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Optimal spatial-temporal capacity allocation for morning commute with carpool considering parking supply constraint

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Abstract

Allocating urban highway capacity to prioritise carpooling traffic is a well-studied subject. However, one area of limited research so far is on the impact of parking constraint at workplace on its spatial-temporal design. This paper examines a multi-modal morning commute scenario, where commuters can freely choose solo driving, carpooling, or public transit to go to work. The highway bottleneck capacity is shared between a general-purpose lane for all drivers and a time-limited carpool lane for carpooling vehicles only. Analytical derivations of all possible commute patterns are provided, considering different parking supply levels and capacity allocation strategy. Results show that all bottleneck capacity should be allocated to carpool lane spatially, but the optimal temporal allocation of carpool lane may not be unique for minimizing total trip cost. This implies, the accurate estimation of extra carpool cost to optimize the temporal lane reservation may be no longer necessary for the traffic authority due to the limited parking supply, whilst it is really required when the parking supply is abundant. We also incorporate the traffic authority's social cost budget for both temporal and spatial allocations into the optimization problem, explaining why general-purpose lanes are still prevalent in reality. The research has implications on policy makers as well as workplace parking managers by providing them a framework to design optimized road-space allocation with workplace parking supply.

Keywords: Bottleneck model; parking supply; capacity allocation; carpooling

1. Introduction

With increasing demand for travel and traffic congestion, especially in urban areas, there has been a major shift in transport planning away from building more roads and towards using technology to better manage traffic

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and to influence travel behavior. One form of traffic management policies is lane reservation, a strategy employed to provide dedicated road space to selected road users. The strategy works by converting some of the existing general purpose (GP) lanes into specified lanes for certain vehicle/user types, such as carpooled cars, buses and/or high occupancy vehicles (HOV). The terms, carpool lane, bus lane, and HOV lane, are examples of such converted lanes designed to improve travel time savings or reliability of those specified users and have been utilized in many cities around the world (Stomos et al., 2012; Fosgerau, 2011; Yuan et al., 2024). Workspace parking management, by limiting parking spaces at work, is another strategy to encourage more sustainable travel, by using public transport, or car-sharing (carpooling) (Zhou et al., 2023; Wang et al., 2019).

Both strategies aim at restricting travel by solo driving, and promoting the use of more sustainable modes of transport. It is conceivable that, when both strategies are put in place, their combined impact on commuters' travel choices can be complex. Understanding such complex behavioral responses would help the policy makers to best design and optimize the combined strategies and address questions such as: how to incorporate parking management in the design of highway lane reservation? and what would be their impact on travelers' choices and on network performance? To our best knowledge, however, existing studies tend to deal with lane reservations and parking management as separate and individual strategies and have not addressed the two strategies in combination, including earlier works on modelling a time-limited carpooling lane (Xiao, et al., 2021b; Wei et al., 2022).

This paper addresses the combined effect of highway lane reservation and workplace parking management on the commuters travel choices and on the system performances. To realistically represent the travel behavior and traffic management scenarios described above, we establish a mathematical model of a typical commute corridor, based on the classical bottleneck formulation. Along the corridor, there is a congested highway whose capacity can be shared between a GP lane and a carpooling lane, either on an all-time basis or for a time-limited carpool lane. Running in parallel with the highway is a public transit line whose cost varies with the crowding conditions on the line. The commuters can choose to travel as solo driver, in carpool, or by public transit, and they choose their departure times to minimize travel and schedule delay costs, as well as cost in case of failed to park. The bottleneck model is based on the framework developed in Xiao et al. (2021b) for modeling carpool lanes, and is extended in this paper to include the crowded public transit line and constraint in workplace parking supply.

From the newly developed bottleneck model, we derive analytically the morning commute patterns for the combination of spatial-temporal highway capacity reallocation with parking supply constraint. There exist the complex variations of the commute patterns with regard to the different spatial-temporal designs of the highway capacity to carpool lane, and to the different levels of parking supply constraint. Based on the derived commute patterns, we analytically investigate the optimization problem of spatial-temporal capacity allocation according to the total trip cost, and discuss their implications on policy designs on carpooling and parking management strategies. It is found that the entire bottleneck capacity should be spatially allocated to carpool lane, and the optimal time allocation for carpool lane may not be unique, if only total trip cost of all commuters is minimized.

This implies, the accurate estimation of extra carpool cost to optimize the temporal lane reservation may be no longer necessary for the traffic authority due to the limited parking supply (the underestimation of extra carpool cost can be allowed at optimum), whilst it is really required when the parking supply is abundant as analyzed in Xiao et al. (2021b). Considering the spatial-temporal capacity allocation and operations require large social efforts or financial support, we also incorporate the traffic authority's social cost budget for both temporal and spatial allocations into the optimization problem, explaining why GP lanes are still prevalent in reality. The research has implications on policy makers as well as workplace parking managers by providing them a framework to design optimized road-space allocation with workplace parking supply.

The rest of this paper is organized as follows. A literature review focusing on carpooling and parking management are presented in Section 2. The multi-mode morning commute setting, spatial-temporal bottleneck capacity allocation scheme and corresponding commute patterns without parking supply constraint are briefly outlined in Section 3. Sections 4 investigates the impacts of parking supply constraint on a temporal-only bottleneck capacity allocation and a joint spatial-temporal one for managing the multi-mode morning commute. The optimal spatial-temporal bottleneck capacity allocation is discussed in Section 5 with the constraints of the traffic authority's social cost budget and parking supply. Section 6 concludes the paper.

2. Related works

This paper examines two traffic management strategies: carpooling and parking management. There have been extensive studies on the two subjects, although separately as individual strategies. In this section, we review the key literature and research focuses reported on each.

(1) Carpooling

By reducing the proportion of single occupant vehicles on the road, carpooling can be an effective measure to alleviate traffic congestion and parking scarcity (Yang and Huang, 1999; Xiao et al., 2024a; Tang et al., 2021). However, empirical studies have showed that carpooling as a measure on its own is found to be less prevalent than would be expected, due to a range of factors including incompatible work schedules between the carpool people, lack of independence and privacy, and loss of convenience (Ferguson, 1997). To enhance the attractiveness of carpooling, highway authorities have started to introduce exclusive high occupancy vehicle (HOV) or carpool lanes on some urban highways, whereby carpoolers' travel time savings and reliability can be directly improved (Menendez and Daganzo, 2007; Xiao et al., 2021b). The benefits of HOV or carpool lane are clear, although questions have also been raised on whether HOV lanes fulfill the objective to attract solo commuters to carpooling or create new demand for driving alone (e.g., Giuliano et al., 1990).

The ridesharing effects of HOV lanes have been examined by a number of theoretical studies, even in conjunction with other demand management measures such as congestion pricing. Yang and Huang (1999) considered two-person carpool and solo-driving and analyze the optimal congestion pricing to incentive carpool. Qian and Zhang (2011) examined a multi-mode morning commute problem, and found that congestion pricing has a significant impact on travel decision making among peak period commuters. When the parking provision is restricted, Xiao et al. (2016) considered a one-to-one corridor, in which commuters could choose driving alone, carpooling or public transit. The commute patterns considering the extra carpool cost were determined, and the optimal spatial capacity allocation between a carpool lane and a GP lane was investigated. Furthermore, Xiao et al. (2021a) extended to a many-to-one network, and two different tradable parking permit strategies were proposed to manage the multi-mode morning commute problem.

Often in practice, carpooling/HOV lanes are operating during a time limited window in order to maximize lane utilization. For example, the carpool lane in operation in the city of Shenzhen, China, is operating during two 2-hour peak periods between 7:30-9:30 am and between 5:30-7:30 pm only. Xiao et al. (2021b) developed a bottleneck model to analyze such a time-limited carpooling scheme, also known as spatial-temporal allocation of bottleneck capacity. They showed that to optimize the spatial-temporal capacity allocation can be designed as long as we have accurate estimation of the additional cost of carpooling. What Xiao et al. (2021b) have not considered, however, is the impact of workplace parking constrains on the design of capacity allocation.

The carpool model introduced in the next section is based on that in Xiao et al. (2021b), i.e., a time-limited carpool lane and a GP lane share the bottleneck capacity. The difference is that, in the current paper, a crowded public transit line is assumed to run in parallel with a bottleneck constraint highway along the corridor connecting the residential to the workplace. Moreover, this paper will consider parking supply constraint on the travel choices of the commuters and the resulted commute patterns. The focus is on the combined effect of carpooling and parking management, and we provide policy insights on how to optimize the spatial-temporal capacity allocation in a parking scarcity environment.

(2) Parking management

Many cities suffer from a chronical shortage of parking spaces in downtown areas, due to the rapid growth of vehicle trips and land values (Shoup, 1999). There have been various parking management strategies proposed to counter this problem (Zhou et al., 2024). These strategies can be broadly categorized into two methods.

The first method aims to maximize the usage of existing parking facilities. Current shared parking (also known as GA parking from gap and parking; From Wikipedia, the free encyclopedia) emerged recently as a new notion of making more efficient use of parking facilities, can significantly improve the overall parking supply turnover rate, reduce the parking demand, and maximize the use of parking spaces (Ardeshiri et al., 2021; Jiang and Fan, 2020; Xiao et al., 2020). Various advanced parking management services, such as parking information, navigation, and reservations, have been designed to help drivers find parking spots quickly. The information service allows drivers to get real time parking vacancy and prices of parking spaces, the navigation service provides convenience and optimize the driving routes to them (Lu et al., 2024), and the reservation service improves the efficiency by providing parking spots reservation before departure or on the go (Liu et al., 2014). Yang and Lam (2019) conduced a driver perception survey on the effect of parking guidance, and they found parking time and parking App usage have positive impacts on Willingness-To-Pay, whilst the influence of driving experience inversion.

The second method is to reduce the demand for parking by reducing the overall vehicles on the road. This important approach includes ridesharing, ride-hailing and public transit service improvements to encourage drivers to reduce automobile ownership and use, and therefore parking demands (see a review by Litman (2006)). Road pricing policies that levy charges for the use of private cars on the road network have also been promoted (Arnott and Inci, 2010; Xiao et al., 2024a), and have been shown to improve welfare and traffic efficiency and meanwhile alleviate competition for limited parking resources (Zhang et al., 2011; Yang et al., 2013).

This paper follows the second method, by reducing or limiting the road space/capacity set for solo drivers in order to manage parking demand. More specifically, we explore ways to optimize the spatial-temporal capacity allocation while considering parking constraint and investigate the impact of parking competition on the commuting behavior of solo drivers and carpoolers.

3. Multi-modal commuting under spatial-temporal capacity allocation

We consider a multi-modal morning commute scenario, where the commuters travelling along a corridor can choose to travel by public transport, solo driving, or carpooling. The highway capacity along the corridor will undergo spatial-temporal capacity allocation strategy. At their destination, there is a limited parking supply. The commuters choose their mode and departure-time, to minimize their overall trip cost and the risk of not finding a parking space. The modeling framework is built on that developed by Xiao et al. (2021b), but extended in this paper to incorporate public transit line as an additional commute mode, and to include destination parking-supply constraint.

3.1. Notation and variables

Table 1 summarizes the notation and variables used throughout this paper.

Table 1. List of notations

Variables	Definitions
α	Value of travel time
β	Value of schedule delay early
γ	Value of schedule delay late
T^w	Waiting time at the bottleneck
T^f	Free-flow travel time is set to be zero
N	Number of commuters
W_f , N_f	Number of vehicles with parking supply constraint or not ($W_f < N_f$)
W_a , N_a	Number of auto commutes with parking supply constraint or not
W_s , N_s	Number of vehicles for solo-driving with parking supply constraint or not
W_c , N_c	Number of vehicles for carpooling with parking supply constraint or not

k	Average number of carpoolers per vehicle
θ	Ratio ($0<\theta<1$)
s	Bottleneck capacity
t	Time
t^*	Work time at the CBD
t_i^o , t_i^e	Arrival time of first/last commuter for role i , $i=s,c$
t^+ , t^-	Starting/ending time for carpool lane temporal reservation
Δ	Extra carpool cost
Δ_x	Estimated extra carpool cost
C	Trip cost for auto users
H	Trip cost for transit commuters
TTC	Total trip cost

It is assumed that N commuters travel from home (a residential area) to the CBD each morning with three modes to choose from: taking the public transit line which has a dedicated right-of-way, carpooling, or solo driving. If driving (either solo or carpooling), they experience bottleneck congestion and spatial-temporal road capacity restrictions. Commuters are assumed to have an identical desired arrival time, t^*, and early or late arrival at their destination will be penalized. Define Δ is the extra carpool cost, i.e., an integration cost considering the disadvantages and advantages of carpooling, to differentiate the travel cost experienced by solo driving and carpooling[3].

Let s be the bottleneck capacity on the highway, which is spatially divided by a carpool lane and a GP lane. For simplicity, we denote θs as the capacity of the carpool lane with $\theta\in[0,1]$, thus $(1-\theta)s$ is the capacity of the GP lane. Further, let N_s and N_c denote the number of vehicles for solo driving and carpooling, respectively. Thus, we easily have the numbers of auto commuters and total vehicles, $N_a=kN_c+N_s$ and $N_f=N_s+N_c$, respectively. Then the number of transit commuters is $N_t=N-N_a$.

3.2. Trip costs

Under the time-limited carpooling scheme, the carpool lane is assumed to be reserved for carpoolers between t^- and t^+ according to the estimated extra carpool cost Δ_x, which may either be smaller or larger than the actual one Δ. Beyond the time-limited window, all commuters are allowed during the peak period (interested readers are referred to Xiao et al. (2021b) for detailed description of the scenario settings). The trip cost of an auto commuter consists of travel time cost, schedule delay cost, and an extra carpool cost if they are carpooling. Departing at time t, the trip costs for solo drivers and carpoolers are defined as, respectively

[3] The extra carpool cost is assumed to be constantly positive, thus we focus on the situation that all auto commuters would choose solo driving when there is no capacity for carpool lane. The reader can refer to Xiao et al. (2016, 2021b) for more situations and discussions.

$$C_s(t) = \alpha \cdot T^w(t) + \beta \cdot \max\{0, t^* - t - T^w(t)\} + \gamma \cdot \max\{0, T^w(t) + t - t^*\}, \tag{1}$$

$$C_c(t) = \alpha \cdot T^w(t) + \beta \cdot \max\{0, t^* - t - T^w(t)\} + \gamma \cdot \max\{0, T^w(t) + t - t^*\} + \Delta_x^+, \tag{2}$$

where $\Delta_x^+ = \max\{\Delta_x, \Delta\}$ and $\beta < \alpha < \gamma$ based on empirical evidence (Small, 1982).

The number of solo-driving vehicles that travel on the GP lane and those that travel on the carpool lane but outside of carpool-limited period are denoted by N_s^1 and N_s^2, respectively, and $N_s = N_s^1 + N_s^2$. Based on separated waiting (SW) behavioral assumption (Laih, 1994; Laih, 2004), we can get $N_s^2 = \Delta_x s/\delta$ with $\delta = \beta\gamma/(\beta+\gamma)$. According to ADL model (Arnott et al. 1990), the auto trip costs for solo driving and carpooling are $C_s = \delta N_s^1/((1-\theta)s)$ and $C_c = \delta N_c/(\theta s) + \Delta_x^+$, respectively.

The trip cost of riding on the public transit is assumed to be a strictly increasing function of the number of transit users, e.g., $H = H(N_t)$ and $N_t = N - N_a$. If both lanes on the highway are used, the auto commuters N_a, N_s^1 and N_c are determined by

$$H(N_t) = C_s = C_c, \; N_t = N - N_a \text{ and } kN_c + N_s^1 + N_s^2 = N_a. \tag{3}$$

To guarantee that some carpoolers would use the carpool lane, the value of the extra carpool cost Δ_x must be smaller than a critical value Δ_c, where $\Delta_c = H(N - s\Delta_x/\delta)$. Xiao et al. (2016) investigated the spatial-only capacity allocation case, in which solo drivers are not allowed to use the carpool lane during the entire rush hours, i.e., $N_s^2 = 0$. It should be noticed that, the introduction of transit as an alternative mode to solo-driving and carpooling is more natural although it has little influence on the commute patterns of auto users, since the parking supply is limited at workplace in the following sections.

Let W_f be the parking supply at workplace. Obviously, if W_f is less than the potential parking demand, N_f, auto commuters have to compete for parking, by changing their departure time and/or commute mode. In the following section, we will focus on the equilibrium analysis with parking supply constraint to derive the commute patterns and explore the effects of spatial-temporal designs on them.

4. Commute patterns with parking supply constraint and effects of spatial-temporal designs

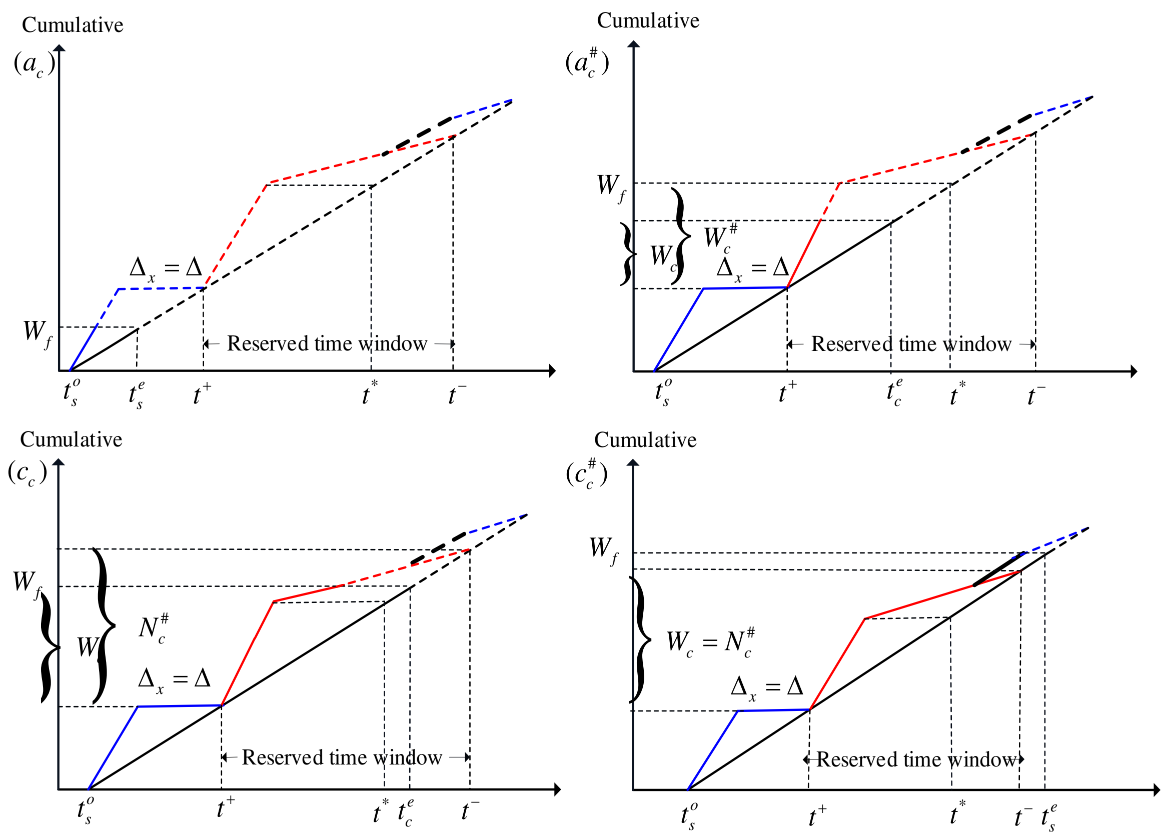
Define W_s and W_c as the numbers of solo-driving and carpooling vehicles, respectively. Let W_a be the total number of auto users travelling on the highway, then $W_a = W_s + kW_c$. It can be expected that, some auto users may have to switch to the transit mode due to the parking supply constraint. For the convenience of analyzing commute patterns, giving parking supply W_f, we first define the four reference numbers. Let $W_s^{\#}, W_c^{\#}$ be the respective numbers of solo-driving vehicles on the GP lane and carpooling vehicles on the carpool lane which can get through the bottleneck before t^*. And use $N_s^{\#}$ and $N_c^{\#}$ as the numbers of solo-driving vehicles on the

GP lane and carpooling vehicles on the carpool lane at the virtual standard bottleneck commuting equilibrium without parking supply constraint (See Appendix A.1 for reference number formulas).

4.1. Commute patterns with parking supply constraint

4.1.1 A special case

For special case that $\theta = 1$ and $\Delta_x = \Delta$, i.e., the highway capacity is full designed to the time-limited carpool lane. The dashed lines in Fig. 1 simply plot the virtual extensions of cumulative departures and arrivals. The four commute patterns, (a_c), $(a_c^\#)$, (c_c) and $(c_c^\#)$ are differentiated by the given parking supply W_f. Obviously, when the parking supply is very little that $W_f \le W_0 = \Delta_x \theta s/\beta$, there is no carpooler and the last commuters arrive earlier than the preferred arrival time, illustrated in Fig. 1 in Pattern (a_c). If $W_f > W_0$, according to the relationships among W_c, $W_c^\#$ and $N_c^\#$, three scenarios are further depicted in Fig. 1. For pattern $(a_c^\#)$, i.e., $W_c \le \bar{W}_c^\#$, all carpoolers would arrive early at workplace. For pattern (c_c), i.e., $\bar{W}_c^\# < W_c < \bar{N}_c^\#$, some carpoolers endure the late arrivals with queue. For pattern $(c_c^\#)$, i.e., $W_c = \bar{N}_c^\#$, some carpoolers endure the late arrivals but the last carpoolers arrive with no queue[4].



[4] Based on separated waiting (SW) behavioral assumption (Laih, 1994; Laih, 2004), in Fig 1 in pattern $c_c^\#$ the black line represents the collection of solo drivers and carpoolers, while the red line indicates carpoolers.

Fig. 1. Possible commute patterns for $\Delta_x = \Delta$ and $\theta = 1$.

4.1.1 General case

A more general allocation scheme is to simultaneously consider the temporal and spatial characteristics of morning commute in allocating the highway capacity. This corresponds to the scenario with given Δ_x (either smaller or larger than Δ) and $0 < \theta < 1$. If the parking supply is unable to accommodate the potential demand, $W_f \le N_f$, there are, in total, four possible commute patterns, i.e., $(a_g)+(a_c)$, $(a_g)+(a_c^{\#})$, $(c_g)+(c_c)$ and $(c_g)+(c_c^{\#})$ for the multi-mode case where transit and both lanes may be used. As before, subscripts g is for GP lane and c is for carpool lane, respectively. Table 2 summarizes the respective required conditions for each of the four commute patterns[5].

Table 2. Conditions and multi-mode equilibrium commute patterns under spatial-temporal allocation scheme.

	Patterns	Conditions	
Multi-mode: Transit and both lanes	$(a_g)+(a_c)$	$0 < W_f \le W_0$	$H(N-W_f) < \Delta_x^+$
	$(a_g)+(a_c^{\#})$	$W_0 < W_f \le W_1$	
	$(c_g)+(c_c)$	$W_1 < W_f \le W_2$	$H(N-W_f) \ge \Delta_x^+$
	$(c_g)+(c_c^{\#})$	$W_2 < W_f \le W_3$	

Note: $W_0 = \left(\left(1-\theta\right)\Delta_x^+ + \theta\Delta_x\right)s/\beta$, $W_1 = \tilde{W}_s^{\#} + \Delta_x\theta s/\beta + \tilde{W}_c^{\#}$, $W_2 = W_f^{\#}$, $W_3 = N_f$.

4.2. Combined effect of spatial-temporal designs

As seen before, the occurrence of each of the four commute patterns in Table 2 depends on the different values of Δ_x and θ. Hereafter, with given W_f, we define $\Delta_0(\theta)$, $\Delta_1(\theta)$, $\Delta_2(\theta)$ and $\Delta_3(\theta)$ as the watershed lines to distinguish the four different patterns from each other. The detailed derivation of the watershed lines can be found in Appendix A.

Pattern $(a_g)+(a_c)$

This is the commute pattern where the last solo drivers arrive earlier than t^* and there is no carpooler. In this pattern, $W_c = 0$, and the parking supply W_f should be smaller than $W_0 = \left(\left(1-\theta\right)\Delta_x^+ + \theta\Delta_x\right)s/\beta$. Thus, we get the watershed line

[5] For GP lane, (a_g) indicates that all solo drivers arrive before t^*, whilst (c_g) means that some solo drivers arrive after t^*.

$$\Delta_0(\theta)=\begin{cases}\beta W_f/(\theta s)-(1-\theta)\Delta/\theta, & \text{if } 0\le\Delta_0<\Delta,\\ \beta W_f/s, & \text{if } \Delta\le\Delta_0\le\Delta_a,\end{cases} \tag{4}$$

where the critical constant $\Delta_a = H(N - s\Delta_a/\beta)$, with which only solo drivers pass the bottleneck and there is just no carpooler, i.e., $W_f = W_s = s\Delta_a/\beta$. For pattern $(a_g)+(a_c)$, all solo drivers arrive early, and thus the trip cost equals to $\beta W_f/s$, which is not larger than Δ_x^+. For $0\le\Delta_0<\Delta$, using Eq. (4), we can get the watershed line $\Delta_0(\theta)$ increasing with $\theta\in[\theta_0,1]$ and $\theta_0 = 1-\beta W_f/(s\Delta)$, whist for $\Delta\le\Delta_0<\Delta_a$, the watershed line $\Delta_0(\theta)$ is constant in $[0,1]$.

Pattern $(a_g)+(a_c^\#)$

The commute pattern that the last carpoolers and solo drivers both arrive earlier than the desired arrival time, i.e., t^*. Then the boundary condition is that the parking supply should satisfy $W_1=\tilde{W}_g^\#+\Delta_x s_h/\beta+\tilde{W}_c^\#$. Using the reference numbers defined in Section 4.1, we can get $\min_{\theta,\Delta_x} W_1=\tilde{W}_c^\#\big|_{\theta=1,\Delta_x=0}=\bar{W}_c^\#$ and $\max_{\theta,\Delta_x} W_1=\tilde{W}_s^\#\big|_{\theta=0}=\bar{W}_s^\#$. Hence, given W_f, the watershed line $\Delta_1(\theta)$ should satisfy

$$W_f=\frac{s}{\beta}H\left(N-W_f-(k-1)\tilde{W}_c^\#\right)+\frac{\theta s}{\beta}\left(\Delta_x-\Delta_x^+\right). \tag{5}$$

According to Eq. (5), if $W_f\in[\bar{W}_c^\#,\bar{W}_s^\#]$, there exists a capacity allocation ratio θ_1 to make the watershed line $\Delta_1(\theta_1)=0$. For $W_f\in[W_1^\#,\bar{W}_s^\#)$ with $W_1^\#=\bar{W}_c^\#+s\Delta/\beta$, there exists a capacity allocation ratio $\theta_1^\#$ to make the watershed line $\Delta_1(\theta_1^\#)=\Delta$. Specially, when M_f equals to $M_1^\#$, then $\Delta_1(1)=\Delta$. We discuss the properties of $\Delta_1(\theta)$ in Proposition 1 (See Appendix A.2 for detailed proofs).

Proposition 1. At multi-modal equilibrium with spatial-temporal capacity allocation,

(i) For the parking supply $W_f\in[\bar{W}_c^\#,W_1^\#]$, the watershed line $\Delta_1(\theta)$ increases with θ and it is no more than the actual extra carpool cost in $[\theta_1,1]$, i.e., $\mathrm{d}\Delta_1/\mathrm{d}\theta>0$ and $\Delta_1(\theta)\le\Delta$, with $\theta\in[\theta_1,1]$.

(ii) For the parking supply $W_f\in(W_1^\#,\bar{W}_s^\#]$, the watershed line $\Delta_1(\theta)$ increases with θ in $(\theta_1,1]$, i.e., $\mathrm{d}\Delta_1(\theta)/\mathrm{d}\theta>0$ for $\theta\in(\theta_1,1]$.

One can easily deduce from Proposition 1 that there exists a capacity allocation ratio $\theta_1^\#$ to make the watershed line $\Delta_1(\theta_1^\#)=\Delta$, and thus we can get $\Delta_1(\theta)<\Delta$ for $\theta\in[\theta_1,\theta_1^\#)$ and $\Delta_1(\theta)\ge\Delta$ for $\theta\in[\theta_1^\#,1]$.

Pattern $(c_g)+(c_c)$

The commute pattern that the last commuters from both lanes arrive late and experience queue. Then, the boundary condition is that the parking supply satisfy $W_2=\tilde{N}_s^*+\theta s\Delta_x/\beta+\tilde{N}_c^\#$. Using the reference numbers defined in Section 4.1, we can get $\min_{\theta,\Delta_x} W_2=\tilde{N}_c^\#\big|_{\theta=1,\Delta_x=0}=\bar{N}_c^\#$ and $\max_{\theta,\Delta_x} W_2=\tilde{N}_s^*\big|_{\theta=0,\Delta_x=\Delta}=\bar{N}_s^*$. Hence, given W_f, the watershed line $\Delta_2(\theta)$ should satisfy

$$W_f=\frac{s}{\delta}H\left(N-W_f-(k-1)\tilde{N}_c^\#\right)+\frac{\Delta_x^+(1-\theta)s+\Delta_x\theta s}{\beta}-\frac{\Delta_x^+ s}{\delta}. \tag{6}$$

According to Eq. (6), for $W_f\in[\bar{N}_c^\#,\bar{N}_s^*]$, there exists a capacity allocation ratio θ_2 to make the watershed line $\Delta_2(\theta_2)=0$. For $W_f\in[W_2^\#,\bar{N}_s^*)$ with $W_2^\#=\bar{N}_c^\#+s\Delta/\beta$, there exists a capacity allocation ratio $\theta_2^\#$ to make the watershed line $\Delta_2(\theta_2^\#)=\Delta$. Specially, when W_f equals to $W_2^\#$, then $\Delta_2(1)=\Delta$. We can accordingly get Proposition 2 (See Appendix A.3 for detailed proofs).

Proposition 2. At multi-modal equilibrium with spatial-temporal capacity allocation,

(i) For the parking supply $W_f\in[\bar{N}_c^\#,W_2^\#]$, the watershed line $\Delta_2(\theta)$ increases with θ and it is no more than the actual extra carpool cost in $[\theta_2,1]$, i.e., $\mathrm{d}\Delta_2/\mathrm{d}\theta>0$ and $\Delta_2(\theta)\le\Delta$, for $\theta\in[\theta_2,1]$.

(ii) For the parking supply $W_f\in(W_2^\#,\bar{N}_s^*)$, if $\mathrm{d}\ln\tilde{N}_c^\#/\mathrm{d}\ln\theta<1$, the watershed line $\Delta_2(\theta)$ increases with $\theta\in(\theta_2,1]$, whilst if $\mathrm{d}\ln\tilde{N}_c^\#/\mathrm{d}\ln\theta>1$, the watershed line $\Delta_2(\theta)$ is a piecewise function, increasing with $\theta\in[\theta_2,\theta_2^\#]$ for $\Delta_2(\theta)<\Delta$ and decreasing with $\theta\in(0,\theta_2^\#)$ for $\Delta_2(\theta)\ge\Delta$.

The condition $\mathrm{d}\ln\tilde{N}_c^\#/\mathrm{d}\ln\theta<1$ means the parking demand for carpooling vehicles, to make the last carpoolers arrive at workplace with no queue except for those possibly by overestimating the extra carpool cost, is inelastic in regard to θ, i.e., the capacity allocation ratio to carpool lane. This condition is important for the monotonic increasing property of Δ_2 in Proposition 2. In reverse, for $\mathrm{d}\ln\tilde{N}_c^\#/\mathrm{d}\ln\theta>1$, $\Delta_2(\theta)$ increases first with θ for $\Delta_2(\theta)<\Delta$, and then decreases with θ for $\Delta_2(\theta)\ge\Delta$.

Pattern $(c_g)+(c_c^\#)$

This is the commute pattern where the last commuters to carpool always arrive at workplace with no queue except for those possibly by overestimating the extra carpool cost. However, for GP lane, the last commuters not only arrive late but also have to queue. Thus, the watershed line is such that the last commuters on both lanes arrive with no excess queue. In other words, the parking supply should satisfy $W_3 = \tilde{N}_s^{\#} + \tilde{N}_c^{\#} + \Delta_x s_c / \delta$. Using the reference numbers defined in Section 4.1, we can get $\min_{\theta,\Delta_x} W_3 = \tilde{N}_c^{\#}\big|_{\theta=1,\Delta_x=0} = \bar{N}_c^{\#}$ and $\max_{\theta,\Delta_x} W_3 = \tilde{N}_s^{\#}\big|_{\theta=0} = \bar{N}_s^{\#}$. Given $W_f = W_3$, we can get the watershed line $\Delta_3(\theta)$ by solving

$$W_f = \frac{s}{\delta} H\left(N - W_f - (k-1)N_c\right) + \frac{\theta s}{\delta}\left(\Delta_x - \Delta_x^+\right). \tag{7}$$

According to Eq. (7), if $W_f \in [\bar{N}_c^{\#}, \bar{N}_s^{\#}]$, there exists a capacity allocation ratio θ_3 to make the watershed line $\Delta_3(\theta_3) = 0$. For $W_f \in [W_3^{\#}, \bar{N}_s^{\#})$ and $W_3^{\#} = \bar{N}_c^{\#} + s\Delta/\delta$, there exists a capacity allocation ratio $\theta_3^{\#}$ to make the watershed line $\Delta_3(\theta_3^{\#}) = \Delta$. Specially, when W_f equals to $W_3^{\#}$, then $\Delta_3(1) = \Delta$. We discuss the properties of $\Delta_3(\theta)$ in Proposition 3 (See Appendix A.4 for detailed proofs).

Proposition 3. At multi-mode equilibrium with spatial-temporal capacity allocation,

(i) For the parking supply $W_f \in [\bar{N}_c^{\#}, W_3^{\#}]$, the watershed line $\Delta_3(\theta)$ increases with θ and it is no more than the extra carpool cost in $[\theta_3, 1]$, i.e., $\mathrm{d}\Delta_3/\mathrm{d}\theta > 0$ and $\Delta_3(\theta) \le \Delta$, for $\theta \in [\theta_3, 1]$.

(ii) For the parking supply $W_f \in (W_3^{\#}, \bar{N}_s^{\#}]$, $(\bar{N}_s^{\#}, \bar{N}_s^{*}]$ the watershed line $\Delta_3(\theta)$ increases with θ in $(\theta_3, 1]$, i.e., $\mathrm{d}\Delta_3(\theta)/\mathrm{d}\theta > 0$ for $\theta \in (\theta_3, 1]$.

One can easily deduce from Proposition 3 that there exists a capacity allocation ratio $\theta_3^{\#}$ to make the watershed line $\Delta_3(\theta_3^{\#}) = \Delta$, and thus we can get $\Delta_3(\theta) < \Delta$ for $\theta \in [\theta_3, \theta_3^{\#})$ and $\Delta_3(\theta) \ge \Delta$ for $\theta \in [\theta_3^{\#}, 1]$. Based on the monotonicity of each watershed line in Propositions 1–3, their relationships are presented in Lemma 1 (See Appendix A.5 for detailed proofs).

Lemma 1. For given parking supply W_f, if there exist θ_1, θ_2 and θ_3 such that $\Delta_1(\theta_1) = \Delta_2(\theta_2) = \Delta_3(\theta_3) = 0$, then the inequalities $\Delta_3(\theta) < \Delta_2(\theta) < \Delta_1(\theta)$ always hold.

Using Lemma 1 we can order different commute patterns in the two-dimensional space (θ, Δ_x) to avoid the tedious consideration of undesirable ones, which is useful for optimal design of the spatial-temporal allocation scheme of bottleneck capacity later.

Hereafter, the above-mentioned critical numbers of parking supply and watershed lines will be used to distinguish the possible commute patterns corresponding to each panel in Fig. 2 and Fig. 3. For convenience, we summarize these conditions for each panel in Table 3. The two rightmost columns in Table 3 depict two extreme cases with $\Delta_x = 0$ ($0 < \theta < 1$) and $\theta = 1$ (Δ_x either smaller or larger than Δ), respectively. For the case with $\Delta_x = 0$ ($0 < \theta < 1$), the carpool lane is only for carpooling during the whole peak period, and all possible commute patterns are consistent with that explored in Xiao et al. (2016), where the spatial-only capacity allocation scheme is introduced. For the case with $\theta = 1$ (Δ_x either smaller or larger than Δ), the spatial-temporal capacity allocation scheme is degenerated into the temporal-only one.

Table 3. Conditions for distinguishing each panel in Fig. 2 and Fig. 3

Partitions	Parking supply W_f	Figures	Watershed line	Patterns		
				Spatial-temporal allocation	Spatial allocation ($\Delta_x = 0$)	Temporal allocation ($\theta = 1$)
1	$[0, \bar{W}_c^{\#}]$	Fig. 2(i)	$\Delta_0(\theta)$	$(a_g)+(a_c)$ $(a_g)+(a_c^{\#})$	$(a_g)+(a_c)$	$(a_g)+(a_c)$ $(a_g)+(a_c^{\#})$
2	$(\bar{W}_c^{\#}, \bar{N}_c^{\#}]$	Fig. 2(ii)	$\Delta_0(\theta)$ $\Delta_1(\theta)$	$(a_g)+(a_c)$ $(a_g)+(a_c^{\#})$ $(c_g)+(c_c)$	$(a_g)+(a_c)$ $(c_g)+(c_c)$	$(a_g)+(a_c)$ $(a_g)+(a_c^{\#})$ $(c_g)+(c_c)$
3	$(\bar{N}_c^{\#}, W_1^{\#}]$	Fig. 3(i)	$\Delta_0(\theta)$ $\Delta_1(\theta)$ $\Delta_2(\theta)$ $\Delta_3(\theta)$	$(a_g)+(a_c)$ $(a_g)+(a_c^{\#})$ $(c_g)+(c_c)$ $(c_g)+(c_c^{\#})$	$(a_g)+(a_c)$ $(c_g)+(c_c)$ $(c_g)+(c_c^{\#})$	$(a_g)+(a_c)$ $(a_g)+(a_c^{\#})$ $(c_g)+(c_c)$ $(c_g)+(c_c^{\#})$
4	$(W_1^{\#}, W_2^{\#}]$	Fig. 3(ii)				
5	$(W_2^{\#}, \bar{W}_s^{\#}]$	Fig. 3(iii)				
6	$(\bar{W}_s^{\#}, W_3^{\#}]$	Fig. 3(iv)	$\Delta_2(\theta)$ $\Delta_3(\theta)$	$(c_g)+(c_c)$ $(c_g)+(c_c^{\#})$	$(c_g)+(c_c)$ $(c_g)+(c_c^{\#})$	$(c_g)+(c_c^{\#})$
7	$(W_3^{\#}, \bar{N}_s^{*}]$	Fig. 3(v)				
8	$(\bar{N}_s^{*}, \bar{N}_s^{\#}]$	Fig. 3(vi)	$\Delta_3(\theta)$	$(c_g)+(c_c^{\#})$	$(c_g)+(c_c^{\#})$	$(c_g)+(c_c^{\#})$

Given that W_f is smaller than $\bar{N}_c^{\#}$, Fig. 2 illustrates watershed lines with different spatial-temporal allocations, differentiating three specific commute patterns based on parking supply constraints and commuter arrival times. For Fig. 2(i), corresponding to the partition that $W_f \le \bar{W}_c^{\#}$, no auto commuters will arrive later than the desired arrival time. The watershed line $\Delta_0(\theta)$ separates pattern $(a_g)+(a_c)$ from pattern $(a_g)+(a_c^{\#})$. This phenomenon arises from severe parking shortages, which compel commuters to depart earlier to secure limited parking spaces. For Fig. 2(ii), corresponding to the partition that $\bar{W}_c^{\#} < W_f \le \bar{N}_c^{\#}$, the parking supply constraint is so severe that the last auto commuters may arrive later than the desired arrival time if $\Delta_x < \Delta_1(\theta)$, where the watershed line $\Delta_1(\theta)$ separates pattern $(a_g)+(a_c^{\#})$ from pattern $(c_g)+(c_c)$. This highlights how the partial alleviation of parking scarcity shifts commuter behavior and arrival dynamics.

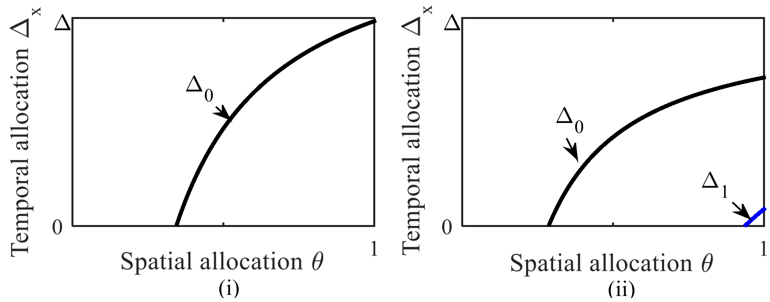


Fig. 2. Watershed lines with different Δ_x and θ for small W_f

When the parking supply belongs to $(\bar{N}_c^{\#}, \bar{N}_s^{\#}]$, Fig. 3 classifies six different combinations of commute patterns by watershed lines. The solid line represents the case that $0 < \Delta_x \le \Delta$ and the dashed line denotes the case that $\Delta < \Delta_x < \Delta_a$. The blue line Δ_1 and the green line Δ_3 are both increasing with θ. The pink line Δ_2 increases with θ when $\Delta_2 < \Delta$, otherwise its monotonicity depends on the parking demand for carpooling vehicles. This aligns with Propositions 1–3, where carpooling alters parking competition dynamics.

Fig. 3(i) depicts the commute pattern domain with the partition that $\bar{N}_c^{\#} < W_f \le W_1^{\#}$, where the watershed lines $\Delta_0(\theta)$, $\Delta_1(\theta)$ and $\Delta_2(\theta)$ separate patterns $(a_g)+(a_c)$, $(a_g)+(a_c^{\#})$, $(c_g)+(c_c)$ and $(c_g)+(c_c^{\#})$ from each other. Further when $\Delta_x \le \Delta_3(\theta)$ in this panel, the parking supply is sufficient to assure the occurrence of the commute pattern that the last auto drivers endure no queue. It should be noted that with the partition $\bar{N}_c^{\#} < W_f \le W_1^{\#}$, the watershed line $\Delta_1(\theta)$ is always equal or less than Δ, and it equals to Δ at $\theta = 1$ when $W_f = W_1^{\#}$. This implies that, for the partition $\bar{N}_c^{\#} < W_f \le W_1^{\#}$, the last auto commuters can always arrive earlier than the desired arrival time regardless of θ, if $\Delta_x \ge \Delta$.

Fig. 3(ii) depicts the commute pattern domain with the partition that $W_1^{\#} < W_f \le W_2^{\#}$, where the watershed lines $\Delta_0(\theta)$, $\Delta_1(\theta)$ and $\Delta_2(\theta)$ also separate patterns $(a_g)+(a_c)$, $(a_g)+(a_c^{\#})$, $(c_g)+(c_c)$ and $(c_g)+(c_c^{\#})$ from each other. Similar to Fig. 3(i), when $\Delta_x \le \Delta_3(\theta)$, the parking supply is also sufficient to assure the last auto drivers commute with no queue. The only difference between Fig. 3(i) and Fig. 3(ii) is that, in Fig. 3(ii), $\Delta_1(\theta)$ can be larger than Δ when θ takes a large value. It can also be seen in Fig. 3(ii) that the watershed line $\Delta_2(\theta)$

is also less or equal to Δ, and it equals to Δ at $\theta = 1$ when $W_f = W_2^\#$. This implies that, when $\Delta_x \geq \Delta$, there exists a capacity allocation ratio $\theta \in [\theta_1^\#, 1]$, where $\theta_1^\#$ satisfies $\Delta_1(\theta_1^\#) = \Delta$, such that the last auto commuters can arrive at workplace on time.

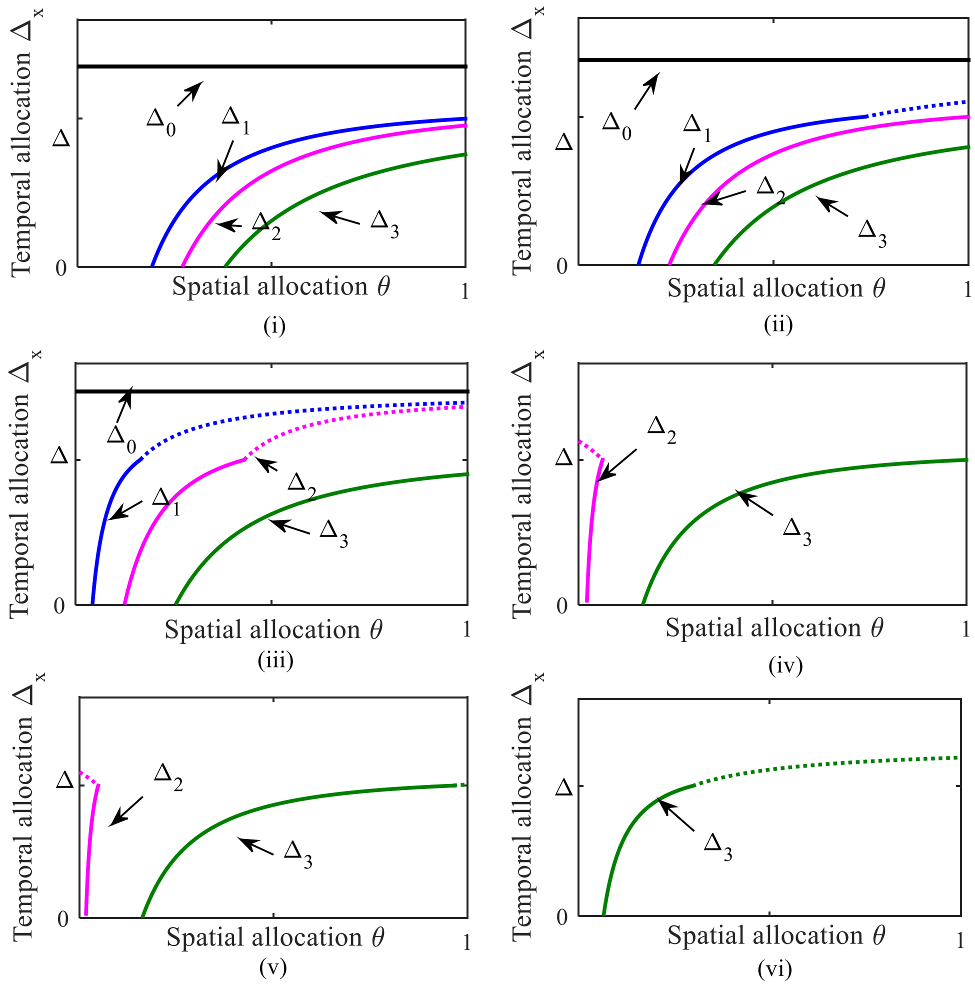


Fig. 3. Watershed line corresponding to each commute pattern for large W_f with different Δ_x and θ

Fig. 3(iii) corresponding to the partition $W_2^\# < W_f \leq \bar{W}_s^\#$ is similar to Fig. 3(i) and Fig. 3(ii), except that both $\Delta_1(\theta)$ and $\Delta_2(\theta)$ can be larger than Δ when θ takes a large value. It can be seen that, with this partition, $\Delta_2(\theta)$ is always increasing with θ. This is because the parameter values in the illustrated example satisfy $d \ln \tilde{N}_c^\# / d \ln \theta < 1$. This is in contrary for the cases illustrated in Fig. 3(iv) and Fig. 3(v), where $\Delta_2(\theta)$ is decreasing with θ when $\Delta_2(\theta) \geq \Delta$ because the parameter values in the illustrated example satisfy $d \ln \tilde{N}_c^\# / d \ln \theta > 1$ (e.g. Proposition 3).

Fig. 3(iv) depicts the commute pattern domain with the partition $W_s^{\#} < W_f \le W_3^{\#}$, where the watershed line $\Delta_2(\theta)$ separates pattern $(c_g)+(c_c)$ from pattern $(c_g)+(c_c^{\#})$. When $\Delta_x \le \Delta_3(\theta)$, the parking supply is sufficient to assure the last auto drivers with no queue. The watershed line $\Delta_3(\theta)$ is always no greater than Δ, and equals to Δ at $\theta=1$ when $W_f = W_3^{\#}$. This implies that, when $\Delta_x \ge \Delta$, no matter how the highway capacity is spatially allocated, the parking supply will always fall short of potential demand. However, there exists a capacity allocation ratio $\theta \in [0,\theta_2^{\#}]$, where $\theta_2^{\#}$ satisfies $\Delta_2(\theta_2^{\#}) = \Delta$, such that the last auto commuters can arrive at workplace on time.

Fig. 3(v) corresponding to the partition $W_3^{\#} < W_f \le \bar{N}_s^{*}$ and is similar to Fig. 3(iv), except that $\Delta_3(\theta)$ can be larger than Δ when θ takes a larger value. Fig. 3(vi) depicts the commute pattern domain with the partition $\bar{N}_s^{*} < W_f \le \bar{N}_s^{\#}$, where the only watershed line $\Delta_3(\theta)$ separates pattern $(c_g)+(c_c^{\#})$ from the commute pattern without parking supply constraint. This implies that, no matter how the highway capacity is spatially allocated, the parking supply will always guarantee that the last carpoolers arrive with no excess queue.

5. Optimizing spatial-temporal capacity allocation with parking supply constraint

In this section, we first present two optimization models, with and without consideration of traffic authority's social cost constraints. Then a numerical example is presented to illustrate the analytical results and the two models are compared.

5.1. Optimization models

The objective of our optimal design is to minimize the total trip cost:

$$\min_{\Delta_x,\theta} TTC(\Delta_x,\theta) = H(N - W_a)N\,, \tag{8}$$

where W_a is calculated from Eq. (8) and Eq. (18) for each situation. As assumed before, the transit cost function H increases (or decreases) with the number of transit (auto) commuters, i.e., $H_a' = dH/dW_a < 0$. Thus, given N and W_f, we just have to analyze about how does the number of auto commuters vary with parameters, Δ_x and θ, for each panel depicted in Fig. 2 and Fig. 3. Next, the optimal set of (Δ_x^o, θ^o) to the minimization problem (8) is the focus of our attention.

To facilitate our analysis, we summarize three possible commute situations at equilibrium in Table 4.

Table 4. Three possible situations at commute equilibrium.

Categories	Definition
Situation (I)	The last drivers on both lanes were not to queue and only endure the schedule delay late cost, and the relationship between t_s^e and t_c^e satisfies $t_c^e < t_s^e$.
Situation (II)	The last carpoolers arrive with no excess queue (except for those possibly be owing to overestimating the

	extra carpool cost) but the last solo drivers queue, and t_s^e, t_c^e satisfies the inequality $t_c^e < t_s^e$.
Situation (III)	The last carpoolers arrive with queue, and the relationship between t_s^e and t_c^e satisfies $t_s^e = t_c^e$.

Given N and W_f, for Situation (II) with $t_c^e < t_s^e$, we have $TTC'_{\Delta_x} \geq 0$ and $TTC'_{\theta} < 0$. For Situation (III) with $t_c^e = t_s^e$, we get $TTC'_{\Delta_x} > 0$ and $TTC'_{\theta} < 0$ (Details are given in Appendix B.1). Thus, we have the following Proposition 4 (See Appendix B.2 for detailed proofs).

Proposition 4 – local optimal spatial-temporal allocations: With given parking supply W_f, for Situation (II), the local optimal spatial capacity allocation is $\theta^o = 1$, and the local optimal temporal allocation satisfies:

$$\Delta^o = \begin{cases} \Delta_x \in \left[\Delta_3(1),\ \min\{\Delta_2(1), \Delta\}\right], & \text{if } \Delta_3(1) < \Delta, \\ \Delta_3(1), & \text{elseif } \Delta_3(1) \geq \Delta. \end{cases} \tag{9}$$

For Situation (III), if $\Delta_2(1) < \Delta$, the local optimal spatial-temporal scheme is achieved at $\theta^o = 1$ and $\Delta^o = \Delta_2(1)$, whilst if $\Delta_2(1) \geq \Delta$, the optimal allocation is $\theta^o = \theta_2^\#$ and $\Delta^o = \Delta_2(\theta_2^\#) = \Delta$.

For Situation (I), i.e., the situation without parking supply constraint, the optimal spatial-temporal scheme is achieved at $\theta^o = 1$ and $\Delta^o = \Delta$, with which all the highway capacity should be designated to carpool use and the commute pattern endures no excess queue and no capacity waste (Details are given in Appendix B.2.3, and also refer to Xiao et al. (2021b)). Together with the optimal result in Situation (I), Proposition 4 gives the local optimal set of spatial-temporal allocation for two commute situations, i.e., Situation (II) and Situation (III), respectively. Clearly, the local optimal set depends on the conditions for each commute situation in Section 4. Considering these, the following Proposition 5 presents the (globally) optimal spatial-temporal allocation (Δ_x^o, θ^o) with given parking supply (See Appendix B.3 for detailed proofs).

Proposition 5 – global optimal spatial-temporal allocation. With given parking supply W_f, the global optimal spatial allocation is always $\theta^o = 1$, i.e., all the highway capacity should be designed spatially to carpool use.

(i) If $W_f \in [0, \bar{N}_c^\#]$, corresponding to the panels in Fig. 2(i)–Fig. 2(ii), the optimal temporal allocation is $\Delta_x^o = 0$, i.e., driving alone is not allowed on the carpool lane.

(ii) If $W_f \in (\bar{N}_c^\#, W_2^\#]$, corresponding to the panels as shown in Fig. 3(i)–Fig. 3(iii), the optimal temporal allocation is $\Delta_x^o \in [\Delta_3(1), \min\{\Delta_2(1), \Delta\}]$.

(iii) If $W_f \in [W_3^{\#}, \bar{N}_g^{\#}]$, corresponding to the panels as shown in Fig. 3(vi)–Fig. 3(iv), the optimal temporal allocation is $\Delta_x^o = \Delta$.

As demonstrated in Proposition 5 (ii), the optimal time reservation window for the carpool lane is no longer necessarily unique. This finding implies that, under limited parking supply, precise estimation of the extra carpool cost is not a prerequisite for optimizing temporal lane reservation by traffic authorities. Even an underestimation of the extra carpool cost may yield optimal outcomes, as parking limitations inherently suppress excessive carpool demand, thereby reducing sensitivity to cost miscalibrations. This is significantly different from the case without parking supply constraint studied in Xiao et al. (2021b), in which the optimal temporal lane reservation must be based on accurate estimation of actual extra carpool cost.

Proposition 5 also tells us that when the total trip cost is minimized with no social cost constraint but with parking supply constraint, the total highway capacity should be allocated spatially to the carpool lane. However, in reality, the spatial-temporal capacity allocation and operations require large social efforts and financial support from the traffic authority, which are generally limited. Hence, the optimal design to minimize the total trip cost should be constrained to the traffic authority's social cost budget, G_f. In general, the more the highway capacity is allocated to carpool lane and the longer time window for carpool lane would bring higher social and operating cost to the traffic authority. This is because the traffic authority would need to put in greater effort in monitoring, managing and controlling the traffic on the carpool lane. Without loss of generality, we assume that $G_f \geq g_1(\Delta_x) + g_2(\theta)$, where g_1 and g_2 are the economic costs for operating temporal and spatial capacity allocation, respectively, and that $\mathrm{d}g_1/\mathrm{d}\Delta_x < 0$ and $\mathrm{d}g_2/\mathrm{d}\theta > 0$ holds. If the traffic authority's social cost budget constraint is binding, i.e., the set (Δ_x, θ) satisfies $G_f = g_1(\Delta_x) + g_2(\theta)$, we can get a function $\Delta_f(\theta)$. Then considering the binding budget constraint, the minimization of the total trip cost can be expressed as:

$$\min_{\theta} TTC^f\left(\Delta_f(\theta), \theta\right) = H\left(N - W_a\right)N. \tag{10}$$

Using the budget constraint function $\Delta_f(\theta)$, we obtain:

$$\frac{\mathrm{d}TTC^f}{\mathrm{d}\theta} = NH_a'\left(k-1\right)\frac{\mathrm{d}W_a}{\mathrm{d}\theta}. \tag{11}$$

Given the parking supply W_f and the budget constraint function $\Delta_f(\theta)$, varying Δ_x and θ, the commute patterns can be identified by the relationships between $\Delta_f(\theta)$ and Δ_0, Δ_1, Δ_2 or Δ_3. For the parking supply W_f, the total trip cost can be calculated from Situation (II) and Situation (III) in Section 4.2. From Eq. (11), the monotonicity of TTC^f with θ subject to the government's binding social cost budget is complicated (Details are given in Appendix B.4). In the next section, we will numerically illustrate why GP lanes are still prevalent in reality.

5.2. A numerical example

In the examples presented in this section, unless otherwise specified, we adopt the following parameters (Small, 1982): $\alpha = 6.4$ (\$/h), $\beta = 3.9$ (\$/h), and $\gamma = 15.21$ (\$/h). The other parameters are set as: $N = 6000$ (person), $s = 3000$ (veh/h), $k = 2$ (person/veh), $\Delta = 3$ (\$). Unless otherwise specified, the transit trip cost function is set as $H(W_t) = 2 + 0.001 \cdot W_t$, and the traffic authority's social cost budget is given as $G_f = 20 \times \theta - 2.5 \times \Delta_x = 10$ (\$).

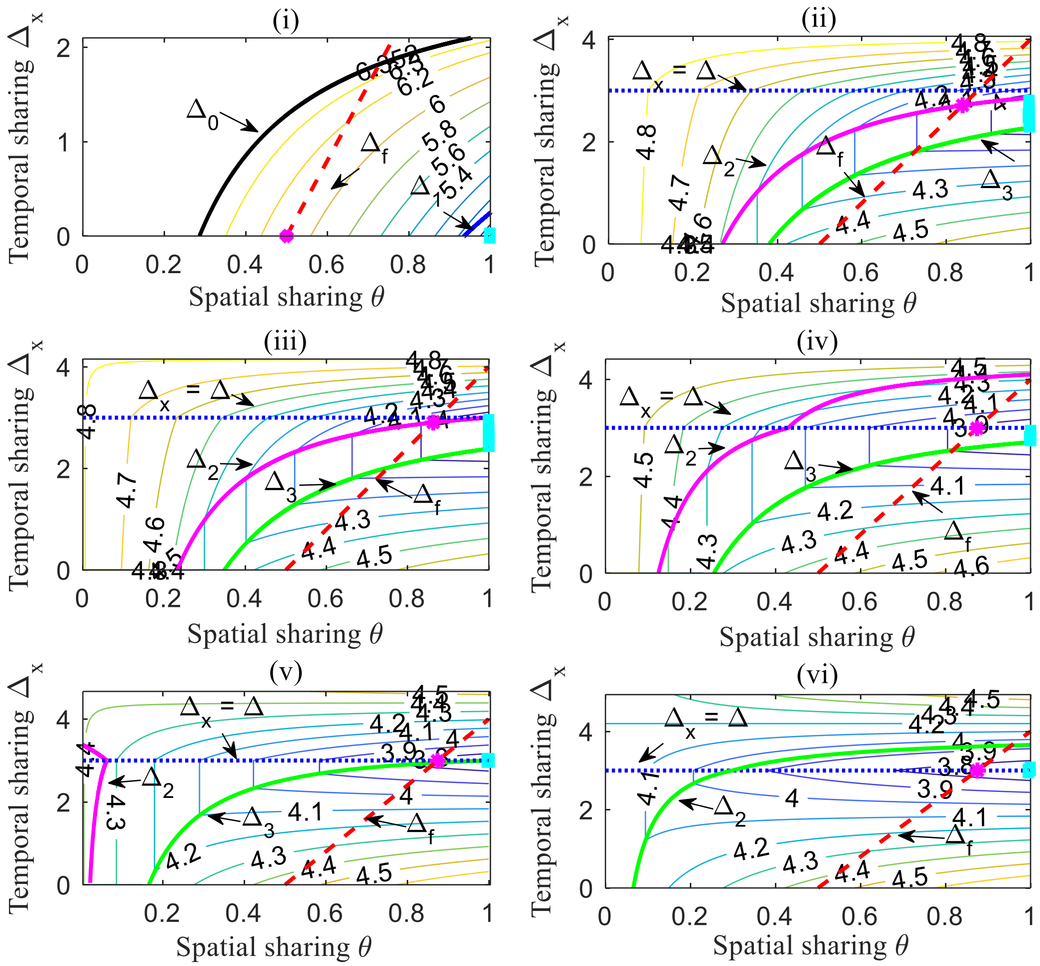


Fig. 4. Indifference curves of equilibrium trip cost for each commute pattern and budget line Δ_f

Letting the parking supply W_f take different values of 1647, 3123, 3195, 3400, 3592 and 3800, respectively, the indifference curves of equilibrium trip cost are depicted in Fig. 4(i)–(vi). As mentioned before, all commute situations at equilibrium can be classified as Situation (I), Situation (II) and Situation (III) in terms of watershed lines Δ_2 and Δ_3. If the traffic authority's social cost budget is ignored, the light blue squares in each panel of

Fig. 4 denote the optimal spatial-temporal allocation in term of total trip cost with given W_f. The optimal spatial allocation is always $\theta^o = 1$, and the optimal temporal allocation may not be unique when W_f is valued by 3123, 3195 or 3400, which are consistent with Proposition 5.

Using Eq. (5), we can get the unique optimal temporal allocation $\Delta_x{}^o = 0$ for Fig. 4(i). It means that solo drivers are forbid access to the carpool lane to guarantee the carpoolers preferentially under the optimal setting, due to very few parking available. If the parking supply increase gradually, Using Eqs. (6)–(7), there is no unique optimal temporal allocations, i.e., $\Delta_x{}^o \in [2.28, 2.86]$ for Fig. 4(ii), $\Delta_x{}^o \in [2.39, 3.00]$ for Fig. 4(iii) and $\Delta_x{}^o \in [2.70, 3.00]$ for Fig. 4(iv), respectively. It indicates that it is not necessary to accurately estimate the extra cost of carpooling to achieve system optimization, but only within a certain range. However, when the parking supply exceeds 3592, then there exists a unique optimal temporal allocation $\Delta_x{}^o = \Delta = 3.00$ for Figs. 4(v)–(vi), i.e., the temporal window for carpool lane should be designed through accurately estimating of the extra carpool cost. Furthermore, Table 5 compares the optimal temporal allocation results for $\theta =$ 0.1, 0.3, 0.5 and 0.7. The last column in Table 5 is the optimal spatial-temporal allocation for each figure as the bench mark. It shows clearly that when the given θ is smaller than θ_2, the optimal temporal allocation is unique and equal to $\Delta_x^o = 0$, whilst if θ is no less than $\theta_3^\#$, then the optimal temporal allocation is unique and equal to $\Delta_x^o = \Delta = 3$. However, when $\theta \in (\theta_2, \theta_3^\#)$, the optimal temporal allocation is not unique and as θ increase, the non-unique results gradually approach $\Delta = 3$.

Table 5. Comparison of the optimal allocation results for different θ

Fig.4	Critical points	$\Delta_x{}^o$				
		$\theta = 0.1$	$\theta = 0.3$	$\theta = 0.5$	$\theta = 0.7$	$\theta^o = 1$
(i)	$\theta_0 = 0.29,\ \theta_1 = 0.94$	0	0	0	0	0
(ii)	$\theta_2 = 0.27,\ \theta_3 = 0.38$	0	$[0, 0.44]$	$[0.93, 1.94]$	$[1.73, 2.51]$	$[2.28, 2.86]$
(iii)	$\theta_2 = 0.24,\ \theta_3 = 0.35,\ \theta_2^\# = 1$	0	$[0, 0.98]$	$[1.18, 2.25]$	$[1.90, 2.72]$	$[2.39, 3]$
(iv)	$\theta_2 = 0.12,\ \theta_3 = 0.26,\ \theta_2^\# = 0.43$	0	$[0.59, 2.54]$	$[1.89, 3]$	$[2.38, 3]$	$[2.70, 3]$
(v)	$\theta_2 = 0.02,\ \theta_3 = 0.17,\ \theta_2^\# = 0.06,\ \theta_3^\# = 1$	$[0, 3]$	$[1.75, 3]$	$[2.53, 3]$	$[2.82, 3]$	3
(vi)	$\theta_3 = 0.07,\ \theta_3^\# = 0.3$	$[1.38, 3]$	3	3	3	3

Note: $\Delta_i(\theta_i) = 0$, $\Delta_j(\theta_j^\#) = \Delta$, $i = 0,1,2,3$, $j = 2,3$.

However, if the traffic authority's social cost budget has to be considered, some capacity can be allocated to the GP lane for solo driving. The optimal spatial-temporal allocation points with social cost budget constraint are the red dots in each panel of Fig. 4. As we can see, Fig. 4(i) plots the scenario where the parking supply is so low that both lines Δ_2 and Δ_3 do not occur and all commute patterns belong to Situation (III). The traffic authority's binding social cost budget is denoted by the red virtual line $\Delta_f(\theta)=8\theta-4$. In this scenario, the optimal spatial-temporal allocation point is the red dot in the red virtual line, i.e., $\Delta_x^o=0$ and $\theta^o=0.5$, at which the total trip cost is minimal.

In panels (ii) and (iii) of Fig. 4, lines Δ_2 and Δ_3 co-occur, but Δ_2 is no larger than Δ. In these scenarios, the optimal spatial-temporal allocation point with social cost budget constraint is where line Δ_f crosses with line Δ_2, implying that the optimal commute pattern follows Situation (II). Specifically for panels (ii) and (iii) of Fig. 4, the corresponding points are (0.84, 2.71) and (0.86, 2.90). Against that, panels (iv) and (v) of Fig. 4 show the scenarios where line Δ_2 may be larger than Δ. The key difference between the two panels is whether the potential parking demand for carpooling vehicles, to make the last carpoolers arrive at workplace with no queue except for those possibly by overestimating the extra carpool cost, is inelastic in terms of the allocation ratio to the carpool lane (Refer to the analysis of Fig. 3 in Section 4.3). Fig. 4(vi) illustrates the scenario where only line Δ_3 occurs. Clearly, in Fig. 4(iv)–(vi), the optimal point is where line Δ_f crosses with line Δ, which is independent of W_f. In other words, the optimal points are identical in Fig. 4(iv)–(vi), depicted by (0.87, 3). However, it should be noticed that the optimal commute patterns are different in Fig. 4(iv)–(vi). The optimal commute patterns belong to Situation (II) in panels (iv) and (v) of Fig. 4 whilst belong to Situation (I) in Fig. 4(vi). These are consistent with the analysis in Appendix B.4, where given the parking supply, the total trip cost with the binding social cost budget constraint decreases with the carpool lane's capacity allocation ratio θ for $\Delta_x<\Delta$, whilst increasing with θ for $\Delta_x\geq\Delta$.

6. Conclusions

This paper investigates how the limited parking supply at workplace impacts the design of spatial-temporal allocation of bottleneck capacity for general and carpool purposes. A one-to-one corridor is considered with three alternative modes for commuters, i.e., solo-driving, carpooling and taking public transit. Along the corridor, there are a congested highway with single bottleneck and a crowded transit line. In addition, a GP lane for all drivers and a time-limited carpool lane for carpooling vehicles jointly divided the bottleneck capacity. The carpool use on the carpool lane is reserved only within a limited time window which is set based on the estimated extra carpool cost by the traffic authority. Outside of the reserved window, the carpool lane is available for all commuters. We formulate a bottleneck model to analyze the multi-mode morning commute equilibrium with parking supply constraint, incorporating a spatial-temporal capacity allocation scheme.

According to the different spatial-temporal designs of the highway capacity to carpool lane, and to the different levels of parking supply constraint, we derive the complex variations of the commuter patterns.

Based on derived commute patterns with parking supply constraint, we further demonstrate what are the optimal spatial-temporal capacity allocations in terms of only total trip cost. it is found that, if the traffic authority's social cost budget is ignored, at optimum, allocating 100% of highway capacity to carpool lanes. This strategy eliminates GP lane utilization, as carpool prioritization reduces parking competition and queuing delays. Additionally, the optimal time window for carpool lane may not be unique under parking scarcity. Limited parking inherently suppresses carpool demand, allowing traffic authorities to tolerate underestimation of extra carpool cost without compromising system efficiency. This contrasts with the findings of Xiao et al. (2021b), who demonstrated that the optimal temporal allocation is unique without parking supply constraint. Accurate estimation of the extra carpool cost is crucial for achieving optimal temporal allocation; inaccuracies can lead to capacity waste or excess queue, increasing trip costs when the parking supply is reserved. After incorporating traffic authority's social cost budget, partial GP lane allocation becomes necessary. In addition, expanding carpool operations increases monitoring and infrastructure costs, requiring trade-offs between trip cost minimization and fiscal feasibility.

The key findings in this paper have not only theoretical, but also practical and policy implications. Firstly, as far as we are aware, this work pioneers the integration of spatial-temporal carpool lane design with workplace parking constraints. By deriving all equilibrium commute patterns under UE conditions, it quantifies the combined effects of parking scarcity, lane allocation, and commuter mode choices. Also, these analytical derivations provide a solid foundation and practical tool for policy design such as congestion tolling, travel incentives and subsidies to induce commuters' travel, and modeling extensions such as incorporating the use of autonomous unmanned vehicles (AUVs), the competition and shifts between private vehicle and public transit. Secondly, we address a critical challenge: governments often struggle to accurately estimate the commuters' extra carpool costs for optimal temporal-spatial capacity allocation. Our findings reveal that under parking supply constraints, the optimal temporal allocation for carpool lane is non-unique, allowing for underestimation of extra carpool cost without sacrificing system efficiency. Furthermore, when the traffic authority's social cost budget has to be considered, it is more realistic that not all highway capacity is allocated to the carpool lane. The derivations of the optimizing scheme provide a possible tool for the government agencies to identify the optimal temporal-spatial capacity allocation scheme in different situations and to analysis the impacts of some other operational and behavioral parameters.

Further study could be explored the implications of allowing solo drivers access to this lane after paying tolls (Di et al., 2017; Zang et al., 2020; Zhong et al., 2020). The carpoolers' matching process can be incorporated to formulate the equilibrium commute patterns and further evaluate its impacts (Ma and Zhang, 2017; Li and Liu, 2021). The cursing time (Su and Wang, 2019; Tang et al., 2025) can be considered to increase the parking supply constraint and formulate the impacts on commuters' travel decision making. Besides, all commuters are assumed to be homogeneous for this paper's focus. In reality, commuters may exhibit wide behavioral differences in their travel choices. It is of importance to explore how commuter heterogeneity affects the scheme of spatial-temporal capacity allocation and resultant commute patterns in the future (Wang et al., 2012; Yu et al., 2019; Wei et al., 2022).

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