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Towards to Analysis in $\mathbb{R}^{pq}$

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Abstract

We are looking for a possible development of analysis in indefinite space $\mathbb{R}^{pq}$ from their group of Möbius automorphisms.

1 Introduction

Applications of Möbius transformations in Clifford analysis have attracted serious consideration recently [1, 5, 6]. The goal of the paper is to make a first step towards analysis in $\mathbb{R}^{pq}$ based on the scheme for analytic function theory described in [3] (see Section 3). In Section 4 we describe the structure of the group of Möbius transformations for positive unit sphere in $\mathbb{R}^{pq}$. Our result allows to construct a function theory in such spheres from the described scheme (to be done elsewhere).

2 Preliminaries

Let $\mathbb{R}^{pq}$ be a real n-dimensional vector space, where $n = p + q$ with a fixed frame $e_1, e_2, \ldots, e_p, e_{p+1}, \ldots, e_n$ and with the nondegenerate bilinear form $B(\cdot,\cdot)$ of signature (p, q), which is diagonal in the frame e_i, i.e.:

$$B(e_i, e_j) = \epsilon_i \delta_{ij}, \text{ where } \epsilon_i = \begin{cases} -1, & i = 1, \ldots, p \\ 1, & i = p+1, \ldots, n \end{cases}$$

and δ_{ij} is the Kronecker delta. Particularly, the usual Euclidean space $\mathbb{R}^n$ is $\mathbb{R}^{0n}$. Let $\mathbf{Cl}(p, q)$ be the *real Clifford algebra* generated by 1, e_j, $1 \le j \le n$ and the relations

$$e_i e_j + e_j e_i = -2B(e_i, e_j).$$

We put $e_0 = 1$ also. Then there is the natural embedding $\mathfrak{i} : \mathbb{R}^{pq} \to \mathbf{Cl}(p, q)$. We identify $\mathbb{R}^{pq}$ with its image under $\mathfrak{i}$ and call its elements *vectors*. There are two linear anti-automorphisms $*$ (reversion) and $-$ (main anti-automorphisms) and automorphism $'$ of $\mathbf{Cl}(p, q)$ defined on its basis $A_\nu = e_{j_1} e_{j_2} \cdots e_{j_r}$, $1 \le j_1 < \cdots < j_r \le n$ by the rule:

$$(A_\nu)^* = (-1)^{\frac{r(r-1)}{2}} A_\nu, \qquad \bar{A}_\nu = (-1)^{\frac{r(r+1)}{2}} A_\nu, \qquad A'_\nu = (-1)^r A_\nu.$$

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In particular, for vectors, $\bar{\mathbf{x}} = \mathbf{x}' = -\mathbf{x}$ and $\mathbf{x}^* = \mathbf{x}$.

It is easy to see that $\mathbf{x}\mathbf{y} = \mathbf{y}\mathbf{x} = 1$ for any $\mathbf{x} \in \mathbb{R}^{pq}$ such that $B(\mathbf{x}, \mathbf{x}) \neq 0$ and $\mathbf{y} = \bar{\mathbf{x}} \left\|\mathbf{x}\right\|^{-2}$, which is the *Kelvin inverse* of $\mathbf{x}$. Finite products of invertible vectors are invertible in $\mathbf{Cl}(p,q)$ and form the *Clifford group* $\Gamma(p,q)$. Elements $a \in \Gamma(p,q)_n$ such that $a\bar{a} = \pm 1$ form the $\mathrm{Pin}(p,q)$ group—the double cover of the group of orthogonal rotations $\mathrm{O}(p,q)$. We also consider [1, § 5.2] $T(p,q)$ to be the set of all products of vectors in $\mathbb{R}^{pq}$.

Let (a,b,c,d) be a quadruple from $T(p,q)$ with the properties:

1. $(ad^* - bc^*) \in \mathbb{R} \setminus 0$;

2. a^*b, c^*d, ac^*, bd^* are vectors.

Then [1, Theorem 5.2.3] 2×2-matrixes $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ form the group $\Gamma(p+1, q+1)$ under the usual matrix multiplication. It has a representation $\pi_{\mathbb{R}^{pq}}$ by transformations of $\overline{\mathbb{R}^{pq}}$ given by:

$$\pi_{\mathbb{R}^{pq}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathbf{x} \mapsto (a\mathbf{x} + b)(c\mathbf{x} + d)^{-1}, \tag{2.1}$$

which form the *Möbius* (or the *conformal*) group of $\overline{\mathbb{R}^{pq}}$. Here $\overline{\mathbb{R}^{pq}}$ the compactification of $\mathbb{R}^{pq}$ by the "necessary number of points" (which form the light cone) at infinity (see [1, § 5.1]). The analogy with fractional-linear transformations of the complex line $\mathbb{C}$ is useful, as well as representations of shifts $\mathbf{x} \mapsto \mathbf{x} + y$, orthogonal rotations $\mathbf{x} \mapsto k(a)\mathbf{x}$, dilatations $\mathbf{x} \mapsto \lambda\mathbf{x}$, and the Kelvin inverse $\mathbf{x} \mapsto \mathbf{x}^{-1}$ by the matrixes $\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} a & 0 \\ 0 & a^{*-1} \end{pmatrix}$, $\begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ respectively.

3 Groups of Symmetries and Analytic Function Theories

Let G be a group that acts via transformation of a closed domain $\bar{\Omega}$. Moreover, let $G : \partial\Omega \to \partial\Omega$ and G act on Ω and $\partial\Omega$ transitively. Let us fix a point $x_0 \in \Omega$ and let $H \subset G$ be a stationary subgroup of point x_0. Then domain Ω is naturally identified with the homogeneous space G/H. And let *there exist a H-invariant measure $d\mu$ on $\partial\Omega$.*

We consider the Hilbert space $L_2(\partial\Omega, d\mu)$. Then geometrical transformations of $\partial\Omega$ give us the representation π of G in $L_2(\partial\Omega, d\mu)$. Let $f_0(x) \equiv 1$ and $F_2(\partial\Omega, d\mu)$ be the closed liner subspace of $L_2(\partial\Omega, d\mu)$ with the properties:

1. $f_0 \in F_2(\partial\Omega, d\mu)$;

2. $F_2(\partial\Omega, d\mu)$ is G-invariant;

3. $F_2(\partial\Omega, d\mu)$ is G-irreducible.

The *standard wavelet transform* W is defined by

$$W : F_2(\partial\Omega, d\mu) \to L_2(G) : f(x) \mapsto \widehat{f}(g) = \langle f(x), \pi(g) f_0(x) \rangle_{L_2(\partial\Omega, d\mu)}$$

Due to the property $[\pi(h) f_0](x) = f_0(x)$, $h \in H$ and identification $\Omega \sim G/H$ it could be translated to the embedding:

$$\widetilde{W} : F_2(\partial\Omega, d\mu) \to L_2(\Omega) : f(x) \mapsto \widehat{f}(y) = \langle f(x), \pi(g) f_0(x) \rangle_{L_2(\partial\Omega, d\mu)}, \tag{3.1}$$

where $y \in \Omega$ for some $h \in H$. The imbedding (3.1) is *an abstract analog of the Cauchy integral formula*. Let functions V_α be the *special functions* generated by the representation of H. Then the decomposition of $\widehat{f_0}(y)$ by V_α gives us the Taylor series.

The Bergman kernel in our approach is given by the formula

$$K(x, y) = c \int_G [\pi_g f_0](x)\overline{[\pi_g f_0](y)}\, dg, \tag{3.2}$$

where c is a constant.

The interpretation of complex analysis based on the given scheme could be found in [3].

4 Möbius Transformations of the Positive Unit Sphere in $\mathbb{R}^{pq}$

One usually says that the conformal group in $\mathbb{R}^{pq}$, $n > 2$ is not so rich as the conformal group in $\mathbb{R}^2$. Nevertheless, the conformal covariance has many applications in Clifford analysis [1, 6]. Notably, groups of conformal mappings of unit spheres $\mathbb{S}^{pq} = \{\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^{pq}, B(\mathbf{x}, \mathbf{x}) = 1\}$ onto itself are similar for all (p, q) and as sets can be parametrized by the product of $\mathbb{B}^{pq} := \mathbb{R}^{pq} \setminus \mathbb{S}^{pq}$ and the group of isometries of $\mathbb{S}^{pq}$.

Proposition 4.1 *The group S_{pq} of conformal mappings of the open unit sphere $\mathbb{S}^{pq}$ onto itself represented by matrixes*

$$\begin{pmatrix} \alpha & \beta \\ \beta' & \alpha' \end{pmatrix}, \qquad \alpha, \beta \in T(p, q), \quad \alpha\beta^* \in \mathbb{R}^{pq}, \quad \alpha\bar{\alpha} - \beta\bar{\beta} = \pm 1. \tag{4.1}$$

Alternatively, let $a \in \mathbb{B}^{pq}$, $b \in \Gamma(p, q)$ then the Möbius transformations of the form

$$\phi_{(a,b)} = \begin{pmatrix} 1 & a \\ a' & 1 \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & b' \end{pmatrix} = \begin{pmatrix} b & ab' \\ a'b & b' \end{pmatrix},$$

constitute S_{pq}. S_{pq} acts on $\mathbb{B}^{pq}$ transitively. Transformations of the form $\phi_{(0,b)}$ constitute a subgroup isomorphic to $\mathrm{O}(p, q)$. The homogeneous space $B_{pq}/\mathrm{O}(p, q)$ is isomorphic as a set to $\mathbb{B}^{pq}$. Moreover:

1. *$\phi^2_{(a,1)} = -1$ on $\mathbb{B}^{pq}$ ($\phi^{-1}_{(a,1)} = -\phi_{(a,1)}$).*
2. *$\phi_{(a,1)}(0) = a$, $\phi_{(a,1)}(a) = 0$.*
3. *$\phi_{(a,1)}\phi_{(c,1)} = \phi_{(d,f)}$ where $d = \phi_{(a,1)}(c)$ and $f = a - c$.*

PROOF. We are using here the notations and results of [1, § 5.1–5.2]. As any Möbius transformations B_{pq} is represented via fractional-linear transformations associated to matrixes $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Its characteristic property is that it preserves (up to projectivity) the unit sphere $\mathbb{S}^{pq}$. $\mathbb{B}^{pq}$ is described in the Fillmore-Springer construction [1, § 5.1] by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. So we are looking for matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with the property

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \bar{d} & \bar{b} \\ \bar{c} & \bar{a} \end{pmatrix} = \begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix}, \qquad \text{for some } r \in \mathbf{Cl}(p, q).$$

This gives us two equations

$$b\bar{d} + a\bar{c} = 0, \qquad d\bar{d} + c\bar{c} = b\bar{b} + a\bar{a}. \tag{4.2}$$

Within different matrices, which satisfy to (4.2) and define the same transformation, there exist exactly one of the form

$$\begin{pmatrix} \alpha & \beta \\ \beta' & -\alpha' \end{pmatrix}, \qquad \alpha\bar{\alpha}+\beta\bar{\beta}=1, \quad \alpha,\beta\in \mathbf{Cl}(p,q).$$

This condition defines the group S_{pq} analogously to $SL(2,\mathbb{R})$.

Now we are looking for the stationary subgroup $S \subset S_{pq}$ of the origin 0. The simple equation

$$\frac{a0+b}{\bar{b}0+\bar{a}}=0$$

convinces us that S consists of matrices $\begin{pmatrix} a & 0 \\ 0 & -a' \end{pmatrix}$, $a\bar{a}=1$ or equivalently $S=\mathrm{Pin}(p,q)$, which acts on the ball $\mathbb{B}^{pq}$ and the sphere $\mathbb{S}^{pq}$ by the isometries. As well known any homogeneous space X is topologically equivalent to the quotient of the symmetry group G with respect to the stationary subgroup G_0 of a point x_0: $X \sim G/G_0$. In our case this means $\mathbb{B}^{pq}=B_{pq}/\mathrm{O}(p,q)$.

Identities 4.1.1– 4.1.3 could be checked by the direct calculations. □

REMARK 4.2 The above Proposition is a generalization of Lemma 2.1 from [4], which was given without proof. Related results for Euclidean spaces are considered in [1, § 6.1].

For Euclidean space one could split $\mathbb{R}^{0q}\setminus\mathbb{S}^{0q}$ on the unit ball $\{x \mid x^2>-1\}$ and its exterior $\{x \mid x^2<-1\}$. This also splits group S_{0n} onto two subgroups. In the general pq case this could not be done because sphere $\mathbb{S}^{pq}$ is not orientable. The details to be given elsewhere [2].

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