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How Many Essentially Different Functional Theories Exist?

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Vladimir V. Kisil Institute of Mathematics, Economics and Mechanics, Odessa State University ul. Petra Velikogo, 2, Odessa June 12, 1996 **Abstrat** The question in the title is ambiguous. At least the understanding of words essentially different and function theory should be clarified. We discuss approaches to do that. We also present a new framework for analytic function theories based on group representations. **1 Introduction** The classic heritage of complex analysis is contested between several complex variables theory and hy-percomplex analysis. The first one was founded long ago by Cauchy and Weierstrass themselves and sometime thought to be the only crown-prince. The hypercomplex analysis is not a single theory but a family of related constructions discovered quite recently [3, 6, 7] (and rediscovered up to now) under hypercomplex framework. Such a variety of theories puts the question on their classification. One could dream about a Mendeleev-like periodic table for hypercomplex analysis, which clearly explains properties of different theories, relationship between them and indicates how many blank cells are waiting for us. Moreover, because hypercomplex analysis is the recognized background for classic and quantum theories like the Maxwell and Dirac equations, such a table could play the role of *the Mendeleev table for elementary particles and fields*. We will return to this metaphor and find it is not very superficial. Maxwell and Dirac equations, such a table could play the role of the Mendeleev table for elementary *particles and fields.* We will return to this metaphor and find it is not very superficial.

To make a step in the desired direction we should specify the notion of *function theory* and define the concept of essential difference. Probably many people agree that the core of complex analysis consist of

1.1. The Cauchy-Riemann equation and complex derivative $\frac{\partial}{\partial z}$;

1.2. The Cauchy theorem;

1.3. The Cauchy integral formula;

1.4. The Plemeli-Sokhotski formula;

1.5. The Taylor and Laurent series.

Any development of several complex variables theory or hypercomplex analysis is beginning from analogies to these notions and results. Thus we adopt the following

Definition 1.1 A function theory is a collection of notions and results, which includes at least analogies of 1.1–1.5.

Of course the definition is more philosophical than mathematical. For example, the understanding of an *analogy* and especially the *right* analogy usually generates many disputes.

Again as a first approximation we propose the following

Definition 1.2 Two function theories is said to be *similar* if there is a correspondence between their objects such that analogies of 1.1–1.5 in one theory follow from their counterparts in another theory. Two function theories are *essentially different* if they are not similar.

Unspecified "correspondence" should probably be a linear map or something else and we will look for its meaning soon. But it is already clear that the *similarity* is an equivalence relation and we are looking for quotient sets with respect to it.

The layout is following. In Subsection 2.1 the classic scheme of hypercomplex analysis is discussed and a possible variety of function theories appears. But we will see in Subsection 2.2 that not all of them are very different. Connection between group representations and (hyper)complex analysis is presented in Section 3. It could be a base for classification of essentially different theories.

2 Abstract Nonsense about Function Theory

In this Section we repeat shortly the scheme of development of Clifford analysis as it could be found in [3, 6]. We examine different options arising on this way and demonstrate that some differences are only apparent not essential.

2.1 Factorizations of the Laplacian

We would like to see how the contents of 1.1–1.5 could be realized in a function theory. We are interested in function theories defined in \mathbb{R}^d . The Cauchy theorem and integral formula clearly indicates that the behavior of functions inside a domain should be governed by their values on the boundary. Such a property is particularly possessed by solutions to the second order elliptic differential operator P

$$P(x,\partial_x) = \sum_{i,j=1}^d a_{ij}(x)\partial_i\partial_j + \sum_{i=1}^d b_i(x)\partial_i + c(x)$$

with some special properties. Of course, the principal example is the Laplacian

$$\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}.$$
(2.1)

2.1. Choice of different operators (for example, the Laplacian or the Helmholtz operator) is the first option which brings the variety in the family of hypercomplex analysis.

The next step is called *linearization*. Namely we are looking for two *first order* differential operators D and D' such that

$$DD' = P(x, \partial_x).$$

The Dirac motivation to do that is to "look for an equation linear in in time derivative $\frac{\partial}{\partial t}$, because the Schrödinger equation is". From the function theory point of view the Cauchy-Riemann operator should be linear also. But the most important gain of the step is an introduction of the Clifford algebra. For example, to factorize the Laplacian (2.1) we put

$$D = \sum_{i=1}^{d} e_i \partial_i \tag{2.2}$$

where e_i are the *Clifford algebra* generators:

$$e_i e_j + e_j e_i = 2\delta_{ij}, \qquad 1 \le i, j \le d. \tag{2.3}$$

2.2. Different linearizations of a second order operator multiply the spectrum of theories.

Mathematicians and physicists are looking up to now new factorization even for the Laplacian. The essential uniqueness of such factorization was already felt by Dirac himself but it was never put as a theorem. So the idea of the *genuine* factorization becomes the philosophers' stone of our times.

After one made a choice 2.1 and 2.2 the following turns to be a routine. The equation

$$D'f(x) = 0,$$

plays the role of the Cauchy-Riemann equation. Having a fundamental solution F(x) to the operator $P(x, \partial_x)$ the Cauchy integral kernel defined by

$$E(x) = D'F(x)$$

with the property $DE(x) = \delta(x)$. Then the Stocks theorem implies the Cauchy theorem and Cauchy integral formula. A decomposition of the Cauchy kernel of the form

$$C(x-y) = \sum_{\alpha} V_{\alpha}(x) W_{\alpha}(y),$$

where $V_{\alpha}(x)$ are some polynomials, yields via integration over the ball the Taylor and Laurent series¹. In such a way the program-minimum 1.1–1.5 could be accomplished.

Thus all possibilities to alter function theory concentrated in 2.1 and 2.2. Possible universal algebras arising from such an approach were investigated by F. SOMMEN [22]. In spite of the apparent wide selection, for operator D and D' with constant coefficients it was found "nothing dramatically new" [22]:

Of course one can study all these algebras and prove theorems or work out lots of examples and representations of universal algebras. But in the constant coefficient case the most important factorization seems to remain the relation $\Delta = \sum x_j^2$, i.e., the one leading to the definition of the Clifford algebra.

We present an example that there is no dramatical news not only on the level of universal algebras but also for function theory (for the constant coefficient case). We will return to non constant case in Section 3.

2.2 Example of Connection

We give a short example of similar theories with explicit connection between them. The full account could be found in [9], another example was considered in [20].

Due to physical application we will consider equation

$$\frac{\partial f}{\partial y_0} = \left(\sum_{j=1}^n e_j \frac{\partial}{\partial y_j} + M\right) f,\tag{2.4}$$

where e_j are generators (2.3) of the Clifford algebra and $M = M_{\lambda}$ is an operator of multiplication from the *right*-hand side by the Clifford number λ . Equation (2.4) is known in quantum mechanics as the *Dirac equation for a particle with a non-zero rest mass* [1, §20], [2, §6.3] and [14]. We will specialize our results for the case $M = M_{\lambda}$, especially for the simplest (but still important!) case $\lambda \in \mathbb{R}$.

 $^{^{1}}$ Not all such decompositions give interesting series. The scheme from Section 3 gives a selection rule to distinguish them.

Theorem 2.1 The function f(y) is a solution to the equation

$$\frac{\partial f}{\partial y_0} = \left(\sum_{j=1}^n e_j \frac{\partial}{\partial y_j} + M_1\right) f$$

if and only if the function

$$g(y) = e^{y_0 M_2} e^{-y_0 M_1} f(y)$$

is a solution to the equation

$$\frac{\partial g}{\partial y_0} = \left(\sum_{j=1}^n e_j \frac{\partial}{\partial y_j} + M_2\right)g,$$

where M_1 and M_2 are bounded operators commuting with e_j .

Corollary 2.2 The function f(y) is a solution to the equation (2.4) if and only if the function $e^{y_0M}f(y)$ is a solution to the generalized Cauchy-Riemann equation (2.2).

In the case $M = M_{\lambda}$ we have $e^{y_0 M_{\lambda}} f(y) = f(y) e^{y_0 \lambda}$ and if $\lambda \in \mathbb{R}$ then $e^{y_0 M_{\lambda}} f(y) = f(y) e^{y_0 \lambda} = e^{y_0 \lambda} f(y)$.

In this Subsection we construct a function theory (in the sense of 1.1-1.5) for *M*-solutions of the generalized Cauchy-Riemann operator based on Clifford analysis and Corollary 2.2.

The set of solutions to (2.2) and (2.4) in a nice domain Ω will be denoted by $\mathfrak{M}(\Omega) = \mathfrak{M}_0(\Omega)$ and $\mathfrak{M}_M(\Omega)$ correspondingly. In the case $M = M_\lambda$ we use the notation $\mathfrak{M}_\lambda(\Omega) = \mathfrak{M}_{M_\lambda}(\Omega)$ also. We suppose that all functions from $\mathfrak{M}_\lambda(\Omega)$ are continuous in the closure of Ω . Let

$$E(y-x) = \frac{\Gamma(\frac{n+1}{2})}{2\pi^{(n+1)/2}} \frac{\overline{y-x}}{|y-x|^{n+1}}$$
(2.5)

be the Cauchy kernel [6, p. 146] and

$$d\sigma = \sum_{j=0}^{n} (-1)^{j} e_{j} dx_{0} \wedge \ldots \wedge [dx_{j}] \wedge \ldots \wedge dx_{m}$$

be the differential form of the "oriented surface element" [6, p. 144]. Then for any $f(x) \in \mathfrak{M}(\Omega)$ we have the Cauchy integral formula [6, p. 147]

$$\int_{\partial\Omega} E(y-x) \, d\sigma_y \, f(y) = \begin{cases} f(x), & x \in \Omega\\ 0, & x \notin \overline{\Omega} \end{cases}$$

Theorem 2.3 (Cauchy's Theorem) Let $f(y) \in \mathfrak{M}_M(\Omega)$. Then

$$\int_{\partial\Omega} d\sigma_y \, e^{-y_0 M} f(y) = 0$$

Particularly, for $f(y) \in \mathfrak{M}_{\lambda}(\Omega)$ we have

$$\int_{\partial\Omega} d\sigma_y f(y) e^{-y_0 \lambda} = 0,$$

and

$$\int_{\partial\Omega} d\sigma_y e^{-y_0\lambda} f(y) = 0$$

if $\lambda \in \mathbb{R}$.

Theorem 2.4 (Cauchy's Integral Formula) Let $f(y) \in \mathfrak{M}_M(\Omega)$. Then

$$e^{x_0 M} \int_{\partial \Omega} E(y-x) \, d\sigma_y \, e^{-y_0 M} f(y) = \begin{cases} f(x), & x \in \Omega \\ 0, & x \notin \overline{\Omega} \end{cases}.$$

$$(2.6)$$

Particularly, for $f(y) \in \mathfrak{M}_{\lambda}(\Omega)$ we have

$$\int_{\partial\Omega} E(y-x) \, d\sigma_y \, f(y) e^{(x_0-y_0)\lambda} = \begin{cases} f(x), & x \in \Omega\\ 0, & x \notin \overline{\Omega} \end{cases}$$

and

$$\int_{\partial\Omega} E(y-x)e^{(x_0-y_0)\lambda} \, d\sigma_y \, f(y) = \begin{cases} f(x), & x \in \Omega\\ 0, & x \notin \overline{\Omega} \end{cases}.$$

if $\lambda \in \mathbb{R}$.

It is hard to expect that formula (2.6) may be rewritten as

$$\int_{\partial\Omega} E'(y-x) \, d\sigma_y \, f(y) = \begin{cases} f(x), & x \in \Omega\\ 0, & x \notin \overline{\Omega} \end{cases}$$

with a simple function E'(y-x).

Because an application of the bounded operator $e^{y_0 M}$ does not destroy uniform convergency of functions we obtain (cf. [6, Chap. II, § 0.2.2, Theorem 2])

Theorem 2.5 (Weierstrass' Theorem) Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathfrak{M}_M(\Omega)$, which converges uniformly to f on each compact subset $K \in \Omega$. Then

- 2.1. $f \in \mathfrak{M}_M(\Omega)$.
- 2.2. For each multi-index $\beta = (\beta_0, \dots, \beta_m) \in \mathbb{N}^{n+1}$, the sequence $\{\partial^\beta f_k\}_{k \in \mathbb{N}}$ converges uniformly on each compact subset $K \in \Omega$ to $\partial^\beta f$.

Theorem 2.6 (Mean Value Theorem) Let $f \in \mathfrak{M}_M(\Omega)$. Then for all $x \in \Omega$ and R > 0 such that the ball $\mathbb{B}(x, R) \in \Omega$,

$$f(x) = e^{x_0 M} \frac{(n+1)\Gamma(\frac{n+1}{2})}{2R^{n+1}\pi^{(n+1)/2}} \int_{\mathbb{B}(x,R)} e^{-y_0 M} f(y) \, dy$$

Such a reduction of theories could be pushed even future [9] up to the notion of hypercomplex differentiability [15], but we will stop here.

3 Hypercomplex Analysis and Group Representations — Towards a Classification

To construct a classification of non-equivalent objects one could use their groups of symmetries. Classical example is Poincaré's proof of bi-holomorphic non-equivalence of the unit ball and polydisk via comparison their groups of bi-holomorphic automorphisms. To employ this approach we need a construction of hypercomplex analysis from its symmetry group. The following scheme is firstly presented here (up to the author knowledge) and has its roots in [13, 11, 10].

Let G be a group which acts via transformation of a closed domain $\overline{\Omega}$. Moreover, let $G : \partial \Omega \to \partial \Omega$ and G act on Ω and $\partial \Omega$ transitively. Let us fix a point $x_0 \in \Omega$ and let $H \subset G$ be a stationary subgroup of point x_0 . Then domain Ω is naturally identified with the homogeneous space G/H. Till the moment we do not request anything untypical. Now let • there exist a H-invariant measure $d\mu$ on $\partial\Omega$.

We consider the Hilbert space $L_2(\partial\Omega, d\mu)$. Then geometrical transformations of $\partial\Omega$ give us the representation π of G in $L_2(\partial\Omega, d\mu)$. Let $f_0(x) \equiv 1$ and $F_2(\partial\Omega, d\mu)$ be the closed liner subspace of $L_2(\partial\Omega, d\mu)$ with the properties:

3.1. $f_0 \in F_2(\partial\Omega, d\mu);$

3.2. $F_2(\partial\Omega, d\mu)$ is G-invariant;

3.3. $F_2(\partial\Omega, d\mu)$ is *G*-irreducible.

The standard wavelet transform W is defined by

$$W: F_2(\partial\Omega, d\mu) \to L_2(G): f(x) \mapsto f(g) = \langle f(x), \pi(g) f_0(x) \rangle_{L_2(\partial\Omega, d\mu)}$$

Due to the property $[\pi(h)f_0](x) = f_0(x), h \in H$ and identification $\Omega \sim G/H$ it could be translated to the embedding:

$$\widetilde{W}: F_2(\partial\Omega, d\mu) \to L_2(\Omega): f(x) \mapsto \widehat{f}(y) = \langle f(x), \pi(g) f_0(x) \rangle_{L_2(\partial\Omega, d\mu)},$$
(3.1)

where $y \in \Omega$ for some $h \in H$. The imbedding (3.1) is an abstract analog of the Cauchy integral formula. Let functions V_{α} be the special functions generated by the representation of H. Then the decomposition of $\hat{f}_0(y)$ by V_{α} gives us the Taylor series.

The scheme is inspired by the following interpretation of complex analysis.

Example 3.1 Let the domain Ω be the unit disk \mathbb{D} , $\partial \mathbb{D} = \mathbb{S}$. We select the group $SL(2, \mathbb{R}) \sim SU(1, 1)$ acting on \mathbb{D} via the fractional-linear transformation:

$$\left(\begin{array}{cc}a&b\\c&d\end{array}\right):z\mapsto\frac{az+b}{cz+d}$$

We fix $x_0 = 0$. Then its stationary group is U(1) of rotations of \mathbb{D} . Then the Lebesgue measure on \mathbb{S} is U(1)-invariant. We obtain $\mathbb{D} \sim SL(2, \mathbb{R})/U(1)$. The subspace of $L_2(\mathbb{S}, dt)$ satisfying to 3.1–3.3 is the Hardy space. The wavelets transform(3.1) give exactly the Cauchy formula. The proper functions of U(1) are exactly z^n , which provide the base for the Taylor series. The Riemann mapping theorem allows to apply the scheme to any connected, simply-connected domain.

The conformal group of the Möbius transformations plays the same role in Clifford analysis. One usually says that the conformal group in \mathbb{R}^n , n > 2 is not so rich as the conformal group in \mathbb{R}^2 . Nevertheless, the conformal covariance has many applications in Clifford analysis [4, 19]. Notably, groups of conformal mappings of open unit balls $\mathbb{B}^n \subset \mathbb{R}^n$ onto itself are similar for all n and as sets can be parametrized by the product of \mathbb{B}^n itself and the group of isometries of its boundary \mathbb{S}^{n-1} .

Theorem 3.2 [12] Let $a \in \mathbb{B}^n$, $b \in \Gamma_n$ then the Möbius transformations of the form

$$\phi_{(a,b)} = \begin{pmatrix} b & 0 \\ 0 & b^{*-1} \end{pmatrix} \begin{pmatrix} 1 & -a \\ a^* & -1 \end{pmatrix} = \begin{pmatrix} b & -ba \\ b^{*-1}a^* & -b^{*-1} \end{pmatrix},$$

constitute the group B_n of conformal mappings of the open unit ball \mathbb{B}^n onto itself. B_n acts on \mathbb{B}^n transitively. Transformations of the form $\phi_{(0,b)}$ constitute a subgroup isomorphic to O(n). The homogeneous space $B_n/O(n)$ is isomorphic as a set to \mathbb{B}^n . Moreover:

3.1. $\phi_{(a,1)}^2 = 1$ identically on \mathbb{B}^n $(\phi_{(a,1)}^{-1} = \phi_{(a,1)}).$

3.2. $\phi_{(a,1)}(0) = a, \phi_{(a,1)}(a) = 0.$

Obviously, conformal mappings preserve the space of null solutions to the *Laplace* operator (2.1) and null solutions the *Dirac* operator (2.2). The group B_n is sufficient for construction of the Poisson and the Cauchy integral representation of harmonic functions and Szegö and Bergman projections in Clifford analysis by the formula [11]

$$K(x,y) = c \int_G [\pi_g f](x) \overline{[\pi_g f](y)} \, dg, \qquad (3.2)$$

where π_g is an irreducible unitary square integrable representation of a group G, f(x) is an arbitrary non-zero function, and c is a constant.

The scheme gives a correspondence between *function theories* and *group representations*. The last are rather well studded and thus such a connection could be a foundation for a classification of function theories. Particularly, the *constant coefficient* function theories in the sense of F. SOMMEN[22] corresponds to the groups acting only on the function domains in the Euclidean space. Between such groups the Moebius transformations play the leading role. On the contrary, the *variable coefficient* case is described by groups acting on the function space in the non-point sense (for example, combining action on the functions domain and range, see [13]). The set of groups of the second kind should be more profound.

REMARK 3.3 It is known that many results in real analysis [16] several variables theory [17] could be obtained or even explained via hypercomplex analysis. One could see roots of this phenomenon in relationships between groups of geometric symmetries of two theories: the group of hypercomplex analysis is wider.

Returning to our metaphor on the Mendeleev table we would like recall that it began as linear ordering with respect to atomic masses but have received an explanation only via representation theory of the rotation group.

Our consideration provides a ground for the following

Conjecture 3.4 Most probably there is the only constant coefficient function theory on the Euclidean space or at most there are two of them.

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The bibliography to such a paper should be definitely more complete and representative. Unfortunately, I mentioned only a few papers among deserving it.

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