



Cross-Toeplitz operators on the Fock–Segal–Bargmann spaces and two-sided convolutions on the Heisenberg group

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Abstract

We introduce an extended class of cross-Toeplitz operators which act between Fock–Segal–Bargmann spaces with different weights. It is natural to consider these operators in the framework of representation theory of the Heisenberg group. Our main technique is representation of cross-Toeplitz by two-sided relative convolutions from the Heisenberg group. In turn, two-sided convolutions are reduced to usual (one-sided) convolutions on the Heisenberg group of the doubled dimensionality. This allows us to utilise the powerful group-representation technique of coherent states, co- and contra-variant transforms, twisted convolutions, symplectic Fourier transform, etc. We discuss connections of (cross-)Toeplitz operators with pseudo-differential operators, localisation operators in time–frequency analysis, and characterisation of kernels in terms of ladder operators. The paper is written in a detailed and reasonably self-contained manner to be suitable as an introduction into group-theoretical methods in phase space and time–frequency operator theory.

Keywords Heisenberg group · Fock–Segal–Bargmann space · Toeplitz operator · Covariant and contravariant transforms · Phase space · Time–frequency analysis · Berezin calculus · Localisation operators · Coherent states · Two-sided convolutions · Pseudo-differential operators · Berezin quantisation

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1 Introduction

Motivated by applications [70, 71] this paper starts a systematic treatment of mixed coherent state transforms (aka time–frequency analysis) with various Gaussian windows. In particular, we introduce cross-Toeplitz operators acting between two Fock–Segal–Bargmann spaces with different Gaussian weights, which are studied through two-sided relative convolutions on the phase space. The paper contains a comprehensive description of the theory out of the main objects of the representation theory. The next subsection provides a summary for readers familiar with the field. A sloping introduction in a wider context starts from Sect. 1.2.

1.1 Brief summary for specialists

Let \mathcal{F}_2 be the Fock–Segal–Bargmann (FSB) space [7, 34, 79, 92] on the phase space, which can be defined as a certain irreducible component of the unitary representation of the Heisenberg group in \mathcal{L}_2 space [18, 23]. Let $P : \mathcal{L}_2 \rightarrow \mathcal{F}_2$ be the respective FSB orthoprojection. For a function ψ on the phase space one defines the Toeplitz operator $T_\psi : \mathcal{F}_2 \rightarrow \mathcal{F}_2$ by the identity

$$T_\psi f = P(\psi f), \quad \text{for } f \in \mathcal{F}_2. \quad (1.1)$$

A study of Toeplitz operators [8, 21, 23, 92], their connections to PDOs [40] and a parallel theory of localisation operators on the phase space [1, 16, 19, 38] is a vivid research area.

Coburn discovered [20] some fundamental limitations for calculus of Toeplitz operators. For example, a composition operator $[C_\theta f](z) = f(e^{i\theta} z)$ with $\pi/4 < \theta < \pi/2$ can be represented as a Toeplitz operator T_ϕ for a Gaussian type function ϕ . However, its square C_θ^2 cannot be given in the form (1.1) for any ψ .

This paper includes Toeplitz operators into a larger class of *two-sided* relative convolutions from the Heisenberg group [57, 64, 66]. Convolutions with certain families of kernels form algebras closed under composition. They are also naturally linked to pseudo-differential operators (PDO) as will be shown in details. Yet, the Coburn counterexample C_θ from the previous paragraph is not a PDO with a symbol in a classical Hörmander class. Thus, a proper treatment of C_θ requires more general Fourier integral operators [41], which is beyond the scope of the present paper.

There are clear advantages of mixed coherent states decompositions over bounded domains [70, 71, 73, Chap. 9]. To this end we introduce and study in this work a new class of cross-Toeplitz operators. They act between FSB spaces with different Gaussian weights [92] or different Gaussian windows in terms time–frequency analysis [38]. Justification of cross-Toeplitz operators from a wider context is discussed in subsequent subsections of this Introduction. To work with cross-Toeplitz operator we need numerous adjustments to the framework, starting from the definition of pre-FSB spaces. We systematically present the theory here from the group-representation perspective, which is not yet dominating in the field (with the some notable exceptions, e.g. [11, 18, 23, 34]).

While Toeplitz operators admit representation as usual (one-sided) convolutions on the Heisenberg group, cross-Toeplitz operators need two-sided convolutions because they mix different irreducible components under the Heisenberg group action. Yet, our main tool is a transformation of two-sided convolutions to one-sided convolutions on the Heisenberg group of the doubled dimensionality. This is an example of the Heisenberg group universality for relative convolutions [57, Thm 3.8]. A more costly alternative is a consideration of the nilpotent *step 3* Dynin [66] (aka Dynin–Folland [78]) group build on top of the simplest nilpotent *step 2* Heisenberg group. Another alternative is a solvable group, which is a semi-direct product of the Heisenberg group and its one-parameter group of automorphisms by inhomogeneous dilations.

1.1.1 The paper outline

In Sect. 2 we recall the minimal background on the Heisenberg group and its representation theory. The standard construction of the induced representations puts all FSB spaces with different Gaussian weights (or rather their unitary equivalent counterparts) as irreducible subspaces of $\mathcal{L}_2(\mathbb{R}^{2n})$. In the traditional approach from holomorphic function perspective all these FSB spaces are detained in differently weighted \mathcal{L}_2 -spaces, which prevents a natural consideration of cross-Toeplitz operators so far. The key ingredients for two-sided convolutions—the left and right pulled action—are introduced here as well.

In Sect. 3 we collect fundamental facts on the co- and contra-variant transforms also known under numerous other names in various fields, e.g. coherent states, voice, Berezin, Wigner, FSB, etc. transforms. Together with twisted convolution and symplectic Fourier transform these will be our main tools. We review complexification of the theory and appearance of (poly-)analytic functions from the representation theory in Sect. 4. Ladder operators for the left and right actions play different but equally important rôles here.

We turn to our study of (cross-)Toeplitz operators in Sect. 5. To larger extend results of this section go back to original works of Howe [42, 43] and Guillemin [40]. Yet we are able to add some more group-theoretic perspectives to this material as well. We discuss connections to the theory of PDO, localisation operators in time–frequency analysis and a significance of the heat flow on symbols.

The final Sect. 6 contains most of the original material. Here we represent cross-Toeplitz operators as two-sided relative convolutions from the Heisenberg group. Two-sided convolutions are reduced to one-sided convolutions from the Heisenberg group of double dimensionality. Thus, we can re-cycle all the theory presented in earlier sections for new needs. Cross-Toeplitz operators are treated through the symbolic calculus of PDO and their symbols are characterised through certain identities with ladder operators. A point of interest is that a PDO symbol of T_ψ is expressed through the invertible FSB-like transform of ψ instead of smoothing heat flow considered in the literature so far.

Now we turn to a discussion of cross-Toeplitz operators within a wider context of harmonic analysis.

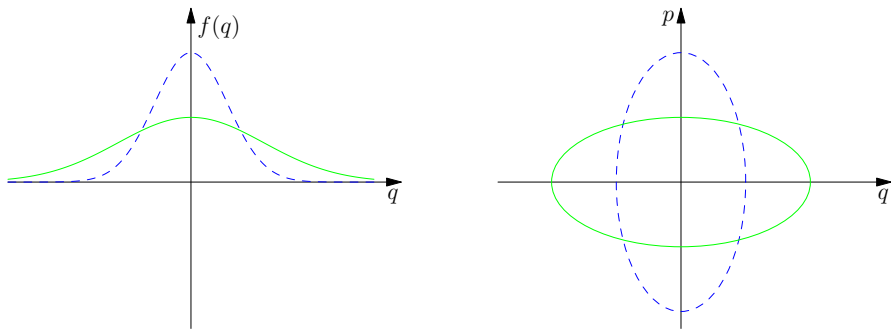


Fig. 1 Two differently squeezed Gaussians on the configurational space (left) and their phase space footprints (right). Green-solid and blue-dashed pencils are used for the respective drawings on both graphs (Color figure online)

1.2 Uncertainty relation and squeeze transformation

The fundamental idea of the Fourier analysis is that a function $f(t)$ of a real variable can be represented as a superposition of simple harmonics $e^{2\pi ikt}$ with all real wave numbers k . The original application to differential equations—due to the fact that harmonics are eigenfunctions of the derivative—was joined later by numerous others.

There is a fundamental restriction to combine the Fourier analysis with another common technique—localisation: a smaller support of a function $f(t)$ implies a wider bandwidth of representing harmonics. The phenomenon was manifested as the Heisenberg uncertainty relation in quantum mechanics, which assigns certain de Broglie wave to every elementary object. It is equally relevant for time–frequency analysis of voice/signal functions. Since these theories are mathematically equivalent we will interchangeably use terms from both areas.

For quantitative description of the uncertainty we need a coefficient to make coordinates and wave numbers to be co-measurable. In quantum mechanics it is known as the Planck constant \hbar . Then, for the coordinate Q and momentum P operators there is the Heisenberg–Kennard uncertainty relation [34, Sect. 1.3]:

$$\Delta_{\phi}(Q) \cdot \Delta_{\phi}(P) \geq \frac{\hbar}{2}, \quad (1.2)$$

where $\Delta_{\phi}(A)$ is the dispersion of an operator A on the state ϕ . The equality holds if and only if ϕ is the Schrödinger coherent state:

$$\phi_{qp}(t) = \sqrt[4]{\tau} e^{2\pi i \hbar p t} e^{-\pi \hbar (t-q)^2 / (2\tau)}. \quad (1.3)$$

which are (q, p) -translations in the phase space of the squeezed Gaussian $\phi_{\tau}(t) = \sqrt[4]{\tau} e^{-\pi \hbar t^2 / (2\tau)}$, see (2.14) below. Fig. 1 shows two differently squeezed Gaussians on the configuration space (on the left graph) and the respective regions of their essential support in the phase space (on the right graph). Green-solid and blue-dashed pencils are used for the respective drawings on both graphs.

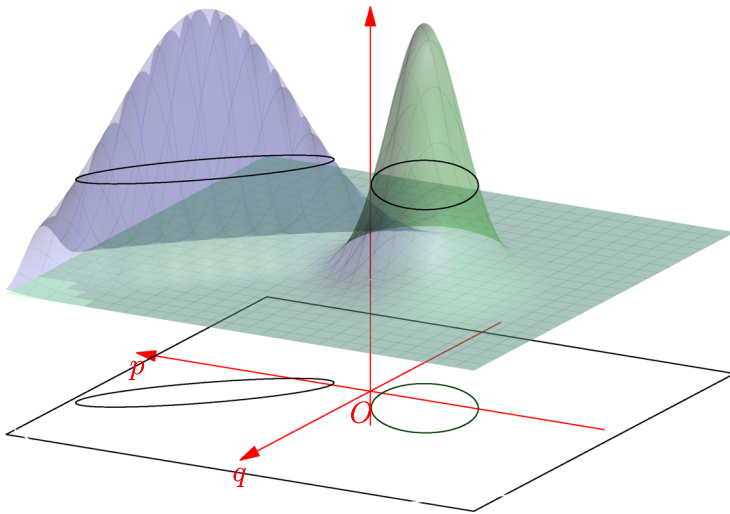


Fig. 2 Squeezed states on the phasespace and their blobs

Clearly, squeezed Gaussians are fully determined by these types of ellipses—their footprints on the phase space, see Fig. 2. This concept (branded as quantum blobs [27, 28]) nicely visualises the phase space uncertainty. The squeeze parameter τ allows us to trade a better localisation in the configuration space for a more defused bandwidth. Thus a possibility to interchangeably operate with differently squeezed Gaussians is an advantage.

1.3 Fock–Segal–Bargmann spaces and Toeplitz operators

The Fourier transform through simple harmonics can be seen as the limiting case of a decomposition into a superposition of the Schrödinger coherent states (1.3) for $\tau \rightarrow 0$. It achieves a Dirac delta-like localisation in the frequency scale on the expense of no localisation whatsoever in coordinates. For a fixed $\tau \neq 0$ such a decomposition is known as the Fock–Segal–Bargmann (FSB) transform ([7, 33, Sect. 28-3]; [79]) with a quantum mechanical origin. Alternatively it appears in time–frequency analysis as the Gabor transform of a signal into simple harmonics $e^{2\pi i \hbar p t}$ modulated by the Gaussian window $\sqrt{\tau} e^{-\pi \tau (t-q)^2/2}$ [38]. FSB transform is an archetypal source of numerous developments: coherent states [4, 77], atomic decompositions [29, 30], etc.

It is common to adjust the construction in such a way that the image space of the FSB transform consists of holomorphic functions. Specifically, define Gaussian-weighted measure on \mathbb{C}^n :

$$d\mu_\tau(z) = (4\tau)^{-n} \exp(-\pi |z|^2 / (4\tau)) dz \wedge d\bar{z}, \quad \text{where } \tau > 0.$$

The FSB space \mathcal{F}^τ is commonly defined as the closed subspace of $\mathcal{L}_2(\mathbb{C}^n, d\mu_\tau)$ consisting of holomorphic functions. There is the associated orthoprojection P_τ :

$\mathcal{L}_2(\mathbb{C}^n, d\mu_\tau) \rightarrow \mathcal{F}^\tau$. Then any bounded function ψ on \mathbb{C}^n defines the Toeplitz operator $T_\psi : \mathcal{F}^\tau \rightarrow \mathcal{F}^\tau$ by the rule $T_\psi f = P_\tau(\psi f)$, for $f \in \mathcal{F}^\tau$.

The correspondence of functions ψ on the phase space (classical observables) to Toeplitz operators T_ψ (quantum observables) is known as Berezin–Toeplitz quantization—a physically significant [10, 34] and mathematically rich [23, 92] concept. Another mainstream quantisation employs the Fourier transform in a different way: a pseudo-differential operator (PDO) $a(X, D)$ with a Weyl symbol $a(x, \lambda)$ is defined by:

$$[a(x, d)f](x) = \int \int a\left(\frac{1}{2}(x+y), \lambda\right) f(y) e^{2\pi i \lambda(x-y)} d\lambda dy.$$

Relations between Berezin–Toeplitz and Weyl quantisation was already addressed by Guillemin [40] and continues to be in the focus of current research [23]. Effectively, the Toeplitz operator T_ψ is unitary equivalent to PDO $\psi_\tau(X, D)$ for

$$\psi_\tau = \psi * \Phi_\tau, \quad (1.4)$$

which is a smoothing of ψ by the convolution with the Gaussian $\Phi_\tau(q, p) = e^{-\pi \hbar(q^2/\tau + \tau p^2)/2}$ [34, (3.5)]. It can be also interpreted as a heat/diffusion flow transformation [13]. Of course, the degree of smoothing depends on τ which again put this parameter in a focus of our attention.

1.4 Variable squeezing

We have seen that the Planck constant \hbar limits the joint localisability in the phase space (1.2) while the squeeze parameter τ controls shares of uncertainty between coordinates and momentum (or time and frequency). While the fundamental physical constants, e.g. the Planck constant, cannot be affected, we can and sometime want to change parameters like squeezing. Here are few illustrations:

1. An upgraded version of FSB transform which treats τ as a parameter on a par with p and q is known as FBI (Fourier–Bros–Iagolnitzer) transform. It is used, for example, to analyse wave fronts [34, Sect. 3.3].
2. An adaptation of FSB transform with τ being a power function of p was employed [25, 34, (3.6)] for a better approximation of a PDO by a Toeplitz operator. Indeed the smoothing (1.4) harder affects high frequencies. To reduce this effect we need a more narrow Gaussian window with its width to be much smaller than targeted wavelengths.
3. There is a possibility to geometrize a quantum dynamics by extending the classical phase space with additional coordinates and the squeeze parameter τ is a suitable option. Examples of coherent states with oscillating squeeze were described in [5, 6].
4. Adopting coherent states decomposition (aka time–frequency analysis) on bounded domains we can use Gaussian windows of narrowing widths through the Archimedes’ method of exhaustion [70, 73, Chap. 9].

The present paper widens the study of Toeplitz operators between FSB spaces with different squeeze parameters. Recall, that for each value τ the FSB space \mathcal{F}^τ is unitary equivalent (through FSB transform) to the space $\mathcal{L}_2(\mathbb{R}^n)$ of functions on the configurational space. Thus, for two squeeze parameters τ and ς there is a natural unitary equivalence of \mathcal{F}^τ and \mathcal{F}^ς . That is, this map recalculates the FSB transform with a squeeze parameter τ into the FSB transform of the same function in $\mathcal{L}_2(\mathbb{R}^n)$ for a different squeeze parameter ς . Alternatively, the equivalence of \mathcal{F}^τ and \mathcal{F}^ς corresponds to the unitary dilation $f(t) \mapsto r^{n/2} f(rt)$ with $r = \sqrt{\tau/\varsigma}$ on $\mathcal{L}_2(\mathbb{R}^n)$.

Furthermore, for a function ψ on the phase space we consider cross-Toeplitz operator $T_\psi : \mathcal{F}^\tau \rightarrow \mathcal{F}^\varsigma$ by the identity:

$$T_\psi f = P_\varsigma(\psi f), \quad \text{where } f \in \mathcal{F}^\tau.$$

We develop some basic result on the calculus of cross-Toeplitz operators using group representation technique notably two-sided relative convolutions from the Heisenberg group [49–54, 84, 86, 87]. Interestingly, some obtained results are new even for the case $\tau = \varsigma$ of the traditional Toeplitz operators. One of applications of cross-Toeplitz operators is related to the above item 4. Zones with different Gaussian widths are overlapping and cross-Toeplitz operators are a right tool to account the imbrication [70].

1.5 Coherent states and group representations

Our choice of group representations technique is well-motivated and historically based. Connections of the Heisenberg group to quantum mechanics are rooted in the Stone-von Neumann theorem. It established the coincidence of formalisms based on the Heisenberg matrix mechanics and the Schrödinger equation through the classification of the unitary irreducible representations of the Heisenberg group. Various values of the Planck constants parameterise non-equivalent classes of representations. Furthermore, the Schrödinger coherent states (1.3) are obtained from the vacuum vector—the squeezed Gaussian $\phi_\tau(t) = \tau^{-1/4} e^{-\pi \hbar t^2 / (2\tau)}$ —by the irreducible Schrödinger representation of the Heisenberg group on $\mathcal{L}_2(\mathbb{R})$. Correspondingly, the FSB transform is the prime example of the coherent states transform generated by a square-integrable unitary group representation [4, 34, 77].

On the other hand, a link of the calculus of PDO to the Heisenberg group was indicated as early as [39]. It was spectacularly developed in [42, 43] and spelt in details in [34]. Toeplitz operators on the Fock space from the Heisenberg group appeared in [42, 43] implicitly (see Sect. 5.2 below), their connection to PDO (and therefore to the Heisenberg group) was explicitly revealed by Guillemin [40]. Folland elaborated this in his step-stone monograph [34]. Further usage of the Heisenberg group for Toeplitz operators can be found in [18–20, 22, 23].

To enable various values of the squeeze parameter we extend the Heisenberg group using the symplectic transformation $(q, p) \mapsto (rq, p/r)$ of the phase space, see the right graph on Fig. 1 for a visualisation. The Heisenberg group acts on the phase-space by shifts in a reducible way, in particular every FSB space \mathcal{F}^τ is invariant (and

irreducible). Thereafter, the von Neumann algebra of operators generated by the phase space shifts contains Toeplitz operators (with suitable class of symbols). However, it can not contain any cross-Toeplitz operators $\mathcal{F}^\tau \rightarrow \mathcal{F}^\varsigma$ for $\tau \neq \varsigma$. On the other hand, the extension of the phase shift with squeeze transformations acts on $\mathcal{L}_2(\mathbb{C}^n)$ irreducibly and cross-Toeplitz operators become covered.

From general principles, we can use other extensions of phase space shifts to achieve irreducibility on $\mathcal{L}_2(\mathbb{C}^n)$, see the mentioned in Sect. 1.1 Dynin group as an example. Our choice allows to achieve results virtually at no extra cost (e.g. without increments of nilpotency of the group), we combine the initial (left) phase space shifts by their right counterparts. This allows us to study cross-Toeplitz operators as two-sided relative convolutions from the Heisenberg group. The latter can be reduced to usual (one-sided) convolutions on a Heisenberg group of the doubled dimensionality [49–54]. This technique successfully treats cross-Toeplitz operators and also gives some new perspective on the traditional Toeplitz operators.

2 The Heisenberg group essentials

This section contains a minimal presentation of the Heisenberg group required for this paper. We refer to other works [2, 34, 48, 69] for further details.

An element of the n -dimensional *Heisenberg group* \mathbb{H}^n is $(s, x, y) \in \mathbb{R}^{2n+1}$, where $s \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$. The group law on \mathbb{H}^n is given as follows:

$$(s, x, y) \cdot (s', x', y') = \left(s + s' + \frac{1}{2}\omega(x, y; x', y'), x + x', y + y' \right), \quad (2.1)$$

where ω is the symplectic form:

$$\omega(x, y; x', y') = xy' - x'y. \quad (2.2)$$

It is often convenient to identify $(x, y) \in \mathbb{R}^{2n}$ with $z = x + iy \in \mathbb{C}^n$ with the group law presented by:

$$(s, z) \cdot (s', z') = \left(s + s' + \frac{1}{2}\Im(z\bar{z}'), z + z' \right), \quad (2.3)$$

with the symplectic form $\omega(z; z') = \Im(z\bar{z}')$ in the complex coordinates $z = x + iy$ and $z' = x' + iy' \in \mathbb{C}^n$. However, some additional considerations are required for this, see Sect. 4.2.

The two-sided Haar (the left and the right invariant) measure on the Heisenberg group coincides with the Lebesgue measure on \mathbb{R}^{2n+1} .

2.1 Induced representations

The bare minimum of the induced representation construction for the Heisenberg group is as follows, see numerous sources for detailed treatments of the induced representation technique [35, 47, 48].

For a real $\hbar \neq 0$ we denote by χ_{\hbar} the non-trivial unitary character of the group $(\mathbb{R}, +)$:

$$\chi_{\hbar}(s) = e^{2\pi i \hbar s}. \quad (2.4)$$

Consider functions on \mathbb{H}^n having the following covariant property with the respect to the central action on \mathbb{H}^n :

$$F(s' + s, x, y) = \overline{\chi}_{\hbar}(s) F(s', x, y) = e^{-2\pi i \hbar s} F(s', x, y) \quad \text{for all } s \in \mathbb{R}. \quad (2.5)$$

This characteristic is preserved under the left $\Lambda(g)$ and right shifts $R(g)$:

$$\Lambda(g) : F(g') \mapsto F(g^{-1}g'), \quad R(g) : F(g') \mapsto F(g'g), \quad g, g' \in \mathbb{H}^n. \quad (2.6)$$

Clearly, a function $F(s, x, y)$ satisfying (2.5) is completely defined by its values $F(0, x, y)$ on $\mathbb{R}^{2n} \subset \mathbb{H}^n$. Thus, for functions F and f on \mathbb{H}^n and \mathbb{R}^{2n} respectively we define the *lifting* L_{\hbar} and *pulling* P as follows:

$$[L_{\hbar}f](s, x, y) = e^{-2\pi i \hbar s} f(x, y), \quad [PF](x, y) = F(0, x, y). \quad (2.7)$$

Obviously, the pulling is a left inverse of the lifting: $[(P \circ L_{\hbar})f](x, y) = f(x, y)$. Also $[(L_{\hbar} \circ P)F](s, x, y) = F(s, x, y)$ if F satisfies (2.5). Thus, we can pull the regular actions (2.6) on \mathbb{R}^{2n} :

$$\Lambda_{\hbar}(g) := P \circ \Lambda(g) \circ L_{\hbar}, \quad R_{\hbar}(g) := P \circ R(g) \circ L_{\hbar}, \quad (2.8)$$

or, explicitly:

$$[\Lambda_{\hbar}(s, x, y)f](x', y') = e^{\pi i \hbar (2s + xy' - yx')} f(x' - x, y' - y), \quad (2.9)$$

$$[R_{\hbar}(s, x, y)f](x', y') = e^{\pi i \hbar (-2s + xy' - yx')} f(x' + x, y' + y). \quad (2.10)$$

These are infinite-dimensional representations of \mathbb{H}^n which are induced in the sense of Mackey [48, App. V.2] from the character $\chi_{\hbar}(s, 0, 0) = e^{2\pi i \hbar s}$ (2.4) of the centre of \mathbb{H}^n . To distinguish Λ_{\hbar} and R_{\hbar} (2.9)–(2.10) from Λ and R (2.6) we call formers as *pulled actions*.

Remark 2.1 (General scheme of induced representation) For a later use in Sect. 5.3 we need a bit more general setup of induced representations ([35, Chap. 6]; [47, Sect. 13.2]; [48, Sect. V.2]; [72]; [82, Chap. 5]). Let G be a locally compact group and $*$ denote its multiplication, H be its closed subgroup and $X = G/H$ —the respective homogeneous space. Let $s : X \rightarrow G$ be a section of the bundle defined by the natural projection $p : G \rightarrow X$. Then, any $g \in G$ has a unique decomposition of the form $g = s(x) * h$ where $x = p(g) \in X$ and $h \in H$. We also define the map $r : G \rightarrow H$:

$$r(g) = s(x)^{-1} * g, \quad \text{where } x = p(g). \quad (2.11)$$

This map is our substitution ([56, Sect. 3.1], [62, 65, 66]) for the so-called master equation [48, Sect. V.2]. Then, the left action of $G : X \rightarrow X$ is given by

$$g : x \mapsto g \cdot x = \mathbf{p}(g * \mathbf{s}(x)), \quad \text{where } g \in G, \ x \in X. \quad (2.12)$$

A character χ of H induces a representation ρ_χ of G in a space of function on X by the formula:

$$\rho_\chi(g) : f(x) \rightarrow \overline{\chi}(\mathbf{r}(g^{-1} * \mathbf{s}(x))) f(g^{-1} \cdot x), \quad (2.13)$$

The complex conjugation of the character χ is used here to make our formula in line the general induced representation construction. If X posses a G -invariant measure $\mathrm{d}x$ then this representation acts by isometries of $\mathcal{L}_p(X, \mathrm{d}x)$. There is a suitable adaptation for quasi-invariant measures as well.

The representations (2.9)–(2.10) of \mathbb{H}^n are reducible. An irreducible representation of \mathbb{H}^n can be induced from the character $\chi_{\hbar}(s, 0, y) = e^{2\pi i \hbar s}$ of the continuous two-dimensional maximal commutative subgroup

$$H_x = \{(s, 0, y) \in \mathbb{H}^n : s \in \mathbb{R}, y \in \mathbb{R}^n\}$$

Using the above steps with the respective lifting and pulling (or maps $\mathbf{p}, \mathbf{s}, \mathbf{r}$ from Rem. 2.1), the *Schrödinger representation* ρ_{\hbar} of \mathbb{H}^n on $\mathcal{L}_2(\mathbb{R}^n)$ can be written as:

$$[\rho_{\hbar}(s, x, y)f](t) = e^{\pi i \hbar (2s - 2yt + xy)} f(t - x). \quad (2.14)$$

By the Stone–von Neumann theorem, any irreducible infinite-dimensional representation of \mathbb{H}^n —in particular, an irreducible subrepresentation of (2.9)–(2.10)—is equivalent to (2.14) with the same value of \hbar . The intertwining operator between representations (2.14) and (2.9) will be constructed as coherent states transform in Ex. 4.2.

In the context of representations of Lie groups the corresponding derived representations of their Lie algebras are important. Take a basis:

$$S = (1, 0, 0), \quad X = (0, 1, 0), \quad Y = (0, 0, 1) \quad (2.15)$$

of the Lie algebra \mathfrak{h}_1 of \mathbb{H}^1 . For the representations (2.9), (2.10) and (2.14) the derived representation is expressed through the following differential operators:

$$\mathrm{d}\Lambda_{\hbar}^S = 2\pi i \hbar I, \quad \mathrm{d}\Lambda_{\hbar}^X = \pi i \hbar y - \partial_x, \quad \mathrm{d}\Lambda_{\hbar}^Y = -\pi i \hbar x - \partial_y; \quad (2.16)$$

$$\mathrm{d}R_{\hbar}^S = -2\pi i \hbar I, \quad \mathrm{d}R_{\hbar}^X = \pi i \hbar y + \partial_x, \quad \mathrm{d}R_{\hbar}^Y = -\pi i \hbar x + \partial_y; \quad (2.17)$$

$$\mathrm{d}\rho_{\hbar}^S = 2\pi i \hbar I, \quad \mathrm{d}\rho_{\hbar}^X = -\partial_t, \quad \mathrm{d}\rho_{\hbar}^Y = -2\pi i \hbar t. \quad (2.18)$$

Each of these derived representations represents the canonical (aka Heisenberg) *commutator relation*

$$[X, Y] = S.$$

It is the source of the Heisenberg–Kennard uncertainty relation (1.2), see [34, Sect. 1.3] for further details and [68] for some recent developments.

2.2 Pulled actions and the phase space

There are several important links between the left and right actions:

1. In general, the left and right regular representations for any group G are equivalent and are intertwined by the reflection $g \mapsto g^{-1}$, $g \in G$. For the Heisenberg group this implies that the reflection of the function *domain* $R : f(x, y) \mapsto f(-x, -y)$ intertwines the left and the right pulled actions with *opposite* Planck constants:

$$\Lambda_{\hbar}(s, x, y) \circ R = R \circ R_{-\hbar}(s, x, y). \quad (2.19)$$

2. The complex conjugation (that is a reflection in the function *range*) intertwines the left and right actions with near-inverse elements:

$$\Lambda_{\hbar}(s, -x, -y)\overline{f} = \overline{R_{\hbar}(s, x, y)f}. \quad (2.20)$$

3. Finally, a specific feature of nilpotent step two Lie groups and \mathbb{H}^n particularly implies:

$$\Lambda_{\hbar}(-s, -x, -y)f(x', y') = R_{\hbar}(s, x', y')f(x, y). \quad (2.21)$$

Note the *swap* of primed and unprimed variables in the last expression in comparison to (2.19). This will be used later for representing the FSB projection as a right integrated representation in (6.4). Also combination of (2.20) and (2.21) yields:

$$\overline{\Lambda_{\hbar}(s, x, y)f(x', y')} = \Lambda_{\hbar}(-s, x', y')\overline{Rf(x, y)}. \quad (2.22)$$

The pulled actions Λ_{\hbar} and R_{\hbar} are also connected to main operators of the classical abelian harmonic analysis—the Euclidean shift S and multiplication E_{\hbar} by a unimodular plane wave:

$$S(x, y) : f(x', y') \mapsto f(x' - x, y' - y), \quad (2.23)$$

$$E_{\hbar}(x, y) : f(x', y') \mapsto e^{\pi i \hbar (xy' - yx')} f(x', y'). \quad (2.24)$$

Both S and E_{\hbar} are reducible representations of the Abelian group $(\mathbb{R}^{2n}, +)$ by invertible isometries in $\mathcal{L}_p(\mathbb{R}^{2n})$. More accurately, E_{\hbar} is a representation of the *dual group* \hat{G} of $G = (\mathbb{R}^{2n}, +)$ by (symplectic) *phase shifts*. Of course, \hat{G} is isomorphic to G but

there is no a canonical isomorphism between two groups and the Planck constant \hbar labels different identifications. Operators S and E_{\hbar} have the maximal localisation in the configuration and frequency spaces respectively, cf. Sect. 1.2.

The direct product of groups $G \times \hat{G}$ is called *phase space*. For $n = 1$ this space is also known as *time–frequency space* with E_{\hbar} called *frequency shift* or *modulation* [38]. In this context the Heisenberg group is the central extension of the phase space which contains different automorphisms of the phase space interchanging G and \hat{G} :

$$(x, y) \mapsto (\hbar y, -\hbar x), \quad (x, y) \in G,$$

parametrised by the non-zero Planck constants.

Since spatial and phase shifts do not commute with each other they are able to represent the non-commutative pulled actions:

$$\Lambda_{\hbar}(s, x, y) = \chi_{\hbar}(s) \cdot E_{\hbar}(x, y) \circ S(x, y), \quad (2.25)$$

$$R_{\hbar}(s, x, y) = \overline{\chi}_{\hbar}(s) \cdot E_{\hbar}(x, y) \circ S(-x, -y). \quad (2.26)$$

Expressions in the opposite direction are:

$$S(x, y) = \Lambda_{\hbar}\left(s, \frac{1}{2}x, \frac{1}{2}y\right) \circ R_{\hbar}\left(s, -\frac{1}{2}x, -\frac{1}{2}y\right) \quad (2.27)$$

$$= \exp\left(\frac{1}{2}x(d\Lambda^X - dR^X) + \frac{1}{2}y(d\Lambda^Y - dR^Y)\right), \quad (2.28)$$

$$E_{\hbar}(x, y) = \Lambda_{\hbar}\left(s, \frac{1}{2}x, \frac{1}{2}y\right) \circ R_{\hbar}\left(s, \frac{1}{2}x, \frac{1}{2}y\right) \quad (2.29)$$

$$= \exp\left(\frac{1}{2}x(d\Lambda^X + dR^X) + \frac{1}{2}y(d\Lambda^Y + dR^Y)\right) \quad (2.30)$$

for an arbitrary $s \in \mathbb{R}$.

Of course, \mathcal{L}_p -norms, $1 \leq p \leq \infty$, are invariant under spatial and phase shift. There are other invariant norms, which are yet less studied, for example [45, 74]:

$$\|f\|_J = \sup_{(x, y) \in \mathbb{R}^{2n}} \left(\int_Q |f(x + x', y + y')|^p dx' dy' \right)^{\frac{1}{p}}, \quad (2.31)$$

where Q is the unit cube (or, equivalently, any compact set) in \mathbb{R}^{2n} . The expressions (2.25)–(2.26) of pulled actions through the spatial and phase shifts imply:

Lemma 2.2 *Let V be a space of functions on \mathbb{R}^{2n} with a norm which is invariant under the spatial and phase shifts. Then, the pulled actions (2.9)–(2.10) are invertible isometries V .*

Remark 2.3 Since in both cases (2.9)–(2.10) the centre’s action reduces to multiplication by a constant it is tempting to use only the “essential” actions $\Lambda_{\hbar}(0, x, y)$ and $R_{\hbar}(0, x, y)$. However, this removes the advantages of the group structure, e.g. a composition $\Lambda_{\hbar}(0, x, y)\Lambda_{\hbar}(0, x', y')$ cannot be presented as $\Lambda_{\hbar}(0, x'', y'')$ without an additional factor.

Remark 2.4 (Spaces of analytical functions) The raw representation (2.9) of the Heisenberg group are rarely used in harmonic analysis. Researchers often prefer to deal with some spaces of analytic functions. This is easily achieved by intertwining Λ_{\hbar} with an operator of multiplication:

$$E_d : f(x, y) \mapsto e^{d(x, y)} f(x, y) \quad (2.32)$$

by the exponent of a function $d(x, y) : \mathbb{R}^{2n} \rightarrow \mathbb{C}$. In an obvious way, E_d is unitary as a map:

$$E_d : \mathcal{L}_2(\mathbb{R}^{2n}) \rightarrow \mathcal{L}_2(\mathbb{R}^{2n}, e^{-2\Re d(x, y)} dx dy)$$

for the appropriate weighted measures. We call E_d *peeling* [66] and will discuss the choice of $d(x, y)$ in Example 4.3.

3 Covariant and contravariant transforms

3.1 The Fourier–Wigner and the covariant transforms

Let ρ be a strongly continuous unitary representation of a group G in a Hilbert space \mathcal{H} , a pair of vectors $u, v \in \mathcal{H}$ defines the *matrix coefficients* of ρ :

$$W(u, v)(g) = \langle u, \rho(g)v \rangle, \quad g \in G, \quad (3.1)$$

which is a continuous bounded function on G . Since the construction occurs in many seemingly disconnected contexts there is an extensive list of its names: wavelet transform [4], voice transform [32], coherent state transform [77], covariant transform [9], etc. To add a diversity we recall that all sorts of special functions are matrix coefficients for various group representations [15, 88, 89]. Furthermore, many classical function spaces can be defined as coorbits [31] with natural norms transported through covariant transforms to new settings [66, 67].

For the case of $G = \mathbb{H}^n$, the elementary action of the centre Z can be omitted. The corresponding matrix coefficients for the Schrödinger representation (2.14) are:

$$\begin{aligned} W(f, \phi)(x, y) &= \langle f, \rho_{\hbar}(0, x, y)\phi \rangle \\ &= \int_{\mathbb{R}^n} e^{2\pi i \hbar y(t - \frac{1}{2}x)} f(t) \overline{\phi}(t - x) dt \\ &= \langle \rho_{\hbar}(0, -\frac{1}{2}x, -\frac{1}{2}y) f, \rho_{\hbar}(0, \frac{1}{2}x, \frac{1}{2}y) \phi \rangle \\ &= \int_{\mathbb{R}^n} e^{2\pi i \hbar y t'} f(t' + \frac{1}{2}x) \overline{\phi}(t' - \frac{1}{2}x) dt'. \end{aligned} \quad (3.2)$$

This was called the *Fourier–Wigner transform* in [34, Sect. I.4], since the Fourier transform $(x, y) \mapsto (q, p)$ of $W(\phi, \phi)(x, y)$ produces the *Wigner transform* of a quantum state ϕ , see also Prop. 5.9.

The Fourier–Wigner transform is initially defined on $\mathcal{L}_2(\mathbb{R}^n) \times \mathcal{L}_2(\mathbb{R}^n)$ and can be naturally extended by linearity to a unitary map on $\mathcal{L}_2(\mathbb{R}^n) \otimes \mathcal{L}_2(\mathbb{R}^n) \sim \mathcal{L}_2(\mathbb{R}^{2n})$. In particular, W possesses the crucial property of “sesqui-unitary” [34, (1.42)]:

$$\langle W(f_1, \phi_1), W(f_2, \phi_2) \rangle_{\mathbb{R}^{2n}} = \langle f_1 f_2 \rangle_{\mathbb{R}^n} \overline{\langle \phi_1, \phi_2 \rangle_{\mathbb{R}^n}}. \quad (3.3)$$

This identity is also known as the *orthogonality relation* and valid for a more general coherent states transform [4, Sect. 8.2]. Also, the identity (3.3) does not contain anything specific for the Schrödinger representation (2.14) on $\mathcal{L}_2(\mathbb{R}^n)$ and holds for a unitary irreducible representation of \mathbb{H}^n on a Hilbert space \mathcal{H} . For a fixed non-zero $\phi \in \mathcal{L}_2(\mathbb{R}^n)$ the linear map:

$$W_\phi : \mathcal{L}_2(\mathbb{R}^n) \rightarrow \mathcal{L}_2(\mathbb{R}^{2n}) : f \mapsto W(f, \phi) \quad (3.4)$$

is a particular case of the general construction of *covariant transform*. Some other names (e.g. coherent states transform, wave packet expansion, voice transform, etc.) are also used in the literature. In physical language the overcomplete system of vectors $\phi_{(x,y)} = \rho_{\hbar}(x, y)\phi$ are known as *Schrödinger coherent states* generated by the *ground* (or *vacuum*, or *fiducial*, etc.) state ϕ .

Let ρ_{\hbar} be an irreducible unitary representation of \mathbb{H}^n in a Hilbert space \mathcal{H} (for example, the Schrödinger representation (2.14) on $\mathcal{L}_2(\mathbb{R}^n)$) and $\theta \in \mathcal{H}$ be a unit vector. The main properties of the covariant transform (3.4) follow from (3.3) and can be summarised as follows ([4, Thm. 7.3.1], [77, Sect. 1.2]).

1. The map W_θ (3.4) is a unitary operator into the space

$$\mathcal{F}^\theta := \{W_\theta(f) : f \in \mathcal{H}\}, \quad (3.5)$$

which is a closed subspace of $\mathcal{L}_2(\mathbb{R}^{2n})$.

2. \mathcal{F}^θ is invariant and irreducible under the pulled action Λ_{\hbar} (2.9) of \mathbb{H}^n .
3. The map W_θ intertwines ρ_{\hbar} and Λ_{\hbar} (2.9) restricted to \mathcal{F}^θ :

$$W_\theta \circ \rho_{\hbar} = \Lambda_{\hbar} \circ W_\theta. \quad (3.6)$$

4. The orthogonal *FSB projection* $P_\theta : \mathcal{L}_2(\mathbb{R}^{2n}) \rightarrow \mathcal{F}^\theta$ is an integral operator:

$$\begin{aligned} [P_\theta f](x, y) &= \langle f, \Lambda_{\hbar}(0, x, y)\Theta \rangle \\ &= \int_{\mathbb{R}^{2n}} f(x', y') e^{-\pi i \hbar(xy' - yx')} \overline{\Theta(x' - x, y' - y)} dx' dy', \end{aligned} \quad (3.7)$$

where $\Theta = W_\theta(\theta) \in \mathcal{F}^\theta$ and $f \in \mathcal{L}_2(\mathbb{R}^{2n})$. An immediate consequence is the following commutativity:

$$P_\theta \circ \Lambda_{\hbar}(0, x, y) = \Lambda_{\hbar}(0, x, y) \circ P_\theta. \quad (3.8)$$

5. The vector Θ is *cyclic* in \mathcal{F}^θ in the sense that \mathcal{F}^θ is the closed linear span of all $\Lambda_{\hbar}(0, x, y)\Theta$, for $(x, y) \in \mathbb{R}^{2n}$.
6. In particular, \mathcal{F}^θ is a *reproducing kernel* Hilbert space with the reproducing kernel

$$\begin{aligned} K_{(x,y)}(x', y') &= \overline{\Lambda_{\hbar}(0, x, y)\Theta(x', y')} = \langle \rho_{\hbar}(x', y')\theta, \rho_{\hbar}(x, y)\theta \rangle \\ &= \Lambda_{\hbar}(0, x', y')\Theta(x, y). \end{aligned} \quad (3.9)$$

For reasons elaborated in Example 4.2, cf. also [34, Sect. 4.5], the important case is the ground state θ defined by the *squeezed Gaussian* $\varphi_\tau(t) = \tau^{-n/4} e^{-\pi \hbar t^2 / \tau}$ for some constant $\tau > 0$. Then:

$$\Phi_\tau(x, y) := W_{\phi_\tau}(\phi_\tau) = e^{-\pi \hbar / (2\tau)(x^2 + \tau^2 y^2)}, \quad (3.10)$$

$$K_{(x,y)}(x', y') = e^{-\pi i \hbar (x y' - y x')} e^{-\pi \hbar / (2\tau)((x' - x)^2 + \tau^2 (y' - y)^2)}. \quad (3.11)$$

7. The contravariant transform $M_\psi^\theta : \mathcal{F}^\theta \rightarrow \mathcal{H}$ is provided the reconstruction formula:

$$M_\psi^\theta : f \mapsto \int_{\mathbb{R}^{2n}} f(x, y) \rho(x, y) \psi \, dx \, dy \quad \text{where } f \in \mathcal{F}^\theta, \quad (3.12)$$

see Defn. 3.5 below for details. By the orthogonality relation (3.3) the identity on \mathcal{H} :

$$M_\psi \circ W_\theta = \langle \psi, \theta \rangle I. \quad (3.13)$$

Remark 3.1 Since the Planck constant \hbar is fixed within this paper (and, possibly, in the physical reality as well) we are not indicating it in our notation, e.g. Φ_τ . On the other hand, we will consider several different values of the squeeze parameter τ simultaneously, in most cases we denote two such values as τ and ς . To make our notation simpler the dependence on ϕ_τ will be indicated by τ alone, e.g. $\mathcal{F}^\tau := \mathcal{F}^{\phi_\tau}$, $M_\tau := M_{\phi_\tau}$, etc.

3.2 Irreducible components for the pulled action

Because the representation Λ_{\hbar} (2.9) is not too far from the left regular representation it is highly reducible on $\mathcal{L}_2(\mathbb{R}^{2n})$. For example, this follows from the observation that Λ_{\hbar} commutes with $R_{\hbar}(0, x, y)$ (2.10), for all $(x, y) \in \mathbb{R}^{2n}$ and the latter operators are not scalar multiplies of the identity on $\mathcal{L}_2(\mathbb{R}^{2n})$.

The intertwining property $W_\theta \circ \rho_{\hbar} = \Lambda_{\hbar} \circ W_\theta$ (3.6) implies that \mathcal{F}^θ (3.5) is invariant. Furthermore, the irreducibility of the Schrödinger representation implies

irreducibility of the restriction of Λ_{\hbar} to \mathcal{F}^{θ} . Together with the standard properties of unitary representations we conclude.

Lemma 3.2 *For any $\theta \in \mathcal{L}_2(\mathbb{R}^n)$, the both spaces \mathcal{F}^{θ} and its orthogonal complement $\mathcal{F}_{\theta}^{\perp}$ are Λ_{\hbar} invariant subspaces of $\mathcal{L}_2(\mathbb{R}^{2n})$. The space \mathcal{F}^{θ} is irreducible and its complement $\mathcal{F}_{\theta}^{\perp}$ is not.*

In the obvious way, two orbits of a group actions either are disjoint or coincide. We can obtain a similar conclusion for two Λ_{\hbar} -invariant subspaces.

Lemma 3.3 *Two irreducible components \mathcal{F}^{θ} and \mathcal{F}^{ψ} have non-trivial intersection if and only if θ and ψ are linearly dependent. In the latter case $\mathcal{F}^{\theta} = \mathcal{F}^{\psi}$.*

Proof Sufficiency is trivial. For necessity, without loss of generality we assume that $\|\theta\| = \|\psi\| = 1$ and a non-zero common element $F \in \mathcal{F}^{\theta} \cap \mathcal{F}^{\psi}$ has unit norm. Then, by the unitarity of the covariant transform $F = W_{\theta} f_1$ and $F = W_{\psi} f_2$ for some $f_{1,2} \in \mathcal{L}_2(\mathbb{R}^n)$ with unit norm. Then from sesqui-unitarity (3.3) and the Cauchy–Schwartz inequality:

$$1 = \langle W_{\theta} f_1, W_{\psi} f_2 \rangle_{\mathbb{R}^{2n}} = \langle f_1, f_2 \rangle_{\mathbb{R}^n} \overline{\langle \theta, \psi \rangle_{\mathbb{R}^n}} \leq \|f_1\| \cdot \|f_2\| \cdot \|\theta\| \cdot \|\psi\| = 1.$$

The identity is achieved only if $\theta = \lambda\psi$ and $f_1 = \lambda f_2$ for an unimodular $\lambda \in \mathbb{C}$. \square

A systematic usage of representation theory requires the following concept of the induced covariant transform [61, 66], which is based on Perelomov's coherent states [77].

Definition 3.4 Let ρ be a unitary representation of a group G in a space \mathcal{H} . Let there exist a common eigenvector $\phi \in \mathcal{H}$ for all operators $\rho(h)$ for all $h \in H$ for some subgroup $H \subset G$. For a fixed section $s : G/H \rightarrow G$ the induced covariant transform $W_{\phi}^{\rho} : \mathcal{H} \rightarrow \mathcal{L}(G/H)$ is

$$W_{\phi}^{\rho} : f \mapsto [W_{\phi}^{\rho} f](x) = \langle f, \rho(s(x))\phi \rangle, \quad \text{where } f \in \mathcal{H} \text{ and } x \in G/H. \quad (3.14)$$

Here $\mathcal{L}(G/H)$ is a certain linear space of functions on G/H depending on properties of ϕ . For example, for admissible ϕ it is $\mathcal{L}_2(G/H)$, however other cases are of interest as well [58, 66, 67].

One can check that (3.4) is a special case of (3.14) for the Heisenberg group $G = \mathbb{H}^n$, its centre as the subgroup H and the Schrödinger representation. Then $\phi \in \mathcal{L}_2(\mathbb{R}^n)$ is a common eigenvector for all operators $\rho(s, 0, 0)$, $s \in \mathbb{R}$.

The adjoint notion to covariant transform is:

Definition 3.5 Let ρ be a unitary representation of a group G in a Hilbert space \mathcal{H} . For some subgroup $H \subset G$ and a fixed Borel section $s : G/H \rightarrow G$ the contravariant transform $M_{\psi}^{\rho} : \mathcal{L}(G/H) \rightarrow \mathcal{H}$ is

$$M_{\psi}^{\rho} : f(x) \mapsto M_{\psi}^{\rho}(f) = \int_{G/H} f(x) \rho(s(x))\psi \, dx. \quad (3.15)$$

Here $\psi \in \mathcal{H}$ is known as *reconstruction vector* and integration is performed over an invariant measure on G/H .

It is easy to check the relation between covariant and contravariant transforms:

$$(W_\phi^\rho)^* = M_\phi^\rho. \quad (3.16)$$

Remark 3.6 To unload our notations we will continue to denote the co-/contravariant transforms generated by the Schrödinger representation by W_ψ and M_ψ respectively. On the contrast, co-/contravariant transforms for the left pulled action will be denoted as W_ψ^Λ and M_ψ^Λ (omitting a reference to the fixed Planck constant \hbar).

Besides the generic relation $(W_\phi^\Lambda)^* = M_\phi^\Lambda$ (3.16) there is a specific connection for the pulled action of the Heisenberg group:

$$W_\Psi^\Lambda = M_{R\Psi}^\Lambda, \quad (3.17)$$

where $[Rf](x, y) = f(-x, -y)$ as before. Indeed, using identities (2.20)–(2.21) between the left and right pulled actions we find:

$$\begin{aligned} [W_\Psi^\Lambda \Theta](x, y) &= \langle \Theta, \Lambda_{\hbar}(0, x, y) \Psi \rangle \\ &= \int_{\mathbb{R}^{2n}} \Theta(x', y') \overline{\Lambda_{\hbar}(0, x, y) \Psi(x', y')} \, dx' \, dy' \\ &= \int_{\mathbb{R}^{2n}} \Theta(x', y') R_{\hbar}(0, -x, -y) \overline{\Psi}(x', y') \, dx' \, dy' \\ &= \int_{\mathbb{R}^{2n}} \Theta(x', y') \Lambda_{\hbar}(0, x', y') [R\overline{\Psi}](x, y) \, dx' \, dy' = [M_{R\Psi}^\Lambda \Theta](x, y). \end{aligned}$$

Corollary 3.7 For $f, \Psi \in \mathcal{L}_2(\mathbb{R}^{2n})$ we have $W_\Psi^\Lambda f \in \mathcal{F}^{R\overline{\Psi}}$.

In particular, since $R\overline{\Phi} = \Phi$, we recover already seen result: $W_{\Phi_\tau}^\Lambda f \in \mathcal{F}^\tau$ for all $f \in \mathcal{L}_2(\mathbb{R}^{2n})$.

Corollary 3.8 For $f \in \mathcal{F}^\theta$, and $\Psi \in \mathcal{L}_2(\mathbb{R}^{2n})$ we have

$$W_\Psi^\Lambda f = W_{\Psi_1}^\Lambda f, \quad \text{where } \Psi_1 = P_\theta \Psi. \quad (3.18)$$

Proof Let $\Psi = \Psi_1 + \Psi_0$ for $\Psi_1 \in \mathcal{F}^\tau$ and $\Psi_0 \in \mathcal{F}_\phi^\perp$. Then from the Λ_{\hbar} -invariance of \mathcal{F}_ϕ^\perp follows that $\langle f, \Lambda_{\hbar}(0, x, y) \Psi_0 \rangle = 0$ for all $(x, y) \in \mathbb{R}^{2n}$. Thus:

$$[W_\Psi^\Lambda f](x, y) = \langle f, \Lambda_{\hbar}(0, x, y) \Psi \rangle = \langle f, \Lambda_{\hbar}(0, x, y) \Psi_1 \rangle = [W_{\Psi_1}^\Lambda f](x, y).$$

□

A Λ_{\hbar} -covariant unitary operator $U_{\theta\psi} : \mathcal{F}^{\theta} \rightarrow \mathcal{F}^{\psi}$ can be constructed as the composition $U = W_{\psi} \circ M_{\theta}$ of contra- and covariant transforms:

$$\begin{aligned} [U_{\theta\psi} f](x, y) &= [W_{\psi} \circ M_{\theta} f](x, y) \\ &= \langle M_{\theta} f, \rho_{\hbar}(0, x, y) \psi \rangle \\ &= \left\langle \int_{\mathbb{R}^{2n}} f(x', y') \rho_{\hbar}(0, x', y') \theta \, dx' \, dy', \rho_{\hbar}(0, x, y) \psi \right\rangle \\ &= \int_{\mathbb{R}^{2n}} f(x', y') \langle \rho_{\hbar}(0, x', y') \theta, \rho_{\hbar}(0, x, y) \psi \rangle \, dx' \, dy', \quad (3.19) \end{aligned}$$

assuming we can change the order of integration, e.g. due to the Fubini theorem. The last formula includes the reproducing kernel (3.9) as the special case for $\psi = \theta$. For $\psi \in \mathcal{F}^{\theta}$, operator $U_{\theta\psi}$ is a multiple of identity by the Schur lemma and irreducibility of the pulled action Λ_{\hbar} on \mathcal{F}^{θ} .

Example 3.9 We may consider the map $\mathcal{F}^{\tau} \rightarrow \mathcal{F}^{\varsigma}$ as a composition $W_{\varsigma} \circ M_{\tau}$ of the contravariant and covariant transform for the respective reconstructing ϕ_{τ} and analysing ϕ_{ς} vectors. It turns out to be an integral operator:

$$\begin{aligned} f_{\varsigma}(x, y) &= \sqrt[4]{\frac{\tau\varsigma}{(\tau+\varsigma)^2}} \int_{\mathbb{R}^{2n}} f_{\tau}(x_1, y_1) \\ &\quad e^{-\frac{\pi\hbar}{\tau+\varsigma}((x-x_1)^2 + \tau\varsigma(y-y_1)^2 + 2i(\varsigma xy_1 - \tau x_1 y) + i(xy + x_1 y_1)(\varsigma - \tau))} \, dx_1 \, dy_1 \\ &= \int_{\mathbb{R}^{2n}} f_{\tau}(x_1, y_1) \Lambda_{\hbar}(x_1, y_1) \Phi_{\tau\varsigma}(x, y) \, dx_1 \, dy_1 = [M_{\Phi_{\tau\varsigma}}^{\Lambda_{\hbar}} f_{\tau}](x, y) \\ &= \int_{\mathbb{R}^{2n}} f_{\tau}(x_1, y_1) \overline{\Lambda_{\hbar}(x, y) \Phi_{\tau\varsigma}(x_1, y_1)} \, dx_1 \, dy_1 = [W_{\Phi_{\tau\varsigma}}^{\Lambda_{\hbar}} f_{\tau}](x, y). \quad (3.20) \end{aligned}$$

where $f_{\tau} \in \mathcal{F}^{\tau}$, $f_{\varsigma} \in \mathcal{F}^{\varsigma}$ and

$$\Phi_{\tau\varsigma}(x, y) = \langle \phi_{\varsigma}, \rho(g) \phi_{\tau} \rangle = \sqrt[4]{\frac{\tau\varsigma}{(\tau+\varsigma)^2}} e^{-\pi\hbar/(\tau+\varsigma)(ixy(\tau-\varsigma) + x^2 + \tau\varsigma y^2)}. \quad (3.21)$$

Obviously, for $\varsigma = \tau$ the identity (3.20) coincides with the reproducing formula (3.7) on \mathcal{F}^{τ} . We will call

$$K_{\tau\varsigma}(x_1, y_1; x, y) = \overline{\Lambda_{\hbar}(x_1, y_1) \Phi_{\tau\varsigma}(x, y)} \quad (3.22)$$

the *intertwining kernel*. Also, the function $\Phi_{\tau\varsigma}$ (3.21) will be named *mixed Gaussian*. We present its main properties in Lem. 4.11 once some additional notions will be introduced.

3.3 Twisted convolution and symplectic Fourier transform

Many operators in analysis (e.g. (3.12) and many further examples in [57, 66]) appear as *integrated representations* or *relative convolution*

$$\rho(k) = \int_X k(x) \rho(s(x)) dx \quad (3.23)$$

for a suitable representation ρ of a group G , its homogeneous space $X = G/H$, a Borel section $s : X \rightarrow G$ and a function (distribution) k defined on X [57, 66]. As usual, the above integral is understood in the weak sense. In particular, (3.15) can be restated as $M_\psi(f) = \rho(f)\psi$.

For an irreducible representation ρ , a composition of two integrated representations is an integrated representation again

$$\rho(k_1) \circ \rho(k_2) = \rho(k_1 \natural k_2), \quad (3.24)$$

for a kernel $k_1 \natural k_2$.

Definition 3.10 The function (distribution) $k_1 \natural k_2$ defined by (3.24) is called the *twisted convolution* of k_1 and k_2 (produced by the representation ρ).

Note, that the twisted convolution (3.24) depends on the equivalence class of ρ but *not* on its specific realisation. In particular, for the left Λ_{\hbar} (2.9), right R_{\hbar} (2.10) and the Schrödinger representation ρ_{\hbar} (2.14) the twisted convolution on the homogeneous space \mathbb{H}^n/Z is given by [34, 43, Sect. 1.3]

$$\begin{aligned} (f_1 \natural f_2)(x, y) &= [\Lambda_{\hbar}(f_1)f_2](x, y) \\ &= \int_{\mathbb{R}^{2n}} f_1(x', y') f_2(x - x', y - y') e^{\pi \hbar i(x' y - y' x)} dx' dy'. \end{aligned} \quad (3.25)$$

More accurately, we shall account the dependence on the Planck constant and denote the twisted convolution by $f_1 \natural_{\hbar} f_2$. However, the Planck constant \hbar is systematically unloaded from notations throughout this paper. As can be expected for an important concept, (3.25) is known under many different names, e.g. *Moyal star product* [75, 90].

Using unitarity of the left regular representation and (2.20) (alternatively, make a change of variables) we can express the twisted convolution through the integrated right regular representation:

$$\begin{aligned} f_1 \natural f_2 &= \Lambda_{\hbar}(f_1)f_2 \\ &= R_{\hbar}(\check{f}_2)f_1, \quad \text{where } \check{f}_2(x, y) := [Rf_2](x, y) = f_2(-x, -y). \end{aligned} \quad (3.26)$$

Example 3.11 Note that the FSB projection $P_\tau : \mathcal{L}_2(\mathbb{R}^{2n}) \rightarrow \mathcal{F}^\tau$ (3.7) is the twisted convolution with the Gaussian:

$$P_\tau f = f \natural \Phi_\tau. \quad (3.27)$$

For reasons emerging soon the following transformation can be a suitable modification of the Fourier transform.

Definition 3.12 The *symplectic Fourier transform* (again depending on \hbar) ([34, Sect. 2.1]; [43, Sect. 2.1]) is defined by:

$$\widehat{\psi}(x, y) = (\hbar/2)^n \int_{\mathbb{R}^{2n}} \psi(x', y') e^{\pi i \hbar (x' y - y' x)} dx' dy'. \quad (3.28)$$

A systematic usage of the symplectic Fourier transform in the context of the FSB spaces can be seen in [18, 20].

Example 3.13 (Symplectic Fourier transform of the Gaussian) For the Gaussian $\Phi(x, y) = e^{-\pi \hbar / (2\tau)(x^2 + \tau^2 y^2)}$ (3.10) the standard calculation shows:

$$\widehat{\Phi}(x, y) = \Phi(x, y). \quad (3.29)$$

An alternative demonstration based on group representations is given in Example 4.2.

The symplectic Fourier transform is a reflection, that is of order two ($\widehat{\widehat{\psi}} = \psi$) in contrast to the usual Fourier transform, which is of order four: $(\widehat{\widehat{\psi}})\widehat{\widehat{t}} = \psi(-t)$. An “intriguing fact” [43, Sect. 2.1] is that the symplectic Fourier transform can be represented (up to a scaling) as integrated representations of $\Lambda_\hbar(0, x', y')$ and $R_\hbar(0, x', y')$ applied to the constant function $\mathbf{1}(x, y) \equiv 1$:

$$\begin{aligned} \widehat{a} &= a \natural \mathbf{1} = \Lambda_\hbar(a) \mathbf{1} = R_\hbar(a) \mathbf{1} \\ &= R_\hbar(\mathbf{1}) a = \Lambda_\hbar(\mathbf{1}) a, \end{aligned} \quad (3.30)$$

where the last line is based on (3.26). Some immediate consequences are:

1. Commutation of left regular representation and the symplectic Fourier transform:

$$\widehat{\circ} \circ \Lambda_\hbar(0, x, y) = \Lambda_\hbar(0, x, y) \circ \widehat{\circ}. \quad (3.31)$$

2. The intertwining property for the right regular representation:

$$\widehat{\circ} \circ R_\hbar(0, x, y) = R_\hbar(0, -x, -y) \circ \widehat{\circ}. \quad (3.32)$$

3. Combination of two previous intertwining properties with (2.27)–(2.29) implies the fundamental intertwining properties of the (symplectic) Fourier transform:

$$\widehat{\circ} \circ S(x, y) = E_\hbar(x, y) \circ \widehat{\circ}, \quad \widehat{\circ} \circ E_\hbar(x, y) = S(x, y) \circ \widehat{\circ}, \quad (3.33)$$

where are operators of the Euclidean shift S (2.23) and multiplication E_{\hbar} (2.24).
 4. The symplectic Fourier transform of a twisted convolution is:

$$(f_1 \natural f_2)^{\wedge} = (f_1 \natural f_2) \natural \mathbf{1} = f_1 \natural (f_2 \natural \mathbf{1}) = f_1 \natural \widehat{f_2}. \quad (3.34)$$

Remark 3.14 There is a dilemma discussed in [34, Prologue]: which Fourier transform to use—the ordinary or symplectic—in the context of \mathbb{H}^n ? In our opinion, these maps are clearly distinguished by their ranges. The symplectic transform (3.28) maps function on \mathbb{H}^n/Z to functions on the same set. The ordinary Fourier transform

$$\widehat{\psi}(q, p) = \int_{\mathbb{R}^{2n}} \psi(x, y) e^{-2\pi i(qx + py)} dx dy \quad (3.35)$$

sends a function ψ on \mathbb{H}^n/Z to a function $\widehat{\psi}$ on the coadjoint orbit in the dual space \mathfrak{h}_n^* of the Lie \mathfrak{h}_n of \mathbb{H}^n . Here coordinates (\hbar, q, p) on \mathfrak{h}_n^* have the physical meaning of the Planck constant, coordinate and momentum on the phase space, respectively. More accurately, (x, y) in (3.35) are coordinates on the Lie algebra \mathfrak{h}_n transferred to \mathbb{H}^n by means of the exponential map $\mathfrak{h}_n \rightarrow \mathbb{H}^n$. Then, (3.35) written in full as

$$\widehat{\psi}(\hbar, q, p) = \int_{\mathbb{H}^n} \psi(s, x, y) e^{-2\pi i(\hbar s + qx + py)} ds dx dy \quad (3.36)$$

is the basic example of the *Fourier–Kirillov transform* [48, Sect. 4.1.4].

Remark 3.15 (Physical units and their mathematical usage) The described difference between two transforms is also dictated by the physical dimensionality ([69, Sect. 1.2]; [59, Sect. 2.1]). Let M be a unit of mass, L —of length, T —of time. Then coordinate q is measured in L , momentum p in LM/T and \hbar in their product $L \times LM/T = L^2M/T$ —a unit of the action. Then dual variables s, x and y are measured in the respective reciprocal units $T/(L^2M)$, $1/L$ and $T/(LM)$ respectively. It is easy to see that formulae (3.28), (3.35), (3.36) (as well as all other formulae in this paper) follow the following rules:

1. Only physical quantities of the *same dimension* can be added or subtracted. However, there is no restrictions on multiplication/division.
2. Therefore, mathematical functions, e.g. $\exp(u) = 1 + u + u^2/2! + \dots$ or $\sin(u)$, can be naturally constructed out of a dimensionless number u only. For example, Fourier dual variables, say x and q , should possess reciprocal dimensions because they have to form the expression like $e^{2\pi i x q}$.

These rules have a natural physical origin and are mathematically valuable as well: validation of formulae is one example, the above discussion of ordinary and symplectic Fourier transforms—another.

In the case of the time-frequency analysis units are simpler. A signal is described by a function $f(t)$, where a variable the “coordinate” q is time in units T , “momentum”

p is frequency measured in $1/T$. The “Plank constant” is dimensionless. The dual variables s , x and y again have reciprocal units 1 , $1/T$ and T of \hbar , q and p .

The squeeze parameter τ shall have units of x/y , that is M/T in quantum mechanics and T^2 in time-frequency analysis.

Finally we establish the connection between the twisted convolution and the wavelet transform, cf. [34, (1.47)]:

Lemma 3.16 *Let ϕ_1 and ϕ_2 be admissible analysing vectors and f_2 be an admissible reconstructing vector. Then for any $f_1 \in \mathcal{H}$:*

$$W_{\phi_1}(f_1) \natural W_{\phi_2}(f_2) = \langle f_2, \phi_1 \rangle W_{\phi_2}(f_1). \quad (3.37)$$

Proof Note, that the reconstruction property (3.13) can be written through the integrated representation as:

$$\rho_{\hbar}(W_{\phi}f)\psi = \langle \psi, \phi \rangle f, \quad f, \phi, \psi \in \mathcal{L}_2(\mathbb{R}^n). \quad (3.38)$$

Then, under the lemma’s assumptions and for an admissible reconstructing vector $\psi \in \mathcal{L}_2(\mathbb{R}^n)$:

$$\begin{aligned} \rho_{\hbar}(W_{\phi_1}(f_1) \natural W_{\phi_2}(f_2))\psi &= \rho_{\hbar}(W_{\phi_1}(f_1))\rho_{\hbar}(W_{\phi_2}(f_2))\psi \\ &= \rho_{\hbar}(W_{\phi_1}(f_1))\langle \psi, \phi_2 \rangle f_2 \\ &= \langle \psi, \phi_2 \rangle \rho_{\hbar}(W_{\phi_1}(f_1))f_2 \\ &= \langle \psi, \phi_2 \rangle \langle f_2, \phi_1 \rangle f_1 \\ &= \langle f_2, \phi_1 \rangle \langle \psi, \phi_2 \rangle f_1 \\ &= \langle f_2, \phi_1 \rangle \rho_{\hbar}(W_{\phi_2}(f_1))\psi. \end{aligned}$$

Since ρ_{\hbar} is faithful and ψ is arbitrary we obtain (3.37). \square

4 Complex variables, analyticity and right shifts

The reasons which make the Gaussian a preferred vacuum vector are linked to the complex structure and analyticity.

4.1 Right shifts and analyticity

Here we use the following general result from [61, Sect. 5], see also further developments in [3, 5, 66–69].

Let G be a locally compact group and ρ be its representation in a Hilbert space \mathcal{H} . Let $[W_{\theta}v](g) = \langle v, \rho(g)\theta \rangle$ be the wavelet transform defined by a vacuum state $\theta \in \mathcal{H}$. Then, the right shift $R(g) : [W_{\theta}v](g') \mapsto [W_{\theta}v](g'g)$ for $g \in G$ coincides with the wavelet transform $[W_{\theta_g}v](g') = \langle v, \rho(g')\theta_g \rangle$ defined by the vacuum state

$\theta_g = \rho(g)\theta$. In other words, the covariant transform intertwines right shifts on the group G with the associated action ρ on vacuum states, cf. (3.6):

$$R(g) \circ W_\theta = W_{\rho(g)\theta}. \quad (4.1)$$

This elementary observation has many fundamental consequences.

Corollary 4.1 (Analyticity of the wavelet transform [61, Sect. 5]) *Let G be a group and dg be a measure on G . Let ρ be a unitary representation of G , which can be extended by integration to a vector space V of functions or distributions on G . Let a mother wavelet $\theta \in \mathcal{H}$ satisfy the equation*

$$\int_G a(g) \rho(g)\theta \, dg = 0,$$

for a fixed distribution $a(g) \in V$. Then any wavelet transform $\tilde{v}(g) = \langle v, \rho(g)\theta \rangle$ obeys the condition:

$$D\tilde{v} = 0, \quad \text{where } D = \int_G \bar{a}(g) R(g) \, dg, \quad (4.2)$$

with R being the right regular representation of G .

Some applications (including discrete ones) produced by the $ax + b$ group can be found in [67, Sect. 6], usage in quantum mechanics is demonstrated in [5, 6]. We turn to the particular case of the Heisenberg group now.

Example 4.2 (Gaussian and FSB transform) Let us consider the squeezed Gaussian $\varphi(t) = e^{-\pi \hbar t^2 / \tau}$. The parameter τ has the physical dimension of mass times frequency, cf. Rem. 3.15. In other words, the (reduced) Planck constant \hbar is in units reciprocal to the product xy and the parameter τ is in units reciprocal to the ratio y/x . The physical meaning of τ is squeeze parameter for coherent states, see [5, 6] and references therein. It is common to put $\tau = 1$ in mathematical texts.

The Gaussian is a null-solution of the operator. cf. (2.18).

$$d\rho_h^{-\tau X + iY} := -\tau d\rho_h^X + i d\rho_h^Y = \tau \partial_t + 2\pi \hbar t$$

For the centre $Z = \{(s, 0, 0) : s \in \mathbb{R}\} \subset \mathbb{H}$, we define the section $s : \mathbb{H}/Z \rightarrow \mathbb{H}$ by $s(x, y) = (0, x, y)$. Then, the corresponding induced wavelet transform (3.4) with the measure renormalised by the factor $(\hbar/\tau)^{n/2}$ is:

$$\begin{aligned} \tilde{f}(x, y) &= \langle f, \rho(s(x, y))\varphi \rangle \\ &= \left(\frac{\hbar}{\tau}\right)^{n/2} \int_{\mathbb{R}^n} f(t) e^{\pi i \hbar (2yt - xy)} e^{-\pi \hbar (t-x)^2 / \tau} \, dt. \end{aligned} \quad (4.3)$$

Note, that the normalising factor makes integration dimensionless in agreement with the Rem. 3.15 as well. The transformation intertwines the Schrödinger representation (2.14) and the left pulled action (2.9). Cor. 4.1 ensures that the operator, cf. (2.17)

$$\begin{aligned} dR^{\tau X+iY} &= \tau(\pi i \hbar y I + \partial_x) + i(-\pi i \hbar x I + \partial_y) \\ &= \pi \hbar(x + i \tau y) I + (\tau \partial_x + i \partial_y) \end{aligned} \quad (4.4)$$

annihilates any $\tilde{f}(x, y)$ from the image space \mathcal{F}^φ of transformation (4.3). Following the idea from [43] we use the above intertwining properties to evaluate $\Phi(g) = \langle \varphi \rho(g) \varphi \rangle$. Indeed, $\Phi(g)$ shall be annihilated by both $dR^{\tau X+iY}$ and $d\Lambda^{\tau X-iY}$ and therefore is the simultaneous null-solution of the operators cf. (2.16)–(2.17):

$$d\Lambda^{\tau X-iY} + dR^{\tau X+iY} = \tau \pi i \hbar y + i \partial_y \quad \text{and} \quad d\Lambda^{\tau X-iY} - dR^{\tau X+iY} = -\tau \partial_x - \pi \hbar x. \quad (4.5)$$

This determines $\Phi(g) = e^{-\pi \hbar/(2\tau)(x^2 + \tau^2 y^2)}$ (3.10) up to a constant factor. Furthermore, $\widehat{\Phi}(x, y)$ shall be the null solution of the same operators (4.5) as $\Phi(x, y)$ due to the intertwining properties (3.31) and (3.32) of the symplectic Fourier transform. Thus, $\widehat{\Phi}(x, y) = \Phi(x, y)$ once the constant factor is confirmed.

Example 4.3 (Gaussian and a peeling map) As described in Rem. 2.4, we may look for a peeling $E_d : f(x, y) \mapsto e^{d(x,y)} f(x, y)$ (2.32), which shall intertwine operator (4.4) with the Cauchy–Riemann operator. Then, a simple differential equation implies $d(x, y) = \pi \hbar/(2\tau)(x^2 + \tau^2 y^2) = \frac{\pi}{2} |z|^2$ for $z = \sqrt{\frac{h}{2\tau}}(x + i \tau y)$ with $h > 0$ and $\tau > 0$ (see (4.21) for motivation of this normalisation). Thereafter, the *peeling* map [66]

$$\tilde{f}(z) \mapsto F(z) = e^{|z|^2/2} \tilde{f}(z) = e^{\pi \hbar/(2\tau)(x^2 + \tau^2 y^2)} \tilde{f}(x, y) \quad (4.6)$$

produces the function F satisfying the Cauchy–Riemann equation

$$\bar{\partial}_z F(z) = (\tau \partial_x + i \partial_y) F(z) = 0.$$

The composition of the coherent state transform (4.3) and the peeling (4.6) is:

$$\begin{aligned} \tilde{f}(x, y) &= \left(\frac{\hbar}{\tau}\right)^{n/2} e^{\pi \hbar/(2\tau)(x^2 + \tau^2 y^2)} \int_{\mathbb{R}^n} f(t) e^{\pi i \hbar(2yt - xy)} e^{-\pi \hbar(t-x)^2/\tau} dt \\ &= \left(\frac{\hbar}{\tau}\right)^{n/2} \int_{\mathbb{R}^n} f(t) e^{\pi \hbar(-t^2 + 2t(x + i \tau y) - (x + i \tau y)^2/\tau)} dt \\ &= \left(\frac{\hbar}{\tau}\right)^{n/2} \int_{\mathbb{R}^n} f(t) e^{-ht^2/(2\tau) + \sqrt{2h/\tau} \cdot t z - z^2/2} dt, \end{aligned} \quad (4.7)$$

where $z = \sqrt{\frac{h}{2\tau}}(x + i\tau y)$. Further discussion of peeling can be found in [2, 3].

The integral (4.7) is known as Fock–Segal–Bargmann (FSB) transform. Many sources use this formula for particular values $\hbar = 1$ and $\tau = 1$ only. With variable value of τ (4.7) becomes the Fourier–Bros–Iagolnitzer (FBI) transform, see [34, Sect. 3.3] for introduction. The image \mathcal{F}^τ of $\mathcal{L}_2(\mathbb{R}^n)$ under FSB transform is called the *Fock–Segal–Bargmann (FSB) space*. More general, \mathcal{F}_p^τ is the closed subspace of $\mathcal{L}_p(\mathbb{C}^n, \Phi_{\hbar\tau}^{-2}(z) dz)$ consisting of analytic functions. The Stone–von Neumann theorem implies the following result.

Corollary 4.4 *The action $\tilde{\Lambda}_{\hbar}(g) = E_d \circ \Lambda_{\hbar}(g) \circ E_d^{-1}$ with $e^{-d(z)} = \Phi_\tau(z, \bar{z})$ is a unitary irreducible representations of \mathbb{H}^n in the FSB space \mathcal{F}^τ . Two such actions $\tilde{\Lambda}_{\hbar}$ and $\tilde{\Lambda}_{\hbar'}$ are unitary equivalent if and only if $\hbar = \hbar'$.*

Furthermore, $\tilde{\Lambda}_{\hbar}(g)$ is an invertible isometric transformation of $\mathcal{F}_p^\tau \rightarrow \mathcal{F}_p^\tau$, cf. [91] or any other space with shift-modulation invariant norm, e.g. (2.31).

In this paper we do not require a peeling and will refer to the coherent state transformation (4.3) and its image \mathcal{F}^φ as *pre-FSB transform* and *pre-FSB space* respectively. This prefix “pre-” is used to distinguish versions of FSB transform with and without peeling, see [2, 3] for further discussions.

Remark 4.5 The Gaussian φ is the preferred vacuum state because

- it produces analytic functions through FSB transform;
- it is the minimal uncertainty state.

Interestingly, the both properties are derived from the identity $d\rho_{\hbar}^{-\tau X+iY}\varphi = 0$, see [68] for further discussion.

4.2 Ladder operators and complex variables

The analyticity considered in Sect. 4.1 suggests that a complexification of the derived representation can be useful. To avoid discussion of the complex Lie algebras we postpone a complexification till a particular representation in a complex Hilbert is already selected and complex scalars are consequently present.

Definition 4.6 Let ρ be a unitary irreducible representation of the Heisenberg group \mathbb{H}^n with a non-zero Planck constant $\hbar = 2\pi\hbar = -i d\rho^S$. The *ladder operators*, that is the pair of the *creation* $L_{\rho,j}^+$ and *annihilation* $L_{\rho,j}^-$ operators, are the following complex linear combination in the derived representation of the Weyl algebra:

$$L_{\rho,j}^{\pm} = \frac{1}{\sqrt{2|\hbar|\tau}} \left(\pm\tau d\rho^{X_j} + i d\rho^{Y_j} \right), \quad \text{for } j = 1, \dots, n, \quad (4.8)$$

where $\tau > 0$. We will incorporate the parameter τ into the notation $L_{\rho,\tau}^{\pm}$ due to course once the subscript j will be discharged in the one-dimensional situation (or implicitly used in multiindex setup).

For a unitary representation ρ the derived representation is skew-symmetric: $(d\rho^U)^* = -d\rho^U$ for all U in the Weyl algebra \mathfrak{h}_n . Thus the ladder operators are adjoint of each other:

$$(L_{\rho,j}^+)^* = L_{\rho,j}^- . \quad (4.9)$$

The main motivation for the above definition is that the Heisenberg commutator relations $[d\rho_{\hbar}^X, d\rho_{\hbar}^Y] = i\hbar I$ imply a simple commutator of ladder operators:

$$[L_{\rho,l}^-, L_{\rho,m}^+] = \delta_{lm} \operatorname{sign}(\hbar) I \quad \text{for any } \tau > 0 \text{ and } 1 \leq l, m \leq n \quad (4.10)$$

with the dimensionless right hand side. For τ of the dimensionality (mass) \times (frequency) the agreement from Rem. 3.15 is satisfied in (4.8) and we do not meet meaningless fractional powers of physical units. Furthermore, the ladder operators are dimensionless and it is also useful to introduce the respective dimensionless complex variable

$$z_\tau = \sqrt{\frac{|\hbar|}{2\tau}}(x + i\tau y) \in \mathbb{C}^n . \quad (4.11)$$

Accordingly our standard convention we will simply denote z_τ by z unless its dependence on τ needs to be indicated.

As usual with the Heisenberg group, everything essential already happens for $n = 1$ and cases $n > 1$ have only minor technical distinctions. To minimise those we will use the convenient notation:

$$\begin{aligned} z \cdot w &= z_1 w_1 + z_2 w_2 + \dots + z_n w_n , \\ z \cdot L_\rho^\pm &= z_1 L_{\rho,1}^\pm + z_2 L_{\rho,2}^\pm + \dots + z_n L_{\rho,n}^\pm . \end{aligned}$$

Example 4.7 Without loss of generality let us assume that $n = 1$ and $\hbar > 0$ for expressions in Sect. 2.1. Recall from (2.16)–(2.17), that the left Λ_\hbar and the right R_\hbar pulled actions have the opposite signs of the derived representation for $S \in \mathfrak{h}_1$, thus creation/annihilation rôles of ladder operators for the left and right pulled actions are opposite. In terms of the above complex variable z and the respective derivatives:¹

$$\partial_z = \frac{1}{\sqrt{2|\hbar|\tau}}(\tau \partial_x - i \partial_y) , \quad \bar{\partial}_z = \frac{1}{\sqrt{2|\hbar|\tau}}(\tau \partial_x + i \partial_y) . \quad (4.12)$$

We rewrite (2.16)–(2.18) as:

$$L_\Lambda^+ = zI - \bar{\partial}_z , \quad L_\Lambda^- = \bar{z}I + \partial_z ; \quad (4.13)$$

$$L_R^+ = zI + \bar{\partial}_z , \quad L_R^- = \bar{z}I - \partial_z . \quad (4.14)$$

¹ Among two possible notations $\bar{\partial}_z$ and $\partial_{\bar{z}}$ we prefer the former for a better visibility of complex conjugation.

Also for the Schrödinger representation ladder operators are:

$$L_{\rho}^{\pm} = \frac{1}{\sqrt{2|h|}\tau} (h\tau t I \mp \partial_t). \quad (4.15)$$

These expressions for ladder operators look rather similar. Interestingly, the peeled actions $\tilde{\Lambda}_{\hbar} = \Phi_{\tau}^{-1} \circ \Lambda_{\hbar} \circ \Phi_{\tau}$ and $\tilde{R}_{\hbar} = \Phi_{\tau}^{-1} \circ R_{\hbar} \circ \Phi_{\tau}$ from Cor. 4.4 highlight their different structures:

$$L_{\tilde{\Lambda}}^{+} = 2zI - \bar{\partial}_z, \quad L_{\tilde{\Lambda}}^{-} = \partial_z; \quad L_{\tilde{R}}^{+} = \bar{\partial}_z, \quad L_{\tilde{R}}^{-} = 2\bar{z}I - \partial_z; \quad (4.16)$$

Furthermore, the restriction of $L_{\tilde{\Lambda}}^{+}$ to the irreducible subspace of functions annihilated by $L_{\tilde{R}}^{+} = \bar{\partial}_z$ is the operator of multiplication $2zI$.

Consequently, for $(0, x, y) \in \mathbb{H}^n$ and the respective $z = \sqrt{\frac{|h|}{2\tau}}(x + i\tau y) \in \mathbb{C}^n$ we can express the representation ρ as exponentiation of ladder operators:

$$\begin{aligned} \rho_{\hbar}(0, x, y) &= \exp(x \cdot d\rho^X + y \cdot d\rho^Y) \\ &= \exp(\bar{z} \cdot L_{\rho}^{+} - z \cdot L_{\rho}^{-}) =: \rho_{\hbar}(z, \bar{z}). \end{aligned} \quad (4.17)$$

Here $\rho_{\hbar}(z, \bar{z})$ is known² as the *displacement operator* in quantum optics [36, 37, Sect. 3.2] and the *Weyl operator* in mathematical physics. Its explicit expression in complex coordinates is:

$$[\Lambda_{\hbar}(s, z)f](z', \bar{z}') = e^{2\pi i \hbar s - (z \cdot \bar{z}' - \bar{z} \cdot z')/2} f(z' - z, \bar{z}' - \bar{z}), \quad (4.18)$$

$$[R_{\hbar}(s, z)f](z', \bar{z}') = e^{-2\pi i \hbar s - (z \cdot \bar{z}' - \bar{z} \cdot z')/2} f(z' + z, \bar{z}' + \bar{z}). \quad (4.19)$$

The composition formula for displacement operators is fully determined by the commutator (4.10) and is representation-independent, i.e. does not contain \hbar :

$$\begin{aligned} &\exp(\bar{z}_1 \cdot L_{\rho}^{+} - z_1 \cdot L_{\rho}^{-}) \exp(\bar{z}_2 \cdot L_{\rho}^{+} - z_2 \cdot L_{\rho}^{-}) \\ &= \exp\left(\frac{1}{2}\Im(z_1 \cdot \bar{z}_2)\right) \exp((\bar{z}_1 + \bar{z}_2) \cdot L_{\rho}^{+} - (z_1 + z_2) \cdot L_{\rho}^{-}). \end{aligned} \quad (4.20)$$

A helpful technique [46] is the separation of the ladder operators in the regular representation by the *Kermack–McCrae identity*³:

$$\begin{aligned} R_{\hbar}(0, z) &= \exp(\bar{z} \cdot L_R^{+} - z \cdot L_R^{-}) \\ &= \Phi(z, \bar{z})^{-1} \exp(\bar{z} \cdot L_R^{+}) \exp(-z \cdot L_R^{-}) \end{aligned} \quad (4.21)$$

$$= \Phi(z, \bar{z}) \exp(-z \cdot L_R^{-}) \exp(\bar{z} \cdot L_R^{+}). \quad (4.22)$$

² We shall point out that our formula (4.17) swaps positions of z and \bar{z} in comparison to the established physics sources.

³ See [26] for an interesting discussion of the Kermack–McCrae's papers as another example of lost opportunities.

Here

$$\Phi(z, \bar{z}) := e^{-|z|^2/2} = e^{-\pi \hbar / (2\tau)(x^2 + \tau^2 y^2)} \quad (4.23)$$

is a complexified form of $\Phi(x, y)$ (3.10). In the same fashion we have:

$$\Lambda_{\hbar}(z, \bar{z}) = \Phi(z, \bar{z}) \exp(-\bar{z} \cdot L_{\Lambda}^+) \exp(z \cdot L_{\Lambda}^-) \quad (4.24)$$

$$= \Phi(z, \bar{z})^{-1} \exp(z \cdot L_{\Lambda}^-) \exp(-\bar{z} \cdot L_{\Lambda}^+). \quad (4.25)$$

Recall, that the difference between (4.21)–(4.22) and (4.24)–(4.25) is due to opposite signs of dR_{\hbar}^S and $d\Lambda_{\hbar}^S$ echoing in the commutator (4.10).

An analytic meaning of operators L_{Λ}^{\pm} is the action by shifts

$$L_{\Lambda}^+ : \Phi_m \mapsto \sqrt{m+1} \Phi_{m+1}, \quad L_{\Lambda}^- : \Phi_m \mapsto \sqrt{m} \Phi_{m-1}. \quad (4.26)$$

on the orthonormal basis $\Phi_m = (\pi^m m!)^{-1/2} (L_{\Lambda}^+)^m \Phi$ within an irreducible component \mathcal{F}^{φ} of the representation Λ_{\hbar} . On the other hand, operators L_R^{\pm} acts in the similar fashion by “shifting” different irreducible components of Λ_{\hbar} one into another. Namely:

$$L_R^{\pm} : \mathcal{F}_m^{\varphi} \rightarrow \mathcal{F}_{m \mp 1}^{\varphi} \quad \text{where } \mathcal{F}_m^{\varphi} = (L_R^-)^m \mathcal{F}^{\varphi} \quad (m = 0, 1, \dots). \quad (4.27)$$

For a consistence we set $\mathcal{F}_{-1}^{\varphi} = \{0\}$ in (4.27). From commutativity of the left and right pulled actions \mathcal{F}_m^{φ} are Λ_{\hbar} -invariant irreducible components of the orthogonal decomposition $\mathcal{L}_2(\mathbb{C}^n) = \bigoplus_{m=0}^{\infty} \mathcal{F}_m^{\varphi}$, cf. [85]. Also these spaces

$$\mathcal{F}_m^{\varphi} := \mathcal{F}^{\varphi_m}$$

are image spaces (3.5) of the covariant transform for the Hermite functions $\varphi_m = (L_{\hbar}^+)^m \varphi$ as the respective ground vectors. Obviously, spaces $\mathcal{F}_m^{\varphi} = (L_R^-)^m \mathcal{F}^{\varphi}$ are annihilated by powers of the right ladder operator $(L_R^+)^n$ with $n \geq m$. Spaces \mathcal{F}_m^{φ} were named true poly-analytic in [85]. The concept was recently revised in [83] from the representation theory viewpoint, where the authors employed the semidirect product of the Heisenberg group and $\mathrm{SL}_2(\mathbb{R})$, known as the Schrödinger [34, Sect. 1.2] or Jacobi [14, Sect. 8.5] group. Although the latter and its various subgroups [5, 6, 71] are very interesting and important objects, they may be excessive for the discussed poly-analytic function decomposition, which is completely manageable by the two-sided action of the Heisenberg group alone.

Lemma 4.8 *The collection of functions $\Phi_{jk} = (L_{\Lambda}^+)^j (L_{\Lambda}^-)^k \Phi$ forms an orthonormal basis of $\mathcal{L}_2(\mathbb{R}^{2n})$ and, thereafter, an arbitrary $v \in \mathcal{L}_2(\mathbb{R}^{2n})$ admits the presentation*

$$v = \sum_{j,k=0}^{\infty} v_{jk} (L_{\Lambda}^+)^j (L_{\Lambda}^-)^k \Phi, \quad \text{where } v_{jk} = \langle v, \Phi_{jk} \rangle = \left\langle v, (L_{\Lambda}^+)^j (L_{\Lambda}^-)^k \Phi \right\rangle \in \mathbb{C}. \quad (4.28)$$

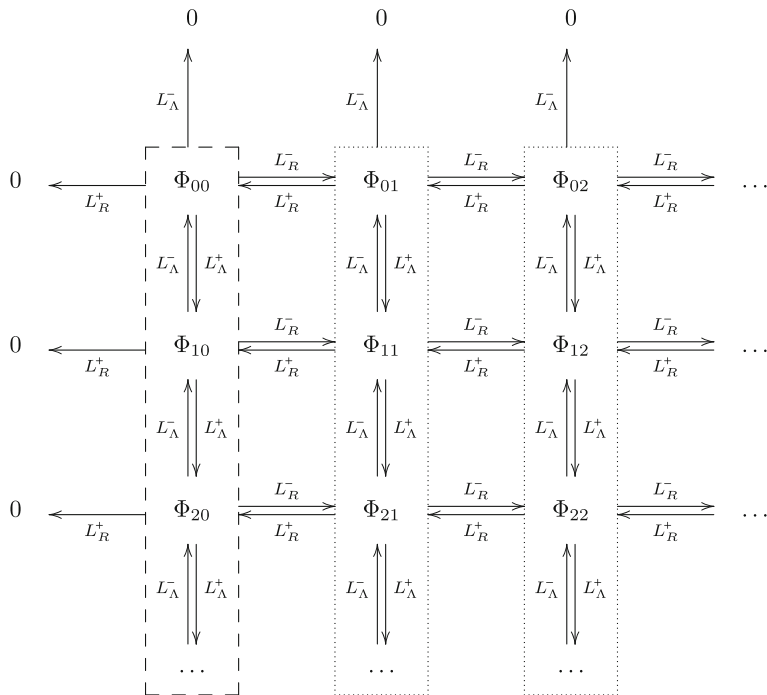


Fig. 3 The action of the ladder operators on the lattice of orthogonal vectors. Boxed columns form orthonormal bases of Λ_{\hbar} -invariant subspaces \mathcal{F}_m^{φ} . Left ladder operators shift elements within a basis. Right ladder operators map elements of one basis to the respective elements of an adjacent basis

See Fig. 3 for an illustration. Note, that the order of ladder operators in (4.28) is not important since they commute. This commutativity is also behind the following result.

Proposition 4.9 *Let a bounded operator $A : \mathcal{L}_2(\mathbb{C}^n) \rightarrow \mathcal{L}_2(\mathbb{C}^n)$ commute with all right ladder operators, i.e. $[L_R^{\pm}, A] = 0$. Then all spaces \mathcal{F}_m^{φ} are A -invariant and A is a left relative convolution on $\mathcal{L}_2(\mathbb{C}^n)$.*

Proof By induction we can show that $[L_R^{\pm}, A] = 0$ is equivalent to $[(L_R^{\pm})^m, A] = 0$ for all natural numbers m . Assume towards a contradiction that for some m the space \mathcal{F}_m^{φ} is not invariant, that is there are exist $v_m \in \mathcal{F}_m^{\varphi}$ and $v_k \in \mathcal{F}_k^{\varphi}$ for some $k \neq m$ such that $\langle Av_m, v_k \rangle \neq 0$. Then:

1. If $k < m$, let $v \in \mathcal{F}_{m-k}^{\varphi}$ be such that $(L_R^{-})^k v = v_m$, Then:

$$\begin{aligned}
 0 &\neq \langle Av_m, v_k \rangle \\
 &= \langle A(L_R^{-})^k v, v_k \rangle \\
 &= \langle (L_R^{-})^k Av, v_k \rangle \\
 &= \langle Av, (L_R^{+})^k v_k \rangle \\
 &= \langle Av, (L_R^{+})^k v_k \rangle
 \end{aligned}$$

$$\begin{aligned}
&= \langle Av, 0 \rangle \\
&= 0.
\end{aligned}$$

2. If $k > m$, let $v \in \mathcal{F}_{k-m}^\varphi$ be such that $(L_R^-)^m v = v_k$, Then:

$$\begin{aligned}
0 &\neq \langle Av_m, v_k \rangle \\
&= \langle Av_m, (L_R^-)^m v \rangle \\
&= \langle (L_R^+)^m Av_m, v_k \rangle \\
&= \langle A(L_R^+)^m v_m, v_k \rangle \\
&= \langle 0, v_k \rangle \\
&= 0.
\end{aligned}$$

We had obtained the contradiction.

To show that A is a relative convolution, the Schur lemma implies that the restriction of A to the irreducible component \mathcal{F}_0^φ belongs to the C^* -algebra spanned by the left action of Λ_{\hbar} of \mathbb{H}^n restricted to \mathcal{F}_0^φ . Thus the restriction of A is equal to a relative convolution on \mathcal{F}_0^φ with some kernel $k(g)$. By commutativity with ladder operators, restrictions of A to any \mathcal{F}_m^φ , $m \in \mathbb{N}$ is again the relative convolution with the same kernel k . The same is true for $\mathcal{L}_2(\mathbb{C}^n)$ —the direct sum of all \mathcal{F}_m^φ , $m \in \mathbb{N}$. \square

The vanishing commutators $[L_R^\pm, A] = 0$ is a sufficient but is not necessary condition for A -invariance of subspaces. A counterexample is a diagonal-type operator $A : \mathcal{L}_2(\mathbb{C}^n) \rightarrow \mathcal{L}_2(\mathbb{C}^n)$ such that restriction of A to \mathcal{F}_m^φ is the identity operator times x_m for a bounded non-constant sequence x_m . We state the following simple result:

Lemma 4.10 *The space \mathcal{F}^φ is invariant under an operator A if and only if \mathcal{F}^φ is in the kernel of the commutator $[L_R^+, A]$.*

Proof Let $Af \in \mathcal{F}^\varphi$ for any $f \in \mathcal{F}^\varphi$. Then $L_R^+ f = 0$ implies $L_R^+ Af = 0$ and $[L_R^+, A]f = 0$. Conversely, if $[L_R^+, A]f = 0$ for some $f \in \mathcal{F}^\varphi$, then $L_R^+ Af = 0$, that is $f \in \mathcal{F}^\varphi$ is an invariant subspace of A . \square

To conclude this subsection we use complex variables to state the following straightforward properties of the mixed Gaussian $\Phi_{\tau\varsigma}$ (3.21):

Lemma 4.11 1. For $\tau = \varsigma$ the mixed Gaussian coincides with the Gaussian: $\Phi_{\tau\tau} = \Phi_\tau$.

2. Complex conjugations swaps parameters τ and ς , that is $\overline{\Phi_{\tau\varsigma}} = \Phi_{\varsigma\tau}$.

3. $\Phi_{\tau\varsigma}$ is fixed by the symplectic Fourier transform. $\Phi_{\tau\varsigma} = \widehat{\Phi_{\tau\varsigma}}$.

4. Complex form of the mixed Gaussian in notations (4.11) is:

$$\Phi_{\tau\varsigma} = \sqrt[4]{\frac{\tau\varsigma}{(\tau+\varsigma)^2}} \exp\left(-\frac{\sqrt{\tau\varsigma}}{\tau+\varsigma} \bar{z}_\varsigma z_\tau\right).$$

5. $\Phi_{\tau\varsigma}$ is annihilated by the right ladder operator $L_{R,\tau}^+$ with squeeze τ .

6. $\Phi_{\tau\varsigma}$ is annihilated by the left ladder operator $L_{\Lambda,\varsigma}^-$ with squeeze ς .

4.3 Twisted convolution and symplectic Fourier transform

Exponentiation of ladder operators in the peeled form (4.16) has an explicit meaning. Indeed, for a polynomial $p(z, \bar{z})$ an algebraic manipulation with the Taylor expansion of the exponent produces:

$$[\exp(w \cdot L_{\Lambda}^-)p](z, \bar{z}) = p(z + w, \bar{z}), \quad [\exp(\bar{w} \cdot L_R^+)p](z, \bar{z}) = p(z, \bar{z} + \bar{w}). \quad (4.29)$$

These formulae can be extended by continuity to any space of functions such that polynomials in z and \bar{z} form a dense subspace. Due to properties from Lem. 4.11.5–6 the respective exponents act on Φ_{τ_S} trivially:

$$\exp(w \cdot L_{\Lambda, S}^-)\Phi_{\tau_S} = \Phi_{\tau_S}, \quad \exp(\bar{w} \cdot L_{R, \tau}^+)\Phi_{\tau_S} = \Phi_{\tau_S}. \quad (4.30)$$

The symplectic Fourier transform (3.28) in complex variables is:

$$\widehat{\psi}(z, \bar{z}) = (\hbar/2)^n \int_{\mathbb{C}^n} \psi(z', \bar{z}') \exp(\pi i (\bar{z}'z - z'\bar{z})) dz' \wedge d\bar{z}'. \quad (4.31)$$

Of course, the intriguing relation (3.30), i.e. $\widehat{a} = a \lrcorner \mathbf{1} = \Lambda_{\hbar}(a)\mathbf{1} = R_{\hbar}(a)\mathbf{1}$, is equally true in complex coordinates as it was in real ones. Therefore, for ladder operators we have the following intertwining properties:

$$\frown \circ L_{\Lambda}^{\pm} = L_{\Lambda}^{\pm} \circ \frown \quad \text{and} \quad \frown \circ L_R^{\pm} = -L_R^{\pm} \circ \frown. \quad (4.32)$$

Furthermore, the intertwining properties (3.31) and (3.32) implying the following complexified versions:

$$\frown \circ S(z, \bar{z}) = E(z, \bar{z}) \circ \frown \quad \frown \circ E(z, \bar{z}) = S(z, \bar{z}) \circ \frown, \quad (4.33)$$

where complexified versions of spatial (2.23) and frequency (2.24) shifts are:

$$S(z, \bar{z}) = \Lambda_{\hbar}(0, \tfrac{1}{2}z) \circ R_{\hbar}(0, -\tfrac{1}{2}z) : f(z', \bar{z}') \mapsto f(z' - z, \bar{z}' - \bar{z}), \quad (4.34)$$

$$E(z, \bar{z}) = \Lambda_{\hbar}(0, \tfrac{1}{2}z) \circ R_{\hbar}(0, \tfrac{1}{2}z) : f(z', \bar{z}') \mapsto e^{-(z\bar{z}' - \bar{z}z')/2} f(z', \bar{z}'). \quad (4.35)$$

Note, that in contrast to (2.24) the complexified frequency shift does not explicitly depend on the Planck constant, the dependence is incorporated in the complex variable z . The infinitesimal version of (4.33) is:

$$\frown \circ \partial_z = -\tfrac{1}{2}\bar{z}I \circ \frown, \quad \frown \circ \bar{\partial}_z = \tfrac{1}{2}zI \circ \frown, \quad \frown \circ \bar{z}I = -2\partial_z \circ \frown, \quad \frown \circ zI = 2\bar{\partial}_z \circ \frown. \quad (4.36)$$

5 Co- and contra-variant transforms of operators

Based on the previous consideration we are ready to describe relations within the family of operators: localisation, Toeplitz, pseudodifferential and some their further generalisations.

5.1 Localisation and Toeplitz operators

The fundamental identity (3.13) implies that $M_\theta \circ W_\theta = I$ for a vector θ with unit norm. This identity is a source of a more sophisticated object, known as *localisation operator* [1, 16, 19, 23, 24]:

$$L_\psi = M_\theta \circ \psi I \circ W_\theta, \quad (5.1)$$

where ψ is a function on the phase space called *symbol* or *weight function*. In a weak form the operator can be written as:

$$\langle L_\psi u, v \rangle = \int_{\mathbb{R}^{2n}} \psi(x, y) \langle u, \rho(x, y)\theta \rangle \langle \rho(x, y)\theta, v \rangle dx dy, \quad (5.2)$$

where ρ is a unitary representation on a Hilbert space \mathcal{H} and u, v are arbitrary vectors in \mathcal{H} . Of course, identities similar to (5.1)–(5.2) can be used to define localisation operators for representations of other groups as well, but we remain with the Heisenberg group within this paper.

For the special choice of $\rho = \Lambda_{\hbar}$ and $\theta = \Phi_\tau$, the operator in (5.1) is more known as the *Toeplitz operators* T_ψ on (pre-)FSB space. Equivalently, Toeplitz operators T_ψ and \tilde{T}_ψ on pre-FSB space \mathcal{F}^τ and FSB spaces $\tilde{\mathcal{F}}_\tau$ respectively are defined by:

$$T_\psi = P_\tau \psi I \quad \text{and} \quad \tilde{T}_\psi = \tilde{P}_\tau \psi I, \quad (5.3)$$

where $P_\tau : \mathcal{L}_2(\mathbb{C}^n) \rightarrow \mathcal{F}^\tau$ and $\tilde{P}_\tau : \mathcal{L}_2(\mathbb{C}^n, e^{-|z|^2} dz \wedge d\bar{z}) \rightarrow \tilde{\mathcal{F}}_\tau$ are corresponding orthoprojections.

Our study of Toeplitz operators is preceded by a simple useful observation on their relations on FSB and pre-FSB spaces. Recall, that a unitary equivalence of \mathcal{F}^τ and $\tilde{\mathcal{F}}_\tau$ is provided by the peeling map $E_\Phi : f \mapsto \Phi f$ (4.6) and $P_\tau = E_\Phi \circ \tilde{P}_\tau \circ E_\Phi^{-1}$. The peeling operators E_Φ obviously commutes with an operator of multiplication $\psi : f \mapsto \psi f$. Thereafter,

Lemma 5.1 *For a bounded function ψ , the Toeplitz operators (5.3) are unitary equivalent through the peeling:*

$$T_\psi = E_\Phi \circ \tilde{T}_\psi \circ E_\Phi^{-1}.$$

In other words, the main properties of Toeplitz operators in terms of their symbol ψ are identical in both spaces. Furthermore, consideration of pre-FSB spaces open the door for the following generalisation.

Definition 5.2 For given positive reals τ, ς and a function $\psi(x, y)$ on \mathbb{R}^{2n} we define *cross-Toeplitz operator* $T_\psi : \mathcal{F}^\tau \rightarrow \mathcal{F}^\varsigma$ by:

$$T_\psi f = P_\varsigma(\psi f), \quad \text{where } f \in \mathcal{F}^\tau. \quad (5.4)$$

The function ψ is called the *symbol* of the operator T_ψ . If parameters τ, ς need to be explicitly indicated we will use the longer notation $T_\psi^{\tau\varsigma}$ for the operator.

Obviously, the traditional Toeplitz operators correspond to the case $\tau = \varsigma$. We can also consider $P_\varsigma \psi I$ as a map $\mathcal{L}_2(\mathbb{R}^{2n}) \rightarrow \mathcal{F}^\varsigma$ which serves as an umbrella for all cross-Toeplitz operators $T_\psi^{\tau\varsigma}$ with any $\tau > 0$.

A natural family of symbols ψ for cross-Toeplitz operators T_ψ (5.4) requires that the set of functions $f \in \mathcal{F}^\tau$, such that $\psi f \in \mathcal{L}_2(\mathbb{R}^{2n})$, is dense in \mathcal{F}^τ . The further discussion will give alternative expressions of cross-Toeplitz operators which is applicable for a wider set of symbols.

Remark 5.3 Our cross-Toeplitz operators belongs to a more general class of operators prompted by the identity (3.13). Let representations ρ_j act in Hilbert spaces \mathcal{H}^j with $j = 1, 2$ and $\theta_j \in \mathcal{H}^j$ be two fixed admissible vectors. Then for a bounded function ψ a generalised Anti-Wick operator $\mathcal{H}^1 \rightarrow \mathcal{H}^2$ is defined by [16]:

$$L_\psi = M_{\theta_2} \circ \psi I \circ W_{\theta_1}, \quad (5.5)$$

That is, our cross-Toeplitz operator corresponds to the representations of the Heisenberg group and θ_1, θ_2 being the Gaussians (with possibly different squeeze parameters). Other groups and analysing/reconstructing vectors are out of our current scope.

5.2 Toeplitz operators and PDO with heat flow

If a representation ρ is induced by a character of a subgroup $H \subset G$, then integration in (3.23) can be reduced to the homogeneous space $X = G/H$ with the essentially same set of resulting operators. More specifically, for the natural projection $\mathfrak{p} : G \rightarrow X$ we fix a Borel section $\mathfrak{s} : X \rightarrow G$ [47, Sect. 13.2], which is a right inverse to \mathfrak{p} . Recall, we define an operator of *relative convolution* on V [57, 66], cf. (3.23):

$$\rho(k) = \int_X k(x) \rho(\mathfrak{s}(x)) \, dx \quad (5.6)$$

with a kernel k being a function on $X = G/H$. For example, if G is the Heisenberg group and ρ is its Schrödinger representation (2.14), then $\rho(\widehat{a})$ (5.6), where \widehat{a} is the symplectic Fourier transform of a (3.28), is a *pseudodifferential operator* (PDO)

$a(D, X)$ with the symbol a [34, 43, 66, Sect. 2.1]:

$$[a(D, X)f](t) = \int_{\mathbb{R}^{2n}} a(\xi, \tfrac{1}{2}(t+r)) e^{\pi i \hbar \xi(t-r)} f(r) dr d\xi$$

with the Weyl (symmetrized) symbol $a(\xi, x)$. For the sake of completeness we provide a calculation parallel to [34, Sect. 2.1] in our notations

$$\begin{aligned} [Af](t) &= [\rho(\widehat{a})f](t) = \int_{\mathbb{R}^{2n}} \widehat{a}(x, y) \rho_{\hbar}(x, y) f(t) dx dy \\ &= \int_{\mathbb{R}^{2n}} \widehat{a}(x, y) e^{\pi i \hbar x(2t-y)} f(t-y) dx dy \\ &= \hbar^n \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} a(x', y') e^{\pi i \hbar(x'y-y'x)} e^{\pi i \hbar x(2t-y)} f(t-y) dx' dy' dx dy \\ &= \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} a(x', y') e^{\pi i \hbar x'y} \delta\left(\hbar\left(\tfrac{1}{2}(y+y')-t\right)\right) f(t-y) dx' dy' dy \\ &= \hbar^{-n} \int_{\mathbb{R}^{2n}} a\left(x', t - \tfrac{1}{2}y\right) e^{\pi i \hbar x'y} f(t-y) dx' dy \\ &= \hbar^{-n} \int_{\mathbb{R}^{2n}} a\left(x', \tfrac{1}{2}(t+r)\right) e^{\pi i \hbar x'(t-r)} f(r) dx' dr. \end{aligned}$$

Now we revise the method used in [43, Sect. 3.1] to prove the Calderón–Vaillancourt estimations. It was described as “rather magical” in [34, Sect. 2.5]. Use of the covariant transform dispels the mystery without undermining the power of the method. Relevantly for the present topic, the demonstration [43, Sect. 3.1] implicitly expresses a PDO as a Toeplitz operator—the result which commonly attributed to a later work of Guillemin [40, (8.20)], see also [34, Sect. 2.7].

We start from the following lemma, which has a transparent proof in terms of covariant transform, cf. earlier presentations in [43, Sect. 3.1] and [34, (2.75)] in the case of $\theta = \phi$.

Lemma 5.4 *Let ρ be an irreducible square integrable representation a Lie group G in V and mother wavelets ϕ and θ are admissible. Then*

$$\Phi(g)\rho(g) = M_{\theta} \circ (\Lambda \otimes R)(g, g) \circ W_{\phi} \quad \text{for all } g \in G \quad (5.7)$$

and $\Phi(g) = \langle \theta, \rho(g)\phi \rangle$.

Proof We know from (3.13) that $M_\theta \circ W_{\rho(g)\phi} = \langle \theta, \rho(g)\phi \rangle I$ on V , thus:

$$M_\theta \circ W_{\rho(g)\phi} \circ \rho(g) = \langle \theta, \rho(g)\phi \rangle \rho(g) = \Phi(g)\rho(g).$$

On the other hand, the intertwining properties (3.6) and (4.1) of the wavelet transform imply:

$$M_\theta \circ W_{\rho(g)\phi} \circ \rho(g) = M_\theta \circ (\Lambda \otimes R)(g, g) \circ W_\phi.$$

A combination of the above two identities yields (5.7). \square

We will use a specialisation of (5.7) to the Schrödinger representation ρ_h and the Gaussian ϕ .

Proposition 5.5 *The cross-Toeplitz operator $P_\zeta \psi I : \mathcal{F}^\tau \rightarrow \mathcal{F}^\zeta$ is unitary equivalent to PDO $a(D, X) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with the symbol:*

$$a_\psi(x, y) = \hbar^{2n} \int_{\mathbb{R}^{2n}} \psi(x', y') \Phi_{\tau\zeta}(x - 2x', y - 2y') dx' dy', \quad (5.8)$$

where $\Phi_{\tau\zeta}$ is the intertwining kernel (3.21).

Proof As we observed in (2.29) that $(\Lambda \otimes R)(g, g)$ is symplectic phase shift $E_h(x, y)$:

$$[\Lambda_h(s, x, y) R_h(s, x, y) f](x', y') = E_h(2x, 2y) f(x', y') = e^{2\pi i h(xy' - yx')} f(x', y'). \quad (5.9)$$

Thus, integrating the identity (5.7) with the function $(2h)^n \widehat{\psi}(2x, 2y)$ over G/H we obtain:

$$a_\psi(D, X) = M_\theta \circ \psi I \circ W_\phi \quad (5.10)$$

where $\widehat{a}_\psi(x, y) = (2h)^n \widehat{\psi}(2x, 2y) \Phi_{\tau\zeta}(x, y)$. Therefore, the standard manipulations based on intertwining properties (3.33) of the symplectic Fourier transform present a_ψ as a sort of convolution:

$$\begin{aligned} a_\psi(x, y) &= (2h)^n \left(\widehat{\psi}(2x, 2y) \Phi_{\tau\zeta}(x, y) \right)^\wedge \\ &= \hbar^{2n} \int_{\mathbb{R}^{2n}} \psi(x', y') \widehat{\Phi}_{\tau\zeta}(x - 2x', y - 2y') dx' dy' \end{aligned} \quad (5.11)$$

$$= \hbar^{2n} \int_{\mathbb{R}^{2n}} \psi(x', y') \Phi_{\tau\zeta}(x - 2x', y - 2y') dx' dy', \quad (5.12)$$

since $\Phi_{\tau\zeta}$ is fixed by the symplectic transform: $\widehat{\Phi}_{\tau\zeta} = \Phi_{\tau\zeta}$.

Finally, we rewrite the right-hand side of (5.10). Since $M_\zeta \circ P_\zeta = M_\zeta$ as operators $\mathcal{L}_2(\mathbb{R}^{2n}) \rightarrow \mathcal{L}_2(\mathbb{R}^n)$ we obtain:

$$M_\zeta \circ \psi I \circ W_\phi = M_\zeta \circ (P_\zeta \psi I) \circ W_\tau,$$

where $P_\zeta \psi I : \mathcal{F}^\tau \rightarrow \mathcal{F}^\zeta$ is the cross-Toeplitz operator. Since M_ζ and W_τ are unitary isomorphisms of $\mathcal{L}_2(\mathbb{R}^n)$ with \mathcal{F}^ζ and \mathcal{F}^τ , respectively, we proved the statement. \square

The convolution with a Gaussian is known as *Weierstrass–Gauss transform* (also called Weierstrass or Gauss transforms separately). Since the Gaussian provides the fundamental solution to the *heat equation*, the Weierstrass–Gauss transform can be interpreted as a heat flow over a fixed period of time. Its appearance in the context of FSB spaces can be traced back to [44] at least. Being a smoothing operator the Weierstrass–Gauss transform is not invertible on many function spaces. This puts restrictions on applications of PDO calculus for Toeplitz operators as was pointed in [40] and extensively discussed afterwards, cf. [8, 12, 13, 20].

The proof of Prop. 5.5 is a rectification of ideas [42, 43] originally employed in the opposite direction: after multiplication of both sides of (5.7) by Φ^{-1} , a given PDO can be estimated through some Toeplitz operator. Thereafter, the elementary bound of the norm of a Toeplitz operator $P_\tau(\psi \Phi^{-1})I$ by $\|\psi \Phi^{-1}\|_\infty$ leads to the Calderón–Vaillancourt theorem [81, Chap. XIII], which limits $\|a(D, X)\|$ by \mathcal{L}_∞ -norms of a finite number of partial derivatives of a . Prop. 5.5 is also a specialisation of [16, Lem. 2.4].

As another development of ideas from [43] we present a representation-theoretic derivation of the fundamental formula for composition of PDOs. Our treatment again relays on a simultaneous consideration of the left and right action of \mathbb{H}^n .

Proposition 5.6 *The composition of two PDOs with symbols a_1 and a_2 has the symbol:*

$$\begin{aligned} a &= \sum_{n,m=0}^{\infty} \frac{1}{n! m! (2\pi i \hbar)^{n+m}} (d\Lambda_h^X - dR_h^X)^m (d\Lambda_h^Y - dR_h^Y)^n a_1 \\ &\quad \times (d\Lambda_h^X - dR_h^X)^n (d\Lambda_h^Y - dR_h^Y)^m a_2 \end{aligned} \quad (5.13)$$

$$= \sum_{n,m=0}^{\infty} \frac{i^{m-n}}{n! m! (2\pi \hbar)^{n+m}} \partial_x^m \partial_y^n a_1 \partial_x^n \partial_y^m a_2. \quad (5.14)$$

Proof The following computation is similar to one in [43, (2.2.5)], but all elements are expressed in terms of representations of \mathbb{H}^n now. For $k_{1,2} = \widehat{a}_{1,2}$, the symbol of a composition of two PDOs is:

$$\begin{aligned} (k_1 \natural k_2)^\wedge &= k_1 \natural \widehat{k}_2 \\ &= \Lambda_h(k_1) \widehat{k}_2 \\ &= \left(\int k_1(x, y) \exp \left(x d\Lambda_h^X + y d\Lambda_h^Y \right) dx dy \right) \widehat{k}_2 \end{aligned}$$

$$= \left(\int k_1(x, y) \exp \left(\frac{1}{2} x (d\Lambda_h^X - dR_h^X) + \frac{1}{2} y (d\Lambda_h^Y - dR_h^Y) \right) \right. \\ \left. \times \exp \left(\frac{1}{2} x (d\Lambda_h^X + dR_h^X) + \frac{1}{2} y (d\Lambda_h^Y + dR_h^Y) \right) dx dy \right) \widehat{k}_2$$

where we used $e^{A/2+B/2} = e^{A/2}e^{B/2}$ for commuting operators $A = x(d\Lambda_h^X - dR_h^X) + y(d\Lambda_h^Y - dR_h^Y)$ and $B = x(d\Lambda_h^X + dR_h^X) + y(d\Lambda_h^Y + dR_h^Y)$. Furthermore, commutativity of the left and right actions and vanishing commutator $[d\Lambda_h^X - dR_h^X, d\Lambda_h^Y - dR_h^Y] = 0$ imply:

$$= \left(\int k_1(x, y) \exp \left(\frac{1}{2} x (d\Lambda_h^X - dR_h^X) \right) \exp \left(\frac{1}{2} y (d\Lambda_h^Y - dR_h^Y) \right) \right. \\ \left. \times \Lambda_h \left(\frac{1}{2} x, \frac{1}{2} y \right) \circ R_h \left(\frac{1}{2} x, \frac{1}{2} y \right) dx dy \right) \widehat{k}_2 \\ = \left(\int k_1(x, y) \sum_{n,m=0}^{\infty} \frac{1}{n! m! 2^{n+m}} x^n (d\Lambda_h^X - dR_h^X)^n y^m (d\Lambda_h^Y - dR_h^Y)^m \right. \\ \left. \times \Lambda_h \left(\frac{1}{2} x, \frac{1}{2} y \right) \circ R_h \left(\frac{1}{2} x, \frac{1}{2} y \right) dx dy \right) \widehat{k}_2 \\ = \sum_{n,m=0}^{\infty} \frac{1}{n! m! 2^{n+m}} \int k_1(x, y) x^n y^m \Lambda_h \left(\frac{1}{2} x, \frac{1}{2} y \right) \circ R_h \left(\frac{1}{2} x, \frac{1}{2} y \right) dx dy \\ \times (d\Lambda_h^X - dR_h^X)^n (d\Lambda_h^Y - dR_h^Y)^m \widehat{k}_2$$

The last integral is the symplectic Fourier transform of $x^n y^m k_1$ under the derived representation, thus the intertwining properties (4.33) imply:

$$= \sum_{n,m=0}^{\infty} \frac{i^{m-n}}{n! m! (2\pi \hbar)^{n+m}} (d\Lambda_h^X - dR_h^X)^m (d\Lambda_h^Y - dR_h^Y)^n \widehat{k}_1 \\ \times (d\Lambda_h^X - dR_h^X)^n (d\Lambda_h^Y - dR_h^Y)^m \widehat{k}_2,$$

that is the first form (5.13) of the composition. Using the expressions for the derived representations (2.16)–(2.17) we obtain (5.14). \square

A parallel computation with complex variables implies:

Corollary 5.7 *The composition of two PDOs with symbols a_1 and a_2 has the symbol:*

$$a = \sum_{n,m=0}^{\infty} \frac{(-1)^m}{n! m! 2^{n+m}} \left(L_{R_h}^- - L_{\Lambda_h}^- \right)^n \left(L_{R_h}^+ - L_{\Lambda_h}^+ \right)^m a_1 \\ \times \left(L_{\Lambda_h}^+ - L_{R_h}^+ \right)^n \left(L_{\Lambda_h}^- - L_{R_h}^- \right)^m a_2 \quad (5.15)$$

$$= \sum_{n,m=0}^{\infty} \frac{(-2)^{n+m}}{n! m!} \partial_z^n \bar{\partial}_z^m a_1 \cdot \bar{\partial}_z^n \partial_z^m a_2. \quad (5.16)$$

5.3 Bounded operators and integrated representation

It does not worth to be locked within the particular case of the Heisenberg group while using group representation techniques. In fact, it is often possible to get a more general statement at no extra cost, Sect. 4.1 provides an example. On the other hand, the ultimate generality is not our purpose here either. Thus, we are working in the comfortable assumption of a unimodular group G and an *irreducible square integrable* (possibly, *modulo a subgroup* H) representation ρ in a Hilbert space \mathcal{H} . Then, the orthogonality relation (3.3) for $W(f, \phi)(g) = \langle f, \rho(g)\phi \rangle$ holds ([4, Thm. 8.2.1]; [32, Thm. 1]).

For an irreducible representation ρ of a group G on a Hilbert space \mathcal{H} , the Schur lemma implies that the von Neumann algebra generated by $\rho(g)$, $g \in G$ contains all bounded linear operators on \mathcal{H} . Thus, any such operator can be considered as an integrated representation $\rho(k)$ with a suitable kernel k (possibly being a distribution). To find the kernel explicitly consider the induced covariant transform of the representation ρ :

$$b(g) = W(u, v)(g) = \langle u, \rho(g)v \rangle_{\mathcal{H}}, \quad \text{where } u, v \in \mathcal{H} \quad (5.17)$$

as a map from $\mathcal{H} \times \mathcal{H}^*$ to a space of function $\mathcal{L}(X)$ on the homogeneous space $X = G/H$. This linear map $\mathcal{H} \times \mathcal{H}^* \rightarrow \mathcal{L}(X)$ can be extended to the tensor product $\mathcal{H} \otimes \mathcal{H}^* \rightarrow \mathcal{L}(X)$.

Proposition 5.8 *Let G , H , \mathcal{H} and ρ be as above. For a Hilbert–Schmidt operator K considered as an element of $\mathcal{H} \otimes \mathcal{H}^*$, let us define a function b_K by the extension of (5.17) from $\mathcal{H} \times \mathcal{H}^*$ to $\mathcal{H} \otimes \mathcal{H}^*$. Then, K is equal to the integrated representation: $K = \rho(b_K)$.*

Proof For $u, v \in \mathcal{H}$, let $K : \mathcal{H} \rightarrow \mathcal{H}$ be the rank one operator $K : f \mapsto \langle f, v \rangle_{\mathcal{H}} u$. For the corresponding integrated representation $\rho(b)$ and arbitrary $f_1, f_2 \in \mathcal{H}$ with square-integrable matrix coefficients:

$$\begin{aligned} \langle \rho(b)f_1, f_2 \rangle_{\mathcal{H}} &= \left\langle \int_X b(g) \rho(g)f_1 \, dg, f_2 \right\rangle_{\mathcal{H}} \\ &= \int_X b(g) \langle \rho(g)f_1, f_2 \rangle_{\mathcal{H}} \, dg \\ &= \int_X b(g) \overline{\langle f_2, \rho(g)f_1 \rangle_{\mathcal{H}}} \, dg \\ &= \langle W(u, v), W(f_2, f_1) \rangle_{\mathcal{L}_2(X)} \end{aligned}$$

then using the orthogonality relation (3.3) for wavelet transform we continue:

$$= \langle u, f_2 \rangle_{\mathcal{H}} \overline{\langle v, f_1 \rangle_{\mathcal{H}}}$$

$$\begin{aligned}
&= \langle u, f_2 \rangle_{\mathcal{H}} \langle f_1, v \rangle_{\mathcal{H}} \\
&= \langle \langle f_1, v \rangle_{\mathcal{H}} u, f_2 \rangle_{\mathcal{H}} \\
&= \langle K f_1, f_2 \rangle.
\end{aligned}$$

Since f_1 and f_2 with the square-integrable property are dense in \mathcal{H} , the identity $K = \rho(b)$ is shown in the simplest case $K \in \mathcal{H} \times \mathcal{H}^*$. Then, this identity generalises to any Hilbert–Schmidt operator by the respective linear extension (5.17). \square

The above correspondence between operators and integrated representations can be pushed beyond Hilbert–Schmidt and bounded operators on Hilbert spaces through the *coorbit theory* (see [29, 30] and the recent survey [32]) or wavelets in Banach spaces technique [57, 58, 66]. A motivating example is the case of $G = \mathbb{H}^n$, H —its centre Z and ρ —the Schrödinger representation (2.14) in $\mathcal{H} = \mathcal{L}_2(\mathbb{R}^n)$, which leads to the following statement.

Corollary 5.9 ([43, Sect. 2.3]; [34, Sect. 2.1]) *Let $K : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ be a linear operator with Schwartz kernel k :*

$$K : f(x) \mapsto [Kf](x) = \int_{\mathbb{R}^n} k(x, y) f(y) dy. \quad (5.18)$$

Then K is PDO $a_K(D, X)$ with the symbol a_K such that:

$$a_K(x, y) = \int_{\mathbb{R}^n} e^{-\pi i \hbar (2t - y)x} k(t, y - t) dt. \quad (5.19)$$

Proof The extension of (5.17) to $\mathcal{S}(\mathbb{R}^n) \otimes \mathcal{S}'(\mathbb{R}^n)$ for the integral operator K (5.18) obviously is:

$$b(g) = \int_{\mathbb{R}^n} \rho(g) k(x, x) dx,$$

where the representation $\rho(g)$ acts on the second variable of the kernel $k(x, y)$. Then, for the Schrödinger representation (2.14) it becomes:

$$b_K(x', y') = \int_{\mathbb{R}^n} e^{-\pi i \hbar (2t - y')x'} k(t, t - y') dt.$$

By the previous Proposition, K is the integrated representation of the function b_K . An application of the symplectic Fourier transform $a_K = \widehat{b}_K$ produces (5.19), with tiny differences between two expressions. \square

For applications to cross-Toeplitz operators we again use a more general setup and notations ($G, H, X, \chi, s, \phi, \dots$) of induced representations from Rem. 2.1. An additional assumption is: there are ρ_χ -irreducible unitary equivalent components \mathcal{F}^τ and \mathcal{F}^ς of $\mathcal{L}_2(X)$ with an intertwining integral kernel $K_{\tau\varsigma}(w, z)$, which also provides an orthogonal projection $P_\varsigma : \mathcal{L}_2(X) \rightarrow \mathcal{F}^\varsigma$. A reader, who is not interested in group representations, may safely remain within the FSB space framework described in subsection 3.1. However, there are other interesting implementations of the scheme, which include the Hardy and weighted Bergman spaces on the unit disk and upper half-plane [17, 32, 56, 61, 65, 66].

Corollary 5.10 *In the above assumptions and using notations of Rem. 2.1, for a bounded function ψ on X , the Toeplitz operator $P_\tau \psi I : \mathcal{F}^\tau \rightarrow \mathcal{F}^\tau$ is an integrated representation $\rho_\chi(b_\psi)$ for the function:*

$$b_\psi(g) = \int_X \chi \left(r(g^{-1} * s(z)) \right) K_\tau(z, g^{-1} \cdot z) \psi(g^{-1} \cdot z) dz, \quad (5.20)$$

where $K_\tau(w, z)$ is the reproducing kernel (3.11). In particular, for FSB spaces:

$$\begin{aligned} b_\psi(g) = & \int_X \chi \left(r(g^{-1} * s(z)) \right) \overline{\chi} \left(r(s(g^{-1} \cdot z)^{-1} * s(z)) \right) \\ & \times \Phi_\tau((s(z))^{-1} * g \cdot z) \psi(g^{-1} \cdot z) dz, \end{aligned} \quad (5.21)$$

Proof Obviously, the Toeplitz operator $P_\tau \psi I$ is an integral operator on \mathbb{C}^n with the kernel $k_\psi(w, z) = K_\tau(w, z) \psi(z)$. Then, for the induced representation ρ_χ (2.13) the Wigner-type transform (5.17) produces the first form (5.20). The second form (5.21) follows from the expression of the reproducing kernel

$$K_\tau(w, z) = \langle \rho(s(z)) \phi_\tau, \rho(s(w)) \phi_\tau \rangle = \overline{\chi} \left(r((s(z))^{-1} * s(w)) \right) \Phi_\tau((s(z))^{-1} \cdot w)$$

through the wavelet transform of vacuum vectors (e.g. Gaussians, cf. (3.9)). \square

The above formula (5.21) greatly simplifies for the Heisenberg group and the (pre-)FSB space. The respective substitution produces the next result:

Corollary 5.11 *The Toeplitz operator $P_\tau \psi I : \mathcal{F}^\tau \rightarrow \mathcal{F}^\tau$ with a symbol ψ is a relative convolution with the kernel:*

$$k_\psi(x, y) = \int_{\mathbb{R}^{2n}} e^{2\pi i \hbar(x y' - y x')} \Phi_\tau(x, y) \psi(x' - x, y' - y) dx' dy' \quad (5.22)$$

$$= \Phi_\tau(x, y) \widehat{\psi}(2x, 2y). \quad (5.23)$$

The result is a particular form for $\tau = \varsigma$ of the identity (5.8) since $\widehat{k}_\psi(x, y)$ is equal to $a(x, y)$ from (5.8)—the symbol of PDO representing the cross-Toeplitz operator

$P_\tau \psi I$ (with all its deficiencies discussed above). A related expression can be found in [16, Sect. 2.3]. We cannot get a cross-Toeplitz operator as an integrated representation of the left action since it preserves spaces \mathcal{F}^τ . Thus, for a better description of cross-Toeplitz operators we need an expanded group action discussed in the next section.

Remark 5.12 This section's methods and results are related to co- and contra-variant Berezin calculus, also known as Wick and anti-Wick symbolic calculus [10], cf. [16, 20, 34, Sect. 2.7]. In the approach advocated here it falls into the general framework of operator covariant transform [57, 58, 60, 66], yet we keep this technique in low profile to avoid unnecessary abstraction.

6 Two-sided convolutions and cross-Toeplitz operators

The previous consideration repeatedly used combinations of the left and right pulled actions (2.8)–(2.9). It is time to employ their union in a systematic way.

6.1 Two-sided relative convolutions from the Heisenberg group

The relation (2.29) (repeated in (5.9)) suggests to present a general operator of multiplication $\psi I : f \mapsto \psi f$ as an integrated representation, cf. (5.6):

$$\begin{aligned} \psi &= (\Lambda_{\hbar} \otimes R_{\hbar})(\widehat{\psi}) \\ &= (2/\hbar)^n \int_{\mathbb{R}^{2n}} \widehat{\psi}(2x, 2y) \Lambda_{\hbar}(0, x, y) R_{\hbar}(0, x, y) dx dy, \end{aligned} \quad (6.1)$$

where $\widehat{\psi}$ is the symplectic Fourier transform (3.28) of ψ . Motivated by (6.1) we introduce the main tool of our investigation.

Definition 6.1 For a function k on \mathbb{R}^{4n} , a *two-sided relative convolution* $D(k)$ is defined by:

$$D(k) = \int_{\mathbb{R}^{4n}} k(x_1, y_1, x_2, y_2) \Lambda_{\hbar}(0, x_1, y_1) R_{\hbar}(0, x_2, y_2) dx_1 dy_1 dx_2 dy_2. \quad (6.2)$$

Such operators also arise as representations of two-sided convolutions on the Heisenberg group studied in [49–54, 84, 86, 87] as local (irreducible) representations of certain C^* -algebras. The connection between these two sources was studied in [55, 63].

Since operators Λ_{\hbar} and R_{\hbar} are isometries in $\mathcal{L}_p(\mathbb{R}^{2n})$, $1 \leq p \leq \infty$ the integral (6.2) defines a bounded operator in $\mathcal{L}_p(\mathbb{R}^{2n})$ for any $k \in \mathcal{L}_1(\mathbb{R}^{4n})$. Alternatively, one can consider any shift-invariant norm from Lem. 2.2. Furthermore, the Schwartz space $\mathcal{S}(\mathbb{R}^{2n})$ of smooth rapidly decreasing functions is invariant under Λ_{\hbar} and R_{\hbar} . Thus,

for any $u, v \in \mathcal{S}(\mathbb{R}^{2n})$, the function

$$\tilde{v}_u(x_1, y_1, x_2, y_2) = \langle v, \Lambda_{\hbar}(0, x_1, y_1) R_{\hbar}(0, x_2, y_2) u \rangle$$

is in $\mathcal{S}(\mathbb{R}^{4n})$. Therefore, a distribution $k \in \mathcal{S}'(\mathbb{R}^{4n})$ defines a bounded operator $D(k) : \mathcal{S}(\mathbb{R}^{2n}) \rightarrow \mathcal{S}'(\mathbb{R}^{2n})$ in the weak sense: $\langle D(k)u, v \rangle = \langle k, \tilde{v}_u \rangle$.

Example 6.2 Several operators relevant to our consideration can be treated as two-sided convolutions.

1. The operator of multiplication $f \mapsto \psi f$ by a function ψ is a two-sided convolution $D(k)$ with the distribution, cf. (6.1) and (6.2)

$$k(x_1, y_1, x_2, y_2) = (2/\hbar)^n \widehat{\psi}(2x_1, 2y_1) \delta(x_1 - x_2, y_1 - y_2), \quad (6.3)$$

where $\widehat{\psi}$ is the symplectic Fourier transform of ψ and δ is the Dirac delta function.

2. Integrated representations $\Lambda_{\hbar}(k)$ or $R_{\hbar}(k)$ with a kernel k on \mathbb{R}^{2n} are two-sided convolutions $D(k \otimes \delta)$ or $D(\delta \otimes k)$, respectively. In particular, the symplectic Fourier transform (3.30) is $D(\mathbf{1} \otimes \delta) = D(\delta \otimes \mathbf{1})$.
3. The orthogonal projection $P_{\tau} : \mathcal{L}_2(\mathbb{R}^{2n}) \rightarrow \mathcal{F}^{\tau}$ (3.7) can be rearranged as an integrated representation R_{\hbar} , cf. (2.21) :

$$\begin{aligned} [P_{\tau} f](x, y) &= \langle f, \Lambda_{\hbar}(0, x, y) \Phi_{\tau} \rangle \\ &= \langle \Lambda_{\hbar}(0, -x, -y) f, \Phi_{\tau} \rangle \\ &= [(R_{\hbar}(\overline{\Phi}_{\tau}) f)(x, y) \\ &= [(R_{\hbar}(\Phi_{\tau}) f)(x, y), \end{aligned} \quad (6.4)$$

because $\overline{\Phi}_{\tau} = \Phi_{\tau}$, see (3.21). Therefore, P_{τ} is a two-sided convolution

$$P_{\tau} = D(\delta \otimes \Phi_{\tau}). \quad (6.5)$$

4. Similarly the intertwining operator $P_{\tau\zeta} : \mathcal{F}^{\tau} \rightarrow \mathcal{F}^{\zeta}$ is a two-sided convolution with the kernel

$$P_{\tau\zeta} = D(\delta \otimes \Phi_{\zeta\tau}). \quad (6.6)$$

Note the swapped order of squeeze parameters due to $\overline{\Phi}_{\tau\zeta} = \Phi_{\zeta\tau}$.

5. A (cross-)Toeplitz operator $T_{\psi} : f \mapsto P_{\zeta}(\psi f)$ is the composition of two-sided convolutions with kernels (6.3) and (6.5), which will be explicitly evaluated in Sect. 6.3.

A simple change of variables in the integral shows that

Lemma 6.3 *A two-sided convolution $D(k)$ with the kernel $k(x_1, y_1; x_2, y_2)$ is an integral operator*

$$[D(k)f] = \int_{\mathbb{R}^{2n}} \tilde{k}(x, y; x', y') f(x', y') dx' dy'$$

with the Schwartz kernel

$$\begin{aligned} & \tilde{k}(x, y; x', y') \\ &= \int_{\mathbb{R}^{2n}} e^{\pi i \hbar ((2x_2 - x')y - (2y_2 - y')x)} k(x - x' + x_2, y - y' + y_2; x_2, y_2) dx_2 dy_2. \end{aligned} \quad (6.7)$$

We can transit between Schwartz and convolution kernels in the opposite direction using Prop. 5.8, cf. Cor. 5.9:

Corollary 6.4 *An integral operator K on \mathbb{R}^{2n} with a Schwartz kernel $\tilde{k}(x_1, y_1; x_2, y_2)$ is equal to two-sided convolution $D(k)$ with the kernel:*

$$k(x_1, y_1; x_2, y_2) = \int_{\mathbb{R}^{2n}} \Lambda_{\hbar}(-x_2, -y_2) R_{\hbar}(-x_1, -y_1) \tilde{k}(x, y; x, y) dx dy, \quad (6.8)$$

there both representations act on the second half of variables in \tilde{k} .

Reciprocity of formulae (6.7) and (6.8) can be checked by a direct calculation.

6.2 Reduction of two-sided convolutions

The above examples indicate that an effective calculus of two-sided convolutions is useful for the theory of cross-Toeplitz operators. To this end we will reduce two-sided convolutions from \mathbb{H}^n to one-sided convolution on the Heisenberg group \mathbb{H}^{2n} of the doubled size [53], cf. [80] for applications of this method.

To begin we note, that composition of two-sided convolutions on the entire group G is related to the left convolutions on the Cartesian product group $G \times G$, that is we have $D(k_1)D(k_2) = D(k)$ for:

$$k(g_1, g_2) = \int_G \int_G k_1(g_3, g_4) k_1(g_3^{-1} g_1, g_4^{-1} g_2) dg_3 dg_4. \quad (6.9)$$

However twisted convolutions for homogeneous spaces require a more accurate treatment shown below.

Let us introduce the action of \mathbb{H}^{2n} on $\mathcal{L}_2(\mathbb{R}^{2n})$ by:

$$\tilde{\Xi}_{\hbar} = (\Lambda_{\hbar} \otimes R_{\hbar}) \circ Y. \quad (6.10)$$

Here the section

$$Y : \mathbb{H}^{2n} \rightarrow \mathbb{H}^n \times \mathbb{H}^n : (s, x_1, x_2, y_1, y_2) \mapsto (s, x_1, y_1) \times (0, y_2, x_2/v) \quad (6.11)$$

is a right inverse of the group homomorphism $\mathbb{H}^n \times \mathbb{H}^n \rightarrow (\mathbb{H}^n \times \mathbb{H}^n)/Z_d \simeq \mathbb{H}^{2n}$, where

$$Z_d = \{(s, 0, 0) \times (s, 0, 0) \mid s \in \mathbb{R}\}$$

is the diagonal of $Z \times Z \subset \mathbb{H}^n \times \mathbb{H}^n$. In other words:

$$\begin{aligned} \tilde{\Xi}_{\hbar}(s, x_1, x_2, y_1, y_2) &= \Lambda_{\hbar}(s, x_1, y_1) R_{\hbar}(0, \nu y_2, x_2/\nu) \\ &= \Lambda_{\hbar}(0, x_1, y_1) R_{\hbar}(-s, \nu y_2, x_2/\nu), \end{aligned} \quad (6.12)$$

where $s \in \mathbb{R}$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$. Note, that the above formulae contain an arbitrary constant squeeze parameter $\nu \neq 0$. From the physical point of view it adjusts units of the respective components of the phase space, cf. Rem. 3.15. The role of ν is rather technical and it will disappear at later stage, cf. Rem. 6.11.

It is a straightforward calculation that $\tilde{\Xi}_{\hbar}$ is an irreducible unitary representation of \mathbb{H}^{2n} on $\mathcal{L}_2(\mathbb{R}^{2n})$. The Stone–von Neumann theorem guarantees equivalence of $\tilde{\Xi}_{\hbar}$ to the Schrödinger representation and this is completely transparent from the unitary map of a partial dilation

$$S_{\nu} : \mathcal{L}_2(\mathbb{R}^{2n}, dx dy) \rightarrow \mathcal{L}_2(\mathbb{R}^{2n}, \nu^n dt_1 dt_2) : f(x, y) \mapsto \tilde{f}(t_1, t_2) = f(\nu t_1, t_2). \quad (6.13)$$

We conjugate the action $\tilde{\Xi}_{\hbar}$ of \mathbb{H}^{2n} on $\mathcal{L}_2(\mathbb{R}^{2n})$ by S_{ν}

$$\Xi_{\hbar}(g) = S_{\nu} \tilde{\Xi}_{\hbar}(g) S_{\nu}^{-1}, \quad \text{for } g \in \mathbb{H}^{2n}. \quad (6.14)$$

We do not indicate dependence of Ξ_{\hbar} on ν to keep notation simpler. Substituting the above expressions in (2.9)–(2.10) we rewrite $\Xi_{\hbar}(g)$ (6.14) explicitly as:

$$\begin{aligned} &[\Xi_{\hbar}(s, x_1, x_2, y_1, y_2)f](t_1, t_2) \\ &= \chi_{\hbar}(2s - (\nu y_1 + x_2)t_1 + (x_1 + \nu y_2)t_2 + x_1 x_2/\nu - \nu y_1 y_2) \\ &\quad \times f(t_1 - x_1/\nu + y_2, t_2 - y_1 + x_2/\nu). \end{aligned} \quad (6.15)$$

The essential properties of Ξ_{\hbar} are summarised as follows.

Lemma 6.5 *The action Ξ_{\hbar} (6.12) is a linear unitary irreducible representation of \mathbb{H}^{2n} in $\mathcal{L}_2(\mathbb{R}^{2n})$. Moreover, the Schrödinger representation (2.14) of \mathbb{H}^{2n} in $\mathcal{L}_2(\mathbb{R}^{2n})$ is the composition of Ξ_{\hbar} with the symplectic automorphism A of \mathbb{H}^{2n} :*

$$A : (t, x_1, x_2, y_1, y_2) \mapsto (t, -\frac{1}{2}x_2 - \frac{1}{2}\nu y_1, \frac{1}{2}x_1 + \frac{1}{2}\nu y_2, x_1/\nu - y_2, -x_2/\nu + y_1). \quad (6.16)$$

The above symplectic map $A : (x_1, x_2, y_1, y_2) \mapsto (\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2)$ in matrix form is:

$$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2} & -\frac{\nu}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{\nu}{2} \\ \frac{1}{\nu} & 0 & 0 & -1 \\ 0 & -\frac{1}{\nu} & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix}. \quad (6.17)$$

Proof It follows from the group law (2.1) that a symplectic transformation of \mathbb{R}^{4n} —in particular (6.16)—produces an automorphism of \mathbb{H}^{2n} [34, Sect. 1.2]. It is another straightforward check, that the composition of A (6.16) and the action (6.15) produces the Schrödinger representation (2.14) of \mathbb{H}^{2n} in $\mathcal{L}_2(\mathbb{R}^{2n})$. A composition of a group automorphism with a representation produces again a representation of the group, thus Ξ_{\hbar} as a composition of the inverse of (6.16) and the Schrödinger representation is again a representation of \mathbb{H}^{2n} . The rest follows from the properties of the Schrödinger representation. \square

A two-sided convolution can be written as the integrated representation $\tilde{\Xi}_{\hbar}$ using a change of variables:

$$\begin{aligned} D(k) &= \int_{\mathbb{R}^{4n}} k(x_1, y_1, x_2, y_2) \Lambda_{\hbar}(0, x_1, y_1) R_{\hbar}(0, x_2, y_2) dx_1 dx_2 dy_1 dy_2 \\ &= \int_{\mathbb{R}^{4n}} k(x_1, y_1, x_2, y_2) \tilde{\Xi}_{\hbar}(0, x_1, \nu y_2, y_1, x_2/\nu) dx_1 dx_2 dy_1 dy_2 \\ &= \int_{\mathbb{R}^{4n}} k(x_1, y_1, \nu y'_2, x'_2/\nu) \tilde{\Xi}_{\hbar}(0, x_1, x'_2, y_1, y'_2) dx_1 dx'_2 dy_1 dy'_2. \end{aligned} \quad (6.18)$$

We denote by B the above change of variables in the integration kernel.⁴

$$B : \mathbb{R}^{4n} \rightarrow \mathbb{R}^{4n} : (x_1, x_2, y_1, y_2) \mapsto (x_1, y_1, \nu y_2, x_2/\nu). \quad (6.19)$$

Its inverse map obviously is:

$$B^{-1} : \mathbb{R}^{4n} \rightarrow \mathbb{R}^{4n} : (x_1, x_2, y_1, y_2) \mapsto (x_1, \nu y_2, x_2, y_1/\nu). \quad (6.20)$$

Then, computation (6.18) implies:

Lemma 6.6 *The two-sided convolution with a kernel $k(x_1, y_1, x_2, y_2)$ is the integrated representation $\tilde{\Xi}$ preceded by B (6.19):*

$$D(k) = \tilde{\Xi}_{\hbar}(k \circ B). \quad (6.21)$$

⁴ We do not overload the notation B by the parameter ν since it will be redundant later, cf. Rem. 6.11.

The uniqueness (up to unitary equivalence) of the irreducible unitary representation of \mathbb{H}^{2n} for a given Planck constant \hbar implies the following:

Proposition 6.7 *The Schrödinger representation $\rho_{\hbar}(g)$ (2.14) on $\mathcal{L}_2(\mathbb{R}^{2n})$ and the representation $\tilde{\Xi}_{\hbar}$ are intertwined by a unitary operator $U : \mathcal{L}_2(\mathbb{R}^{2n}) \rightarrow \mathcal{L}_2(\mathbb{R}^{2n})$:*

$$\rho_{\hbar}(g) = U \circ \tilde{\Xi}_{\hbar}(g) \circ U^{-1}, \quad \text{for all } g \in \mathbb{H}^{2n}. \quad (6.22)$$

Proof The composition of a symplectic automorphism A and the Schrödinger representation ρ_{\hbar} is intertwined with ρ_{\hbar} by the unitary operator $U(A)$ —the metaplectic representation of the double cover of the symplectic group ([76, Chap. 1]; [34, Chap. 4]). \square

It is known that operator U is the Gaussian integral operator ([76, Chap. 1]; [34, Sect. 4.4]), but its explicit form will not be used in this paper, merely its existence is important for us. The proposition implies the same relation for the integrated representations.

Theorem 6.8 *The unitary operator U (6.22) conjugates the operators of two-sided convolution $D(k)$ with the kernel k and the pseudodifferential operator $a(D, X)$*

$$U \circ D(k) \circ U^{-1} = a(D, X),$$

where the Weyl symbol of $a(D, X)$ is:

$$a(x_1, x_2, y_1, y_2) = \widehat{k}(x_1, y_1, \nu y_2, x_2/\nu), \quad (6.23)$$

and \widehat{k} is the symplectic Fourier transform of k .

Proof A combination of (6.21) and the intertwining property (6.22) implies:

$$\begin{aligned} U \circ D(k) \circ U^{-1} &= \int_{\mathbb{R}^{4n}} k(x_1, y_1, \nu y_2, x_2/\nu) \rho_{\hbar}(0, x_1, x_2, y_1, y_2) dx_1 dx_2 dy_1 dy_2 \\ &= \int_{\mathbb{R}^{4n}} \widehat{a}(x_1, x_2, y_1, y_2) \rho_{\hbar}(0, x_1, x_2, y_1, y_2) dx_1 dx_2 dy_1 dy_2. \end{aligned} \quad (6.24)$$

where $\widehat{a}(x_1, x_2, y_1, y_2) = k(x_1, y_1, \nu y_2, x_2/\nu)$ and (6.23) follows. \square

We can now review Example 6.2.

Example 6.9 1. For the kernel k (6.3) of operator of multiplication by ψ , we obtain $\widehat{k}(x_1, y_1, x_2, y_2) = \psi(\frac{1}{2}(y_1 + y_2), \frac{1}{2}(x_1 + x_2))$. Thus

$$\begin{aligned} a(x_1, x_2, y_1, y_2) &= \widehat{k}(x_1, y_1, \tau y_2, x_2/\tau) \\ &= \psi\left(\frac{1}{2}(y_1 + x_2/\tau), \frac{1}{2}(x_1 + \tau y_2)\right). \end{aligned}$$

2. Integrated representations $\Lambda_{\hbar}(k)$ and $R_{\hbar}(k)$, which are two-sided convolutions $D(k \otimes \delta)$ and $D(\delta \otimes k)$, respectively, are unitary equivalent to PDO's with symbols

$$a(x_1, x_2, y_1, y_2) = \widehat{k}(x_1, y_1) \quad \text{and} \quad a(x_1, x_2, y_1, y_2) = \widehat{k}(\tau y_2, x_2/\tau),$$

respectively.

3. For $\Phi_{\tau\varsigma}(x, y) = \langle \phi_{\varsigma}, \rho_{\hbar}(x, y) \phi_{\tau} \rangle$ (3.21), Lem. 4.11 states $\widehat{\Phi}_{\tau\varsigma} = \Phi_{\tau\varsigma}$. Also, $\Phi_{\varsigma\tau}(y_2, x_2/\nu) = \Phi_{\varsigma\tau}(x_2, y_2)$ for $\nu = \sqrt{\tau\varsigma}$ by inspection. Then, the intertwining operator $\mathcal{F}^{\tau} \rightarrow \mathcal{F}^{\varsigma}$ from Ex. 3.9 is unitary equivalent to PDO with the symbol

$$a(x_1, x_2, y_1, y_2) = \Phi_{\tau\varsigma}(x_2, y_2). \quad (6.25)$$

In the special case $\tau = \varsigma$ we note, that in the decomposition $\mathcal{L}_2(\mathbb{R}^{2n}) = \mathcal{L}_2(\mathbb{R}^n) \otimes \mathcal{L}_2(\mathbb{R}^n)$, PDO with symbol (6.25) acts as $I \otimes P_{\phi}$ —the identity in the first component and one-dimensional projection on the subspace spanned by the Gaussian ϕ_{τ} on the second. A similar representation of the projection $\mathcal{L}_2(\mathbb{C}^n) \rightarrow \mathcal{F}_{\phi}$ was obtained in [85, Thm. 2.1] by a chain of explicit integral transformations.

6.3 Composition and twisted convolution

Now we can use the composition of integrated representations (3.24) to establish a calculus of two-sided representations. This replaces the composition formula (6.9) of two-sided convolutions on groups.

Lemma 6.10 *The composition $D(k_1)D(k_2)$ of two-sided convolutions with kernels k_1 and k_2 is a two-sided convolution $D(k)$ with the kernel $k = k_1 \tilde{\natural} k_2$ obtained by a twisted convolution type formula:*

$$k_1 \tilde{\natural} k_2 := ((k_1 \circ B) \natural (k_2 \circ B)) \circ B^{-1}. \quad (6.26)$$

Explicitly it is given by:

$$\begin{aligned} k(x'_1, y'_1, x'_2, y'_2) &= \int_{\mathbb{R}^{4n}} k_1(x_1, y_1, x_2, y_2) k_2(x'_1 - x_1, y'_1 - y_1, x'_2 - x_2, y'_2 - y_2) \\ &\quad \times e^{\pi \hbar i (y_1 x'_1 + x_2 y'_2 - x_1 y'_1 - y_2 x'_2)} dx_1 dx_2 dy_1 dy_2. \end{aligned} \quad (6.27)$$

Remark 6.11 Note, that the explicit expression (6.27) is independent from the parameter ν , which formally participates in the abstract formula (6.26) through the map B (6.19).

Proof Using the relation (6.21) in both directions we obtain:

$$\begin{aligned} D(k_1) D(k_2) &= \widetilde{\Xi}_{\hbar}(k_1 \circ B) \widetilde{\Xi}_{\hbar}(k_2 \circ B) \\ &= \widetilde{\Xi}_{\hbar}((k_1 \circ B) \natural (k_2 \circ B)) \end{aligned}$$

$$= D \left(((k_1 \circ B) \natural (k_2 \circ B)) \circ B^{-1} \right).$$

Thus, for the composition kernel defined by the identity $k \circ B = (k_1 \circ B) \natural (k_2 \circ B)$ we compute:

$$\begin{aligned} & (k \circ B)(x'_1, y'_1, x'_2, y'_2) \\ &= \int_{\mathbb{R}^{4n}} k_1(x_1, y_1, \tau y_2, x_2/\tau) k_2(x'_1 - x_1, y'_1 - y_1, \tau(y'_2 - y_2), (x'_2 - x_2)/\tau) \\ & \quad \times e^{\pi \hbar i (y_1 x'_1 + y_2 x'_2 - x_1 y'_1 - x_2 y'_2)} dx_1 dx_2 dy_1 dy_2. \end{aligned}$$

Applying transformation B^{-1} (6.20) to the last expression we obtain (6.27). \square

Remark 6.12 Formula (6.27) is different from the twisted convolution on the Heisenberg group \mathbb{H}^{2n} by the signs in the exponent for terms with x_2 or y_2 . Thus, the twisted convolution part for (x_2, y_2) phase space is intertwined by complex conjugation.

Proposition 6.13 A Toeplitz operator $T_\psi : f \mapsto P_\zeta(\psi f)$ considered as an operator $\mathcal{L}_2(\mathbb{C}^n) \rightarrow \mathcal{F}^\zeta$ is a two-sided convolution with the kernel $k_\psi^\#$ given by either of the following expressions:

$$\begin{aligned} k_\psi^\#(x_1, y_1, x_2, y_2) &= (2/\hbar)^n \widehat{\psi}(2x_1, 2y_1) e^{\pi \hbar i (y_1 x_2 - x_1 y_2)} \Phi_\zeta(x_2 - x_1, y_2 - y_1) \\ &= (2/\hbar)^n \widehat{\psi}(2x_1, 2y_1) \overline{[\Lambda_{\hbar}(x_1, y_1) \Phi_\zeta](x_2, y_2)} \end{aligned} \quad (6.28)$$

$$= (2/\hbar)^n \widehat{\psi}(2x_1, 2y_1) [\Lambda_{\hbar}(x_2, y_2) \Phi_\zeta](x_1, y_1) \quad (6.29)$$

$$= (2/\hbar)^n \widehat{\psi}(2x_1, 2y_1) K_\zeta(x_1, y_1; x_2, y_2). \quad (6.30)$$

Also, a cross-Toeplitz operator $T_\psi : f \mapsto P_\zeta(\psi P_\tau f)$ considered as an operator $\mathcal{F}^\tau \rightarrow \mathcal{F}^\zeta$ is a two-sided convolution with the kernel:

$$k^\#(x_1, y_1, x_2, y_2) = (2/\hbar)^n \widehat{\psi}(2x_1, 2y_1) \overline{[\Lambda_{\hbar}(x_1, y_1) \Phi_{\tau\zeta}](x_2, y_2)}. \quad (6.31)$$

Proof For the kernel (6.5) of the Bargmann projection and the kernel (6.3) of the multiplication operator we use (6.27) to calculate the kernel of the composition:

$$\begin{aligned} k_\psi^\#(x'_1, y'_1, x'_2, y'_2) &= (2/\hbar)^n \int_{\mathbb{R}^{4n}} \delta(x_1, y_1) \Phi_\zeta(x_2, y_2) \widehat{\psi}(2(x'_1 - x_1), 2(y'_1 - y_1)) \\ & \quad \times \delta(x'_1 - x'_2 - (x_1 - x_2), y'_1 - y'_2 - (y_1 - y_2)) \\ & \quad \times e^{\pi \hbar i (y_1 x'_1 + x_2 y'_2 - x_1 y'_1 - y_2 x'_2)} dx_1 dx_2 dy_1 dy_2 \\ &= (2/\hbar)^n \int_{\mathbb{R}^{2n}} \Phi_\zeta(x_2, y_2) \widehat{\psi}(2x'_1, 2y'_1) \end{aligned}$$

$$\begin{aligned}
& \times \delta(x'_1 - x'_2 + x_2, y'_1 - y'_2 + y_2) e^{\pi \hbar i (x_2 y'_2 - y_2 x'_2)} dx_2 dy_2 \\
& = (2/\hbar)^n \widehat{\psi}(2x'_1, 2y'_1) \Phi_{\varsigma}(x'_2 - x'_1, y'_2 - y'_1) e^{\pi \hbar i (y'_1 x'_2 - x'_1 y'_2)} .
\end{aligned} \quad (6.32)$$

That is the first form the kernel. Its comparison with (3.11) gives formula (6.30). The expression (2.9) of Λ_{\hbar} gives two other forms of the kernel.

Composing (6.32) with the Bargmann projection P_{τ} on the right we obtain:

$$\begin{aligned}
k_{\psi}^{\#}(x'_1, y'_1, x'_2, y'_2) &= (2/\hbar)^n \int_{\mathbb{R}^{4n}} \widehat{\psi}(2x_1, 2y_1) \Phi_{\varsigma}(x_2 - x_1, y_2 - y_1) e^{\pi \hbar i (y_1 x_2 - x_1 y_2)} \\
&\quad \times \delta(x'_1 - x_1, y'_1 - y_1) \Phi_{\tau}(x'_2 - x_2, y'_2 - y_2) \\
&\quad \times e^{\pi \hbar i (y_1 x'_1 + x_2 y'_2 - x_1 y'_1 - y_2 x'_2)} dx_1 dx_2 dy_1 dy_2 \\
&= (2/\hbar)^n \int_{\mathbb{R}^{2n}} \widehat{\psi}(2x'_1, 2y'_1) \Phi_{\varsigma}(x'_1 - x_2, y'_1 - y_2) \Phi_{\tau}(x'_2 - x_2, y'_2 - y_2) \\
&\quad \times e^{\pi \hbar i (y'_1 x_2 - x'_1 y_2 + x_2 y'_2 - y_2 x'_2)} dx_2 dy_2 \\
&= (2/\hbar)^n \widehat{\psi}(2x'_1, 2y'_1) e^{\pi \hbar i (y_1 x_2 - x_1 y_2)} \Phi_{\tau\varsigma}(x'_1 - x'_2, y'_1 - y'_2) ,
\end{aligned}$$

which is equivalent to (6.31). \square

It is clear, that for $\tau = \varsigma$ relation (6.28) is an expanded version of (5.23):

$$k_{\psi}(x, y) = (\hbar/2)^n k_{\psi}^{\#}(x, y, 0, 0) .$$

On the other hand (6.31) extends (5.23) for the cross-Toeplitz operators.

6.4 PDO from two-sided convolutions

The form of cross-Toeplitz operators as two-sided convolution gives a new connection to PDO.

Theorem 6.14 *A Toeplitz operator $T_{\psi} : \mathcal{L}_2(\mathbb{R}^{2n}) \rightarrow \mathcal{F}^{\tau}$ is unitary equivalent to PDO with the symbol:*

$$a_{\psi}^{\#}(x_1, x_2, y_1, y_2) = \int_{\mathbb{R}^{2n}} \psi(x, y) \Lambda_{\hbar}(2x - x_1, 2y - y_1) \Phi_{\tau}(x_2, y_2) dx dy \quad (6.33)$$

$$= \int_{\mathbb{R}^{2n}} \psi(x, y) \overline{\Lambda_{\hbar}(x_2, y_2) \Phi_{\tau}(2x - x_1, 2y - y_1)} dx dy . \quad (6.34)$$

Proof We need to evaluate the symplectic Fourier transform of the kernel $k = k_{\phi}^{\#}$ (6.30) to be substituted in (6.23). The transform can be explicitly calculated, this reduces to a

chain of manipulations with Gauss-type integrals. However, it is more inspiring to split symplectic Fourier transform in (x_1, y_1, x_2, y_2) into a composition of two commuting partial symplectic Fourier transforms in coordinates (x_1, y_1) and (x_2, y_2) , we denote them by $\overset{\sim}{\wedge}^1$ and $\overset{\sim}{\wedge}^2$ respectively. For $\overset{\sim}{\wedge}^1$, a calculation identical to (5.11) produces the convolution type expression:

$$\begin{aligned}\widehat{k}^1(x_1, y_1, x_2, y_2) &= \left(\widehat{\psi}(2x_1, 2y_1) K_{(x_2, y_2)}(x_1, y_1) \right)^{\sim 1} \\ &= \int_{\mathbb{R}^{2n}} \psi(x, y) \widehat{K_{(x_2, y_2)}}^1(x_1 - 2x, y_1 - 2y) dx dy.\end{aligned}$$

Now we shall use the following properties, which were already stated:

1. $K_{(x_2, y_2)} = \overline{\Lambda_{\hbar}(x_2, y_2) \Phi_{\tau}}$, cf. (3.9);
2. the symplectic Fourier transform propagates through the complex conjugation as follows: $\widehat{(\overline{f})}(x, y) = \widehat{f}(-x, -y)$;
3. the symplectic Fourier transform commutes with left shifts (3.31); and
4. the symplectic Fourier transform fixes the Gaussian: $\widehat{\Phi}_{\tau}(x, y) = \Phi_{\tau}(x, y)$, cf. (3.29).

A combination of those produces:

$$\begin{aligned}\widehat{K_{(x_2, y_2)}}^1(x_1, y_1) &= \overline{\Lambda_{\hbar}(x_2, y_2) \widehat{\Phi}_{\tau}(-x_1, -y_1)} \\ &= \overline{\Lambda_{\hbar}(x_2, y_2) \Phi_{\tau}(-x_1, -y_1)} \\ &= \Lambda_{\hbar}(-x_1, -y_1) \Phi_{\tau}(x_2, y_2).\end{aligned}$$

Then, the symplectic Fourier transform of k as the composition of $\overset{\sim}{\wedge}^1$ and $\overset{\sim}{\wedge}^2$ is:

$$\begin{aligned}\widehat{k}(x_1, y_1, x_2, y_2) &= \left(\widehat{k}^1(x_1, y_1, x_2, y_2) \right)^{\sim 2} \\ &= \left(\int_{\mathbb{R}^{2n}} \psi(x, y) \Lambda_{\hbar}(2x - x_1, 2y - y_1) \Phi_{\tau}(x_2, y_2) dx dy \right)^{\sim 2} \\ &= \int_{\mathbb{R}^{2n}} \psi(x, y) \Lambda_{\hbar}(2x - x_1, 2y - y_1) \Phi_{\tau}(x_2, y_2) dx dy,\end{aligned}$$

where we again use properties (3.31) and (3.29) and linearity of the symplectic Fourier transform. Thus, the PDO symbol in accordance with (6.23) is:

$$\begin{aligned}a_{\psi}^{\#}(x_1, x_2, y_1, y_2) &= \widehat{k}(x_1, y_1, \tau y_2, x_2/\tau) \\ &= \int_{\mathbb{R}^{2n}} \psi(x, y) \Lambda_{\hbar}(2x - x_1, 2y - y_1) \Phi_{\tau}(x_2, y_2) dx dy,\end{aligned}$$

because $\Phi_\tau(\tau y_2, x_2/\tau) = \Phi_\tau(x_2, y_2)$. To obtain the second form we swap variables using (2.21) and note that Φ_τ is an even function.

□

In a similar way (or by CAS computation) we can demonstrate:

Theorem 6.15 *The cross-Toeplitz operator $T_\psi : \mathcal{F}^\tau \rightarrow \mathcal{F}^\varsigma$ is unitary equivalent to PDO with the symbol:*

$$a_\psi^\#(x_1, x_2, y_1, y_2) = \int_{\mathbb{R}^{2n}} \psi(x, y) \Lambda_{\hbar}(2x - x_1, 2y - y_1) \Phi_{\tau\varsigma}(\varsigma y_2, x_2/\varsigma) dx dy \quad (6.35)$$

$$= \int_{\mathbb{R}^{2n}} \psi(x, y) \overline{\Lambda_{\hbar}(\varsigma y_2, x_2/\varsigma) \Phi_{\varsigma\tau}(2x - x_1, 2y - y_1)} dx dy. \quad (6.36)$$

Remark 6.16 We note that $a_\psi^\#$ contains all information from the Guillemin symbol a_ψ (5.8) since $a_\psi(x, y) = \hbar^{2n} a_\psi^\#(x, 0, y, 0)$. However, the important difference of the obtained formula (6.33) and the Guillemin result (5.8) is that the new symbol map $\psi \mapsto a_\psi^\#$ is based on the invertible transformation. In fact, (6.33) is a case of quadratic Fourier transform (aka short time Fourier transform) with the Gaussian window [76, Sect. 4.3.5].

To confirm the invertibility of the symbol map $\psi \mapsto a_\psi^\#$ we re-write (3.7) in order to express a similarity with (6.34) more explicitly:

$$\begin{aligned} a_\psi^\#(x_1, x_2, y_1, y_2) &= \int_{\mathbb{R}^{2n}} \psi(x, y) \overline{\Lambda_{\hbar}(\varsigma y_2, x_2/\varsigma) \Phi_{\varsigma\tau}(2x - x_1, 2y - y_1)} dx dy \\ &= \int_{\mathbb{R}^{2n}} \psi(x, y) \overline{[\Lambda_{\hbar}(s, x_l, y_l) \otimes R(0, x_r, y_r) \Phi_2](x, y)} dx dy \\ &= e^{\pi i \hbar (x_1 x_2 / \varsigma - \varsigma y_1 y_2) / 2} \\ &\quad \times \int_{\mathbb{R}^{2n}} \psi(x, y) \overline{[\widetilde{\Xi}_{\hbar}(0, x_l, \varsigma y_r, y_l, x_r / \varsigma) \Phi_2](x, y)} dx dy, \end{aligned} \quad (6.37)$$

where $\Phi_2(x, y) = \Phi_{\varsigma\tau}(2x, 2y) = \overline{\Phi_{\tau\varsigma}(2x, 2y)}$ and:

$$(s, x_l, x_r, y_l, y_r) = \frac{1}{4}(x_1 x_2 / \varsigma - \varsigma y_1 y_2, x_1 + 5\varsigma y_2, -x_1 + 3\varsigma y_2, 5x_2 / \varsigma + y_1, 3x_2 / \varsigma - y_1), \quad (6.38)$$

$$(x_1, x_2, y_1, y_2) = \frac{1}{8}(3x_l - 5x_r, \varsigma(y_l + y_r), 3y_l - 5y_r, (x_l + x_r)/\varsigma). \quad (6.39)$$

From (6.37) we can directly deduce the following

Corollary 6.17 *PDO symbol $a_\psi^\#$ (6.34) of the Toeplitz operator T_ψ is expressed through the FSB transform of ψ :*

$$\begin{aligned} a_\psi^\#(x_1, x_2, y_1, y_2) &= e^{\pi i \hbar(x_1 y_2 - y_1 x_2)} [W_{\Phi_2}^{\tilde{\Xi}_\hbar} \psi](g) \\ &= e^{\pi i \hbar(x_1 y_2 - y_1 x_2)} \langle \psi, \tilde{\Xi}_\hbar(g) \Phi_2 \rangle, \end{aligned} \quad (6.40)$$

where

$$g = \frac{1}{4}(0, x_1 + 5\varsigma y_2, 3x_2 - \varsigma y_1, 5x_2/\varsigma + y_1, -x_1/\varsigma + 3y_2).$$

This allows us to convert our knowledge of (pre-)FSB transform to information on the symbol map $\psi \mapsto a_\psi^\#$.

6.5 Characterising the kernels of cross-Toeplitz operators

Using the annihilation properties for the mixed Gaussian from Lem. 4.11 we observe from (6.31) that kernels of a two-sided convolution generated by a cross-Toeplitz operator vanished under some derived representation. Obviously,

$$L_{R,\tau}^+[2]k_\psi^\#(x_1, y_1, x_2, y_2) = 0,$$

where the index 2 above the ladder operator indicates that it acts on the pair (x_2, y_2) . Also, the identity $L_{\Lambda,\varsigma}^- \Phi_{\tau\varsigma} = 0$ implies that $k_\psi^\#(x_1, y_1, x_2, y_2)$ is annihilated under the family of operators

$$\begin{aligned} \Lambda_{\hbar}^2(x_1, y_1) \circ L_{\Lambda,\varsigma}^-[2] \circ \Lambda_{\hbar}^2(-x_1, -y_1) &= L_{\Lambda,\varsigma}^-[2] - L_{\Lambda,\varsigma}^-[1] - L_{R,\varsigma}^-[1] \\ &= L_{\Lambda,\varsigma}^-[2] + 2\bar{z}_1 I, \end{aligned} \quad (6.41)$$

for each fixed $(x_1, y_1) \in \mathbb{R}^{2n}$ (with the above agreement on the numeric supersets). Obviously, an operator from the family (6.41) is invariant under the right shift in (x_2, y_2) . Therefore we are naturally arriving to the following definition.

Definition 6.18 A kernel $k(x_1, y_1, x_2, y_2)$ on \mathbb{R}^{4n} is of *cross-Toeplitz type* if it is annihilated by

1. the left-invariant ladder operator $L_{R,\tau}^+[2]$; and
2. the right-invariant operators $L_{\Lambda,\varsigma}^-[2] - L_{\Lambda,\varsigma}^-[1] - L_{R,\varsigma}^-[1]$, which is equal to:

$$\Lambda_{\hbar,2}(x_1, y_1) \circ L_{\Lambda,\varsigma}^-[2] \circ \Lambda_{\hbar,2}(-x_1, -y_1) = L_{\Lambda,\varsigma}^-[2] + 2\bar{z}_1 I,$$

$$\text{where } z_1 = \sqrt{\frac{|h|}{2\tau}}(x_1 + i\varsigma y_1).$$

Lemma 6.19 Any differentiable cross-Toeplitz type kernel k is associated to a symbol ψ of Toeplitz operator for

$$\widehat{\psi}(2x_1, 2y_1) = \frac{k(x_1, y_1, x_2, y_2)}{[\Lambda_{\hbar}(x_1, y_1)\Phi_{\tau\zeta}](x_2, y_2)}. \quad (6.42)$$

Proof It is enough to show that the right-hand side of (6.42) is independent of variables (x_2, y_2) . Let $k_{(x_1, y_1)}(x, y) = k(x_1, y_1, x, y)$ and $\Phi_{(x_1, y_1)} = \Lambda_{\hbar}(x_1, y_1)\Phi_{\tau\zeta}$. Since $L_{R, \tau}^+ k_{(x_1, y_1)} = 0$ and $L_{R, \tau}^+ \Phi_{(x_1, y_1)} = 0$, we verify (omitting the parameter τ of the complexification) that:

$$\begin{aligned} 0 &= \frac{L_R^+ k_{(x_1, y_1)}}{\Phi_{(x_1, y_1)}} - \frac{k_{(x_1, y_1)} \cdot L_R^+ \Phi_{(x_1, y_1)}}{\Phi_{(x_1, y_1)}^2} \\ &= \frac{(z + \bar{\partial}_z)k_{(x_1, y_1)}}{\Phi_{(x_1, y_1)}} - \frac{k_{(x_1, y_1)} \cdot (z + \bar{\partial}_z)\Phi_{(x_1, y_1)}}{\Phi_{(x_1, y_1)}^2} \\ &= \frac{\bar{\partial}_z k_{(x_1, y_1)}}{\Phi_{(x_1, y_1)}} - \frac{k_{(x_1, y_1)} \cdot \bar{\partial}_z \Phi_{(x_1, y_1)}}{\Phi_{(x_1, y_1)}^2} \\ &= \bar{\partial}_z \left(\frac{k_{(x_1, y_1)}}{\Phi_{(x_1, y_1)}} \right). \end{aligned} \quad (6.43)$$

Similarly, from the second condition on right-invariant operators on cross-Toeplitz kernels we obtain that

$$\partial_z \left(\frac{k_{(x_1, y_1)}}{\Phi_{(x_1, y_1)}} \right) = 0, \quad (6.44)$$

with the complexification for the parameter ζ . Although, partial derivatives $\bar{\partial}_z$ in (6.43) and ∂_z in (6.44) have different parameters of complexification we can conclude that $k_{(x_1, y_1)}/\Phi_{(x_1, y_1)}$ is a constant for every fixed (x_1, y_1) . \square

Remark 6.20 Many results of this section remain true (with necessary adjustments) for the general localisation operators (5.5). For example, the kernel of a two-sided convolution representing $L_\psi = M_{\Theta_2} \circ \psi I \circ W_{\Theta_1}$ (5.5) is cf. (6.31):

$$k^\#(x_1, y_1, x_2, y_2) = (2/\hbar)^n \widehat{\psi}(2x_1, 2y_1) [\Lambda_{\hbar}(x_1, y_1)\Theta](x_2, y_2)$$

where $\Theta(x, y) = \langle \theta_1, \rho(x, y)\theta_2 \rangle = W_{R\Theta_1} \Theta_2(x, y)$.

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