ARTICLE TEMPLATE

Adjustment process of decentralised signal control policies with route choices: a case study with the P_0 policy

Koki Satsukawa[a], Takamasa Iryo[b], Naoki Yoshizawa[b], Michael J. Smith[c] and David Watling[d]

[a]Institute of Transdisciplinary Sciences for Innovation,, Kanazawa University, Japan; [b]Graduate School of Information Sciences, Tohoku University, Japan; [c]Department of Mathematics, University of York, United Kingdom of Great Britain and Northern Ireland; [d]Institute for Transport Studies, University of Leeds, United Kingdom of Great Britain and Northern Ireland

ARTICLE HISTORY
Compiled April 28, 2025

ABSTRACT

This study investigates the mathematical properties and network performance of the P_0 policy through numerical analysis. We consider a static traffic assignment problem with a P_0 policy in a simple signal-controlled network. We then calculate the equilibrium states while changing the demand level from unsaturated to near-saturated to investigate whether uniqueness holds for various traffic demand levels. After investigating the uniqueness through numerical analysis, we examine the stability of the computed equilibrium states using a graphical approach. This approach enables us to assess stability under natural evolutionary dynamics without explicitly specifying the dynamical system. Through these analyses, we demonstrate the occurrence of multiple equilibria with different total travel times at certain moderate traffic demand levels and investigate their stability. We demonstrate that such multiple stable equilibria cause a hysteresis loop, implying that different stable equilibria with different total travel times emerge when traffic demand increases and then decreases.

KEYWORDS
Decentralised signal control; the P_0 policy ; user equilibrium; route choice; uniqueness and stability

1. Introduction

In modern traffic engineering, decentralised signal control has emerged as an important solution for efficiently managing urban traffic on large-scale road networks. Traffic lights at intersections are a major means of control in urban road networks (Papageorgiou et al. 2003), and their signal settings are adjusted to reduce congestion. When adjusting signal settings, a centralised signal control method is expected to significantly improve the overall network performance by coordinating all traffic lights simultaneously (Diakaki, Papageorgiou, and Aboudolas 2002; Aboudolas et al. 2010; Aslani, Mesgari, and Wiering 2017). However, as highlighted by many studies (e.g. Chow, Sha, and Li 2020;

CONTACT K. S. Email: satsukawa@staff.kanazawa-u.ac.jp

Yu et al. 2023), this method has several limitations, such as scalability and robustness, because it is difficult to calculate an optimal pattern of signal settings in a short time, and mistakes in a single central control system can lead to significant degradation of network performance. Decentralised signal control methods distribute decision-making processes across individual traffic lights, allowing them to adjust their signal settings based on local traffic conditions. This method can improve the network performance while enhancing the scalability and robustness when adapted to fluctuating traffic conditions. For this reason, decentralised signal control has attracted widespread attention, and numerous studies have been conducted on this topic thus far (e.g. Robertson and Bretherton 1991; Varaiya 2013; Chen et al. 2020; Chu et al. 2020; Su, Chow, and Zhong 2021; Mo et al. 2022).

Among decentralised signal control methods, the P_0 policy, proposed by Smith (1979b), is distinctive in that it aims to guide the transport system toward a desirable state in the long run while taking users' route choice behaviour into account. Signal adjustments in a road network influence users' day-to-day route choices as they seek to minimise travel times, which in turn affects the control system and creates a dynamic interaction that shapes overall network performance. In particular, myopic control strategies that allocate green time solely based on immediate travel time reductions may inadvertently induce excessive demand on lower-capacity routes, leading to network inefficiencies. To address this, the P_0 policy defines a congestion measure, known as 'pressure', based on link delay times and saturation flow rates. By preferentially allocating green time to links with higher pressure, the policy balances improvements in travel times and overall network throughput. Consequently, through repeated daily adjustments in route choices and green time allocations, an equilibrium is reached in which both user equilibrium and signal control conditions are simultaneously satisfied, meaning that users have no incentive to switch routes and signal controllers no longer need to adjust green times. Numerous studies suggest that such equilibrium exists even under high traffic demand, where the network approaches saturation. This property is referred to as 'capacity maximisation', which underscores a key advantage of the P_0 policy in maintaining efficient network performance in near-saturated conditions.

Existing research on the P_0 policy defines equilibrium states as stationary points of the dynamical systems that govern users' route choices and signal control adjustments. These studies have analysed the mathematical properties of the stationary points, seeking to demonstrate the effectiveness of the P_0 policy. Smith (1979b) formulated a static user equilibrium assignment problem with capacity constraints in a simple two-route network. The study showed that when traffic demand is sufficiently high, a unique and stable equilibrium emerges in which both signal control and users' route choices are in equilibrium. Moreover, this equilibrium maximises network capacity by optimally balancing signal timings and traffic distribution. Smith, Liu, and Mounce (2015) extended this analysis to networks with general topology by explicitly formulating day-to-day dynamics that describes the adjustment of route flows and green times from disequilibrium. They proved the global convergence of the day-to-day dynamics to an equilibrium state under natural conditions. Beyond static traffic assignment models, subsequent studies have explored alternative frameworks, including quasi-dynamic assignment models incorporating queueing effects and models with congestion pricing. A more detailed review is provided in Section A.

Despite these advances, the properties of the dynamical system under the P_0 control policy remain incompletely understood, even within static traffic assignment models. In particular, the uniqueness of stationary points and their individual stability are still unclear. Stationary points are not necessarily unique, and multiple equilibria with different traffic performance levels may exist for the same traffic demand. When multiple equilibria arise, analysing the stability of each stationary point is important if we wish to determine which equilibrium state is most likely to be realised. However, most existing studies do not examine the uniqueness of stationary points or conduct stability analyses for individual equilibria. While some studies analyse traffic performance characteristics under the P_0 policy, they typically focus on specific equilibrium types at particular traffic demand levels. In other words, a comprehensive investigation of equilibrium properties across a wide range of traffic demands is still lacking. Consequently, it remains unclear under which demand conditions equilibrium points are unique and whether stable equilibria with differing traffic performance exist.

Such analyses would be particularly valuable for assessing the effectiveness of the P_0 policy in scenarios where traffic demand fluctuates significantly over extended periods, such as over a 24-hour cycle. Analysing the uniqueness and stability of equilibria at various OD demand levels provides insights into whether different traffic states can emerge and which among the efficient and inefficient traffic states is stably realised as traffic demand changes. They also shed light on how complementary control measures should be combined to avoid inefficient traffic states. Therefore, to develop effective implementation strategies for the P_0 policy, it is important to conduct comprehensive mathematical analyses of stationary points, incorporating quantitative evaluations of traffic performance.

This study revisits static traffic assignment problems and conducts a numerical analysis to investigate the mathematical properties and network performance at equilibrium under the P_0 policy. We consider a simple signal-controlled network with two routes, as commonly employed in previous studies. An equilibrium state is one in which both the static user equilibrium and the pressure equilibrium conditions of the P_0 policy are simultaneously satisfied. We then compute and enumerate the equilibrium states over a range of traffic demand levels, from unsaturated to near-saturated, thereby assessing the uniqueness of the equilibrium and its associated traffic performance. To examine the stability of the computed equilibrium states, we employ a graphical approach using a phase diagram (Iryo and Watling 2019). This approach enables us to analyse a generic dynamical process that satisfies fundamental conditions, such as positive correlation, for the equilibrium problem involving users' route choices and signal control. Our analyses reveal that multiple stable equilibria with different total travel times can occur at moderate traffic demand levels. We also find that the presence of multiple equilibria may induce a hysteresis loop, whereby the particular stable equilibrium attained depends on whether the total travel demand is increasing or decreasing. We show the underlying mechanism of this hysteresis loop and provide insights into the design of decentralised signal control strategies under fluctuating traffic conditions.

Furthermore, we conduct these uniqueness and stability analyses using three different types of link travel time functions. Specifically, we employ a variation of Webster's delay formula (Webster 1958) and the Bureau of Public Roads (BPR) function, which are steady-state travel-time functions with and without capacity constraints. We also employ the time-dependent link travel time function proposed by Burrow (1989). We found multiple stable equilibria under various settings by conducting numerical analyses with these different link travel time functions. These results confirm that the non-uniqueness and stability of equilibria occur in a broad range of situations.

Regarding the interaction between decentralised signal control and route choices, Xiao and Lo (2015) analyse the emergence and stability of multiple equilibria, similar to our study. However, the previous study employed Webster's signal control policy, unlike the P_0 policy investigated in this study. Additionally, it specifies a link travel time function

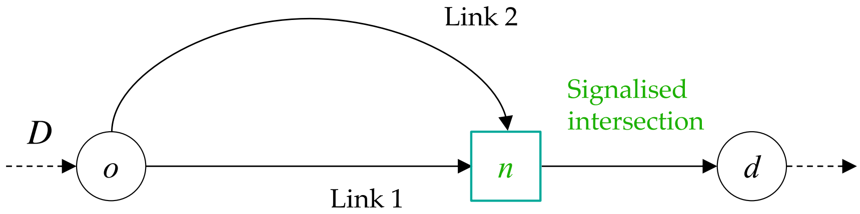


Figure 1. One-to-one origin-destination network

and a dynamical system aiming to utilise the convenience of the stability analysis using Jacobian matrices. Therefore, compared to existing studies, our study investigates the characteristics of equilibrium states across a much broader range of settings while employing a different decentralised traffic signal control policy. This complements the knowledge of signal control accumulated in the existing body of research.

The contributions of this study can be summarised as follows. First, we demonstrate that multiple equilibria can occur at moderate traffic demand levels due to the interplay between decentralised traffic signal control and users' route choices. Second, our phase diagram analysis shows that these equilibria–each exhibiting distinct traffic performance characteristics–are generally stable under natural evolutionary dynamics. Our numerical analyses employing various link travel time functions reveal that the non-uniqueness and stability properties persist across the diverse scenarios considered. Finally, we show the emergence of hysteresis loops, whereby the stable equilibrium state realised depends on whether traffic demand is increasing or decreasing. These findings provide valuable insights into the long-term behaviour of decentralised signal control systems, such as these, particularly during periods of significant fluctuations in daily traffic demand.

2. Model settings

2.1. *Network*

We consider a signal-controlled network with a one-to-one origin-destination (OD) demand (Figure 1). The network consists of three nodes and three links. We denote by s_i ($\forall i \in \{1, 2\}$) the saturation flow rate of a link i. Note that we do not consider explicitly the link 3 since it does not affect users' route choices and signal control, which will be explained later. The free-flow travel time of a link i is denoted by t_i. We assume that the free-flow travel time of link 2 is larger than that of link 1, i.e. $t_2 - t_1 > 0$.

Travellers who depart from node o and travel to node d choose one route by considering the route travel times (details are shown later). Two routes exist between the OD, denoted by routes 1 and 2. Route 1 consists of links 1 and 3 whereas route 2 consists of links 2 and 3; hence, the free-flow travel time of route 2 is longer than that of route 1. The flow on link i ($\forall i \in \{1, 2\}$) is denoted by x_i. The flow on route 1 (or route 2) is equal to that on link 1 (or link 2), as can be seen from the figure. Thus, we sometimes interchangeably use the words 'link flow' and 'route flow'. The total travel demand is denoted by D, and the equation $x_1 + x_2 = D$ is satisfied.

A traffic signal is located at node n, which is the intersection of links 1 and 2. The traffic signal controls the green splits assigned to these two links according to the P_0 policy (Smith 1979b), whose details are presented later. Let g_i ($\forall i \in 1, 2$) represent the green split for link i. For simplicity, we assume that there is no lost time and that the sum of the green splits equals one, i.e. $g_1 + g_2 = 1$. The minimum and maximum green

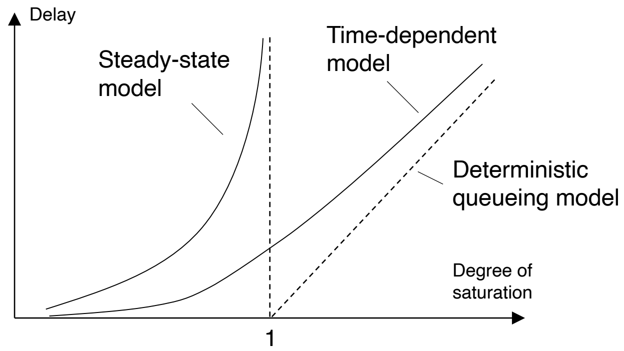


Figure 2. Steady-state and time-dependent link travel time functions

splits for each link are denoted by $g_{\min}$ and $g_{\max}$ $(= 1 - g_{\min})$, respectively.

2.2. *Link delay functions*

In a static traffic assignment problem in a signalised network, we consider a link delay function $d_i(x_i, g_i)$ that determines the delay time of link i according to the link flow and the assigned green split. This study considers the following three types of link delay functions to investigate whether and how mathematical properties (e.g. uniqueness and stability) change according to the different types of functions.

We first consider the variation in the well-known Webster's delay formula (Webster 1958), which was proposed by Hutchinson (1972). This link travel time function is a steady-state model constructed based on steady-state queueing theory. This link delay function is represented as follows:

$$\frac{9}{20}\left[\frac{c(1-g_i)^2}{1-y_i} + I\frac{y_i^2}{x_i g_i (g_i - y_i)}\right], \tag{1}$$

$$\text{where} \quad y_i = x_i/s_i, \tag{2}$$

where c represents the cycle time, y_i represents the degree of saturation. The variable I represents the ratio of the variance of the number of arrivals per cycle to the mean number of arrivals per cycle; Hutchinson (1972) introduced this variable, which improved the accuracy of Webster's delay formula while retaining its simplicity (Cheng et al. 2016). The original Webster's delay formula is a special case of this formula, in which I is set to be one. This delay function is valid only if the under-saturation condition is satisfied, i.e. $x_i < s_i g_i$, $\forall i \in \{1, 2\}$.

Next, we consider the Bureau of Public Roads (BPR) function widely used in static traffic assignment problems. To incorporate the effect of signal control, we assume that the capacity of link i is calculated as $g_i s_i$, which has often been employed in previous studies (e.g. Yang and Yagar 1995). The link delay function is expressed as follows:

$$t_i \alpha \left(\frac{x_i}{g_i s_i}\right)^{\beta}, \tag{3}$$

where α and β are parameters. Unlike Webster's delay formula, link flows exceeding the capacity are allowed.

Finally, we consider the time-dependent model proposed by Burrow (1989). Time-dependent models were developed to handle an oversaturated situation in which traditional link delay functions overestimate the delay; for example, Webster's delay function produces an infinite delay time as the degree of saturation approaches one, thereby providing unreasonable results. To address this issue, time-dependent models were constructed by shifting the steady-state model from an asymptotic to a deterministic queueing model, as shown in Figure 2 (for details, see Cheng et al. 2016). While several studies have constructed time-dependent models such ,as Australian (Akcelik 1981), Canadian delay (Teply 1997) and Akcelik's models (Akcelik 1988), Burrow (1989) developed the following generalised model that includes them:

$$\frac{c(1-g_i)^2}{2(1-x_i/s_i)} + 900T\left(\frac{x_i}{g_i s_i}\right)^n \left[\frac{x_i}{g_i s_i} - 1 + a + \sqrt{\left(\frac{x_i}{g_i s_i} - 1\right)^2 + \frac{m(x_i/(g_i s_i) + b)}{C_i T}}\right], \quad (4)$$

where a, b, m and n are parameters. Variable c represents the cycle time, variable T represents the flow period, and variable C_i is the capacity; we assume that $C_i = g_i s_i$.

Importantly, the link delay calculated using Webster's delay formula or Burrow's formula does not become zero, even when the link flow is zero. Specifically, the first term of Webster's delay formula produces the following non-zero delay:

$$d_i(0, g_i) = \frac{9}{20}c(1-g_i)^2. \quad (5)$$

Also, the first term of Burrow's formula produces the following delay:

$$d_i(0, g_i) = \frac{1}{2}c(1-g_i)^2. \quad (6)$$

These positive delays affect the signal control and network performance, as shown in the numerical experiments.

3. Equilibrium conditions and stability analysis method

In the network shown in Figure 1, we consider a situation where users' route choices and the signal control parameters (green splits) evolve day-to-day according to reasonable adjustment processes. We define an equilibrium state as a stationary point of these processes, where aggregated route flows and green splits remain unchanged as days pass. The mathematical properties of equilibria, focusing on uniqueness and stability, are examined through numerical experiments. Below, we first describe the users' route choice behaviour and the signal control policy which we consider. We then formulate the equilibrium conditions. We also introduce the stability analysis method, which is particularly important when multiple equilibria exist.

3.1. *Route choice equilibrium*

We assume that travellers adjust their routes on a day-to-day basis to reduce their individual travel times. The route flow, representing the aggregation of travellers' route choices, evolves based on the following principle: *each day, a portion of the flow on* *any route with positive flow shifts toward a less costly route if there is one.* Following this principle, the system will often eventually reach a state where no user has an incentive to unilaterally switch routes. This condition corresponds to Wardrop's first principle (Wardrop 1952), which states that the travel times on all routes used by travellers are equal and not greater than those on unused routes.

We define route choice equilibrium as the Wardrop equilibrium. Since this study focuses on the two-route network, the equilibrium conditions are formulated as follows:

$$\begin{cases} \text{(A) } d_1(x_1, g_1) = d_2(x_2, g_2) + \Delta\tau & \text{if } x_1 > 0 \text{ and } x_2 > 0, \\ \text{(B) } d_1(x_1, g_1) \leq d_2(x_2, g_2) + \Delta\tau & \text{if } x_2 = 0, \\ \text{(C) } d_1(x_1, g_1) \geq d_2(x_2, g_2) + \Delta\tau & \text{if } x_1 = 0, \end{cases} \tag{7}$$

where $\Delta\tau = t_2 - t_1$ represents the difference in free-flow travel times between links 1 and 2. This also corresponds to the difference in free-flow travel times between routes 1 and 2. Case (A) represents the situation where the travel times of both routes are equal and both routes are used by travellers (i.e. interior equilibrium); Case (B) represents the situation where the travel time of Route 1 is not larger than that of Route 2, and only Route 1 is used by travellers (i.e. textcolorblackboundary equilibrium). Case (C) represents the situation where the travel time of Route 2 is not larger than that of Route 1 and the traveller uses Route 2 only.

3.2. *Signal control equilibrium under the P_0 policy*

The P_0 policy is a pressure-driven signal control method that adjusts green splits based on *pressure*, a variable representing the traffic stress on a link. Pressure is determined by link parameters and link delay. Under pressure-driven policies, green splits are continuously adjusted by shifting green time toward the link with higher pressure and away from the link with lower pressure. This adjustment process continues until the pressures of all links at a signal-controlled intersection are equal or until the green split for the lower-pressure link reaches its minimum allowable value. Apart from the P_0 policy, other pressure-driven policies include the equisaturation policy, the MaxPressure policy, and their variations (Webster 1958; Varaiya 2013; Gregoire et al. 2015; Le et al. 2015).

Under the P_0 policy, the pressure of link i is calculated as:

$$p_i(x_i, g_i) = s_i d_i(x_i, g_i). \tag{8}$$

In the Figure 1 network, the signal controller at node n reduces the green split for the link with lower pressure while increasing it for the link with higher pressure. As a result, the signal control equilibrium, where the adjustment process terminates, is defined by the following condition:

$$\begin{cases} \text{(1) } p_1(x_1, g_1) = p_2(x_2, g_2) & \text{if } g_{\min} < g_1 < g_{\max}, \\ \text{(2) } p_1(x_1, g_1) \leq p_2(x_2, g_2) & \text{if } g_1 = g_{\min}, \\ \text{(3) } p_1(x_1, g_1) \geq p_2(x_2, g_2) & \text{if } g_1 = g_{\max}. \end{cases} \tag{9}$$

Case (1) represents a situation where the pressures of the two links are equal. The other cases represent situations where the pressure of one link remains larger than that of the other, even when the green split is maximised. This indicates that the traffic state is at a corner solution.

We define equilibrium as a state that satisfies both the route choice and signal control equilibrium conditions. At equilibrium, the traffic state corresponds to a stationary point of the adjustment processes governing users' route choices and signal control. Thus, an equilibrium traffic state remains unchanged as day follows day.

Adding the route choice and signal control equilibrium conditions to the feasibility constraints on traffic states, the equilibrium condition is formulated as follows:

Definition 3.1. We consider $(\mathbf{x}, \mathbf{g})$ where $\mathbf{x} \equiv (x_1, x_2, x_3)$ and $\mathbf{g} \equiv (g_1, g_2)$. $(\mathbf{x}, \boldsymbol{g})$ is an equilibrium consistent with the P_0 policy if and only if

- $\mathbf{x}$ is demand feasible: $x_1 + x_2 = D = x_3$ and $x_1 \geq 0$, $x_2 \geq 0$.
- $\mathbf{g}$ is green-time feasible: $g_1 + g_2 = 1$ and $g_{\min} \leq g_1 \leq g_{\max}$.
- $(\mathbf{x}, \mathbf{g})$ is supply feasible if the applied link delay function requires it: $x_1 < s_1 g_1$ and $x_2 < s_2 g_2$.
- $(\mathbf{x}, \mathbf{g})$ satisfies the route choice equilibrium condition (7).
- $(\mathbf{x}, \mathbf{g})$ satisfies the P_0 policy (9).

In conventional static traffic assignment problems, equilibrium properties, particularly uniqueness, are well established due to the monotonicity of the route travel time function (Iryo 2013). However, in our problem, monotonicity may not hold because green split adjustments under the signal control policy alter the shape of link delay functions. As a result, route travel times may exhibit non-monotonic behaviour with respect to route traffic flows, potentially leading to the existence of multiple equilibria. When multiple equilibria exist, stability analysis is essential to determine which equilibrium is likely to be realised.

3.3. *A method for stability analysis using a phase diagram*

In stability analysis, we typically consider some form of evolutionary (or day-to-day) dynamics that describe how traffic states, such as route flows and green splits, evolve day by day. By analysing its behaviour, we examine whether a traffic state, when slightly deviated from a given equilibrium state, returns to the original equilibrium state. If the traffic state does not return to an equilibrium state, it is considered unstable and regarded as an extreme state that rarely occurs (Beckmann, McGuire, and Winsten 1956). Thus, stability analysis enables us to determine which equilibrium states are more likely to emerge and persist in a given transport system.

This study examines the stability of equilibrium states under a broad class of dynamics rather than restricting the analysis to a specific model. Following Sandholm (2010), we consider evolutionary dynamics that satisfies the following conditions: (i) the flow on routes with shorter travel times increases and the flow on routes with longer travel times decreases; (ii) the green split for routes with higher pressure increases and the green split for routes with lower pressure decreases; and (iii) the stationary points correspond to the equilibrium states defined in **Theorem 3.1**. Conditions (i) and (ii) are known as the *positive correlation properties*, while condition (iii) is referred to as *Nash stationarity*. These properties reflect rational decision-making behaviours aligned with the fundamental principles of users' route choices and signal control introduced earlier. Thus, they are intrinsic to any natural dynamical system describing the adjustment processes of traffic flow. Specific examples of such dynamics include the Smith dynamics (Smith 1984) and the projection dynamics (Zhang and Nagurney 1996).

To examine stability, we employ a graphical approach utilising a *phase diagram*,

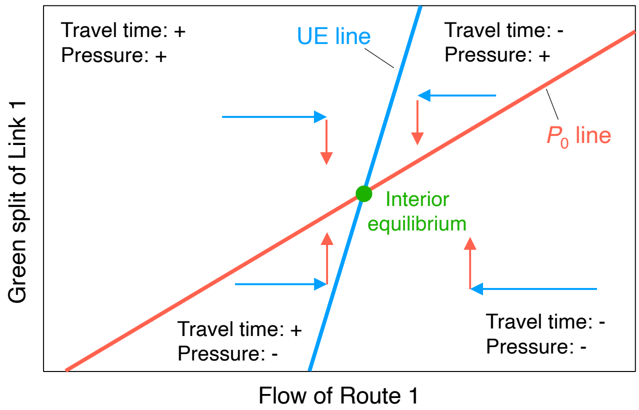


Figure 3. Example of the phase diagram

referring to Iryo and Watling (2019). An example is shown in Figure 3, where the horizontal axis represents x_1 and the vertical axis represents g_1. Since the other variables, x_2 and g_2, are uniquely determined by the conservation conditions in **Theorem 3.1**, the pair x_1 and g_1 can be interpreted as the state variable. The phase diagram contains two lines, each of which represents the traffic state (x_1, g_1) satisfying the interior equilibrium condition of the route choice equilibrium (A) in Eq. (7) or the signal control equilibrium (1) in Eq. (9). The intersection points of these lines correspond to interior equilibria, where the route travel times and pressures on both routes (links) are equalised. Each region divided by these lines is marked with signs indicating whether the travel time and pressure on Route (Link) 2 are greater (+) or smaller (−) than those on Route (Link) 1 at a given traffic state.

The phase diagram enables us to understand the directions of change of x_1 and g_1 under natural evolutionary dynamics that satisfy the previously mentioned conditions. Specifically, x_1 increases in regions where the sign of the travel time is +, shifting the traffic state to the right. Similarly, g_1 decreases in regions where the sign of the pressure is +, shifting the traffic state downward. Thus, the stability of each equilibrium can be assessed graphically by examining the directions of change around the corresponding intersection points for interior equilibria. Additionally, the stability of boundary equilibria can be analysed by evaluating the directional changes around the respective boundary points.

4. Numerical experiments

4.1. *Preparation*

In this section, we present numerical experiments to investigate how the equilibrium flow, split, and cost patterns change in response to a change in the total travel demand D. The study network is illustrated in Figure 1. The network parameters are set as follows:

- $s_1 = 1$ and $s_2 = 2$.
- $t_1 = 30$ and $t_2 = 60$: $\therefore \Delta\tau = 30$.
- $g_{\min} = 0.01$ and $g_{\max} = 0.99$.

In addition, the parameters for the link delay functions are set as follows:

Webster's delay formula (or Hutchinson's extended formula)

- $c = 60$.
- $I = \{1, 1.5, 2.0\}$; we calculate equilibrium states under each parameter.

BPR function

- $(\alpha, \beta) = \{(0.15, 4), (0.96, 1.2)\}$; we calculate equilibrium states under each parameter. The former parameter setting is commonly used in the United States, whereas the latter is often employed in Japan.

Burrow's formula of time-dependent delay

- $(a, b, m, n) = (0, 0, 4, 2)$. These parameter settings are used in the Highway Capacity Manual (Burrow 1989).
- $T = 0.25$.

We change D from 0.1 to 2.0 by 0.01 and calculate the equilibrium traffic states for each D under the implementation of Webster's delay and Burrow's formulas. When implementing the BPR functions, we change D to 3.0, considering that link flow is allowed to exceed the saturation rate. The equilibrium states are numerically derived using the `vpasolve` function in MATLAB R2023a.

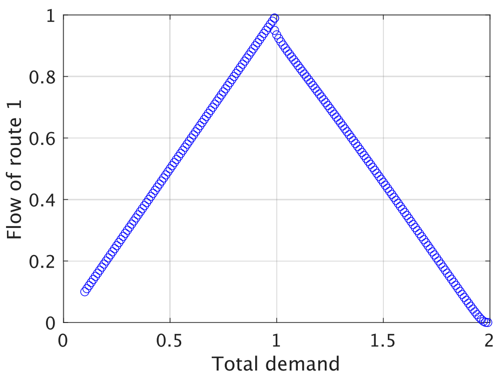
4.2. ***Results: examination of the uniqueness***

4.2.1. Webster's delay formula

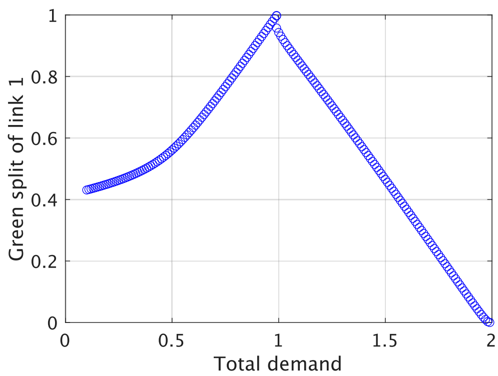
Figure 4 shows the equilibrium traffic states when implementing Webster's delay function. Each mark represents the traffic variable x_1 or g_1 at equilibrium for a given total travel demand D. There is no equilibrium when $D > 2$ because the supply feasible conditions are no longer satisfied. Below, we examine how the equilibrium states change with D increasing from 0.

First, we confirm that a unique equilibrium exists when D is small. Figures 4(a), 4(c) and 4(e) suggest that x_1 equals D in those equilibrium states, which means that all travellers use route (link) 1; that is, equilibrium condition (B) is satisfied. This is because link delays are small for low demand levels. Nevertheless, Figures 4(b), 4(d) and 4(f) suggest that the green splits of link 1 in the equilibrium states are not maximised and increase according to the increase in D; the green split of link 2 is not minimised even though the link flow is zero, and the equilibrium condition (1) is satisfied. This is because the delay on link 2, calculated using Webster's delay formula, does not become zero at zero flow, as we have mentioned in previous sections. Therefore, the pressure of link 2 becomes positive, and the green split is assigned to the link over the minimum green split to equalise the pressures; that is, an unnecessary assignment of the green split occurs.

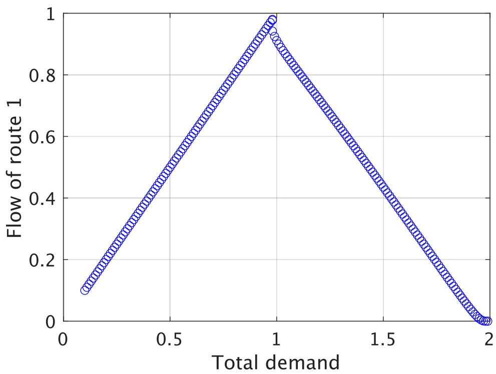
Interestingly, we confirm that multiple equilibrium states exist when D is moderate. One boundary equilibrium state satisfies route choice condition (B) and signal control condition (1): all travellers use Route 1, and the green splits are assigned to both links to equalise the pressures. Two interior equilibrium states also satisfy conditions (A) and (1): travellers use both routes such that the travel times are equalised, and the green splits are assigned to both links. The figures also suggest that non-uniqueness still holds even if the variable I is changed, but the threshold value of the demand at which the



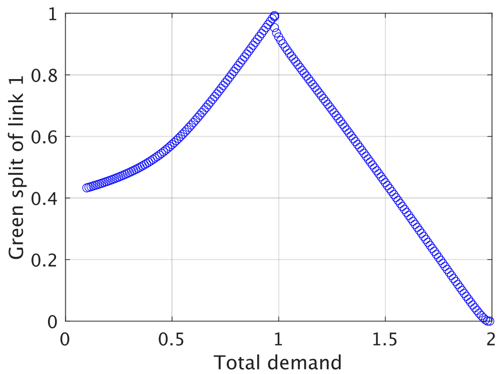
(a) Flow of route 1 when $I = 1.0$



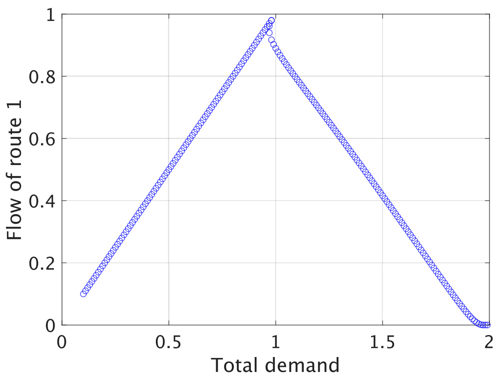
(b) Green split of link 1 when $I = 1.0$



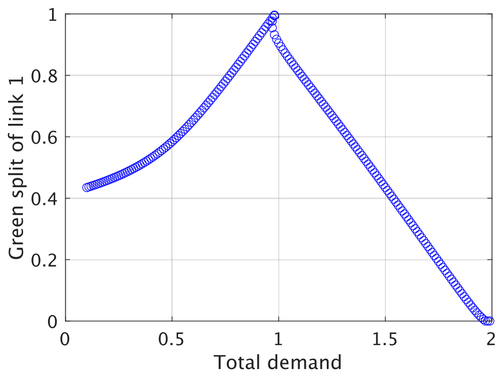
(c) Flow of route 1 when $I = 1.5$



(d) Green split of link 1 when $I = 1.5$



(e) Flow of route 1 when $I = 2.0$

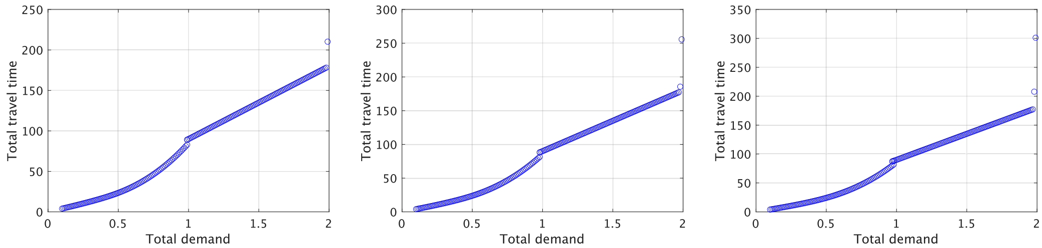


(f) Green split of link 1 when $I = 2.0$

Figure 4. Flow of route 1 and green split of link 1 under Webster's delay formula

property begins to emerge decreases as I increases. This is because the delay time for the same inflow increases with an increase in I, resulting in a lower total travel demand value at which the travel time on route 1 (when $x_1 = D$) equals the travel time on route 2 when $x_2 = 0$.

We observe that the equilibrium becomes unique when D further increases. There exists an interior equilibrium that satisfies conditions (A) and (1). At the equilibrium, x_1 and g_1 decrease as D increases, which implies that the traffic flow and green split are allocated to link 2 with a higher saturation flow rate to handle the high traffic demand. Eventually, all travellers use route 2, and the green split of link 2 is maximised in near-saturated conditions, that is, $x_1 = 0$ and $g_1 = g_{\min}$ when D is approximately 2.



(a) $I = 1.0$ (b) $I = 1.5$ (c) $I = 2.0$

Figure 5. Total travel times under Webster's delay formula

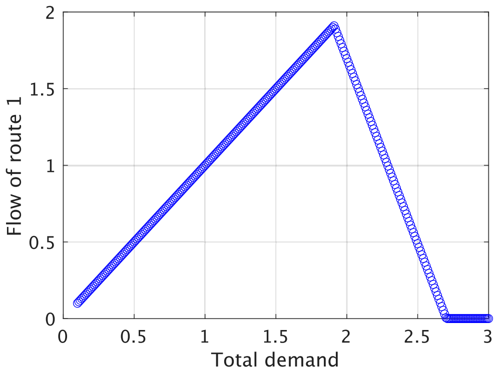
This confirms that the P_0 policy achieves capacity maximisation under near-saturated conditions. Note that the threshold value of the demand at which all travellers begin to use route 2 decreases as I increases for the same reason for the non-uniqueness: the travel time for the same inflow increases with an increase in variable I.

Figure 5 shows the change in total travel times in the equilibrium states. Like the flow and green split variables, the total travel time is unique when the demand is low; however, multiple total travel times emerge at moderate demand levels and become unique again with further demand increases. More specifically, there are two total travel times at moderate demand levels; the total travel time at the boundary equilibrium and the total travel time at the interior equilibrium. The total travel time is smaller in the boundary equilibrium state than in the interior equilibrium state, which implies that it is efficient for the transport system performance to allocate the green split to one link as much as possible, in this case. Note that the interior equilibrium has the same total travel time, which can be easily confirmed through a mathematical analysis of the equilibrium conditions.

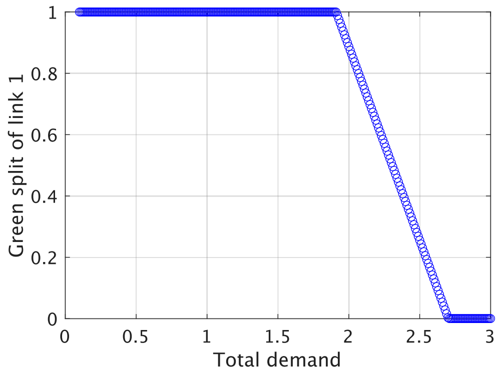
4.2.2. BPR function and Burrow's formula

Figure 6 shows the equilibrium traffic states when implementing the BPR functions. Figure 7 shows the total travel times at equilibrium. As shown in these figures, the uniqueness of the equilibrium depends on the parameters of the functions. When $\alpha = 0.15$ and $\beta = 4$ (Figures 6(a) and 6(b)), the equilibrium becomes unique for a given D. More specifically, there exists an equilibrium state in which travellers use only route 1, and the green split of link 1 is maximised; conditions (B) and (3) are satisfied for a low demand. The unnecessary assignment of the green split to link 2 does not occur because the BPR function does not produce a positive delay when the link flow is zero. Then, there exists an interior equilibrium satisfying the conditions (A) and (1) for a moderate demand. An equilibrium satisfying the conditions (A) and (2) emerges for a high demand. Most travellers use route 2, and the green split of link 2 is maximised to handle such high demand, but a very small number of travellers use route 1. The corresponding equilibrium total travel times are also unique for a given D, as shown in Figure 7(a).

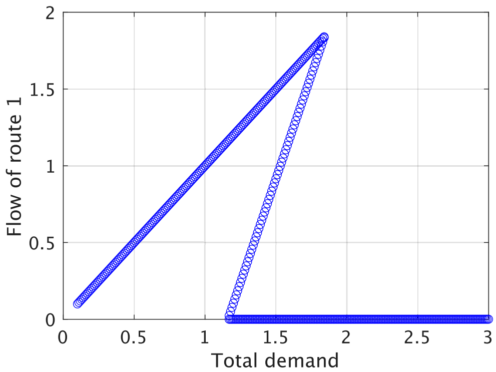
Meanwhile, Figures 6(c) and 6(d) show that the equilibrium is not always unique for a moderate demand when $\alpha = 0.96$ and $\beta = 1.2$. In this case, one equilibrium satisfies the conditions of (B) and (3), another satisfies the conditions of (A) and (2), and yet another satisfies the conditions of (A) and (1); that is, two boundary equilibrium states and one interior equilibrium state exist. Figure 7(b) shows the three corresponding patterns of the total travel time. The highest one corresponds to the equilibrium satisfying (B) and (3); the lowest one corresponds to that satisfying (A) and (2); the medium one



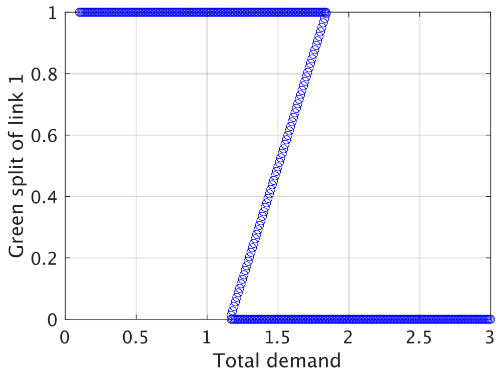
(a) Flow when $(\alpha, \beta) = (0.15, 4)$



(b) Green split when $(\alpha, \beta) = (0.15, 4)$

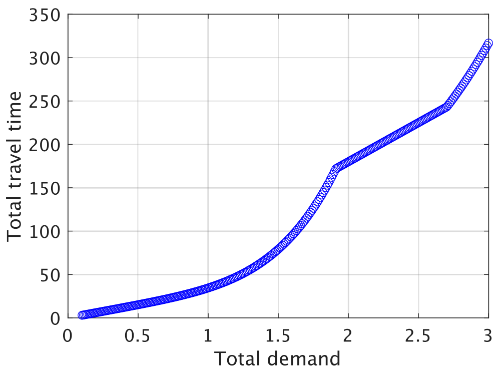


(c) Flow when $(\alpha, \beta) = (0.96, 1.2)$

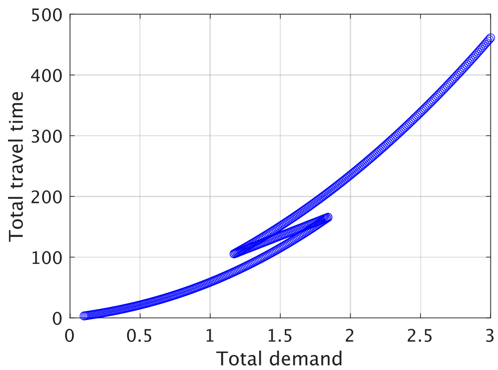


(d) Green split when $(\alpha, \beta) = (0.96, 1.2)$

Figure 6. Flow and green split under BPR functions



(a) Flow when $(\alpha, \beta) = (0.15, 4)$

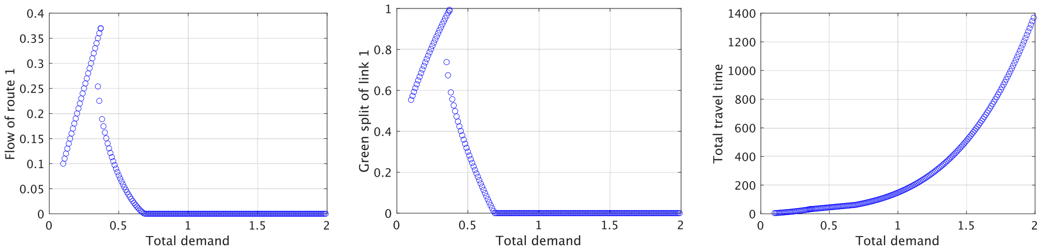


(b) Green split when $(\alpha, \beta) = (0.96, 1.2)$

Figure 7. Total travel times under BPR functions

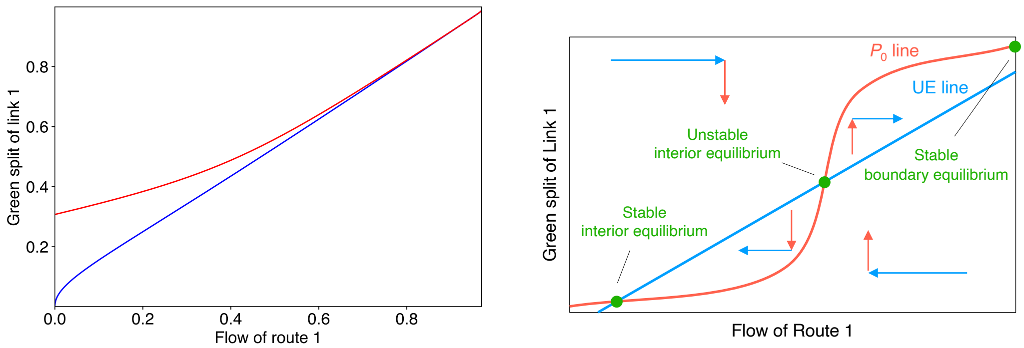
corresponds to that satisfying (A) and (1). Thus, the mathematical properties and emergence transport performance can be significantly different, even when the same form of the delay function is implemented.

Figure 8 shows the results of the numerical experiments using Burrow's formula. Again, we confirm that equilibrium is not always unique in this case. The equilibrium pattern is similar to that observed when Webster's delay formula is implemented. Specifically, an equilibrium state satisfying conditions (B) and (1) emerges when D is low. We also see that the green split of link 2 is not minimised in equilibrium even though



(a) Flow (b) Green split (c) Total travel time

Figure 8. Flow, split and total travel time under Burrow's formula



(a) Overall view (b) Enlarged conceptual diagram

Figure 9. Phase diagram under Webster's delay function

the link flow is zero; this is because the delay does not become zero, as is the case in implementing Webster's delay formula. An increase in D results in the emergence of multiple equilibria, followed by a return to a unique equilibrium. Note that the types of equilibrium states (i.e. which equilibrium conditions are satisfied) are the same as those in the case with Webster's delay formula. The change in the total travel time is also qualitatively similar.

4.3. *Results: examination of the stability*

Let us examine the stability of multiple equilibria using a graphical approach with a phase diagram. Figure 9 shows the phase diagram when Webster's delay formula with $I = 2.0$ is implemented and $D = 0.97$. To clarify how the two lines intersect, we prepare an enlarged conceptual diagram focusing on the vicinity of the intersection. The blue and red lines indicate the traffic states that satisfy the route choice and signal control equilibrium conditions, respectively. The two intersection points correspond to the interior equilibrium states. There also exists one point corresponding to the boundary equilibrium state in the upper-right corner of the diagram.

By investigating the change directions around the equilibrium states, we see that one of the interior equilibrium states is unstable, and the other equilibrium states (i.e. the other interior and boundary equilibrium states) are stable. The unstable equilibrium is the middle one; once a traffic state deviates from the equilibrium, the natural adjustment process of travellers leads the traffic state away from the equilibrium. A natural adjustment process eventually causes the traffic state to reach a stable equilibrium state. The realisation of either traffic state depends primarily on the initial state. Simply put,

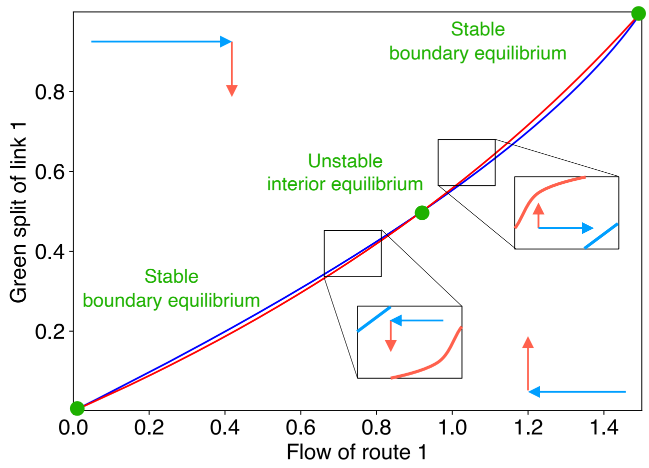
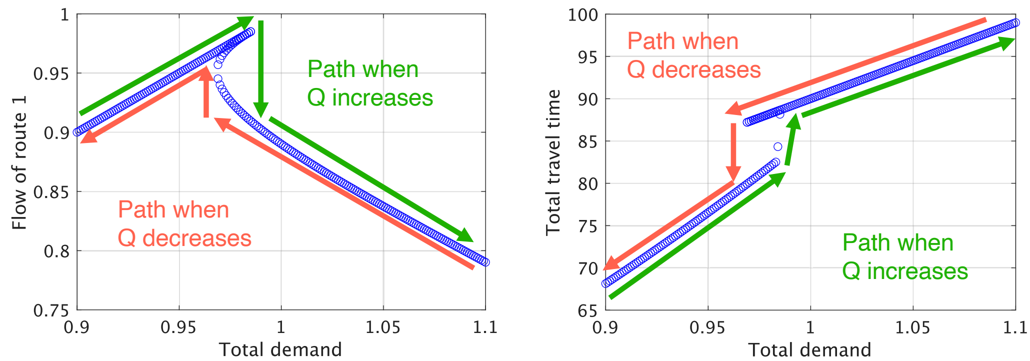


Figure 10. Phase diagram under the BPR function



(a) Flow (b) Total travel times

Figure 11. Hysteresis loops under the Webster's delay formula with $I = 2.0$

boundary equilibrium is realised if the initial state is located in the upper right of the unstable equilibrium in this figure; the stable interior equilibrium is realised if it is located in the lower left. Note that we qualitatively have the same phase diagram when Burrow's formula is implemented; one unstable equilibrium and two stable equilibria exist.

Figure 10 shows the phase diagram when the BPR function with $\alpha = 0.96$ and $\beta = 1.2$ is implemented and $D = 1.5$. Unlike the previous two cases, there exist one unstable interior equilibrium and two stable boundary equilibria in which almost all travellers use Route 1 or Route 2. Thus, either of such an extreme traffic state will be realised in the transport system. This implies that the performance of the transport system varies significantly depending on which stable equilibrium is realised, as discussed in the next section.

4.4. *Discussion*

The existence of multiple stable equilibria causes a *hysteresis loop*, where different stable equilibrium states emerge at the same demand level depending on the direction of change in total travel demand D. Figure 11 illustrates the formation of a clockwise hysteresis loop in the flow on Route 1. Suppose that the total OD demand is slowly increasing or decreasing over time from its initial level, and the joint system of route choices and signal control adjusts to the positive or negative demand perturbation and thereby re-equilibrates at any given OD demand. When the demand increases and enters the range of total demand where multiple equilibria exist, the traffic state settles in the boundary equilibrium, where most of the traffic flow concentrates on Route 1 (see the green trajectory). Once the total demand exceeds this range, the traffic state shifts to the unique interior equilibrium, resulting in a sharp decrease in the flow on Route 1 and a corresponding increase in the flow on Route 2. Conversely, when the total demand decreases from a high value, a different trajectory will emerge: even after re-entering the range of multiple equilibria, the traffic state remains in the interior equilibrium and does not return to the boundary equilibrium (see the red trajectory). Finally, when the demand falls below this range, the traffic state shifts to the boundary equilibrium, causing a sharp increase in the flow on Route 1.

The mechanism behind the hysteresis loop lies in the fact that the traffic state is attracted to different stable equilibria when demand increases or decreases. When total demand D marginally increases and enters the range of multiple equilibria, the route flow lies near the stable boundary equilibrium. This suggests that the traffic state belongs to the domain of attraction of the boundary equilibrium, as demonstrated in the previous section. Therefore, the boundary equilibrium is realised through the natural adjustment process of users' route choice behaviour and signal control. Conversely, when the total demand marginally decreases and enters the range of multiple equilibria, the route flow lies near the stable interior equilibrium, indicating that the traffic state belongs to the domain of attraction of the interior equilibrium. Thus, the interior equilibrium is realised through the natural adjustment process. In this way, different stable equilibria are realised at the same demand level depending on the previous traffic states. Qualitatively, a similar clockwise hysteresis loop can be observed for the green splits, as the relationship between total demand and green splits resembles that between total demand and route flows.

The presence of hysteresis loops in traffic states implies that a similar hysteresis effect occurs in total travel time. Figure 11(b) illustrates an anti-clockwise loop in total travel time. When D increases, equilibrium traffic states with lower total travel times are realised. Conversely, when D decreases, equilibrium traffic states with higher total travel times persist. This phenomenon can be explained as follows. Initially, travellers choose Route 1, which has a shorter free-flow travel time, and green splits are primarily allocated to Link 1. This results in an efficient traffic state with a smaller total travel time. However, as traffic demand increases beyond the range of multiple equilibria, the travel time on Link 1 becomes significantly longer, prompting travellers to shift to Route 2, which has a large capacity but a longer free-flow travel time. Consequently, the pressure on Link 2 increases, and green splits are allocated to Link 2. Conversely, when traffic demand decreases and re-enters the range of multiple equilibria, the pressure on Link 2 remains high because travellers continue to choose Route 2. This pressure keeps green splits allocated to Link 2. As a result, travellers persist in using Route 2, and the traffic state gets stuck in an inefficient equilibrium. Only when traffic demand decreases beyond the range of multiple equilibria, all travellers shift to Route 1, ultimately resolving the inefficient state.

These findings provide valuable insights into the efficient implementation of decentralised signal control during periods of significant daily traffic demand fluctuations. Traffic demand typically increases and decreases significantly within a 24-hour period, with pronounced peaks during the day. The P_0 policy was originally designed to maximise network capacity under near-saturated conditions, such as when traffic demand reaches its peak, and this desirable property is indeed expressed in our numerical results. However, our numerical results also reveal that decentralised signal control can lead to traffic states with varying performance levels under unsaturated conditions. Specifically, as demand approaches its peak, efficient flow and green split patterns with lower total travel times are realised. In contrast, as demand decreases from its peak, inefficient patterns with higher total travel times may emerge. To address this, it is desirable to introduce complementary measures during unsaturated conditions as demand decreases from its peak, guiding the traffic state toward efficient equilibrium. For instance, strategies to direct travellers toward routes with shorter free-flow travel times or perimeter controls to prevent oversaturation can sustain an efficient traffic state and enhance overall network performance. Modifications of the signal control policy itself may also be able to help travellers choose the better routeing alternative. These findings highlight the importance of further investigating the behaviour of decentralised signal control under moderate traffic demand and unsaturated conditions, which is crucial for developing robust and effective decentralised signal control systems.

5. Conclusion

This study investigated the mathematical properties of the P_0 policy under various demand levels and link travel time functions. We first defined the equilibrium state as one that satisfies the route choice condition of the user equilibrium and the signal control condition of the P_0 policy. The equilibrium states for a simple network were numerically derived through computational analysis under three types of link delay functions while changing the total travel demand. From numerical analysis, we found that uniqueness, satisfied in a conventional static traffic assignment problem, does not always hold for a traffic assignment problem with signal control. We also examined the stability of multiple equilibrium states using the phase diagram and showed that two stable equilibrium states sometimes exist where the total travel times are different. We then demonstrated the occurrence of hysteresis loops in response to changes in the total travel demand. Furthermore, we confirmed that the non-uniqueness and stability hold robustly in various situations with different total travel time functions.

In future research, it is important to investigate the stability and performance of multiple equilibria in more general networks, since this study uses just the simple Figure 1 network. We suggest that further research, including systematic numerical experiments, should be conducted to explore how stable traffic states differ across various types of networks, such as grid or ring networks. New methods may be needed to achieve this since in higher-dimensional problems with many state variables (e.g. many routes and many complicated signalised intersections), the phase diagram becomes complex, making the direct application of the graphical approach utilised here unsuitable. Analysing the mathematical properties of the decentralised signal control problem with within-day dynamics would also be an interesting direction for future work. Such considerations are crucial for effectively implementing decentralised signal control and ensuring robust improvements in network performance.

Acknowledgement

We would like to thank the anonymous referees for their constructive comments, which helped to improve this paper.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This work was supported by the JSPS, Japan Grants-in-Aid for Scientific Research under Grant numbers JP20H00265 and JP24K01002.

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Appendix A. A comprehensive review of the P_0 signal control policy

The P_0 signal control policy was first proposed by Smith (1979b), who formulated a static user equilibrium assignment problem under the P_0 policy as a variational inequality in a simple two-route network. This variational inequality identifies stable equilibrium points in the vector field governing green time adjustments under P_0 and route flow changes due to users' route choice behaviour. As shown in Smith (1979a), these stable points correspond to the Wardrop equilibrium. By analysing the vector field, Smith (1979b) demonstrated that when traffic demand is sufficiently high and the network approaches saturation, a unique and stable equilibrium exists. A similar analysis was conducted for Webster's equisaturation signal control policy (Webster 1958). The findings revealed that this method may fail to converge to an equilibrium state where both route choice and signal control conditions are satisfied, potentially degrading network performance. These results underscore the advantages of the P_0 policy over the equisaturation policy. Building on this work, Smith (1980) extended the analysis to a four-route network. Theoretical investigations further demonstrated that the P_0 policy effectively guides traffic flows toward higher-capacity routes through green time adjustments, thereby maximising network throughput.

Subsequent studies have investigated the effectiveness of the P_0 policy in static traffic assignment problems by analysing equilibrium properties such as existence and stability. Smith (1981a) examined a user equilibrium problem with capacity constraints in networks with general topologies and derived sufficient conditions for equilibrium existence, focusing on the conditions that link cost functions must satisfy. These conditions align with those established in Smith (1979a) for the existence of solutions to the user equilibrium assignment problem. Building on this work, Smith (1981b) formulated the user equilibrium problem with capacity constraints under decentralised signal control as a variational inequality. The study further demonstrated that the link cost functions under Webster's equisaturation control policy fail to meet these conditions, but these conditions are satisfied under the P_0 signal control policy.

Several studies have examined the convergence and stability of equilibrium under the P_0 policy. Smith and van Vuren (1993) proposed an iterative algorithm to identify equilibrium states. The algorithm alternates between determining the equilibrium route flow pattern for a given green time pattern and computing the equilibrium green time pattern for a given route flow pattern. The study established sufficient conditions for the algorithm to converge to an equilibrium state where both conditions are simultaneously satisfied and demonstrated that the P_0 policy meets these conditions. Smith and Ghali (1990) and Smith, Liu, and Mounce (2015) defined explicitly dynamical systems that describe the adjustment processes of both users' route choices and signal controls at disequilibrium, examining their stability. Smith and Ghali (1990) formulated a continuous-time system using differential equations and, employing a Lyapunov approach (Smith 1984), proved that equilibrium states in this system are asymptotically stable. Similarly, Smith, Liu, and Mounce (2015) introduced a discrete-time system to explicitly model the day-to-day dynamics of route flows and green times, demonstrating the asymptotic stability of equilibrium states.

The properties of the P_0 signal control policy have also been investigated in quasi-dynamic traffic assignment problems, which assume steady traffic demand while explicitly dealing with queueing delays. Smith (1987) considered the Payne-Thomson traffic assignment model and examined the P_0 policy under which pressure was defined based on vertical queueing delay. Smith (2015) extended this work by explicitly considering spatial queueing delays while disregarding blocking-back effects. The study formulated a dynamical system describing route flow and green time adjustments under the P_0 policy and demonstrated that equilibrium is achieved if the cost functions are monotonic. Smith, Liu, and Mounce (2015) further analysed a network where four routes merge at a single signalised intersection, with signal controllers allocating green times to each route. This study investigated the existence of equilibrium states under the P_0 policy and assessed traffic performance under other decentralised control policies, including Max pressure (Varaiya 2013).

In quasi-dynamic traffic assignment problems, variations of the P_0 policy have also been proposed. Smith, Huang, and Viti (2013); Smith et al. (2019b) introduced two variations: the biased spatial P_0 policy (P_h) and the spatial P_0 policy (P_{hk}), both designed to diminish queue formation and blocking-back effects. These studies demonstrated that equilibrium states consistent with these policies exist under suitable conditions. Additionally, equilibrium analysis in the two-route network in Smith (1979b) revealed that among the three policies, the spatial biased P_0 policy achieved the greatest reduction in route travel times. However, while these policies often improved travel costs, they did not eliminate queues. To address this, Smith et al. (2022) proposed a control method that integrates congestion pricing with signal control. This method imposes congestion charges on each link at a level sufficient to eliminate queues, incorporating these charges into the pressure calculation instead of queueing delays. The study demonstrated that this approach could stably maximise network capacity while removing queues. Smith and Mounce (2024) introduced a new signal control model that integrates the concept of "backpressure" (Varaiya 2013) into the P_0 policy. This model considers the traffic states of both upstream and downstream links when calculating pressure. However, numerical experiments indicated that such pressure-based signal control could, in some cases, increase delay times at equilibrium.

Several studies have highlighted that in quasi-dynamic traffic assignment problems, the original P_0 policy may fail to achieve desirable traffic states, emphasising the need for appropriate modifications. Smith et al. (2019a) demonstrated that under very high traffic demand, the original P_0 policy may lead to a reduction in network capacity.

However, they also showed that a simple modification of the P_0 policy could maximise long-run network throughput in two signal-controlled networks, even when demand exceeds capacity. Similarly, Smith et al. (2023) found that in models incorporating spatial queueing, equilibria consistent with the P_0 policy do not always exist, indicating that the policy may fail to maximise network capacity. However, they also demonstrated that equilibria consistent with a modified P_{hk} policy can exist for any feasible demand, under suitable conditions.

While the mathematical properties of equilibrium states under the P_0 policy have been extensively studied, the uniqueness of equilibrium solutions remains unclear, even within the framework of static traffic assignment problems. Furthermore, comprehensive analyses of traffic performance at equilibrium points have yet to be fully explored despite work initiates in Smith (1979b). For low to moderate traffic demand levels on general networks, it remains uncertain whether equilibrium solutions are unique or what traffic performance characteristics they might exhibit. Several studies have analysed equilibrium traffic performance from the perspectives of total travel time and network throughput. However, these analyses focus on interior equilibria, where at least two routes are utilised, and the pressures on links connected to signalised intersections are balanced. As shown in Smith (1979b), the P_0 policy does not necessarily lead to interior equilibria; rather, it can yield boundary equilibria as stable solutions, where only a single route is utilised. A more comprehensive understanding of these equilibrium properties is important for evaluating the effectiveness of decentralised signal control under fluctuating traffic demand conditions, such as those observed throughout the day. Therefore, revisiting static traffic assignment problems and conducting detailed numerical analyses under the P_0 policy are essential steps toward designing more effective signal control strategies.