# Comment on "Product states and Schmidt rank of mutually unbiased bases in dimension six"

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**Abstract**

A lemma by Chen *et al.* [*J. Phys. A: Math. Theor.* **50**, 475304 (2017)] provides a necessary condition on the structure of any complex Hadamard matrix in a set of four mutually unbiased bases in $\mathbb{C}^6$. The proof of the lemma is shown to contain a mistake, ultimately invalidating three theorems derived in later publications.

It is unknown whether complete sets of $(d+1)$ mutually unbiased (MU) bases exist in a complex Hilbert space $\mathbb{C}^d$ if the dimension $d \in \mathbb{N}$ does *not* equal the power of a prime. The simplest case of this long-standing open problem arises when $d = 6$, the smallest composite dimension that is not a prime-power. It has been conjectured that no more than three MU bases exist for $d = 6$. In the space $\mathbb{C}^6$, any orthonormal basis MU to the standard basis corresponds to a complex Hadamard matrix of order six. A potential strategy to prove the conjecture is to derive an exhaustive[1] list of $6 \times 6$ Hadamard matrices and show that none of them can be part of a *quadruple* of MU bases.

Along these lines, Chen *et al.* [2] have ruled out the existence of quadruples of MU bases that contain Hadamard matrices of a specific form. Unfortunately, one part of a lemma they present is marred by an erroneous proof.

**Lemma 1** ([2], Lemma 11(v) Part 6)**.** *If a set of four MU bases in dimension six exists, none of the Hadamard matrices from the set contains a real* $3 \times 2$ *submatrix.*

In its original formulation, the lemma considers a "MUB trio", which is a set of three mutually unbiased Hadamard matrices. Equivalently, we consider a set of four MU bases and assume that one basis is the standard one.

The proof proceeds by contradiction: a set of four MU bases is assumed to exist that contains both the identity matrix $\mathbb{I}$ and a complex Hadamard matrix $H$ with a real submatrix of size $3 \times 2$. By applying suitable row and column permutations to the four MU bases, as well as rephasing individual basis vectors, one can ensure that the first row and column of $H$ have entries all equal to $1/\sqrt{6}$ and that its upper left $3 \times 2$ matrix is real, i.e.

$$\frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 1 \\ 1 & y \\ 1 & x \end{pmatrix}, \qquad y, x \in \{\pm 1\}. \tag{1}$$

Let $y = 1$. If the submatrix (1) has rank one, which occurs for $x = 1$, the matrix $H$ cannot be a member of a MU quadruple, as shown in Part 2 of Lemma 11(v) of Ref. [2]. In this case, the matrix $H$ must contain a $3 \times 3$ unitary matrix. If it does, the pair $\{\mathbb{I}, H\}$ would be equivalent to a pair of MU

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[1]A (not necessarily exhaustive) list of the known complex Hadamard matrices of order six can be found in the online catalogue [1]. It contains the definitions of the Hadamard matrices appearing throughout this note, such as $D_6^{(1)}$, $B_6^{(1)}$, and $M_6^{(1)}$.

product bases which, however, cannot be extended to four MU bases as shown in [3, 4]. Therefore, one must have $x = -1$. If $y = -1$, both choices $x = \pm 1$ also lead, upon suitably permuting rows and rephasing the second vector, to the case $(y, x) = (1, -1)$.

Now, given $(y, x) = (1, -1)$, orthogonality of the first two columns of $H$ implies that the last three elements of the second column must be given by $(-1, s, -s)/\sqrt{6}$, where $s$ is a complex number of modulus one. The third column vector (labelled $v$) of $H$ must be orthogonal to both the first and the second column of $H$. According to Ref. [2], these conditions on the vector $v$ imply that its "third and sixth elements must be zero" [2, p. 24]. Since $H$ had been assumed to be a Hadamard matrix, a contradiction is reached that is sufficient to complete the proof of Lemma 1. However, we are unable to confirm that two components of the vector $v$ must vanish. Therefore, it is not possible to rule out the existence of MU quadruples containing Hadamard matrices with real $3 \times 2$ submatrices.

An example shows explicitly that the final step of the argument cannot be correct. Consider the one-parameter family of *symmetric* Hadamard matrices

$$M_6^{(1)} \equiv M_6(a) = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & a & a & -a & -a \\ 1 & a & b & c & d & e \\ 1 & a & c & b & e & d \\ 1 & -a & d & e & f & g \\ 1 & -a & e & d & g & f \end{pmatrix}, \tag{2}$$

where $a = e^{it}$ and $t \in (\pi/2, \pi] \cup (3\pi/2, 2\pi]$. The entries $b, c, d, e, f$ and $g$ are functions of $a$ defined in Ref. [1]. Multiplying the second column of this matrix by the complex conjugate of $a$ produces a real $4 \times 2$ submatrix in the lower left corner. Suitable row permutations turn the upper left $3 \times 2$ matrix into the expression in (1) with $(y, x) = (1, -1)$. Multiplying the remaining columns by suitable phase factors, we arrive at a matrix $H$ that, according to the analysis given in the previous paragraph, cannot exist. In addition, there is no proof that excludes the matrices $M_6(a)$ from appearing in a set of four MU bases (although numerical evidence suggests that for some values of the parameter $a$, pairs of the form $\{\mathbb{I}, M_6(a)\}$ cannot even be extended to a triple of MU bases [5]). Note that $M_6(a)$ cannot, in general, contain three columns that form product vectors, including after row and column permutations, therefore Part 4 of Lemma 11(v) of Ref. [2] does not apply.

We are aware of at least three theorems on the existence of MU quadruples that build on the now unverified lemma, derived in Refs. [6, 7, 8]. Thus, their validity is called into question. As before, we assume that any set of four MU bases contains the standard basis.

**Theorem 1** ([6], Theorem 10)**.** *If a set of four MU bases in dimension six exists, none of the Hadamard matrices from the set contains more than 22 real entries.*

The proof of this theorem explicitly uses Lemma 1 to exclude Hadamard matrices with more than 22 real entries from appearing in quadruples of MU bases.[2]

A restriction on the type of $H_2$-reducible Hadamard matrices that are permitted in MU quadruples was derived in Ref. [7]. A complex Hadamard matrix of order six is *$H_2$-reducible* if it can be partitioned into nine $2 \times 2$ blocks, each proportional to a Hadamard matrix of order two. The complete set of $H_2$-reducible matrices is known as the three-parameter *Karlsson* family (cf. [1]).

**Theorem 2** ([7], Theorem 12 & Lemma 13)**.** *A $H_2$-reducible matrix in a set of four MU bases contains exactly nine or eighteen $2 \times 2$ submatrices proportional to Hadamard matrices.*

This claim relies directly on Lemma 1: the non-existence of specific submatrices of size $3 \times 2$ is used to limit the number of $2 \times 2$ Hadamard submatrices.[3]

Finally, a theorem in Ref. [8] further limits the number of $2 \times 2$ Hadamard submatrices of any $H_2$-reducible matrix in a quadruple of MU bases. This property is then used to severely restrict the types of Hadamard matrices that may figure in such a set.

[2]Some of the authors of [6] have privately communicated an alternative proof of Thm. 1 that is independent of Lemma 1 and awaits publication.

[3]The proof of Thm. 2 also contains an inconsistency unrelated to Lemma 1. Consider, for example, the one-parameter family of self-adjoint Hadamard matrices $B_6^{(1)}$. Members of this family contain no $3 \times 2$ real submatrix but more than eighteen $2 \times 2$ Hadamard submatrices, as can be seen by inspection. However, by the argument applied to prove Thm. 2 (which relies only on Lemma 1 to restrict the bases), $B_6^{(1)}$ should not be excluded from appearing in an MU quadruple. It is therefore feasible—regardless of the veracity of Lemma 1—that an MU quadruple contains a Hadamard matrix with more than eighteen $2 \times 2$ Hadamard submatrices.

**Theorem 3** ([8], Theorems 7, 8 & 9). *A $H_2$-reducible matrix in a set of four MU bases contains exactly nine $2 \times 2$ submatrices each proportional to a Hadamard matrix. Thus, members of the families $D_6^{(1)}$, $B_6^{(1)}$, $M_6^{(1)}$ and $X_6^{(2)}$ do not figure in a quadruple of MU bases.*

Thm. 2 and Lemma 1 are required to prove Thm. 3. Hence, the restrictions claimed in Thm. 3 cannot be upheld.

Without a proof of Thm. 3, only a few constraints on the types of complex Hadamard matrices that may figure in an MU quadruple remain known. The isolated matrix $S_6$ does not extend to a triple, let alone a quadruple. Quadruples do not contain members of the two-parameter Fourier family $F_6^{(2)}$, as shown rigorously by a combination of analytic estimates and numerical evidence [9] as well as a proof using Delsarte's bound [10]. To the best of our knowledge [11], no non-existence proofs for other families are known.

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