_E$$

is an L_E -linear map.

Suppose that there exists $h \in \mathbb{U}(\mathbb{V})$ such that $r^{-1} \circ h \circ r'$ is an L_E -linear map. Then $h \circ r' = r \circ m_y$ for some $y \in L_E^\times$. For any $x_1, x_2 \in L_E$, we have

$$\langle hr'x_1, hr'x_2 \rangle_{\mathbb{V}} = \langle r'x_1, r'x_2 \rangle_{\mathbb{V}} = \langle x_1, x_2 \rangle_{\lambda'}.$$

and

$$\langle rm_yx_1, rm_yx_2 \rangle_{\mathbb{V}} = \langle m_yx_1, m_yx_2 \rangle_{\lambda} = \langle x_1, x_2 \rangle_{N_{L_E/E}(y)\lambda}.$$

Hence, we have for every $x_1, x_2 \in L_E$,

$$\langle x_1, x_2 \rangle_{\lambda'} = \langle x_1, x_2 \rangle_{N_{L_E/E}(y)\lambda},$$

which implies that $\lambda' = N_{L_E/E}(y)\lambda$, and therefore

$$\lambda^{-1}\lambda' = N_{L_E/E}(y),$$

as required.

We move to prove the other direction. Suppose that $\lambda' = N_{L_E/E}(y)\lambda$, where $y \in L_E^\times$. Define $h = r \circ m_y \circ r'^{-1}$. Then $r^{-1} \circ h \circ r' = m_y$ is an L_E -linear map. We check that the element h we have constructed lies in the unitary group $\mathbb{U}(\mathbb{V})$. Repeating similar steps to before, we have

$$\langle hr'x_1, hr'x_2 \rangle_{\mathbb{V}} = \langle rm_yx_1, rm_yx_2 \rangle_{\mathbb{V}} = \langle x_1, x_2 \rangle_{N_{L_E/E}(y)\lambda}.$$

Since $N_{L_E/E}(y)\lambda = \lambda'$, we get

$$\langle hr'x_1, hr'x_2 \rangle_{\mathbb{V}} = \langle x_1, x_2 \rangle_{\lambda'} = \langle r'x_1, r'x_2 \rangle_{\mathbb{V}}.$$

Therefore, we get that $h \in U(V)$, as required. \square

Remark 2.7 If $i_1, i_2: L_E^1 \rightarrow U(V)$ are embeddings, we say that i_1 and i_2 are *stably conjugate* if there exists $g \in U(V)(\bar{F})$, such that for any $x \in L_E^1$,

$$g \circ i_1(x) \circ g^{-1} = i_2(x).$$

Here, \bar{F} is an algebraic closure of F , and we refer to Section 2.5 for the definition of $U(V)(\bar{F})$. We have that the set of admissible embeddings $i: L_E^1 \rightarrow U(V)$ is a stable conjugacy class. We will show this only in the case where K/F is a quadratic field extension, as the other case is simpler. In this case, we choose \bar{F} to be an algebraic closure of K . We have that $U(V)(\bar{F})$ is naturally isomorphic to $GL_{\bar{F}}(V \otimes_K \bar{F})$, the group of invertible \bar{F} -linear maps $V \otimes_K \bar{F} \rightarrow V \otimes_K \bar{F}$. If i_1 and i_2 are admissible embeddings corresponding to the data $r_1: V_{E, \lambda_1} \rightarrow V$ and $r_2: V_{E, \lambda_2} \rightarrow V$ with $\lambda_1, \lambda_2 \in E^\times$, then for any $x \in L_E^1$, we have $g \circ i_1(x) \circ g^{-1} = i_2(x)$, where $g = r_2 \circ r_1^{-1} \in GL_K(V)$ (where $GL_K(V)$ is the group of invertible K -linear maps $V \rightarrow V$). Hence, i_1 and i_2 are stably conjugate.

On the other hand, if $i_1: L_E^1 \rightarrow U(V)$ is an admissible embedding, corresponding to the data $\lambda \in E^\times$ and $r: V_{E, \lambda} \rightarrow V$, and $i_2: L_E^1 \rightarrow U(V)$ is an embedding, such that i_1 and i_2 are stably conjugate, then there exists an invertible \bar{F} -linear map $g: V \otimes_K \bar{F} \rightarrow V \otimes_K \bar{F}$ such that $g \circ (i_1(x) \otimes \text{id}_{\bar{F}}) \circ g^{-1} = i_2(x) \otimes \text{id}_{\bar{F}}$ for any $x \in L_E^1$. Since $i_2(L_E^1)$ is a maximal torus in $U(V)$, there exists an étale algebra E' of rank n over F , a $\lambda' \in (E')^\times$, and an isomorphism of hermitian spaces $r': V_{E', \lambda'} \rightarrow V$, such that the image of the map $L_{E'}^1 \rightarrow U(V)$ given by $x \mapsto r' \circ \text{m}_x \circ (r')^{-1}$ is $i_2(L_E^1)$. Consider the map $T = (r^{-1} \otimes \text{id}_{\bar{F}}) \circ g^{-1} \circ (r' \otimes \text{id}_{\bar{F}}): L_{E'} \otimes_K \bar{F} \rightarrow L_E \otimes_K \bar{F}$. Since we have for any $x \in L_E^1$, $T \circ (\text{m}_x \otimes \text{id}_{\bar{F}}) = (\text{m}_{\tau(x)} \otimes \text{id}_{\bar{F}}) \circ T$ for some $\tau(x) \in L_{E'}^1$, by Proposition A.2 for any $x \in L_E$, there exists a unique $\tau(x) \in L_{E'}$, such that $T \circ (\text{m}_{\tau(x)} \otimes \text{id}_{\bar{F}}) = (\text{m}_x \otimes \text{id}_{\bar{F}}) \circ T$. It follows that $\tau: L_E \rightarrow L_{E'}$ is an isomorphism of étale algebras over K , that is, it is an isomorphism of rings that fixes K . We have that for any $x \in L_E^1$,

$$\begin{aligned} i_2(x) \otimes \text{id}_{\bar{F}} &= (r' \otimes \text{id}_{\bar{F}}) \circ T^{-1} \circ (\text{m}_x \otimes \text{id}_{\bar{F}}) \circ T \circ ((r')^{-1} \otimes \text{id}_{\bar{F}}) \\ &= (r' \circ \text{m}_{\tau(x)} \circ (r')^{-1}) \otimes \text{id}_{\bar{F}}. \end{aligned}$$

This implies that i_2 is an admissible embedding with respect to the data $\tau^{-1}(\lambda') \in E^\times$ and $r'': V_{E, \tau^{-1}(\lambda')} \rightarrow V$, given by $r''(y) = r'(\tau(y))$, as required.

It is clear that if $\lambda, \lambda' \in E^\times$ are such that $\lambda = N_{L_E/E}(y) \cdot \lambda'$, for some $y \in L_E^\times$, then the hermitian spaces $V_{E, \lambda}$ and $V_{E, \lambda'}$ are isomorphic by the map $V_{E, \lambda} \rightarrow V_{E, \lambda'}$ given by $x \mapsto \text{m}_y x$. The following theorem establishes a bijection between admissible embeddings of L_E^1 modulo conjugation and certain classes $\lambda \in E^\times / N_{L_E/E}(L_E^\times)$.

Theorem 2.8 *There exists a natural bijection between the set*

$$\{\lambda \in E^\times / N_{L_E/E} (L_E^\times) \mid V_{E,\lambda} \cong V \text{ as hermitian spaces}\}$$

and the set

$$\Sigma_{E,V} = \left\{ i: L_E^1 \rightarrow U(V) \mid i \text{ is admissible} \right\} / U(V)\text{-conjugation}.$$

This bijection is given as follows. For any $\lambda \in E^\times$ such that $V \cong V_{E,\lambda}$, choose an isomorphism of hermitian spaces $r: V_{E,\lambda} \rightarrow V$ and define an admissible embedding $i_r: L_E^1 \rightarrow U(V)$ by the formula

$$i_r(x) = r \circ m_x \circ r^{-1},$$

where $x \in L_E^1$. The bijection sends the class $[\lambda] \in E^\times / N_{L_E/E} (L_E^\times)$ to the conjugacy class $c_\lambda = [i_r] \in \Sigma_{E,V}$ of i_r .

Proof We claim that the conjugacy class of i_r does not depend on the choice of r . Indeed, if $r_2: V_{E,\lambda} \rightarrow V$ is another isomorphism, then for any $x \in L_E^1$

$$i_{r_2}(x) = (r \circ r_2^{-1})^{-1} \circ i_r(x) \circ r \circ r_2^{-1},$$

and we have that $r \circ r_2^{-1} \in U(V)$ as a composition of two isometries.

We show that $\lambda \mapsto c_\lambda$ is a bijection as in the theorem.

The map is injective: given $\lambda, \lambda' \in E^\times$, and isomorphisms $r: V_{E,\lambda} \rightarrow V$ and $r': V_{E,\lambda'} \rightarrow V$, we have for any $x \in L_E^1$,

$$i_{r'}(x) = (r \circ r'^{-1})^{-1} \circ i_r(x) \circ r \circ r'^{-1}.$$

It follows from Lemma 2.6 that the embeddings i_r and $i_{r'}$ are conjugate if and only if $(\lambda')^{-1} \lambda \in N_{L_E/E} (L_E^\times)$.

The map is surjective: let $i: L_E^1 \rightarrow U(V)$ be an admissible embedding corresponding to the data $r: V_{E,\lambda} \rightarrow V$. We have that for any $x \in L_E^1$,

$$i(x) = r \circ m_x \circ r^{-1}.$$

Therefore, c_λ is the conjugacy class of i , as required. \square

Remark 2.9 When F is a non-archimedean local field and K/F is a quadratic field extension, there exist exactly two isomorphism classes of non-degenerate hermitian spaces of dimension n over K . The isomorphism class of such hermitian space is determined by its discriminant. Using Lemma 2.2, we can rewrite the first set in the proposition as

$$\{\lambda \in E^\times / N_{L_E/E} (L_E^\times) \mid \text{disc } V = N_{E/F}(\lambda) \text{disc}_F(E)\}.$$

Remark 2.10 Let $\lambda, \lambda' \in E^\times$. If F is a non-archimedean local field then, as in the previous remark, we have that the hermitian spaces $V_{E,\lambda}$ and $V_{E,\lambda'}$ are isomorphic if and only if $N_{E/F}(\lambda^{-1}\lambda') \in N_{K/F}(K^\times)$.

If F is an archimedean local field, then the hermitian spaces $V_{E,\lambda}$ and $V_{E,\lambda'}$ are isomorphic if and only if the number of non-trivial components of λ and λ' as elements of $E^\times/N_{L_E/E}(L_E^\times)$ is the same.

2.5 Functoriality for admissible embeddings

In this section, we explain how an admissible embedding of a torus $i: L_E^1 \rightarrow T \subset U(V)$, gives rise to a family of embeddings $i(R): L_E^1(R) \rightarrow U(V)(R)$ for any F -algebra R .

Let R be a commutative F -algebra, and consider the ring $L_R = K \otimes_F R$, equipped with the involution $\sigma: L_R \rightarrow L_R$ defined on pure tensors by

$$\sigma(k \otimes h) = k^c \otimes h,$$

where $k \in K$ and $h \in R$. Henceforth, we will write x^c instead of $\sigma(x)$ for $x \in L_R$.

As before, one can define the notion of an ϵ -hermitian space over L_R , as in Section 2.1 by replacing K with L_R in the definitions.

Let $(V, \langle \cdot, \cdot \rangle_V)$ be an ϵ -hermitian space over K . Then the space $(V(R), \langle \cdot, \cdot \rangle_{V(R)})$ is an ϵ -hermitian space over L_R , where $V(R) = V \otimes_F R$ and $\langle \cdot, \cdot \rangle_{V(R)}$ is defined on pure tensors by

$$\langle v_1 \otimes h_1, v_2 \otimes h_2 \rangle_{V(R)} = \langle v_1, v_2 \rangle_V \otimes (h_1 h_2),$$

where $v_1, v_2 \in V$ and $h_1, h_2 \in R$. We denote by $U(V)(R)$ the group consisting of invertible L_R -linear maps that preserve the form $\langle \cdot, \cdot \rangle_{V(R)}$. Suppose that E is an étale algebra over F of degree $n = \dim V$, and that $i: L_E^1 \rightarrow U(V)$ is an admissible embedding corresponding to the data $\lambda \in E^\times$ and $r: V_{E,\lambda} \rightarrow V$. Since r is an isomorphism of ϵ -hermitian spaces over K , we have that the map $r(R) = r \otimes \text{id}_R: V_{E,\lambda}(R) \rightarrow V(R)$ is an isomorphism of ϵ -hermitian spaces over L_R .

We may define an embedding $i(R): \text{Res}_{E/F} L_E^1(R) \rightarrow U(V)(R)$, corresponding to the data $\lambda \in E^\times$ and $r(R): V_{E,\lambda}(R) \rightarrow V(R)$, by the formula

$$i(R)(x) = r(R) \circ m_x \circ r(R)^{-1},$$

where

$$\text{Res}_{E/F} L_E^1(R) = \{x \in (L_E \otimes_F R)^\times \mid x \cdot x^c = 1\},$$

and for $x \in (L_E \otimes_F R)^\times$, the map $m_x: L_E \otimes_F R \rightarrow L_E \otimes_F R$ is the multiplication by x map. Here, as usual, $x \mapsto x^c$ is the involution on $L_E \otimes_F R$, defined on pure tensors by

$$(y \otimes h)^c = y^c \otimes h,$$

where $y \in L_E$ and $h \in R$.

Note that if $k \in K^1$, we always have that $i(R)(k \otimes 1) = m_k \otimes \text{id}_R$, where $m_k: V \rightarrow V$ is the map $m_k v = kv$, i.e., the multiplication by the scalar k map. In the special case where V is one-dimensional, and hence $E = F$, we get that $i(R): L_E^1(R) \rightarrow U(V)(R)$ is an isomorphism for every R .

In the sequel, given an admissible embedding $i: L_E^1 \rightarrow U(V)$, we will often write i for $i(R)$, especially when $F = \mathbb{F}$ is a number field and $R = \mathbb{A}_{\mathbb{F}}$ is its ring of adèles.

3 Local theory

In this section, we recall the local theta correspondence. We use it to define a theta correspondence for 1-dimensional L_E -hermitian spaces. We then discuss a seesaw identity satisfied by the theta lift we defined and the usual theta lift. Finally, we recall the definition of local root numbers, and the classical results regarding theta lifting from $U(1)$ to $U(1)$ in terms of them. We use these results to determine when our theta lift for 1-dimensional L_E -hermitian spaces does not vanish. These results are needed for Section 5.

3.1 The local theta correspondence

Let F be a local field of characteristic $\neq 2$ and let K/F be a quadratic étale algebra. Let V and W be non-degenerate finite-dimensional hermitian and skew-hermitian spaces over K , respectively.

Consider the tensor product $\text{Res}_{K/F}(V \otimes_K W)$. We equip this space with a symplectic form defined on pure tensors by

$$\langle \langle v \otimes w, v' \otimes w' \rangle \rangle = \text{tr}_{K/F}(\langle v, v' \rangle_V \cdot \langle w, w' \rangle_W),$$

where $v, v' \in V$ and $w, w' \in W$.

For any non-trivial character $\psi: F \rightarrow \mathbb{C}^\times$, we have a unique (up to isomorphism) irreducible (smooth) representation $\omega_{\psi, F}$ of the Heisenberg group associated with $\text{Res}_{K/F}(V \otimes_K W)$, such that the central character of $\omega_{\psi, F}$ is ψ . The representation $\omega_{\psi, F}$ gives rise to an \mathbb{S}^1 -metaplectic cover $\text{Mp}_\psi(\text{Res}_{K/F}(V \otimes_K W))$ of $\text{Sp}(\text{Res}_{K/F}(V \otimes_K W))$, where $\mathbb{S}^1 \subset \mathbb{C}^\times$ is the unit circle. We denote this group by $\text{Mp}_\psi(V, W)$ for short. The representation $\omega_{\psi, F}$ above gives rise to an irreducible representation of the metaplectic group $\text{Mp}_\psi(V, W)$, which we also denote by $\omega_{\psi, F}$. We call $\omega_{\psi, F}$ the *Weil representation* associated with ψ .

We have an embedding $\iota: U(V) \times U(W) \rightarrow \text{Sp}(\text{Res}_{K/F}(V \otimes_K W))$, where for $g_V \in U(V)$ and $g_W \in U(W)$, the map $\iota(g_V, g_W)$ is defined on pure tensors by

$$\iota(g_V, g_W)(v \otimes w) = g_V v \otimes g_W w,$$

where $v \in V$ and $w \in W$.

When referring to representations of $U(V)$ (or $U(W)$), we will always mean smooth admissible representations. Let us denote by $\text{Irr } U(V)$ and $\text{Irr } U(W)$ the set of equivalence classes of irreducible (smooth) representations of $U(V)$ and of $U(W)$, respectively.

The theta correspondence allows us to transfer irreducible representations of $U(V)$ to irreducible representations of $U(W)$, and vice versa. In order to describe it, we need a lifting of ι to the metaplectic group

$$\tilde{\iota}: U(V) \times U(W) \rightarrow \text{Mp}_{\psi}(V, W).$$

The existence of such liftings, usually called splittings, is due to Kudla [19]. We postpone the discussion regarding this splitting to the next subsection. Given such a splitting $\tilde{\iota}$, we may pullback $\omega_{\psi, F}$ to a representation $\Omega_{V, W, \tilde{\iota}, \psi}$ of $U(V) \times U(W)$.

We proceed by describing the theta correspondence. Let π be an irreducible representation of $U(V)$. The big theta lift $\Theta(\pi)$ is defined as follows. Consider the maximal π -isotypic quotient of $\Omega_{V, W, \tilde{\iota}, \psi}$:

$$(\Omega_{V, W, \tilde{\iota}, \psi})_{\pi, U(V)} := \Omega_{V, W, \tilde{\iota}, \psi} / \bigcap_f \ker f,$$

where the intersection is over all

$$f \in \text{Hom}_{U(V) \times 1}(\Omega_{V, W, \tilde{\iota}, \psi}, \pi).$$

By construction, we have that $(\Omega_{V, W, \tilde{\iota}, \psi})_{\pi, U(V)}$ is of the form $\pi \otimes \sigma$, where σ is a representation of $U(W)$. We write $\Theta_{V, W, \tilde{\iota}, \psi}(\pi)$ for σ and call this representation the *big theta lift* of π . The big theta lift $\Theta_{V, W, \tilde{\iota}, \psi}(\pi)$ satisfies the following functorial property: for any irreducible representation τ of $U(W)$ we have that

$$\text{Hom}_{U(V) \times U(W)}(\Omega_{V, W, \tilde{\iota}, \psi}, \pi \otimes \tau) \cong \text{Hom}_{U(W)}(\Theta_{V, W, \tilde{\iota}, \psi}(\pi), \tau).$$

More generally, for any subgroup $H \leq U(W)$ and any irreducible representation τ of H , we have that

$$\text{Hom}_{U(V) \times H}(\Omega_{V, W, \tilde{\iota}, \psi}, \pi \otimes \tau) \cong \text{Hom}_H(\Theta_{V, W, \tilde{\iota}, \psi}(\pi) \upharpoonright_H, \tau). \quad (4)$$

We move to discuss the *small* theta lift. Howe and Kudla proved that if the big theta lift defined above is non-zero, then it is of finite length. It follows that $\Theta_{V, W, \tilde{\iota}, \psi}(\pi)$ has a maximal semisimple quotient, which we denote by $\theta_{V, W, \tilde{\iota}, \psi}(\pi)$ and call the *small theta lift* of π .

The following two theorems were proved by Howe in the archimedean case [17], by Waldspurger in the non-archimedean case for fields with residue field of odd characteristic [32], and by Gan–Takeda in the non-archimedean case in general [12].

Theorem 3.1 *If the big theta lift $\Theta_{V, W, \tilde{\iota}, \psi}(\pi)$ is not zero, then it has a unique irreducible quotient.*

Therefore, it follows that if $\Theta_{V,W,\tilde{\iota},\psi}(\pi)$ is not zero, then $\theta_{V,W,\tilde{\iota},\psi}(\pi)$ is the unique irreducible quotient of $\Theta_{V,W,\tilde{\iota},\psi}(\pi)$.

Moreover, if π and π' are irreducible representations of $U(V)$ with the same non-zero small theta lift, then π and π' are isomorphic. These results combined yield the following statement, which is known as *Howe duality*.

Theorem 3.2 *We have a map $\text{Irr } U(V) \rightarrow \text{Irr } U(W) \cup \{0\}$, given by $\pi \mapsto \theta_{V,W,\tilde{\iota},\psi}(\pi)$. The restriction of this map to the set of representations with non-zero theta lift is an injective map.*

We remark that we started with an irreducible representation π of $U(V)$ and constructed its big and small theta lifts. Similarly, we can start with an irreducible representation σ of $U(W)$ and construct its big and small theta lifts. We have analogous results by exchanging the roles of V and W , and of π and σ , respectively. The above results yield the following multiplicity one theorem:

Theorem 3.3 *For any $\pi \in \text{Irr } U(V)$ and $\sigma \in \text{Irr } U(W)$ we have*

$$\dim \text{Hom}_{U(V) \times U(W)}(\Omega_{V,W,\tilde{\iota},\psi}, \pi \otimes \sigma) \leq 1.$$

Let us mention a useful fact relating the big theta lift and the small theta lift in a special case. By [23, Page 69, Theoreme principal], if π is supercuspidal and $\Theta_{V,W,\tilde{\iota},\psi}(\pi)$ is not zero, then $\Theta_{V,W,\tilde{\iota},\psi}(\pi)$ is irreducible, and we have that it equals $\theta_{V,W,\tilde{\iota},\psi}(\pi)$. In particular, if V is one-dimensional, then π is a character and hence $\Theta_{V,W,\tilde{\iota},\psi}(\pi)$ coincides with $\theta_{V,W,\tilde{\iota},\psi}(\pi)$.

3.1.1 Splitting of the embedding ι

In this subsection, we discuss the details regarding the splitting provided by Kudla's work [19]. We will explain the data needed in order to define a splitting

$$\tilde{\iota}: U(V) \times U(W) \rightarrow \text{Mp}_{\psi}(V, W)$$

for the embedding

$$\iota: U(V) \times U(W) \rightarrow \text{Sp}(\text{Res}_{K/F}(V \otimes_F W))$$

described above.

The splitting $\tilde{\iota}$ depends on a choice of two characters χ_V, χ_W of K^\times such that

$$\chi_W|_{F^\times} = \omega_{K/F}^{\dim W} \quad \text{and} \quad \chi_V|_{F^\times} = \omega_{K/F}^{\dim V}.$$

For example, we can choose a character $\mu: K^\times \rightarrow \mathbb{C}^\times$ such that $\mu|_{F^\times} = \omega_{K/F}$ and define $\chi_V = \mu^{\dim V}$ and $\chi_W = \mu^{\dim W}$.

Given such χ_V , Kudla constructs an embedding $\tilde{\iota}_{\psi,\chi_V}: U(W) \rightarrow \text{Mp}_{\psi}(V, W)$. Similarly, given χ_W , Kudla constructs an embedding $\tilde{\iota}_{\psi,\chi_W}: U(V) \rightarrow \text{Mp}_{\psi}(V, W)$.

It turns out that the images of \tilde{t}_{ψ, χ_W} and of \tilde{t}_{ψ, χ_W} commute. It also turns out that the images of these embeddings have mutual center. Hence, we get a splitting $\tilde{t} = \tilde{t}_{\psi, \chi_V, \chi_W} : U(V) \times U(W) \rightarrow \text{Mp}_{\psi}(V, W)$, as desired.

3.1.2 Notation for theta lifts of characters of K^1

We introduce another notation for lifting of characters of K^1 that uses a trace zero element δ instead of a skew-hermitian space W .

Let $\delta \in K^\times$ be a trace zero element, and let $\mu : K^\times \rightarrow \mathbb{C}^\times$ be a character such that $\mu|_{F^\times} = \omega_{K/F}$. Suppose that V is a hermitian space over K and that $\beta : K^1 \rightarrow \mathbb{C}^\times$ is a character. We denote

$$\Theta_{\delta, V, \mu, \psi}(\beta) := \Theta_{W_{F,1}^\delta, V, \tilde{t}_\mu, \psi} \left(\beta \circ i'_{W_{F,1}^\delta} \right)$$

and

$$\theta_{\delta, V, \mu, \psi}(\beta) := \theta_{W_{F,1}^\delta, V, \tilde{t}_\mu, \psi} \left(\beta \circ i'_{W_{F,1}^\delta} \right)$$

where $i'_{W_{F,1}^\delta} : U(W_{F,1}^\delta) \rightarrow K^1$ is the obvious isomorphism and where \tilde{t}_μ is the splitting associated to the characters $(\mu, \mu^{\dim V})$.

3.1.3 Theta lifting for unitary groups of 1-dimensional spaces over étale algebras

Let E be an étale algebra of rank n over F . Choose a trace zero element $\delta \in K^\times$. Let $\lambda, \lambda' \in L_E^\times$ and consider the L_E -hermitian space $L_{E,\lambda}$ and the L_E -skew-hermitian space $W_{E,\lambda'}^\delta$. In this section, we describe the theta correspondence for the groups $U(L_{E,\lambda})$ and $U(W_{E,\lambda'}^\delta)$.

As before, we write $E = \prod_{j=1}^m F_j$, where for every j , F_j/F is a field extension. Then for $\lambda = (\lambda_1, \dots, \lambda_m) \in E^\times$ and $\lambda' = (\lambda'_1, \dots, \lambda'_m) \in E^\times$, we have that

$$U(L_{E,\lambda}) = \prod_{j=1}^m U(L_{F_j,\lambda_j}) \quad \text{and} \quad U(W_{E,\lambda'}^\delta) = \prod_{j=1}^m U(W_{F_j,\lambda'_j}^\delta).$$

For every $1 \leq j \leq m$, we denote $V_j = L_{F_j,\lambda_j}$ and $W_j = W_{F_j,\lambda'_j}^\delta$. Then every character $\alpha : U(L_{E,\lambda}) \rightarrow \mathbb{C}^\times$ is equivalent to a tuple $(\alpha_1, \dots, \alpha_m)$, where for every j , the map $\alpha_j : U(V_j) \rightarrow \mathbb{C}^\times$ is a character. Therefore, we may use the usual local theta correspondence to define a local theta correspondence for the groups $U(L_{E,\lambda})$ and $U(W_{E,\lambda'}^\delta)$. Let us describe this correspondence.

Let $\chi_{L_{E,\lambda}}, \chi_{W_{E,\lambda'}^\delta} : L_E^\times \rightarrow \mathbb{C}^\times$ be multiplicative characters, such that

$$\chi_{L_{E,\lambda}}|_{E^\times} = \chi_{W_{E,\lambda'}^\delta}|_{E^\times} = \omega_{L_E/E}.$$

We have that $\chi_{L_{E,\lambda}}$ and $\chi_{W_{E,\lambda}^\delta}$ correspond to tuples $(\chi_{V_1}, \dots, \chi_{V_m})$ and $(\chi_{W_1}, \dots, \chi_{W_m})$, respectively, where for every $1 \leq j \leq m$, $\chi_{V_j}, \chi_{W_j}: L_{F_j}^\times \rightarrow \mathbb{C}^\times$ are characters, such that

$$\chi_{V_j} \upharpoonright_{L_{F_j}^\times} = \chi_{W_j} \upharpoonright_{L_{F_j}^\times} = \omega_{L_{F_j}/F_j}.$$

Therefore, we get a splitting

$$\tilde{\iota}_j: U(V_j) \times U(W_j) \rightarrow \mathrm{Mp}_{\psi_j}(V_j, W_j),$$

where $\psi_j = \psi \circ \mathrm{tr}_{F_j/F}$. We write $\tilde{\iota} = (\tilde{\iota}_1, \dots, \tilde{\iota}_m)$.

We define the big theta lift of α as above by the formula

$$\Theta_{L_{E,\lambda}, W_{E,\lambda'}^\delta, \tilde{\iota}, \psi}(\alpha) = \Theta_{V_1, W_1, \tilde{\iota}_1, \psi_1}(\alpha_1) \otimes \cdots \otimes \Theta_{V_m, W_m, \tilde{\iota}_m, \psi_m}(\alpha_m).$$

It is a representation of $U(W_{E,\lambda'}^\delta)$ (might be the zero representation).

Let $\omega_{\psi,E} = \bigotimes_{j=1}^m \omega_{\psi_j, F_j}$ and $\Omega_{L_{E,\lambda}, W_{E,\lambda'}^\delta, \tilde{\iota}, \psi} = \bigotimes_{j=1}^m \Omega_{V_j, W_j, \tilde{\iota}_j, \psi_j}$. Once again, consider the maximal α -isotypic quotient of $\Omega_{L_{E,\lambda}, W_{E,\lambda'}^\delta, \tilde{\iota}, \psi}$:

$$\left(\Omega_{L_{E,\lambda}, W_{E,\lambda'}^\delta, \tilde{\iota}, \psi} \right)_{\alpha, U(L_{E,\lambda})} = \Omega_{L_{E,\lambda}, W_{E,\lambda'}^\delta, \tilde{\iota}, \psi} / \bigcap_f \ker f,$$

where the intersection is over all $f \in \mathrm{Hom}_{U(L_{E,\lambda}) \times 1}(\Omega_{L_{E,\lambda}, W_{E,\lambda'}^\delta, \tilde{\iota}, \psi}, \alpha)$. Then, similarly to before, we have that

$$\left(\Omega_{L_{E,\lambda}, W_{E,\lambda'}^\delta, \tilde{\iota}, \psi} \right)_{\alpha, U(L_{E,\lambda})} \cong \alpha \otimes \Theta_{L_{E,\lambda}, W_{E,\lambda'}^\delta, \tilde{\iota}, \psi}(\alpha).$$

As before, for any subgroup $H \subset U(W_{E,\lambda}^\delta)$ and any irreducible representation τ of H , we have that

$$\mathrm{Hom}_{U(L_{E,\lambda}) \times H}(\Omega_{L_{E,\lambda}, W_{E,\lambda'}^\delta, \tilde{\iota}, \psi}, \alpha \otimes \tau) \cong \mathrm{Hom}_H(\Theta_{L_{E,\lambda}, W_{E,\lambda'}^\delta, \tilde{\iota}, \psi}(\alpha) \upharpoonright_H, \tau). \quad (5)$$

We make the following remark which will be useful later. Suppose that $\lambda' = 1$. We have the following decomposition of symplectic spaces

$$\mathrm{Res}_{K/F}(V_{E,\lambda} \otimes_K W_{F,1}^\delta) = \bigoplus_{j=1}^m \mathrm{Res}_{L_{F_j}/F}(V_j \otimes_{L_{F_j}} W_j).$$

Hence, we get a natural map (see [26, Remark 2.7])

$$\prod_{j=1}^m \mathrm{Mp}_{\psi_j}(\mathbf{V}_j, \mathbf{W}_j) \rightarrow \mathrm{Mp}_{\psi}(\mathbf{V}_{E,\lambda}, \mathbf{W}_{F,1}^{\delta}).$$

This map is not injective, but its restriction to $\mathrm{Mp}_{\psi_j}(\mathbf{V}_j, \mathbf{W}_j)$ is injective for every j . Therefore, we may regard \tilde{t} as a map

$$\tilde{t}: \mathrm{U}(L_{E,\lambda}) \times \mathrm{U}(\mathbf{W}_{E,1}^{\delta}) \rightarrow \mathrm{Mp}_{\psi}(\mathbf{V}_{E,\lambda}, \mathbf{W}_{F,1}^{\delta}).$$

Furthermore, since the restriction of the Weil representation $\omega_{\psi,E}$ to $\mathrm{Mp}_{\psi_j}(\mathbf{V}_j, \mathbf{W}_j)$ is ω_{ψ_j,F_j} , we have that the restriction of $\Omega_{\mathbf{V}_{E,\lambda}, \mathbf{W}_{F,1}^{\delta}, \tilde{t}, \psi}$ to $\mathrm{U}(\mathbf{V}_j) \times \mathrm{U}(\mathbf{W}_j)$ is $\Omega_{\mathbf{V}_j, \mathbf{W}_j, \tilde{t}, \psi_j}$. This compatibility is important for the local seesaw identity, which we will describe in the next section.

Remark 3.4 By [23, Pages 36-37, Remarque (6)], the kernel of the map $\prod_{j=1}^m \mathrm{Mp}_{\psi_j}(\mathbf{V}_j, \mathbf{W}_j) \rightarrow \mathrm{Mp}_{\psi}(\mathbf{V}_{E,\lambda}, \mathbf{W}_{F,1}^{\delta})$ is given by all tuples (g_1, \dots, g_m) such that for every j the projection of g_j to $\mathrm{Sp}(\mathrm{Res}_{L_{F_j}/F_j}(\mathbf{V}_j \otimes_{L_{F_j}} \mathbf{W}_j))$ is the identity, and such that if t_j is the projection of g_j to \mathbb{S}^1 then $\prod_{j=1}^m t_j = 1$.

3.2 A local seesaw identity

3.2.1 Splitting set up

Suppose we are in the setup of Section 3.1.3 with $\lambda' = 1$. We will consider the following seesaw diagram:

$$\begin{array}{ccc} \mathrm{U}(\mathbf{W}_{E,1}^{\delta}) & & \mathrm{U}(\mathbf{V}_{E,\lambda}) \\ | & \searrow & \swarrow \\ \mathrm{U}(\mathbf{W}_{F,1}^{\delta}) & & \mathrm{U}(L_{E,\lambda}). \end{array}$$

Here $\mathrm{U}(\mathbf{W}_{F,1}^{\delta})$ is realized as a subgroup of $\mathrm{U}(\mathbf{W}_{E,1}^{\delta})$ diagonally, that is, an element $x \in K^1 \cong \mathrm{U}(\mathbf{W}_{F,1}^{\delta})$ acts on $\mathbf{W}_{E,1}^{\delta}$ by the multiplication by x map m_x , given by $m_x w = xw$, for $w \in \mathbf{W}_{E,1}^{\delta}$.

In order to write down a seesaw identity, we need to fix compatible splittings. We explain this now.

Given characters $\chi_{L_{E,\lambda}}, \chi_{\mathbf{W}_{E,1}^{\delta}} : L_E^{\times} \rightarrow \mathbb{C}^{\times}$, such that

$$\chi_{L_{E,\lambda}} \upharpoonright_{E^{\times}} = \chi_{\mathbf{W}_{E,1}^{\delta}} \upharpoonright_{E^{\times}} = \omega_{L_E/E},$$

we constructed a map

$$\tilde{t}: U(L_{E,\lambda}) \times U(W_{E,1}^\delta) \rightarrow \mathrm{Mp}_\psi(V_{E,\lambda}, W_{F,1}^\delta).$$

By Kudla's construction, this splitting is of the form $\tilde{t} = \tilde{t}_{\chi_{W_{E,1}^\delta}} \times \tilde{t}_{\chi_{L_{E,\lambda}}}$, where

$$\tilde{t}_{\chi_{W_{E,1}^\delta}}: U(L_{E,\lambda}) \rightarrow \mathrm{Mp}_\psi(V_{E,\lambda}, W_{F,1}^\delta) \quad \text{and} \quad \tilde{t}_{\chi_{L_{E,\lambda}}}: U(W_{E,1}^\delta) \rightarrow \mathrm{Mp}_\psi(V_{E,\lambda}, W_{F,1}^\delta).$$

On the other hand, given characters $\chi_{V_{E,\lambda}}, \chi_{W_{F,1}^\delta}: K^\times \rightarrow \mathbb{C}^\times$, such that

$$\chi_{V_{E,\lambda}} \upharpoonright_{F^\times} = \omega_{K/F}^{\dim_K V_{E,\lambda}} \quad \text{and} \quad \chi_{W_{F,1}^\delta} \upharpoonright_{F^\times} = \omega_{K/F},$$

we have a splitting

$$\tilde{t}': U(V_{E,\lambda}) \times U(W_{F,1}^\delta) \rightarrow \mathrm{Mp}_\psi(V_{E,\lambda}, W_{F,1}^\delta).$$

Once again, by Kudla's construction, this embedding is of the form $\tilde{t}' = \tilde{t}'_{\chi_{W_{F,1}^\delta}} \times \tilde{t}'_{\chi_{V_{E,\lambda}}}$, where

$$\tilde{t}'_{\chi_{W_{F,1}^\delta}}: U(V_{E,\lambda}) \rightarrow \mathrm{Mp}_\psi(V_{E,\lambda}, W_{F,1}^\delta) \quad \text{and} \quad \tilde{t}'_{\chi_{V_{E,\lambda}}}: U(W_{F,1}^\delta) \rightarrow \mathrm{Mp}_\psi(V_{E,\lambda}, W_{F,1}^\delta).$$

We say that these splitting \tilde{t} and \tilde{t}' are *compatible* if they agree on the subgroup $U(L_{E,\lambda}) \times U(W_{F,1}^\delta)$. This is equivalent to requiring the following equalities between the characters involved:

$$\chi_{W_{E,1}^\delta} = \chi_{W_{F,1}^\delta} \circ N_{L_E/K} \quad \text{and} \quad \chi_{L_{E,\lambda}} \upharpoonright_{K^\times} = \chi_{V_{E,\lambda}}.$$

We refer to the discussion in [15, Section 1] for more details.

3.2.2 The local seesaw identity

Let $\alpha: U(L_{E,\lambda}) \rightarrow \mathbb{C}^\times$ and $\beta: U(W_{F,1}^\delta) \rightarrow \mathbb{C}^\times$ be characters.

Choose compatible splittings \tilde{t} and \tilde{t}' as above. Let $\Theta(\beta)$ be the big theta lift of β to $U(V_{E,\lambda})$ with respect to the splitting \tilde{t}' . By (4), we have the following equality, where we take $H = U(L_{E,\lambda})$:

$$\mathrm{Hom}_{U(L_{E,\lambda}) \times U(W_{F,1}^\delta)} \left(\Omega_{V_{E,\lambda}, W_{F,1}^\delta, \tilde{t}', \psi}, \alpha \otimes \beta \right) \cong \mathrm{Hom}_{U(L_{E,\lambda})} \left(\Theta(\beta) \upharpoonright_{U(L_{E,\lambda})}, \alpha \right).$$

Similarly, Let $\Theta(\alpha)$ be the big theta lift of α to $U(W_{E,1}^\delta)$, with respect to the splitting \tilde{t} . Similarly to before, by (5), we have the following equality, where this time we take $H = U(W_{F,1}^\delta)$:

$$\mathrm{Hom}_{U(L_{E,\lambda}) \times U(W_{F,1}^\delta)} \left(\Omega_{V_{E,\lambda}, W_{F,1}^\delta, \tilde{t}, \psi}, \alpha \otimes \beta \right) \cong \mathrm{Hom}_{U(W_{F,1}^\delta)} \left(\Theta(\alpha) \upharpoonright_{U(W_{F,1}^\delta)}, \beta \right).$$

Since the splittings $\tilde{\iota}'$ and $\tilde{\iota}$ are compatible, they agree on the subgroup $U(L_{E,\lambda}) \times U(W_{F,1}^\delta)$. Hence, we get the following identity, which is called *the local seesaw identity*:

$$\mathrm{Hom}_{U(L_{E,\lambda})}(\Theta(\beta) \upharpoonright_{U(L_{E,\lambda})}, \alpha) \cong \mathrm{Hom}_{U(W_{F,1}^\delta)}(\Theta(\alpha) \upharpoonright_{U(W_{F,1}^\delta)}, \beta). \quad (6)$$

This identity will serve as a key ingredient in the proof of our main result.

3.3 Theta lifting for unitary groups of one-dimensional spaces

In this section, we recall results regarding theta lifting of characters of $U(W)$ to $U(V)$, where W and V are one-dimensional. These results are stated in terms of a relation between the root number of a character and the discriminants of V and W . We begin with recalling the definition of the root number of a character of K^\times and defining a similar notion for a conjugate-dual character of L_E^\times , where E/F is an étale algebra. We then state results of [15] and [25] regarding the non-vanishing of a theta lift of a character of $U(W)$. We finish with using our definition for root numbers of characters of L_E^\times in order to deduce a similar statement for theta lifts of characters of unitary groups of one-dimensional L_E -skew-hermitian spaces.

3.3.1 Vector of root numbers of characters of L_E^\times

Let $\psi: F \rightarrow \mathbb{C}^\times$ be a non-trivial character. For any trace zero element $\delta \in K^\times$ we define $\psi_\delta: K \rightarrow \mathbb{C}^\times$ by the formula $\psi_\delta(x) = \psi(\mathrm{tr}_{K/F}(\delta x))$. Note that for any $x \in K^\times$, we have $\psi_\delta(x^c) = \psi_\delta^{-1}(x)$.

For any character $\chi: L_E^\times \rightarrow \mathbb{C}^\times$ such that $\chi(x^c) = \chi(x^{-1})$, we will define a vector of root numbers $\varepsilon(\chi, \psi, \delta)$.

Assume first that E is a field extension of F . The étale algebra $L_E = E \otimes_F K$ is either a field or is isomorphic to $E \times E$. Let $\chi: L_E^\times \rightarrow \mathbb{C}^\times$ be a character as above. If L_E is a field, we set

$$\varepsilon_{L_E/E}(\chi, \psi, \delta) = \varepsilon_{L_E}^{\mathrm{Tate}}\left(\frac{1}{2}, \chi, \psi_\delta \circ \mathrm{tr}_{L_E/K}\right),$$

where for a complex number s , the factor $\varepsilon_{L_E}^{\mathrm{Tate}}(s, \chi, \psi_\delta \circ \mathrm{tr}_{L_E/K})$ is the epsilon factor defined by Tate [20, 30]. If $L_E = E \times E$, then we define $\varepsilon_{L_E/E}(\chi, \psi, \delta) = 1$.

We move to the general case. Let E be a finite-dimensional étale algebra. As before, we may write $E = \prod_{j=1}^m F_j$, where F_j/F is a field extension. Given a character $\chi: L_E^\times \rightarrow \mathbb{C}^\times$, we may regard it as a tuple (χ_1, \dots, χ_m) , where $\chi_j: L_{F_j}^\times \rightarrow \mathbb{C}^\times$ is a character satisfying $\chi_j(x^c) = \chi_j^{-1}(x)$ for every j and $x \in L_{F_j}^\times$. We define $\varepsilon_{L_E/E}(\chi, \psi, \delta)$ as the following tuple:

$$\varepsilon_{L_E/E}(\chi, \psi, \delta) = \left(\varepsilon_{L_{F_1}/F_1}(\chi_1, \psi, \delta), \dots, \varepsilon_{L_{F_m}/F_m}(\chi_m, \psi, \delta) \right).$$

Recall that for any $x \in L_E^\times$ we have that $\chi(x^c) = \chi^{-1}(x)$ and $\psi_\delta(x^c) = \psi_\delta^{-1}(x)$. This implies that $\varepsilon_{L_E/E}(\chi, \psi, \delta)$ is a tuple of signs.

3.3.2 Base change for characters of L_E^1

We have an isomorphism $j_E: L_E^\times/E^\times \rightarrow L_E^1$ given by $j_E(x) = \frac{x}{x^c}$.

Given a character $\beta: L_E^1 \rightarrow \mathbb{C}^\times$, we define a character $\beta_{L_E}: L_E^\times \rightarrow \mathbb{C}^\times$ by the formula

$$\beta_{L_E}(x) = (\beta \circ j_E)(x) = \beta\left(\frac{x}{x^c}\right).$$

Notice that $\beta_{L_E}(x^c) = \beta_{L_E}^{-1}(x)$. Therefore, if $\chi: L_E^\times \rightarrow \mathbb{C}^\times$ is a character such that $\chi|_{N_{L_E/E}(L_E^\times)} = 1$, then for any $x \in L_E^\times$,

$$\left(\chi^{-1}\beta_{L_E}\right)(x^{-1}) = \left(\chi^{-1}\beta_{L_E}\right)(x^c),$$

and the vector of root numbers $\varepsilon_{L_E/E}(\chi^{-1} \cdot \beta_{L_E}, \psi, \delta)$ is defined.

3.3.3 Non-vanishing of theta lifts

Suppose that K/F is a quadratic field extension, and let $\delta \in K^\times$ be a trace zero element. Let V and W be non-degenerate one-dimensional hermitian and skew hermitian spaces over K , respectively.

The non-vanishing of a theta lift of a character of $U(V)$ is treated separately for the archimedean case and the non-archimedean case. When F is non-archimedean, the statement is given by [15, Theorem 6.1]. There are subtle differences between the versions of this result presented in [15] and the version we state below. Such differences are explained in [8, Section 9]. When $F = \mathbb{R}$, the result is given by [25, Theorem 6.1]. Once again, it is written in a different language, and we refer to [34, Section 3.2] for the translation¹. Let $i'_V: U(V) \rightarrow K^1$ and $i'_W: U(W) \rightarrow K^1$ be the obvious isomorphisms. Let $\alpha: K^1 \rightarrow \mathbb{C}^\times$ be a character. The following result determines the theta lift $\Theta_{V,W,\tilde{\iota},\psi}(\alpha \circ i'_V)$.

Theorem 3.5 (*Epsilon Dichotomy*) *The theta lift $\Theta_{V,W,\tilde{\iota},\psi}(\alpha \circ i'_V)$ with respect to the splitting $\tilde{\iota}$ associated to the characters (χ_V, χ_W) is non-zero if and only if*

$$\varepsilon_{K/F}\left(\chi_W^{-1} \cdot \alpha_K, \psi, \delta\right) = \epsilon(V) \cdot \epsilon_\delta(W).$$

Moreover, in this case

$$\Theta_{V,W,\tilde{\iota},\psi}(\alpha \circ i'_V) = \left(\left(\chi_W^{-1} \cdot \chi_V\right) \circ j_F^{-1} \cdot \alpha\right) \circ i'_W.$$

¹ We warn the reader that the characters χ_V and $\chi_{V'}$ in [34] are χ_W and χ_V , respectively, in our notation.

We remark that this theorem is also true in the split case, i.e., it is true when $K = F \times F$. In this case, the characters χ_W and χ_V , are trivial, and all the invariants specified in the theorem are also trivial. Hence, the condition is always satisfied. By [5, 7, 22], we have in this case that $\Theta_{V,W,\tilde{\iota},\psi}(\alpha \circ i'_V) = \alpha \circ i'_W$, which is the same statement as in the theorem, since the characters χ_V and χ_W are trivial.

3.3.4 Non-vanishing of theta lifts for one-dimensional spaces over an étale algebra

Let E be an étale algebra of degree n over F . Let $\lambda \in E^\times$ and let $\delta \in K^\times$ be a trace zero element. Consider the same setup as in Section 3.2. Theorem 3.5 has an obvious extension for the spaces $W_{E,1}^\delta$ and $V_{E,\lambda}$, which will be useful for the proof of our main result.

Let $i'_V: U(L_{E,\lambda}) \rightarrow L_E^1$ and $i'_W: U(W_{E,1}^\delta) \rightarrow L_E^1$ be the obvious isomorphisms. Let $\alpha: L_E^1 \rightarrow \mathbb{C}^\times$ be a character.

Theorem 3.6 *The theta lift $\Theta(\alpha \circ i'_V) = \Theta_{L_{E,\lambda}, W_{E,1}^\delta, \tilde{\iota}, \psi}(\alpha \circ i'_V)$ is non-zero if and only if*

$$\varepsilon_{L_E/E} \left(\chi_{W_{F,1}^\delta}^{-1} \circ N_{L_E/K} \cdot \alpha_{L_E}, \psi, \delta \right) = \epsilon(V_{E,\lambda}) \cdot \epsilon_\delta(W_{E,1}^\delta) = \omega_{L_E/E}(\lambda),$$

and in this case

$$\Theta(\alpha \circ i'_V) = \left(\left(\chi_{W_{F,1}^\delta}^{-1} \circ N_{L_E/K} \cdot \chi_{L_{E,\lambda}} \right) \circ j_E^{-1} \cdot \alpha \right) \circ i'_W.$$

4 Global theory

We now consider the global analogs of the previous section. We introduce the global theta correspondence and a global seesaw identity that we will need in Section 5. Finally, we recall a result of Yamana regarding the non-vanishing of the global theta lift.

4.1 The global theta correspondence

Let \mathbf{F} be a number field and let \mathbf{K}/\mathbf{F} be a quadratic field extension with involution $x \mapsto x^c$, whose set of fixed points is \mathbf{F} .

Let V and W be non-degenerate finite dimensional hermitian and skew-hermitian spaces over \mathbf{K} , respectively. As in the local case, we consider the tensor product $\text{Res}_{\mathbf{K}/\mathbf{F}}(V \otimes_{\mathbf{F}} W)$. Let $\text{Sp}(V, W)(\mathbf{F}) = \text{Sp}(\text{Res}_{\mathbf{K}/\mathbf{F}}(V \otimes_{\mathbf{K}} W))$.

Let $\mathbb{A}_{\mathbf{F}}$ be the adèles of \mathbf{F} . For an algebraic group G defined over \mathbf{F} , denote by $[G] = G(\mathbf{F}) \backslash G(\mathbb{A}_{\mathbf{F}})$ its automorphic quotient. Let $\psi: \mathbf{F} \backslash \mathbb{A}_{\mathbf{F}} \rightarrow \mathbb{C}^\times$ be a non-trivial character. Write $\psi = \bigotimes_v \psi_v$. For every place v of \mathbf{F} , we denote $V_v = V \otimes_{\mathbf{F}} \mathbf{F}_v$ and $W_v = W \otimes_{\mathbf{F}} \mathbf{F}_v$. We also denote $\mathbf{K}_v = \mathbf{K} \otimes_{\mathbf{F}} \mathbf{F}_v$.

For almost all v , the covering

$$1 \longrightarrow \mathbb{S}^1 \longrightarrow \mathrm{Mp}_{\psi_v}(V_v, W_v) \longrightarrow \mathrm{Sp}(\mathrm{Res}_{\mathbf{K}_v/\mathbf{F}_v}(V_v \otimes_{\mathbf{K}_v} W_v)) \longrightarrow 1$$

splits uniquely over the maximal hyperspecial subgroup \mathcal{K}_v of $\mathrm{Sp}(\mathrm{Res}_{\mathbf{K}_v/\mathbf{F}_v}(V_v \otimes_{\mathbf{K}_v} W_v))$.
Let

$$\mathrm{Sp}(V, W)(\mathbb{A}_{\mathbf{F}}) = \prod'_v \mathrm{Sp}(\mathrm{Res}_{\mathbf{K}_v/\mathbf{F}_v}(V_v \otimes_{\mathbf{K}_v} W_v))$$

be the restricted product with respect to $\mathcal{K}_v \subset \mathrm{Sp}(\mathrm{Res}_{\mathbf{K}_v/\mathbf{F}_v}(V_v \otimes_{\mathbf{K}_v} W_v))$. Consider the restricted product $\prod'_v \mathrm{Mp}_{\psi_v}(V_v, W_v)$ with respect to $\mathcal{K}_v \subset \mathrm{Mp}_{\psi_v}(V_v, W_v)$. We denote by $\mathrm{Mp}_{\psi}(V, W)(\mathbb{A}_{\mathbf{F}})$ the quotient of the latter restricted product by the central subgroup

$$Z = \left\{ (z_v)_v \in \bigoplus_v \mathbb{S}^1 \mid \prod_v z_v = 1 \right\}.$$

Then $\mathrm{Mp}_{\psi}(V, W)(\mathbb{A}_{\mathbf{F}})$ fits into the following exact sequence

$$1 \longrightarrow \mathbb{S}^1 \longrightarrow \mathrm{Mp}_{\psi}(V, W)(\mathbb{A}_{\mathbf{F}}) \longrightarrow \mathrm{Sp}(V, W)(\mathbb{A}_{\mathbf{F}}) \longrightarrow 1.$$

We have that $\mathrm{Mp}_{\psi}(V, W)(\mathbb{A}_{\mathbf{F}})$ splits canonically over $\mathrm{Sp}(V, W)(\mathbf{F})$. Thus, we may regard $\mathrm{Sp}(V, W)(\mathbf{F})$ as a subgroup of $\mathrm{Mp}_{\psi}(V, W)(\mathbb{A}_{\mathbf{F}})$ and define

$$[\mathrm{Mp}_{\psi}(V, W)] = \mathrm{Sp}(V, W)(\mathbf{F}) \setminus \mathrm{Mp}_{\psi}(V, W)(\mathbb{A}_{\mathbf{F}}).$$

Let $\mathrm{Res}_{\mathbf{K}/\mathbf{F}}(V \otimes_{\mathbf{K}} W) = X \oplus Y$ be a polarization, and for every v let $X_v = X \otimes_{\mathbf{F}} \mathbf{F}_v$ and $Y_v = Y \otimes_{\mathbf{F}} \mathbf{F}_v$. For every v , we realize the Weil representation $\omega_{\psi_v, \mathbf{F}_v}$ of $\mathrm{Mp}_{\psi_v}(V_v, W_v)$ via its Schrödinger model, acting on the space $\mathcal{S}(Y_v)$ consisting of Schwartz functions on Y_v . We denote by $\mathcal{S}(Y, \mathbb{A}_{\mathbf{F}}) = \bigotimes'_v \mathcal{S}(Y_v)$ the restricted tensor product and by $\omega_{\psi, \mathbb{A}_{\mathbf{F}}} = \bigotimes_v \omega_{\psi_v, \mathbf{F}_v}$ the global Weil representation of $\mathrm{Mp}_{\psi}(V, W)$.

For a function $\varphi \in \mathcal{S}(Y, \mathbb{A}_{\mathbf{F}})$, we consider its theta series, defined for $g \in \mathrm{Mp}_{\psi}(V, W)(\mathbb{A}_{\mathbf{F}})$ by

$$\theta(\varphi)(g) = \sum_{y \in Y} (\omega_{\psi, \mathbb{A}_{\mathbf{F}}}(g)\varphi)(y).$$

Then it is well-known that $\theta(\varphi)$ is an automorphic form of $[\mathrm{Mp}_{\psi}(V, W)]$.

The embedding $\iota: \mathrm{U}(V) \times \mathrm{U}(W) \rightarrow \mathrm{Sp}(V, W)$ discussed in Section 3.1 has a global analog

$$\iota: \mathrm{U}(V)(\mathbb{A}_{\mathbf{F}}) \times \mathrm{U}(W)(\mathbb{A}_{\mathbf{F}}) \rightarrow \mathrm{Sp}(V, W)(\mathbb{A}_{\mathbf{F}}).$$

As in the local case, in order to describe the theta correspondence, we need a lifting of ι to the metaplectic group

$$\tilde{\iota}: U(V)(\mathbb{A}_{\mathbf{F}}) \times U(W)(\mathbb{A}_{\mathbf{F}}) \rightarrow \mathrm{Mp}_{\psi}(V, W)(\mathbb{A}_{\mathbf{F}}),$$

such that the image of $U(V)(\mathbf{F}) \times U(W)(\mathbf{F})$ under $\tilde{\iota}$ lies in $\mathrm{Sp}(V, W)(\mathbf{F})$. Such a lifting exists, and we postpone the discussion regarding the data needed in order to construct it to the next subsection.

For a cuspidal automorphic form $f: [U(V)] \rightarrow \mathbb{C}$, a Schwartz function $\varphi \in \mathcal{S}(Y, \mathbb{A}_{\mathbf{F}})$, and an element $g_W \in U(W)(\mathbb{A}_{\mathbf{F}})$, we denote

$$\theta_{\tilde{\iota}}(\varphi, f)(g_W) = \int_{[U(V)]} \theta(\varphi)(\tilde{\iota}(g_V, g_W)) \overline{f(g_V)} dg_V.$$

It is well-known that $\theta_{\tilde{\iota}}(\varphi, f)$ is an automorphic form of $[U(W)]$. Given an irreducible cuspidal automorphic representation π of $U(V)$, we denote

$$\Theta_{V, W, \tilde{\iota}, \psi}(\pi) = \mathrm{Span}_{\mathbb{C}} \{ \theta_{\tilde{\iota}}(\varphi, f) \mid \varphi \in \mathcal{S}(Y, \mathbb{A}_{\mathbf{F}}), f \in \pi \},$$

and call $\Theta_{V, W, \tilde{\iota}, \psi}(\pi)$ the *global theta lift* of π .

By [21, Corollary 7.3] if $\Theta_{V, W, \tilde{\iota}, \psi}(\pi)$ lies in the space of square-integrable automorphic forms, then $\Theta_{V, W, \tilde{\iota}, \psi}(\pi) = \bigotimes'_v \theta_{V_v, W_v, \tilde{\iota}_v, \psi_v}(\pi_v)$.

4.1.1 Splitting of the embedding ι

The goal of this subsection is to describe the data needed in order to construct a splitting $\tilde{\iota}: U(V)(\mathbb{A}_{\mathbf{F}}) \times U(W)(\mathbb{A}_{\mathbf{F}}) \rightarrow \mathrm{Mp}_{\psi}(V, W)(\mathbb{A}_{\mathbf{F}})$ of ι . Let $\mathbf{A}_{\mathbf{K}}$ be the adèles of \mathbf{K} , and let $\omega_{\mathbf{K}/\mathbf{F}}$ be the quadratic character attached to the field extension \mathbf{K}/\mathbf{F} by global class field theory.

Similarly to Section 3.1.1, in order to construct a splitting, we need to choose automorphic characters $\chi_V = \bigotimes_v \chi_{V_v}$ and $\chi_W = \bigotimes_v \chi_{W_v}$ of $\mathbf{K}^{\times} \backslash \mathbf{A}_{\mathbf{K}}^{\times}$, such that

$$\chi_W|_{\mathbf{A}_{\mathbf{F}}^{\times}} = \omega_{\mathbf{K}/\mathbf{F}}^{\dim W} \quad \text{and} \quad \chi_V|_{\mathbf{A}_{\mathbf{F}}^{\times}} = \omega_{\mathbf{K}/\mathbf{F}}^{\dim V}.$$

By choosing such characters, for any place v we get an embedding $\tilde{\iota}_{\psi_v, \chi_{W_v}}: U(V_v) \rightarrow \mathrm{Mp}_{\psi_v}(V_v, W_v)$ and an embedding $\tilde{\iota}_{\psi_v, \chi_{V_v}}: U(W_v) \rightarrow \mathrm{Mp}_{\psi_v}(W_v, V_v)$. We obtain the desired embedding $\tilde{\iota} = \tilde{\iota}_{\psi, \chi_V, \chi_W}$ by forming the tensor products $\tilde{\iota}_{\psi, \chi_W} = \bigotimes_v \tilde{\iota}_{\psi_v, \chi_{W_v}}$ and $\tilde{\iota}_{\psi, \chi_V} = \bigotimes_v \tilde{\iota}_{\psi_v, \chi_{V_v}}$ and setting $\tilde{\iota}_{\psi, \chi_V, \chi_W} = \tilde{\iota}_{\psi, \chi_W} \times \tilde{\iota}_{\psi, \chi_V}$.

4.1.2 Notation for theta lifts of automorphic characters of $[L_{\mathbf{F}}^1]$

We introduce the global counterpart of the notation in Section 3.1.2.

Let $\delta \in \mathbf{K}^{\times}$ be a trace zero element, and let $\mu: \mathbf{K}^{\times} \backslash \mathbf{A}_{\mathbf{K}}^{\times} \rightarrow \mathbb{C}^{\times}$ be an automorphic character such that $\mu|_{\mathbf{A}_{\mathbf{F}}^{\times}} = \omega_{\mathbf{K}/\mathbf{F}}$. Suppose that V is a hermitian space over \mathbf{K} and that $\beta: [L_{\mathbf{F}}^1] \rightarrow \mathbb{C}^{\times}$ is an automorphic character (see Section 2.5). We denote

$$\Theta_{\delta, V, \mu, \psi}(\beta) = \Theta_{W_{F,1}^\delta, V, \tilde{\iota}_\mu, \psi} \left(\beta \circ i'_{W_{F,1}^\delta} \right),$$

where $i'_{W_{F,1}^\delta} : U(W_{F,1}^\delta)(\mathbb{A}_F) \rightarrow L_F^1(\mathbb{A}_F)$ is the obvious isomorphism and where $\tilde{\iota}_\mu$ is the splitting associated to the characters $(\mu, \mu^{\dim V})$.

4.1.3 Theta lifting for unitary groups of 1-dimensional spaces over étale algebras

Similarly to Section 3.1.3, we define a theta correspondence for unitary groups of one-dimensional spaces over an étale algebra.

We use the definitions in Section 2.2 with $F = \mathbf{F}$ and $K = \mathbf{K}$. Let $\delta \in \mathbf{K}^\times$ be a trace zero element.

If \mathbf{E}/\mathbf{F} is a finite field extension, then $\mathbb{A}_{\mathbf{E}} = \mathbf{E} \otimes_{\mathbf{F}} \mathbb{A}_{\mathbf{F}}$. Recall that in this case, if $\lambda \in \mathbf{E}^\times$ and if R is a ring over \mathbf{F} , then

$$U(L_{\mathbf{E}, \lambda})(R) \cong L_{\mathbf{E}}^1(R) = \{x \in (\text{Res}_{L_{\mathbf{E}}/\mathbf{F}} L_{\mathbf{E}}^\times)(R) \mid x \cdot x^c = 1\}.$$

In particular, we have that the \mathbf{F} -adelic points of the unitary group $U(L_{\mathbf{E}, \lambda})$ are the same as the \mathbf{E} -adelic points of the unitary group of $L_{\mathbf{E}, \lambda}$ defined over \mathbf{E} , which we denote $U_{\mathbf{E}}(L_{\mathbf{E}, \lambda})$, i.e.,

$$U(L_{\mathbf{E}, \lambda})(\mathbb{A}_{\mathbf{F}}) = U_{\mathbf{E}}(L_{\mathbf{E}, \lambda})(\mathbb{A}_{\mathbf{E}}) \cong \{x \in (L_{\mathbf{E}} \otimes_{\mathbf{E}} \mathbb{A}_{\mathbf{E}})^\times \mid x \cdot x^c = 1\}.$$

This relation will allow us to make use of statements about theta lifts of characters of $U_{\mathbf{E}}(L_{\mathbf{E}, \lambda})(\mathbb{A}_{\mathbf{E}})$.

Let \mathbf{E}/\mathbf{F} be an étale algebra of rank n over \mathbf{F} . As before, we write $\mathbf{E} = \prod_{j=1}^m \mathbf{F}_j$, where for every j , \mathbf{F}_j/\mathbf{F} is a finite field extension. We will assume that $L_{\mathbf{E}}^1$ is anisotropic, i.e., we will assume that $L_{\mathbf{F}_j}$ is a field for every j . This is equivalent to the assumption that there is no embedding of \mathbf{F} -algebras $\mathbf{K} \hookrightarrow \mathbf{E}$.

As in Section 3.1.3, for $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbf{E}^\times$ and $\lambda' = (\lambda'_1, \dots, \lambda'_m) \in \mathbf{E}^\times$, we have that

$$U(L_{\mathbf{E}, \lambda})(\mathbb{A}_{\mathbf{F}}) = \prod_{j=1}^m U(L_{\mathbf{F}_j, \lambda_j})(\mathbb{A}_{\mathbf{F}}) \quad \text{and} \quad U(W_{\mathbf{E}, \lambda'}^\delta)(\mathbb{A}_{\mathbf{F}}) = \prod_{j=1}^m U(W_{\mathbf{F}_j, \lambda'_j}^\delta)(\mathbb{A}_{\mathbf{F}}).$$

For every $1 \leq j \leq m$, let $V_j = L_{\mathbf{F}_j, \lambda_j}$ and $W_j = W_{\mathbf{F}_j, \lambda'_j}^\delta$. Every automorphic character $\alpha : [U(L_{\mathbf{E}, \lambda})] \rightarrow \mathbb{C}^\times$ is equivalent to a choice $(\alpha_1, \dots, \alpha_m)$, where $\alpha_j : [U(V_j)] \rightarrow \mathbb{C}^\times$ is an automorphic character for $1 \leq j \leq m$. We will use the usual global theta correspondence to define a theta correspondence for the adelic groups $U(L_{\mathbf{E}, \lambda})(\mathbb{A}_{\mathbf{F}})$ and $U(W_{\mathbf{E}, \lambda'}^\delta)(\mathbb{A}_{\mathbf{F}})$.

Let $\chi_{L_{\mathbf{E}, \lambda}}, \chi_{W_{\mathbf{E}, \lambda'}^\delta} : L_{\mathbf{E}}^\times \setminus (L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbb{A}_{\mathbf{F}})^\times \rightarrow \mathbb{C}^\times$ be automorphic characters such that

$$\chi_{L_{\mathbf{E}, \lambda}} \upharpoonright_{\mathbb{A}_{\mathbf{F}}^\times} = \chi_{W_{\mathbf{E}, \lambda'}^\delta} \upharpoonright_{\mathbb{A}_{\mathbf{F}}^\times} = \omega_{L_{\mathbf{E}}/\mathbf{E}}.$$

By this we mean that $\chi_{L_{E,\lambda}}$ and $\chi_{W_{E,\lambda'}^\delta}$ correspond to tuples $(\chi_{V_1}, \dots, \chi_{V_m})$ and $(\chi_{W_1}, \dots, \chi_{W_m})$, respectively, where $\chi_{V_j}, \chi_{W_j}: L_{\mathbf{F}_j}^\times \setminus (L_{\mathbf{F}_j} \otimes_{\mathbf{F}_j} \mathbb{A}_{\mathbf{F}_j})^\times \rightarrow \mathbb{C}^\times$ are automorphic characters satisfying

$$\chi_{V_j}|_{\mathbb{A}_{\mathbf{F}_j}^\times} = \chi_{W_j}|_{\mathbb{A}_{\mathbf{F}_j}^\times} = \omega_{L_{\mathbf{F}_j}/\mathbf{F}_j}.$$

As in the local case, for every j , we get a splitting

$$\tilde{t}_j: \mathrm{U}_{\mathbf{F}_j}(V_j)(\mathbb{A}_{\mathbf{F}_j}) \times \mathrm{U}_{\mathbf{F}_j}(W_j)(\mathbb{A}_{\mathbf{F}_j}) \rightarrow \mathrm{Mp}_{\psi_j}(V_j, W_j)(\mathbb{A}_{\mathbf{F}_j}).$$

Here $\psi_j: \mathbf{F}_j \setminus \mathbb{A}_{\mathbf{F}_j} \rightarrow \mathbb{C}^\times$ is the character $\psi_j = \psi \circ \mathrm{tr}_{\mathbf{F}_j/\mathbf{F}}$, where $\mathrm{tr}_{\mathbf{F}_j/\mathbf{F}}: \mathbf{F}_j \otimes_{\mathbf{F}} \mathbb{A}_{\mathbf{F}} \rightarrow \mathbb{A}_{\mathbf{F}}$ is the trace map. Denote $\tilde{t} = (\tilde{t}_1, \dots, \tilde{t}_m)$.

We define the global theta lift of α as above by the formula

$$\Theta_{L_{E,\lambda}, W_{E,\lambda'}^\delta, \tilde{t}, \psi}(\alpha) = \Theta_{V_1, W_1, \tilde{t}_1, \psi_1}(\alpha_1) \otimes \cdots \otimes \Theta_{V_m, W_m, \tilde{t}_m, \psi_m}(\alpha_m).$$

Suppose that $\lambda' = 1$. For every j , let

$$\mathrm{Res}_{L_{\mathbf{F}_j}/\mathbf{F}_j}(V_j \otimes_{L_{\mathbf{F}_j}} W_j) = X_j \oplus Y_j$$

be a polarization. Let $X = \bigoplus_{j=1}^m X_j$ and $Y = \bigoplus_{j=1}^m Y_j$. Then

$$\mathrm{Res}_{\mathbf{K}/\mathbf{F}}(V_{E,\lambda} \otimes_{\mathbf{K}} W_{\mathbf{F},1}^\delta) = X \oplus Y$$

is a polarization. As explained in the local case, we have a natural map

$$\prod_{j=1}^m \mathrm{Mp}_{\psi_j}(V_j, W_j)(\mathbb{A}_{\mathbf{F}_j}) \rightarrow \mathrm{Mp}_{\psi}(V_{E,\lambda}, W_{\mathbf{F},1}^\delta)(\mathbb{A}_{\mathbf{F}}),$$

which is not injective, but its restriction to $\mathrm{Mp}_{\psi_j}(V_j, W_j)(\mathbb{A}_{\mathbf{F}_j})$, for every j , is injective. Hence, we may regard \tilde{t} as a map

$$\tilde{t}: \mathrm{U}(L_{E,\lambda})(\mathbb{A}_{\mathbf{F}}) \times \mathrm{U}(W_{\mathbf{F},1}^\delta)(\mathbb{A}_{\mathbf{F}}) \rightarrow \mathrm{Mp}_{\psi}(V_{E,\lambda}, W_{\mathbf{F},1}^\delta)(\mathbb{A}_{\mathbf{F}}).$$

Remark 4.1 Similarly to Remark 3.4, we have that the kernel of the map $\prod_{j=1}^m \mathrm{Mp}_{\psi_j}(V_j, W_j)(\mathbb{A}_{\mathbf{F}_j}) \rightarrow \mathrm{Mp}_{\psi}(V_{E,\lambda}, W_{\mathbf{F},1}^\delta)(\mathbb{A}_{\mathbf{F}})$ consists of tuples (g_1, \dots, g_m) such that for every j , the projection of g_j to $\mathrm{Sp}(\mathrm{Res}_{L_{\mathbf{F}_j}/\mathbf{F}_j}(V_j \otimes_{L_{\mathbf{F}_j}} W_j))$ is the identity and such that if t_j is the projection of g_j to \mathbb{S}^1 then $\prod_{j=1}^m t_j = 1$.

Let $\varphi \in \mathcal{S}(Y, \mathbb{A}_{\mathbf{F}})$ be a decomposable Schwartz function, that is, $\varphi = \bigotimes_{j=1}^m \varphi_j$, where $\varphi_j \in \mathcal{S}(Y_j, \mathbb{A}_{\mathbf{F}_j})$. Denote for $h \in U(W_{\mathbf{E},1}^{\delta})(\mathbb{A}_{\mathbf{F}})$,

$$\theta_t(\varphi, \alpha)(h) = \int_{[U(L_{\mathbf{E},\lambda})]} \theta(\varphi)(\tilde{t}(g, h)) \overline{\alpha(g)} dg.$$

Then $\theta_t(\varphi, \alpha) \in \theta_{L_{\mathbf{E},\lambda}, W_{\mathbf{E},1}^{\delta}, \tilde{t}, \psi}(\alpha)$. Analogously to the local case, for any $(g_1, h_1), \dots, (g_m, h_m)$ such that $(g_j, h_j) \in U(V_j)(\mathbb{A}_{\mathbf{F}}) \times U(W_j)(\mathbb{A}_{\mathbf{F}})$, we have

$$\begin{aligned} & \omega_{\psi, \mathbb{A}_{\mathbf{F}}}(\tilde{t}((g_1, \dots, g_m), (h_1, \dots, h_m))) \varphi \\ &= \omega_{\psi_1, \mathbb{A}_{\mathbf{F}_1}}(\tilde{t}_1(g_1, h_1)) \varphi_1 \otimes \dots \otimes \omega_{\psi_m, \mathbb{A}_{\mathbf{F}_m}}(\tilde{t}_m(g_m, h_m)) \varphi_m. \end{aligned}$$

This implies that for h_1, \dots, h_m , where $h_j \in U(W_j)(\mathbb{A}_{\mathbf{F}})$, we have that

$$\theta_t(\varphi, \alpha)(h_1, \dots, h_m) = \theta_{t_1}(\varphi_1, \alpha_1)(h_1) \cdots \theta_{t_m}(\varphi_m, \alpha_m)(h_m). \quad (7)$$

This compatibility will be important for the seesaw identity which we will discuss in the next section.

4.2 A global seesaw identity

The goal of this section is to introduce a global seesaw identity, analogous to the local one we described in Section 3.2. This identity will be a key ingredient for the proof of our main global theorem.

4.2.1 Splitting setup

Similarly to Section 3.2.1, we first need set up our splittings in a way that they are compatible. Let us be in the setup of Section 4.1.3 with $\lambda' = 1$. We consider the following seesaw diagram:

$$\begin{array}{ccc} U(W_{\mathbf{E},1}^{\delta})(\mathbb{A}_{\mathbf{F}}) & & U(V_{\mathbf{E},\lambda})(\mathbb{A}_{\mathbf{F}}) \\ | & \searrow & | \\ U(W_{\mathbf{F},1}^{\delta})(\mathbb{A}_{\mathbf{F}}) & & U(L_{\mathbf{E},\lambda})(\mathbb{A}_{\mathbf{F}}). \end{array}$$

Given automorphic characters $\chi_{L_{\mathbf{E},\lambda}}, \chi_{W_{\mathbf{E},1}^{\delta}} : (L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbb{A}_{\mathbf{F}})^{\times} \rightarrow \mathbb{C}^{\times}$, such that

$$\chi_{L_{\mathbf{E},\lambda}} \upharpoonright_{\mathbb{A}_{\mathbf{F}}^{\times}} = \chi_{W_{\mathbf{E},1}^{\delta}} \upharpoonright_{\mathbb{A}_{\mathbf{F}}^{\times}} = \omega_{L_{\mathbf{E}}/\mathbf{F}},$$

we constructed a map

$$\tilde{t} : U(L_{\mathbf{E},\lambda})(\mathbb{A}_{\mathbf{F}}) \times U(W_{\mathbf{E},1}^{\delta})(\mathbb{A}_{\mathbf{F}}) \rightarrow \mathrm{Mp}_{\psi}(V_{\mathbf{E},\lambda}, W_{\mathbf{F},1}^{\delta})(\mathbb{A}_{\mathbf{F}}).$$

Similarly, given automorphic characters $\chi_{W_{F,1}^\delta}, \chi_{V_{E,\lambda}} : \mathbb{A}_K^\times \rightarrow \mathbb{C}^\times$, such that

$$\chi_{W_{F,1}^\delta} \upharpoonright_{\mathbb{A}_F^\times} = \omega_{K/F} \quad \text{and} \quad \chi_{V_{E,\lambda}} \upharpoonright_{\mathbb{A}_F^\times} = \omega_{K/F}^{\dim_K V_{E,\lambda}},$$

we constructed a map

$$\tilde{\iota}' : U(V_{E,\lambda})(\mathbb{A}_F) \times U(W_{F,1}^\delta)(\mathbb{A}_F) \rightarrow \mathrm{Mp}_\psi(V_{E,\lambda}, W_{F,1}^\delta)(\mathbb{A}_F).$$

We say that the splittings $\tilde{\iota}$ and $\tilde{\iota}'$ are *compatible* if their restrictions to the subgroup $U(L_{E,\lambda})(\mathbb{A}_F) \times U(W_{F,1}^\delta)(\mathbb{A}_F)$ coincide. As in Section 3.2.1, this is equivalent to requiring the following relations between the characters defining the splittings:

$$\chi_{W_{E,1}^\delta} = \chi_{W_{F,1}^\delta} \circ N_{L_E/K} \quad \text{and} \quad \chi_{L_{E,\lambda}} \upharpoonright_{\mathbb{A}_K^\times} = \chi_{V_{E,\lambda}},$$

where $N_{L_E/K} : (L_E \otimes_F \mathbb{A}_F)^\times \rightarrow (K \otimes_F \mathbb{A}_F)^\times = \mathbb{A}_K^\times$ is the norm map.

4.2.2 The global seesaw identity

We are ready to state our global seesaw identity. Choose compatible splittings $\tilde{\iota}$ and $\tilde{\iota}'$ as above. Let $\beta : [U(W_{F,1}^\delta)] \rightarrow \mathbb{C}^\times$ be an automorphic character, and let $\varphi \in \mathcal{S}(Y, \mathbb{A}_F)$. Consider the element $\theta_{\tilde{\iota}'}(\varphi, \beta)$ in the global theta lift of β from $U(W_{F,1}^\delta)(\mathbb{A}_F)$ to $U(V_{E,\lambda})(\mathbb{A}_F)$. Given an automorphic character $\alpha : [U(L_{E,\lambda})] \rightarrow \mathbb{C}^\times$, we consider the α -period of $\theta_{\tilde{\iota}'}(\varphi, \beta)$:

$$\begin{aligned} & \int_{[U(L_{E,\lambda})]} \theta_{\tilde{\iota}'}(\varphi, \beta)(g) \overline{\alpha(g)} dg \\ &= \int_{[U(L_{E,\lambda})]} \int_{[U(W_{F,1}^\delta)]} \theta(\varphi)(\tilde{\iota}'(g, h)) \overline{\alpha(g)\beta(h)} dh dg. \end{aligned} \quad (8)$$

The latter integral converges absolutely because the automorphic quotients $[U(L_{E,\lambda})]$ and $[U(W_{F,1}^\delta)]$ are compact (the former due to our assumption that L_E^1 is anisotropic). By exchanging the order of integration and using the fact that $\tilde{\iota}$ and $\tilde{\iota}'$ are compatible, we get that (8) is equal to

$$\int_{[U(W_{F,1}^\delta)]} \int_{[U(L_{E,\lambda})]} \theta(\varphi)(\tilde{\iota}(g, h)) \overline{\alpha(g)\beta(h)} dg dh = \int_{[U(W_{F,1}^\delta)]} \theta_{\tilde{\iota}}(\varphi, \alpha)(h) \overline{\beta(h)} dh.$$

Hence, we obtained the global seesaw identity

$$\int_{[U(L_{E,\lambda})]} \theta_{\tilde{\iota}'}(\varphi, \beta)(g) \overline{\alpha(g)} dg = \int_{[U(W_{F,1}^\delta)]} \theta_{\tilde{\iota}}(\varphi, \alpha)(h) \overline{\beta(h)} dh.$$

If φ is decomposable, i.e., $\varphi = \bigotimes_{j=1}^m \varphi_j$, where $\varphi_j \in \mathcal{S}(Y_j, \mathbb{A}_{\mathbf{F}_j})$, we may use (7) to decompose further and get the identity

$$\int_{[U(L_{\mathbf{E}, \lambda})]} \theta_{\tau'}(\varphi, \beta)(g) \overline{\alpha(g)} dg = \int_{[U(W_{\mathbf{F}, j}^s)]} \overline{\beta(h)} \cdot \prod_{j=1}^m \theta_{\tau_j}(\varphi_j, \alpha)(h) dh.$$

4.3 Global theta lifts for unitary groups of one-dimensional spaces

4.3.1 Central L -function values of automorphic characters of $(L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbb{A}_{\mathbf{F}})^{\times}$

In this section, we discuss the definition of the central value of an L -function associated with an automorphic character of $(L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbb{A}_{\mathbf{F}})^{\times}$.

Assume first that \mathbf{E}/\mathbf{F} is a field extension. Then $L_{\mathbf{E}} = \mathbf{E} \otimes_{\mathbf{F}} \mathbf{K}$ is a field extension (as we assume that $L_{\mathbf{E}}^1$ is anisotropic). We have that $L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbb{A}_{\mathbf{F}} = \mathbb{A}_{L_{\mathbf{E}}}$, and therefore an automorphic character of $(L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbb{A}_{\mathbf{F}})^{\times}$ is the same as an automorphic character of $\mathbb{A}_{L_{\mathbf{E}}}^{\times}$. For an automorphic character $\chi: L_{\mathbf{E}}^{\times} \backslash \mathbb{A}_{L_{\mathbf{E}}}^{\times} \rightarrow \mathbb{C}^{\times}$, we define

$$\mathcal{L}(\chi) = L\left(\frac{1}{2}, \chi\right).$$

Next, suppose that \mathbf{E} is an étale algebra of degree n over \mathbf{F} , such that $L_{\mathbf{E}}^1$ is anisotropic. As before, write $\mathbf{E} = \prod_{j=1}^m \mathbf{F}_j$. Let $\chi: L_{\mathbf{E}}^{\times} \backslash (L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbb{A}_{\mathbf{F}})^{\times} \rightarrow \mathbb{C}^{\times}$ be an automorphic character. As before, χ corresponds to a tuple (χ_1, \dots, χ_m) , where for every j , $\chi_j: \mathbf{F}_j^{\times} \backslash (L_{\mathbf{F}_j} \otimes_{\mathbf{F}} \mathbb{A}_{\mathbf{F}})^{\times} \rightarrow \mathbb{C}^{\times}$ is an automorphic character. We define

$$\mathcal{L}(\chi) = \prod_{j=1}^m \mathcal{L}(\chi_j).$$

4.3.2 Base change for characters of $\text{Res}_{\mathbf{E}/\mathbf{F}} L_{\mathbf{E}}^1(\mathbb{A}_{\mathbf{F}})$

Let us write

$$(L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbb{A}_{\mathbf{F}})^{\times} = \prod_v' (L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbf{F}_v)^{\times}.$$

Recall the definition of $\text{Res}_{\mathbf{E}/\mathbf{F}} L_{\mathbf{E}}^1(\mathbb{A}_{\mathbf{F}})$ from Section 2.5. We have that

$$\text{Res}_{\mathbf{E}/\mathbf{F}} L_{\mathbf{E}}^1(\mathbb{A}_{\mathbf{F}}) = \prod_v' \text{Res}_{\mathbf{E}/\mathbf{F}} L_{\mathbf{E}}^1(\mathbf{F}_v) = \prod_v' (L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbf{F}_v)^1.$$

By Section 3.3.2, for every v we have an isomorphism

$$j_{\mathbf{E} \otimes_{\mathbf{F}} \mathbf{F}_v}: (L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbf{F}_v)^{\times} / (\mathbf{E} \otimes_{\mathbf{F}} \mathbf{F}_v)^{\times} \rightarrow (L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbf{F}_v)^1$$

given by $j_{\mathbf{E} \otimes_{\mathbf{F}} \mathbf{F}_v}(x) = \frac{x}{x^c}$. Hence, the map

$$j_{\mathbf{E} \otimes_{\mathbf{F}} \mathbf{A}_{\mathbf{F}}}: (L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbf{A}_{\mathbf{F}})^{\times} / (\mathbf{E} \otimes_{\mathbf{F}} \mathbf{A}_{\mathbf{F}})^{\times} \rightarrow (L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbf{A}_{\mathbf{F}})^1 = \text{Res}_{\mathbf{E}/\mathbf{F}} L_{\mathbf{E}}^1(\mathbf{A}_{\mathbf{F}})$$

given by $j_{\mathbf{E} \otimes_{\mathbf{F}} \mathbf{A}_{\mathbf{F}}}(x) = \frac{x}{x^c}$ decomposes as $j_{\mathbf{E} \otimes_{\mathbf{F}} \mathbf{A}_{\mathbf{F}}} = \bigotimes_v j_{\mathbf{E} \otimes_{\mathbf{F}} \mathbf{F}_v}$, and therefore is an isomorphism.

Analogously to Section 3.3.2, given an automorphic character $\beta: [L_{\mathbf{E}}^1] \rightarrow \mathbb{C}^{\times}$, we define an automorphic character $\beta_{L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbf{A}_{\mathbf{F}}}: L_{\mathbf{E}}^{\times} \setminus (L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbf{A}_{\mathbf{F}})^{\times} \rightarrow \mathbb{C}^{\times}$ by the formula

$$\beta_{L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbf{A}_{\mathbf{F}}}(x) = (\beta \circ j_{\mathbf{E} \otimes_{\mathbf{F}} \mathbf{A}_{\mathbf{F}}})(x) = \beta\left(\frac{x}{x^c}\right).$$

4.3.3 Non-vanishing of global theta lifts

In this section, we recall a result regarding the non-vanishing of the global theta lift. This result serves as an analog of Theorem 3.5. It is established using the Rallis inner product formula. Since we do not need the generality of the Rallis inner product formula, we will just state the non-vanishing result in the generality we need.

Let V and W be non-degenerate one-dimensional hermitian and skew-hermitian spaces over \mathbf{K} , respectively. Let $i'_W: U(W) \rightarrow L_{\mathbf{F}}^1$ and $i'_V: U(V) \rightarrow L_{\mathbf{F}}^1$ be the obvious isomorphisms. Let $\alpha: [L_{\mathbf{F}}^1] \rightarrow \mathbb{C}^{\times}$ be an automorphic character. The following theorem due to Yamana follows from [35, Lemma 10.2]².

Theorem 4.2 *The global theta lift $\Theta_{V,W,\tilde{t},\psi}(\alpha \circ i'_V)$ with respect to the splitting \tilde{t} associated to the characters (χ_V, χ_W) is non-zero if and only if the following two conditions are satisfied.*

- (1) *For every place v , the big theta lift $\Theta_{V_v, W_v, \tilde{t}_v, \psi_v}(\alpha_v \circ i'_{V_v})$ does not vanish.*
- (2) *The central L -function value $\mathcal{L}\left(\chi_W^{-1} \cdot \alpha_{\mathbf{K} \otimes_{\mathbf{F}} \mathbf{A}_{\mathbf{F}}}\right)$ is non-zero.*

If the global theta lift is not zero, we may use the compatibility with the local theta lift to describe it.

Proposition 4.3 *If the global theta lift $\Theta_{V,W,\tilde{t},\psi}(\alpha \circ i'_V)$ is not zero, then it is given by*

$$\Theta_{V,W,\tilde{t},\psi}(\alpha \circ i'_V) = \left(\left(\chi_W^{-1} \cdot \chi_V \right) \circ j_{\mathbf{A}_{\mathbf{F}}}^{-1} \cdot \alpha \right) \circ i'_W.$$

4.3.4 Non-vanishing of global theta lifts for one-dimensional spaces over an étale algebra

Let \mathbf{E} be an étale algebra of degree n over \mathbf{F} , such that $L_{\mathbf{E}}^1$ is anisotropic, and let $\lambda \in \mathbf{E}^{\times}$. Choose a trace zero element $\delta \in \mathbf{K}^{\times}$. Consider the same setup as in Section 4.2. The non-vanishing result described in Section 4.3.3 has a straightforward extension that

² We warn the reader that in [35], $G = U(W)$ and $H = U(V)$ in our notation.

allows us to determine whether the global theta lift of an automorphic character of $U(W_{\mathbf{F},1}^\delta)(\mathbb{A}_{\mathbf{F}})$ to $U(V_{\mathbf{E},\lambda})(\mathbb{A}_{\mathbf{F}})$ is non-zero. This extension will be important for our main result.

Let $i'_V: U(V_{\mathbf{E},\lambda}) \rightarrow L_{\mathbf{E}}^1$ and $i'_W: U(W_{\mathbf{E},1}^\delta) \rightarrow L_{\mathbf{E}}^1$ be the obvious isomorphisms. Let $\alpha: [L_{\mathbf{E}}^1] \rightarrow \mathbb{C}^\times$ be an automorphic character.

Theorem 4.4 *The global theta lift $\Theta(\alpha \circ i'_V) = \Theta_{L_{\mathbf{E},\lambda}, W_{\mathbf{E},1}^\delta, \tilde{\iota}, \psi}(\alpha \circ i'_V)$ is non-zero if and only if the following two conditions hold:*

- *For every place v , the big theta lift*

$$\Theta_{L_{\mathbf{E},\lambda,v}, W_{\mathbf{E},1,v}^\delta, \tilde{\iota}_v, \psi_v}(\alpha_v \circ i'_{V,v})$$

does not vanish.

- *The central L -function value*

$$\mathcal{L}\left(\chi_{W_{\mathbf{F},1}^\delta}^{-1} \circ N_{L_{\mathbf{E}}/\mathbf{K}} \cdot \alpha_{L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbb{A}_{\mathbf{F}}}\right)$$

is non-zero.

Moreover, in this case, we have that

$$\Theta(\alpha \circ i'_V) = \left(\left(\chi_{W_{\mathbf{F},1}^\delta}^{-1} \circ N_{L_{\mathbf{E}}/\mathbf{K}} \cdot \chi_{L_{\mathbf{E},\lambda}} \right) \circ j_{\mathbf{E} \otimes_{\mathbf{F}} \mathbb{A}_{\mathbf{F}}}^{-1} \cdot \alpha \right) \circ i'_W.$$

5 Toric periods of Weil representations

In this section we prove our main results on toric periods of Weil representations.

5.1 Local problem

Let F be a local field (either archimedean or non-archimedean of characteristic $\neq 2$) and let K/F be a quadratic étale algebra. Let V be a non-degenerate n -dimensional hermitian space over K , and let W be a non-degenerate one-dimensional skew-hermitian space over K . Let $i'_W: U(W) \rightarrow K^1$ be the obvious isomorphism. Fix a character $\beta: K^1 \rightarrow \mathbb{C}^\times$, and let $\Theta(\beta \circ i'_W) = \Theta_{W,V,\tilde{\iota},\psi}(\beta \circ i'_W)$ be the big theta lift of $\beta \circ i'_W$ to $U(V)$, where $\tilde{\iota}$ is the splitting associated with the characters (χ_W, χ_V) . Recall that in this case $\Theta(\beta \circ i'_W)$ coincides with the small theta lift $\theta(\beta \circ i'_W) = \theta_{W,V,\tilde{\iota},\psi}(\beta \circ i'_W)$ because $\beta \circ i'_W$ is supercuspidal.

Given a maximal torus $T \subset U(V)$ and a character $\alpha': T \rightarrow \mathbb{C}^\times$, we would like to investigate whether the space $\text{Hom}_T(\Theta(\beta \circ i'_W), \alpha')$ is non-zero.

By Theorem 2.3, we have that if $T \subset U(V)$ is a maximal torus, then there exists an n -dimensional étale algebra E over F , an element $\lambda \in E^\times$, and an isomorphism $r: V_{E,\lambda} \rightarrow V$ of hermitian spaces, such that

$$T = T_{E,r} = \left\{ i(x) = r \circ m_x \circ r^{-1} \mid x \in L_E^1 \right\}.$$

We formulate an answer to our question in the following theorem.

Theorem 5.1 *Let E be an étale algebra of degree n over F , $\alpha: L_E^1 \rightarrow \mathbb{C}^\times$ be a character, and let $i: L_E^1 \rightarrow U(V)$ be an admissible embedding, corresponding to the data $\lambda \in E^\times$ and $r: V_{E,\lambda} \rightarrow V$. Then $\text{Hom}_{i(L_E^1)}(\Theta(\beta \circ i'_W), \alpha \circ i^{-1})$ is non-zero if and only if*

$$\beta = (\chi_W^{-n} \cdot \chi_V) \circ j_F^{-1} \cdot \alpha|_{K^1}$$

and

$$\omega_{L_E/E}(\lambda) = \varepsilon_{L_E/E}(\alpha_{L_E} \cdot \chi_W^{-1} \circ N_{L_E/K}, \psi, \delta),$$

where $\delta \in K^\times$ is a trace zero element, such that $\delta = \text{disc } W \pmod{N_{K/F}(K^\times)}$. Moreover, in the case that the space $\text{Hom}_{i(L_E^1)}(\Theta(\beta \circ i'_W), \alpha \circ i^{-1})$ is non-zero, it is one-dimensional.

Proof By the choice of δ , we have that $W \cong W_{F,1}^\delta$, as hermitian spaces, where we recall that $W_{F,1}^\delta = (K, \langle \cdot, \cdot \rangle_{W_{F,1}^\delta})$ is the one-dimensional space equipped with the skew-hermitian form

$$\langle x, y \rangle_{W_{F,1}^\delta} = \delta xy^c.$$

Henceforth, we will identify W with $W_{F,1}^\delta$.

Let us be in the setup of Section 3.2. We will use the following seesaw diagram:

$$\begin{array}{ccc} U(W_{E,1}^\delta) & & U(V) \\ | & \searrow & | \\ U(W) & & U(L_{E,\lambda}), \end{array}$$

where $U(W)$ is embedded in $U(W_{E,1}^\delta)$ diagonally, i.e., $U(W)$ acts on elements of $W_{E,1}^\delta$ by scalar multiplication via the obvious isomorphism $i'_W: U(W) \rightarrow K^1$.

By the local seesaw identity (6), we have that

$$\text{Hom}_{i(L_E^1)}\left(\Theta(\beta \circ i'_W)|_{i(L_E^1)}, \alpha \circ i^{-1}\right) \cong \text{Hom}_{U(W)}\left(\Theta(\alpha \circ i^{-1})|_{U(W)}, \beta \circ i'_W\right),$$

where $\Theta(\alpha \circ i^{-1})$ is the theta lift of $\alpha \circ i^{-1}$ from $U(L_{E,\lambda})$ to $U(W_{E,1}^\delta)$. By Theorem 3.6, we have that $\Theta(\alpha \circ i^{-1})$ is non-zero if and only if

$$\varepsilon_{L_E/E}\left(\chi_W^{-1} \circ N_{L_E/K} \cdot \alpha_{L_E}, \psi, \delta\right) = \omega_{L_E/E}(\lambda),$$

and in this case

$$\Theta(\alpha \circ i^{-1}) = \left(\left(\chi_W^{-1} \circ N_{L_E/K} \cdot \chi_{L_E, \lambda} \right) \circ j_E^{-1} \cdot \alpha \right) \circ i'_{W_{E,1}^\delta}, \quad (9)$$

where $i'_{W_{E,1}^\delta} : U(W_{E,1}^\delta) \rightarrow L_E^1$ is the obvious isomorphism. Since $i'_{W_{E,1}^\delta}$ is the obvious isomorphism, we have that $i'_{W_{E,1}^\delta} \upharpoonright_{U(W)}$ agrees with i'_W .

Since the theta lift $\Theta(\alpha \circ i^{-1})$ is a character, given that it is not zero, we have that the space $\text{Hom}_{U(W)} \left(\Theta(\alpha \circ i^{-1}) \upharpoonright_{U(W)}, \beta \circ i'_W \right)$ is non-zero if and only if $\Theta(\alpha \circ i^{-1}) \upharpoonright_{U(W)}$ is the same as $\beta \circ i'_W$. By (9), this is equivalent to

$$\left(\left(\chi_W^{-1} \circ N_{L_E/K} \cdot \chi_{L_E, \lambda} \right) \circ j_E^{-1} \cdot \alpha \right) \upharpoonright_{K^1} = \beta.$$

Since $\chi_{L_E, \lambda} \upharpoonright_{K^\times} = \chi_V$, we get that this condition is equivalent to

$$\beta = (\chi_W^{-n} \cdot \chi_V) \circ j_F^{-1} \cdot \alpha \upharpoonright_{K^1},$$

as required.

Finally, if $\text{Hom}_{U(W)} \left(\Theta(\alpha \circ i^{-1}) \upharpoonright_{U(W)}, \beta \circ i'_W \right)$ is non-zero, it has to be one-dimensional, since all the representations involved are characters. \square

Remark 5.2 By substituting $W = W_{F,1}^\delta$ and $(\chi_W, \chi_V) = (\mu, \mu^n)$ as in Section 3.1.2, we obtain Theorem 1.2.

Remark 5.3 If F is non-archimedean and K/F is a quadratic field extension, then there exist exactly two isomorphism classes of non-degenerate hermitian spaces over K of dimension n . The isomorphism class of such hermitian space is determined by its discriminant. We may use this to determine when there exists an admissible embedding $i : L_E^1 \rightarrow U(V)$ with non-zero Hom-space, where E is an arbitrary étale algebra of degree n over F .

Let $\alpha : L_E^1 \rightarrow \mathbb{C}^\times$ be a character and let $\lambda \in E^\times$ be such that

$$\omega_{L_E/E}(\lambda) = \varepsilon_{L_E/E}(\alpha_{L_E} \cdot \chi_W^{-1} \circ N_{L_E/K}, \psi, \delta).$$

(By Theorem 5.1, this is the only possible class $\lambda \in E^\times / N_{L_E/E}(L_E^\times)$, such that there exists an admissible embedding corresponding to λ with non-vanishing Hom-space).

Then there exists an admissible embedding $i : L_E^1 \rightarrow U(V)$ corresponding to an isomorphism $r : V_{E, \lambda} \rightarrow V$ if and only if

$$\text{disc } V = N_{E/F}(\lambda) \cdot \text{disc}_F(E). \quad (10)$$

In this case,

$$\text{Hom}_{i(L_E^1)} \left(\Theta(\beta \circ i'_W), \alpha \circ i^{-1} \right) \neq 0, \quad (11)$$

if and only if

$$\beta = (\chi_W^{-n} \cdot \chi_V) \circ j_F^{-1} \cdot \alpha|_{K^1} \quad (12)$$

Here we used Lemma 2.2 and Theorem 5.1.

If $F = \mathbb{R}$ and $K = \mathbb{C}$, then (10) is only a necessary condition for the existence of an isomorphism $r: V_{E,\lambda} \rightarrow V$. Given that such isomorphism exists, we have by Theorem 5.1 that (11) holds if and only if (12) holds.

Corollary 5.4 *Let E be an étale algebra of degree n over F , and let $\alpha: L_E^1 \rightarrow \mathbb{C}^\times$ be a character. Then we have that*

$$\begin{aligned} & \sum_H \sum_i \dim \operatorname{Hom}_{i(L_E^1)} \left(\Theta_{W,H} (\beta \circ i'_W)|_{i(L_E^1)}, \alpha \circ i^{-1} \right) \\ &= \begin{cases} 1 & \text{if } \beta = (\chi_W^{-n} \cdot \chi_V) \circ j_F^{-1} \cdot \alpha|_{K^1}, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where the sum over H is over a set of representatives of isomorphism classes of non-degenerate hermitian spaces of degree n , and the sum over i is over a set of representatives for $\Sigma_{E,H}$. Here, $\Theta_{W,H} (\beta \circ i'_W)$ is the big theta lift from $U(W)$ to $U(H)$, with respect to a splitting corresponding to a prescribed choice of characters (χ_W, χ_V) .

Proof By Theorem 2.8, for a fixed non-degenerate hermitian space H of dimension n , the set $\Sigma_{E,H}$ is in bijection with $\lambda \in E^\times / N_{L_E/E} (L_E^\times)$, such that $V_{E,\lambda} \cong H$. Therefore, we have that

$$\begin{aligned} & \sum_H \sum_{i \in \Sigma_{E,H}} \dim \operatorname{Hom}_{i(L_E^1)} \left(\Theta_{W,H} (\beta \circ i'_W)|_{i(L_E^1)}, \alpha \circ i^{-1} \right) \\ &= \sum_\lambda \dim \operatorname{Hom}_{i_{r_\lambda}(L_E^1)} \left(\Theta_{W,H_\lambda} (\beta \circ i'_W), \alpha \circ i_{r_\lambda}^{-1} \right), \end{aligned}$$

where the summation is over a set of representatives $\lambda \in E^\times$ for $E^\times / N_{L_E/E} (L_E^\times)$, where H_λ is the representative for the class of the hermitian space $V_{E,\lambda}$, and where $i_{r_\lambda}: L_E^1 \rightarrow T_{E,r_\lambda} \subset U(H_\lambda)$ is an arbitrary admissible embedding corresponding to the data $r_\lambda: V_{E,\lambda} \rightarrow H_\lambda$. Given such λ , we have by Theorem 5.1 that the space $\operatorname{Hom}_{i_{r_\lambda}(L_E^1)} (\Theta_{W,H_\lambda} (\beta \circ i'_W), \alpha \circ i_{r_\lambda}^{-1})$ can be non-zero only when $\beta = (\chi_W^{-n} \cdot \chi_V) \circ j_F^{-1} \cdot \alpha|_{K^1}$, and in this case the space is non-zero only for one class in $E^\times / N_{L_E/E} (L_E^\times)$, and for that class it is one-dimensional. Therefore, we get the result. \square

Remark 5.5 By substituting the same data as in Remark 5.2, we get Theorem 1.3.

5.2 Global problem

Let \mathbf{F} be a number field, and let \mathbf{K}/\mathbf{F} be a quadratic field extension. Let V be a non-degenerate n -dimensional hermitian space over \mathbf{K} , and let W be a non-degenerate one-dimensional skew-hermitian space over \mathbf{K} . Let $i'_W: U(W) \rightarrow L_{\mathbf{F}}^1$ be the obvious isomorphism. Fix an automorphic character $\beta: [L_{\mathbf{F}}^1] \rightarrow \mathbb{C}^\times$ and let $\Theta(\beta \circ i'_W) = \Theta_{W,V,\tilde{i},\psi}(\beta \circ i'_W)$ be the global theta lift of $\beta \circ i'_W$ to $U(V) (\mathbb{A}_{\mathbf{F}})$, where \tilde{i} is the splitting associated with the characters (χ_W, χ_V) .

Given a maximal anisotropic torus $T \subset U(V)$ and an automorphic character $\alpha: [T] \rightarrow \mathbb{C}^\times$, we would like to investigate whether the α -period of T is identically zero on $\Theta(\beta \circ i'_W)$, that is, we would like to check whether the integral

$$\mathcal{P}_{T,\alpha}(f) = \int_{[T]} f(t) \overline{\alpha(t)} dt$$

is zero for every $f \in \Theta(\beta \circ i'_W)$.

As before, given such T , by Theorem 2.3, we may find an étale algebra \mathbf{E} of degree n over \mathbf{F} , an element $\lambda \in \mathbf{E}^\times$, and an isomorphism $r: V_{E,\lambda} \rightarrow V$ of hermitian spaces over \mathbf{K} , such that

$$T = \left\{ r \circ m_x \circ r^{-1} \mid x \in L_{\mathbf{E}}^1 \right\}.$$

Analogously to the local case, we formulate an answer to this problem in the following theorem.

Theorem 5.6 *Let \mathbf{E} be an étale algebra of degree n over \mathbf{F} , such that $L_{\mathbf{E}}^1$ is anisotropic, $\alpha: [L_{\mathbf{E}}^1] \rightarrow \mathbb{C}^\times$ be an automorphic character, and let $i: L_{\mathbf{E}}^1 \rightarrow U(V)$ be an admissible embedding corresponding to the data $\lambda \in \mathbf{E}^\times$ and $r: V_{E,\lambda} \rightarrow V$. Then $\mathcal{P}_{i(L_{\mathbf{E}}^1), \alpha \circ i^{-1}}$ is not identically zero on $\Theta(\beta \circ i'_W)$ if and only the three following conditions are satisfied:*

- (1) $\beta = (\chi_W^{-n} \cdot \chi_V) \circ j_{\mathbb{A}_{\mathbf{F}}}^{-1} \cdot \alpha|_{L_{\mathbf{F}}^1(\mathbb{A}_{\mathbf{F}})}$.
- (2) For every place v ,

$$\omega_{L_{\mathbf{E} \otimes_{\mathbf{F}} \mathbf{F}_v} / \mathbf{E} \otimes_{\mathbf{F}} \mathbf{F}_v}(\lambda) = \varepsilon_{L_{\mathbf{E} \otimes_{\mathbf{F}} \mathbf{F}_v} / \mathbf{E} \otimes_{\mathbf{F}} \mathbf{F}_v} \left(\alpha_{v, L_{\mathbf{E} \otimes_{\mathbf{F}} \mathbf{F}_v}} \cdot \chi_{W,v}^{-1} \circ N_{L_{\mathbf{E} \otimes_{\mathbf{F}} \mathbf{F}_v} / \mathbf{K}_v}, \psi_v, \delta \right).$$

- (3) The following central L -function value does not vanish:

$$\mathcal{L} \left(\chi_W^{-1} \circ N_{L_{\mathbf{E}} / \mathbf{K}} \cdot \alpha_{L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbb{A}_{\mathbf{F}}} \right) \neq 0.$$

Here, $\delta \in \mathbf{K}^\times$ is a trace zero element, such that $\delta = \text{disc } W \pmod{N_{\mathbf{K}/\mathbf{F}}(\mathbf{K}^\times)}$.

Remark 5.7 If $\mathcal{P}_{i(L_{\mathbf{E}}^1), \alpha \circ i^{-1}}$ is non-zero then for every v , we must have that

$$\text{Hom}_{i(L_{\mathbf{E}})(\mathbf{F}_v)} \left(\Theta(\beta \circ i'_W)_v, \alpha_v \circ i_v^{-1} \right) \neq 0. \quad (13)$$

If $\Theta(\beta \circ i'_W)$ lies in the space of square-integrable automorphic forms, then by Theorem 5.1 and the compatibility between the global and local theta lifts, (13) is equivalent to the first two conditions of the theorem. We mention that if $\dim V \neq 2$, then $\Theta(\beta \circ i'_W)$ is square-integrable. When V is one-dimensional this follows from the fact that the groups are anisotropic, and when $\dim V \geq 3$, this follows from [35, Corollary 10.1 part (4)] (in the notations of [35], $\rho_n = 1$, $\dim V = m \geq 3$ and $0 \leq r \leq [\frac{m}{2}]$ so $m - r > 1 = \rho_n$).

Proof We will use the global seesaw identity. By our choice of δ , we have that $W \cong W_{\mathbf{F},1}^\delta$ as hermitian spaces. Henceforth, we will identify W with $W_{\mathbf{F},1}^\delta$. Let us be in the setup of Section 4.2. Consider the following seesaw diagram:

$$\begin{array}{ccc} U(W_{\mathbf{E},1}^\delta)(\mathbb{A}_{\mathbf{F}}) & & U(V)(\mathbb{A}_{\mathbf{F}}) \\ | & \searrow & | \\ U(W)(\mathbb{A}_{\mathbf{F}}) & & U(L_{\mathbf{E},\lambda})(\mathbb{A}_{\mathbf{F}}). \end{array}$$

As in Section 3.2.1, $U(W)$ is realized as a subgroup of $U(W_{\mathbf{E},1}^\delta)$ diagonally.

By the results of Section 4.2, we have that

$$\mathcal{P}_{i(L_{\mathbf{E}}^1), \alpha \circ i^{-1}}(\theta_i(\varphi, \beta \circ i'_W)) = \int_{[U(W_{\mathbf{F},1}^\delta)]} \theta_i(\varphi, \alpha \circ i^{-1})(h) \overline{\beta(i'_W(h))} dh, \quad (14)$$

where $\varphi \in \mathcal{S}(Y, \mathbb{A}_{\mathbf{F}})$. Hence, the period $\mathcal{P}_{i(L_{\mathbf{E}}^1), \alpha \circ i^{-1}}$ is non-zero if and only if the global theta lift $\Theta_i(\alpha \circ i^{-1})$ from $U(L_{\mathbf{E},\lambda})(\mathbb{A}_{\mathbf{F}})$ to $U(W_{\mathbf{E},1}^\delta)(\mathbb{A}_{\mathbf{F}})$ is non-zero, and the integral (14) is non-zero. By Theorems 4.4 and 3.6, the global theta lift $\Theta_i(\alpha)$ is non-zero if and only if conditions (2) and (3) hold. In this case, we have that

$$\Theta_i(\alpha \circ i^{-1}) = \left(\left(\chi_W^{-1} \circ N_{L_{\mathbf{E}}/\mathbf{K}} \cdot \chi_{L_{\mathbf{E},\lambda}} \right) \circ j_{\mathbf{E} \otimes_{\mathbf{F}} \mathbb{A}_{\mathbf{F}}}^{-1} \cdot \alpha \right) \circ i'_{W_{\mathbf{E},1}^\delta},$$

where $i'_{W_{\mathbf{E},1}^\delta} : U(W_{\mathbf{E},1}^\delta) \rightarrow L_{\mathbf{E}}^1$ is the obvious isomorphism. Hence, by choosing φ such that $\theta_i(\varphi, \alpha) \neq 0$, we get from the fact that two different characters of a group are orthogonal, that if $\Theta_i(\alpha \circ i^{-1})$ is not zero, then $\mathcal{P}_{i(L_{\mathbf{E}}^1), \alpha \circ i^{-1}}$ is not identically zero if and only if

$$\beta \circ i'_W = \left(\left(\chi_W^{-1} \circ N_{L_{\mathbf{E}}/\mathbf{K}} \cdot \chi_{L_{\mathbf{E},\lambda}} \right) \circ j_{\mathbf{E} \otimes_{\mathbf{F}} \mathbb{A}_{\mathbf{F}}}^{-1} \cdot \alpha \right) \circ i'_{W_{\mathbf{E},1}^\delta} \Big|_{U(W_{\mathbf{F},1}^\delta)(\mathbb{A}_{\mathbf{F}})}.$$

Since i'_W and $i'_{W_{\mathbf{E},1}^\delta}$ are the obvious isomorphisms, we have that they agree on $U(W_{\mathbf{F},1}^\delta)(\mathbb{A}_{\mathbf{F}})$, and therefore the condition is equivalent to

$$\beta = \left(\left(\chi_W^{-1} \circ N_{L_{\mathbf{E}}/\mathbf{K}} \cdot \chi_{L_{\mathbf{E},\lambda}} \right) \circ j_{\mathbf{E} \otimes_{\mathbf{F}} \mathbb{A}_{\mathbf{F}}}^{-1} \cdot \alpha \right) \Big|_{L_{\mathbf{F}}^1(\mathbb{A}_{\mathbf{F}})}.$$

Using the relations $j_{\mathbb{A}_F} = j_{\mathbb{E} \otimes_F \mathbb{A}_F} \upharpoonright_{L_F^1(\mathbb{A}_F)}$, $\chi_{L_{\mathbb{E}, \lambda}} \upharpoonright_{\mathbb{A}_K^\times} = \chi_V$, and the fact that $N_{L_{\mathbb{E}}/\mathbb{K}}(x) = x^n$ for $x \in (\mathbb{K} \otimes_F \mathbb{A}_F)^\times$, we get this condition is equivalent to

$$\beta = (\chi_W^{-n} \cdot \chi_V) \circ j_{\mathbb{A}_F}^{-1} \cdot \alpha \upharpoonright_{L_F^1(\mathbb{A}_F)},$$

which is condition (1). Hence, we proved the theorem. \square

If two admissible embeddings $i_1, i_2: L_{\mathbb{E}}^1 \rightarrow \mathrm{U}(\mathrm{V})$ are conjugate, then there exists $h \in \mathrm{U}(\mathrm{V})$ such that $i_1(x) = h^{-1}i_2(x)h$ for every $x \in L_{\mathbb{E}}^1$. For an automorphic form $f: [\mathrm{U}(\mathrm{V})] \rightarrow \mathbb{C}$ we have that $f(h^{-1}g) = f(g)$, for any $g \in \mathrm{U}(\mathrm{V})(\mathbb{A}_F)$, and hence we have the relation $\mathcal{P}_{i_1(L_{\mathbb{E}}^1), \alpha \circ i_1^{-1}}(f) = \mathcal{P}_{i_2(L_{\mathbb{E}}^1), \alpha \circ i_2^{-1}}(\rho(h)f)$, where $\rho(h)$ represents right translation by h . Therefore, the non-vanishing of the period $\mathcal{P}_{i(L_{\mathbb{E}}^1), \alpha \circ i^{-1}}$ does not depend on the representative i of a class of $\Sigma_{\mathbb{E}, \mathrm{V}}$. The following corollary describes when there exists a class in $\Sigma_{\mathbb{E}, \mathrm{V}}$ with non-vanishing period, and shows that if this class exists, it is unique.

Corollary 5.8 *Let \mathbb{E} be an étale algebra of degree n over \mathbb{F} , such that $L_{\mathbb{E}}^1$ is anisotropic, and let $\alpha: [L_{\mathbb{E}}^1] \rightarrow \mathbb{C}^\times$ be an automorphic character. Then there exists a non-degenerate hermitian space H of degree n , and an admissible embedding $i: L_{\mathbb{E}}^1 \rightarrow \mathrm{U}(\mathrm{H})$, such that the period $\mathcal{P}_{i(L_{\mathbb{E}}^1), \alpha \circ i^{-1}}$ is not identically zero on $\Theta_{\mathrm{W}, \mathrm{H}}(\beta \circ i'_{\mathrm{W}})$, if and only if the following two conditions hold*

- (1) $\beta = (\chi_W^{-n} \cdot \chi_V) \circ j_{\mathbb{A}_F}^{-1} \cdot \alpha \upharpoonright_{L_F^1(\mathbb{A}_F)}$.
- (2) The following central L -function value does not vanish:

$$\mathcal{L}\left(\chi_W^{-1} \circ N_{L_{\mathbb{E}}/\mathbb{K}} \cdot \alpha_{L_{\mathbb{E}} \otimes_F \mathbb{A}_F}\right) \neq 0.$$

Moreover, when these conditions hold, the isomorphism class of H as a hermitian space, and the class $[i] \in \Sigma_{\mathbb{E}, \mathrm{H}}$ are unique.

Here, $\Theta_{\mathrm{W}, \mathrm{H}}(\beta \circ i'_{\mathrm{W}})$ is the global theta lift of $\beta \circ i'_{\mathrm{W}}$ from $\mathrm{U}(\mathrm{W})(\mathbb{A}_F)$ to $\mathrm{U}(\mathrm{H})(\mathbb{A}_F)$, taken with respect to the splitting defined by a prescribed choice of characters (χ_W, χ_V) .

Proof By Theorem 5.6, the conditions in the theorem are necessary. Assuming these conditions, we will show the existence and uniqueness of a non-degenerate hermitian space H of dimension n and class $[i] \in \Sigma_{\mathbb{E}, \mathrm{H}}$, such that the period $\mathcal{P}_{i(L_{\mathbb{E}}^1), \alpha \circ i^{-1}}$ is not identically zero.

Let us begin with uniqueness. By Theorem 2.8, for every H , a choice of a class $[i]$ corresponds to an element $\lambda \in \mathbb{E}^\times / N_{L_{\mathbb{E}}/\mathbb{E}}(L_{\mathbb{E}}^\times)$, such that there exists an isomorphism of \mathbb{K} -hermitian spaces $r_\lambda: \mathrm{V}_{E, \lambda} \rightarrow \mathrm{H}$. If $\lambda_1, \lambda_2 \in \mathbb{E}^\times$ are such that there exist isomorphisms of hermitian spaces $r_{\lambda_1}: \mathrm{V}_{E, \lambda_1} \rightarrow \mathrm{H}$ and $r_{\lambda_2}: \mathrm{V}_{E, \lambda_2} \rightarrow \mathrm{H}$, such that the corresponding admissible embeddings i_{λ_1} and i_{λ_2} admit a non-zero period, then by condition (2) of Theorem 5.1, we must have that for every place v ,

$$\omega_{L_{\mathbb{E}} \otimes_F \mathbb{F}_v / \mathbb{E} \otimes_F \mathbb{F}_v}(\lambda_1) = \omega_{L_{\mathbb{E}} \otimes_F \mathbb{F}_v / \mathbb{E} \otimes_F \mathbb{F}_v}(\lambda_2).$$

This implies that for every v ,

$$\omega_{L_{\mathbf{E} \otimes_{\mathbf{F}} \mathbf{F}_v} / \mathbf{E} \otimes_{\mathbf{F}} \mathbf{F}_v} (\lambda_2 \lambda_1^{-1}) = 1,$$

and therefore for every place v , we have that $\lambda_2 \lambda_1^{-1} \in N_{L_{\mathbf{E} \otimes_{\mathbf{F}} \mathbf{F}_v} / \mathbf{E} \otimes_{\mathbf{F}} \mathbf{F}_v} ((L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbf{F}_v)^{\times})$. By the Hasse norm principle for quadratic extensions, this implies that $\lambda_2 \lambda_1^{-1} \in N_{L_{\mathbf{E}} / \mathbf{E}} (L_{\mathbf{E}}^{\times})$. Hence, i_{λ_1} and i_{λ_2} are conjugate.

Suppose that H and H' are both non-degenerate hermitian space of dimension n , and suppose that there exist $\lambda, \lambda' \in \mathbf{E}^{\times}$ and isomorphisms of hermitian spaces $r_{\lambda}: V_{\mathbf{E}, \lambda} \rightarrow H$ and $r_{\lambda'}: V_{\mathbf{E}, \lambda'} \rightarrow H'$, such that the corresponding admissible embeddings i_{λ} and $i_{\lambda'}$ admit a non-zero period. Then, as above, we get that $\lambda' \lambda^{-1} \in N_{L_{\mathbf{E}} / \mathbf{E}} (L_{\mathbf{E}}^{\times})$. This implies that $V_{\mathbf{E}, \lambda}$ is isomorphic to $V_{\mathbf{E}, \lambda'}$, and therefore H is isomorphic to H' .

We move to show the existence of a non-degenerate hermitian space H of dimension n and a class $[i] \in \Sigma_{\mathbf{E}, H}$ with non-zero period $\mathcal{P}_{i(L_{\mathbf{E}}^{\times}), \alpha \circ i^{-1}}$. We need to find an element $\lambda \in \mathbf{E}^{\times}$ such that for every place v , condition (2) of Theorem 5.6 holds. By writing $\mathbf{E} = \prod_{j=1}^m \mathbf{F}_j$, where $\mathbf{F}_j / \mathbf{F}$ is a finite field extension, the problem is reduced to the case where \mathbf{E} / \mathbf{F} is a field extension, and that $L_{\mathbf{E}} / \mathbf{E}$ is a quadratic field extension. Consider the diagonal map $\Delta_{\mathbf{E}}: \mathbf{E}^{\times} / N_{L_{\mathbf{E}} / \mathbf{E}} (L_{\mathbf{E}}^{\times}) \rightarrow (\mathbf{E} \otimes_{\mathbf{F}} \mathbb{A}_{\mathbf{F}})^{\times} / N_{L_{\mathbf{E}} / \mathbf{E}} ((L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbb{A}_{\mathbf{F}})^{\times})$. This map has co-kernel

$$\begin{aligned} & ((\mathbf{E} \otimes_{\mathbf{F}} \mathbb{A}_{\mathbf{F}})^{\times} / N_{L_{\mathbf{E}} / \mathbf{E}} ((L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbb{A}_{\mathbf{F}})^{\times})) / (\mathbf{E}^{\times} / N_{L_{\mathbf{E}} / \mathbf{E}} (L_{\mathbf{E}}^{\times})) \\ & \cong ((\mathbf{E} \otimes_{\mathbf{F}} \mathbb{A}_{\mathbf{F}})^{\times} / \mathbf{E}^{\times}) / (N_{L_{\mathbf{E}} / \mathbf{E}} ((L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbb{A}_{\mathbf{F}})^{\times} / L_{\mathbf{E}}^{\times})), \end{aligned}$$

which by global class field theory is isomorphic to the Galois group $\text{Gal}(L_{\mathbf{E}} / \mathbf{E}) \cong \{\pm 1\}$. It follows that the image of $\Delta_{\mathbf{E}}$ is the kernel of the quadratic character $\omega_{L_{\mathbf{E}} / \mathbf{E}}$, that is,

$$\text{Im} \Delta_{\mathbf{E}} = \left\{ (x_v)_v \in \prod'_v (\mathbf{E} \otimes_{\mathbf{F}} \mathbf{F}_v)^{\times} / N_{L_{\mathbf{E}} / \mathbf{E}} ((L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbf{F}_v)^{\times}) \mid \prod_v \omega_{L_{\mathbf{E}} / \mathbf{E}} (x_v) = 1 \right\}.$$

Since the central L -function value $\mathcal{L}(\chi_W^{-1} \circ N_{L_{\mathbf{E}} / \mathbf{K}} \cdot \alpha_{L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbb{A}_{\mathbf{F}}})$ is non-zero, we must have that the global root number $\prod_v \varepsilon_{L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbf{F}_v / \mathbf{E} \otimes_{\mathbf{F}} \mathbf{F}_v} (\alpha_{v, L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbf{F}_v} \cdot \chi_{W, v}^{-1} \circ N_{L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbf{F}_v / \mathbf{K}_v}, \psi_v, \delta)$ is equal to 1. Hence, the sequence

$$\left(\varepsilon_{L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbf{F}_v / \mathbf{E} \otimes_{\mathbf{F}} \mathbf{F}_v} \left(\alpha_{v, L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbf{F}_v} \cdot \chi_{W, v}^{-1} \circ N_{L_{\mathbf{E}} \otimes_{\mathbf{F}} \mathbf{F}_v / \mathbf{K}_v}, \psi_v, \delta \right) \right)_v$$

lies in the image of $\Delta_{\mathbf{E}}$, and we can find $\lambda \in \mathbf{E}^{\times}$ as desired. \square

Remark 5.9 Similarly to Remark 5.2, by substituting $W = W_{\mathbf{F}, 1}^{\delta}$ and $(\chi_w, \chi_v) = (\mu, \mu^n)$ as in Section 4.1.2, we get Theorem 1.5.

Appendix A. Morphisms of norm one tori

In this appendix, we prove some technical statements regarding extensions of morphisms of L_E^1 to L_E .

Let F be an infinite field with characteristic different than 2, and let K/F be a quadratic étale algebra, equipped with an involution $x \mapsto x^c$ whose set of fixed points is F . Let $0 \neq \delta \in K$ be a trace zero element. Any element $x \in K$ can be written in the form $x = a + b\delta$, where $a, b \in F$. We have that $\delta^c = -\delta$, and therefore $N_{K/F}(a + b\delta) = a^2 - b^2\delta^2 \in F$.

Consider the map $j_{K/F}: K^\times \rightarrow K^1$ given by

$$j_F(x) = \frac{x}{x^c}.$$

Denote for $b \in F$ with $b^2\delta^2 \neq 1$ (that is, $N_{K/F}(1 + b\delta) \neq 0$),

$$q_F(b) = \frac{j_F(1 + b\delta) - j_F(1 - b\delta)}{j_F(1 + b\delta) + j_F(1 - b\delta) + 2}. \quad (15)$$

Then a simple computation yields $q_F(b) = b\delta$.

Let V be an n -dimensional non-degenerate hermitian space over K . We are ready to prove our results.

Proposition A.1 *Let E be an étale algebra of degree n over F , and let $\lambda \in E^\times$ be such that $V_{E,\lambda}$ and V are isomorphic as hermitian spaces. Let $r_1, r_2: V_{E,\lambda} \rightarrow V$ be isomorphisms of hermitian spaces. For $j = 1, 2$, let $i'_j: L_E \rightarrow \text{End}(V)$ be the map*

$$i'_j(x) = r_j \circ m_x \circ r_j^{-1}.$$

Suppose that there exists $g \in \text{GL}(V)$ such that for any $x \in L_E^1$,

$$i'_1(x) = g \circ i'_2(x) \circ g^{-1}. \quad (16)$$

Then (16) holds for any $x \in L_E$.

Proof Write $E = \prod_{j=1}^m F_j$, where F_j/F is a field extension. Suppose first that $m = 1$. Then E/F is a finite field extension and L_E/E is a quadratic étale algebra, and $L_E = E \oplus E\delta$. Let $a, b \in E$, and choose $c \in F^\times$, such that $(ac)^2, c^2, (bc)^2 \neq \delta^{-2}$ (if L_E/E is a field extension, any $c \in F^\times$ satisfies this). Then

$$a + b\delta = \frac{q_E(ac)}{q_E(c)} + \frac{q_E(bc)}{c}. \quad (17)$$

We have that

$$i'_j(a + b\delta) = r_j \circ m_{a+b\delta} \circ r_j^{-1}$$

and

$$m_{a+b\delta} = m_{q_E(ac)} \circ m_{q_E(c)^{-1}} + c^{-1} \cdot m_{q_E(bc)}.$$

Therefore,

$$i'_j(a + b\delta) = i'_j(q_E(ac)) \circ i'_j(q_E(c)^{-1}) + c^{-1} \cdot i'_j(q_E(bc)),$$

and it suffices to explain why for any $b \in F$ with $\delta^2 b^2 \neq 1$, we have the equality

$$i'_1(q_E(b)) = g \circ i'_2(q_E(b)) \circ g^{-1}. \quad (18)$$

Using the definition of q_E , and the fact that the assignment $E \rightarrow \text{End}(V)$ given by $x \mapsto i'_j(x)$ is an isomorphism for $j = 1, 2$, we get, similarly to above, that $i'_j(q_E(b))$ is given by the formula

$$(i'_j(j_E(1 + b\delta)) - i'_j(j_E(1 - b\delta))) \circ (i'_j(j_E(1 + b\delta)) + i'_j(j_E(1 - b\delta)) + 2 \text{id}_V)^{-1}.$$

The equality (18) now follows from the fact that $j_E(1 \pm b\delta)$ lies in L_E^1 , and from the assumption that (16) holds for elements in L_E^1 .

If $m > 1$, then by restricting to L_{F_j} , we get from the proof above that (16) holds for every $x \in L_{F_j}$, for every $1 \leq j \leq m$. Using linearity, this implies that (16) holds for every $x \in L_E$. \square

Proposition A.2 *Let E and E' be étale algebras of degree n over F . Suppose that there exists an invertible \bar{F} -linear map $T: L_{E'} \otimes_F \bar{F} \rightarrow L_E \otimes_F \bar{F}$, such that for any $x \in L_E^1$ there exists $\tau(x) \in L_{E'}^1$, such that*

$$T \circ (m_{\tau(x)} \otimes \text{id}_{\bar{F}}) \circ T^{-1} = m_x \otimes \text{id}_{\bar{F}}. \quad (19)$$

Then for any $x \in L_E$ there exists $\tau(x) \in L_{E'}^1$, such that (19) holds. Moreover, such $\tau(x)$ is unique.

Proof Uniqueness follows from writing $m_{\tau(x)} \otimes_F \text{id}_{\bar{F}} = T^{-1} \circ (m_x \otimes_F \text{id}_{\bar{F}}) \circ T$ and applying both sides to $1 \otimes 1 \in L_{E'} \otimes_F \bar{F}$.

To show existence, first write $E = \prod_{j=1}^m F_j$, where F_j/F is a field extension. Define for $b \in F_j$ with $\delta^2 b^2 \neq 1$,

$$\tau(q_{F_j}(b)) = \frac{\tau(j_{F_j}(1 + b\delta)) - \tau(j_{F_j}(1 - b\delta))}{\tau(j_{F_j}(1 + b\delta)) + \tau(j_{F_j}(1 - b\delta)) + 2}.$$

The fact that (19) holds for $x = q_{F_j}(b)$, follows from (15), from the fact that $q_{F_j}(b) = b\delta$, and from the fact that (19) holds for elements in $L_{F_j}^1$. For any $a, b \in F_j$, choose $c \in F^\times$, such that $(ac)^2, c^2, (bc)^2 \neq \delta^{-2}$. Define

$$\tau(a + b\delta) = \frac{\tau(q_{F_j}(ac))}{\tau(q_{F_j}(c))} + \frac{\tau(q_{F_j}(bc))}{c}.$$

It follows from (17) that (19) holds for $x = a + b\delta$. Finally, extend τ to a general element of $L_E = \prod_{j=1}^m L_{F_j}$ by linearity. \square

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