Article:
Giovenzana, F., Giovenzana, L. orcid.org/0000-0002-2192-0215, Graffeo, M. et al. (1 more author) (2025) A counterexample to the parity conjecture. Algebraic Geometry, 12 (2). pp. 173-188. ISSN 2313-1691

https://doi.org/10.14231/ag-2025-005

Algebraic Geometry 12 (2) (2025) 173–188
doi:10.14231/AG-2025-005

A counterexample to the parity conjecture

Franco Giovenzana, Luca Giovenzana, Michele Graffeo and Paolo Lella

Abstract

Let $[Z] \in \operatorname{Hilb}^d \mathbb{A}^3$ be a zero-dimensional subscheme of the affine 3-dimensional complex space of length $d > 0$. Okounkov and Pandharipande have conjectured that the dimension of the tangent space to $\operatorname{Hilb}^d \mathbb{A}^3$ at $[Z]$ and d have the same parity. The conjecture was proven by Maulik, Nekrasov, Okounkov and Pandharipande for points $[Z]$ defined by monomial ideals and very recently by Ramkumar and Sammartano for homogeneous ideals. In this paper we exhibit a family of zero-dimensional schemes in $\operatorname{Hilb}^{12} \mathbb{A}^3$ which disproves the conjecture in the general non-homogeneous case.

1. Introduction

Given a quasi-projective variety X defined over the field of complex numbers and a positive integer $d > 0$, the Hilbert scheme $\operatorname{Hilb}^d X$ of d points on X is the scheme parametrising zero-dimensional subschemes of length d of X. It is a quasi-projective scheme, and it was introduced by Grothendieck in [Gro62].

Over the last decades the study of Hilbert schemes of points has been a central topic of research, and even though a number of results have been proven, several questions about their geometry remain open. By a classical result of Fogarty, we know that the Hilbert scheme $\operatorname{Hilb}^d X$ of a connected variety X is connected for all d; see [Fog68]. If X is smooth and irreducible, then $\operatorname{Hilb}^d X$ is smooth, and hence irreducible, as long as $\dim X \leqslant 2$; see [Fog68]. In higher dimension $\operatorname{Hilb}^d X$ is smooth for $\dim X \geqslant 3$ and $d \leqslant 3$, and singular otherwise. Recently, in [Jel20] Jelisiejew showed that its singularities are pathological, proving that Hilbert schemes of points satisfy Vakil's Murphy's law up to retraction [Vak06]. Another open question concerns the irreducibility of these schemes. It is known that, when $\dim X \geqslant 4$, the Hilbert scheme $\operatorname{Hilb}^d X$ is irreducible for $d \leqslant 7$ and reducible otherwise [IE78, Maz80, CEVV09]. On the other hand,

Received 30 January 2024, accepted in final form 6 June 2023.
2020 Mathematics Subject Classification 14C05, 13D10, 13F55, 13P10.
Keywords: parity conjecture, Hilbert schemes of points, 12 points.

This work is partially supported by the Thematic Research Programme "Tensors: geometry, complexity and quantum entanglement", University of Warsaw, Excellence Initiative – Research University and the Simons Foundation Award No. 663281 granted to the Institute of Mathematics of the Polish Academy of Sciences for the years 2021-2023. F.G.'s research is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation), Projektnummer 509501007 and partially supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (ERC-2020-SyG-854361-HyperK). L.G. was supported by Engineering and Physical Sciences Research Council (EPSRC) New Investigator Award EP/V005545/1 "Mirror Symmetry for Fibrations and Degenerations". L.G. was also supported by the DFG research grant Le 3093/3-2. M.G. and P.L. were partially supported by the project PRIN 2020 "Squarefree Gröbner degenerations, special varieties and related topics" (MUR, project number 2020355B8Y). All authors are members of GNSAGA.

the problem of determining the irreducibility of $\operatorname{Hilb}^d X$ for a smooth irreducible threefold X is only partially solved. Indeed, it has been recently proved that it is irreducible for $d \leqslant 11$, see [Šiv12, HJ18, DJNT17], and by a classical result of Iarrobino, it is reducible for $d \geqslant 78$, see [Iar84], but nothing is known for the intermediate cases.

It is worth concluding this historical introduction by mentioning that, although it is known that when $\dim X \geqslant 4$ and d is big enough, $\operatorname{Hilb}^d X$ has non-reduced components, nothing is known about the reducedness of the Hilbert schemes of points on a smooth threefold [Jel24, Problem XV]; see also [Sza21] for more examples.

The study of the singularities of $\operatorname{Hilb}^d X$ has benefited from renewed interest in the mathematical community due to the connection with enumerative geometry established by Maulik, Nekrasov, Okounkov, and Pandharipande [MNOP06]. In particular, Okounkov and Pandharipande formulate the following conjecture.

Conjecture 1 (Parity conjecture [Pan22]). *Let $d \in \mathbb{N}$ be a positive integer, and let X be an irreducible smooth threefold. Then, for any $[Z] \in \operatorname{Hilb}^d X$, one has*

$$\dim_{\mathbb{C}} \mathsf{T}_{[Z]} \operatorname{Hilb}^d X \equiv d \pmod 2,$$

where $\mathsf{T}_{[Z]} \operatorname{Hilb}^d X$ denotes the tangent space of $\operatorname{Hilb}^d X$ at $[Z]$.

Conjecture 1 was proven to hold for monomial ideals in [MNOP06] and very recently for homogeneous ideals in [RS25]. In this paper, we exhibit counterexamples to Conjecture 1, proving that the parity conjecture does not hold for $\operatorname{Hilb}^d \mathbb{A}^3$ whenever $d \geqslant 12$. For instance, the ideal

$$\begin{aligned} I &:= \big(x + (y,z)^2\big)^2 + (y^3 - xz) \\ &= \big(x^2, xy^2, xyz, xz^2, y^2z^2, yz^3, z^4, y^3 - xz\big) \end{aligned} \tag{1.1}$$

is immediately checked to be of colength 12, and standard routines on a computer algebra software like *Macaulay2* [GS] can compute that $\dim T_{[I]} \operatorname{Hilb}^{12} \mathbb{A}^3 = 45$, hence showing the main result of the paper.

Theorem (Corollary 3.2). *The parity conjecture is false for any $d \geqslant 12$.*

Let us make some comments. Our main result shows that the parity conjecture fails for $d \geqslant 12$, and at the moment we do not know any counterexample of smaller length. At the same time the irreducibility of $\operatorname{Hilb}^d \mathbb{A}^3$ is known for $d \leqslant 11$, while for $d = 12$ it is not known whether the Hilbert scheme is irreducible. From this observation one might wonder if irreducibility is related to the parity conjecture. In this direction it is worth mentioning that the counterexample we present in equation (3.1) is smoothable, as we show in Proposition 3.8. Indeed, it can be deformed to the disjoint union of four fat points as depicted in Figure 1.

Further, even if it is possible to produce counterexamples of greater length by adding k-tuples of distinct points disjoint from the support of our example, we emphasise that we can exhibit other counterexamples, not obtained in this way. We give an example in Remark 3.4.

The parity conjecture is also linked to a long-standing open problem in enumerative geometry, namely the constancy of the Behrend function $\nu_{\operatorname{Hilb}^d \mathbb{A}^3}$ on $\operatorname{Hilb}^d \mathbb{A}^3$; see [Beh09, Ric24]. Recently, in [JKS23] it was shown how the failure of the parity conjecture implies the non-constancy of $\nu_{\operatorname{Hilb}^d \mathbb{A}^3}$. In a future project we will address the problem of computing the Behrend function at points disproving the parity conjecture by generalising the techniques introduced in [GR23].

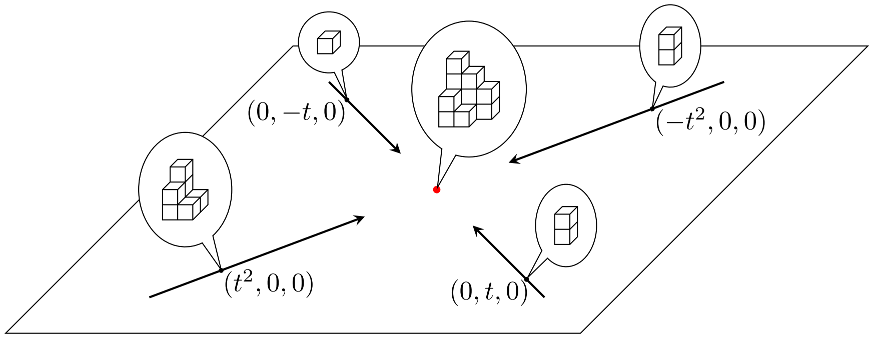

Figure 1. A degeneration of four fat points leading to a counterexample to the parity conjecture.

Plan of the paper

After recalling basic notions of Hilbert schemes and marked families in Section 2, we prove the main result in Section 3. Also, we investigate the family of counterexamples given by ideals modelled on the one given in equation (1.1), and we show in Proposition 3.8 that all of them lie in the smoothable component. In Section 4 we describe the locus in $\mathrm{Hilb}^{12}\mathbb{A}^3$ that the family of counterexamples determine. We are not aware if these are all the ideals of length 12 for which the parity conjecture fails. In the appendix we provide and explain the computations necessary for the proof of the main results.

2. Preliminaries

In this section we recall some well-known results, and we settle some notation.

Let $Z \hookrightarrow \mathbb{A}^n$ be a closed subscheme defined by the ideal $I_Z \subset \mathbb{C}[x_1,\dots,x_n]$. Recall that the *dimension* of Z is defined as the Krull dimension (see [Eis95, Chapter II] for more details on dimension theory) of the ring $\mathcal{O}_Z = \mathbb{C}[x_1,\dots,x_n]/I_Z$. When $\mathcal{O}_Z$ is zero-dimensional as a ring, it is a semilocal Artinian $\mathbb{C}$-algebra of finite type, and, as a consequence, it is a finite-dimensional vector space over the complex numbers. The complex dimension of $\mathcal{O}_Z$ is called the *length* of Z or the *colength* of I_Z,

$$\operatorname{len} Z = \operatorname{colen} I_Z = \dim_{\mathbb{C}} \mathcal{O}_Z\,.$$

When Z is a zero-dimensional closed subscheme of $\mathbb{A}^n$ and $\mathcal{O}_Z$ is a local $\mathbb{C}$-algebra, we will say that Z is a *fat point*.

Let d be a positive integer, and let X be a smooth quasi-projective variety. Recall that the *Hilbert functor* of d points in X is the association $\underline{\mathrm{Hilb}}^d X\colon \mathrm{Schemes}^{\mathrm{op}} \to \mathrm{Sets}$ defined by

$$\big(\underline{\mathrm{Hilb}}^d X\big)(S) = \{\mathcal{Z} \hookrightarrow X\times S \text{ closed subscheme} \mid \mathcal{Z}\to S \text{ is flat and finite of degree } d\}\,.$$

By a celebrated result of Grothendieck, the functor $\underline{\mathrm{Hilb}}^d X$ is representable, and the fine moduli space $\mathrm{Hilb}^d X$ representing it is a quasi-projective scheme called a *Hilbert scheme* [Gro62]. As there is a bijection between closed subschemes Z and their ideal sheaves I_Z, we will denote points of the Hilbert scheme by $[Z]$ or $[I_Z]$.

Although it is not known, in general, if the Hilbert scheme $\mathrm{Hilb}^d\mathbb{A}^3$ is irreducible, there is a component which can always be defined. Precisely, the *smoothable component* is defined as the closure of the open subscheme $U \subset \mathrm{Hilb}^d\mathbb{A}^3$ parametrising closed and reduced zero-dimensional subschemes of length d of $\mathbb{A}^3$.

Definition 2.1. A point $[Z] \in \operatorname{Hilb}^d \mathbb{A}^3$ is *smoothable* if it belongs to the smoothable component.

We now report the description of the tangent space of $\operatorname{Hilb}^d X$ in terms of first-order deformations. Let $\mathbb{C}[\epsilon] = \mathbb{C}[t]/(t^2)$ be the ring of *dual numbers*, where ϵ stands for the equivalence class of t in the quotient ring, and let $D = \operatorname{Spec} \mathbb{C}[\epsilon]$ be its spectrum. A *first-order deformation* of a scheme $Z \subset \mathbb{A}^n$ is a commutative diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & \mathcal{D} \arrow[d, "f"] \\ \operatorname{Spec} \mathbb{C} \arrow[r, hook] & D \,, \end{tikzcd}$$

where f is a flat morphism, such that the induced morphism $Z \to \mathcal{D} \times_D \operatorname{Spec} \mathbb{C}$ is an isomorphism. As a consequence of the universal property of the Hilbert scheme, we can associate with each first-order deformation of Z a morphisms of schemes $D \to \operatorname{Hilb}^d \mathbb{A}^n$ mapping the unique closed point of D to $[I_Z]$. We denote the collection of these morphisms by $\operatorname{Hom}_{I_Z}(D, \operatorname{Hilb}^d \mathbb{A}^n)$. Recall that this set has a canonical structure of vector space and that it is canonically isomorphic to the tangent space $\mathsf{T}_{[I_Z]} \operatorname{Hilb}^d \mathbb{A}^n$. The following result gives a different characterisation of the tangent space of the Hilbert scheme at a given point $[I] \in \operatorname{Hilb}^d \mathbb{A}^n$.

Theorem 2.2 ([FGI+05, Corollary 6.4.10]). *Let* $[I] \in \operatorname{Hilb}^d \mathbb{A}^n$ *be any point, and let* $\mathsf{T}_{[I]} \operatorname{Hilb}^d \mathbb{A}^n$ *denote the tangent space of* $\operatorname{Hilb}^d \mathbb{A}^n$ *at* $[I]$*. Then,*

$$\mathsf{T}_{[I]} \operatorname{Hilb}^d \mathbb{A}^n \simeq \operatorname{Hom}_{\mathbb{C}[x_1,\ldots,x_n]}(I, \mathbb{C}[x_1,\ldots,x_n]/I) \simeq \operatorname{Hom}_I\big(D, \operatorname{Hilb}^d \mathbb{A}^n\big)\,.$$

In Section 3 we give an explicit description of the tangent space $\mathsf{T}_{[I]} \operatorname{Hilb}^d \mathbb{A}^n$ for an ideal I disproving the parity conjecture. Our main tools are the *marked bases*[1] (see [CR11, BCLR13, BLR13, LR16, CMR15, BCR17] and references therein). We recall here the main definitions and properties, and we postpone more details to the appendix.

In this paper, we describe monomials in $\mathbb{C}[x_1,\ldots,x_n]$ via the standard multi-index notation. Namely, for any $\underline{\alpha} = (\alpha_1,\ldots,\alpha_n) \in \mathbb{Z}^n_{\geqslant 0}$, the symbol $x^{\underline{\alpha}}$ stands for $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$.

A set of monomials $\mathcal{N}$ in $\mathbb{C}[x_1,\ldots,x_n]$ is called an *order ideal* if it is closed under subdivision, that is,

$$x^{\underline{\alpha}} \in \mathcal{N} \quad \text{and} \quad x^{\underline{\beta}} \mid x^{\underline{\alpha}} \quad \Longrightarrow \quad x^{\underline{\beta}} \in \mathcal{N}.$$

The monomials in the complementary set of $\mathcal{N}$ generate a monomial ideal in $\mathbb{C}[x_1,\ldots,x_n]$ that we denote by $J_{\mathcal{N}}$. Notice that $J_{\mathcal{N}} \cap \mathcal{N} = \emptyset$.

If the order ideal is finite, then the Krull dimension of the quotient ring $\mathbb{C}[x_1,\ldots,x_n]/J_{\mathcal{N}}$ is 0 and $\mathcal{N}$ is the unique monomial basis of $\mathbb{C}[x_1,\ldots,x_n]/J_{\mathcal{N}}$ as $\mathbb{C}$-vector space; that is, $J_{\mathcal{N}}$ defines a zero-dimensional scheme in $\mathbb{A}^n$ of length $|\mathcal{N}|$.

Definition 2.3 ([BCR17, Definition 6.2]). Let $\mathcal{N} \subset \mathbb{C}[x_1,\ldots,x_n]$ be a finite-order ideal. The marked family functor associated with $\mathcal{N}$ is the covariant functor $\underline{\mathrm{Mf}}_{\mathcal{N}}\colon \mathbb{C}\text{-Algebras} \to \text{Sets}$ defined by

$$\underline{\mathrm{Mf}}_{\mathcal{N}}(A) = \{I \subset A[x_1,\ldots,x_n] \mid A[x_1,\ldots,x_n] = I \oplus \langle \mathcal{N} \rangle\}\,. \tag{2.1}$$

[1]The theory of marked bases has been introduced for homogeneous ideals and the projective setting. The affine version we refer to has been developed in [BCR17]. For the zero-dimensional case, see also [BC22].

The marked family functor $\underline{\mathrm{Mf}}_{\mathcal{N}}$ is a subfunctor of the Hilbert functor since for every ideal $I \in \underline{\mathrm{Mf}}_{\mathcal{N}}(A)$, the quotient algebra $A[x_1, \dots, x_n]/I$ turns out to be a free A-module with a monomial basis given by the order ideal $\mathcal{N}$. Moreover, the functor $\underline{\mathrm{Mf}}_{\mathcal{N}}$ is an open representable subfunctor of the Hilbert functor [BCR17, Theorem 6.6 and Proposition 6.13]. Hence, the fine moduli space $\mathrm{Mf}_{\mathcal{N}}$ representing this functor is an open subscheme of $\mathrm{Hilb}^{|\mathcal{N}|} \mathbb{A}^n$.

The collection of schemes

$$\{\mathrm{Mf}_{\mathcal{N}} \mid \mathcal{N} \subset \mathbb{C}[x_1, \dots, x_n] \text{ is an order ideal of cardinality } d\}$$

is an atlas of $\mathrm{Hilb}^d \mathbb{A}^n$; see [BC22, Proposition 5]. Thus, given a point $[I] \in \mathrm{Hilb}^d \mathbb{A}^n$, we can always compute $\mathsf{T}_{[I]} \mathrm{Hilb}^d \mathbb{A}^n$ as $\mathsf{T}_{[I]} \mathrm{Mf}_{\mathcal{N}} \simeq \mathrm{Hom}_I(D, \mathrm{Mf}_{\mathcal{N}})$ for a suitable order ideal $\mathcal{N}$.

Since we work with closed zero-dimensional subschemes of $\mathbb{A}^3$, we fix once and for all the ring R to be the polynomial ring in three variables and complex coefficients, $R = \mathbb{C}[x, y, z]$, and the ideal $\mathfrak{m} \subset R$ to be the unique maximal homogeneous ideal $\mathfrak{m} = (x, y, z)$. Moreover, we denote the kth graded piece of a graded module M by M_k. For instance, we have

$$R_k = \{p \in \mathbb{C}[x, y, z] \mid p \text{ is homogeneous of degree } \deg p = k\} \cup \{0\}\,.$$

We conclude this section by warning that, with abuse of notation, we denote both the elements of R and the elements of R/I in the same way, tacitly assuming that we are choosing representatives.

3. Main results

In the following, given a subspace W of a vector space V, we refer to any subspace U of V such that $V = W \oplus U$ as a complement of W.

Theorem 3.1. *Let $\ell \in R_1 \setminus \{0\}$ be a non-zero linear form, and let $I \subset R$ be an ideal of the form*

$$I = \big((\ell) + \mathfrak{m}^2\big)^2 + (v)\,, \tag{3.1}$$

where $v \in \big(\mathrm{Sym}^3(L) + \ell \cdot L\big) \setminus \{0\}$ for some $L \in \mathrm{Gr}(2, R_1)$ complement of $\langle \ell \rangle$. Then, I has colength $\operatorname{colen} I = 12$ and, for a general choice of v, we have

$$\dim_{\mathbb{C}} \mathsf{T}_{[I]} \mathrm{Hilb}^{12} \mathbb{A}^3 = 45\,.$$

Proof. Without loss of generality we can suppose $\ell = x$ and $L = \langle y, z \rangle$. Under these assumptions the ideal I takes the form

$$I = \big((x) + (y, z)^2\big)^2 + \big(b_0 y^3 + b_1 y^2 z + b_2 y z^2 + b_3 z^3 + b_4 xy + b_5 xz\big)$$

for some $[b_0 : \cdots : b_5] \in \mathbb{P}^5$.

In order to prove the first part of the statement, first notice that the ideal

$$J = \big((x) + (y, z)^2\big)^2$$

has colength $\operatorname{colen} J = 13$. Now, we also have

$$b_0 y^3 + b_1 y^2 z + b_2 y z^2 + b_3 z^3 + b_4 xy + b_5 xz \notin J\,,$$

which implies $\operatorname{colen} I = 12$ for all I of the form (3.1).

We now move to the computation of the tangent space $\mathsf{T}_{[I]} \mathrm{Hilb}^{12} \mathbb{A}^3$ for a general I of the form (3.1). By the generality of v, we can set $b_0 = 1$, and the ideal I is minimally generated by

$$x^2\,, \quad xy^2\,, \quad xyz\,, \quad xz^2\,, \quad y^2z^2\,, \quad yz^3\,, \quad z^4\,, \quad v = y^3 + b_1 y^2 z + b_2 y z^2 + b_3 z^3 + b_4 xy + b_5 xz\,.$$

Thus, the quotient algebra $\mathbb{C}[x,y,z]/I$ admits the monomial basis
$$\mathcal{N} = \left\{y^2z, yz^2, z^3, xy, y^2, xz, yz, z^2, x, y, z, 1\right\},$$
and $[I] \in \mathrm{Mf}_{\mathcal{N}}$. We can compute the tangent space $\mathsf{T}_{[I]} \operatorname{Hilb}^{12} \mathbb{A}^3$ as $\mathsf{T}_{[I]} \mathrm{Mf}_{\mathcal{N}} \simeq \operatorname{Hom}_I(D, \mathrm{Mf}_{\mathcal{N}})$. If the polynomial $\mathfrak{B} := b_3b_4^3 - b_2b_4^2b_5 + b_1b_4b_5^2 - b_5^3$ is different from zero, the first-order deformations are described by the ideal generated by the polynomials
$$\begin{aligned}
&x^2 + \epsilon_1 y^2z + \epsilon_2 yz^2 + \epsilon_3 z^3 + \epsilon_4 xy + \epsilon_5 xz + 2\epsilon_{10} x\,,\\
&xy^2 + \epsilon_6 y^2z + \epsilon_7 yz^2 + \epsilon_8 z^3 + \epsilon_9 xy + \epsilon_{10} y^2 + \epsilon_{11} xz\,,\\
&xyz + \epsilon_{12} y^2z + \epsilon_{13} yz^2 + \epsilon_{14} z^3 + \epsilon_{15} xy + \epsilon_{16} xz + \epsilon_{10} yz\,,\\
&xz^2 + \epsilon_{17} y^2z + \epsilon_{18} yz^2 + \epsilon_{19} z^3 + \epsilon_{20} xy + \epsilon_{21} xz + \epsilon_{10} z^2\,,\\
&y^2z^2 + \epsilon_{22} y^2z + \epsilon_{23} yz^2 + \epsilon_{24} z^3 + \epsilon_{25} xy + \epsilon_{26} xz\,,\\
&yz^3 + \epsilon_{27} y^2z + \epsilon_{28} yz^2 + \epsilon_{29} z^3 + \epsilon_{30} xy + \epsilon_{31} xz\,,\\
&z^4 + \epsilon_{32} y^2z + \epsilon_{33} yz^2 + \epsilon_{34} z^3 + \epsilon_{35} xy + \epsilon_{36} xz\,,\\
&v + \epsilon_{37} y^2z + \epsilon_{38} yz^2 + \epsilon_{39} z^3 + \epsilon_{40} xy + \epsilon_{41} y^2 + \epsilon_{42} xz + \epsilon_{43} yz + \epsilon_{44} z^2 +\\
&\quad + \epsilon_{45} x + b_4\epsilon_{10} y + b_5 \epsilon_{10} z
\end{aligned} \tag{3.2}$$
in $\mathbb{C}[\epsilon_1, \ldots, \epsilon_{45}][x,y,z]$, where ϵ_i is the equivalence class of t_i in $\mathbb{C}[t_1, \ldots, t_{45}]/(t_1, \ldots, t_{45})^2$ (see Appendix A.1 for the computational details).[2] Thus, we have
$$\dim_{\mathbb{C}} \mathsf{T}_{[I]} \operatorname{Hilb}^{12} \mathbb{A}^3 = 45\,. \qquad \square$$

COROLLARY 3.2. *The parity conjecture (Conjecture 1) is false for any $d \geqslant 12$.*

Proof. For $d = 12$ a counterexample is provided by Theorem 3.1. The case $d = 12 + k$ with $k \geqslant 1$ is treated by adding a k-tuple of distinct points disjoint from the support of the counterexample of length 12. $\square$

Remark 3.3 (cf. [RS25, Remark 11 and Example 12]). In [RS25] the authors prove that the parity conjecture holds for any ideal homogeneous with respect to a grading taking values in a torsion-free abelian group such that $\deg(x) + \deg(y) + \deg(z)$ is not divisible by 2. Our example (1.1) is in fact homogeneous with respect to the grading $\deg\colon R \to \mathbb{Z}^2$ defined by $\deg(x) = (3,0)$, $\deg(y) = (1,1)$, and $\deg(z) = (0,3)$. However, $\deg(x) + \deg(y) + \deg(z) = 2(2,2)$.

Remark 3.4. It is worth mentioning that with the help of a computer, we are able to produce many counterexamples in addition to the one in the proof of Corollary 3.2. For instance, the following is a counterexample of length 78 and tangent space of complex dimension 263:
$$\begin{aligned}
I = \big(&y^2z^4, x^2z^4 + z^6, y^3z^3, x^3z^3, xy^3z^2, xy^4z + x^3yz^2 - y^2z^3, x^4yz - z^6 + y^3z^2\,,\\
&x^3y^3 - x^2yz^2 - yz^4, x^5z + x^3y^2z + xyz^4 - x^2z^3 - z^5, y^6 + x^4z^2 + xy^2z^3 + xz^5\,,\\
&x^6 - y^4z - xz^4, y^5z^2 + xyz^5, x^2y^5 + xyz^5 + z^7\,,\\
&x^5y^2 + x^3y^2z^2 + xyz^5 + x^2y^4 - y^4z^2 + z^6 - xy^2z^2\big)\,.
\end{aligned}$$
In particular, it is not of the form described in the proof of Corollary 3.2.

COROLLARY 3.5. *The ideal*
$$\mathcal{I}_x = \big((x + a_1y + a_2z) + (y,z)^2\big)^2 + \big(b_0y^3 + b_1y^2z + b_2yz^2 + b_3z^3 + b_4xy + b_5xz\big) \subset R[a_1,a_2][b_0,\ldots,b_5]$$
defines a family $\mathcal{Z}_{\mathcal{I}_x} \subset \mathbb{A}^3 \times \mathbb{A}^2 \times \mathbb{P}^5$ of zero-dimensional subschemes of $\mathbb{A}^3$ flat over $\mathbb{A}^2 \times \mathbb{P}^5$.

[2] Notice that in the appendix, the polynomial v is denoted by f_{y^3}.

Analogously, there are ideals $\mathcal{I}_y$ and $\mathcal{I}_z$ defining families $\mathcal{Z}_{\mathcal{I}_y}, \mathcal{Z}_{\mathcal{I}_z} \subset \mathbb{A}^3 \times \mathbb{A}^2 \times \mathbb{P}^5$ of zero-dimensional subschemes of $\mathbb{A}^3$ flat over $\mathbb{A}^2 \times \mathbb{P}^5$.

We now move to the study of the collection of ideals of the form (3.1).

Lemma 3.6. *Let us denote by $\Lambda_{\ell,L}$ the set of ideals*

$$\Lambda_{\ell,L} = \left\{ \left((\ell) + \mathfrak{m}^2\right)^2 + (v) \mid v \in \left(\mathrm{Sym}^3(L) + \ell \cdot L\right) \smallsetminus \{0\} \right\},$$

where $\ell \in R_1 \smallsetminus \{0\}$ is a non-zero linear form and $L \in \mathrm{Gr}(2, R_1)$ is a complement of $\langle \ell \rangle$. Then, the set $\Lambda_{\ell,L}$ does not depend on the choice of L; that is, for any two complements $L, L' \in \mathrm{Gr}(2, R_1)$ of $\langle \ell \rangle$, we have

$$\Lambda_{\ell,L} = \Lambda_{\ell,L'} .$$

We denote this set by Λ_ℓ.

Proof. Let $\ell \in R_1 \smallsetminus \{0\}$ be a non-zero linear form, and let $L, L' \in \mathrm{Gr}(2, R_1)$ be two complements of $\langle \ell \rangle$. Then, there are linear forms $\ell_1, \ell_2 \in R_1$ and values $(\alpha, \beta) \in \mathbb{C}^2$ such that

$$\{\ell_1, \ell_2\} \quad \text{and} \quad \{\ell_1' = \ell_1 + \alpha\ell, \ell_2' = \ell_2 + \beta\ell\}$$

are bases of L and L', respectively.

Now, the two ideals

$$\begin{aligned} &\left((\ell) + \mathfrak{m}^2\right)^2 + \left(b_0\ell_1^3 + b_1\ell_1^2\ell_2 + b_2\ell_1\ell_2^2 + b_3\ell_2^3 + b_4\ell\ell_1 + b_5\ell\ell_2\right), \\ &\left((\ell) + \mathfrak{m}^2\right)^2 + \left(b_0\ell_1'^3 + b_1\ell_1'^2\ell_2' + b_2\ell_1'\ell_2'^2 + b_3\ell_2'^3 + b_4\ell\ell_1' + b_5\ell\ell_2'\right) \end{aligned} \tag{3.3}$$

are the same for any choice of $(\alpha, \beta) \in \mathbb{C}^2$. This completes the proof of the lemma. □

Remark 3.7. Let $\ell \in R_1 \smallsetminus \{0\}$ be a non-zero linear form. Let us consider Δ_ℓ defined as

$$\Delta_\ell = \mathbb{P}\left(\mathrm{Sym}^3(R_1/\langle \ell \rangle) \oplus (\langle \ell \rangle \underset{\mathbb{C}}{\otimes} (R_1/\langle \ell \rangle))\right) \cong \mathbb{P}^5 .$$

Now, given a complement L of $\langle \ell \rangle$, the restriction of the natural projection $\pi_\ell \colon R_1 \to R_1/\langle \ell \rangle$ to L induces a bijection $\gamma_\ell \colon \Delta_\ell \to \Lambda_\ell$. Precisely, we have

$$\begin{aligned} \Delta_\ell &\xrightarrow{\quad\quad\gamma_\ell\quad\quad} \Lambda_\ell \\ [(p, \ell \otimes q)] &\longmapsto \left(\ell + \mathfrak{m}^2\right)^2 + \left(\overline{p} + \ell\overline{q}\right), \end{aligned}$$

where $\overline{p} = (\mathrm{Sym}\,\pi_\ell)^{-1}(p) \cap \mathrm{Sym}(L)$ and $\overline{q} = \pi_\ell^{-1}(q) \cap L$. Equation (3.3) shows that this map does not depend on the choice of the complement L.

We abuse notation and interpret an element in Δ_ℓ as the corresponding ideal in Λ_ℓ. As a consequence of Theorem 3.1, the map γ_ℓ introduced in Remark 3.7 induces a morphism $\Delta_\ell \to \mathrm{Hilb}^{12} \mathbb{A}^3$ that, with abuse of notation, we denote by γ_ℓ as well. The image of γ_ℓ coincides with the image of $\{(a_1, a_2)\} \times \mathbb{P}^5$ under the classifying morphism induced by the family $\mathcal{Z}_{\mathcal{I}_x}$ from Corollary 3.5.

We conclude this section by showing that the ideals of the form (3.1) are smoothable.

Proposition 3.8. *Any ideal $I \subset R$ of the form (3.1) is smoothable.*

Proof. Since being smoothable is a closed condition, it is enough to prove the statement for the general $I \subset R$ of the form (3.1). Without loss of generality, up to the $\mathrm{GL}(3, \mathbb{C})$-action, we can suppose $\ell = x$. Then, I takes the form

$$I = \left((x) + (y, z)^2\right)^2 + \left(y^3 + b_1y^2z + b_2yz^2 + b_3z^3 + b_4xy + b_5xz\right).$$

In order to prove the statement, it is enough to exhibit a flat family $\mathcal{Z} \hookrightarrow \mathbb{A}^1 \times \mathbb{A}^3$ of zero-dimensional subschemes of $\mathbb{A}^3$ such that the fiber over the origin $\mathcal{Z}_0$ is I and such that the general fiber $\mathcal{Z}_t$ is supported in at least two points and hence is smoothable (see [DJNT17]).

For every counterexample $I \subset R$ to the parity conjecture discussed in Theorem 3.1, the tangent vector

$$\big(x^2, xy^2, xyz + \boldsymbol{\epsilon y^2 z}, xz^2, y^2z^2, yz^3, z^4, y^3 + b_1y^2z + b_2yz^2 + b_3z^3 + b_4xy + b_5xz\big)$$

lifts to the deformation given by the polynomials

$$\begin{gathered} x^2 \frac{b_4^2(b_3b_4 - b_2b_5)}{\mathfrak{B}} t^2y^2 - \frac{b_4(b_3b_4 - b_2b_5)(b_1b_4 - b_5)}{\mathfrak{B}} t^2yz + \frac{b_3b_4b_5(b_1b_4 - b_5)}{\mathfrak{B}} t^2\, z^2 \\ - \frac{b_4\big(2\mathfrak{B} - b_5^2(b_1b_4 - b_5)\big)}{\mathfrak{B}} t^2x + \frac{b_4^4(b_3b_4 - b_2b_5)}{\mathfrak{B}} t^4 , \quad xy^2 , \quad xyz + ty^2z - b_4t^2yz , \qquad (3.4) \\ xz^2 - b_4t^2z^2 , \quad y^2z^2 , \quad yz^3 , \quad z^4 , \quad y^3 + b_1y^2z + b_2yz^2 + b_3z^3 + b_4xy + b_5xz - b_4^2t^2y - b_4b_5t^2z , \end{gathered}$$

whenever the polynomial $\mathfrak{B} = b_3b_4^3 - b_2b_4^2b_5 + b_1b_4b_5^2 - b_5^3 = b_4^2(b_3b_4 - b_2b_5) + b_5^2(b_1b_4 - b_5)$ is different from zero. Notice that away from the zero locus of the polynomial $\mathfrak{B}$, the ideal I has tangent space of dimension 45, as proven in Theorem 3.1. If $b_4 \neq 0$, the generic fiber is the disjoint union of four schemes of length 7, 2, 2, and 1 (see Appendix A.2 for the details). $\square$

3.1 Comments

In this subsection we make some comments regarding our main result.

The first observation is that Theorem 3.1 and Proposition 3.8 admit more direct proofs that does not require the use of a computer. We decided to present them in this form because, as an outcome, we obtain a more accurate description of the locus parametrising the counterexamples. Precisely, in Theorem 3.1 we obtain the polynomial $\mathfrak{B}$ that defines the complement of this locus. In Proposition 3.8, instead, we obtain an explicit deformation of our counterexample into a scheme with support consisting of four distinct points. It is worth mentioning that we found the first counterexamples via *Macaulay2* computations using a marked basis as main tool. Also for this reason we have presented the proofs in this form.

The Hilbert scheme of 12 points on a smooth threefold is interesting for several reasons. For instance, 12 is the minimal number of points for which the irreducibility of the Hilbert scheme is not known. It is then natural to ask whether the failure of the parity conjecture is unique for $\mathrm{Hilb}^d \mathbb{A}^3$ for $d \geqslant 12$ or if it already fails for $\mathrm{Hilb}^{11} \mathbb{A}^3$. Our method does not produce any counterexample of length smaller than 12. Nevertheless, we tested a large number of ideals of colength 11 without finding new counterexamples. Precisely, starting from the description of the so-called Hilbert–Samuel stratification of the punctual Hilbert scheme of 11 points described in [DJNT17] and using the marked basis technology, we have generated some examples of non-homogeneous ideals in each of the Hilbert–Samuel strata, certifying the veracity of the conjecture at each of these points. A complete proof of the parity conjecture for the Hilbert scheme of 11 points is outside the scope of this article. Hence, we leave this question open.

We conclude this section with a remark regarding the characteristic of the base field. A direct check shows that our counterexamples are valid in any characteristic other than 2 (see Remark A.5). We have tested in characteristic 2 all the counterexamples we found (see Remark 3.4), and none of them disproves the conjecture. Therefore, the parity conjecture remains open in characteristic 2. However, in the recent paper [GGGL24] we also provide a counterexample for the Quot scheme $\mathrm{Quot}_2^8\mathbb{A}^3$ using the same techniques.

4. Explicit description of the locus of counterexamples

We now describe the locus in $\mathrm{Hilb}^{12}\mathbb{A}^3$ of the counterexamples to the parity conjecture given in Theorem 3.1. It is worth pointing out that we do not know if the described locus comprises all of the counterexamples to Conjecture 1 of length less than or equal to 12.

We consider the projective space $\mathbb{P}(R_1)$ parametrising 1-dimensional linear subspaces of R_1 and the tautological exact sequence on it

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{O}_{\mathbb{P}(R_1)} \otimes R_1 \longrightarrow \mathcal{Q} \longrightarrow 0\,,$$

where $\mathcal{S}$ is the universal subbundle and $\mathcal{Q}$ is the universal quotient bundle. We set E to be the vector bundle

$$E = \mathrm{Sym}^3(\mathcal{Q}) \oplus \big(\mathcal{S} \underset{\mathcal{O}_{\mathbb{P}(R_1)}}{\otimes} \mathcal{Q}\big)\,,$$

and we consider the associated projective bundle $\mathbb{P}(E)$ over $\mathbb{P}(R_1)$. Its fiber $\mathbb{P}(E)_\ell$ over the point $[\ell] \in \mathbb{P}(R_1)$ is the space Δ_ℓ introduced in Remark 3.7. The morphism defined by the association

$$\begin{aligned} \mathbb{P}(E) &\longrightarrow \mathrm{Hilb}^{12}\mathbb{A}^3 \\ ([\ell],[v]) &\longmapsto \gamma_\ell([v])\,, \end{aligned}$$

where γ_ℓ is the bijection defined in Remark 3.7, induces a family $\mathcal{Z}_E \subset \mathbb{P}(E) \times \mathbb{A}^3$ of zero-dimensional schemes of length 12.

Consider the coordinate atlas on $\mathbb{P}(E)$ induced by the basis $\{x, y, z\}$ of R_1; then the restrictions of the family to the coordinate charts agree with the families $\mathcal{Z}_{\mathcal{I}_x}, \mathcal{Z}_{\mathcal{I}_y}, \mathcal{Z}_{\mathcal{I}_z}$ in Corollary 3.5.

Finally, we enlarge our family by acting via translations, and we get a new family $\mathcal{Z} \subset \mathbb{P}(E) \times \mathbb{A}^3 \times \mathbb{A}^3$ of fat points not necessarily supported at the origin $0 \in \mathbb{A}^3$.

In what follows we shall denote by V the product $\mathbb{P}(E) \times \mathbb{A}^3$ and by $\theta\colon V \to \mathrm{Hilb}^{12}\mathbb{A}^3$ the classifying morphism.

Remark 4.1. Notice that set-theoretically, the image $\theta(V)$ consists of the ideals described in

$$\theta(V) = \big\{[I] \in \mathrm{Hilb}^{12}\mathbb{A}^3 \mid I \text{ is of the form (3.1)}\big\}\,.$$

Theorem 4.2. *The morphism $\theta\colon V \to \mathrm{Hilb}^{12}\mathbb{A}^3$ described above is a closed immersion of the smooth and irreducible 10-dimensional variety V.*

Proof. We start by proving that the map θ is universally closed. Let $\mathrm{Hilb}^{12}_0\mathbb{A}^3$ be the projective subscheme of $\mathrm{Hilb}^{12}\mathbb{A}^3$ parametrising fat points $Z \subset \mathbb{A}^3$ of length 12 supported at the origin $0 \in \mathbb{A}^3$. Also let $\mathbb{A}^3 \times \mathrm{Hilb}^{12}_0\mathbb{A}^3 \subset \mathrm{Hilb}^{12}\mathbb{A}^3$ be the closed subscheme of $\mathrm{Hilb}^{12}\mathbb{A}^3$ parametrising all the fat points $Z \subset \mathbb{A}^3$ of length 12. The map $\theta|_{\mathbb{P}(E)\times\{0\}}$ takes values in $\mathrm{Hilb}^{12}_0\mathbb{A}^3$; it is proper because $\mathbb{P}(E)$ is projective. Finally, the map θ is universally closed because the square in the following diagram is Cartesian:

$$\begin{tikzcd} V \arrow[r] \arrow[rr, bend left, "\theta"] \arrow[d] & \mathbb{A}^3 \times \mathrm{Hilb}^{12}_0\mathbb{A}^3 \arrow[r] \arrow[d] & \mathrm{Hilb}^{12}\mathbb{A}^3 \\ \mathbb{P}(E) \arrow[r] & \mathrm{Hilb}^{12}_0\mathbb{A}^3\,. & \end{tikzcd}$$

The map θ is injective because different points in V lead to different ideals in $\mathrm{Hilb}^{12}\mathbb{A}^3$, as can be seen as follows. It is enough to treat fat points supported at the origin, that is, points of

$\mathbb{P}(E)\times\{0\}$ that we identify with $\mathbb{P}(E)$. Now to conclude, the bijection γ_ℓ maps different points of Δ_ℓ to different ideals, and any ideal $\gamma_\ell(\Delta_\ell)$ contains, up to scalar, the square of a unique linear form, namely ℓ.

In order to prove the statement, we show that the differential $d\theta$ is of full rank at every point. It is enough to prove the claim for ideals belonging to the open $U_x \subset \mathbb{P}(E)\times\{0\}\subset V$ consisting of ideals of the form

$$\big((x+a_1y+a_2z)+(y,z)^2\big)^2+\big(b_0y^3+b_1y^2z+b_2yz^2+b_3z^3+b_4xy+b_5xz\big)\,.$$

The proof now consists of a direct and standard computation, but it is too lengthy to be reported on paper. Hence, we exhibit the result only for the ideals lying in the open subset defined by the condition $b_0\neq 0$. We refer to the *M2* ancillary file [GGGLa] for the complete discussion.

Concretely, given $\ell = x+a_1y+a_2z$ and $v=y^3+b_1y^2z+b_2yz^2+b_3z^3+b_4xy+b_5xz$, we show that the following differential is injective:

$$d\theta_{([\ell],[v],0)}\colon \mathsf{T}_{[\ell]}\mathbb{P}(R_1)\times \mathsf{T}_{[v]}\Delta_\ell\times\mathsf{T}_0\mathbb{A}^3 \longrightarrow \mathsf{T}_{\gamma_\ell([v])}\operatorname{Hilb}^{12}\mathbb{A}^3\,.$$

Let us denote by $\{\partial_{\tau_x},\partial_{\tau_y},\partial_{\tau_z}\}\subset \mathsf{T}_0\mathbb{A}^3$ the basis corresponding to the variables x, y, z. The image of $d\theta$ at the point $([\ell],[v],0)$ is generated by the first-order deformations

$$\begin{aligned}
\mathsf{T}_{[\ell]}\mathbb{P}(R_1)\colon d\theta(\partial_{\alpha_1}) &= \big(x^2+2\epsilon xy, xy^2+\epsilon y^3, xyz+\epsilon y^2z, xz^2+\epsilon yz^2, y^2z^2, yz^3, z^4,\\
&\qquad v+b_4\epsilon y^2+b_5\epsilon yz\big)\,,\\
d\theta(\partial_{\alpha_2}) &= \big(x^2+2\epsilon xz, xy^2+\epsilon y^2z, xyz+\epsilon yz^2, xz^2+\epsilon z^3, y^2z^2, yz^3, z^4,\\
&\qquad v+b_4\epsilon yz+b_5\epsilon z^2\big)\,,\\
\mathsf{T}_{[v]}\Delta_\ell\colon\ d\theta(\partial_{\beta_1}) &= \big(x^2, xy^2, xyz, xz^2, y^2z^2, yz^3, z^4, v+\epsilon y^2z\big)\,,\\
d\theta(\partial_{\beta_2}) &= \big(x^2, xy^2, xyz, xz^2, y^2z^2, yz^3, z^4, v+\epsilon yz^2\big)\,,\\
d\theta(\partial_{\beta_3}) &= \big(x^2, xy^2, xyz, xz^2, y^2z^2, yz^3, z^4, v+\epsilon z^3\big)\,,\\
d\theta(\partial_{\beta_4}) &= \big(x^2, xy^2, xyz, xz^2, y^2z^2, yz^3, z^4, v+\epsilon xy\big)\,,\\
d\theta(\partial_{\beta_5}) &= \big(x^2, xy^2, xyz, xz^2, y^2z^2, yz^3, z^4, v+\epsilon xz\big)\,,\\
\mathsf{T}_0\mathbb{A}^3\colon d\theta(\partial_{\tau_x}) &= \big(x^2+2\epsilon x, xy^2+\epsilon y^2, xyz+\epsilon yz, xz^2+\epsilon z^2, y^2z^2, yz^3, z^4,\\
&\qquad v+b_4\epsilon y+b_5\epsilon z\big)\,,\\
d\theta(\partial_{\tau_y}) &= \big(x^2, xy^2+2\epsilon xy, xyz+\epsilon xz, xz^2, y^2z^2+2\epsilon yz^2, yz^3+\epsilon z^3, z^4,\\
&\qquad v+3\epsilon y^2+2b_1\epsilon yz+b_2\epsilon z^2+b_4\epsilon x\big)\,,\\
d\theta(\partial_{\tau_z}) &= \big(x^2, xy^2, xyz+\epsilon xy, xz^2+2\epsilon xz, y^2z^2+2\epsilon y^2z, yz^3+3\epsilon yz^2,\\
&\qquad z^4+4\epsilon z^3, v+b_1\epsilon y^2+2b_2\epsilon yz+3b_3\epsilon z^2+b_5\epsilon x\big)\,;
\end{aligned}$$

thus θ is a closed immersion of the irreducible and smooth variety V of dimension 10. □

Appendix. Computational methods: Marked bases

In this section we give further details about the computational content of the paper. We also provide ancillary files based on the *M2* Package [GGGLb].

In the polynomial ring $\mathbb{C}[x_1,\dots,x_n]$, we assume that the variables are ordered as $x_1<x_2<\cdots<x_n$. Given a monomial $x^{\underline{\alpha}}\neq 1\in\mathbb{C}[x_1,\dots,x_n]$, we define the *minimum* and the *maximum*

of an $x^{\underline{\alpha}} \neq 1$ as the smallest and largest variables dividing $x^{\underline{\alpha}}$, and we define its *Pommaret cone* as the set of monomials

$$\mathcal{C}(x^{\underline{\alpha}}) = \{x^{\underline{\alpha}}\} \cup \left\{x^{\underline{\alpha}}\, x^{\underline{\beta}} \mid \max x^{\underline{\beta}} \leqslant \min x^{\underline{\alpha}}\right\}.$$

The monomials in the complementary set of an order ideal $\mathcal{N}$ generate a monomial ideal in $\mathbb{C}[x_1, \dots, x_n]$ that we denote by $J_{\mathcal{N}}$. If $\mathcal{N}$ is finite, the ideal $J_{\mathcal{N}}$ is always *quasi-stable* [BCR17, Corollary 2.3], that is, $J_{\mathcal{N}}$ has a finite set of generators $\mathcal{P}_{J_{\mathcal{N}}} = \{x^{\underline{\alpha}_1}, \dots, x^{\underline{\alpha}_k}\}$, called the *Pommaret basis* of $J_{\mathcal{N}}$, such that the set of monomials in $J_{\mathcal{N}}$ decomposes as the union of the Pommaret cones

$$\mathcal{C}(x^{\underline{\alpha}_1}) \sqcup \cdots \sqcup \mathcal{C}(x^{\underline{\alpha}_k})$$

of the elements of the Pommaret basis.

Definition A.1. Let $J \in \mathbb{C}[x_1, \dots, x_n]$ be a monomial ideal of finite colength, $\mathcal{P}_J$ be its Pommaret basis, and $\mathcal{N}$ be the finite-order ideal whose image in the quotient is a monomial basis. A (monic) *J-marked set* is a set of polynomials

$$F := \left\{ f_{\underline{\alpha}} := x^{\underline{\alpha}} + \sum_{x^{\underline{\gamma}} \in \mathcal{N}} c_{\underline{\alpha},\underline{\gamma}} x^{\underline{\gamma}} \;\middle|\; x^{\underline{\alpha}} \in \mathcal{P}_J,\; c_{\underline{\alpha},\underline{\gamma}} \in \mathbb{C} \right\}.$$

A J-marked set F is called a *J-marked basis* if

$$\mathbb{C}[x_1, \dots, x_n] = (F) \oplus \langle \mathcal{N} \rangle$$

as vector spaces.

The notion of a marked basis generalises the notion of a reduced Gröbner basis. In particular, from the definition it follows that $\mathcal{N}$ is a basis of the quotient algebra $\mathbb{C}[x_1, \dots, x_n]/(F)$. Hence, every ideal $I \subset \mathbb{C}[x_1, \dots, x_n]$ generated by a J-marked set defines a zero-dimensional scheme of length $|\mathcal{N}|$.

With the notion of a marked basis, we can rephrase the definition of a marked family functor as follows:

$$\underline{\mathrm{Mf}}_{\mathcal{N}}(A) = \left\{(F) \subset A[x_1, \dots, x_n] \mid F \text{ is a } J_{\mathcal{N}}\text{-marked basis}\right\}.$$

This formulation has a crucial role because there is an effective (and algorithmic) criterion to determine whether a J-marked set is a J-marked basis. Given a J-marked set $F = \{f_{\underline{\alpha}} \mid x^{\underline{\alpha}} \in \mathcal{P}_J\}$, we denote by $\xrightarrow{F}$ the transitive closure of the relation $g \to g - (c x^{\underline{\beta}}) f_{\underline{\alpha}}$, where $x^{\underline{\beta}} x^{\underline{\alpha}}$ is a monomial appearing in g with coefficient $c \neq 0$ and $x^{\underline{\beta}} \in \mathcal{C}(x^{\underline{\alpha}})$. This reduction procedure is Noetherian, see [CMR15, Theorem 5.9] and [BCR17, Proposition 4.3]; that is, starting from any polynomial g, we obtain a polynomial $h \in \langle \mathcal{N} \rangle$ in a finite number of steps. We write $g \xrightarrow{F}_* h$ to denote the beginning and the end of the reduction process, and we call $h \in \langle \mathcal{N} \rangle$ the J-normal form of g with respect to F.

Theorem A.2 ([BCR17, Proposition 5.6]). *Given a zero-dimensional quasi-stable ideal J in $\mathbb{C}[x_1, \dots, x_n]$, a J-marked set $F = \{f_{\underline{\alpha}} \mid x^{\underline{\alpha}} \in \mathcal{P}_J\}$ is a J-marked basis if and only if*

$$x_i f_{\underline{\alpha}} \xrightarrow{F}_* 0 \quad \forall\ f_{\underline{\alpha}} \in F,\ x_i > \min x^{\underline{\alpha}}.$$

Example A.3. In the polynomial ring $\mathbb{C}[x, y]$ (with $x > y$), consider the monomial ideal $J = (x^2, y)$. The monomial basis $\mathcal{N}$ of $\mathbb{C}[x, y]/J$ is the pair $\{x, 1\}$, and the Pommaret basis of J is $\mathcal{P}_J = \{x^2, xy, y\}$. A J-marked set F is a triple of polynomials

$$f_{x^2} = x^2 + c_1 x + c_2, \quad f_{xy} = xy + c_3 x + c_4, \quad f_y = y + c_5 x + c_6.$$

In order to have J-marked basis, the J-normal forms of xf_{xy} and xf_y have to be zero. One has

$$\begin{aligned} xf_{xy} = x^2y + c_3x^2 + c_4x &\xrightarrow{-yf_{x^2}} c_3x^2 - c_1xy + c_4x - c_2y \\ &\xrightarrow{-c_3f_{x^2}} -c_1xy + (c_4 - c_1c_3)x - c_2y - c_2c_3 \\ &\xrightarrow{+c_1f_{xy}} c_4x - c_2y + c_1c_4 - c_2c_3 \\ &\xrightarrow{+c_2f_y} (c_4 + c_2c_5)x + (c_1c_4 - c_2c_3 + c_2c_6)\,, \\ xf_y = xy + c_5x^2 + c_6x &\xrightarrow{-f_{xy}} c_5x^2 - (c_3 - c_6)x - c_4 \\ &\xrightarrow{-c_5f_{x^2}} -(c_3 - c_6 + c_1c_5)x - (c_4 + c_2c_5)\,. \end{aligned}$$

Hence, F is a marked basis if and only if

$$c_3 - c_6 + c_1c_5 = 0 \quad \text{and} \quad c_4 + c_2c_5 = 0$$

(the third equation is redundant). These equations define the open subset of $\mathrm{Mf}_{\mathcal{N}} \subset \mathrm{Hilb}^2 \mathbb{A}^2$ and guarantee that the generator f_{xy} is redundant, as expected. In fact, $f_{xy} = xy + (c_6 - c_1c_5)x - c_2c_5 = xf_y - c_5f_{x^2}$.

If we want to describe the tangent space $\mathsf{T}_{[(F)]} \mathrm{Hilb}^2 \mathbb{A}^2 \simeq \mathsf{T}_{[(F)]} \mathrm{Mf}_{\mathcal{N}} \simeq \mathrm{Hom}_{(F)}(D, \mathrm{Mf}_{\mathcal{N}})$, we can compute the set of flat families $\underline{\mathrm{Mf}}_{\mathcal{N}}(D)$ with the unique closed point corresponding to (F). We start with the marked set $\tilde{F}$ consisting of the following polynomials in $\mathbb{C}[\epsilon][x, y]$:

$$\begin{aligned} &\tilde{f}_{x^2} = x^2 + (c_1 + T_1\epsilon)x + (c_2 + T_2\epsilon)\,, \quad \tilde{f}_{xy} = xy + (c_3 + T_3\epsilon)x + (c_4 + T_4\epsilon)\,, \\ &\tilde{f}_y = y + (c_5 + T_5\epsilon)x + (c_6 + T_6\epsilon)\,, \end{aligned}$$

where $T_1, \ldots, T_6$ are complex parameters, and we impose the flatness via Theorem A.2. Assuming that F is a J-marked basis, the same holds for $\tilde{F}$ if

$$\begin{aligned} x\tilde{f}_{xy} &\xrightarrow{\tilde{F}}_* \big(c_5T_2 + T_4 + c_2T_5\big)\epsilon\, x + \big(c_4T_1 - (c_3 - c_6)T_2 - c_2T_3 + c_1T_4 + c_2T_6\big)\epsilon = 0\,, \\ x\tilde{f}_y &\xrightarrow{\tilde{F}}_* \big(c_5T_1 + T_3 + c_1T_5 - T_6\big)\epsilon\, x + \big(c_5T_2 + T_4 + c_2T_5\big)\epsilon = 0\,. \end{aligned}$$

By solving the linear system

$$\begin{bmatrix} 0 & c_5 & 0 & 1 & c_2 & 0 \\ c_5 & 0 & 1 & 0 & c_1 & -1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(the third equation is again redundant), one obtains a complete description of first-order deformations of the ideal generated by F.

A.1 Computational details of the proof of Theorem 3.1

The detailed computation of the tangent space of ideals of type (3.1) is available in the *Macaulay2* ancillary file [GGGLc] Here, we report a summary of the procedure and the results.

Assume that in $R = \mathbb{C}[x, y, z]$ we have $x > y > z$, and consider the quasi-stable ideal

$$J = \big(x + (y, z)^2\big)^2 + \big(y^3\big) = \big(x^2, xy^2, y^3, xyz, xz^2, y^2z^2, yz^3, z^4\big)\,.$$

The minimal set of generators of J is its Pommaret basis $\mathcal{P}_J$. An ideal I of the form (3.1) is generated by the J-marked basis[3]

$$\begin{aligned} &f_{x^2} = x^2, \quad f_{xy^2} = xy^2, \quad f_{y^3} = y^3 + b_1 y^2 z + b_2 yz^2 + b_3 z^3 + b_4 xy + b_5 xz,\\ &f_{xyz} = xyz, \quad f_{xz^2} = xz^2, \quad f_{y^2z^2} = y^2z^2, \quad f_{yz^3} = yz^3, \quad f_{z^4} = z^4. \end{aligned}$$

Thus, $[I] \in \mathrm{Mf}_{\mathcal{N}}$ with $\mathcal{N}$ the order ideal

$$\mathcal{N} = \left\{1, x, y, z, xy, y^2, xz, yz, z^2, y^2z, yz^2, z^3\right\}.$$

We compute $\mathsf{T}_{[I]} \operatorname{Hilb}^{12} \mathbb{A}^3 = \mathsf{T}_{[I]} \mathrm{Mf}_{\mathcal{N}_J}$ as the set of morphisms $\mathrm{Hom}_I(D, \mathrm{Mf}_{\mathcal{N}})$ mapping the unique closed point of D to $[I]$. This is equivalent to classifying all J-marked sets $\widetilde{F}$ consisting of polynomials in $\mathbb{C}[\epsilon][x,y,z]$ of the form

$$\begin{aligned} \tilde{f}_\bullet = f_\bullet &+ T_{\bullet,1}\epsilon\, y^2 z + T_{\bullet,2}\epsilon\, yz^2 + T_{\bullet,3}\epsilon\, z^3 + T_{\bullet,4}\epsilon\, xy + T_{\bullet,5}\epsilon\, y^2 + T_{\bullet,6}\epsilon\, xz + \\ &+ T_{\bullet,7}\epsilon\, yz + T_{\bullet,8}\epsilon\, z^2 + T_{\bullet,9}\epsilon\, x + T_{\bullet,10}\epsilon\, y + T_{\bullet,11}\epsilon\, z + T_{\bullet,12}\epsilon \qquad \forall\ f_\bullet \text{ generator of } I \end{aligned}$$

that are J-marked bases.

Imposing that

$$\begin{aligned} &x\tilde{f}_{xy^2} \xrightarrow{F}_* 0, \quad x\tilde{f}_{y^3} \xrightarrow{F}_* 0, \quad x\tilde{f}_{xyz} \xrightarrow{F}_* 0, \quad y\tilde{f}_{xyz} \xrightarrow{F}_* 0, \quad x\tilde{f}_{xz^2} \xrightarrow{F}_* 0, \quad y\tilde{f}_{xz^2} \xrightarrow{F}_* 0,\\ &x\tilde{f}_{y^2z^2} \xrightarrow{F}_* 0, \quad y\tilde{f}_{y^2z^2} \xrightarrow{F}_* 0, \quad x\tilde{f}_{yz^3} \xrightarrow{F}_* 0, \quad y\tilde{f}_{yz^3} \xrightarrow{F}_* 0, \quad x\tilde{f}_{z^4} \xrightarrow{F}_* 0, \quad y\tilde{f}_{z^4} \xrightarrow{F}_* 0 \end{aligned}$$

gives rise to a linear system of 98 equations in 96 variables. Looking at the minors of the system matrix S, we get that $42 \leqslant \mathrm{rk}(S) \leqslant 51$ and

$$\mathrm{rk}(S) < 43 \quad \text{if } b_4 = b_5 = 0, \quad \mathrm{rk}(S) < 51 \quad \text{if } \mathfrak{B} = 0,$$

where $\mathfrak{B} = b_3 b_4^3 - b_2 b_4^2 b_5 + b_1 b_4 b_5^2 - b_5^3$ is the same polynomial that appears in the proof of Theorem 3.1. Hence,

$$\dim_{\mathbb{C}} \mathsf{T}_{[I]} \operatorname{Hilb}^{12} \mathbb{A}^3 = \begin{cases} 54 & \text{if } b_4 = b_5 = 0,\\ 48 & \text{if } \mathfrak{B} = 0 \text{ and } |b_4| + |b_5| \neq 0,\\ 45 & \text{if } \mathfrak{B} \neq 0. \end{cases} \tag{A.1}$$

Remark A.4. Under the condition $b_4 = b_5 = 0$, the ideal I is homogeneous. Hence, the parity conjecture holds true for I; see [RS25].

For ideals with tangent space of dimension 48, we notice that $b_4 \neq 0$ (in fact, $b_4 = b_3 b_4^3 - b_2 b_4^2 b_5 + b_1 b_4 b_5^2 - b_5^3 = 0$ implies $b_5 = 0$). Thus, one has that

$$b_3 = b_2 \frac{b_5}{b_4} - b_1 \left(\frac{b_5}{b_4}\right)^2 + \left(\frac{b_5}{b_4}\right)^3,$$

and the polynomial f_{y^3} becomes reducible:

$$\begin{aligned} &y^3 + b_1 y^2 z + b_2 yz^2 + \left(b_2 \frac{b_5}{b_4} - b_1 \left(\frac{b_5}{b_4}\right)^2 + \left(\frac{b_5}{b_4}\right)^3\right) z^3 + b_4 xy + b_5 xz\\ &\quad = \frac{1}{b_4^3}(b_4 y + b_5 z)\big(b_4^2 y^2 + b_4(b_1 b_4 - b_5) yz + \big(b_2 b_4^2 - b_1 b_4 b_5 + b_5^2\big) z^2 + b_4^3 x\big). \end{aligned}$$

Remark A.5. The classification (A.1) of the dimension of the tangent space $\mathsf{T}_{[I]} \operatorname{Hilb}^{12} \mathbb{A}^3$ holds for every characteristic other than 2. If the characteristic is 2, the rank of the matrix S is at most 50, and for $\mathfrak{B} \neq 0$ the tangent space has dimension 46.

[3] Notice that the polynomial f_{y^3} is called v in Theorem 3.1.

A.2 Computational details of the proof of Proposition 3.8

The detailed computation of the smoothing deformation is available in the *M2* ancillary file [GGGLd]

The ideal (3.4) for $t \neq 0$ is the intersection of the ideals

$$\begin{aligned}
P_1 &= \big((x-b_4t^2)^2, y^2, (b_3b_4-b_2b_5)yz - b_3b_5z^2 - b_5^2(x-b_4t^2), b_3yz^2 + b_5y(x-b_4t^2)\,,\\
&\qquad b_2yz^2 + b_3z^3 + (b_4y+b_5z)(x-b_4t^2), (x-b_4t^2)z^2, (x-b_4t^2)yz, yz^3, z^4\big)\,,\\
P_2 &= \Bigg(x - \tfrac{b_4^3(b_3b_4-b_2b_5)}{\mathfrak{B}}t^2, b_4y+b_5z, \Big(x - \tfrac{b_4^3(b_3b_4-b_2b_5)}{\mathfrak{B}}t^2\Big)\Big(b_5^2y + \big(b_4(b_3b_4-b_2b_5)+b_1b_5^2\big)z\Big)\\
&\qquad + \tfrac{b_4^3b_5^2(b_3b_4-b_2b_5)}{\mathfrak{B}}t^2y + \tfrac{b_4^2b_5^3(b_3b_4-b_2b_5)}{\mathfrak{B}}t^2z, y^2, yz, z^2\Bigg)\,,\\
P_3 &= \big(x, 2b_4(y-b_4t) + (b_1b_4-b_5)z, (2y+b_1z)(y-b_4t) + (b_1b_4-b_5)tz, (y-b_4t)z, z^2\big),\\
P_4 &= \big(x, y+b_4t, z\big)\,,
\end{aligned}$$

of length 7, 2, 2, 1 and supported at the points

$$\big(b_4t^2,0,0\big)\,,\quad \Big(\tfrac{b_4^3(b_3b_4-b_2b_5)}{\mathfrak{B}}t^2,0,0\Big)\,,\quad (0,b_4t,0)\,,\quad (0,-b_4t,0)\,.$$

Figure 1 illustrates the deformation for $b_1 = 0$, $b_2 = -\frac{1}{2}$, $b_3 = 0$, $b_4 = 1$, and $b_5 = 1$. In the figure any arrangement of boxes represents a chosen order ideal describing a monomial basis for R/P_i for $i = 1, \dots, 4$ and R/I.

We note that ideals I corresponding to points in the locus $\mathfrak{B} = 0$ still have a first-order deformation involving the generator xyz perturbed by ϵy^2z. But if $\mathfrak{B} = 0$ and $b_4 \neq 0$, this deformation is obstructed. And if $b_4 = b_5 = 0$, then the first-order deformation lifts to the family

$$\big(x^2, xy^2, xyz + \boldsymbol{t}y^2z, xz^2, y^2z^2, yz^3, z^4, y^3 + b_1y^2z + b_2yz^2 + b_3z^3\big)$$

whose generic fiber is irreducible.

Acknowledgements

We are grateful for the research conditions and the stimulating environment that we were provided with during the workshop “Deformation Theory” held in December 2022 in Warsaw as a part of the semester *“Algebraic Geometry with Applications to TEnsors and Secants”*. Special thanks go to Andrea Ricolfi for his support and guidance during the writing of the paper. It is our pleasure to thank Joachim Jelisiejew and Alessio Sammartano for useful discussions. Moreover, we thank Daniele Faenzi, Martijn Kool, and Reinier F. Schmiermann for the interest shown.

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Franco Giovenzana franco.giovenzana@universite-paris-saclay.fr
Laboratoire de Mathématiques d'Orsay, Université Paris-Saclay, Rue Michel Magat, Bât. 307, 91405 Orsay, France

Luca Giovenzana l.giovenzana@sheffield.ac.uk
Fakultät für Mathematik, Technische Universität Chemnitz, Reichenhainer Straße 39, 09126 Chemnitz, Germany
Current address: Department of Pure Mathematics, University of Sheffield, Hicks Building, Hounsfield Road, Sheffield, S3 7RH UK

Michele Graffeo mgraffeo@sissa.it
Scuola Internazionale Superiore di Studi Avanzati (SISSA), Via Bonomea 265, 34136 Trieste, Italy

Paolo Lella paolo.lella@polimi.it
Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milan, Italy