Bounding Width on Graph Classes of Constant Diameter

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Abstract. We determine if the width of a graph class $\mathcal{G}$ changes from unbounded to bounded if we consider only those graphs from $\mathcal{G}$ whose diameter is bounded. As parameters we consider treedepth, pathwidth, treewidth and clique-width, and as graph classes we consider classes defined by forbidding some specific graph F as a minor, induced subgraph or subgraph, respectively. Our main focus is on treedepth for F-subgraph-free graphs of diameter at most d for some fixed integer d. We give classifications of boundedness of treedepth for $d \in \{4, 5, \ldots\}$ and partial classifications for $d = 2$ and $d = 3$.

1 Introduction

Graph width parameters play a prominent role in modern graph theory. One of the reasons is that large sets of NP-complete graph problems may become polynomial-time solvable on graph classes on which some width parameter is bounded (by a constant). For example, the celebrated meta-theorem of Courcelle [7] states that every problem definable in MSO_2 is polynomial-time solvable on graph classes of bounded treewidth. Another well-known meta-theorem, due to Courcelle, Makowsky and Rotics [8], states that every problem definable in MSO_1 is polynomial-time solvable on graph classes of bounded clique-width. The logic MSO_1 is more restrictive than MSO_2. However, any graph class of bounded treewidth has bounded clique-width, whereas the reverse statement does not hold. That is, clique-width is more *powerful* than treewidth.

Due to the above algorithmic implications and also out of a graph-structural interest, there exist many papers in the literature that research whether certain graph classes have bounded width. The framework of graph containment opens the way for a more systematic approach. For example, the recent treewidth dichotomy of Lozin and Razgon [28] determines exactly for which finite sets $\mathcal{F}$, the class of $\mathcal{F}$-free graphs has bounded treewidth; here, $\mathcal{F}$*-free* means not containing any graph from $\mathcal{F}$ as an *induced* subgraph. Hickingbotham [22] showed that in the treewidth dichotomy we may replace "treewidth" by "pathwidth". In contrast, there still exist some pairs (H_1, H_2) for which boundedness of clique-width is open for (H_1, H_2)-free graphs; see the survey [10] for details.

Our Focus. A width parameter p that is unbounded on a graph class $\mathcal{G}$ may be bounded on a subclass $\mathcal{G}'$ of $\mathcal{G}$. Ideally, we would like $\mathcal{G}'$ to be as large as possible to optimally benefit from the algorithmic benefits if p is bounded. We consider the *diameter* of the input graph and ask:

For a graph class $\mathcal{G}$ of unbounded width, what is the largest d such that the graphs in $\mathcal{G}$ of diameter at most d have bounded width?

This is a natural question, as there exist numerous NP-complete problems that stay NP-complete even on graphs of diameter $d = 2$; see, e.g. the diameter study for k-Colouring [13,32], in particular for $\mathcal{F}$-free graphs in [13,25,30,32]. There are even problems that are NP-complete *only if* $d = 2$ (e.g. Disconnected Cut [29]). Answering the above question will have a wide range of algorithmic consequences, in particular due to meta-theorems, as we discussed above [8,9].

Our Approach. We work within the framework of graph containment and thus focus on graph classes defined by some set $\mathcal{F}$ of forbidden graphs. To get a handle on these, we restrict ourselves to the case when $\mathcal{F}$ consists of a single graph F. To answer our research question, we selected some classical graph containment relations and width parameters. We forbid F as an induced subgraph, subgraph or minor. We say that a graph G is *F-free*, *F-subgraph-free* or *F-minor-free* if it does not contain F as an induced subgraph, subgraph or minor, respectively. Note that F-minor-free graphs are F-subgraph-free and F-subgraph-free graphs are F-free. As width parameters, we will consider pathwidth pw, treewidth tw, clique-width cw and also treedepth td (which has algorithmic applications for many problems where treewidth cannot be used; see e.g. [18,20,21,23,26]). We write $p \rhd q$ if p is more powerful than q. It is well known [2,9] that cw$\rhd$tw$\rhd$pw$\rhd$td.

Known Results. We first describe, in Table 1, the situation without a diameter bound. For $r \geq 1$, the graph P_r is the r-vertex path. The set $\mathcal{S}$ consists of all graphs, every component of which is a path or *subdivided claw* (cubic tree with exactly one vertex of degree 3). The set $\overline{\mathcal{S}}$ consists of all graphs that are subgraphs of any *subdivided star* (any tree with exactly one vertex of degree at least 3).

Table 1 also includes known results on diameter-width. Eppstein [16] defined a graph class $\mathcal{G}$ to have the *diameter-treewidth* property if the treewidth of every graph in $\mathcal{G}$ is bounded by a function of the diameter of G. For graph classes closed under taking subgraphs, this notion coincides with bounded local treewidth, a crucial notion in bidimensionality theory. We define the properties of diameter-clique-width, diameter-pathwidth and diameter-treedepth analogously. A graph G is *apex* $\mathcal{G}$ for a graph class $\mathcal{G}$ if $G - v \in \mathcal{G}$ for some vertex $v \in V(G)$. So, for instance, the class of *apex linear forests* consists of all graphs that become *linear forests* (disjoint unions of paths) after removing at most one vertex.

We first note that by adding a dominating vertex to a wall we obtain a graph of diameter $d = 2$ whose clique-width can be arbitrarily large. This graph only

	minor		induced subgraph		subgraph	
cw	planar [11]	(**apex planar**)	$\subseteq_i P_4$ [9]	($\subseteq_\mathbf{i}$ $\mathbf{P_4}$)	$\mathcal{S}$ [3]	(?)
tw	planar [35]	(apex planar [16])	$\subseteq_i P_2$	($\subseteq_\mathbf{i}$ $\mathbf{P_2}$)	$\mathcal{S}$ [34]	(?)
pw	forest [1]	(apex forest [14])	$\subseteq_i P_2$	($\subseteq_\mathbf{i}$ $\mathbf{P_2}$)	$\mathcal{S}$ [34]	(?)
td	linear forest [33]	(**apex linear forest**)	$\subseteq_i P_2$	($\subseteq_\mathbf{i}$ $\mathbf{P_2}$)	linear forest [33]	($\overline{\mathcal{S}}$)

Table 1. Overview of known and new results. Entries without brackets classify the graphs F such that the width of F-free graphs is bounded. For example, a class of F-minor free graphs has bounded clique-width if and only if F is a planar graph. Entries within brackets classify the graphs F such that the class of F-free graphs has the diameter-width property. Unreferenced results indicate a trivial/folklore result. A "?" indicates an open case. Results in bold/blue are new results proven in this paper.

contains apex planar graphs as minors. Hence, the result of Eppstein [16] implies that a class of F-minor-free graphs of diameter 2 has bounded clique-width if and only if F is apex planar. As this is not true for $d = 1$ (just take $F = K_6$), we say that $d = 2$ is *tight* for diameter-clique-width for minors. By adding a dominating vertex to a full binary tree, we find that $d = 2$ is also tight for diameter-pathwidth for minors.

Our Results. Table 1 also contains several new results, and all the new and known results together already show a clear impact of bounding the diameter. For each of the new results in Table 1 we show that diameter $d = 2$ is tight (see Appendix B) except for one result. Namely, to classify the diameter-treedepth property, we prove the following two results in Section 3, the second of which shows that $d = 5$ is tight. In our proofs, we will exploit a result of Galvin, Rival and Sands [19], who proved that a graph of large treedepth must either contain a large complete bipartite graph as a subgraph or a large induced path; see Section 2 for a more detailed discussion of this result and its consequences.

Theorem 1 (Classification for diameter $d \geq 5$). *Let $d \geq 5$. For a graph F, the class of F-subgraph-free graphs of diameter at most d has bounded treedepth if and only if F is a subgraph of a subdivided star.*

Theorem 2 (Classification for diameter 4). *For a graph F, the class of F-subgraph-free graphs of diameter at most 4 has bounded treedepth if and only if F is a subgraph of a subdivided star or H_2^ℓ for some $\ell \in \mathbb{N}$ (see also Fig. 1).*

Theorems 1 and 2 show that there is a considerable scope for improvement by considering graphs of diameter 2 and 3, in line with our research question. This is unlike the other cases in Table 1, for which $d = 2$ is always tight.

We were not able to give complete classifications for treedepth under the subgraph relation for $d = 2$ and $d = 3$. However, by a deeper exploration of the result of Galvin, Rival and Sands [19] and using a result on polarity graphs, due to Erdős, Rényii and Sós [17], we were able to prove a variety of results summarized in the state-of-the-art theorem below. We again refer to Fig. 1 for an explanation of the various graphs that we define in these statements.

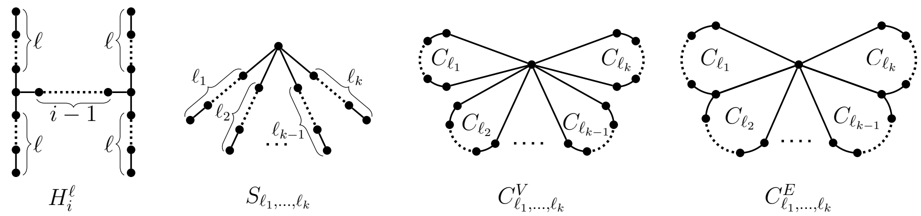

Fig. 1. The subdivided "H"-graph H_i^ℓ, the subdivided star $S_{\ell_1,\dots,\ell_k}$, the V-type graph $C^V_{\ell_1,\dots,\ell_k}$ (set of cycles sharing one common vertex) and the E-type graph $C^V_{\ell_1,\dots,\ell_k}$ (set of consecutive cycles sharing an edge). We write $C^V_{\ell_1,\dots,\ell_k} = C^V_{k\times[\ell_1]}$ and $C^E_{\ell_1,\dots,\ell_k} = C^E_{k\times[\ell_1]}$ if $\ell_1 = \dots = \ell_k$, and $C^V_{\ell_1,\dots,\ell_k} = C^V_{i\times[\ell_1],k-i\times[\ell_k]}$ if $\ell_1 = \dots = \ell_i$, $\ell_{i+1} = \dots = \ell_k$.

Theorem 3 (Partial classification for diameters 2 and 3). *For a graph F and $d \in \{2,3\}$, the class of F-subgraph-free graphs of diameter at most d has:*

(i) bounded treedepth if:
1. *$d = 3$ and F is an acyclic apex linear forest (Theorem 12);*
2. *$d = 3$ and F is C_8 (Theorem 29);*
3. *$d = 2$ and F is a bipartite, C_4-subgraph-free subgraph of $P_n \bowtie K_1$ for some $n \geq 1$ that contains exactly one cycle (Theorem 28);*
4. *$d = 2$ and F is $C^{\text{V}}_{2\ell_1,2\ell_2}$ for some $\ell_1, \ell_2 \geq 3$ (Theorem 30);*
5. *$d = 2$ and F is a subgraph of $C^{\text{V}}_{k*[2\ell]}$ for some $\ell \geq 3$ and $k \geq 1$ (Theorem 31) or*
6. *$d = 2$ and F is $C^{\text{E}}_{2\ell_1,2\ell_2}$ for some $\ell_1, \ell_2 \geq 3$ (Theorem 33);*

(ii) unbounded treedepth if:
1. *$d = 2$ and F is not bipartite (Observation 7);*
2. *$d = 2$ and F is bipartite, but is not a subgraph of $P_n \bowtie K_1$ for any $n \geq 1$ (Observation 7);*
3. *$d = 2$ and F is a supergraph of C_4 (Theorem 9 [17]);*
4. *$d = 2$ and F is a supergraph of $C^{\text{V}}_{12\times[6],12\times[8]}$ (Theorem 32);*
5. *$d = 2$ and F is a supergraph of $C^{\text{E}}_{k*[2\ell]}$ for some $\ell \geq 3$ and $k = 2(2\ell - 3)$ (Theorem 13);*
6. *$d = 3$ and F is a supergraph of C_6 (Theorem 27) or*
7. *$d = 3$ and F is a supergraph of either $C^{\text{V}}_{4\ell_1,4\ell_2}$, $C^{\text{V}}_{2\ell,2\ell}$, $C^{\text{E}}_{4\ell_1,4\ell_2}$ or $C^{\text{E}}_{2\ell,2\ell}$ for $\ell_1, \ell_2 \geq 2$ and $\ell \geq 4$ (Theorem 34).*

From Theorem 3 we note a jump from boundedness for diameter $d = 2$ to unboundedness for diameter $d = 3$ if $F = C_6$. Moreover, there is a difference for $d = 2$ when we forbid a graph F whose cycles share a unique vertex or an edge with a common end-vertex, i.e. a *V-type* graph $C^{\text{V}}_{k*[2\ell]}$ or *E-type* graph $C^{\text{E}}_{k*[2\ell]}$, respectively. The treedepth also becomes unbounded even for $d = 2$ if the cycles of a V-type graph or E-type graph have only two different lengths.

In Section 2, we show Observation 7 and how to derive Theorem 9 from a result on polarity graphs [17] (just like Theorem 27). We prove Theorem 12 in Section 4 and Theorem 13 in Section 5. The proofs of all other results in Theorem 3 are in the appendix. In Section 6 we discuss the open cases in Table 1 as well as other open problems.

2 Preliminaries and Basic Results

We only consider finite, simple, undirected graphs $G = (V(G), E(G))$. Let G be a graph. We denote the *neighbourhood* of a vertex $v \in V(G)$ by $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ (we may also just write $N(v)$). For a subset $S \subseteq V(G)$, we write $N_G(S) = \bigcup_{v \in S} N_G(v)$. A graph H is a *subgraph* of G if H can be obtained from G by a sequence of vertex deletions and edge deletions, whereas H is an *induced subgraph* of G if H can be obtained from G by a sequence of vertex deletions. For a vertex set $S \subseteq V(G)$, we write $G[S]$ to denote the subgraph of G *induced by* S, that is, the graph obtained from G after deleting the vertices not in S. The *contraction* of an edge $e = uv$ in G replaces u and v by a new vertex w that is adjacent to every vertex in $(N_G(u) \cup N_G(v)) \setminus \{u, v\}$ (without creating parallel edges). We let G/e denote the graph obtained from G after contracting e. A graph H is a *minor* of G if H can be obtained from G by a sequence of edge deletions, edge contractions and vertex deletions. For a set of graphs $\mathcal{F}$, we say that G is *$\mathcal{F}$-subgraph free*, *$\mathcal{F}$-free*, or *$\mathcal{F}$-minor free* if G does not contain any graph in $\mathcal{F}$ as a subgraph, induced subgraph or minor, respectively.

We may refer to a path P with vertices $u_0, \dots, u_l$ and edges $u_{i-1}u_i$ for $1 \leq i \leq l$ by the sequence $(u_0, u_1, u_2, \dots, u_l)$. The *length* of P is its number of edges l. The *distance* $d_G(u, v)$ between two vertices u and v of a graph G is the length of a shortest path from u to v. The *line graph* of a graph G is the graph with vertex set $E(G)$ with an edge between two vertices e and f if and only if e and f do not have a common end-vertex in G. For two graphs G and H, we let $G \bowtie H$ denote the graph obtained from the disjoint union of G and H after adding all edges between the vertices in $V(G)$ and the vertices in $V(H)$.

An *elimination forest* of a graph G is a rooted forest T such that $V(G) = V(T)$ and for every $uv \in E(G)$ both u and v are on the same root-to-leaf path of T. The *treedepth* $\mathrm{td}(G)$ is the minimum height of an elimination forest of G.

Fact 4 ([33]) *For a graph G with a longest path of length ℓ, $\log(\ell) \leq \mathrm{td}(G) \leq \ell$.*

The following theorem, combined with Fact 4, has a useful consequence.

Theorem 5 ([19]). *For all $r, s, \ell \in \mathbb{N}$, there is a number $c(r, s, \ell)$ such that every $K_{r,s}$-subgraph-free graph with a path of length $c(r, s, \ell)$ has an induced P_ℓ.*

Corollary 6. *For all $r, s, \ell \in \mathbb{N}$, there is a number $c(r, s, \ell)$ such that every $K_{r,s}$-subgraph-free graph of treedepth $c(r, s, \ell)$ has an induced P_ℓ.*

By Fact 4, complete bipartite graphs and graphs $P_n \bowtie K_1$ have unbounded treedepth. As both classes consist of graphs of diameter at most 2, we obtain:

Observation 7 *Let $d \geq 2$. For a graph F, the class of F-subgraph-free graphs with diameter at most d has unbounded treedepth if there is no integer n such that F is a subgraph of both $K_{n,n}$ and of $P_n \bowtie K_1$.*

Every subgraph of $P_n \bowtie K_1$ can have at most one component which is not a path. Moreover, in a graph of large enough treedepth, we can find any number of disjoint paths as subgraphs. Hence, Observation 7 implies the following:

Observation 8 *For any graph F, there is a component C of F such that the class of F-subgraph-free graphs with diameter d has bounded treedepth if and only if the class of C-subgraph-free graphs with diameter d has bounded treedepth.*

Therefore, from now on we consider all forbidden subgraphs F to be connected.

Erdős, Rényii and Sós [17] showed how a family of C_4-subgraph-free graphs with diameter 2 can be constructed from a polarity of a projective plane. Making the observation that this family has unbounded minimum degree and so also unbounded treewidth we obtain the following.

Theorem 9 ([17]). *The class of C_4-subgraph-free graphs of diameter 2 has unbounded treedepth.*

We also use polarity graphs to show that the class of C_6-subgraph-free graphs of diameter 3 has unbounded treedepth (see Theorem 27 in Appendix D).

3 Proofs of Theorems 1 and 2

We first note that the 1-subdivisions of complete bipartite graphs have diameter 4 and unbounded treedepth alongside the graphs P_n^{1001}, i.e. for $b \in \{0,1\}^*$, where P_n^b is the graph obtained from P_n after adding a new vertex u and making u adjacent to the i-th vertex of P_n if the (i mod $|b|$)-th bit of b is 1. Likewise, the 2-subdivisions of complete graphs have diameter 5 and unbounded treedepth.

If F is a subgraph of the 1-subdivision of a complete bipartite graph and some P_n^{1001}, then F is a subgraph of H_2^ℓ or $S_{\ell_1,\dots,\ell_k}$ for some $\ell \geq 1$ and $\ell_1,\dots,\ell_k \geq 1$. If F is also a subgraph of the 2-subdivision of a complete graph, this leaves only $S_{\ell_1,\dots,\ell_k}$ for some $\ell \geq 1$. Hence, Theorems 1 and 2 follow from the following two lemmas. Forbidding $S_{\ell_1,\dots,\ell_k}$ in the first lemma bounds the treedepth of a graph of any diameter d by $c((k\ell)/2+1,(k\ell)/2,\ell(k^d+1))$ where $\ell = \max\{\ell_1,\dots,\ell_k\}$ and c is the function from Theorem 5. We did not try to optimize this function.

Lemma 10. *Let $d \geq 1$. For all $\ell_1,\dots,\ell_k \in \mathbb{N}$, the class of $S_{\ell_1,\dots,\ell_k}$-subgraph-free graphs of diameter at most d has bounded treedepth.*

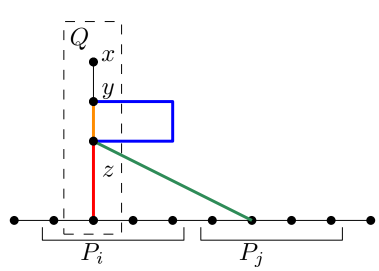

Fig. 2. Path Q. Paths $Q^1, Q^2, \overline{Q}^1$ and $\overline{Q}^2$ are orange, red, blue, and green, resp.

Proof. Let G be some $S_{\ell_1,\ldots,\ell_k}$-subgraph-free graph of diameter at most d and $\ell = \max\{\ell_1,\ldots,\ell_k\}$. We claim $td(G) < c((k\ell)/2+1,(k\ell)/2,\ell(k^d+1))$. Suppose for contradiction $td(G) \geq c((k\ell)/2+1,(k\ell)/2,2\ell(k^d+1))$. As $K_{(k\ell)/2+1,(k\ell)/2}$ contains $S_{\ell_1,\ldots,\ell_k}$ as a subgraph, G cannot contain a large complete bipartite graph as a subgraph. From Corollary 6, G contains some induced path P of length $2\ell(k^d+1)$. We may assume that $k \geq 2$ by Fact 4.

Pick disjoint subpaths $P_1,\ldots,P_{k^d+1}$ of P of length 2ℓ. In the following we prove inductively that for every $x \in V(G)$ and every $\delta \in [d]$ there is a path of length at most δ from x to some vertex on P_i for at most k^δ different $i \in [1,k^d+1]$. For $\delta = d$ this yields a contradiction to G having diameter at most d. First observe that every vertex $x \in V(G)$ can have a neighbour in at most $k-1$ distinct subpaths from $\{P_1,\ldots,P_{k^d+1}\}$ else G contains $S_{\ell_1,\ldots,\ell_k}$. Considering that x might be on P, we obtain that there is a path of length at most 1 from x to some vertex on P_i for at most k different $i \in [k^d+1]$.

Now, let $x \in V(G)$ and $\mathcal{I} \subseteq [k^d+1]$ be all indices such that there is a path of length at most $\delta+1$ from x to some vertex on P_i. Let $\mathcal{Q}$ be a set consisting a single minimum length path Q from x to any vertex q on P_i for every $i \in \mathcal{I}$. Towards a contradiction, assume that $|\mathcal{Q}| > k^{\delta+1}$. As there are paths of length at most δ from any $v \in V(G)$ to some $q \in P_i$ for at most k^δ different $i \in [k^d+1]$ by assumption, there is a subset $\mathcal{Q}_0 \subseteq \mathcal{Q}$ of paths that have length $\delta+1$ with $|\mathcal{Q}_0| \geq (k-1)k^\delta+1$.

Claim 10.1. *For any path $Q \in \mathcal{Q}_0$ there are at most $k^\delta - 1$ other paths $\overline{Q} \in \mathcal{Q}_0$ that intersect Q in more than one vertex.*

Proof of Claim: Let $i \in [k^d+1]$ be the index such that the last vertex of $Q \in \mathcal{Q}$ is on P_i. Let y be the second vertex of Q. Let $j \in [k^d+1]$, $j \neq i$ be any index such that the path $\overline{Q} \in \mathcal{Q}$ from x to some vertex in P_j intersects Q in some vertex $z \neq x$. We argue that there is a path from y to some vertex on P_j of length δ. Let Q^1 be the subpath of Q from y to z, $\overline{Q}^1$ be the subpath of $\overline{Q}$ from y to z and $\overline{Q}^2$ subpath of $\overline{Q}$ from z to some vertex on P_j. As Q is a shortest path from x to any vertex on P_i of length $\delta+1$ and $\overline{Q}$ is a shortest paths from x to any vertex on P_j of length $\delta+1$, Q^1 and $\overline{Q}^1$ have the same length. Hence, the concatenation of Q^1 and $\overline{Q}^2$ yields a path of length δ from y to some vertex on P_j. Since, additionally to i, there can be paths of length δ to some vertex on P_j for at most $k^\delta - 1$ different $j \in [k^d+1]$, $j \neq i$, the claim follows. $\diamond$

We now iteratively choose paths $Q_1,\ldots,Q_m$ in such a way that Q_i is any path from the set $\mathcal{Q}_i \subseteq \mathcal{Q}_0$ of paths that intersect each Q_j, $j \in [i-1]$ in x only. Here m is the minimum integer such that $\mathcal{Q}_{m+1} = \emptyset$. By Claim 10.1 we know that $|\mathcal{Q}_i| \geq |\mathcal{Q}_{i-1}| - k^\delta = (k-i)k^\delta + 1$. Hence, $|\mathcal{Q}_k| \geq 1$ which implies that there are paths $Q_1,\ldots,Q_k$ pairwise only intersecting in vertex x. Therefore, G contains $S_{\ell_1,\ldots,\ell_k}$ as a subgraph, a contradiction. $\square$

The proof of the next lemma is in Appendix C.

Lemma 11. *For any $\ell \in \mathbb{N}$, the class of H_2^ℓ-subgraph-free graphs of diameter at most 4 has bounded treedepth.*

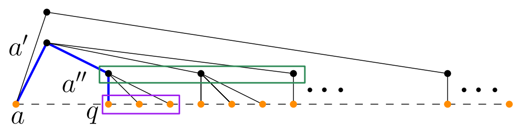

Fig. 3. The dashed line is P with vertices of A in orange. Some vertices from $X(A,a)$ with their respective shortest paths are drawn, with the path (a,a',a'',q) is in blue.

4 Proof of Theorem 12

Theorem 12. *For every tree F, the class of F-subgraph-free graphs of diameter at most 3 has bounded treedepth if and only if F is an acyclic apex linear forest.*

Proof. A graph is an apex linear forest if, and only if it is a subgraph of $P_n \bowtie K_1$ for some integer n. This together with Observation 7 shows the forward direction.

For the backward direction, let F be an acyclic apex linear forest. There exists some k and ℓ such that F is a subgraph of the graph obtained from the disjoint union of a vertex v and k paths of length $\ell-1$ after making v adjacent to exactly one vertex on every path. We claim the class $\mathcal{C}$ of F- subgraph-free graphs of diameter at most 3 have bounded treedepth. Suppose for a contradiction that $\mathcal{C}$ has unbounded treedepth, it must contain some graph G such that $\text{td}(G) \geq c((k\ell+1)/2, (k\ell+1)/2, \gamma(\ell+1)(5k)^7)$, where γ is some function of k and the function c is given by Corollary 6. As F is contained in $K_{(k\ell+1)/2,(k\ell+1)/2}$, G has no large complete bipartite subgraph. Corollary 6 implies that G contains some induced path P with length at least $\gamma(\ell+1)(5k)^7$. Let $P^{\ell+1} = \{p_i : i \bmod (\ell+1) = 0\} \setminus p_0$, where p_i is the i-th vertex of P (starting from $i=0$).

Claim 12.1. *For every γ, if $|P^{\ell+1}| > \gamma(5k)^7$, then G contains the disjoint union of γ copies of $S_{k*[\ell]}$ as an induced subgraph.*

Proof of Claim: Consider some $A \subseteq P^{\ell+1}$ and $a \in A$. Let $X(A,a)$ contain all vertices that lie on some shortest path from a to some $a' \in A$ in G. We claim that if $|A| \geq (k'-1)^3+1$, then there is some vertex $x \in X(A,a)$ with k' disjoint shortest paths to vertices in A for every $k' \geq k$. Note that every vertex in G is adjacent to at most $k-1$ vertices in A; else F is a subgraph of G; see the purple vertices in Figure 3. Consider some $x \in X(A,a)$ together with all shortest paths of length at most 2 from x to some vertex in A. These paths can contain at most $k'-1$ distinct intermediate vertices, else there are k' disjoint paths from x to vertices in A. Each intermediate vertex (green in Figure 3) has a path of length at most 1 to at most $k'-1$ vertices in A, so there are at most $(k'-1)^2$ paths of length at most 2 from x to some vertex in A. As G has diameter 3, a has a path of length at most 3 to every vertex in A. Let (a,a',a'',q) be such a path. There are at most $(k'-1)^2$ shortest paths of length 2 from a' to vertices in A. At most $(k'-1)^2$ shortest paths from a to some vertex in A contain either a' or a'', so at least $(k'-1)^3+1-(k'-1)^2$ paths (among all shortest paths from a to some vertex in A) are disjoint from this a to q path. As $|A| \geq (k'-1)^3+1$, there are at least k' disjoint shortest paths from a to vertices in A.

Let $x \in X(A,a)$ have k' disjoint shortest paths to vertices in A, these paths together with sections of P describe a $S_{k'*[\ell]}$ subgraph, which we call S. Let $X_1(S)$ and $X_2(S)$ be the sets of vertices of S containing all vertices at distance 1 and 2 from x in S, respectively. We refer to edges in $G[S]$ but not in S as cross edges. Given all vertices with distance at least 3 from x in S are in P and P is an induced path, all cross edges have some endpoint in $\{x\} \cup X_1(S) \cup X_2(S)$.

Claim 12.2. *There is a set of at most $4k-3$ branches of S whose removal from S leaves no cross edges with an endpoint in $X_1(S) \cup \{x\}$.*

Proof of Claim: If a cross edge is incident to x, then the other endpoint is in P as $N(x) \cap X_2(S) = \emptyset$. If x has some cross edge to $2k$ different branches, then x has k neighbours in P each with pairwise distance at least $\ell - 1$ along P, which would imply that F is a subgraph of G. Therefore, after removing at most $2k-1$ branches, all cross edges have some endpoint in $X_1(S)$ or $X_2(S)$.

The graph G_{X_1} has one vertex for every branch of S and an edge (b, b') between two distinct branches b and b' if a vertex $x' \in X_1(S)$ laying on b is adjacent to some vertex in b'. If G_{X_1} has a matching of size k, then G has F with centre x as a subgraph. Matching edges indicate k vertices in $X_1(S)$ with a pair of disjoint paths of length ℓ, the first via its respective branch in S and the second via the branch given by the matching. As G_{X_1} has a maximum matching of size at most $k-1$, it has a vertex cover R of size at most $2(k-1)$. By deleting the branches indicated by R, no cross edge is adjacent to $X_1(S)$ either. ⋄

If some $x'' \in X_2(S)$ has cross edges to $5k-3$ branches, let $S_{x''}$ be that $S_{(5k-3)*[\ell]}$ with centre x'' resulting from these cross edges. As $X_2(S_{x''}) \subseteq P$ all cross edges of $S_{x''}$ have an endpoint in $\{x''\} \cup X_1(S_{x''})$. From Claim 12.2, removing at most $4k-3$ branches of $S_{x''}$ gives some induced $S_{k*[\ell]}$. If each vertex in $X_2(S)$ has a cross edge to at most $5k-4$ other branches and S has $k(5k-4) + 5k-4$ branches at least k of these have no cross edges. That is, if $k' \geq (5k-4)(k+1)$, then there is some induced $S_{k*[\ell]}$.

If there are $\gamma k(2k+1)$ disjoint subsets of $P^{\ell+1}$ of size $(5k-4)^3(k+1)^3$ (recall that if $|A| \geq (k'-1)^3 + 1$, then G has a $S_{k'*[\ell]}$ as a subgraph) there must be $\gamma k(2k+1)$ induced stars, $S_1, \cdots, S_{\gamma(2k+1)}$, as described above. It remains to show that at least γ among those stars are pairwise disjoint and have no edges between each other. Recall that every S_i contains at most $2k+1$ vertices that are external to the path P. No vertex can be contained in or adjacent to k of these stars as otherwise G contains F as a subgraph. Therefore, since we are given $\gamma k(2k+1)$ stars, there must be at least γ that are pairwise disjoint and have no edges between each other. ⋄

Let $\mathcal{S} = \{S_1, \cdots, S_\gamma\}$ be the set of pairwise distinct, non-adjacent copies of $S_{k*[\ell]}$ in G obtained from Claim 12.1. For every $S \in \mathcal{S}$, let x_S be the centre of S. For $b \in \{1,2\}$, let $X_b(S)$ be the set of all vertices in S of distance b from x_S in S.

Let $T \subseteq \mathcal{S}$ with $|T| \geq 2k$. For $S \in T$, if there is some bijective mapping between k vertices in $X_1(S)$ and k stars in T, such that there are disjoint paths between these k vertices in $X_1(S)$ and their respective star in T containing no edges from S, then G contains F. For any pair $S, S' \in T$, the shortest path between any pair $u \in X_1(S)$ and $v \in X_1(S')$ has length at most 3 and has at most two vertices not in $V(S) \cup V(S')$. Hence, there is a set $Z(S,T)$ of $2(k-1)$ vertices such that for every $x' \in X_1(S)$ with a neighbour $x'' \in X_2(S)$, the shortest path from x' to each vertex in $X_1(S')$ contains some vertex in $Z(S,T) \cup \{x_S, x''\}$.

For $\rho \leq k$, let $\Gamma_\rho \subseteq \mathcal{S}$ be such that for every $S \in \Gamma_\rho$ there are ρ vertex disjoint shortest paths containing no edges from S each from a vertex in $X_1(S)$ to its own star in $\mathcal{S} \setminus \Gamma_\rho$, i.e. there is a one-to-one mapping between the ρ vertices and the ρ stars. See Figure 4. Note that $\Gamma_0 = \mathcal{S}$, we claim that that if $|\Gamma_\rho| \geq 2^{\rho+2}k^{\rho+2}$, then $|\Gamma_{\rho+1}| \geq \frac{|\Gamma_\rho|}{2^{\rho+1}k^{\rho+1}} - (k-1)^2$. This implies there is a constant γ which is a function of k such that if $|\mathcal{S}| = \gamma$, then $|\Gamma_k| \geq 1$. If $|\Gamma_k| \geq 1$ then G contains F as a subgraph, a contradiction. Therefore, for the proof of this theorem, it remains only to show that $|\Gamma_{\rho+1}| \geq \frac{|\Gamma_\rho|}{2^{\rho+1}k^{\rho+1}} - (k-1)^2$ for every $\rho \in \{0, \ldots, k-1\}$.

For $S \in \Gamma_\rho$, we let $\hat{Z}(S)$ be the set of all vertices that lie on the ρ disjoint paths from $X_1(S)$ to other stars and we denote by x'_S an arbitrary vertex in $X_1(S) \setminus \hat{Z}(S)$; note that x'_S always exists because $k > \rho$. Let x''_S denote the vertex in $N(x'_S) \cap X_2(S)$. For every $S' \in \Gamma_\rho$, the shortest path from x'_S to $x'_{S'}$ must contain some vertex from $Z(S,\Gamma_\rho) \cup \{x_S, x''_S\}$. As $|Z(S,\Gamma_\rho) \cup \{x_S, x''_S\}| \leq 2k$, there exists some $z_S(\Gamma_\rho) \in Z(S,\Gamma_\rho) \cup \{x_S, x''_S\}$ that lies on at least $\frac{|\Gamma_\rho|-1}{2k}$ of these paths. Such a $z_S(T)$ can also be defined for every $T \subseteq \Gamma_\rho$ with size at least $2k$.

Claim 12.3. *For every $r \geq k^2$, if $|\Gamma_\rho| \geq r2^{\rho+1}k^{\rho+1}$, then there is some $\bar{S} \in \Gamma_\rho$ and $\Gamma'_\rho \subseteq \Gamma_\rho$ of size r such that for some $z \notin \bigcup_{S' \in \Gamma'_\rho} \hat{Z}(S')$, z lies on the shortest path from $x'_{\bar{S}}$ to $x'_{S'}$ for every $S' \in \Gamma'_\rho$.*

Proof of Claim: Say, for contradiction, such set and vertex does not exist. We claim there exists a set $Q_{\rho+1} \subseteq \Gamma_\rho$ of size at least r and $\rho+1$ stars $A = \{S_0, \ldots, S_\rho\} \subseteq \Gamma_\rho \setminus Q_{\rho+1}$ such that the following holds. For every $S \in A$ there is a distinct vertex z, such that for every $S' \in Q_{\rho+1}$, $z \in \hat{Z}(S')$, $N(z) \cap V(S') = \emptyset$, and z lies on the shortest path from x'_S to $x_{S'}$: a contradiction as for any $S \in \Gamma_\rho$ there are at most ρ vertices $\hat{z} \in \hat{Z}(S)$ with $\hat{z}$ not adjacent to S.

Consider some arbitrary $S_0 \in \Gamma_\rho$ and its respective $z_{S_0}(\Gamma_\rho)$. Let $T_0 \subseteq \Gamma_\rho$ be those $S \in \Gamma_\rho$ such that $z_{S_0}(\Gamma_\rho)$ lies on the shortest path from x'_{S_0} to x'_S. By assumption there are at most $r-1$ different $S \in T_0$ such that $z_{S_0}(\Gamma_\rho) \notin \hat{Z}(S)$. Given $z_{S_0}(\Gamma_\rho)$ has some neighbour in at most $k-1$ different stars in $\mathcal{S}$, there must exists some $Q_0 \subseteq T_0$ with size at least $|T_0| - (k-1) - r \geq \frac{|\Gamma_\rho|-1}{2k} - (k-1) - r$ such that $z_{S_0}(S) \in \hat{Z}(S)$ and $N(z_{S_0}(S)) \cap V(S) = \emptyset$ for every $S \in Q_0$.

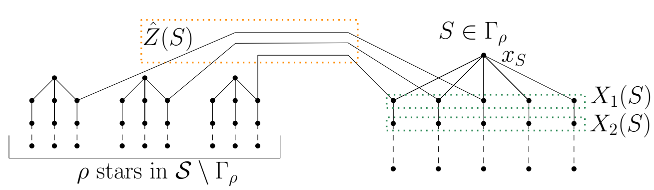

Fig. 4. Some $S \in \Gamma_\rho$. Boxes in green indicate $X_1(S)$ and $X_2(S)$ and in orange $\hat{Z}(S)$.

Assume there exists some $Q_\delta \subseteq \Gamma_\rho$ and stars $A_\delta = \{S_0, \dots, S_\delta\} \subseteq \Gamma_\rho \setminus Q_{\delta-1}$. For every $S \in A_\delta$ there is some distinct vertex z such that for every $S' \in Q_\delta$, $z \in \hat{Z}(S')$; $N(z) \cap V(S') = \emptyset$; and z lies on the shortest path from x'_S to $x_{S'}$.

Consider some arbitrary $S_{\delta+1} \in Q_\delta$ with its respective $z_{S_{\delta+1}}(Q_\delta)$, let $T_{\delta+1} \subseteq Q_\delta$ be those $S \in Q_\delta$ such that $z_{S_{\delta+1}}(S_\delta)$ lies on the shortest path from $x'_{S_{\delta+1}}$ to x'_S. If there are at most r stars $S \in T_{\delta+1}$ such that $z_{S_{\delta+1}}(Q_\delta) \notin \hat{Z}(S)$, then there is some $Q_\delta \subseteq T_\delta$ with size at least $|T_\delta| - (k-1) - r \geq \frac{|Q_\delta| - k(r-1)}{2k}$ such that $z_{S_{\delta+1}}(Q_\delta) \in \hat{Z}(S)$ and $N(z_{S_{\delta+1}}(Q_\delta)) \cap V(S) = \emptyset$ for every $S \in Q_\delta$. If $|\Gamma_\rho| = 2^\rho k^\rho + (r-1)(\sum_{i=1}^{\rho} 2^i k^i) \leq r 2^{\rho+1} k^{\rho+1}$ (for sufficiently large r and ρ), then $|Q_{\rho+1}| \geq r$. Let $\Gamma'_\rho = Q_{\rho+1}$ which concludes the proof of this claim. $\diamond$

Let $S \in \Gamma_\rho$, $\Gamma'_\rho \subseteq \Gamma_\rho$ and z be those obtained from Claim 12.3. Let $\mathcal{P}$ be the set of shortest paths containing z from x'_S to $x'_{S'}$ for every $S' \in \Gamma'_\rho$. There are at most $k-1$ intermediate vertices q on the paths in $\mathcal{P}$ with $q \in \hat{Z}(S')$ for some $S' \in \Gamma'_\rho$. Each such vertex q is adjacent to some star, so q can lie on at most $k-1$ of these paths from z to some $S' \in \Gamma'_\rho$. Hence, there is some $\Gamma_{\rho+1} \subseteq \Gamma'_\rho$ of size at least $|\Gamma'_\rho| - (k-1)^2$ such that for each $S' \in \Gamma_{\rho+1}$, the shortest path from z to $x'_{S'}$ has no vertex from $\bigcup_{S'' \in \Gamma'_\rho} \hat{Z}(S'')$, thus concluding our proof. $\square$

5 The Proof of Theorem 13

Theorem 13. *For any $\ell \geq 3$, there exists some $k \leq 2(2\ell-3)$ such that the class of $C^{\mathrm{E}}_{k*[2\ell]}$-subgraph-free graphs of diameter at most 2 has unbounded treedepth.*

Proof. We set $k = 2(2\ell-3)$. For all $n \in \mathbb{N}$ we construct a $C^{\mathrm{E}}_{k\times[2\ell]}$-subgraph-free graph G_n with diameter at most 2 and treedepth at least $\log(n)$ by taking $2\ell-3$ vertices $x_0, \dots, x_{2\ell-4}$ and a path $P = (p_0, \dots, p_n)$. Consider the binary string $R = 1(10)^{\ell-2}$ consisting of 1 followed by $\ell-2$ repetitions of the string 10. We add the edge $p_i x_j$ if the $i+j \mod (2\ell-3)$th bit of R is 1. See Figure 5. By Fact 4, $\mathrm{td}(G_n) \geq \log(n)$.

Next we argue that G_n has diameter 2 by showing that any pair of vertices has distance at most 2. Each x_i is either adjacent to p_{j-1} or p_j for every $j \in [n]$. Next consider p_i and p_j and observe that p_i is both adjacent to $x_{i \mod (2\ell-3)}$ and $x_{i+1 \mod (2\ell-3)}$. Furthermore, p_j must be either adjacent to $x_{i \mod (2\ell-3)}$ or $x_{i+1 \mod (2\ell-3)}$. Finally, consider x_i and x_j. Note that x_i is adjacent to both p_i and p_{i+1} and x_j must be adjacent to either p_i or p_{i+1}.

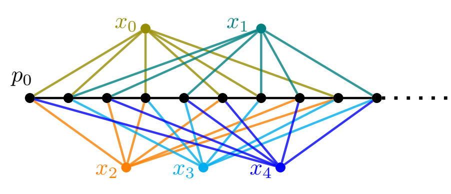

Fig. 5. The construction from Theorem 13 for $\ell = 4$ for which the pattern is $R = 11010$.

Now, for a contradiction, assume G_n contains $C^{\mathrm{E}}_{k\times[2\ell]}$ as a subgraph and $x \in V(G_n)$ is the vertex common to all cycles. As each vertex on P has degree $\ell-1$, we have $x \in \{x_0,\dots,x_{2\ell-4}\}$. Any $C_{2\ell}$ in $G[\{x\}\cup V(P)]$ is of the form $(x,p_i,\dots,p_{2\ell-2},x)$. As R is of length $2\ell-3$, there is either one or two indices $j \in \{i,\dots,2\ell-3\}$ such that x is adjacent to both p_j and p_{j+1}. If there is only one such index j, x cannot be adjacent to both p_i and $p_{i+2\ell-2}$. Hence, there are two. But then there is only one index $j \in \{2\ell-2,\dots,4\ell-5\}$ such that x is adjacent to both p_j and p_{j+1}. Hence, x cannot be adjacent to $p_{i+4\ell-4}$ and thus not both $(x,p_i,\dots,p_{i+2\ell-2},x)$ and $(x,p_{i+2\ell-2},\dots,p_{i+4\ell-4},x)$ are subgraphs of G_n. Therefore, at least every second of the k cycles of the subgraph isomorphic to $C^{\mathrm{E}}_{k\times[2\ell]}$ contains some $y \in \{x_0,\dots,x_{2\ell-3}\}\setminus\{x\}$. As x is not adjacent to any vertex in $\{x_0,\dots,x_{2\ell-3}\}\setminus\{x\}$ we get that each of the vertices $y \in \{x_0,\dots,x_{2\ell-3}\}\setminus\{x\}$ can be in at most one cycle, a contradiction as we may only obtain $2(2\ell-4)+1 < k$ cycles. □

6 Conclusions

We showed that bounding the diameter has a significant impact on the boundedness of width of a graph class, in particular for the subgraph relation and treedepth. We pose some open problems. ", First, recall that Table 1 still contains three open cases: which classes of F-subgraph-free graphs have the diameter-width property for clique-width, treewidth or pathwidth? Towards solving these questions, we can show there are graphs F such that F-subgraph-free graphs of diameter 2 have bounded pathwidth but unbounded treedepth; and also that there are graphs F such that F-subgraph-free graphs of diameter 2 have bounded clique-width but unbounded treewidth (see Appendix L for proofs).

Second, we note that Demaine and Hajiaghayi [12] proved the diameter-treewidth property with even a linear (and optimal) diameter bound. We leave it as an open problem whether our diameter bound in Theorem 1 can be optimized.

Third, recall that we classified boundedness of treedepth of F-subgraph-free graphs of diameter at most d for every constant $d \geq 4$. Completing the classification for $d = 2$ is challenging due to the cases where F is of V-type C^V_x or E-type C^E_x. For $d = 3$, we must consider $F = C_{2r}$ for $r \geq 2$. We showed the treedepth is unbounded for $r \in \{2,3\}$ and $d = 2$, but bounded for $r = 4$ and $d = 3$. Our proof for the latter is an involved case analysis (see Appendix F), which seems not easy to extend to larger values of r. But by use of a computer [15] we can also prove boundedness for $r \in \{5,\dots,12\}$ and $d = 3$. We therefore conjecture:

Conjecture 14. For every $r \geq 4$, the class of C_{2r}-subgraph-free graphs of diameter at most 3 has bounded treedepth.

If a stronger version of Conjecture 14 involving all graphs with one cycle (analogous to Theorem 28) is true, just like the following conjecture (which is supported by all our results so far), then we will obtain a complete classification for $d = 3$.

Conjecture 15. For a graph F with at least two cycles, the class $\mathcal{C}$ of F-subgraph-free graphs of diameter at most 3 has unbounded treedepth.

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A Supplementary Preliminaries

All concepts defined in this section are used in the appendix exclusively.

Treewidth and Pathwidth. Treewidth is an important decompositional parameter for (sparse) graphs that informally measures how close a graph is to being a tree and that has many algorithmic applications, see, e.g. [31] for a brief introduction into treewidth. Moreover, pathwidth is strongly related to treewidth and informally measures the similarity of a graph to a path. Here, we will not formally define treewidth or pathwidth, since we will only use the following well-known facts about treewidth and pathwidth.

Fact 16 *Let $\mathcal{C}$ be any class of graphs that contains an arbitrary large clique or bi-clique as a minor. Then, $\mathcal{C}$ has unbounded treewidth and unbounded pathwidth. Moreover, if $\mathcal{C}$ contains all trees, then it has unbounded pathwidth.*

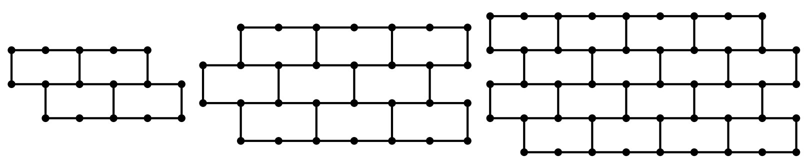

Fig. 6. A wall of height 2, 3 and 4, respectively.

Clique-Width. We will not formally define clique-width, but refer to e.g. the survey [10] for more details. We will need the following facts. The first one is well known and follows immediately from the definition of clique-width. Here, we denote the complement of a graph G by $\overline{G}$.

Fact 17 *For every graph G, it holds that* $\mathrm{cw}(G) \leq 2\mathrm{cw}(\overline{G})$.

Finally, we need the well known notion of a wall, which is illustrated in Figure 6 which we will not formally define (see, e.g. [5] for a formal definition). It is well known that walls have unbounded clique-width (see e.g. [24]). A k-subdivided wall is the graph obtained from a wall by subdividing it each edge k times.

Fact 18 ([27]) *For every $k \geq 0$, the class of k-subdivided walls has unbounded clique-width.*

Finally, we need the following facts, providing the relationships between the width parameters defined thus far.

Fact 19 ([6]) *For every graph G, it holds that* $\mathrm{cw}(G) \leq 3 \times 2^{\mathrm{tw}(G)-1}$ *and* $\mathrm{tw}(G) \leq \mathrm{pw}(G) \leq \mathrm{td}(G)$.

B The Missing Proofs from Table 1

We first consider the induced subgraph relation. Let F be a graph and let $\mathcal{C}$ be the class of F-free graphs with diameter at most d. In this section, we first provide dichotomies characterizing exactly when $\mathcal{C}$ has bounded treedepth, pathwidth, treewidth, or clique-width. Note that the theorem shows that $d = 2$ is tight.

Theorem 20. *Let F be a graph. The class of F-free graphs of diameter at most d, for $d \geq 1$, has bounded treedepth/pathwidth/treewidth if and only if either:*

- *$d = 1$ and F is a clique or*
- *$d \geq 2$ and $F \in \{K_1, K_2\}$.*

Proof. Let F be a graph such that the class $\mathcal{F}$ of F-free graphs of diameter at most d has bounded treedepth/pathwidth/treewidth. If F is not a clique, then K_n is F-free, and therefore $\mathcal{F}$ contains graphs with diameter 1 and unbounded treedepth/pathwidth/treewidth. Note that this completes the proof for $d = 1$, since there are finitely many connected F-free graphs with diameter at most 1, if F is a clique. If $F = K_i$ for $i \geq 3$, then $K_{n,n}$ is F-free, and therefore $\mathcal{F}$ contains graphs of diameter 2 with arbitrarily large treedepth/pathwidth/treewidth. Hence, $F \in \{K_1, K_2\}$. Since the class of connected K_i-free graphs is finite if $i \in \{1, 2\}$, the theorem follows. □

The following theorem now provides our dichotomy result for clique-width. Note that $d = 2$ is tight.

Theorem 21. *Let $d \geq 2$. For a graph F, the class of F-free graphs of diameter at most d has bounded clique-width if and only if F is an induced subgraph of P_4.*

Proof. Suppose that F contains an induced cycle C_k. Let G be a k-subdivided wall and let $G' = G \bowtie K_1$ and note that G' has diameter 2. Observe that G is C_k-free and if $k > 3$, then G' is also C_k-free and therefore F-free. By Fact 18, the class of k-subdivided walls has unbounded clique-width, so the clique-width of G' can be arbitrarily large. Therefore, if F contains a cycle, then we may assume that this cycle is a C_3.

We will now construct a class of C_3-free graphs of diameter 2 with unbounded clique-width. Namely, for every h, we construct the graph G_h that is obtained from the wall W_h of height h as follows. Let $\{R, B\}$ be a proper 2-colouring of W_h. We add two new vertices r and b together with the edges (r, v) for every $v \in R$, (b, v) for every $v \in B$ and the edge (r, b). Moreover, for every $w_i \in V(W_h)$ we add a vertex x_i. If $w_i \in R$, then x_i is adjacent to every $v \in B \setminus N_{W_h}(w_i)$, to x_j for every $w_j \in N_{W_h}(w_i)$ and to w_i; see Figure 7 for an illustration. We describe the neighbourhood of x_i for $w_i \in B$ similarly interchanging the sets R and B. This completes the construction of G_h.

We first show that G_h does not contain an induced C_3. This is because the graph $G_h[V(W_h) \cup \{r, b\}]$ is bipartite with bipartition $\{R \cup \{b\}, B \cup \{r\}\}$ and

	w_i	x_i
w_n	adjacent	w_i
x_n	w_n	adjacent
w_b	x_i	adjacent
x_b	adjacent	w_i
w_r	r	$\in N_{W_h}(w_r) \setminus N_{W_h}(w_i)$
x_r	$\in N_{W_h}(w_i) \setminus N_{W_h}(w_r)$	$\in B \setminus (N_{W_h}(w_i) \cup N_{W_h}(w_j))$
r	adjacent	x_i
b	w_n	w_b

Table 2. Table used in the proof of Theorem 21 to show that G_h has diameter 2. Here, $w_i \in R$ and w_n, w_b, and w_r are arbitrary vertices in $N_{W_h}(w_i)$, $B \setminus N_{W_h}(w_i)$, and $R \setminus \{w_i\}$, respectively. Moreover, x_i, x_n, x_b, x_r are the vertices corresponding to w_i, w_n, w_b, and w_r in X, respectively. Each cell of the table either states that the two vertices corresponding to the row and column are adjacent or provides a common neighbour of both vertices.

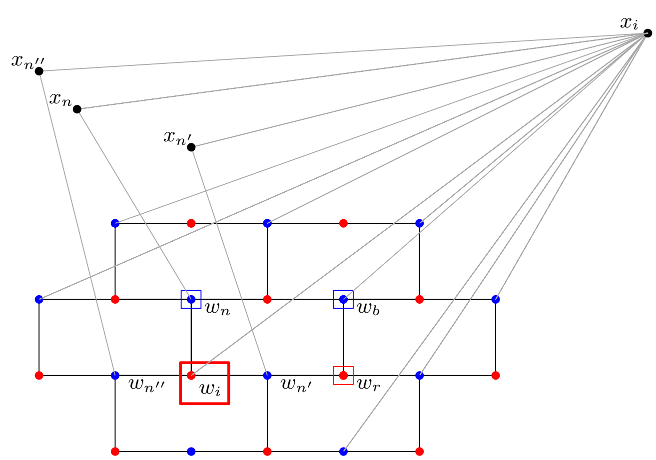

Fig. 7. The construction used in Theorem 21 to provide a C_3-free class of graphs with diameter 2 and unbounded clique-width. The illustration shows a wall W with a proper 2-colouring using colours red and blue together with one additional vertex x_i for every $w_i \in V(W)$.

therefore C_3-free. Therefore, any C_3 in G_h would have to have at least one vertex from $X = \{ x_i \mid w_i \in V(W_h) \}$. Since no vertex in X is adjacent to both end-points edge of W_h, there can be no triangle containing exactly one vertex from X. Similarly, since no two adjacent vertices in X have a common neighbour in W_h, there is no C_3 containing exactly two vertices from X. Finally, any C_3 containing three vertices from X would give rise to a C_3 in W_h, which cannot exist since W_h is bipartite.

We show next that G_h has diameter 2. To see this first consider a vertex w_i and assume without loss of generality that $w_i \in R$. In relation to w_i, every vertex of W_h is in one of the following three sets: $N_{W_h}(w_i)$, $B \setminus N_{W_h}(w_i)$ or $R \setminus \{w_i\}$. Let w_n, w_b and w_r be arbitrary vertices from each of these sets respectively. Table 2 now gives, for each pair of vertices, either a common neighbour or shows that they are adjacent, which shows that every pair of vertices in G_h (apart from r and b, which are adjacent) have distance at most 2. Since G_h is C_3-free and therefore F-free, this completes the case when F contains an induced C_3.

It remains to consider the case when F does not contain an induced cycle, i.e. when it is a forest. If F is not an induced subgraph of P_4, then it contains an induced $2P_2$ or $3P_1$. A wall is $\{C_3, C_4\}$-free, so the complement of a wall is $\{2P_2, 3P_1\}$-free, and therefore F-free. Complements of walls have diameter 2 and they have arbitrarily large clique-width by Facts 17 and 18.

We may therefore assume that F is an induced subgraph of P_4. It is readily seen and well-known that P_4-free graphs have clique-width at most 2 even without the restriction on diameter. ◻

We next present our dichotomy for minor-closed classes of graphs of bounded diameter. That is, we provide dichotomies characterizing exactly for which graph classes $\mathcal{F}$, the $\mathcal{F}$-minor-free class of graphs of diameter at most d has bounded treedepth or bounded clique-width, respectively. W

Recall that a graph G is an apex planar graph if there is a vertex $v \in V(G)$ such that $G - v$ is planar. Analogously, G is an *apex forest* or *apex linear forest* if for some $v \in V(G)$, $G - v$ is a forest or linear forest, respectively. Note that a forest is an cyclic undirected graph, and a linear forest is a disjoint union of paths.

Recall that a class of graphs $\mathcal{C}$ has the diameter-treewidth property if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that every subgraph of a graph in $\mathbb{C}$ with diameter at most d has treewidth at most $f(d)$.

Theorem 22 (Theorem 1 in [16]). *Let $\mathcal{C}$ be a minor-closed family of graphs. Then, $\mathcal{C}$ has the diameter-treewidth property if and only if $\mathcal{C}$ does not contain all apex planar graphs.*

We obtain the following as a corollary. Note that $d = 2$ is tight.

Corollary 23. *Let $d \geq 2$. For a class of graphs $\mathcal{F}$, the class $\mathcal{C}$ of $\mathcal{F}$-minor-free graphs of diameter at most d has bounded treewidth/clique-width if and only if $\mathcal{F}$ contains some apex planar graph.*

Proof. First assume that $\mathcal{F}$ contains some apex planar graph. Then $\mathcal{C}$ has bounded treewidth (and therefore also bounded clique-width Fact 19) by Theorem 22. On the other hand, if $\mathcal{F}$ does not contain an apex planar graph, then $\mathcal{C}$ contains all apex planar graphs of diameter at most d and therefore $\mathcal{C}$ contains any wall with one added universal vertex, which due to Fact 18 and Fact 19 implies that $\mathcal{C}$ does not have bounded treewidth or bounded clique-width. □

Theorem 24 (Theorem 1.5 in [14]). *Let $\mathcal{C}$ be a minor-closed family of graphs. Then $\mathcal{C}$ has bounded local pathwidth if and only if $\mathcal{C}$ does not contain all apex forests.*

We obtain the following corollary, where we note that $d = 2$ is tight.

Corollary 25. *Let $d \geq 2$. For class of graphs $\mathcal{F}$, the class $\mathcal{C}$ of $\mathcal{F}$-minor-free graphs of diameter at most d has bounded pathwidth if and only if $\mathcal{F}$ contains some apex forest.*

Proof. First assume that $\mathcal{F}$ contains some apex forest. Then $\mathcal{C}$ has bounded pathwidth by Theorem 24. On the other hand, if $\mathcal{F}$ does not contain an apex forest, then $\mathcal{C}$ contains all apex forests of diameter at most d and therefore $\mathcal{C}$ contains $T \bowtie K_1$, where T is any tree, which has diameter at most 2 and which due to Fact 16 implies that $\mathcal{C}$ does not have bounded pathwidth. □

An analogous result holds for treedepth as well, where again $d = 2$ is tight.

Theorem 26. *Let $d \geq 2$. For a class of graphs $\mathcal{F}$, the class $\mathcal{C}$ of $\mathcal{F}$-minor-free graphs of diameter at most d has bounded treedepth if and only if $\mathcal{F}$ contains an apex linear forest.*

Proof. We start by showing the reverse direction of the statement. Let F be a linear apex forest contained in $\mathcal{F}$. We claim that for every d, there is a constant $c(d, |V(F)|)$ such that every graph with diameter at most d and treedepth at least $c(d, |V(F)|)$ contains $P_{|V(F)|} \bowtie K_1$ as a minor. Since $P_{|V(F)|} \bowtie K_1$ contains every apex linear forest of size at most $|V(F)| + 1$ as a minor (and therefore also F), this shows the reverse direction of the statement given in the theorem.

By Corollary 6, there is a constant $c(|V(F)|, |V(F)|, d|V(F)|)$ such that every graph with treedepth at least $c(|V(F)|, |V(F)|, d|V(F)|)$ either contains $K_{|V(F)|,|V(F)|}$ or $P_{d|V(F)|}$ as an induced subgraph. Clearly, $K_{|V(F)|,|V(F)|}$ contains $P_{|V(F)|} \bowtie K_1$ as a minor, as required. So suppose that G is a graph with diameter at most d containing $P = P_{d|V(F)|}$ as an induced subgraph.

Now consider any vertex $v \in V(G) \setminus V(P)$, which must exists because G has diameter at most d, and let R_v be the set of all vertices reachable from v in $G \setminus V(P)$. Then, R_v is connected and, moreover, if P contains a subpath P' of d vertices that have no neighbour in R_v, then v has distance at least $d + 1$ to some vertex on P', contradicting our assumption that G has diameter at most d. But then, P can have length at most $d|V(P)|$, since otherwise we can obtain $K_1 \bowtie P_{|V(P)|}$ as a minor of G by first contracting all edges in R_v and then contracting edges on P adjacent to vertices not adjacent to R_v.

If instead $\mathcal{F}$ does not contains any apex linear forests, then it contains $P_n \bowtie K_1$ for every n, which has diameter at most 2 and unbounded treedepth because of Fact 4. □

C The Missing Proof from Section 3

Lemma 11. *For any $\ell \in \mathbb{N}$, the class of H_2^ℓ-subgraph-free graphs of diameter at most 4 has bounded treedepth.*

Proof. For some $\ell \geq 0$, let G be some H_2^ℓ-subgraph-free graph of diameter at most 4. We claim $td(G) \leq c((4\ell+1)/2,(4\ell+1)/2,16(\ell+1)+1)$. Suppose for contradiction $td(G) > c((4\ell+1)/2,(4\ell+1)/2,16(\ell+1)+1)$. As $K_{(4\ell+1)/2,(4\ell+1)/2}$ contains H_2^ℓ as a subgraph, G cannot contain a large complete bipartite. From Corollary 6, G contains some induced path $P = (p_1, \dots, p_{16(\ell+1)})$ of length $16(\ell+1)$.

We first observe the following.

Claim 11.1. *For any two indices i, j with $\ell \leq i < j \leq 16(\ell+1) - \ell$ and $|i-j| \geq 2\ell+1$ the shortest path between p_i and p_j must have length at least 3.*

Proof of Claim: As P is induced there cannot be a path of length 1 from p_i to p_j. Furthermore, any path of length 2 from p_i to p_j yields H_2^ℓ as a subgraph with degree 3 vertices p_i and p_j. ◇

We now argue that the shortest path between p_i and p_j for i, j of sufficient distance has to be of length 4. To prove this we use the following claim.

Claim 11.2. *For any path (p_i, x, x', p_j) with $2\ell \leq i < j \leq 16(\ell+1) - 2\ell$ and $|i-j| \geq 3\ell+1$ it holds that x has no neighbours besides p_i on P and x' has no neighbours beside p_j on P.*

Proof of Claim: As P is induced $x, x' \notin P$. Furthermore, we have that $N(x) \cap \{p_1, \dots, p_{i-(2\ell+1)}, p_{i+2\ell+1}, \dots, p_{16(\ell+1)}\} = \emptyset$ by Claim 11.1.

Suppose for contradiction x is adjacent to p_k with $k \neq i$ and $i-2\ell \leq k \leq i+2\ell$. Observe that we can choose two disjoint path P^i and P^ℓ of length $\ell-1$ within the vertices $\{p_{i-2\ell}, \dots, p_{i+2\ell}\}$. Using P^i and P^ℓ it is easy to observe that we found H_2^ℓ as a subgraph with degree 3 vertices x and p_j. Hence, x cannot have any neighbours beside x_i on P. The proof for x' is symmetric. ◇

Claim 11.3. *For any two indices i, j with $2\ell \leq i < j \leq 16(\ell+1) - 5\ell - 1$ and $|i-j| \geq 3\ell+1$ the shortest path between p_i and p_j must have length 4.*

Proof of Claim: Suppose for a contradiction that i, j are two indices with $2\ell \leq i < j \leq 16(\ell+1) - 5\ell - 1$ and $|i-j| \geq 3\ell+1$ and there is a path (p_i, x, x', p_j). Let $k = j+3\ell+1$ and hence $k \leq 16(\ell+1)-2\ell$. The shortest path Q from x to p_k must contain some vertex $p_m \in \{p_{i-\ell}, \dots, p_{i+\ell}\} \cup \{p_{j-\ell}, \dots, p_{j+\ell}\}$ else there is a H_2^ℓ with degree 3 vertices x and p_j. As m and k satisfy the conditions of Claim 11.1, the subpath of Q from p_m to p_k must have length at least 3. As Q is a shortest path this implies that x is adjacent to p_m and hence $k = i$ by Claim 11.2. Hence, there is a path of the form (x, p_i, y, y', p_k). We argue in a similar way for x'. Let Q' be the shortest path from x' to p_{k+2}. To avoid H_2^ℓ with degree 3 vertices x' and p_i, Q' must contain a vertex $p_{m'} \in \{p_{i-\ell}, \ldots, p_{i+\ell}\} \cup \{p_{j-\ell}, \ldots, p_{j+\ell}\}$. We get that x' is adjacent to $p_{m'}$ as Q' has length at most 4 and the subpath of Q' from $p_{m'}$ to p_{k+2} must have length at least 3 by Claim 11.1. By Claim 11.2 we get that $j = m'$ and hence there is a path $(x', p_j, z, z', p_{k+2})$. As y, y', z, z' must be pairwise disjoint by Claim 11.2 this yields a H_2^ℓ as a subgraph with degree 3 vertices p_k and p_{k+2}. Hence, the shortest path from p_i to p_j cannot be 3 and therefore it must be 4 as G has diameter at most 4. ⋄

By Claim 11.3 the shortest path from $p_{2\ell}$ to $p_{5\ell+1}$ must have length 4 and contain 3 vertices not on P. The same holds for the path from $p_{5\ell+3}$ to $p_{8\ell+4}$ and the path from $p_{8\ell+6}$ to $p_{11\ell+7}$. Let us denote these paths by $(p_{2\ell}, x, x', x'', p_{5\ell+1})$ and $(p_{5\ell+3}, y, y', y'', p_{8\ell+4})$ and $(p_{8\ell+6}, z, z', z'', p_{11\ell+7})$. If x, x', x'', y, y', y'' are all distinct then there is some H_2^ℓ with degree 3 vertices $p_{5\ell+1}$ and $p_{5\ell+3}$. This means $x'' = y$ as all other combinations lead to a shortest path of length at most 3 between some pair of vertices of P with distance at least $3\ell + 1$ contradicting Claim 11.3. This also holds for y, y', y'', z, z', z'' meaning $y'' = z$. However, this leads to a H_2^ℓ with degree 3 vertices y and z. Thus a contradiction. □

D The Missing Proof of Theorem 27

We recall that Erdős, Rényii and Sós [17] showed how a family of C_4-subgraph-free graphs with diameter 2 can be constructed from a polarity of a projective plane. Making the observation that this family has unbounded minimum degree and so also unbounded treewidth we observed that the class $\mathcal{C}$ of C_4-subgraph-free graphs of diameter 2 has unbounded treedepth. Considering geometries of higher dimensions an analogous result can be shown for diameter 3.

Theorem 27. *The class $\mathcal{C}$ of C_6-subgraph-free graphs of diameter 3 has unbounded treedepth.*

Proof. Let $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a geometry defined by a set of points $\mathcal{P}$, lines $\mathcal{L}$ and incidence relation $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{L}$. The corresponding instance graph G_I has vertices $\mathcal{P} \cup \mathcal{L}$ with $E(G_I) = \mathcal{I}$. In particular we consider consider regular generalised m-gons, these are finite geometries such that their incidence graph is r-regular with diameter m and girth $2m$. In particular we consider where $m = 4$, these are called generalised quadrangles. A polarity π is a bijective function mapping points to lines and lines to points which both an involution and incidence is preserved i.e $\forall p \in \mathcal{P}, \forall l \in \mathcal{L}$ then $(\pi(l), \pi(p)) \in \mathcal{I}$ if and only if $(p, l) \in \mathcal{I}$. From a finite geometry $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ and polarity π the polarity graph G_π has the set of vertices $\mathcal{P}$ and edges $\{\{p, q\} : p, q \in \mathcal{P}, p \neq q, (p, \pi(q)) \in \mathcal{I}\}$. While such a polarity does not exist for all $(q + 1)$-regular m-gons, a generalised quadrangles with a polarity exists where $q = p^{2\alpha+1}$, see [4,36]. We denote this family of polarity graphs by $\mathcal{G}_{GQ}$ respectively. As the polarity of a $(q + 1)$-regular m-gon has minimum degree q, $\mathcal{G}_{GQ}$ has unbounded treewidth. We claim $\mathcal{G}_{GQ}$ is

C_6-subgraph-free with diameter 3, more generally, if π is a polarity of some regular generalised m-gon $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ then its corresponding polarity graph G_π is $C_{2(m-1)}$-subgraph-free with diameter $m-1$. If G_π contained some $C_{2(m-1)}$ with vertices $(v_1, v_2, \ldots, v_{2(m-1)-1}, v_{2(m-1)})$ then G_I contains the cycle with vertices $(v_1, \pi(v_2), \ldots, \pi(v_{2(m-1)-1}), v_{2(m-1)})$ as G_I has girth m, G_π is $C_{2(m-1)}$-subgraph-free. As G_I is a bipartite graph of diameter m, for any $\ell \in \mathcal{L}$ and $p \in \mathcal{P}$ if $(p, \ell) \notin \mathcal{I}$, there must be some path of length at most $m-1$ between p and ℓ. Without loss this path can be given by $p, \ell_1, p_2, \ldots, p_{m-2}, \ell$. Let u, v be a pair of non-adjacent vertices of G_π, we claim there must be a path of length at most $m-1$ between them. As $(u, \pi(v)) \notin \mathcal{I}$ from above there must exist points and lines forming the path $u, \ell_1, p_2, \ldots, p_{m-2}, \pi(v)$ in G_I. This leads to the path $u, \pi(\ell_1), p_2, \ldots, p_{m-2}, v$ in G_π with length at most $m-1$. □

E The Missing Proof of Theorem 28

Theorem 5 allows (apart from the exception of C_4) to classify for which cycles F the class of F-subgraph-free graphs of bounded diameter 2 has bounded treedepth.

Theorem 28. *Let F be any graph containing exactly one cycle. The class of F-subgraph-free graphs of diameter at most 2 has bounded treedepth if and only if F does not contain C_4, is bipartite and a subgraph of $P_n \bowtie K_1$ for some large $n \in \mathbb{N}$.*

Proof. First note that the forward direction follows directly from Observation 7 and Theorem 9.

Any graph F that contains exactly one cycle of even length larger than 4 that is a subgraph of $P_n \bowtie K_1$ for some $n \in \mathbb{N}$ can be constructed by taking a single vertex v, and k paths of lengths at most ℓ, making v adjacent to one vertex on each path and choosing one paths for which v has a second neighbour of distance $m-2$ from the first for some integers $k, \ell \geq 0$ and even $m > 4$. We claim that any class of F-subgraph-free graphs of diameter at most 2 has treedepth $< c(k(\ell+1)+1, k(\ell+1), 2\ell(k+1)^2$.

Towards a contradiction assume that there exists some graph F-subgraph-free graph, G, with diameter 2 and treedepth at least $c(k(\ell+1)+1, k(\ell+1), 2\ell(k+1)^2)$. As $K_{k(\ell+1)+1,k(\ell+1)}$ contains F, G must contain some induced path $P = (p_0, \cdots, p_{2\ell(k+1)^2})$ by Corollary 6. Let $P^\ell = \{p_i : i \equiv 0 \mod (2\ell), i \neq 0\}$ and note that $|P^\ell| = (k+1)^2$.

Claim 28.1. *There exists some vertex of G with at least $k+1$ neighbours in P^ℓ.*

Proof of Claim: We proof this by contradiction and hence assume that every vertex of G has at most k neighbours in P^ℓ. Let x be the common neighbour of the pair $(p_\ell, p_{\ell+m-2})$. For any vertex $p_i \in P^\ell$ we define a vertex y_i which is a neighbour of x and for any distinct vertices $p_i, p_j \in P^\ell$ we define a vertex z_{ij} and a path Q_{ij} of length at least $2\ell-1$ with middle vertex y_i as follows. Note that there is a path of length at most 2 from p_i to x. We set y_i to be p_i in case x is adjacent to p_i or we choose y_i to be the common neighbour of x and p_i. There is also a path of length at most 2 from y_i to p_j. We set z_{ij} to be either p_j if y_i is adjacent to p_j or to be the common vertex of y_i and p_j. Notice that $y_i \neq z_{ij}$ and both y_i and z_{ij} are adjacent to some vertex in P^ℓ. We can choose $Q_{i,j}$ fitting the criteria containing vertices from $\{p_i, \dots, p_{i+\ell}), y_i, z_{i,j}, p_j, \dots, p_{j+\ell}\}$. Given that no vertex is adjacent to $k+1$ vertices in P^ℓ, there are at least $|P^\ell| - 2k$ vertices $p_{i'} \in P^\ell$ such that $y_{i'} \neq y_i$ and $y_{i'} \neq z_{ij}$. Furthermore, there are at least $|P^\ell| - 2k - 1$ vertices $p_{j'} \neq p_{i'}$ such that $z_{i'j'} \neq y_i$, $z_{i'j'} \neq z_{ij}$ and Q_{ij} and $Q_{i'j'}$ are disjoint. Inductively, we obtain that there must be at least $k-1$ pairs $p_i, p_j \in P^\ell$ for which Q_{ij} are pairwise disjoint and are disjoint from $(p_0, \dots, p_{2\ell})$ as $|P^\ell| \geq (k+1)^2$. Hence, we obtain F as a subgraph with high degree vertex x. ◇

Let x be some vertex in G with at least $k+1$ neighbours in P^ℓ and X be the set of neighbours of x in S^ℓ. If $p_i \in X$ then $p_{i+m-2} \notin X$ else F is contained as a subgraph. We claim that $p_{i+2} \notin X$. Assume otherwise. As p_i and p_{i+m-2} must have a common neighbour $x' \neq x$ we obtain $(x', p_i, x, p_{i+2}, \dots, p_{i+m-2}, x')$ as a subgraph. As this cycle of length m contains x and does not overlap with $(p_{j-\ell+1}, \dots, p_{j+\ell})$ for any $p_j \in X$, $p_j \neq p_i$. Hence, choose any $p_i, p_{i'} \in X$. By our previous argument, the common neighbour of p_{i+2} and $p_{i'+m-6}$ must have a common neighbour $y \neq x$. But then $(x, p_i, p_{i+1}, p_{i+2}, y, p_{i'+m-6}, \dots, p_{i'}, x)$ is a C_m which is disjoint from any $(p_{j-\ell+1}, \dots, p_{j+\ell})$ for $p_j \in X$ different from p_i and $p_{i'}$. Hence, G contains a copy of F with high degree vertex x. □

F The Missing Proof of Theorem 29

Theorem 29. *The class of C_8-subgraph-free graphs of diameter d at most 3 has bounded treedepth.*

Proof. Assume G is a C_8-subgraph-free graph of diameter at most 3 and treedepth at least $c(4,4,42)$. Note that G cannot contain a large complete bipartite subgraph as $K_{4,4}$ contains C_8 as a subgraph. Hence, by Corollary 6, G must contain $P_\ell = (p_0, \dots, p_\ell)$ where $\ell = 42$ as an induced subgraph.

First observe that for any $i \in [\ell - 6]$ there cannot be a vertex x not on P which is adjacent to both p_i and p_{i+6} since G is C_8-subgraph free. Additionally, for any $i \in [\ell - 5]$ there cannot be x, y not on P such that (p_i, x, y, p_{i+5}) is a path in G as $(p_{i+5}, x, y, p_i, \dots, p_{i+5})$ would yield a C_8. We say that $i \in [\ell - 5]$ is of

distance 5 **type** 1 if there is x such that (p_i, x, p_{i+5}) is a path in G;
distance 5 **type** 2 if there is x such that (p_{i+1}, x, p_{i+5}) is a path in G;
distance 5 **type** 3 if there is x such that (p_i, x, p_{i+4}) is a path in G.

Since G has diameter 3 we have to ensure that the distance between p_i and p_{i+5} is at most 3. Therefore, any $i \in [\ell - 5]$ has to be either of distance 5 type 1, 2 or 3.

Similarly, if we consider any two vertices on P of distance 6 we get the following types. We say that $i \in [2, \ell - 7]$ has

distance 6 **type** 1 There are x, y not on P such that (p_i, x, y, p_{i+6}) is a path in G;
distance 6 **type** 2 There is x such that (p_i, x, p_{i+5}) is a path in G;
distance 6 **type** 3 There is x such that (p_i, x, p_{i+7}) is a path in G;
distance 6 **type** 4 There is x such that (p_{i-1}, x, p_{i+6}) is a path in G;
distance 6 **type** 5 There is x such that (p_{i+1}, x, p_{i+6}) is a path in G.

From our above observation it also follows that every $i \in [2, \ell - 7]$ has to be of distance 6 type 1, 2, 3, 4 or 5.

We now use the types defined above to effectively consider all possible cases of how the neighbourhood (on P) of vertices not contained on P can look like. In the following we show three claims forbidding certain configurations. Using the claims below, it is straight forward to prove that G must have bounded treedepth.

Claim 29.1. *There is no vertex $x \in V(G)$ such that x is adjacent to p_i, p_{i+5} and p_{i+7} or x is adjacent to p_i, p_{i+2}, p_{i+7} for some $i \in [10, \ell - 10]$.*

Proof of Claim: Assume the statement is not true and there is $x \in V(G)$ and $i \in [10, \ell - 10]$ such that $p_i, p_{i+5}, p_{i+7} \in N_G(x)$. The case when $p_i, p_{i+2}, p_{i+7} \in N_G(x)$ is symmetric. We show that G must contain C_8, a contradiction. Note that $p_{i-1}, p_{i+1}, p_{i+6} \notin N_G(x)$ as otherwise $(x, v_{i-1}, \ldots, v_{i+5}, x)$ or $(x, v_{i+1}, \ldots, v_{i+7}, x)$ or $(x, v_i, \ldots, v_{i+6}, x)$ is a C_8 in G.

First, assume that $i + 1$ has distance 5 type 1. As p_{i+1} is not adjacent to x, there exists a vertex $y \neq x$ not on P such that (p_{i+1}, y, p_{i+6}) is a path in G. In this case we have $p_{i+4} \notin N_G(y)$ as otherwise $(x, p_i, p_{i+1}, y, p_{i+4}, \ldots, p_{i+7}, x)$ is a C_8 in G; $p_{i+4} \notin N_G(x)$ as otherwise $(y, p_{i+1}, \ldots, p_{i+4}, x, p_{i+5}, p_{i+6}, y)$ is a C_8 in G; $p_{i-2} \notin N_G(x)$ as otherwise $(x, p_{i-2}, \ldots, p_{i+1}, y, p_{i+6}, p_{i+7}, x)$ is a C_8 in G. Now consider the distance 6 type of $i-2$ (see Figure 8 for an illustration of the different cases). If $i-2$ has distance 6 type 1, there are z_1, z_2 not on P and different from x (as $p_{i-2} \notin N_G(x)$ and $p_{i+4} \notin N_G(x)$) such that $(p_{i-2}, z_1, z_2, p_{i+4})$ is a path in G. Then $(p_{i-2}, z_1, z_2, p_{i+4}, p_{i+5}, x, p_i, p_{i-1}, p_{i-2})$ is a C_8 in G. In case $i - 2$ has distance 6 type 2, there is $z \neq x$ (as $p_{i-2} \notin N_G(x)$) such that (p_{i-2}, z, p_{i+3}) is a path in G. Then $(p_{i-2}, z, p_{i+3}, p_{i+4}, p_{i+5}, x, p_i, p_{i-1}, p_{i-2})$ is a C_8 in G. If $i - 2$ has distance 6 type 3 then there is $z \neq x$ (as $p_{i-2} \notin N_G(x)$) such that (p_{i-2}, z, p_{i+5}) is a path in G. Then $(p_{i-2}, z, p_{i+5}, p_{i+6}, p_{i+7}, x, p_i, p_{i-1}, p_{i-2})$ is a C_8 in G. If $i-2$ has distance 6 type 4 then there is $z \neq x$ (as $p_{i+4} \notin N_G(x)$) such that (p_{i-3}, z, p_{i+4}) is a path in G. Then $(p_{i-3}, z, p_{i+4}, p_{i+5}, x, p_i, \ldots, p_{i-3})$ is a C_8 in G. If $i-2$ has distance 6 type 5 then there is $z \neq y$ (as $p_{i+4} \notin N_G(y)$) such that (p_{i-1}, z, p_{i+4}) is a path in G. Then $(p_{i-1}, z, p_{i+4}, p_{i+5}, p_{i+6}, y, p_{i+1}, p_i, p_{i-1})$ is a C_8 in G.

Next, in case $i+1$ has distance 5 type 2, there is $y \neq x$ (as $p_{i+6} \notin N_G(x)$) such that (p_{i+2}, y, p_{i+6}) is a path in G and hence $(y, p_{i+2}, \ldots, p_{i+5}, x, p_{i+7}, p_{i+6}, y)$ is a C_8 in G.

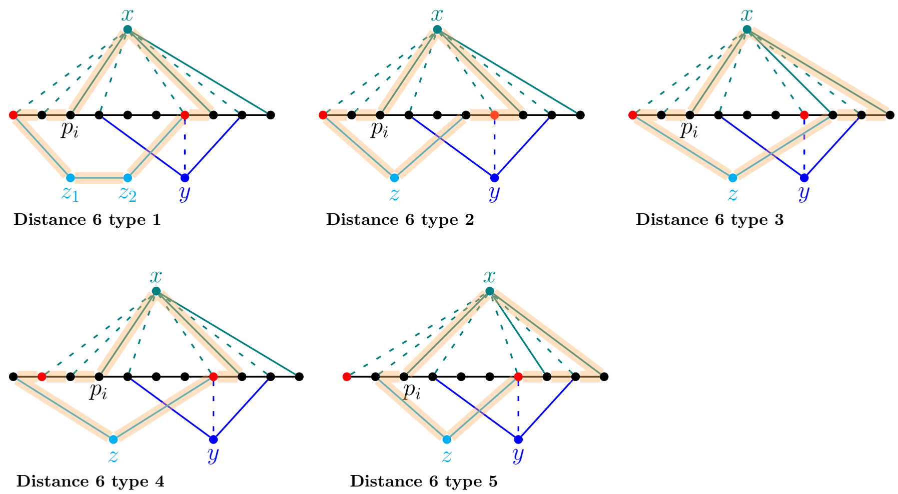

Fig. 8. The five different distance 6 types of $i-2$ in the case when $i+1$ has distance 5 type 1 in the proof of Claim 29.1.

Finally, assume that $i+1$ has distance 5 type 3. As $p_{i+1} \notin N_G(x)$, there is $y \neq x$ such that (p_{i+1}, y, p_{i+5}) is a path in G. Note that in this case $p_{i+2} \notin N_G(x)$ as otherwise $(p_{i+2}, y, p_{i+5}, p_{i+6}, p_{i+7}, x, p_i, p_{i+1}, p_{i+2})$ is a C_8 in G and $p_{i+8} \notin N_G(x)$ as in this case $(p_{i+8}, x, p_i, p_{i+1}, y, p_{i+5}, \ldots, p_{i+8})$ is a C_8 in G. We now consider the distance 6 type of $i+2$ (see Figure 8 for an illustration of the different cases). If $i+2$ has distance 6 type 1, then there are z_1, z_2 not on P and different from x (as $p_{i+2} \notin N_G(x)$ and $p_{i+8} \notin N_G(x)$) such that $(p_{i+2}, z_1, z_2, p_{i+8})$ is a path in G. Then $(p_{i+2}, z_1, z_2, p_{i+8}, p_{i+7}, x, p_i, p_{i+1}, p_{i+2})$ is a C_8 in G. In case $i+2$ has distance 6 type 2, there is $z \neq x$ (as $p_{i+2} \notin N_G(x)$) such that (p_{i+2}, z, p_{i+7}) is a path in G. Then $(p_{i+2}, z, p_{i+7}, p_{i+6}, p_{i+5}, x, p_i, p_{i+1}, p_{i+2})$ is a C_8 in G. In case $i+2$ has distance 6 type 3, there is $z \neq x$ (as $p_{i+2} \notin N_G(x)$) such that (p_{i+2}, z, p_{i+9}) is a path in G. Then $(p_{i+2}, z, p_{i+9}, p_{i+8}, p_{i+7}, x, p_i, p_{i+1}, p_{i+2})$ is a C_8 in G. If $i+2$ has distance 6 type 4, there is $z \neq x$ (as $p_{i+8} \notin N_G(x)$) such that (p_{i+1}, z, p_{i+8}) is a path in G. Then $(p_{i+1}, z, p_{i+8}, \ldots, p_{i+5}, x, p_i, p_{i+1})$ is a C_8 in G. Finally, if $i+2$ has distance 6 type 5, there is $z \neq x$ (as $p_{i+8} \notin N_G(x)$) such that (p_{i+3}, z, p_{i+8}) is a path in G. Then $(p_{i+3}, z, p_{i+8}, p_{i+7}, x, p_i, \ldots, p_{i+3})$ is a C_8 in G. We conclude that the claim holds. ◇

Claim 29.2. *There is no vertex $x \in V(G)$ such that x is adjacent to p_i, p_{i+1} and p_{i+4} or x is adjacent to p_i, p_{i+3} and p_{i+4} for some $i \in [10, \ell-10]$.*

Proof of Claim: Assume the claim is false and there is $x \in V(G)$ and $i \in [10, \ell - 10]$ such that $p_i, p_{i+1}, p_{i+4} \in N_G(x)$. The case where $p_i, p_{i+3}, p_{i+4} \in N_G(x)$ is symmetric. To avoid C_8 we directly obtain that $p_{i-2}, p_{i+6}, p_{i+7} \notin N_G(x)$.

First, assume that $p_{i-1} \notin N_G(x)$. As G has diameter at most 3 there must be a path Q from p_{i-1} and p_{i+6} of length at most 3. As $p_{i-1}, p_{i+6} \notin N_G(x)$ the

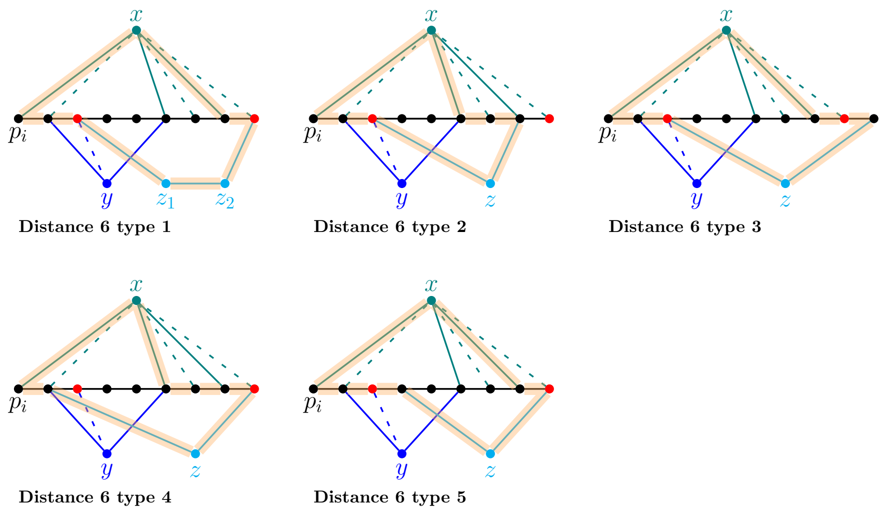

Fig. 9. The five different distance 6 types of $i+2$ in the case when $i+1$ has distance 5 type 3 in the proof of Claim 29.1.

path Q cannot contain x (both inner vertices are adjacent to p_{i-1} or p_{i+6}). As P is induced, at least one vertex of Q is not contained in P. Hence, the union of P and Q must contain a cycle C of length 8,9 or 10. In each case, C contains $(p_i, \dots, p_{i+4})$ as a subpath. If C has length 9, then replacing $(p_{i+1}, \dots, p_{i+4})$ by (p_{i+1}, x, p_{i+4}) yields a C_8. On the other hand, if C has length 10, then replacing $(p_i, \dots, p_{i+4})$ by (p_i, x, p_{i+4}) yields a C_8.

On the other hand, in the case that $p_{i-1} \in N_G(x)$ there must be a path Q from p_{i-2} and p_{i+6} of length at most 3. As $p_{i-2}, p_{i+6} \notin N_G(x)$ the path Q cannot contain x. Additionally, as P is induced, at least one vertex of Q is not contained in P. Therefore, the union of P and Q must contain a cycle C of length 9,10 or 11. Note that C must contain $(p_{i-1}, \dots, p_{i+4})$ as a subpath. If C has length 9, then replacing $(p_{i+1}, \dots, p_{i+4})$ by (p_{i+1}, x, p_{i+4}) yields a C_8. If C has length 10, then replacing $(p_i, \dots, p_{i+4})$ by (p_i, x, p_{i+4}) yields a C_8. Finally, if C has length 11, then replacing $(p_{i-1}, \dots, p_{i+4})$ by (p_{i-1}, x, p_{i+4}) yields a C_8. We conclude that the claimed must be true. $\diamond$

Claim 29.3. *There is no vertex $v \in V(G)$ such that v is adjacent to p_i and p_{i+4} for some $i \in [20, \ell - 20]$.*

Proof of Claim: Assume the statement is not true and there is $v \in V(G)$ and $i \in [20, \ell - 20]$ such that $p_i, p_{i+4} \in N_G(v)$. Note that to avoid C_8 we get that $p_{i-2}, p_{i+6} \notin N_G(v)$ and additionally $p_{i+1}, p_{i+3} \notin N_G(v)$ by Claim 29.2.

Case 1: First assume that $i-3$ has distance 5 type 1. Hence, there is w such that (p_{i-3}, w, p_{i+2}) is a path in G. Note that $w \neq v$ as a consequence of Claim 29.1. Furthermore, to avoid C_8 we have that $p_{i-4}, p_{i+3} \notin N_G(w)$ and

$p_{i-5}, p_{i+4} \notin N_G(w)$ by Claim 29.1. Additionally, $p_{i-2} \notin N_G(w)$ as otherwise $(p_{i-2}, w, p_{i+2}, p_{i+3}, p_{i+4}, v, p_i, p_{i-1}, p_{i-2})$ is a C_8 in G. We now consider the distance 6 type of $i-2$.

Case 1a: First consider $i-2$ has distance 6 type 1. In this case there are x_1, x_2 not on P and different from w (as $p_{i-2}, p_{i+4} \notin N_G(w)$) such that $(p_{i-2}, x_1, x_2, p_{i+4})$ is a path in G. In this case $(p_{i-2}, x_1, x_2, p_{i+4}, p_{i+3}, p_{i+2}, w, p_{i-3}, p_{i-2})$ is a C_8 in G.

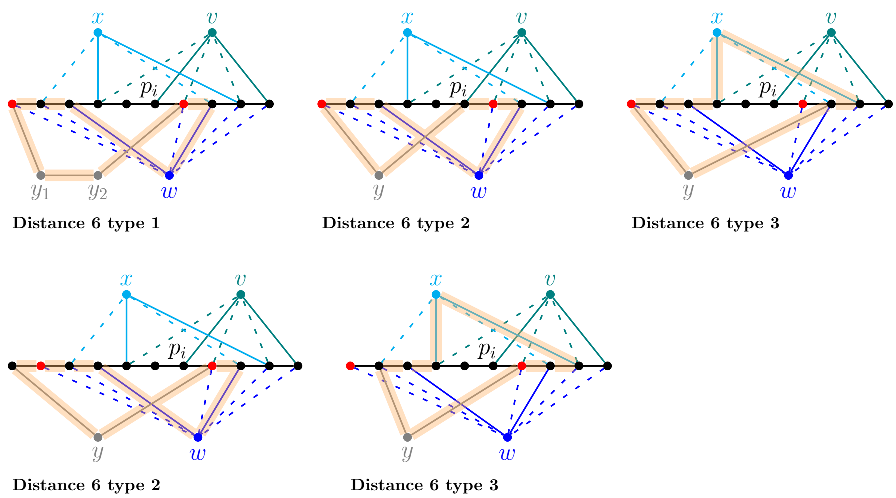

Fig. 10. The five different distance 6 types of $i-5$ in the case that $i-3$ has distance 5 type 1 and $i-2$ has distance 6 type 2 in the proof of Claim 29.3.

Case 1b: Next, assume $i-2$ has distance 6 type 2. In this case there is $x \neq v$, $x \neq w$ (as $p_{i-2} \notin N_G(v)$ and $p_{i-2} \notin N_G(w)$) such that (p_{i-2}, x, p_{i+3}) is a path in G. Note that $p_{i-4} \notin N_G(x)$ by Claim 29.1 and $p_{i+2} \notin N_G(x)$ as otherwise $(p_{i-2}, x, p_{i+2}, p_{i+3}, p_{i+4}, v, p_i, p_{i-1}, p_{i-2})$ is a C_8 in G. Furthermore, in this case $p_{i+1} \notin N_G(w)$ as otherwise $(x, p_{i-2}, \ldots, p_{i+1}, w, p_{i+2}, p_{i+3}, x)$ is a (non-induced) C_8 in G. We now consider the distance 6 type of $i-5$ (see Figure 10 for an illustration of the different cases). First assume $i-5$ has distance 6 type 1. In this case, there are y_1, y_2 not on P and different from w (as $p_{i-5}, p_{i+1} \notin N_G(w)$) such that $(p_{i-5}, y_1, y_2, p_{i+1})$ is a path in G. Then $(p_{i-5}, y_1, y_2, p_{i+1}, p_{i+2}, w, p_{i-3}, p_{i-4}, p_{i-5})$ is a C_8 in G. Hence, consider that $i-5$ has distance 6 type 2. Then there is $y \neq w$ (as $p_{i-5} \notin N_G(w)$) such that (p_{i-5}, y, p_i) is a path in G. Then $(p_{i-5}, y, p_i, p_{i+1}, p_{i+2}, w, p_{i-3}, p_{i-4}, p_{i-5})$ is a C_8 in G. Next, consider $i-5$ has distance 6 type 3. In this case there is $y \neq x$ (as $p_{i+2} \notin N_G(x)$) such that (p_{i-5}, y, p_{i+2}) is a path in G. Then $(p_{i-5}, y, p_{i+2}, p_{i+3}, x, p_{i-2}, \ldots, p_{i-5})$ is a C_8 in G. If $i-5$ has distance 6 type 4, then there is $y \neq w$ (as $p_{i+1} \notin N_G(w)$) such that (p_{i-6}, y, p_{i+1}) is a path in G. Then $(p_{i-6}, y, p_{i+1}, p_{i+2}, w, p_{i-3}, \ldots, p_{i-6})$ is a C_8 in G. Finally, if $i-5$ has distance 6 type 5, there is $y \neq x$ (as $p_{i-4} \notin N_G(x)$) such that (p_{i-4}, y, p_{i+1}) is a path in G. Then $(p_{i-4}, y, p_{i+1}, p_{i+2}, p_{i+3}, x, p_{i-2}, p_{i-3}, p_{i-4})$ is a C_8 in G, a contradiction.

Case 1c: Next, consider the case that $i-2$ has distance 6 type 3. In this case there is $x \neq w$ (as $p_{i-2} \notin N_G(w)$) such that (p_{i-2}, x, p_{i+5}) is a path in G. In this case $(p_{i-2}, x, p_{i+5}, \ldots, p_{i+2}, w, p_{i-3}, p_{i-2})$ is a C_8 in G.

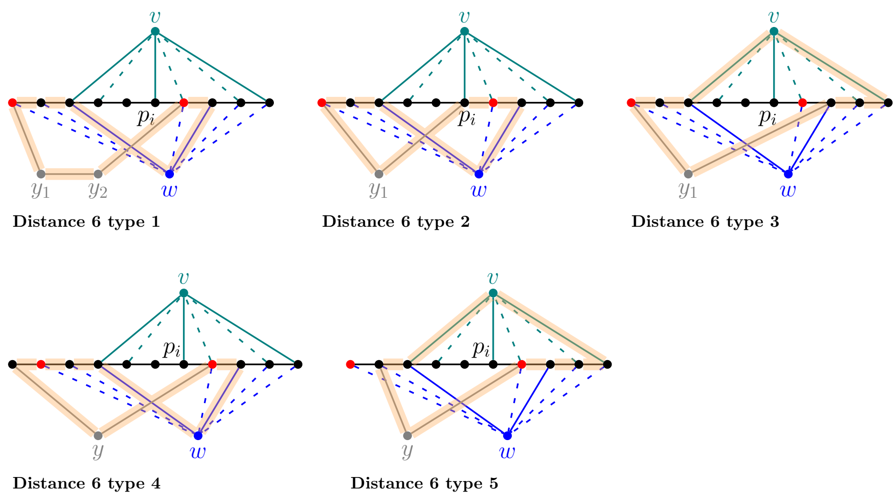

Fig. 11. The five different distance 6 types of $i-5$ in the case that $i-3$ has distance 5 type 1 and $i-2$ has distance 6 type 4 in the proof of Claim 29.3.

Case 1d: In case $i-2$ has distance 6 type 4, there is $x \neq w$ (as $p_{i+4} \notin N_G(w)$) such that (p_{i-3}, x, p_{i+4}) is a path in G. First observe that in case $x \neq v$ we obtain $(p_{i-3}, x, p_{i+4}, v, p_i, p_{i+1}, p_{i+2}, x, p_{i-3})$ as a C_8 in G. Hence, $x = v$ and therefore additionally $p_{i-3} \in N_G(v)$. We now consider the distance 6 type of $i-5$ (see Figure 11 for an illustration of the different cases). First assume $i-5$ has distance 5 type 1. In this case there are y_1, y_2 not on P and different from v (as $p_{i-5}, p_{i+1} \notin N_G(v)$, where the former would yields $(p_{i-5}, v, p_i, p_{i+1}, p_{i+2}, w, p_{i-3}, p_{i-4}, p_{i-5})$ as a C_8 in G) such that $(p_{i-5}, y_1, y_2, p_{i+1})$ is a path in G. But then we get $(p_{i-5}, y_1, y_2, p_{i+1}, p_i, v, p_{i-3}, p_{i-4}, p_{i-5})$ is a C_8 in G. Next, assume that $i-5$ has distance 6 type 2. In this case there is $y \neq w$ (as $p_{i-5} \notin N_G(w)$) such that (p_{i-5}, y, p_i) is a path in G. Hence, $(p_{i-5}, y, p_i, p_{i+1}, p_{i+2}, w, p_{i-3}, p_{i-4}, p_{i-5})$ is a C_8 in G. Next, consider the case that $i-5$ has distance 6 type 3. In this case there is $y \neq v$ (as $p_{i-5} \notin N_G(v)$) such that (p_{i-5}, y, p_{i+2}) is a path in G. But then $(p_{i-5}, y, p_{i+2}, p_{i+1}, p_i, v, p_{i-3}, p_{i-4}, p_{i-5})$ is a C_8 in G. Assume that $i-5$ has distance 6 type 4. Then there is $x \neq v$ (as $p_{i+1} \notin N_G(v)$) such that (p_{i-6}, y, p_{i+1}) is a path in G. Hence, $(p_{i-6}, y, p_{i+1}, p_i, v, p_{i-3}, \ldots, p_{i-6})$ is a C_8 in G. Finally, consider the case that $i-5$ has distance 6 type 5. In this case there is $y \neq v$ (as $p_{i+1} \notin N_G(v)$) such that (p_{i-4}, y, p_{i+1}) is a path in G. Then $(p_{i-4}, y, p_{i+1}, \ldots, p_{i+4}, v, p_{i-3}, p_{i-4})$ is a C_8 in G.

Case 1e: Finally, assume that $i-2$ has distance 6 type 5. Hence, there is $x \neq w$ (as $p_{i+4} \notin N_G(w)$) such that (p_{i-1}, x, p_{i+4}) is a path in G. Then $(p_{i-1}, x, p_{i+4}, p_{i+3}, p_{i+2}, w, p_{i-3}, p_{i-2}, p_{i-1})$ is a C_8 in G.

Case 2: In the case that $i-3$ has distance 5 type 2, there is $x \neq w$ (as $p_{i-2} \notin N_G(w)$) such that (p_{i-2}, x, p_{i+2}) is a path in G. In this case we get that $(p_{i-2}, x, p_{i+2}, p_{i+3}, p_{i+4}, w, p_i, p_{i-1}, p_{i-2})$ is a C_8 in G, a contradiction.

Case 3: It remains to consider the case that $i-3$ has distance 5 type 3. In this case there is $w \neq v$ (as $p_{i+1} \notin N_G(v)$) such that (p_{i-3}, w, p_{i+1}) is a path in G. Note that $p_{i-5}, p_{i+3} \notin N_G(w)$ to avoid C_8 and $p_{i-2}, p_i \notin N_G(w)$ by Claim 29.2. We consider the distance 5 type of $i-2$. First observe that in case $i-2$ has distance type 2, we get $x \neq v$ (as $p_{i-2} \notin N_G(v)$) such that (p_{i-2}, x, p_{i+2}) is a path in G and hence $(p_{i-2}, x, p_{i+2}, p_{i+3}, p_{i+4}, v, p_i, p_{i-1}, p_{i-2})$ is a C_8 in G. Similarly, in case $i-2$ has distance type 3 we get $x \neq w$ (as $p_{i+3} \notin N_G(w)$) such that $(p_{i-1}, x, p_{i+3}, p_{i+2}, p_{i+1}, w, p_{i-3}, p_{i-2}, p_{i-1})$ is a C_8 in G. Hence, $i-2$ must have distance 5 type 1. Therefore, there exists $x \neq v$, $x \neq w$ (as $p_{i-2} \notin N_G(v)$ and $p_{i-2} \notin N_G(w)$) such that (p_{i-2}, x, p_{i+3}) is a path in G. Note that $p_{i-3}, p_{i+4} \notin N_G(x)$ to avoid C_8 and $p_{i-4}, p_{i+5} \notin N_G(x)$ by Claim 29.1. Furthermore, $p_{i+2} \notin N_G(x)$ as otherwise $(p_{i-2}, x, p_{i+2}, p_{i+3}, p_{i+4}, x, p_i, p_{i-1}, p_{i-2})$ is a C_8 in G. We now consider the 6 type of $i-4$.

Case 3a: First, assume that $i-4$ has distance 6 type 1. In this case there are y_1, y_2 not in P and different from x (as $p_{i-4}, p_{i+2} \notin N_G(x)$) such that $(p_{i-4}, y_1, y_2, p_{i+2})$ is a path in G. Then $(p_{i-4}, y_1, y_2, p_{i+2}, p_{i+3}, x, p_{i-2}, p_{i-3}, p_{i-4})$ is a C_8 in G.

Case 3b: Next, assume that $i-4$ has distance 6 type 2. Hence, there is $y \neq x$ (as $p_{i-4} \notin N_G(x)$) such that (p_{i-4}, y, p_{i+1}) is a path in G. In this case $(p_{i-4}, y, p_{i+1}, p_{i+2}, p_{i+3}, x, p_{i-2}, p_{i-3}, p_{i-4})$ is a C_8 in G.

Case 3c: Consider that $i-4$ has distance 6 type 3. Hence, there is $y \neq w$, $y \neq x$ (as $p_{i+3} \notin N_G(w)$ and $p_{i-4} \notin N_G(x)$) such that (p_{i-4}, y, p_{i+3}) is a path in G. Note that $p_{i-2}, p_{i+1} \notin N_G(y)$ by Claim 29.1. Additionally, $p_{i-5} \notin N_G(y)$ as otherwise $(p_{i-5}, y, p_{i+3}, p_{i+2}, p_{i+1}, w, p_{i-3}, p_{i-4}, p_{i-5})$ is a C_8 in G. We consider the distance 6 type of $i-5$ (see Figure 12 for an illustration of the different cases). First assume that $i-5$ has distance 6 type 1. In this case there are z_1, z_2 not on P and different from y (as $p_{i-5}, p_{i+1} \notin N_G(y)$) such that $(p_{i-5}, z_1, z_2, p_{i+1})$ is a path in G. Hence, $(p_{i-5}, z_1, z_2, p_{i+1}, p_{i+2}, p_{i+3}, y, p_{i-4}, p_{i-5})$ is a C_8 in G. Similarly, in case $i-5$ has distance 6 type 2 there is $z \neq y$ (as $p_{i-5} \notin N_G(y)$) such that (p_{i-5}, z, p_i) is a path in G. Then $(p_{i-5}, z, p_i, \ldots, p_{i+3}, y, p_{i-4}, p_{i-5})$ is a C_8 in G. Now assume that $i-5$ has distance 6 type 3. Then there is $z \neq x$ (as $p_{i+2} \notin N_G(x)$) such that (p_{i-5}, z, p_{i+2}) is a path in G. Hence, $(p_{i-5}, z, p_{i+2}, p_{i+3}, x, p_{i-2}, \ldots, p_{i-5})$ is a C_8 in G. In case $i-5$ has distance 6 type 4, there is $z \neq y$ (as $p_{i+1} \notin N_G(y)$) such that such that (p_{i-6}, z, p_{i+1}) is a path in G. Then $(p_{i-6}, z, p_{i+1}, p_{i+2}, p_{i+3}, y, p_{i-4}, p_{i-5}, p_{i-6})$ is a C_8 in G. Finally, assume

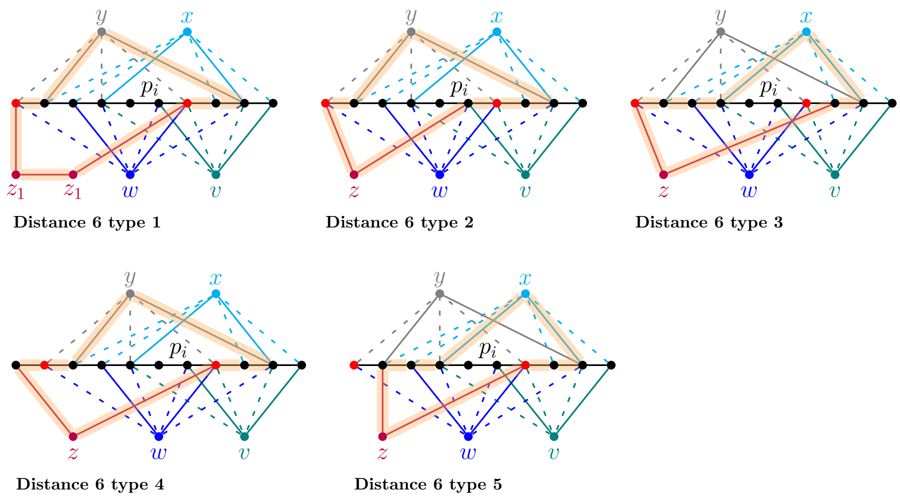

Fig. 12. The five different distance 6 types of $i-5$ in the case that $i-3$ has distance 5 type 3 (which implies that $i-2$ has distance 5 type 1) and $i-4$ has distance 6 type 3 in the proof of Claim 29.3.

that $i-5$ has distance 6 type 5. Hence, there is $z \neq x$ (as $p_{i-4} \notin N_G(x)$) such that (p_{i-4}, z, p_{i+1}) is a path in G. Then $(p_{i-4}, z, p_{i+1}, p_{i+2}, p_{i+3}, x, p_{i-2}, p_{i-3}, p_{i-4})$ is a C_8 in G.

Case 3d: Next assume that $i-4$ has distance 6 type 2. In this case there is $y \neq x$ (as $p_{i+2} \notin N_G(x)$) such that (p_{i-5}, y, p_{i+2}) is a path in G. In this case $(p_{i-5}, y, p_{i+2}, p_{i+3}, x, p_{i-2}, \ldots, p_{i-5})$ is a C_8 in G.

Case 3e: Finally, consider the case that $i-4$ has distance 6 type 5. Hence, there is $y \neq x$ (as $p_{i+2} \notin N_G(x)$) such that (p_{i-3}, y, p_{i+2}) is a path in G. To avoid C_8 we get that $p_{i-4}, p_{i+3} \notin N_G(y)$. Additionally, $p_{i-5}, p_{i+4} \notin N_G(y)$ by Claim 29.1 and $p_{i+1} \notin N_G(y)$ as otherwise $(p_{i+1}, y, p_{i+2}, p_{i+3}, x, p_{i-2}, \ldots, p_{i+1})$ is a (non-induced) C_8 in G. We consider the distance 6 type of $i-5$ (see Figure 13 for an illustration of the different cases). First assume that $i-5$ has distance 6 type 1. In this case there are z_1, z_2 not on P and different from y (as $p_{i-5}, p_{i+1} \notin N_G(y)$) such that $(p_{i-5}, z_1, z_2, p_{i+1})$ is a path in G. Hence, $(p_{i-5}, z_1, z_2, p_{i+1}, p_{i+2}, y, p_{i-3}, p_{i-4}, p_{i-5})$ is a C_8 in G. Similarly, in case $i-5$ has distance 6 type 2 there is $z \neq y$ (as $p_{i-5} \notin N_G(y)$) such that (p_{i-5}, z, p_i) is a path in G. In this case $(p_{i-5}, z, p_i, \ldots, p_{i+2}, y, p_{i-3}, p_{i-4}, p_{i-5})$ is a C_8 in G. Now assume that $i-5$ has distance 6 type 3. Then there is $z \neq x$ (as $p_{i+2} \notin N_G(x)$) such that (p_{i-5}, z, p_{i+2}) is a path in G. Hence, $(p_{i-5}, z, p_{i+2}, p_{i+3}, x, p_{i-2}, \ldots, p_{i-5})$ is a C_8 in G. In case $i-5$ has distance 6 type 4, there is $z \neq y$ (as $p_{i+1} \notin N_G(y)$) such that such that (p_{i-6}, z, p_{i+1}) is a path in G. Then $(p_{i-6}, z, p_{i+1}, p_{i+2}, y, p_{i-3}, p_{i-4}, p_{i-5}, p_{i-6})$ is a C_8 in G. Finally, assume that $i-5$ has distance 6 type 5. Hence, there is $z \neq x$ (as $p_{i-4} \notin N_G(x)$) such

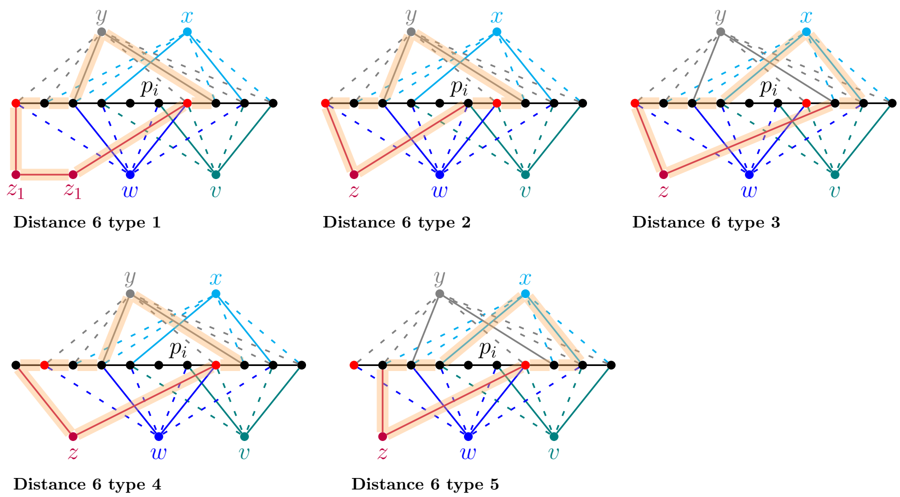

Fig. 13. The five different distance 6 types of $i-5$ in the case that $i-3$ has distance 5 type 3 (which implies that $i-2$ has distance 5 type 1) and $i-4$ has distance 6 type 5 in the proof of Claim 29.3.

that (p_{i-4}, z, p_{i+1}) is a path in G and $(p_{i-4}, z, p_{i+1}, p_{i+2}, p_{i+3}, x, p_{i-2}, p_{i-3}, p_{i-4})$ is a C_8 in G.

As we found a C_8 in each possible case contradicting our assumption that G is C_8-subgraph free, the claim must be true. ⋄

Consider the distance 5 type of 20. By Claim 29.3 we know that 20 must have distance 5 type 1 and hence there is x such that (p_{20}, x, p_{25}) is a path in G. Now consider the distance 5 type of 22. Similarly, by Claim 29.3 we know that 22 has distance 5 type 1 and hence there is y such that (p_{22}, y, p_{27}) is a path in G. Furthermore, we know that $x \neq y$ by Claim 29.1. Therefore, $(p_20, x, p_{25}, p_{26}, p_{27}, y, p_{22}, p_{21}, p_{20})$ is a C_8 in G. We conclude that G has treedepth at most $c(4, 4, 42)$. □

G The Missing Proof of Theorem 30

Theorem 30. *For any $\ell_1, \ell_2 > 2$ the class of all $C^{\mathrm{V}}_{2\ell_1,2\ell_2}$-subgraph-free graphs of diameter at most 2 has bounded treedepth.*

Proof. Let G be some $C^{\mathrm{V}}_{2\ell_1,2\ell_2}$-subgraph-free graphs with diameter at most 2, we claim $td(G) < c(2\ell_1 + 2\ell_2, 2\ell_1 + 2\ell_2, 4(\ell_1 + \ell_2 - 1) + 1)$. Suppose for contradiction $td(G) \geq c(2\ell_1 + 2\ell_2, 2\ell_1 + 2\ell_2, 4(\ell_1 + \ell_2 - 1) + 1)$. As G cannot contain $K_{2\ell_1+2\ell_2,2\ell_1+2\ell_2}$ as a subgraph, Corollary 6 implies G contains an induced path P of length $4(\ell_1 + \ell_2 - 1)$. Let $P = (p_0, \ldots, p_{4(\ell_1+\ell_2-1)})$.

As G has diameter at most 2, p_0 and $p_{2\ell_1-1}$ must have a common neighbour, call this vertex x. The common neighbour of $p_{2\ell_1-1}$ and $p_{2(\ell_1+\ell_2-1)}$ must also be x otherwise $C^{\mathrm{V}}_{2\ell_1,2\ell_2}$ is contained as a subgraph with the pair of cycles of length $2\ell_1$ and $2\ell_2$ sharing the common vertex x. This also implies the common neighbours of $p_{2(\ell_1+\ell_2-1)}$ and $p_{4\ell_1+2\ell_2-3}$ as well as $p_{4\ell_1+2\ell_2-3}$ and $p_{4\ell_1+4\ell_2-4}$ are x. However this leads to cycles of length $2\ell_1$ and $2\ell_2$ with a single common vertex x. □

H The Missing Proof of Theorem 31

Theorem 31. *For any integers $\ell \geq 3$ and $k \geq 1$ the class of all $C^{\mathrm{V}}_{k*[2\ell]}$-subgraph-free graphs of diameter at most 2 has bounded treedepth.*

Proof. Let G be a $C^{\mathrm{V}}_{k*[2\ell]}$-subgraph-free graph of diameter at most 2, we claim $td(G) \leq c(k(2\ell-1)+1, k(2\ell-1), 4^{k+1}k^{k+4}\ell)$. Suppose for contradiction $td(G) > c(k(2\ell-1)+1, k(2\ell-1), 4^{k+1}k^{k+4}\ell)$. G cannot contain a large complete bipartite subgraph as $K_{k(2\ell-1)+1,k(2\ell-1)}$ contains $C^{\mathrm{V}}_{k*[2\ell]}$ as a subgraph. From Corollary 6, G contains an induced path, $P = (p_0, \dots, p_m)$ of length $m = 4^{k+1}k^{k+4}\ell$.

We define the distance of two pairs (p_i, p_j) and $(p_{i'}, p_{j'})$ of vertices on P with $i < j$ and $i' < j'$ as follows. If either $i \leq i' \leq j$ or $i \leq j' \leq j$ we set the distance of (p_i, p_j) and $(p_{i'}, p_{j'})$ to be 0. Otherwise, the distance of (p_i, p_j) and $(p_{i'}, p_{j'})$ is the positive integer d for which $j' + d = i$ if $j' < i$ or $j + d = i'$ if $j < i'$.

Suppose some vertex $v \in V(G)$ has k pairs of neighbours in P, (p_i, p_j) such that $j = i + 2\ell - 2$ and these k pairs have pairwise distance at least 1. Such a vertex results in F as there are k cycles of length 2ℓ each containing the single common vertex v. Hence, no vertex is adjacent to k pairs of neighbours $(p_i, p_{i+2\ell-2})$ of pairwise distance at least 1.

Claim 31.1. *Let x be some vertex in G, then $N(x) \cap \{p_0, \dots, p_{m-\ell}\}$ contains at most $(k-1)^2$ disjoint pairs $\{p_i, p_j\}$ where $j = i + 2\ell - 4$, $j \leq m - \ell$ and each pair has pairwise distance at least ℓ.*

Proof of Claim: Say x has neighbours $p_i, p_{i+2\ell-4} \in \{p_0, \dots, p_{m-\ell}\}$. The vertices $p_{i+\ell-3}$, $p_{i+3\ell-5}$ must have a common neighbour, x'. We call the vertex x', that is the common neighbour of $p_{i+\ell-3}$ and $p_{i+3\ell-5}$ the *connector* of the pair (p_i, p_j) of neighbours of x. If $x' = x$ then there is some $C_{2\ell}$ containing only x and the path vertices $p_{i+\ell-3}, \dots, p_{i+3\ell-5}$, otherwise $x' \neq x$ and there is a $C_{2\ell}$ given by $(x, p_i, \dots, p_{i+\ell-3}, x', p_{i+3\ell-5}, \dots, p_{i+2\ell-4})$. Note there is a cycle of length 2ℓ containing x' and the path vertices $p_{i+\ell-3}, \dots, p_{i+3\ell-5}$.

Suppose $N(x) \cap \{p_0, \dots, p_{m-\ell}\}$ contains $(k-1)^2 + 1$ disjoint pairs with pairwise distance at least ℓ. Let x'_r denote the *connector* for the rth pair with $X' = \{x'_1, \dots, x'_{(k-1)^2+1}\}$. If X' contains k pairwise distinct vertices, then G contains F with k cycles of length 2ℓ each cycle, apart from possibly one, containing x and a distinct vertex $x' \in X'$. Note that one of these distinct connectors could be the vertex x itself, in which case the cycle does not contain an additional vertex $x' \in X'$. On the other hand, consider $|X'| \leq k - 1$. As there are $(k-1)^k + 1$ connector there must be some $x' \in X'$ which is the connector of at least k distinct pairs. Assume without loss of generality, $x'_1 = x'_2 = \dots = x'_k$.

However, this is a contradiction as there are k cycles of length 2ℓ containing vertices from P and the single common vertex x'_1. $\diamond$

Claim 31.2. *For every $x \in V(G)$, there are less than $4^k k^{k+1}$ vertices in $N(x) \cap \{p_0, \ldots, p_{m-3\ell-4}\}$ with pairwise distance at least $3\ell - 4$.*

Proof of Claim: Consider some vertex x_0 and let $Z_0 \subseteq N(x_0) \cap \{p_0, \ldots, p_{m-3\ell-4}\}$ be some set of vertices with pairwise distance at least $3\ell - 4$ along the path. Suppose $|Z_0| \geq 4^k k^{k+1}$. Let $Z_0^+ = \{p_{i+2\ell-4} : p_i \in Z_0\}$, note $|Z_0^+| = |Z_0|$.

In the following we recursively construct vertices $x_0, x_1, \ldots, x_{k-1}$, sets $Z_0 \supseteq Z_1 \supseteq \ldots \supseteq Z_{k-1}$ and sets $\hat{Z}_0 \supseteq \hat{Z}_1 \supseteq \ldots \supseteq \hat{Z}_{\delta-1} \supseteq \hat{Z}_{k-1}$ such that for every $1 \leq i \leq k-1$

1. $|Z_i| \geq \left\lfloor \frac{|Z_{i-1}|-(k-1)^2|}{2(k-1)} \right\rfloor \geq \left\lfloor \frac{|Z_{i-1}|}{4(k-1)} \right\rfloor$, where $|Z_{i-1}| \geq 2(k-1)^2$;
2. $|\hat{Z}_i| \geq |Z_i| - (k-1)^2$;
3. $Z_i \subseteq N(x_0, \ldots, x_i) \cap \{p_0, \ldots, p_m\}$;
4. $\hat{Z}_i \cap N(x_0, \ldots, x_i) = \emptyset$;
5. $N(x_i) \cap \hat{Z}_{i-1} \neq \emptyset$; and
6. $\hat{Z}_i \subseteq Z_0^+$.

Suppose for $\delta \leq k-2$ we have constructed vertices $x_0, x_1, \ldots, x_\delta$ and sets $Z_0 \supseteq Z_1 \supseteq \ldots \supseteq Z_\delta$ and $\hat{Z}_0 \supseteq \hat{Z}_1 \supseteq \ldots \supseteq \hat{Z}_{\delta-1}$ with the properties above. Let $Z_\delta^+ = \{p_{i+2\ell-4} : p_i \in Z_{\delta-1}\}$. Again from Claim 31.1, at most $(k-1)^2$ vertices in Z_δ^+ are adjacent to x_δ. Let $\hat{Z}_\delta = Z_\delta^+ \setminus N(x, x_1, \ldots, x_\delta)$ with $|\hat{Z}_\delta| \geq |Z_\delta| - (k-1)^2$. Thus properties 2, 4 and 6 hold. Let (p_i, p_j) be some pair such that $p_i \in Z_\delta$, $j \neq i + 2\ell - 4$ and $p_j \in \hat{Z}_\delta$. As p_i, p_j must have some common neighbour, $x' \neq x$, there is some $C_{2\ell}$ given by $(x, p_i, x', p_j, \cdots, p_{j-(2\ell-4)})$. Note that we can choose $\left\lfloor \frac{|Z_{\delta-1}-(k-1)^2|}{2} \right\rfloor$ such pairs with distance at least 1. Let (p_{r_1}, p_{r_2}) denote the rth pair and let x'_r denote that common neighbour of p_{r_1}, p_{r_2}. Let $X' = \{x'_1, \ldots, x'_{\left\lfloor \frac{|Z_\delta|-(k-1)^2}{2} \right\rfloor}\}$. If X' contains k distinct vertices, then without loss of generality $x'_1 \neq \ldots \neq x'_k$. For every $1 \leq r \leq k$, there is some $C_{2\ell}$ containing x_0, x'_r and vertices from $\{p_{r_1}, \ldots, p_{r_2}\}$, given the pairs (p_{r_1}, p_{r_2}) have pairwise distance at least 1 this results in F. Therefore, by the pigeon hole principle there is some vertex, we call this $x_{\delta+1}$, which is the common vertex for at least $\left\lfloor \frac{|Z_{|\delta}|-(k-1)^2}{2(k-1)} \right\rfloor$ different pairs. Note that $x_{\delta+1}$ is adjacent to at least $\left\lfloor \frac{|Z_\delta|-(k-1)^2}{2(k-1)} \right\rfloor$ vertices in $\hat{Z}_\delta$, and $\left\lfloor \frac{|Z_\delta|-(k-1)^2}{2(k-1)} \right\rfloor$ vertices in Z_δ that is properties 1, 3 and 5 hold.

Given $|Z_0| \geq 4^k k^{k+1} \geq 4^{k-1}(k-1)^{k-1} \cdot ((k-1)^2 + 1)$ there exist vertices $x_0, \ldots, x_{k-1}$, and sets $Z_0 \supseteq \ldots \supseteq Z_{k-1}$ and $\hat{Z}_0 \supseteq \ldots \supseteq \hat{Z}_{k-1}$ with each of the properties described above. From property 5, for every $1 \leq i \leq k-1$, $N(x_i) \cap \hat{Z}_{i-1} \neq \emptyset$, this implies there is some vertex in $\hat{Z}_{i-1}$ adjacent to x_i, let y_i be some arbitrary vertex in $\hat{Z}_{i-1}$ adjacent to x_i. Let y_k be some arbitrary vertex in $\hat{Z}_{k-1}$. From property 4, $\hat{Z}_i \cap N(x_0, \ldots, x_i) = \emptyset$ meaning, vertices y_i are distinct for each $1 \leq i \leq k$. Further from property 6, $\{y_1, \ldots, y_k\} \subseteq Z_0^+$. That is for every $1 \leq i \leq k$ there is some distinct $p_j \in Z_0$, we call this vertex y'_i, such that $p_{j+2\ell-4} = y_i$. Let $Y_i = \{p_j, \ldots, p_{j+2\ell-4}\}$ where j is chosen to satisfy $p_{j+2\ell-4} = y_i$. Note the sets $Y_1, \ldots, Y_k$ are pairwise disjoint.

Further note $|Z_{k-1}| \geq 2k$, meaning there exist a set of vertices $C = \{c_1, \ldots, c_k\} \subseteq Z_{k-1}$ such that $y'_i \notin C$ for any $1 \leq i \leq k$. Let x_k be the common neighbour of y_k and c_k, as $y_k \in \hat{Z}_{k-1}$, $x_k \neq x_i$ for any $0 \leq i \leq k-1$. However, this is a contradiction as for each $1 \leq i \leq k$, there is some cycle of length 2ℓ containing x_0, c_i, x_i and Y_i, given these cycles have a single common vertex, x_0, this describes F. ⋄

You don't want to define x_k here , right? Shouldn't it be: Recall that by construction x_k is a common neighbour.... TEV: I think this is where I want to define x_k NK: Sorry, my bad. You are right!

We divide P into pairwise disjoint subpaths each containing m' vertices, where $m' = (2\ell-2)((k-1)^2+1)+\ell \geq 10\ell k^2$. We call each of these subpath a segment of P. As $m = 4^{k+1}k^{k+4}\ell$, there are at least $4^{k+1}k^{k+2}$ such segments. Let x_0 be the common neighbour of p_0 and $p_{2(\ell-1)}$. From Claim 31.2, x_0 cannot have neighbours in $4^k k^{k+1}$ different segments (excluding the final segment, i.e. that containing p_m). Given there are at least $4^k k^{k+1}+1$ segments there is some segment which does not contain a neighbour of x_0.

Claim 31.3. *Let X be a set of external vertices and $Q = (q_0, \cdots, q_{m'})$ be some segment which does not contain a neighbour of any vertex in X. Then there is a $C_{2\ell}$ containing p_0 and exactly 2 external vertices $x \notin X$ and $x' \notin X$ and vertices from Q.*

Proof of Claim: Let x be the common neighbour of p_0 and q_0. Let y the common neighbour of p_0 and $q_{2\ell-4}$. As both x and y have a neighbour in Q we know that $x \notin X$ and $y \notin X$ by assumption. If $y \neq x$, we let $x' = y$, our claim holds as the cycle $(p_0, x, q_0, \ldots, q_{2\ell-4}, x')$ is of length 2ℓ and contains p_0 and exactly 2 external vertices $x \neq x_0$ and $x' \neq x_0$ and vertices from Q. Otherwise $y = x$.

Repeating this argument, as $m' > 2(2\ell-4)((k-1)^2+1)+\ell$, either there is some $1 \leq \delta \leq 2((k-1)^2+1)$ and vertex $x' \neq x$, $x' \notin X$ such that x is adjacent to $q_{\delta(2\ell-4)}$ and x' is the common neighbour of $q_{(\delta+1)(2\ell-4)}$ and p_0 or $q_{\delta(2\ell-4)}$ is adjacent to x for all $0 \leq \delta \leq 2((k-1)^2+1)$. In the former case, we have found a cycle $(p_0, x, q_{\delta(2\ell-4)}, \ldots, q_{(\delta+1)(2\ell-4)}, x')$ of length 2ℓ containing p_0, two external vertices $x \notin X$, $x' \notin X$ and vertices from Q as claimed. On the other hand, the latter case contradicts Claim 31.1 as x has $(k-1)^2+1$ pairs of neighbours with pairwise distance at least ℓ. ⋄

In the following we recursively define $k-1$ distinct segments $Q_1, \ldots, Q_{k-1}$ and $2k-2$ distinct external vertices $x_1, \ldots, x_{k-1}$ and $x'_1, \ldots, x'_{k-1}$ such that for every $i \in [k-1]$ we have that $x_i \neq x_0$, $x'_i \neq x_0$ and there is a cycle of length 2ℓ containing p_0, x_i, x'_i and vertices from Q_i.

As highlighted above, as there are at least $4^k k^{k+1}+1$ segments, from Claim 31.2, there is some segment which does not contain a neighbour of x_0. Let Q_1 be some segment which does not contain any neighbour of x_0, from Claim 31.3 there is some pair of vertices, call these $x_1, x'_1 \neq x_0$ such that there is a cycle of length 2ℓ containing p_0, x_1, x'_1 and vertices from Q_1. Suppose now there are segments $Q_1, \ldots, Q_\delta$ and distinct external vertices $x_1, \ldots, x_\delta$ and $x'_1, \ldots, x'_\delta$ for some $\delta < k-1$. Let $X_\delta = \{x_0, x_1, \ldots, x_\delta, x'_1, \ldots, x'_\delta\}$. From Claim 31.2, no vertex in X_δ can have neighbours in $4^k k^{k+1}$ different segments, as $|X_\delta| = 2\delta+1$ and there are at least $(2\delta + 1) \cdot 4^k k^{k+1} + 1$ segments, there is some segment which does not contain a neighbour in X, let $Q_{\delta+1}$ be such a segment. Again from Claim 31.3 there is some pair of vertices, call these $x_{\delta+1}, x'_{\delta+1} \notin X_\delta$ such that there is a cycle of length 2ℓ containing p_0, $x_{\delta+1}, x'_{\delta+1}$ and vertices from $Q_{\delta+1}$.

Given segments $Q_1, \ldots, Q_{k-1}$ and vertices $x_1, \ldots, x_{k-1}$ and $x'_1, \ldots, x'_{k-1}$ as described above, there are $k-1$ cycles with a common vertex p_0. Further the cycle $(x_0, p_0, \ldots, p_{2(\ell-1)})$ has length 2ℓ and contains no vertex from the segments $Q_1, \ldots, Q_{k-1}$, as these segments do not contain a neighbour of x_0. That is, this describes F. □

I The Missing Proof of Theorem 32

Theorem 30 and Theorem 31 might suggest that we obtain bounded treedepth for any class of $C^{\mathrm{V}}_{\ell_1,\ldots,\ell_k}$-subgraph-free graphs for $\ell_1, \ldots, \ell_k > 4$ even number of bounded diameter 2. But in fact, this is not the case. If we allow only two different length (6 and 8) and take sufficiently many C_6 and C_8 sharing a vertex, the treedepth is unbounded. That is the inverse of Observation 7 is not true.

Theorem 32. *The class of $C^{\mathrm{V}}_{12\times[6],12\times[8]}$-subgraph-free graphs of diameter at most 2 has unbounded treedepth.*

Proof. For every $n \in \mathbb{N}$ we construct a graph G_n which is $C^{\mathrm{V}}_{12\times[6],12\times[8]}$-subgraph-free, has diameter at most 2 and treedepth at least $\log(n)$. We construct G_n from the disjoint union of $P_n = (p_0, \ldots, p_n)$ and a complete graph K_{12} on vertex set

$$Z = \{x_{0,1}, x_{1,2}, x_{2,3}, x_{3,0}, y_{0,2}, y_{1,3}, y_{2,4}, y_{3,5}, y_{4,6}, y_{5,7}, y_{6,0}, y_{7,1}\}$$

by adding edges as follows. For all $i, j \in [0, 3]$, $j \equiv i + 1 \mod 4$ and $k \in [0, n]$ we add the edge $x_{i,j}p_k$ if and only if $k \equiv i \mod 4$ or $k \equiv j \mod 4$. Similarly, for all $i, j \in [0, 7]$, $j \equiv i + 2 \mod 8$ and $j \in [0, n]$ we add the edge $y_{i,j}p_k$ if and only if $k \equiv i \mod 8$ or $k \equiv j \mod 8$. For an illustration of the construction see Figure 14.

We first argue that G_n has diameter 2. Trivially, any two vertices in Z are of distance 1 of each other. Furthermore, the distance between any $z \in Z$ and p_k is at most 2 as p_k is adjacent to some vertex in Z and Z is a clique. Therefore, consider p_k, p_ℓ with $k < \ell$.

First assume that $k \equiv \ell \mod 4$. Set $i, j \in [0, 3]$, $j \equiv i + 1 \mod 4$ such that $k \equiv \ell \equiv i \mod 4$ and observe that both p_k and p_ℓ are adjacent to $x_{i,j}$ and hence are of distance 2.

Next assume that $k + 1 \equiv \ell \mod 4$. We set $i, j \in [0, 3]$ such that $k \equiv i \mod 4$ and $\ell \equiv j \mod 4$ and observe that $x_{i,j}$ is a vertex in G_n. Hence, p_k and p_ℓ have distance at most 2 as p_k and p_ℓ are adjacent to $x_{i,j}$.

Next assume that $k + 3 \equiv \ell \mod 4$. We define $i, j \in [0, 3]$ such that $\ell \equiv i \mod 4$ and $k \equiv j \mod 4$ and observe again that $x_{i,j}$ is a vertex in G_n. Hence, p_k and p_ℓ have distance at most 2 as both p_k and p_ℓ are adjacent to $x_{i,j}$.

Finally, assume that $k+2 \equiv \ell \mod 4$. In this case either $k+2 \equiv \ell \mod 8$ or $\ell+6 \equiv \ell \mod 8$. In the former case, let $i,j \in [0,7]$ such that $k \equiv i \mod 8$ and $\ell \equiv j \mod 8$ and observe that $y_{i,j}$ is a vertex in G_n. In the latter case we choose $i,j \in [0,7]$ such that $\ell \equiv i \mod 8$ and $k \equiv j \mod 8$ and remark that $y_{i,j}$ is a vertex in G_n. We conclude that p_k and p_ℓ are of distance at most 2 as in both cases p_k and p_ℓ are adjacent to $y_{i,j}$.

We now argue that G_n is $C^{\mathrm{V}}_{12\times[6],12\times[8]}$-subgraph-free. First observe, that for any $i,j \in [0,3]$, $j \equiv i \mod 4$ any C_8 in G_n which contains $x_{i,j}$ must contain a second vertex from Z. Indeed, if this is not the case then there is a C_8 which comprise of $x_{i,j}$ and a subpath of P of length 7 with start and end vertex adjacent to $x_{i,j}$ which is impossible by construction. Additionally, for any $i,j \in [0,7]$, $j \equiv i \mod 8$ any C_6 in G_n which contain $y_{i,j}$ must contain a second vertex from Z. Indeed, this follows from $y_{i,j}$ not being adjacent to any pair of vertices of distance 4 on P.

Towards a contradiction, assume G_n contains $C^{\mathrm{V}}_{12\times[6],12\times[8]}$ as a subgraph. We distinguish three cases. First assume that the shared vertex in the subgraph $C^{\mathrm{V}}_{12\times[6],12\times[8]}$ is $x_{i,j}$ for some $i,j \in [0,3]$. In this case, each of the 12 C_8's contained in $C^{\mathrm{V}}_{12\times[6],12\times[8]}$ must contain a second vertex in Z and these additional vertices have to be pairwise different for different C_8's. A contradiction as Z contains only 12 vertices in total. We obtain a contradiction in the same way considering the shared vertex to be $y_{i,j}$ for some $i,j \in [0,7]$ with the requirement of the C_6's contained in $C^{\mathrm{V}}_{12\times[6],12\times[8]}$. Finally, in case the shared vertex in the subgraph $C^{\mathrm{V}}_{12\times[6],12\times[8]}$ is on P, each of the 24 cycles of $C^{\mathrm{V}}_{12\times[6],12\times[8]}$ has to contain a vertex not on P and all of them have to pairwise different, a contradiction.

Finally, G_n contains P_n and hence has treedepth at least $\log(n)$ by Fact 4. We conclude that the class of $C^{\mathrm{V}}_{12\times[6],12\times[8]}$-subgraph-free graph of diameter at most 2 has unbounded treedepth. □

J The Missing Proof of Theorem 33

Theorem 33. *For any numbers $\ell_1, \ell_2 > 2$ the class of all $C^{\mathrm{E}}_{2\ell_1,2\ell_2}$-subgraph-free graphs of diameter at most 2 has bounded treedepth.*

Proof. Let $\ell_1, \ell_2 > 2$ be integers and assume G is a $C^{\mathrm{E}}_{2\ell_1,2\ell_2}$-subgraph-free graph of diameter at most 2 and treedepth at least $c(\ell_1+\ell_2, \ell_1+\ell_2, 3\ell_1+2\ell_2-13)$. Note that G cannot contain a large complete bipartite subgraph as $K_{\ell_1+\ell_2,\ell_1+\ell_2}$ contains $C^{\mathrm{E}}_{2\ell_1,2\ell_2}$ as a subgraph. Hence, by Corollary 6, G must contain $P_\ell = (p_0,\dots,p_\ell)$ where $\ell = 6\ell_1+4\ell_2-13$ as an induced subgraph.

First observe that G having diameter at most 2 implies that for any two vertices p,q of distance at least 3 on P there must be a vertex x not on P adjacent to both p and q. We also remark that a vertex x not on P being adjacent to two vertices p,q of distance $k-2$ on P forms a C_k with the part of P from p to q.

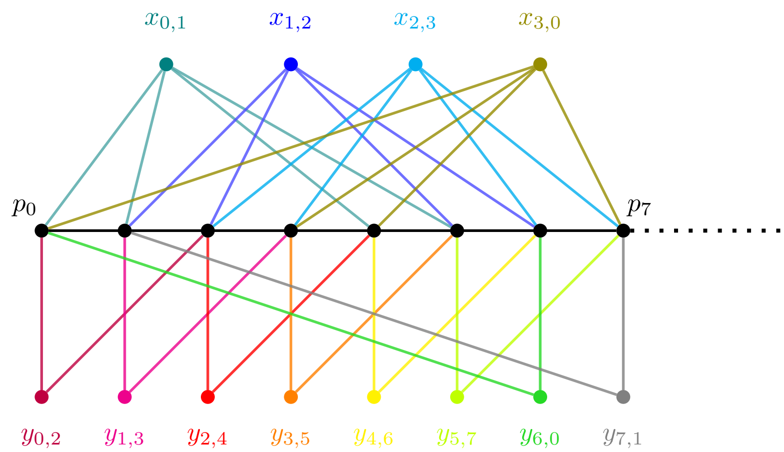

Fig. 14. Construction of the graph G_n from the proof of Theorem 32. The edges within the clique $\{x_{0,1}, x_{1,2}, x_{2,3}, x_{3,0}, y_{0,2}, y_{1,3}, y_{2,4}, y_{3,5}, y_{4,6}, y_{5,7}, y_{6,0}, y_{7,1}\}$ are omitted.

Claim 33.1. *Let $i \in \{1, 2\}$, $j \in [\ell - 2\ell_1 - 2\ell_2 + 5]$ and x a vertex not on P. If x is adjacent to p_j and $p_{j+2\ell_i-2}$ then*

1. *x is adjacent to $p_{j+2\ell_i-3}$ and to $p_{j+2\ell_i+2\ell_{3-i}-5}$, and*
2. *x is not adjacent to $p_{j+2\ell_i+2\ell_{3-i}-4}$ and to p_{j-1} if $j > 1$.*

Proof of Claim: Assume x is adjacent to p_j and $p_{j+2\ell_i-2}$. Let y (possibly equal to x) be a vertex not on P which is adjacent to $p_{j+2\ell_i-3}$ and $p_{j+2\ell_i+2\ell_{3-i}-5}$ (y must exist as $p_{j+2\ell_i-3}$ and $p_{j+2\ell_i+2\ell_{3-i}-5}$ are of distance $(j + 2\ell_i + 2\ell_{3-i} - 5) - (j + 2\ell_i - 3) = 2\ell_{3-i} - 2 \geq 3$). If $x \neq y$ we get the copy of H consisting of cycles $(x, p_j, \ldots, p_{j+2\ell_i-2}, x)$ and $(y, p_{j+2\ell_i-3}, \ldots, p_{j+2\ell_i+2\ell_{3-1}-5}, y)$ with shared edge $\{p_{j+2\ell_i-3}, p_{j+2\ell_i-2}\}$. Hence, $x = y$ and the first statement follows.

To show the second statement simply observe, that x cannot be adjacent to p_{n+m-4} as otherwise we get the copy of H with cycles $(x, p_j, \ldots, p_{j+2\ell_i-2}, x)$ and $(x, p_{j+2\ell_i-2}, \ldots, p_{j+2\ell_i+2\ell_{3-i}-4})$ with shared edge $\{x, p_{j+2\ell_i-2}\}$. The assumption that x is adjacent to p_{j-1} (in case $j > 1$) yields a contradiction with a symmetric argument. ◇

Let x be a vertex not on P which is adjacent to p_0 and $p_{2\ell_1-2}$ (x must exist as the distance of p_0 and $p_{2\ell_1-2}$ is $2\ell_1 - 2 \geq 3$). By Claim 33.1, this implies that $p_{2\ell_1-3}$ and $p_{2\ell_1+2\ell_2-5}$ are also adjacent to x while $p_{2\ell_1+2\ell_2-4}$ is not adjacent to x. Applying Claim 33.1 again (for $p_{2\ell_1-3}$ and $p_{2\ell_1+2\ell_2-5}$), implies that $p_{2\ell_1+2\ell_2-6}$ and $p_{4\ell_1+2\ell_2-8}$ are adjacent to x. Finally, applying Claim 33.1 ($p_{2\ell_1+2\ell_2-6}$ and $p_{4\ell_1+2\ell_2-8}$) implies that $p_{2\ell_1+2\ell_2-7}$ is not adjacent to x.

We now let y (possibly equal to x) be a vertex not on P which is adjacent to p_1 and $p_{2\ell_1-1}$ (which exists as p_1 and $p_{2\ell_1-1}$ are of distance $2\ell_1 - 2 \geq 3$ on P). Applying Claim 33.1 four times sequentially (similar as before) yields that $p_{2\ell_1-2}$, $p_{2\ell_1+2\ell_2-4}$, $p_{2\ell_1+2\ell_2-5}$, $p_{4\ell_1+2\ell_2-7}$, $p_{4\ell_1+2\ell_2-8}$ and $p_{4\ell_1+4\ell_2-10}$ are adjacent to y

while $p_{2\ell_1+2\ell_2-9}$ is not adjacent to y. As y is adjacent to $p_{2\ell_1+2\ell_2-4}$ while x is not adjacent to $p_{2\ell_1+2\ell_2-4}$ we know that $x \neq y$.

Finally, we let z be a vertex not on P which is adjacent to $p_{2\ell_1+2\ell_2-7}$ and $p_{4\ell_1+2\ell_2-9}$ (such a vertex exists as $p_{2\ell_1+2\ell_2-7}$ and $p_{4\ell_1+2\ell_2-9}$ are of distance $2\ell_1 - 2 \geq 3$ on P). Note that $z \neq x$ as x is not adjacent to $p_{2\ell_1+2\ell_2-7}$ and $z \neq y$ as y is not adjacent to $p_{4\ell_1+2\ell_2-9}$. Furthermore, $C_1 = (z, p_{2\ell_1+2\ell_2-7}, \dots, p_{4\ell_1+2\ell_2-9}, z)$ is a cycle of length $2\ell_1$ in G. Additionally, we obtain a second cycle $C_2 = (x, p_{2\ell_1-3}, p_{2\ell_1-2}, y, p_{2\ell_1-1}, \dots, p_{2\ell_1+2\ell_2-6}, x)$ of length $2\ell_2$. As C_1 and C_2 precisely share the edge $p_{2\ell_1+2\ell_2-7}p_{2\ell_1+2\ell_2-6}$ we obtain a copy of $C^{\mathrm{E}}_{2\ell_1,2\ell_2}$. See Figure 15 for illustration. Since this contradicts the assumption that G is $C^{\mathrm{E}}_{2\ell_1,2\ell_2}$-subgraph-free, G has treedepth less than $c(\ell_1+\ell_2, \ell_1+\ell_2, 6\ell_1+4\ell_2-13)$. □

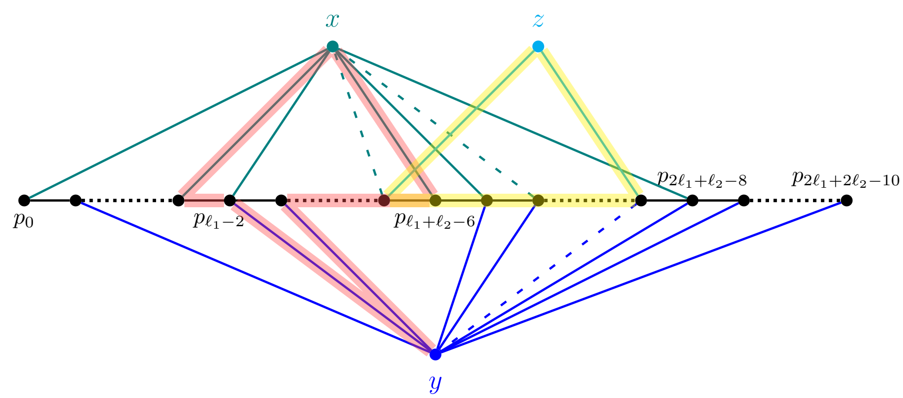

Fig. 15. The construction from the proof of Theorem 33 of a subgraph $C^{\mathrm{E}}_{\ell_1,\ell_2}$ of G under the assumption that G is $C^{\mathrm{E}}_{\ell_1,\ell_2}$-free, has diameter at most 2 and large treedepth.

K The Missing Proof of Theorem 34

Theorem 34. *For any integers $\ell_1, \ell_2 \geq 2$, $\ell \geq 4$ the class of $\{C^{\mathrm{V}}_{4\ell_1,4\ell_2}, C^{\mathrm{E}}_{4\ell_1,4\ell_2}\}$-subgraph-free graphs of diameter at most 3 and the class of $\{C^{\mathrm{V}}_{2\ell,2\ell}, C^{\mathrm{E}}_{2\ell,2\ell}\}$-subgraph-free graphs of diameter at most 3 have unbounded treedepth.*

Proof. We use two different constructions for the two classes of graphs. First consider any pair of integers $\ell_1, \ell_2 \geq 2$. For every $n \in \mathbb{N}$ we construct in the following a $\{C^{\mathrm{V}}_{4\ell_1,4\ell_2}, C^{\mathrm{E}}_{4\ell_1,4\ell_2}\}$-subgraph-free graph G_n. We construct G_n from the disjoint union of a path $P = (p_0, \cdots, p_n)$ and two isolated vertices x and y. We add an edge xp_i whenever the $i \mod 4$ is either 0 or 1 and we add an edge yp_i otherwise. By fact 4 $\mathrm{td}(G_n) \geq \log(n)$ as G_n contains P_n.

To see that G_n has diameter at most 3 observe that x and y have distance at most 3 as x is adjacent to p_1 while y is adjacent to p_2. Furthermore, x (y resp.)

has distance at most 3 to every vertex p_i of the path as x (y resp.) is adjacent to either p_i or p_{i+1} or p_{i+2}. Finally, consider any two vertices p_i and p_j and assume without loss of generality that p_i is adjacent to x. Now either p_j is also adjacent to x or p_j has a neighbour which is adjacent to x by construction. Hence, any pair of vertices on P has distance at most 3.

Finally, we argue that G_n is $\{C^{\mathrm{V}}_{4\ell_1,4\ell_2}, C^{\mathrm{E}}_{4\ell_1,4\ell_2}\}$-subgraph-free. Observe that if x (or y resp.) is adjacent to p_i then x (or y resp.) is not adjacent to p_{i+4m-2} for any $m \geq 1$. Hence, $G_n[\{x\} \cup V(P)]$ and symmetrically $G_n[\{y\} \cup V(P)]$ are $\{C_{4\ell_1}, C_{4\ell_2}\}$-subgraph-free. We conclude that G_n contains neither $C^{\mathrm{V}}_{4\ell_1,4\ell_2}$ nor $C^{\mathrm{E}}_{4\ell_1,4\ell_2}$ as a subgraph.

Next consider any $\ell \geq 2$. For every $n \in \mathbb{N}$ we construct a $\{C^{\mathrm{V}}_{2\ell,2\ell}, C^{\mathrm{E}}_{2\ell,2\ell}\}$-subgraph-free graph $\overline{G}_n$. We construct $\overline{G}_n$ from the disjoint union of a path $P = (p_0, \cdots, p_n)$ and two isolated vertices x and y. Let $R \in \{0,1\}^*$ be the binary word $R = 11(01)^{\ell-2}00(10)^{\ell-2}$ consisting of the word 11 followed by $\ell - 2$ repetitions of the word 01, the word 00 and $\ell - 2$ repetitions of the word 10. We add an edge xp_i whenever the $i \mod 4\ell - 4$th bit of R is 1 and we add an edge yp_i whenever the $i \mod 4\ell - 4$th bit of R is 0. By fact 4 $\mathrm{td}(\overline{G}_n) \geq \log(n)$ as $\overline{G}_n$ contains P_n.

To see that $\overline{G}_n$ has diameter at most 3 observe that x and y have distance at most 3 as x is adjacent to p_1 while y is adjacent to p_2. Furthermore, x (y resp.) has distance at most 3 to every vertex p_i of the path as x (y resp.) is adjacent to either p_i or p_{i+1} or p_{i+2}. Finally, consider any two vertices p_i and p_j and assume without loss of generality that p_i is adjacent to x. Now either p_j is also adjacent to x or p_j has a neighbour which is adjacent to x by construction. Hence, any pair of vertices on P has distance at most 3.

Finally, we argue that $\overline{G}_n$ is $\{C^{\mathrm{V}}_{2\ell,2\ell}, C^{\mathrm{E}}_{2\ell,2\ell}\}$-subgraph-free. To see this, we observe that RR has the property that the ith bit of RR is 1 if and only if the $i+2\ell-2$th bit of RR is 0 for every $i \in [4\ell-4]$. This directly implies that if x (or y resp.) is adjacent to p_i then x (or y resp.) is not adjacent to $p_{i+2\ell-2}$. Hence, both $\overline{G}_n[\{x\} \cup V(P)]$ and $\overline{G}_n[\{y\} \cup V(P)]$ are $C_{2\ell}$-subgraph-free. We conclude that $\overline{G}_n$ can contain neither $C^{\mathrm{V}}_{2\ell,2\ell}$ nor $C^{\mathrm{E}}_{2\ell,2\ell}$ as a subgraph. □

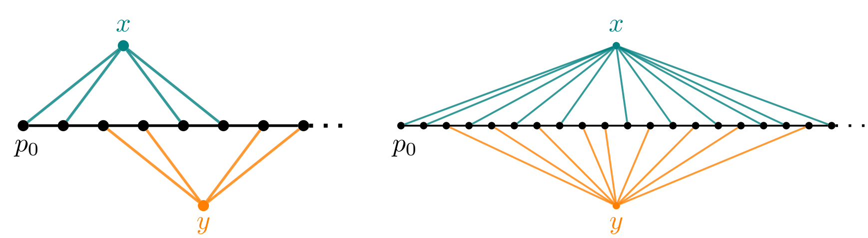

Fig. 16. The constructions from Theorem 34. To the left the first construction with pattern 1100 and to the left the second construction for $\ell = 5$ for which the pattern is $R = 1101010100101010$.

L The Missing Proofs from Section 6

The following theorem gives a contrast between treedepth and pathwidth (or treewidth) in our setting.

Theorem 35. *The class of H_3-subgraph-free graphs of diameter at most 2 has bounded pathwidth but unbounded treedepth.*

Proof. It follows from Observation 7 that the class of H_3-subgraph-free graphs of diameter at most 2 has unbounded pathwidth.

Let G be some H_3-subgraph-free graph of diameter at most 2 with $\mathrm{pw}(G) \geq c(4,4,6)$. As H_3 is a subgraph of $K_{4,4}$, we find that G contains an induced 6-vertex path $P = (p_0, p_1, p_2, p_3, p_4, p_5)$ due to Corollary 6.

As G has diameter at most 2, we find that p_1 and p_4 have a common neighbour x. For the same reason, p_0 and p_4 have a common neighbour y. If $y \neq x$, then G contains H_3 as a subgraph with p_1 and p_4 as its vertices of degree 3. Hence, $x = y$. For the same reason, p_5 must be adjacent to x. If p_2 has some neighbour not in $V(P) \cup \{x\}$ then G contains H_3 as a subgraph with p_2 and x as its vertices of degree 3. Note that p_2 and p_5 must have a common neighbour, as G has diameter at most 2. Hence, p_2 is also adjacent to x, and by symmetry the same holds for p_3. The shortest path from p_2 to every vertex in $G - V(P)$ must contain x implying x dominates G. We also note that p_1, p_3, and p_4 are of degree 3, due to the H_3-subgraph-freeness of G.

We now claim $G - x$ has degree at most 2, which implies $\mathrm{pw}(G) \leq 3$. Namely, if $G - x$ contains some vertex v of degree at least 3 with some neighbour v', then there must be at least three consecutive path vertices of P not contained in $N_G[v]$. Say p_i is the middle of these three path vertices. As p_i has degree 3 and needs to have a common neighbour with v, we find that v' must be adjacent to x. However, we now find that G contains H_3 in which vertices v and p_i have degree 3, a contradiction. ☐

The following theorem gives a contrast between treewidth and clique-width in our setting. We note that C_3-subgraph-free graphs, or equivalently, C_3-free graphs have unbounded clique-width by Theorem 21.

Theorem 36. *For every $r \geq 2$, the class of C_{2r+1}-subgraph-free graphs of diameter at most 2 has bounded clique-width but unbounded treewidth.*

Proof. Let $r \geq 2$. As the class of complete bipartite graphs has unbounded treewidth and is a subclass of the class of C_{2r+1}-subgraph-free graphs of diameter at most 2, we find that C_{2r+1}-subgraph-free graphs have unbounded treewidth. It remains to prove that C_{2r+1}-subgraph-free graphs of diameter at most 2 have bounded clique-width.

Let G be a C_{2r+1}-subgraph-free graphs of diameter at most 2 of arbitrarily large treewidth. Corollary 6 implies that G either has the complete bipartite graph $K_{p,s}$ as a subgraph for arbitrarily large values of r and s, or the path P_ℓ as an induced subgraph for arbitrarily large value of ℓ.

First suppose the latter case holds. Let P be an induced P_ℓ of G. Take two vertices u and v that are of distance $2r-1$ from each other on P. As P is an induced path in G, and G has diameter at most 2, there exists a vertex w not on P that is adjacent to both u and v. This yields a subgraph of G that is isomorphic to C_{2r+1}, a contradiction.

Now suppose the former case holds. Let K be a subgraph of G isomorphic to $K_{p,s}$. Assume that K is a maximal complete bipartite graph of G. Let A and B be the partition classes of K. Note that K is an induced subgraph of G, as otherwise K, and thus G, contains a subgraph isomorphic to C_{2r+1} (assuming we have chosen p and s large enough). We now claim that $G = K$. If not, then G, which is connected as it has diameter at most 2, contains a vertex u not in K that is adjacent to at least one vertex of K. If u is adjacent to both a vertex of A and a vertex of B, we find again that G has a subgraph isomorphic to C_{2r+1}. Hence, we may assume without loss of generality that u is only adjacent to one or more vertices of A. By maximality of K, we find that u is not adjacent to every vertex of A, say u is not adjacent to $x \in A$. We recall that K is an induced subgraph of G and also that u is not adjacent to any vertex of B. Hence, as G has diameter at most 2, there must exist a vertex $y \notin K$ that is adjacent to x and to u. However, now again G has C_{2r+1} as a subgraph, a contradiction. $\square$