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Logics with probabilistic team semantics and the Boolean negation

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Abstract

We study the expressivity and the complexity of various logics in probabilistic team semantics with the Boolean negation. In particular, we study the extension of probabilistic independence logic with the Boolean negation, and a recently introduced logic first-order theory of random variables with probabilistic independence. We give several results that compare the expressivity of these logics with the most studied logics in probabilistic team semantics setting, as well as relating their expressivity to a numerical variant of second-order logic. In addition, we introduce novel entropy atoms and show that the extension of first-order logic by entropy atoms subsumes probabilistic independence logic. Finally, we obtain some results on the complexity of model checking, validity and satisfiability of our logics.

Keywords: probabilistic team semantics, model checking, satisfiability, validity, computational complexity, expressivity of logics.

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1 Introduction

Probabilistic team semantics is a novel framework for the logical analysis of probabilistic and quantitative dependencies. Team semantics, as a semantic framework for logics involving qualitative dependencies and independencies, was introduced by Hodges [19] and popularized by Väänänen [28] via his dependence logic. Team semantics defines truth in reference to collections of assignments called *teams*, and is particularly suitable for the formal analysis of properties, such as the functional dependence between variables, that arise only in the presence of multiple assignments. The idea of generalizing team semantics to the probabilistic setting can be traced back to the works of Galliani [7] and Hyttinen et al. [20]; however, the beginning of a more systematic study of the topic dates back to works of Durand et al. [5].

In *probabilistic team semantics*, the basic semantic units are probability distributions (i.e. *probabilistic teams*). This shift from set-based to distribution-based semantics allows probabilistic notions of dependency, such as conditional probabilistic independence, to be embedded in the framework.[1] The expressivity and complexity of non-probabilistic team-based logics can be related to fragments of (existential) second-order logic and have been studied extensively (see, e.g. [6, 8, 11]). Team-based logics, by definition, are usually not closed under Boolean negation, so adding it can greatly increase the complexity and expressivity of these logics [14, 22]. Some expressivity and complexity results have also been obtained for logics in probabilistic team semantics (see Figure 1 and Table 1). However, richer semantic and computational frameworks are sometimes needed to characterize these logics.

Metafinite model theory, introduced by Grädel and Gurevich [10], generalizes the approach of *finite model theory* by shifting to two-sorted structures, which extend finite structures by another (often infinite) numerical domain and weight functions bridging the two sorts. A particularly important subclass of metafinite structures are the so-called $\mathbb{R}$-*structures*, which extend finite structures with the real arithmetic on the second sort. *Blum-Shub-Smale machines* (BSS machines for short) [2] are essentially register machines with registers that can store arbitrary real numbers and compute rational functions over reals in a single time step. Interestingly, Boolean languages that are decidable by a non-deterministic polynomial-time BSS machine coincide with those languages which are polynomial-time (PTIME) reducible to the true existential sentences of real arithmetic (i.e. the complexity class $\exists\mathbb{R}$) [3, 27].

Recent works have established fascinating connections between second-order logics over $\mathbb{R}$-structures, complexity classes using the BSS-model of computation and logics using probabilistic team semantics. In [16], Hannula et al. establish that the expressivity and complexity of probabilistic independence logic coincide with a particular fragment of existential second-order logic over $\mathbb{R}$-structures and non-deterministic polynomial-time (NP) on BSS-machines. In [12], Hannula and Virtema focus on probabilistic inclusion logic, which is shown to be tractable (when restricted to Boolean inputs), and relate it to linear programming.

In this paper, we focus on the expressivity and model checking complexity of probabilistic team-based logics that have access to Boolean negation. We will see that adding the Boolean negation to probabilistic independence logic increases the expressivity from a numerical variant of existential second-order logic to full second-order logic with numerical terms. We also study the connections between probabilistic independence logic and a logic called FOPT($\leq^{\delta}_{c}$), which is defined via a

[1] Li [24] recently introduced *first-order theory of random variables with probabilistic independence (FOTPI)* whose variables are interpreted by discrete distributions over the unit interval. The paper shows that truth in arithmetic is interpretable in FOTPI, whereas probabilistic independence logic is by our results far less complex.

TABLE 1. Overview of our complexity results of the problem MC for sentences

Logic	Lower bound	Upper bound	Reference
$\mathrm{FOPT}(\leq_c^\delta)$	PSPACE	PSPACE	Corollary 2
$\mathrm{FO}(\perp\!\!\!\perp_c)$	NEXPTIME	EXPSPACE	Theorem 9
$\mathrm{FO}(\approx)$	PSPACE	EXPTIME	Theorem 8
$\mathrm{FO}(\sim, \perp\!\!\!\perp_c)$	AEXPTIME [poly]	3-EXPSPACE	Theorem 10

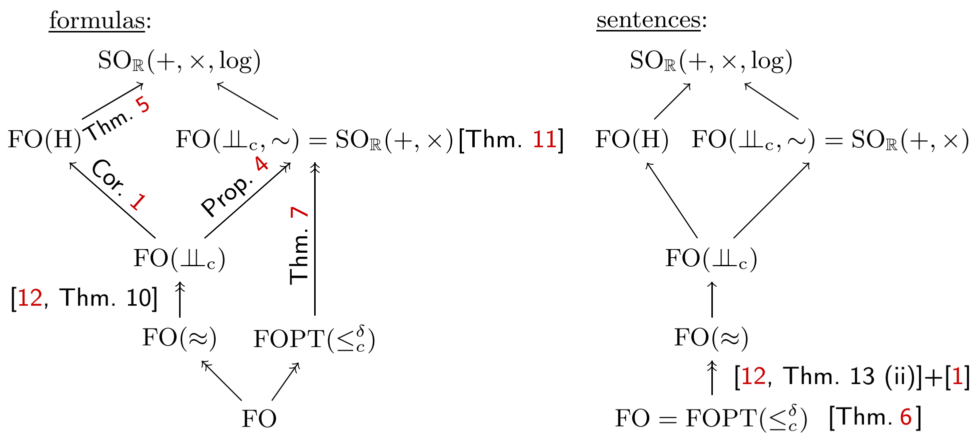

FIGURE 1. Landscape of relevant logics. Single arrows indicate inclusions and double arrows indicate strict inclusions. The non-strict inclusions for sentences follow from the inclusions for formulas depicted on the left

computationally simpler probabilistic semantics [17]. The logic $\mathrm{FOPT}(\leq_c^\delta)$ is the probabilistic variant of a certain team-based logic that can define exactly those dependencies that are first-order definable [23]. The results for the logic $\mathrm{FOPT}(\leq_c^\delta)$ demonstrate that having the Boolean negation in logics with probabilistic team semantics does not necessarily mean that the expressivity goes to the level of full second-order logic. If the quantifiers and the disjunction are weak enough, we can stay at the level of the first-order logic. We also introduce novel entropy atoms and relate the extension of first-order logic with these atoms to probabilistic independence logic. This paper extends the conference paper [18] and includes complete proofs, which were omitted from the conference version.

See Figure 1 for our expressivity results and Table 1 for our complexity results.

2 Preliminaries

We assume the reader is familiar with the basics in complexity theory [25]. In this paper, we will encounter complexity classes PSPACE, EXPTIME, NEXPTIME, EXPSPACE and the class AEXPTIME [poly] together with the notion of completeness under the usual polynomial time many to one reductions. A bit more formally for the latter complexity class, which is more uncommon than the others, AEXPTIME [poly] consists of all languages that can be decided by alternating Turing machines within an exponential runtime of $O(2^{n^{O(1)}})$ and polynomially many alternations between universal and existential states. There exist problems in propositional team logic with generalized dependence atoms that are complete for this class [13]. It is also known that truth evaluation of alternating dependency quantified Boolean formulae is complete for this class [13].

2.1 Probabilistic team semantics

Given a tuple $\vec{a}$, we write $|\vec{a}|$ for its length. The set of variables that appear in a tuple of variables $\vec{x}$ is denoted by $\mathrm{Var}(\vec{x})$. A vocabulary τ is a finite set of relation, function and constant symbols. Each relation symbol R and function symbol f has a prescribed arity, denoted by $\mathrm{Ar}(R)$ and $\mathrm{Ar}(f)$.

Let τ be a finite relational vocabulary, such that $\{=\} \subseteq \tau$, and let $\mathcal{A}$ be a finite τ-structure. We write $\mathrm{Dom}(\mathcal{A})$ for the domain of a structure $\mathcal{A}$, and use the corresponding print (noncursive) letter for it, i.e. $\mathrm{Dom}(\mathcal{A}) = A$, $\mathrm{Dom}(\mathcal{B}) = B$, etc. The intended interpretation of the relation symbol "=" in any τ-structure $\mathcal{A}$ is the first-order equality, i.e. $\{(a, a) \mid a \in A\}$. For a finite τ-structure $\mathcal{A}$ and a finite set of variables D, an *assignment* of $\mathcal{A}$ for D is a function $s\colon D \to A$. A *team* X of $\mathcal{A}$ over D is a finite set of assignments $s\colon D \to A$.

A *probabilistic team* $\mathbb{X}$ is a function $\mathbb{X}\colon X \to \mathbb{R}_{\geq 0}$, where $\mathbb{R}_{\geq 0}$ is the set of non-negative real numbers. The value $\mathbb{X}(s)$ is called the *weight* of assignment s. Since zero weights are allowed, we may, when useful, assume that X is maximal, i.e. it contains all assignments $s\colon D \to A$. The *support* of $\mathbb{X}$ is defined as $\mathrm{supp}(\mathbb{X}) := \{s \in X \mid \mathbb{X}(s) \neq 0\}$. A team $\mathbb{X}$ is *nonempty* if $\mathrm{supp}(\mathbb{X}) \neq \emptyset$.

These teams are called probabilistic because we usually consider teams that are probability distributions, i.e. functions $\mathbb{X}\colon X \to \mathbb{R}_{\geq 0}$ for which $\sum_{s \in X} \mathbb{X}(s) = 1$.[2] In this setting, the weight of an assignment can be thought of as the probability that the values of the variables are as in the assignment. If $\mathbb{X}$ is a probability distribution, we also write $\mathbb{X}\colon X \to [0, 1]$. In this paper, we assume that probabilistic teams are probability distributions, except in the case of logic $\mathrm{FOPT}(\leq^{\delta}_{c})$, for which the team can be any function $\mathbb{X}\colon X \to \mathbb{R}_{\geq 0}$. Note that for technical reasons concerning the disjunction, the empty function is also considered a probability distribution.

For a set of variables V, the restriction of the assignment s to V is denoted by $s \upharpoonright V$. The *restriction of a team* X to V is $X \upharpoonright V = \{s \upharpoonright V \mid s \in X\}$, and the *restriction of a probabilistic team* $\mathbb{X}$ to V is $\mathbb{X} \upharpoonright V\colon X \upharpoonright V \to \mathbb{R}_{\geq 0}$ where

$$(\mathbb{X} \upharpoonright V)(s) = \sum_{\substack{s' \upharpoonright V = s, \\ s' \in X}} \mathbb{X}(s').$$

If ϕ is a first-order formula, then $\mathbb{X}_{\phi}$ is the restriction of the team $\mathbb{X}$ to those assignments in X that satisfy the formula ϕ. The weight $|\mathbb{X}_{\phi}|$ is defined analogously as the sum of the weights of the assignments in X that satisfy ϕ, e.g.

$$|\mathbb{X}_{\vec{x}=\vec{a}}| = \sum_{\substack{s \in X, \\ s(\vec{x})=\vec{a}}} \mathbb{X}(s).$$

Note that for probability distributions $\mathbb{X}$, the value $|\mathbb{X}_{\vec{x}=\vec{a}}|$ corresponds to the marginal probability of that the variables $\vec{x}$ have values $\vec{a}$ in the probabilistic team $\mathbb{X}$. The notion of marginal probability will be important later for defining the semantics of different probabilistic dependency notions.

[2] In some sources, the term probabilistic team only refers to teams that are distributions, and the functions $\mathbb{X}\colon X \to \mathbb{R}_{\geq 0}$ that are not distributions are called *weighted teams*.

For a variable x and $a \in A$, we write $s(a/x)$ for the modified assignment $s(a/x)\colon D \cup \{x\} \to A$ such that $s(a/x)(y) = a$ if $y = x$, and $s(a/x)(y) = s(y)$ otherwise. For a set $B \subseteq A$, the modified team $X(B/x)$ is defined as the set $X(B/x) := \{s(a/x) \mid a \in B, s \in X\}$.

Next, we will define the notions of modified probabilistic teams $\mathbb{X}(B/x)$ and $\mathbb{X}(F/x)$, which will be used later to define semantics for universal and existential quantifiers in probabilistic team semantics. Let $\mathbb{X}\colon X \to \mathbb{R}_{\geq 0}$ be any probabilistic team. Then the probabilistic team $\mathbb{X}(B/x)$ is a function $\mathbb{X}(B/x)\colon X(B/x) \to \mathbb{R}_{\geq 0}$ defined as

$$\mathbb{X}(B/x)(s(a/x)) = \sum_{\substack{t \in X,\\ t(a/x)=s(a/x)}} \mathbb{X}(t) \cdot \frac{1}{|B|}.$$

If x is a fresh variable, the summation can be dropped and the right hand side of the equation becomes $\mathbb{X}(s) \cdot \frac{1}{|B|}$. For singletons $B = \{a\}$, we write $X(a/x)$ and $\mathbb{X}(a/x)$ instead of $X(\{a\}/x)$ and $\mathbb{X}(\{a\}/x)$.

Let then $\mathbb{X}\colon X \to [0,1]$ be a distribution. Let p_B be the set of all probability distributions $d\colon B \to [0,1]$, and let F be a function $F\colon X \to p_B$. Then the probabilistic team $\mathbb{X}(F/x)$ is a function $\mathbb{X}(F/x)\colon X(B/x) \to [0,1]$ defined as

$$\mathbb{X}(F/x)(s(a/x)) = \sum_{\substack{t \in X,\\ t(a/x)=s(a/x)}} \mathbb{X}(t) \cdot F(t)(a)$$

for all $a \in B$ and $s \in X$. If x is a fresh variable, the summation can again be dropped and the right hand side of the equation becomes $\mathbb{X}(s) \cdot F(s)(a)$.

In the following, we define the notion of k-scaled union, which will used later to define the semantics of disjunction in probabilistic team semantics. Let $\mathbb{X}\colon X \to [0,1]$ and $\mathbb{Y}\colon Y \to [0,1]$ be probabilistic teams with common variable and value domains, and let $k \in [0,1]$. The k-scaled union of $\mathbb{X}$ and $\mathbb{Y}$, denoted by $\mathbb{X} \sqcup_k \mathbb{Y}$, is the probabilistic team $\mathbb{X} \sqcup_k \mathbb{Y}\colon X \cup Y \to [0,1]$ defined as

$$\mathbb{X} \sqcup_k \mathbb{Y}(s) := \begin{cases} k \cdot \mathbb{X}(s) + (1-k) \cdot \mathbb{Y}(s) & \text{if } s \in X \cap Y, \\ k \cdot \mathbb{X}(s) & \text{if } s \in X \setminus Y, \\ (1-k) \cdot \mathbb{Y}(s) & \text{if } s \in Y \setminus X. \end{cases}$$

3 Probabilistic independence logic with Boolean negation

In this section, we define probabilistic independence logic with Boolean negation, denoted by $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim)$. The logic extends first-order logic with *probabilistic independence atom* $\vec{y} \perp\!\!\!\perp_{\vec{x}} \vec{z}$, which states that the tuples $\vec{y}$ and $\vec{z}$ are independent given the tuple $\vec{x}$. This corresponds to the notion of conditional independence that is important, e.g. in probability theory and statistics. The syntax for the logic $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim)$ over a vocabulary τ is as follows:

$$\phi ::= R(\vec{x}) \mid \neg R(\vec{x}) \mid \vec{y} \perp\!\!\!\perp_{\vec{x}} \vec{z} \mid {\sim}\phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid \exists x\phi \mid \forall x\phi,$$

where x is a first-order variable, $\vec{x}, \vec{y}$ and $\vec{z}$ are tuples of first-order variables, and $R \in \tau$.

Let ψ be a first-order formula. We write $\psi^{\neg}$ for the formula, which is obtained from $\neg\psi$ by pushing the negation in front of atomic formulas. We also use the shorthand notations $\psi \to \phi := (\psi^{\neg} \vee (\psi \wedge \phi))$ and $\psi \leftrightarrow \phi := \psi \to \phi \wedge \phi \to \psi$.

Let $\mathbb{X}\colon X \to [0,1]$ be a probability distribution. The semantics for the logic is defined as follows:

$\mathcal{A} \models_{\mathbb{X}} R(\vec{x})$ iff $\mathcal{A} \models_s R(\vec{x})$ for all $s \in \mathrm{supp}(\mathbb{X})$.
$\mathcal{A} \models_{\mathbb{X}} \neg R(\vec{x})$ iff $\mathcal{A} \models_s \neg R(\vec{x})$ for all $s \in \mathrm{supp}(\mathbb{X})$.
$\mathcal{A} \models_{\mathbb{X}} \vec{y} \perp\!\!\!\perp_{\vec{x}} \vec{z}$ iff $|\mathbb{X}_{\vec{x}\vec{y}=s(\vec{x}\vec{y})}| \cdot |\mathbb{X}_{\vec{x}\vec{z}=s(\vec{x}\vec{z})}| = |\mathbb{X}_{\vec{x}\vec{y}\vec{z}=s(\vec{x}\vec{y}\vec{z})}| \cdot |\mathbb{X}_{\vec{x}=s(\vec{x})}|$ for all $s\colon \mathrm{Var}(\vec{x}\vec{y}\vec{z}) \to A$.
$\mathcal{A} \models_{\mathbb{X}} {\sim}\phi$ iff $\mathcal{A} \not\models_{\mathbb{X}} \phi$.
$\mathcal{A} \models_{\mathbb{X}} \phi \wedge \psi$ iff $\mathcal{A} \models_{\mathbb{X}} \phi$ and $\mathcal{A} \models_{\mathbb{X}} \psi$.
$\mathcal{A} \models_{\mathbb{X}} \phi \vee \psi$ iff $\mathcal{A} \models_{\mathbb{Y}} \phi$ and $\mathcal{A} \models_{\mathbb{Z}} \psi$ for some $\mathbb{Y}, \mathbb{Z}, k$ such that $\mathbb{Y} \sqcup_k \mathbb{Z} = \mathbb{X}$.
$\mathcal{A} \models_{\mathbb{X}} \exists x \phi$ iff $\mathcal{A} \models_{\mathbb{X}(F/x)} \phi$ for some $F\colon X \to p_A$.
$\mathcal{A} \models_{\mathbb{X}} \forall x \phi$ iff $\mathcal{A} \models_{\mathbb{X}(A/x)} \phi$.

The satisfaction relation $\models_s$ above refers to the Tarski semantics of first-order logic. For a sentence ϕ, we write $\mathcal{A} \models \phi$ if $\mathcal{A} \models_{\mathbb{X}_\emptyset} \phi$, where $\mathbb{X}_\emptyset$ is the distribution that maps the empty assignment to 1.

Note that the semantics of the first-order atoms, connectives and quantifiers coincide with the corresponding notions from the usual (non-probabilistic) team semantics in the sense that for all first-order formulas ϕ, $\mathcal{A} \models_{\mathbb{X}} \phi$ if and only if $\mathcal{A} \models_{\mathrm{supp}(\mathbb{X})} \phi$. Moreover, this means that for all first-order formulas ϕ, it is enough to check the satisfaction for all the assignments in the support individually, i.e. $\mathcal{A} \models_{\mathbb{X}} \phi$ if and only if $\mathcal{A} \models_s \phi$ for all $s \in \mathrm{supp}(\mathbb{X})$.

The existential quantification can be viewed as extending the probabilistic team to a variable whose distribution over the universe of the structure may depend on the values of the other variables in the team. The universal quantification corresponds to an extension with a variable that is uniformly distributed over the universe of the structure.

The logic also has the following useful property called *locality*. Let $\mathrm{Fr}(\phi)$ be the set of the free variables of a formula ϕ. We say that ϕ is an $\mathcal{L}[\tau]$-formula if it is a formula of logic $\mathcal{L}$ over a vocabulary τ.

PROPOSITION 1 (Locality, [5, Prop. 12]).
Let ϕ be any $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim)[\tau]$-formula. Then for any set of variables V, any τ-structure $\mathcal{A}$, and any probabilistic team $\mathbb{X}\colon X \to [0,1]$ such that $\mathrm{Fr}(\phi) \subseteq V \subseteq D$,

$$\mathcal{A} \models_{\mathbb{X}} \phi \iff \mathcal{A} \models_{\mathbb{X}\upharpoonright V} \phi.$$

In addition to probabilistic conditional independence atoms, we may also consider other atoms. If $\vec{x}$ and $\vec{y}$ are tuples of variables, then $=(\vec{x}, \vec{y})$ is a *dependence atom*. If $\vec{x}$ and $\vec{y}$ are also of the same length, $\vec{x} \approx \vec{y}$ is a *marginal identity atom*. The semantics for these atoms are defined as follows:

$\mathcal{A} \models_{\mathbb{X}} =(\vec{x}, \vec{y})$ iff for all $s, s' \in \mathrm{supp}(\mathbb{X})$, $s(\vec{x}) = s'(\vec{x})$ implies $s(\vec{y}) = s'(\vec{y})$,
$\mathcal{A} \models_{\mathbb{X}} \vec{x} \approx \vec{y}$ iff $|\mathbb{X}_{\vec{x}=\vec{a}}| = |\mathbb{X}_{\vec{y}=\vec{a}}|$ for all $\vec{a} \in A^{|\vec{x}|}$.

The dependence atom $=(\vec{x}, \vec{y})$ expresses the notion of so-called functional dependency states that there is a function $f\colon \{s(\vec{x}) \mid s \in \mathrm{supp}(\mathbb{X})\} \to \{s(\vec{y}) \mid s \in \mathrm{supp}(\mathbb{X})\}$ such that $f(s(\vec{x})) = s(\vec{y})$ for all $s \in \mathrm{supp}(\mathbb{X})$.

Marginal identity $\vec{x} \approx \vec{y}$ states that the probability distributions of $\vec{x}$ and $\vec{y}$ are identical. With probabilistic independence and marginal identity, we can express that a collection random variables is *independent and identically distributed* (IID), which is a common assumption in probability theory and statistics. The formula $\bigwedge_{i=2}^{n} (x_1 \approx x_i \wedge x_1 \dots x_{i-1} \perp\!\!\!\perp x_i)$ expresses the property that the finite random variables $x_1, \dots, x_n$ are IID.

We write $\mathrm{FO}(=(\cdot))$ and $\mathrm{FO}(\approx)$ for first-order logic with dependence atoms or marginal identity atoms, respectively. Analogously, for $C \subseteq \{=(\cdot), \approx, \perp\!\!\!\perp_{\mathrm{c}}, \sim\}$, we write $\mathrm{FO}(C)$ for the logic with access to the atoms (or the Boolean negation) from C.

For two logics L and L' over probabilistic team semantics, we write $L \leq L'$ if for any formula $\phi \in L$, there is a formula $\psi \in L'$ such that $\mathcal{A} \models_{\mathbb{X}} \phi \iff \mathcal{A} \models_{\mathbb{X}} \psi$ for all $\mathcal{A}$ and $\mathbb{X}$. The equivalence $\equiv$ and strict inequivalence $<$ are defined from the above relation in the usual way. The next two propositions follow from the fact that dependence atoms and marginal identity atoms can be expressed with probabilistic independence atoms.[3]

PROPOSITION 2 ([4, Prop. 24]).
$\mathrm{FO}(=(\cdot)) \leq \mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}})$.

PROPOSITION 3 ([15, Thm. 10]).
$\mathrm{FO}(\approx) \leq \mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}})$.

On the other hand, it is known that on the level of open formulae $\mathrm{FO}(\approx)$ and $\mathrm{FO}(=(\cdot))$ are incomparable, for only the former is closed under scaled unions [15] and only the latter is relational[4]. Restricted to sentences, $\mathrm{FO}(=(\cdot))$ can express all properties that are in NP [28], while $\mathrm{FO}(\approx)$ can only express properties in P [12]. It is an open problem, whether $\mathrm{FO}(\approx) < \mathrm{FO}(=(\cdot))$ holds over sentences in general; restricted to finite ordered structures, this is equivalent to the question whether P $<$ NP [12].

Omitting the Boolean negation from the probabilistic independence logic strictly decreases the expressivity:

PROPOSITION 4
$\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}) < \mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim)$.

PROOF. Clearly, $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}) \leq \mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim)$. We show that $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}) \not\equiv \mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim)$. For each $m \in \mathbb{N}$, define $P_m = \{(a_1, \ldots, a_m) \in \mathbb{R}^m \mid \sum_{i=1}^{m} a_i = 1 \text{ and} a_i \geq 0 \text{ for all} 1 \leq i \leq m\} \subseteq \mathbb{R}^m$. We show that any open formula of $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}})$ defines a closed subset of P_m for a suitable m depending on the size of the universe and the number of free variables. Here closed means closed with respect to the subspace topology on P_m induced from the standard topology of $\mathbb{R}^m$, i.e. a set $S \subseteq P_m$ is closed in P_m if and only if $S = P_m \cap F$ for some set $F \subseteq \mathbb{R}^m$ that is closed in $\mathbb{R}^m$.

Let $\phi(v_1, \ldots, v_k) \in \mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}})$. Fix a structure $\mathcal{A}$. We may assume that $A = \{1, \ldots, n\}$. By slightly modifying the construction of the formulas in Lemma 3, we can define the formula $\psi_{\phi,\mathcal{A}}(s_1, \ldots, s_m)$, $m = n^k$, so that it is in the existential loose [0, 1]-guarded fragment of real arithmetic (with constants 0 and 1). Then by Theorem 4.5 of [16], the set $S = \{(a_1, \ldots, a_m) \in \mathbb{R}^n \mid (\mathbb{R}, +, \times, \leq, 0, 1) \models \psi_{\phi,\mathcal{A}}(a_1, \ldots, a_m)\}$ is closed in $\mathbb{R}^m$. Note that now for any probability distribution $\mathbb{X}\colon \{s_1, \ldots, s_m\} \to [0, 1]$, we have $\mathcal{A} \models_{\mathbb{X}} \phi(v_1, \ldots, v_k)$ if and only if $(\mathbb{X}(s_1), \ldots, \mathbb{X}(s_m)) \in S$.

We have $P_m = \{(a_1, \ldots, a_m) \in \mathbb{R}^n \mid (\mathbb{R}, +, \times, \leq, 0, 1) \models \sum_{i=1}^{m} a_i = 1 \wedge \bigwedge_{i=1}^{m} a_i \geq 0\}$, so also P_m is closed in $\mathbb{R}^m$ by Theorem 4.5 of [16]. Note also that P_m is the set of probability distributions in the sense that $(a_1, \ldots, a_m) \in P_m$ if and only if $\mathbb{X}\colon \{s_1, \ldots, s_m\} \to [0, 1]$, $\mathbb{X}(s_i) = a_i$ for all $1 \leq i \leq m$ is a probability distribution. Since S and P_m are closed subsets of $\mathbb{R}^m$ and $S \subseteq P_m$, the set S is also closed in P_m. Hence, the formula $\phi(v_1, \ldots, v_k)$ defines a closed subset S of P_m.

We now show that the formula $\sim x \perp\!\!\!\perp_y z$ cannot be translated to $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}})$. Fix a structure $\mathcal{A}$ such that $A = \{1, 2\}$. Suppose for a contradiction that there is a formula $\phi(x, y, z) \in \mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}})$ such that for all probability distributions $\mathbb{X}\colon \{s_1, \ldots, s_8\} \to [0, 1]$, we have $\mathcal{A} \models_{\mathbb{X}} \phi(x, y, z)$ if and only if $\mathcal{A} \models_{\mathbb{X}} \sim x \perp\!\!\!\perp_y z$. Then $\mathcal{A} \models_{\mathbb{X}} \phi(x, y, z)$ iff $\mathcal{A} \not\models_{\mathbb{X}} x \perp\!\!\!\perp_y z$ iff $(\mathbb{X}(s_1), \ldots, \mathbb{X}(s_8)) \in P_m \setminus S$ for some

[3]A dependence atom $=(\vec{x}, \vec{y})$ is actually a special case of probabilistic conditional independence atom, and it can be expressed as $\vec{y} \perp\!\!\!\perp_{\vec{x}} \vec{y}$.

[4]A logic is relational, if satisfaction of a formula depends only on the support of the team.

closed $S \subseteq P_m$ that is neither the empty set $\emptyset$ nor P_m. (This is because the independence $x \perp\!\!\!\perp_y z$ clearly holds for some, but not for all probability distributions.) Recall that a subset $S \subseteq P_m$ is closed in P_m if and only if its complement is open in P_m. Since $P_n \subseteq \mathbb{R}^m$ is convex, it is also connected. Thus, the only sets that are both open and closed in P_m are the sets $\emptyset$ and P_m. Hence, the formula $\phi(x,y,z) \in \mathrm{FO}(\perp\!\!\!\perp_c)$ does not define a closed subset of P_m, and therefore such formula ϕ cannot exist. □

The following lemma shows that we can use the probabilistic independence atom to express that a team is extended with some distribution $d\colon A^k \to [0,1]$. Note that this differs from the usual k-variable existential quantification in probabilistic team semantics, because here the distribution over A^k is the same for all $s \in X$. If we have the Boolean negation, we can also state that a formula must hold when the team is extended with *any* distribution $d\colon A^k \to [0,1]$. The lemma will be used in Section 5 to show that the logics $\mathrm{FO}(\perp\!\!\!\perp_c, \sim)$ and $\mathrm{SO}_{\mathbb{R}}(+,\times)$ are equi-expressive.

LEMMA 1
We use the abbreviations $\forall^* x\phi$ and $\phi \to^* \psi$ for the $\mathrm{FO}(\perp\!\!\!\perp_c, \sim, \approx)$-formulas $\sim\exists x{\sim}\phi$ and $\sim(\phi \wedge \sim\psi)$, respectively. Let $\phi_\exists := \exists\vec{y}(\vec{x} \perp\!\!\!\perp \vec{y} \wedge \psi(\vec{x},\vec{y}))$ and $\phi_\forall := \forall^*\vec{y}(\vec{x} \perp\!\!\!\perp \vec{y} \to^* \psi(\vec{x},\vec{y}))$ be $\mathrm{FO}(\perp\!\!\!\perp_c, \sim, \approx)$-formulas with free variables from $\vec{x} = (x_1,\ldots,x_n)$ such that $\vec{y}$ are disjoint from $\vec{x}$. Then for any structure $\mathcal{A}$ and probabilistic team $\mathbb{X}$ over $\{x_1,\ldots,x_n\}$,

(i) $\mathcal{A} \models_{\mathbb{X}} \phi_\exists$ iff $\mathcal{A} \models_{\mathbb{X}(d/\vec{y})} \psi$ for some distribution $d\colon A^{|\vec{y}|} \to [0,1]$,
(ii) $\mathcal{A} \models_{\mathbb{X}} \phi_\forall$ iff $\mathcal{A} \models_{\mathbb{X}(d/\vec{y})} \psi$ for all distributions $d\colon A^{|\vec{y}|} \to [0,1]$.

PROOF. Let $\mathbb{Y} := \mathbb{X}(\vec{F}/\vec{y})$ for some sequence of functions $\vec{F} = (F_1,\ldots,F_{|\vec{y}|})$ such that $F_i\colon X(A/y_1)\ldots(A/y_i) \to p_A$. Now

$$\mathcal{A} \models_{\mathbb{Y}} \vec{x} \perp\!\!\!\perp \vec{y} \iff |\mathbb{Y}_{\vec{x}\vec{y}=s(\vec{x})\vec{a}}| = |\mathbb{Y}_{\vec{x}=s(\vec{x})}| \cdot |\mathbb{Y}_{\vec{y}=\vec{a}}| \text{ for all } s \in X, \vec{a} \in A^{|\vec{y}|}.$$

Since the variables $\vec{y}$ are fresh, the right-hand side becomes $\mathbb{X}(s) \cdot F_1(s)(a_1) \cdot \ldots \cdot F_{|\vec{y}|}(s(a_1/y_1)\ldots(a_{|\vec{y}|-1}/y_{|\vec{y}|-1}))(a_{|\vec{y}|}) = \mathbb{X}(s) \cdot |\mathbb{Y}_{\vec{y}=\vec{a}}|$ for all $s \in X, \vec{a} \in A^{|\vec{y}|}$, i.e. $\mathbb{X}(\vec{F}/\vec{y}) = \mathbb{X}(d/\vec{y})$ for some distribution $d\colon A^{|\vec{y}|} \to [0,1]$. It is now straightforward to check that the two claims hold. □

4 Metafinite logics

In this section, we consider logics over $\mathbb{R}$-structures. These structures extend finite relational structures with real numbers $\mathbb{R}$ as a second domain and add functions that map tuples from the finite domain to $\mathbb{R}$.

DEFINITION 1 ($\mathbb{R}$-structures).
Let τ and σ be finite vocabularies such that τ is relational and σ is functional. An $\mathbb{R}$-*structure of vocabulary* $\tau \cup \sigma$ is a tuple $\mathcal{A} = (A, \mathbb{R}, F)$ where the reduct of $\mathcal{A}$ to τ is a finite relational structure, and F is a set that contains functions $f^{\mathcal{A}}\colon A^{\mathrm{Ar}(f)} \to \mathbb{R}$ for each function symbol $f \in \sigma$. Additionally,

(i) for any $S \subseteq \mathbb{R}$, if each $f^{\mathcal{A}}$ is a function from $A^{\mathrm{Ar}(f)}$ to S, $\mathcal{A}$ is called an S-*structure*,
(ii) if each $f^{\mathcal{A}}$ is a distribution, $\mathcal{A}$ is called a $d[0,1]$-*structure*.

Next, we will define certain metafinite logics that are variants of functional second-order logic with numerical terms. The numerical σ-terms i are defined as follows:

$$i ::= f(\vec{x}) \mid i \times i \mid i + i \mid \mathrm{SUM}_{\vec{y}} i \mid \log i,$$

where $f \in \sigma$ and $\vec{x}$ and $\vec{y}$ are first-order variables such that $|\vec{x}| = \mathrm{Ar}(f)$. The interpretation of a numerical term i in the structure $\mathcal{A}$ under an assignment s is denoted by $[i]_s^{\mathcal{A}}$. We define

$$[\mathrm{SUM}_{\vec{y}} i]_s^{\mathcal{A}} := \sum_{\vec{a} \in A^{|\vec{y}|}} [i]_{s(\vec{a}/\vec{y})}^{\mathcal{A}}.$$

The interpretations of the rest of the numerical terms are defined in the obvious way.

Suppose that $\{=\} \subseteq \tau$, and let $O \subseteq \{+, \times, \mathrm{SUM}, \log\}$. The syntax for the logic $\mathrm{SO}_{\mathbb{R}}(O)$ is defined as follows:

$$\phi ::= i = j \mid \neg i = j \mid R(\vec{x}) \mid \neg R(\vec{x}) \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid \exists x\phi \mid \forall x\phi \mid \exists f \psi \mid \forall f \psi,$$

where i and j are numerical σ-terms constructed using operations from O, $R \in \tau$, x, y and $\vec{x}$ are first-order variables, f is a function variable and ψ is a $\tau \cup \sigma \cup \{f\}$-formula of $\mathrm{SO}_{\mathbb{R}}(O)$.

The semantics of $\mathrm{SO}_{\mathbb{R}}(O)$ is defined via $\mathbb{R}$-structures and assignments analogous to first-order logic, except for the interpretations of function variables f, which range over functions $A^{\mathrm{Ar}(f)} \to \mathbb{R}$. For any $S \subseteq \mathbb{R}$, we define $\mathrm{SO}_S(O)$ as the variant of $\mathrm{SO}_{\mathbb{R}}(O)$, where the quantification of function variables ranges over $A^{\mathrm{Ar}(f)} \to S$. We write $\mathrm{SO}_{d[0,1]}(O)$ for the logic where the quantification of function variables is restricted to distributions. The existential fragment, in which universal quantification over function variables is not allowed, is denoted by $\mathrm{ESO}_{\mathbb{R}}(O)$.

For metafinite logics L and L' (over the same vocabulary $\tau \cup \sigma$), we define the expressivity comparison relations as in [16]. Let $X \subseteq R$ or $X = d[0,1]$. For a formula $\phi \in L$, let $\mathrm{Struc}^{X,s}(\phi)$ to be the class of X-structures $\mathcal{A}$ of vocabulary $\tau \cup \sigma$ such that $\mathcal{A} \models_s \phi$. We write $L \leq_X L'$ if for all formulas $\phi \in L$, there is a formula $\psi \in L'$ such that $\mathrm{Struc}^{X,s}(\phi) = \mathrm{Struc}^{X,s}(\psi)$ for all s. The relations $L \equiv_X L'$, and $L <_X L'$ are defined in the obvious way. If $X = \mathbb{R}$, we drop the set X from the notation, and just write $L \leq L'$, $L \equiv L'$ and $L < L'$.

Note that the subscript S in $\mathrm{SO}_S(O)$ refers to the class of functions that can be quantified, and the superscript X in $\mathrm{Struc}^{X,s}(\phi)$ is the class of functions available for the function symbols in the vocabulary. This means that we may define the class $\mathrm{Struc}^{X,s}(\phi)$ for a formula $\phi \in \mathrm{SO}_S(O)$ also when $X \neq S$.

PROPOSITION 5
$\mathrm{SO}_{\mathbb{R}}(\mathrm{SUM}, \times) \equiv \mathrm{SO}_{\mathbb{R}}(+, \times)$.

PROOF. First, note that since the constants 0 and 1 are definable in both logics, we may use them when needed. To show that $\mathrm{SO}_{\mathbb{R}}(\mathrm{SUM}, \times) \leq \mathrm{SO}_{\mathbb{R}}(+, \times)$, it suffices to show that any numerical identity $f(\vec{x}) = \mathrm{SUM}_{\vec{y}} g(\vec{x}, \vec{y})$ can also be expressed in $\mathrm{SO}_{\mathbb{R}}(+, \times)$. Suppose that $|\vec{y}| = n$. Since the domain of $\mathcal{A}$ is finite, we may assume that it is linearly ordered: a linear order $\leq_{\mathrm{fin}}$ can be defined with an existentially quantified binary function variable f such that the formulas $f(x,y) = 1$ and $f(x,y) = 0$ correspond to $x \leq_{\mathrm{fin}} y$ and $x \not\leq_{\mathrm{fin}} y$, respectively. Then, without loss of generality, we may assume that we have an n-ary successor function S defined by the lexicographic order induced by the linear order. Thus, we can existentially quantify a function variable h such that

$$\forall \vec{x}\vec{z}(h(\vec{x}, \mathbf{min}) = g(\vec{x}, \mathbf{min}) \wedge h(\vec{x}, S(\vec{z})) = h(\vec{x}, \vec{z}) + g(\vec{x}, S(\vec{z})).$$

Then $f(\vec{x}) = h(\vec{x}, \mathbf{max})$ is as wanted.

To show that $\mathrm{SO}_{\mathbb{R}}(+,\times) \leq \mathrm{SO}_{\mathbb{R}}(\mathrm{SUM},\times)$, we show that any numerical identity $f(\vec{x}\vec{y}) = i(\vec{x}) + j(\vec{y})$ can be expressed in $\mathrm{SO}_{\mathbb{R}}(\mathrm{SUM},\times)$. We can existentially quantify a function variable g such that

$$g(\vec{x}\vec{y}, min) = i(\vec{x}) \wedge g(\vec{x}\vec{y}, max) = j(\vec{y})$$
$$\wedge \forall z((\neg z = min \wedge \neg z = max) \rightarrow g(\vec{x}\vec{y}, z) = 0).$$

Then $f(\vec{x}\vec{y}) = \mathrm{SUM}_z g(\vec{x}\vec{y}, z)$ is as wanted. Note that since no universal quantification over function variables was used, the proposition also holds for existential fragments, i.e. $\mathrm{ESO}_{\mathbb{R}}(\mathrm{SUM},\times) \equiv \mathrm{ESO}_{\mathbb{R}}(+,\times)$. □

PROPOSITION 6
$\mathrm{SO}_{d[0,1]}(\mathrm{SUM},\times) \equiv \mathrm{SO}_{\mathbb{R}}(+,\times)$.

PROOF. Since 1 is definable in $\mathrm{SO}_{\mathbb{R}}(\mathrm{SUM},\times)$ and the formula $\mathrm{SUM}_{\vec{x}} f(\vec{x}) = 1$ states that f is a probability distribution, we have that $\mathrm{SO}_{d[0,1]}(\mathrm{SUM},\times) \leq \mathrm{SO}_{\mathbb{R}}(\mathrm{SUM},\times) \equiv \mathrm{SO}_{\mathbb{R}}(+,\times)$.

Next, we show that

$$\mathrm{SO}_{\mathbb{R}}(+,\times) \leq \mathrm{SO}_{\mathbb{R}_{\geq 0}}(+,\times) \leq \mathrm{SO}_{[0,1]}(+,\times) \leq \mathrm{SO}_{d[0,1]}(\mathrm{SUM},\times).$$

To show that $\mathrm{SO}_{\mathbb{R}}(+,\times) \leq \mathrm{SO}_{\mathbb{R}_{\geq 0}}(+,\times)$, let $\phi \in \mathrm{SO}_{\mathbb{R}}(+,\times)$. Note that any function $f\colon A^{\mathrm{Ar}(f)} \to \mathbb{R}$ can be expressed as $f_+ - f_-$, where f_+ and f_- are functions $A^{\mathrm{Ar}(f)} \to \mathbb{R}_{\geq 0}$ such that $f_+(\vec{x}) = f(\vec{x}) \cdot \chi_{\mathbb{R}_{\geq 0}}(f(\vec{x}))$ and $f_-(\vec{x}) = f(\vec{x}) \cdot \chi_{\mathbb{R}\setminus\mathbb{R}_{\geq 0}}(f(\vec{x}))$, where $\chi_S\colon \mathbb{R} \to \{0,1\}$ is the characteristic function of $S \subseteq \mathbb{R}$. Since numerical terms $i(\vec{x}) - j(\vec{x})$ can clearly be expressed in $\mathrm{SO}_{\mathbb{R}}(+,\times)$ by moving the term $j(\vec{x})$ to the other side of any numerical (in)equality atom in which the term $i(\vec{x}) - j(\vec{x})$ appears, it suffices to modify ϕ as follows: for all quantified function variables f, replace each appearance of term $f(\vec{x})$ with $f_+(\vec{x}) - f_-(\vec{x})$ and instead of f, quantify two function variables f_+ and f_-.

To show that $\mathrm{SO}_{\mathbb{R}_{\geq 0}}(+,\times) \leq \mathrm{SO}_{[0,1]}(+,\times)$, let $\phi \in \mathrm{SO}_{\mathbb{R}_{\geq 0}}(+,\times)$. Note that any positive real number can be written as a ratio $x/(1-x)$, where $x \in [0,1)$. Since numerical terms of the form $i(\vec{x})/(1-i(\vec{x}))$ can clearly be expressed in $\mathrm{SO}_{d[0,1]}(+,\times)$, it suffices to modify ϕ as follows: for all quantified function variables f, replace each appearance of term $f(\vec{x})$ with $f^*(\vec{x})/(1-f^*(\vec{x}))$ and instead of f, quantify a function variable f^* such that $f^*(\vec{x}) \neq 1$ for all $\vec{x}$.

Lastly, to show that $\mathrm{SO}_{[0,1]}(+,\times) \leq \mathrm{SO}_{d[0,1]}(\mathrm{SUM},\times)$, it suffices to see that for any $\phi \in \mathrm{SO}_{[0,1]}(+,\times)$, we can compress each function term into a fraction of size $1/n^k$, where n is the size of the finite domain and k the maximal arity of any function variable appearing in ϕ. We omit the proof, since it is essentially the same as the one for Lemma 6.4 in [16]. □

4.1 *Expressivity comparison between logics with probabilistic team semantics and metafinite logics*

In this section, we define expressivity comparison relations $\leq$, $\equiv$ and $<$ between probabilistic team-based logics and metafinite logics.

In the following, let $C \subseteq \{=\!(\cdot), \approx, \perp\!\!\!\perp_{\mathrm{c}}, \sim, \mathrm{H}\}$, $O = \{+,\times,\mathrm{SUM},\log\}$ and $S \subseteq \mathbb{R}$.

DEFINITION 2
We write $\mathrm{FO}(C) \leq \mathrm{SO}_S(O)$, if for every formula $\phi(\vec{v}) \in \mathrm{FO}(C)$ such that its free variables are from $\vec{v} = (v_1,\ldots,v_k)$, there is a formula $\psi_\phi(f) \in \mathrm{SO}_S(O)$ with exactly one free function variable f such that for all structures $\mathcal{A}$ and all probabilistic teams $\mathbb{X}\colon X \to [0,1]$, $\mathcal{A} \models_{\mathbb{X}} \phi(\vec{v})$ if and only if $(\mathcal{A}, f_{\mathbb{X}}) \models \psi_\phi(f)$, where $f_{\mathbb{X}}\colon A^k \to [0,1]$ is a function such that $f_{\mathbb{X}}(s(\vec{v})) = \mathbb{X}(s)$ for all $s \in X$.

DEFINITION 3
We write $\mathrm{SO}_S(O) \leq \mathrm{FO}(C)$, if for every formula $\phi(p) \in \mathrm{SO}_S(O)$ with exactly one free function variable p, with $\mathrm{Ar}(p) = k$, there is a formula $\psi_\phi(v_1, \ldots, v_k) \in \mathrm{FO}(C)$ such that for all structures $\mathcal{A}$, and all probability distributions $p^{\mathcal{A}}$ over A^k, $\mathcal{A} \models_{\mathbb{X}} \psi_\phi(\vec{v})$ if and only if $(\mathcal{A}, p^{\mathcal{A}}) \models \phi(p)$, where the probabilistic team $\mathbb{X}\colon X \to [0,1]$ is such that $\mathbb{X}(s) = p^{\mathcal{A}}(s(v_1, \ldots, v_k))$.

The relations $\equiv$ and $<$ are then defined from $\leq$ as usual. The definitions can be extended to the $d[0,1]$-fragment and the existential fragment of $\mathrm{SO}_S(O)$ in the obvious way.

5 Equi-expressivity of $\mathbf{FO}(\perp\!\!\!\perp_{\mathbf{c}}, \sim)$ and $\mathbf{SO}_{\mathbb{R}}(+, \times)$

In this section, we show that the expressivity of probabilistic independence logic with the Boolean negation coincides with the second-order logic over $\mathbb{R}$-structures. The expressivity of $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}})$ corresponds to the loose fragment of $\mathrm{ESO}_{d[0,1]}(+, \times)$ [5, 16]. Recall that the logic $\mathrm{ESO}_{d[0,1]}(+, \times)$ is the fragment of $\mathrm{SO}_{\mathbb{R}}(+, \times)$ where universal quantification of function variables is disallowed and the interpretations of existentially quantified functions must be distributions. In the loose fragment, called $\text{L-ESO}_{d[0,1]}(+, \times)$, we have an additional restriction that the negated numerical atoms are not allowed. Our result demonstrates that adding the Boolean negation to probabilistic independence logic increases its expressivity from existential second-order logic to the level of full second-order logic.

THEOREM 1
$\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim) \equiv \mathrm{SO}_{\mathbb{R}}(+, \times)$.

We first show that $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim) \leq \mathrm{SO}_{\mathbb{R}}(+, \times)$. Note that by Proposition 6, we have $\mathrm{SO}_{d[0,1]}(\mathrm{SUM}, \times) \equiv \mathrm{SO}_{\mathbb{R}}(+, \times)$, so it suffices to show that $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim) \leq \mathrm{SO}_{d[0,1]}(\mathrm{SUM}, \times)$. We may assume that every independence atom is in the form $\vec{y} \perp\!\!\!\perp_{\vec{x}} \vec{z}$ or $\vec{y} \perp\!\!\!\perp_{\vec{x}} \vec{y}$ where $\vec{x}, \vec{y}$, and $\vec{z}$ are pairwise disjoint tuples. [5, Lemma 25].

THEOREM 2
$\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim) \leq \mathrm{SO}_{\mathbb{R}}(+, \times)$

PROOF. Let formula $\phi(\vec{v}) \in \mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim)$ be such that its free variables are from $\vec{v} = (v_1, \ldots, v_k)$. We show that there is a formula $\psi_\phi(f) \in \mathrm{SO}_{d[0,1]}(\mathrm{SUM}, \times)$ with exactly one free function variable such that for all structures $\mathcal{A}$ and all probabilistic teams $\mathbb{X}\colon X \to [0,1]$, $\mathcal{A} \models_{\mathbb{X}} \phi(\vec{v})$ if and only if $(\mathcal{A}, f_{\mathbb{X}}) \models \psi_\phi(f)$, where $f_{\mathbb{X}}\colon A^k \to [0,1]$ is a probability distribution such that $f_{\mathbb{X}}(s(\vec{v})) = \mathbb{X}(s)$ for all $s \in X$.

Define the formula $\psi_\phi(f)$ as follows:

1. If $\phi(\vec{v}) = R(v_{i_1}, \ldots, v_{i_l})$, where $1 \leq i_1, \ldots, i_l \leq k$, then $\psi_\phi(f) := \forall \vec{v}(f(\vec{v}) = 0 \vee R(v_{i_1}, \ldots, v_{i_l}))$.
2. If $\phi(\vec{v}) = \neg R(v_{i_1}, \ldots, v_{i_l})$, where $1 \leq i_1, \ldots, i_l \leq k$, then $\psi_\phi(f) := \forall \vec{v}(f(\vec{v}) = 0 \vee \neg R(v_{i_1}, \ldots, v_{i_l}))$.
3. If $\phi(\vec{v}) = \vec{v}_1 \perp\!\!\!\perp_{\vec{v}_0} \vec{v}_2$, where $\vec{v}_0, \vec{v}_1, \vec{v}_2$ are disjoint, then

$$\begin{aligned}\psi_\phi(f) := \forall \vec{v}_0 \vec{v}_1 \vec{v}_2 (&\mathrm{SUM}_{\vec{v} \setminus (\vec{v}_0 \vec{v}_1)} f(\vec{v}) \times \mathrm{SUM}_{\vec{v} \setminus (\vec{v}_0 \vec{v}_2)} f(\vec{v}) \\ &= \mathrm{SUM}_{\vec{v} \setminus (\vec{v}_0 \vec{v}_1)} f(\vec{v}) \times \mathrm{SUM}_{\vec{v} \setminus \vec{v}_0} f(\vec{v})).\end{aligned}$$

4. If $\phi(\vec{v}) = \vec{v}_1 \perp\!\!\!\perp_{\vec{v}_0} \vec{v}_1$, where $\vec{v}_0, \vec{v}_1$ are disjoint, then

$$\psi_\phi(f) := \forall \vec{v}_0 \vec{v}_1 (\mathrm{SUM}_{\vec{v}\setminus(\vec{v}_0\vec{v}_1)} f(\vec{v}) = 0 \vee \mathrm{SUM}_{\vec{v}\setminus(\vec{v}_0\vec{v}_1)} f(\vec{v}) = \mathrm{SUM}_{\vec{v}\setminus\vec{v}_0} f(\vec{v})).$$

5. If $\phi(\vec{v}) = \sim\phi_0(\vec{v})$, then $\psi_\phi(f) := \psi^{\neg}_{\phi_0}(f)$, where $\psi^{\neg}_{\phi_0}$ is obtained from $\neg\psi_{\phi_0}$ by pushing the negation in front of atomic formulas.
6. If $\phi(\vec{v}) = \phi_0(\vec{v}) \wedge \phi_1(\vec{v})$, then $\psi_\phi(f) := \psi_{\phi_0}(f) \wedge \psi_{\phi_1}(f)$.
7. If $\phi(\vec{v}) = \phi_0(\vec{v}) \vee \phi_1(\vec{v})$, then

$$\begin{aligned}\psi_\phi(f) := \;& \psi_{\phi_0}(f) \vee \psi_{\phi_1}(f) \\ & \vee (\exists g_0 g_1 g_2 g_3 (\forall \vec{v} \forall x (x = l \vee x = r \vee (g_0(x) = 0 \wedge g_3(\vec{v}, x) = 0)) \\ & \wedge \forall \vec{v} (g_3(\vec{v}, l) = g_1(\vec{v}) \times g_0(l) \wedge g_3(\vec{v}, r) = g_2(\vec{v}) \times g_0(r)) \\ & \wedge \forall \vec{v} (\mathrm{SUM}_x g_3(\vec{v}, x) = f(\vec{v})) \wedge \psi_{\phi_0}(g_1) \wedge \psi_{\phi_1}(g_2))),\end{aligned}$$

where l and r are the same as *min* and *max*, respectively, and thus they can be defined as in the proof of Proposition 5.
8. If $\phi(\vec{v}) = \exists x \phi_0(\vec{v}, x)$, then $\psi_\phi(f) := \exists g (\forall \vec{v} (\mathrm{SUM}_x g(\vec{v}, x) = f(\vec{v})) \wedge \psi_{\phi_0}(g))$.
9. If $\phi(\vec{v}) = \forall x \phi_0(\vec{v}, x)$, then

$$\psi_\phi(f) := \exists g (\forall \vec{v} (\forall x \forall y (g(\vec{v}, x) = g(\vec{v}, y)) \wedge \mathrm{SUM}_x g(\vec{v}, x) = f(\vec{v})) \wedge \psi_{\phi_0}(g)).$$

Since the above is essentially same as the translation in [5, Theorem 14], but extended with the Boolean negation (for which the claim follows directly from the semantical clauses), it is easy to show that $\psi_\phi(f)$ satisfies the claim.

For items 3 and 4, note that for any $\vec{v}_i$ such that $\mathrm{Var}(\vec{v}_i) \subseteq \mathrm{Var}(\vec{v})$, we have $[\mathrm{SUM}_{\vec{v}\setminus\vec{v}_i} f(\vec{v})]^{\mathcal{A}}_s = \sum_{\vec{a} \in A^{|\vec{v}\setminus\vec{v}_i|}} f_{\mathbb{X}}(s(\vec{a}/(\vec{v}\setminus\vec{v}_i))(\vec{v})) = |\mathbb{X}_{\vec{v}_i = s(\vec{v}_i)}|$, because $f_{\mathbb{X}}(s(\vec{v})) = \mathbb{X}(s)$ for all $s \in X$. Hence, the formula of item 3 just expresses the semantics of the atom $\vec{v}_1 \perp\!\!\!\perp_{\vec{v}_0} \vec{v}_2$, and similarly in the special case of the atom $\vec{v}_1 \perp\!\!\!\perp_{\vec{v}_0} \vec{v}_1$ in item 4.

For item 7, recall that $\mathcal{A} \models_{\mathbb{X}} \phi_0 \vee \phi_1$ iff $\mathcal{A} \models_{\mathbb{Y}} \phi_0$ and $\mathcal{A} \models_{\mathbb{Z}} \phi_1$ for some $\mathbb{Y}, \mathbb{Z}, k$ such that $\mathbb{Y} \sqcup_k \mathbb{Z} = \mathbb{X}$. The purpose of the functions g_1 and g_2 is that they correspond to the probabilistic teams $\mathbb{Y}$ and $\mathbb{Z}$, respectively. The function g_0 is used to express k, in the following way: $g_0(l)$ corresponds to k, $g_0(r)$ corresponds to $1 - k$, and otherwise g_0 is just 0. The function g_3 is used to express that $\mathbb{Y} \sqcup_k \mathbb{Z} = \mathbb{X}$.

In items 8 and 9, the quantification of the function g of arity $\mathrm{Ar}(f) + 1$ corresponds to the way that quantifying a new variable modifies the probabilistic team for the other logic. ☐

We now show that $\mathrm{SO}_{\mathbb{R}}(+, \times) \leq \mathrm{FO}(\perp\!\!\!\perp_c, \sim)$. By Propositions 3 and 6, $\mathrm{FO}(\perp\!\!\!\perp_c, \sim, \approx) \equiv \mathrm{FO}(\perp\!\!\!\perp_c, \sim)$ and $\mathrm{SO}_{\mathbb{R}}(+, \times) \equiv \mathrm{SO}_{d[0,1]}(\mathrm{SUM}, \times)$, so it suffices to show that $\mathrm{SO}_{d[0,1]}(\mathrm{SUM}, \times) \leq \mathrm{FO}(\perp\!\!\!\perp_c, \sim, \approx)$.

Note that even though we consider $\mathrm{SO}_{d[0,1]}(\mathrm{SUM}, \times)$, where only distributions can be quantified, it may still happen that the interpretation of a numerical term does not belong to the unit interval. This may happen if we have a term of the form $\mathrm{SUM}_{\vec{x}} i(\vec{y})$ where $\vec{x}$ contains a variable that does not appear in $\vec{y}$. Fortunately, for any formula containing such terms, there is an equivalent formula without them [12, Lemma 19]. Thus, it suffices to consider formulas without such terms.

To prove that $\mathrm{SO}_{d[0,1]}(\mathrm{SUM}, \times) \leq \mathrm{FO}(\perp\!\!\!\perp_c, \sim, \approx)$, we construct a useful normal form for $\mathrm{SO}_{d[0,1]}(\mathrm{SUM}, \times)$-sentences. The normal form is convenient for our translation, because all the distribution quantifiers are in front of the formula, so we can express them with probabilistic independence atoms as described in Lemma 5. Moreover, the form of the numerical atoms allows us to use marginal identity and probabilistic independence to express them in Theorem 15. The following lemma is based on similar lemmas from [5, Lemma, 16] and [12, Lemma, 20].

DEFINITION 4
We say that a formula $\phi \in \mathrm{SO}_{d[0,1]}(\mathrm{SUM}, \times)$ is in normal form if it is in the form $\phi := Q_1 f_1 \ldots Q_n f_n \forall \vec{x} \theta$, where $Q_i \in \{\exists, \forall\}$, θ is quantifier-free and such that all the numerical identity atoms are in the form $f_i(\vec{u}\vec{v}) = f_j(\vec{u}) \times f_k(\vec{v})$ or $f_i(\vec{u}) = \mathrm{SUM}_{\vec{v}} f_j(\vec{u}\vec{v})$ for distinct f_i, f_j, f_k.

LEMMA 2
For every formula $\phi \in \mathrm{SO}_{d[0,1]}(\mathrm{SUM}, \times)$, there is an equivalent formula ϕ^* that is in the normal form of Definition 4.

PROOF. We begin by defining a formula θ_i for each numerical term $i(\vec{x})$ using fresh function symbols f_i.

1. If $i(\vec{u}) = g(\vec{u})$ where g is a function symbol, then θ_i is defined as $f_i(\vec{u}) = \mathrm{SUM}_{\emptyset} g(\vec{u})$.
2. If $i(\vec{u}\vec{v}) = j(\vec{u}) \times k(\vec{v})$, then θ_i is defined as $\theta_j \wedge \theta_k \wedge f_i(\vec{u}\vec{v}) = f_j(\vec{u}) \times f_k(\vec{v})$.
3. If $i(\vec{u}) = \mathrm{SUM}_{\vec{v}} j(\vec{u}\vec{v})$, then θ_i is defined as $\theta_j \wedge f_i(\vec{u}) = \mathrm{SUM}_{\vec{v}} f_j(\vec{u}\vec{v})$.

Then the formula ϕ^* is defined as follows:

1. If $\phi = i(\vec{u}) = j(\vec{v})$, then $\phi^* := \exists \vec{f}(f_i(\vec{u}) = f_j(\vec{v}) \wedge \theta_i \wedge \theta_j)$ where $\vec{f}$ consists of the function symbols f_k for each subterm k of i or j. The negated case $\phi = \neg i(\vec{u}) = j(\vec{v})$ is analogous; just add negation in front of $f_i(\vec{u}) = f_j(\vec{v})$.
2. If ϕ is an atom or a negated atom (of the first sort), then $\phi^* := \phi$.
3. If $\phi = \psi_0 \circ \psi_1$, where $\circ \in \{\wedge, \vee\}$ and $\psi_i^* = Q^i_1 f^i_1 \ldots Q^i_{m_i} f^i_{m_i} \forall \vec{x}_i \theta_i$ for $i = 0, 1$, then $\phi^* := Q^0_1 f^0_1 \ldots Q^0_{m_0} f^0_{m_0} Q^1_1 f^1_1 \ldots Q^1_{m_1} f^1_{m_1} \forall \vec{x}_0 \vec{x}_1 (\theta_0 \circ \theta_1)$.
4. If $\phi = \exists y \psi$, where $\psi^* = Q_1 f_1 \ldots Q_m f_m \forall \vec{x} \theta$, then
$$\phi^* := \exists g Q_1 f_1 \ldots Q_m f_m \forall \vec{x} \forall \vec{y} (g(y) = 0 \vee \theta).$$
5. Let $\phi = \forall y \psi$, where $\psi^* = Q_1 f_1 \ldots Q_m f_m \forall \vec{x} \theta$. Then define
$$\begin{aligned}\phi^* :=& Q_1 f_1^* \ldots Q_m f_m^* \exists \vec{f}_{id} \exists d \forall y y' \forall \vec{x} (d(y) = d(y') \wedge \\ & (\mathrm{SUM}_{\vec{x}} f_1^*(y, \vec{x}) = d(y) \circ_1 (\mathrm{SUM}_{\vec{x}} f_2^*(y, \vec{x}) = d(y) \circ_2 \ldots \\ & \circ_{m-1} (\mathrm{SUM}_{\vec{x}} f_m^*(y, \vec{x}) = d(y) \circ_m \theta^*) \ldots))),\end{aligned}$$
where each f_i^*, for $1 \leq i \leq n$, is such that $\mathrm{Ar}(f_i^*) = \mathrm{Ar}(f_i) + 1$, $\vec{f}_{id}$ introduces a new function symbol for each multiplication in θ,
$$\circ_i := \begin{cases} \wedge & \text{if } Q_i = \exists, \\ \rightarrow & \text{if } Q_i = \forall, \end{cases}$$
and the formula θ^* is obtained from θ by replacing all second sort identities α of the form $f_i(\vec{u}\vec{v}) = f_j(\vec{u}) \times f_k(\vec{v})$ with
$$f_\alpha(y, \vec{u}\vec{v}) = d(y) \times f_i^*(y, \vec{u}\vec{v}) \wedge f_\alpha(y, \vec{u}\vec{v}) = f_j^*(y, \vec{u}) \times f_k^*(y, \vec{v})$$
and $f_i(\vec{u}) = \mathrm{SUM}_{\vec{v}} f_j(\vec{u}\vec{v})$ with $f_i^*(y, \vec{u}) = \mathrm{SUM}_{\vec{v}} f_j^*(y, \vec{u}\vec{v})$.
6. If $\phi = Qf\psi$, where $Q \in \{\exists, \forall\}$ and $\psi^* = Q_1 f_1 \ldots Q_m f_m \forall \vec{x} \theta$, then $\phi^* := Qf\psi^*$.

It is straightforward to check that ϕ^* is as wanted. In (3), we may assume that no variable in $\vec{x}_i$ appears free in θ_{1-i} for $i = 0, 1$; if this is not the case, we can rename the bound variables in θ_{1-i}. In (4), there must be at least one y for which $g(y) \neq 0$, because otherwise g would not be a distribution. This means that there must exist y for which θ holds. In (5), instead of quantifying for each y a distribution f_y, we quantify a single distribution f^* such that $f^*(y, \vec{x}) = \frac{1}{|A|} \cdot f_y(\vec{x})$, where A is the domain of our structure. For this, we existentially quantify a unary uniform distribution d such that $d(y) = \frac{1}{|A|}$ for all y, and state that $\mathrm{SUM}_{\vec{x}} f^*(y, \vec{x}) = d(y)$ for a fixed y. Each symbol $\circ_i$ is interpreted either as a conjunction or an implication depending on the quantifier Q_i. This ensures that the condition for f_i^* is interpreted appropriately in either case. For the replace second sort identities, we now have $f_y(\vec{u}\vec{v}) = g_y(\vec{u}) \cdot h_y(\vec{v})$ iff $\frac{1}{|A|} f^*(y, \vec{u}\vec{v}) = g^*(y, \vec{u}) \cdot h^*(y, \vec{v})$ iff $d(y) \cdot f^*(y, \vec{u}\vec{v}) = g^*(y, \vec{u}) \cdot h^*(y, \vec{v})$ and $f_y(\vec{u}) = \mathrm{SUM}_{\vec{v}} g_y(\vec{u}\vec{v})$ iff $f^*(y, \vec{u}) = \mathrm{SUM}_{\vec{v}} g^*(y, \vec{u}\vec{v})$. □

THEOREM 3
$\mathrm{SO}_{d[0,1]}(\mathrm{SUM}, \times) \leq \mathrm{FO}(\perp\!\!\!\perp_c, \sim, \approx)$

PROOF. Let $\phi(p) \in \mathrm{SO}_{d[0,1]}(\mathrm{SUM}, \times)$. By Lemma 2, we may assume that the formula is in the form $\phi := Q_1 f_1 \dots Q_n f_n \forall \vec{x} \theta$, where $Q_i \in \{\exists, \forall\}$, θ is quantifier-free and such that all the numerical identity atoms are in the form $f_i(\vec{u}\vec{v}) = f_j(\vec{u}) \times f_k(\vec{v})$ or $f_i(\vec{u}) = \mathrm{SUM}_{\vec{v}} f_j(\vec{u}\vec{v})$ for distinct f_i, f_j, f_k from $\{f_1, \dots, f_n, p\}$. We show that there is a formula $\Phi \in \mathrm{FO}(\perp\!\!\!\perp_c, \sim, \approx)$ such that for all structures $\mathcal{A}$ and probabilistic teams $\mathbb{X} := p^{\mathcal{A}}$,

$$\mathcal{A} \models_{\mathbb{X}} \Phi \text{ if and only if } (\mathcal{A}, p) \models \phi.$$

Define

$$\Phi := \forall \vec{x} Q_1^* \vec{y}_1 (\vec{x} \perp\!\!\!\perp \vec{y}_1 \circ_1 Q_2^* \vec{y}_2 (\vec{x}\vec{y}_1 \perp\!\!\!\perp \vec{y}_2 \circ_2 Q_3^* \vec{y}_3 (\vec{x}\vec{y}_1\vec{y}_2 \perp\!\!\!\perp \vec{y}_3 \circ_3 \dots$$
$$Q_n^* \vec{y}_n (\vec{x}\vec{y}_1 \dots \vec{y}_{n-1} \perp\!\!\!\perp \vec{y}_n \circ_n \Theta) \dots))),$$

where $Q_i^* = \exists$ and $\circ_i = \wedge$, whenever $Q_i = \exists$ and $Q_i^* = \forall^*$ and $\circ_i = \rightarrow^*$, whenever $Q_i = \forall$, and the formula Θ is constructed by induction on θ as described below.

Before we construct Θ, note that by Lemma 1, it suffices to show that for all distributions $f_1, \dots, f_n$, subsets $M \subseteq A^{|\vec{x}|}$, and probabilistic teams $\mathbb{Y} := \mathbb{X}(M/\vec{x})(f_1/\vec{y}_1) \dots (f_n/\vec{y}_n)$, we have

$$\mathcal{A} \models_{\mathbb{Y}} \Theta \iff (\mathcal{A}, p, f_1, \dots, f_n) \models \theta(\vec{a}) \text{ for all } \vec{a} \in M.$$

Now, the formula Θ is constructed by induction on θ such that for every step, we ensure that the above equivalence holds.

1. If θ is an atom or a negated atom (of the first sort), then clearly we may let $\Theta := \theta$.
2. Let $\theta = f_i(\vec{x}_i) = f_j(\vec{x}_j) \times f_k(\vec{x}_k)$. Then define

$$\Theta := \exists \alpha \beta ((\alpha = 0 \leftrightarrow \vec{x}_i = \vec{y}_i) \wedge (\beta = 0 \leftrightarrow \vec{x}_j\vec{x}_k = \vec{y}_j\vec{y}_k) \wedge \vec{x}\alpha \approx \vec{x}\beta).$$

   The idea of the formula is that we quantify two new variables α and β such that the assignments where $\vec{x}_i = \vec{y}_i$ and $\vec{x}_j\vec{x}_k = \vec{y}_j\vec{y}_k$ are marked with the constant 0 using each of the two variables, respectively. Since $f_i(s(\vec{x}_i)) = |\mathbb{Y}_{\vec{y}_i = s(\vec{x}_i)}|$ and $f_j(s(\vec{x}_j)) \cdot f_k(s(\vec{x}_k)) = |\mathbb{Y}_{\vec{y}_j = s(\vec{x}_j)}| \cdot |\mathbb{Y}_{\vec{y}_k = s(\vec{x}_k)}| = |\mathbb{Y}_{\vec{y}_j\vec{y}_k = s(\vec{x}_j\vec{x}_k)}|$ for all $s \in Y$, the distribution of $\vec{y}_i$ encodes the function f_i and the distribution of $\vec{y}_j\vec{y}_k$ encodes the function $f_j \times f_k$, and we can compare the values of the functions by comparing the distributions of $\vec{x}\alpha$ and $\vec{x}\beta$ with marginal identity.

Assume first that $(\mathcal{A},p,f_1,\ldots,f_n) \models \theta(\vec{a})$ for a given $\vec{a} \in M$. Then $f_i(\vec{a}_i) = f_j(\vec{a}_j) \times f_k(\vec{a}_k)$. Define functions $F_\alpha, F_\beta \colon Y \to \{0,1\}$ such that $F_\alpha(s) = 0$ iff $s(\vec{x}_i) = s(\vec{y}_i)$, and $F_\beta(s) = 0$ iff $s(\vec{x}_j\vec{x}_k) = s(\vec{y}_j\vec{y}_k)$. Let $\mathbb{Z} := \mathbb{Y}(F_\alpha/\alpha)(F_\beta/\beta)$. It suffices to show that $\mathcal{A} \models_{\mathbb{Z}} \vec{x}\alpha \approx \vec{x}\beta$. Now, by the definition of $\mathbb{Z}$, we have $|\mathbb{Z}_{\vec{x}\alpha=\vec{a}0}| = |\mathbb{Z}_{\vec{x}\vec{y}_i=\vec{a}\vec{a}_i}| = |\mathbb{Y}_{\vec{x}=\vec{a}}| \cdot f_i(\vec{a}_i)$ and $|\mathbb{Z}_{\vec{x}\beta=\vec{a}0}| = |\mathbb{Z}_{\vec{x}\vec{y}_j\vec{y}_k=\vec{a}\vec{a}_j\vec{a}_k}| = |\mathbb{Y}_{\vec{x}=\vec{a}}| \cdot f_j(\vec{a}_j) \cdot f_k(\vec{a}_k)$. Since $f_i(\vec{a}_i) = f_j(\vec{a}_j) \times f_k(\vec{a}_k)$, we obtain $|\mathbb{Z}_{\vec{x}\alpha=\vec{a}0}| = |\mathbb{Z}_{\vec{x}\beta=\vec{a}0}|$ and $|\mathbb{Z}_{\vec{x}\alpha=\vec{a}1}| = |\mathbb{Y}_{\vec{x}=\vec{a}}| \cdot (1 - f_i(\vec{a}_i)) = |\mathbb{Z}_{\vec{x}\beta=\vec{a}1}|$. Hence, $\mathcal{A} \models_{\mathbb{Y}} \Theta$.

Assume then that $\mathcal{A} \models_{\mathbb{Y}} \Theta$, and define $\mathbb{Z}$ as the extension of $\mathbb{Y}$ such that $\mathbb{Z}_{\alpha=0} = \mathbb{Z}_{\vec{x}_i=\vec{y}_i}$ and $\mathbb{Z}_{\beta=0} = \mathbb{Z}_{\vec{x}_j\vec{x}_k=\vec{y}_j\vec{y}_k}$. Then $|\mathbb{Y}_{\vec{x}=\vec{a}}| \cdot f_i(\vec{a}_i) = |\mathbb{Z}_{\vec{x}\vec{y}_i=\vec{a}\vec{a}_i}| = |\mathbb{Z}_{\vec{x}\vec{x}_i=\vec{a}\vec{y}_i}| = |\mathbb{Z}_{\vec{x}\alpha=\vec{a}0}| = |\mathbb{Z}_{\vec{x}\beta=\vec{a}0}| = |\mathbb{Z}_{\vec{x}\vec{x}_j\vec{x}_k=\vec{a}\vec{y}_j\vec{y}_k}| = |\mathbb{Z}_{\vec{x}\vec{y}_j\vec{y}_k=\vec{a}\vec{a}_j\vec{a}_k}| = |\mathbb{Y}_{\vec{x}=\vec{a}}| \cdot f_j(\vec{a}_j) \cdot f_k(\vec{a}_k)$ for all $\vec{a} \in M$. Hence, $(\mathcal{A},p,f_1,\ldots,f_n) \models \theta(\vec{a})$ for all $\vec{a} \in M$.

The negated case $\neg f_i(\vec{x}_i) = f_j(\vec{x}_j) \times f_k(\vec{x}_k)$ is analogous; just add $\sim$ in front of the existential quantification.

3. Let $\theta = f_i(\vec{x}_i) = \mathrm{SUM}_{\vec{x}_{j0}} f_j(\vec{x}_{j0}\vec{x}_{j1})$. Then define

$$\Theta := \exists\alpha\beta((\alpha = 0 \leftrightarrow \vec{x}_i = \vec{y}_i) \wedge (\beta = 0 \leftrightarrow \vec{x}_{j1} = \vec{y}_{j1}) \wedge \vec{x}\alpha \approx \vec{x}\beta).$$

The idea of the formula is similar to the case (2). The main difference is that the variable β and the constant 0 are now used to mark the assignments where $\vec{x}_{j1} = \vec{y}_{j1}$ instead of the assignments where $\vec{x}_j\vec{x}_k = \vec{y}_j\vec{y}_k$. Since $\mathrm{SUM}_{x_{j0}} f_j(\vec{x}_{j0}s(\vec{x}_{j1})) = |\mathbb{Y}_{\vec{y}_{j1}=\vec{s}(x_{j1})}|$ for all $s \in Y$, we can again compare the values of the functions by comparing the distributions of $\vec{x}\alpha$ and $\vec{x}\beta$ with marginal identity.

Assume first that $(A,p,f_1,\ldots,f_n)| = \theta(\vec{a})$ for a given $\vec{a} \in M$. Then $f_i(\vec{a}_i) = \mathrm{SUM}_{x_{j0}} f_j(\vec{x}_{j0}\vec{a}_{j1})$. Define functions $F_\alpha, F_\beta \colon Y \to \{0,1\}$ such that $F_\alpha(s) = 0$ iff $s(\vec{x}_i) = s(\vec{y}_i)$, and $F_\beta(s) = 0$ iff $s(\vec{x}_{j1}) = s(\vec{y}_{j1})$. Let $\mathbb{Z} := \mathbb{Y}(F_\alpha/\alpha)(F_\beta/\beta)$. It again suffices to show that $\mathcal{A} \models_{\mathbb{Z}} \vec{x}\alpha \approx \vec{x}\beta$. By the definition of $\mathbb{Z}$, we have $|\mathbb{Z}_{\vec{x}\alpha=\vec{a}0}| = |\mathbb{Y}_{\vec{x}\vec{x}_i=\vec{a}\vec{y}_i}| = |\mathbb{Y}_{\vec{x}\vec{y}_i=\vec{a}\vec{a}_i}| = |\mathbb{Y}_{\vec{x}=\vec{a}}| \cdot f_i(\vec{a}_i) = |\mathbb{Y}_{\vec{x}=\vec{a}}| \cdot \mathrm{SUM}_{\vec{x}_{j0}} f_j(\vec{x}_{j0}\vec{a}_{j1}) = |\mathbb{Y}_{\vec{x}=\vec{a}}| \cdot |\mathbb{Y}_{\vec{y}_{j1}=\vec{a}_{j1}}| = |\mathbb{Y}_{\vec{x}\vec{y}_{j1}=\vec{a}\vec{a}_{j1}}| = |\mathbb{Y}_{\vec{x}\vec{x}_{j1}=\vec{a}\vec{y}_{j1}}| = |\mathbb{Z}_{\vec{x}\beta=\vec{a}0}|$. Since $|\mathbb{Z}_{\vec{x}\alpha=\vec{a}1}| = |\mathbb{Y}_{\vec{x}=\vec{a}}| \cdot (1 - f_i(\vec{a}_i)) = |\mathbb{Z}_{\vec{x}\beta=\vec{a}1}|$, we also have $|\mathbb{Z}_{\vec{x}\alpha=\vec{a}1}| = |\mathbb{Z}_{\vec{x}\beta=\vec{a}1}|$. Hence, $\mathcal{A} \models_{\mathbb{Y}} \Theta$.

Assume then that $\mathcal{A} \models_{\mathbb{Y}} \Theta$, and define $\mathbb{Z}$ as the extension of $\mathbb{Y}$ such that $\mathbb{Z}_{\alpha=0} = \mathbb{Z}_{\vec{x}_i=\vec{y}_i}$ and $\mathbb{Z}_{\beta=0} = \mathbb{Z}_{\vec{x}_{j1}=\vec{y}_{j1}}$. Then $|\mathbb{Y}_{\vec{x}=\vec{a}}| \cdot f_i(\vec{a}_i) = |\mathbb{Y}_{\vec{x}\vec{y}_i=\vec{a}\vec{a}_i}| = |\mathbb{Y}_{\vec{x}\vec{x}_i=\vec{a}\vec{y}_i}| = |\mathbb{Z}_{\vec{x}\alpha=\vec{a}0}| = |\mathbb{Z}_{\vec{x}\beta=\vec{a}0}| = |\mathbb{Y}_{\vec{x}\vec{x}_{j1}=\vec{a}\vec{y}_{j1}}| = |\mathbb{Y}_{\vec{x}\vec{y}_{j1}=\vec{a}\vec{a}_{j1}}| = |\mathbb{Y}_{\vec{x}=\vec{a}}| \cdot |\mathbb{Y}_{\vec{y}_{j1}=\vec{a}_{j1}}| = |\mathbb{Y}_{\vec{x}=\vec{a}}| \cdot \mathrm{SUM}_{\vec{x}_{j0}} f_j(x_{j0}\vec{a}_{j1})$ for all $\vec{a} \in M$. Hence, $(\mathcal{A},p,f_1,\ldots,f_n) \models \theta(\vec{a})$ for all $\vec{a} \in M$.

The negated case $\neg f_i(\vec{x}_i) = \mathrm{SUM}_{\vec{x}_k} f_j(\vec{x}_k\vec{x}_j)$ is again analogous; just add $\sim$ in front of the existential quantification.

4. If $\theta = \theta_0 \wedge \theta_1$, then $\Theta = \Theta_0 \wedge \Theta_1$. The claim directly follows from semantics of conjunction.
5. Let $\theta = \theta_0 \vee \theta_1$. Then define

$$\Theta := \exists z(z \perp\!\!\!\perp_{\vec{x}} z \wedge ((\Theta_0 \wedge z = 0) \vee (\Theta_1 \wedge z = 1))).$$

In the formula, we quantify the variable z that is determined by the value of $\vec{x}$, and we state that the disjunction must split the team such that the value of z is constant 0 or 1 on each side. The idea is that the constant $i \in \{0,1\}$ chosen for each value $\vec{a}$ of $\vec{x}$ corresponds to the index of one of the disjoint sets M_0, M_1 such that $\vec{a} \in M_i$, and $\theta_i(\vec{a})$ holds.

Assume first that $(\mathcal{A},p,f_1,\ldots,f_n) \models \theta(\vec{a})$ for all $\vec{a} \in M$. Then there are M_0, M_1 such that $M_0 \cup M_1 = M$, $M_0 \cap M_1 = \emptyset$ and $(\mathcal{A},p,f_1,\ldots,f_n) \models \theta_i(\vec{a})$ for all $\vec{a} \in M_i$, $i \in \{0,1\}$. This is because, by the assumption, $\theta(\vec{a})$ is a disjunction that holds for all $\vec{a} \in M$, and thus the sets M_0

and M_1 can be defined by collecting in each set those elements of M that satisfy corresponding side of the disjunction. Define $F\colon Y \to p_A$ such that $F(s) = c_i$ when $s(\vec{x}) \in M_i$, where c_i is the distribution defined as

$$c_i(a) := \begin{cases} 1 \text{ if } a = i, \\ 0 \text{ otherwise.} \end{cases}$$

Let $\mathbb{Z}_i := \mathbb{X}(M_i/\vec{x})(f_1/\vec{y}_1)\ldots(f_n/\vec{y}_n)(c_i/z)$ and $k = |M_0|/|M|$. Now $\mathbb{Z} = \mathbb{Y}(F/z) = \mathbb{Z}_0 \sqcup_k \mathbb{Z}_1$, and we have $\mathcal{A} \models_{\mathbb{Z}} z \perp\!\!\!\perp_{\vec{x}} z$, $\mathcal{A} \models_{\mathbb{Z}_0} \Theta_0 \wedge z = 0$ and $\mathcal{A} \models_{\mathbb{Z}_1} \Theta_1 \wedge z = 1$. By locality, this implies that $\mathcal{A} \models_Y \Theta$.

Assume then that $\mathcal{A} \models_{\mathbb{Y}} \Theta$. Let $F\colon Y \to p_A$ be such that $\mathcal{A} \models_{\mathbb{Z}} z \perp\!\!\!\perp_{\vec{x}} z \wedge ((\Theta_0 \wedge z = 0)) \vee (\Theta_1 \wedge z = 1)$ for $\mathbb{Z} = \mathbb{Y}(F/z)$. Let then $k\mathbb{Z}'_0 = \mathbb{Z}_{z=0}$ and $(1-k)\mathbb{Z}'_1 = \mathbb{Z}_{z=1}$ for $k = |\mathbb{Z}_{z=0}|$. Now, we also have $\mathcal{A} \models_{\mathbb{Z}'_i} \Theta_i$ for $i = 0, 1$. Since $\mathcal{A} \models_{\mathbb{Z}} z \perp\!\!\!\perp_{\vec{x}} z$, we have either $\mathbb{Z}_{\vec{x}=\vec{a}} = \mathbb{Z}_{\vec{x}z=\vec{a}0}$ or $\mathbb{Z}_{\vec{x}=\vec{a}} = \mathbb{Z}_{\vec{x}z=\vec{a}1}$ for all $a \in M$. We get that $\mathbb{Z}_{\vec{z}=0} = \mathbb{Z}_{\vec{x}\in M_0}$ for some $M_0 \subseteq M$. Thus, $\mathbb{Z}'_0 = \frac{|M|}{|M_0|}(\mathbb{X}(M/\vec{x})(f_1/\vec{y}_1)\ldots(f_n/\vec{y}_n))_{\vec{x}\in M_0} = \mathbb{X}(M_0/\vec{x})(f_1/\vec{y}_1)\ldots(f_n/\vec{y}_n)$. Hence, $(\mathcal{A}, p, f_1, \ldots, f_n) \models \theta_0(\vec{a})$ for all $\vec{a} \in M_0$. We obtain $(\mathcal{A}, p, f_1, \ldots, f_n) \models \theta_1(\vec{a})$ for all $\vec{a} \in M \setminus M_0$ by an analogous argument. As a result, we get that $(\mathcal{A}, p, f_1, \ldots, f_n) \models \theta(\vec{a})$ for all $\vec{a} \in M$. □

6 Probabilistic logics and entropy atoms

In this section, we consider extending probabilistic team semantics with novel entropy atoms. For a discrete random variable X, with possible outcomes $x_1, ..., x_n$ occuring with probabilities $\mathrm{P}(x_1), ..., \mathrm{P}(x_n)$, the Shannon entropy of X is given as:

$$\mathrm{H}(X) := -\sum_{i=1}^{n} \mathrm{P}(x_i) \log \mathrm{P}(x_i),$$

The base of the logarithm does not play a role in this definition (usually it is assumed to be 2). For a set of discrete random variables, the entropy is defined in terms of the vector-valued random variable it defines. Given three sets of discrete random variables X, Y, Z, it is known that X is conditionally independent of Y given Z (written $X \perp\!\!\!\perp Y \mid Z$) if and only if the conditional mutual information $\mathrm{I}(X; Y|Z)$ vanishes. Similarly, functional dependence of Y from X holds if and only if the conditional entropy $H(Y|X)$ of Y given X vanishes. Writing UV for the union of two sets U and V, we note that $\mathrm{I}(X; Y|Z)$ and $H(Y|X)$ can respectively be expressed as $H(ZX) + H(ZY) - H(Z) - H(ZXY)$ and $H(XY) - H(X)$. Thus many familiar dependency concepts over random variables translate into linear equations over Shannon entropies. In what follows, we shortly consider similar information-theoretic approach to dependence and independence in probabilistic team semantics.

Let $\mathbb{X}\colon X \to [0,1]$ be a probabilistic team over a finite structure $\mathcal{A}$ with universe A. Let $\vec{x}$ be a k-ary sequence of variables from the domain of $\mathbb{X}$. Let $P_{\vec{x}}$ be the vector-valued random variable, where $P_{\vec{x}}(\vec{a})$ is the probability that $\vec{x}$ takes value $\vec{a}$ in the probabilistic team $\mathbb{X}$. The *Shannon entropy* of $\vec{x}$ in $\mathbb{X}$ is defined as follows:

$$\mathrm{H}_{\mathbb{X}}(\vec{x}) := -\sum_{\vec{a}\in A^k} \mathrm{P}_{\vec{x}}(\vec{a}) \log \mathrm{P}_{\vec{x}}(\vec{a}). \tag{1}$$

Using this definition, we now define the concept of an entropy atom.

DEFINITION 5 (Entropy atom).
Let $\vec{x}$ and $\vec{y}$ be two sequences of variables from the domain of $\mathbb{X}$. These sequences may be of different lengths. The *entropy atom* is an expression of the form $\mathrm{H}(\vec{x}) = \mathrm{H}(\vec{y})$, and it is given the following semantics:

$$\mathcal{A} \models_{\mathbb{X}} \mathrm{H}(\vec{x}) = \mathrm{H}(\vec{y}) \iff \mathrm{H}_{\mathbb{X}}(\vec{x}) = \mathrm{H}_{\mathbb{X}}(\vec{y}).$$

We then define *entropy logic* FO(H) as the logic obtained by extending first-order logic with entropy atoms. The entropy atom is relatively powerful compared to our earlier atoms, since, as we will show next, it encapsulates many familiar dependency notions such as dependence and conditional independence.

THEOREM 4
The following equivalences hold over probabilistic teams of finite structures with two distinct constants 0 and 1:

1. $=(\vec{x}, \vec{y}) \equiv \mathrm{H}(\vec{x}) = \mathrm{H}(\vec{x}\vec{y})$.
2. $\vec{x} \perp\!\!\!\perp \vec{y} \equiv \phi$, where ϕ is defined as

$$\begin{aligned}\forall z \exists \vec{u}\vec{v} \Big(& \big[z = 0 \to \big(=(\vec{u}, \vec{x}) \wedge =(\vec{x}, \vec{u}) \wedge =(\vec{v}, \vec{x}\vec{y}) \wedge =(\vec{x}\vec{y}, \vec{v})\big)\big] \wedge \\ & \big[z = 1 \to \big(=(\vec{u}, \vec{y}) \wedge =(\vec{y}, \vec{u}) \wedge \vec{v} = \vec{0}\big)\big] \wedge \\ & \big[(z = 0 \vee z = 1) \to \mathrm{H}(\vec{u}z) = \mathrm{H}(\vec{v}z)\big]\Big),\end{aligned}$$

where $|\vec{u}| = \max\{|\vec{x}|, |\vec{y}|\}$ and $|\vec{v}| = |\vec{x}\vec{y}|$.

PROOF. The translation of the dependence atom simply expresses that the conditional entropy of $\vec{y}$ given $\vec{x}$ vanishes, which expresses that $\vec{y}$ depends functionally on $\vec{x}$.

Consider the translation of the independence atom. Observe that ϕ essentially restricts attention to that subteam $\mathbb{Y}$ in which the universally quantified variable z is either 0 or 1. There, the weight distribution of $\vec{u}z$ is obtained by vertically stacking together halved weight distributions of $\vec{x}$ and $\vec{y}$. Similarly, $\vec{v}z$ corresponds to halving and vertical stacking of $\vec{x}\vec{y}$ and a dummy constant distribution $\vec{0}$. Consider now the effect of halving the weights of the entropy function given in (1):

$$\begin{aligned}\mathrm{H}(\frac{1}{2}X) &= -\sum_{i=1}^{n} \frac{1}{2}\mathrm{P}(x_i) \log \frac{1}{2}\mathrm{P}(x_i) \\ &= -\frac{1}{2}\sum_{i=1}^{n} \mathrm{P}(x_i)(\log \frac{1}{2} + \log \mathrm{P}(x_i)) \\ &= -\frac{1}{2}\sum_{i=1}^{n} \mathrm{P}(x_i) \log \frac{1}{2} - \frac{1}{2}\sum_{i=1}^{n} \mathrm{P}(x_i) \log \mathrm{P}(x_i) \\ &= \frac{1}{2} + \frac{1}{2}\mathrm{H}(X).\end{aligned}$$

Let us turn back to our subteam $\mathbb{Y}$, obtained by quantification and split disjunction from some initial team $\mathbb{X}$. This subteam has to satisfy $\mathrm{H}(\vec{u}z) = \mathrm{H}(\vec{v}z)$. What this amounts to, is the following

$$\begin{aligned}\mathrm{H}_{\mathbb{Y}}(\vec{u}z) = \mathrm{H}_{\mathbb{Y}}(\vec{y}z) &\iff \mathrm{H}_{\mathbb{X}}(\frac{1}{2}\vec{x}) + \mathrm{H}_{\mathbb{X}}(\frac{1}{2}\vec{y}) = \mathrm{H}_{\mathbb{X}}(\frac{1}{2}\vec{x}\vec{y}) + \mathrm{H}_{\mathbb{X}}(\frac{1}{2}\vec{0}) \\ &\iff 1 + \frac{1}{2}\mathrm{H}_{\mathbb{X}}(\vec{x}) + \frac{1}{2}\mathrm{H}_{\mathbb{X}}(\vec{y}) = 1 + \frac{1}{2}\mathrm{H}_{\mathbb{X}}(\vec{x}\vec{y}) + \frac{1}{2}\mathrm{H}_{\mathbb{X}}(\vec{0}) \\ &\iff \mathrm{H}_{\mathbb{X}}(\vec{x}) + \mathrm{H}_{\mathbb{X}}(\vec{y}) = \mathrm{H}_{\mathbb{X}}(\vec{x}\vec{y}).\end{aligned}$$

Thus, the translation captures the entropy condition of the independence atom. □

Since conditional independence can be expressed with marginal independence, i.e. $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}) \equiv \mathrm{FO}(\perp\!\!\!\perp)$ [15, Theorem 11], we obtain the following corollary:

COROLLARY 1
$\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}) \leq \mathrm{FO}(\mathrm{H})$.

It is easy to see at this point that entropy logic and its extension with negation are subsumed by second-order logic over the reals with exponentiation.

THEOREM 5
$\mathrm{FO}(\mathrm{H}) \leq \mathrm{ESO}_{\mathbb{R}}(+, \times, \log)$ and $\mathrm{FO}(\mathrm{H}, \sim) \leq \mathrm{SO}_{\mathbb{R}}(+, \times, \log)$.

PROOF. The translation is similar to the one in Theorem 2, so it suffices to notice that the entropy atom $\mathrm{H}(\vec{x}) = \mathrm{H}(\vec{y})$ can be expressed as

$$\mathrm{SUM}_{\vec{x}}(\mathrm{SUM}_{\vec{z}} f(\vec{x}, \vec{z}) \log \mathrm{SUM}_{\vec{z}} f(\vec{x}, \vec{z})) = \mathrm{SUM}_{\vec{y}}(\mathrm{SUM}_{\vec{z}'} f(\vec{y}, \vec{z}') \log \mathrm{SUM}_{\vec{z}'} f(\vec{y}, \vec{z}')).$$

Since SUM can be expressed in $\mathrm{ESO}_{\mathbb{R}}(+, \times, \log)$ and $\mathrm{SO}_{\mathbb{R}}(+, \times, \log)$, we are done. □

Note that since $\mathrm{SO}_{\mathbb{R}}(+, \times) \equiv \mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim) \leq \mathrm{FO}(\mathrm{H}, \sim)$, adding the Boolean negation to the entropy logic also increases the expressivity from the level of existential second-order logic to the level of full second-order logic.

7 Logic for first-order probabilistic dependencies

Here, we define the logic $\mathrm{FOPT}(\leq_c^\delta)$, which was introduced in [17].[5] The logic can be viewed as a probabilistic generalization of the logic FOT [23], which has weaker versions of disjunction and quantifiers than the usual ones in team semantics in order to keep the expressivity on the level first-order logic.

Let δ be a quantifier- and disjunction-free first-order formula, i.e. $\delta ::= \lambda \mid \neg\delta \mid (\delta \wedge \delta)$ for a first-order atomic formula λ of the vocabulary τ. Let x be a first-order variable. The syntax for the logic $\mathrm{FOPT}(\leq_c^\delta)$ over a vocabulary τ is defined as follows:

$$\phi ::= \delta \mid (\delta|\delta) \leq (\delta|\delta) \mid \dot{\sim}\phi \mid (\phi \wedge \phi) \mid (\phi \vee\!\!\!\vee \phi) \mid \exists^1 x\phi \mid \forall^1 x\phi.$$

[5] In the work of Hannula et al. [17], two sublogics of $\mathrm{FOPT}(\leq_c^\delta)$, called $\mathrm{FOPT}(\leq^\delta)$ and $\mathrm{FOPT}(\leq^\delta, \perp\!\!\!\perp_c^\delta)$, were also considered. Note that the results of this section also hold for these sublogics.

Let $\mathbb{X}\colon X \to \mathbb{R}_{\geq 0}$ be any probabilistic team, not necessarily a probability distribution. The semantics for the logic is defined as follows:

$\mathcal{A} \models_{\mathbb{X}} \delta$ iff $\mathcal{A} \models_s \delta$ for all $s \in \mathrm{supp}(\mathbb{X})$.
$\mathcal{A} \models_{\mathbb{X}} (\delta_0|\delta_1) \leq (\delta_2|\delta_3)$ iff $|\mathbb{X}_{\delta_0\wedge\delta_1}| \cdot |\mathbb{X}_{\delta_3}| \leq |\mathbb{X}_{\delta_2\wedge\delta_3}| \cdot |\mathbb{X}_{\delta_1}|$.
$\mathcal{A} \models_{\mathbb{X}} \dot\sim\phi$ iff $\mathcal{A} \not\models_{\mathbb{X}} \phi$ or $\mathbb{X}$ is empty.
$\mathcal{A} \models_{\mathbb{X}} \phi \wedge \psi$ iff $\mathcal{A} \models_{\mathbb{X}} \phi$ and $\mathcal{A} \models_{\mathbb{X}} \psi$.
$\mathcal{A} \models_{\mathbb{X}} \phi \mathbin{\mathbb{v}} \psi$ iff $\mathcal{A} \models_{\mathbb{X}} \phi$ or $\mathcal{A} \models_{\mathbb{X}} \psi$.
$\mathcal{A} \models_{\mathbb{X}} \exists^1 x\phi$ iff $\mathcal{A} \models_{\mathbb{X}(a/x)} \phi$ for some $a \in A$.
$\mathcal{A} \models_{\mathbb{X}} \forall^1 x\phi$ iff $\mathcal{A} \models_{\mathbb{X}(a/x)} \phi$ for all $a \in A$.

The formula $(\delta_0|\delta_1) \leq (\delta_2|\delta_3)$ is called a *conditional probability inequality atom*, and it can be used to compare the conditional probabilities of events described by quantifier-free first-order formulas. The formula $(\delta_0|\delta_1) \leq (\delta_2|\delta_3)$ says that the conditional probability of δ_0 given δ_1 is at most the conditional probability of δ_2 given δ_3. This type of atomic formula can be used to express, e.g. marginal identity and probabilistic conditional independence, as well as many different nonprobabilistic atoms, including dependence, inclusion, exclusion and independence atoms [17]. For example, the marginal identity $\vec{x} \approx \vec{y}$ can be expressed with the formula $\forall^1 u_1 \dots \forall^1 u_k((\vec{x} = \vec{u} \mid u_1 = u_1) \leq (\vec{y} = \vec{u} \mid u_1 = u_1))$, where $|\vec{x}| = |\vec{y}| = k$ and the notation $\vec{x} = \vec{u}$ means the conjunction $\bigwedge_{i=1}^{k} x_i = u_i$.

The weak Boolean negation $\dot\sim$ allows the satisfying team to be empty, because this preserves the so-called "empty-team property" that every formula of $\mathrm{FOPT}(\leq_c^\delta)$ is satisfied by the empty team. This variant of the Boolean negation was chosen for the logic $\mathrm{FOPT}(\leq_c^\delta)$, because the same one is used in the logic FOT, which also has the empty team property.

Next, we present some useful properties of $\mathrm{FOPT}(\leq_c^\delta)$.

PROPOSITION 7 (Locality, [17, Prop. 3.2]).
Let ϕ be any $\mathrm{FOPT}(\leq_c^\delta)[\tau]$-formula. Then for any set of variables V, any τ-structure $\mathcal{A}$, and any probabilistic team $\mathbb{X}\colon X \to \mathbb{R}_{\geq 0}$ such that $\mathrm{Fr}(\phi) \subseteq V \subseteq D$,

$$\mathcal{A} \models_{\mathbb{X}} \phi \iff \mathcal{A} \models_{\mathbb{X}\upharpoonright V} \phi.$$

Over singleton teams the expressivity of $\mathrm{FOPT}(\leq_c^\delta)$ coincides with that of FO. For $\phi \in \mathrm{FOPT}(\leq_c^\delta)$, define ϕ^* as the FO-formula obtained by replacing the symbols $\dot\sim$, $\mathbb{v}$, $\exists^1$ and $\forall^1$ by $\neg, \vee, \exists$ and $\forall$, respectively, and expressions of the form $(\delta_0 \mid \delta_1) \leq (\delta_2 \mid \delta_3)$ by the formula $\neg\delta_0 \vee \neg\delta_1 \vee \delta_2 \vee \neg\delta_3$.

PROPOSITION 8 (Singleton equivalence).
Let ϕ be a $\mathrm{FOPT}(\leq_c^\delta)[\tau]$-formula, $\mathcal{A}$ a τ-structure and $\mathbb{X}$ a probabilistic team of $\mathcal{A}$ with support $\{s\}$. Then $\mathcal{A} \models_{\mathbb{X}} \phi$ iff $\mathcal{A} \models_s \phi^*$.

PROOF. The proof proceeds by induction on the structure of formulas. The cases for literals and Boolean connectives are trivial. The cases for quantifiers are immediate once one notices that interpreting the quantifiers $\exists^1$ and $\forall^1$ maintain singleton supportness. We show the case for $\leq$. Let $\|\delta\|_{\mathcal{A},s} = 1$ if $\mathcal{A} \models_s \delta$, and $\|\delta\|_{\mathcal{A},s} = 0$ otherwise. Then

$$\begin{aligned}\mathcal{A} \models_{\mathbb{X}} (\delta_0 \mid \delta_1) \leq (\delta_2 \mid \delta_3) &\iff |\mathbb{X}_{\delta_0\wedge\delta_1}| \cdot |\mathbb{X}_{\delta_3}| \leq |\mathbb{X}_{\delta_2\wedge\delta_3}| \cdot |\mathbb{X}_{\delta_1}| \\ &\iff \|\delta_0 \wedge \delta_1\|_{\mathcal{A},s} \cdot \|\delta_3\|_{\mathcal{A},s} \leq \|\delta_2 \wedge \delta_3\|_{\mathcal{A},s} \cdot \|\delta_1\|_{\mathcal{A},s} \\ &\iff \mathcal{A} \models_s \neg\delta_0 \vee \neg\delta_1 \vee \delta_2 \vee \neg\delta_3.\end{aligned}$$

The first equivalence follows from the semantics of $\leq$ and the second follows from the induction hypotheses after observing that the support of $\mathbb{X}$ is $\{s\}$. The last equivalence follows via a simple arithmetic observation. □

The following theorem follows directly from Propositions 7 and 8.

THEOREM 6
For sentences we have that $\mathrm{FOPT}(\leq_c^\delta) \equiv \mathrm{FO}$.

For a logic L, we write $\mathsf{MC}(L)$ for the following variant of the model checking problem: given a *sentence* $\phi \in L$ and a structure $\mathcal{A}$, decide whether $\mathcal{A} \models \phi$. We restrict to sentences in the model checking, because otherwise we would have to encode the probabilistic team as a part of the input, which cannot be done finitely, as the weights of the assignments are real numbers.

The above result immediately yields the following corollary.

COROLLARY 2
$\mathsf{MC}(\mathrm{FOPT}(\leq_c^\delta))$ is PSPACE-complete.

PROOF. This follows directly from the linear translation of $\mathrm{FOPT}(\leq_c^\delta)$-sentences into equivalent FO-sentences of Theorem 6 and the well-known fact that the model-checking problem of FO is PSPACE-complete. □

THEOREM 7
$\mathrm{FOPT}(\leq_c^\delta) < \mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim)$ and $\mathrm{FOPT}(\leq_c^\delta)$ is non-comparable to $\mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}})$ for open formulas.

PROOF. We begin the proof of the first claim by showing that $\mathrm{FOPT}(\leq_c^\delta) \leq \mathrm{ESO}_{\mathbb{R}}(\mathrm{SUM}, +, \times)$. Note that we may use numerical terms of the form $i \leq j$ in $\mathrm{ESO}_{\mathbb{R}}(\mathrm{SUM}, +, \times)$, because they can be expressed by the formula $\exists f \exists g (g \times g = f \wedge i + f = j)$.

Let formula $\phi(\vec{v}) \in \mathrm{FOPT}(\leq_c^\delta)$ be such that its free variables are from $\vec{v} = (v_1, \ldots, v_k)$. Then there is a formula $\psi_\phi(f) \in \mathrm{ESO}_{\mathbb{R}}(\mathrm{SUM}, +, \times)$ with exactly one free function variable such that for all structures $\mathcal{A}$ and all probabilistic teams $\mathbb{X}\colon X \to \mathbb{R}_{\geq 0}$, $\mathcal{A} \models_{\mathbb{X}} \phi(\vec{v})$ if and only if $(\mathcal{A}, f_{\mathbb{X}}) \models \psi_\phi(f)$, where $f_{\mathbb{X}}\colon A^k \to \mathbb{R}_{\geq 0}$ is a function such that $f_{\mathbb{X}}(s(\vec{v})) = \mathbb{X}(s)$ for all $s \in X$.

We may assume that the formula is in the form $\phi = Q_1^1 x_1 \ldots Q_n^1 x_n \theta(\vec{v}, \vec{x})$, where $Q_i \in \{\exists, \forall\}$ and θ is quantifier-free. We begin by defining inductively a formula $\theta^*(f, \vec{x})$ for the subformula $\theta(\vec{v}, \vec{x})$. Note that in the following χ_δ refers to the characteristic function of δ, i.e. $\chi_\delta\colon A^{k+n} \to \{0, 1\}$ such that $\chi_\delta(\vec{a}) = 1$ if and only if $\mathcal{A} \models \delta(\vec{a})$. The characteristic functions χ_δ will be defined using formulas ξ_δ, which will be given after we define the formula $\theta^*(f, \vec{x})$. For simplicity, we only write $\theta^*(f, \vec{x})$ despite the fact that θ^* may contain free function variables χ_δ in addition to the variables $f, \vec{x}$.

1. If $\theta(\vec{v}, \vec{x}) = \delta(\vec{v}, \vec{x})$, then $\theta^*(f, \vec{x}) := \forall \vec{v} (f(\vec{v}) = 0 \vee \delta(\vec{v}, \vec{x}))$.
2. If $\theta(\vec{v}, \vec{x}) = (\delta_0 \mid \delta_1) \leq (\delta_2 \mid \delta_3)(\vec{v}, \vec{x})$, then

$$\begin{aligned}\theta^*(f, \vec{x}) :=& \mathrm{SUM}_{\vec{v}}(f(\vec{v}) \times \chi_{\delta_0 \wedge \delta_1}(\vec{v}, \vec{x})) \times \mathrm{SUM}_{\vec{v}}(f(\vec{v}) \times \chi_{\delta_3}(\vec{v}, \vec{x})) \\ &\leq \mathrm{SUM}_{\vec{v}}(f(\vec{v}) \times \chi_{\delta_2 \wedge \delta_3}(\vec{v}, \vec{x})) \times \mathrm{SUM}_{\vec{v}}(f(\vec{v}) \times \chi_{\delta_1}(\vec{v}, \vec{x})).\end{aligned}$$

3. If $\theta(\vec{v}, \vec{x}) = \dot{\sim} \theta_0(\vec{v}, \vec{x})$, then $\theta^*(f, \vec{x}) := \theta_0^{*\neg}(f, \vec{x}) \vee \forall \vec{v} f(\vec{v}) = 0$, where $\theta_0^{*\neg}$ is obtained from $\neg \theta_0^*$ by pushing the negation in front of atomic formulas.
4. If $\theta(\vec{v}, \vec{x}) = (\theta_0 \circ \theta_1)(\vec{v}, \vec{x})$, where $\circ \in \{\wedge, \mathbin{\vee\!\!\!\vee}\}$, then $\theta^*(f, \vec{x}) := (\theta_0^* \star \theta_1^*)(\vec{x})$, where $\star \in \{\wedge, \vee\}$, respectively.

For each δ, we define a formula $\xi_\delta \in \mathrm{ESO}_{\mathbb{R}}(\mathrm{SUM}, +, \times)$, which says that χ_δ is the characteristic function of δ. Let $\vec{y} = (y_1, \ldots, y_{k+n})$ and define ξ_δ as follows:

1. If $\delta(\vec{y}) = R(y_{i_1}, \ldots, y_{i_l})$, where $1 \leq i_1, \ldots, i_l \leq k+n$, then $\xi_\delta := \forall \vec{y}((\chi_\delta(\vec{y}) = 1 \leftrightarrow R(y_{i_1}, \ldots, y_{i_l})) \wedge (\chi_\delta(\vec{y}) = 0 \leftrightarrow \neg R(y_{i_1}, \ldots, y_{i_l}))$.
2. If $\delta(\vec{y}) = \neg\delta_0(\vec{y})$, then $\xi_\delta := \forall \vec{y}(\chi_{\delta_0}(\vec{y}) + \chi_{\neg\delta_0}(\vec{y}) = 1)$.
3. If $\delta(\vec{y}) = (\delta_0 \wedge \delta_1)(\vec{y})$, then $\xi_\delta := \forall \vec{y}(\chi_{\delta_0 \wedge \delta_0}(\vec{y}) = \chi_{\delta_0}(\vec{y}) \times \chi_{\delta_1}(\vec{y}))$

Let $\delta_1, \ldots, \delta_m$ be a list such that each δ_i, $1 \leq i \leq m$, is a subformula of some formula δ that appears in a function symbol χ_δ of the formula $\theta^*(f, \vec{x})$. Now, we can define

$$\psi_\phi(f) := \exists_{1 \leq i \leq m} \chi_{\delta_i} \left(Q_1 x_1 \ldots Q_k x_k \theta^*(f, \vec{x}) \wedge \bigwedge_{1 \leq i \leq m} \xi_{\delta_i}(\chi_{\delta_1}, \ldots, \chi_{\delta_m}) \right).$$

This shows that $\mathrm{FOPT}(\leq^\delta_c) \leq \mathrm{ESO}_{\mathbb{R}}(\mathrm{SUM}, +, \times)$. The first claim now follows, since $\mathrm{ESO}_{\mathbb{R}}(\mathrm{SUM}, +, \times) < \mathrm{SO}_{\mathbb{R}}(+, \times) \equiv \mathrm{FO}(\perp\!\!\!\perp_c, \sim)$.

We will prove the second claim now. In the proof of Proposition 4, it was noted that the formula $\sim x \perp\!\!\!\perp_y z$ cannot be expressed in $\mathrm{FO}(\perp\!\!\!\perp_c)$. This is not the case for $\mathrm{FOPT}(\leq^\delta_c)$ as it contains the Boolean negation, and thus the formula $\sim x \perp\!\!\!\perp_y z$ can be expressed in $\mathrm{FOPT}(\leq^\delta_c)$ by the results of Section 4.2 in [17]. The corresponding formula of $\mathrm{FOPT}(\leq^\delta_c)$ is

$$\dot{\sim} \forall^1 u_1 u_2 u_3 ((x = u_1 | y = u_2) \approx (x = u_1 | y = u_2 \wedge z = u_3)),$$

where $(\delta_0|\delta_1) \approx (\delta_2|\delta_3)$ is a short-hand notation for the formula $(\delta_0|\delta_1) \leq (\delta_2|\delta_3) \wedge (\delta_2|\delta_3) \leq (\delta_0|\delta_1)$. Note that the formula says that it is not the case that for any values u_1, u_2 and u_3 the conditional probabilities $P(x = u_1 | y = u_2)$ and $P(x = u_1 | y = u_2, z = u_3)$ are the same, i.e. it is not the case that the conditional independence $x \perp\!\!\!\perp_y z$ holds.

On the other hand, we have $\mathrm{FO}(=(\ldots)) \leq \mathrm{FO}(\perp\!\!\!\perp_c)$ (Prop. 2). Since on the level of sentences, $\mathrm{FO}(=(\ldots))$ is equivalent to existential second-order logic [28], there is a sentence $\phi \in \mathrm{FO}(\perp\!\!\!\perp_c)$ such that for all $\mathbb{X}\colon X \to [0,1]$, $\mathcal{A} \models_{\mathbb{X}} \phi$ iff the undirected graph $\mathcal{A} = (V, E)$ is 2-colourable. Since over singleton teams the expressivity of $\mathrm{FOPT}(\leq^\delta_c)$ coincides with FO, the sentence ϕ cannot be expressed in $\mathrm{FOPT}(\leq^\delta_c)$, as 2-colourability cannot be expressed in FO. □

8 Complexity of satisfiability, validity and model checking

We now define satisfiability and validity in the context of probabilistic team semantics. Let $\phi \in \mathrm{FO}(\perp\!\!\!\perp_c, \sim, \approx)$. The formula ϕ *is satisfiable in a structure* $\mathcal{A}$ if $\mathcal{A} \models_{\mathbb{X}} \phi$ for some nonempty probabilistic team $\mathbb{X}$, and ϕ *is valid in a structure* $\mathcal{A}$ if $\mathcal{A} \models_{\mathbb{X}} \phi$ for all probabilistic teams $\mathbb{X}$ over $\mathrm{Fr}(\phi)$. The formula ϕ is *satisfiable* if there is a structure $\mathcal{A}$ such that ϕ is satisfiable in $\mathcal{A}$, and ϕ is *valid* if ϕ is valid in $\mathcal{A}$ for all structures $\mathcal{A}$.[6]

For a logic L, the satisfiability problem $\mathsf{SAT}(L)$ and the validity problem $\mathsf{VAL}(L)$ are defined as follows: given a formula $\phi \in L$, decide whether ϕ is satisfiable (or valid, respectively). Note that if the Boolean negation is available in the logic L, the problems $\mathsf{SAT}(L)$ and $\mathsf{VAL}(L)$ are complementary. This is, in general, not the case for logics with team semantics.

For the model checking problem $\mathsf{MC}(L)$, we consider the following variant: given a *sentence* $\phi \in L$ and a structure $\mathcal{A}$, decide whether $\mathcal{A} \models \phi$. We restrict to sentences in the model checking,

[6] Note that our notion of satisfiability (validity, resp.) resembles what is often called finite satisfiability (finite validity, respectively) in the literature. This is because, in the probabilistic team semantics setting, one considers only finite structures.

because a probabilistic team cannot be finitely encoded as a part of the input, as the weights of the assignments are real numbers.

THEOREM 8
MC (FO($\approx$)) is in EXPTIME and PSPACE-hard.

PROOF. First note that FO($\approx$) is clearly a conservative extension of FO, as it is easy to check that probabilistic semantics and Tarski semantics agree on first-order formulas over singleton teams. The hardness now follows from this and the fact that model checking problem for FO is PSPACE-complete.

For upper bound, notice first that any FO($\approx$)-formula ϕ can be reduced, with only a polynomial blow-up in size, to an almost conjunctive $\mathrm{ESO}_{\mathbb{R}}(+, \leq, \mathrm{SUM})$-formula of the form $\phi^*(g) = \exists \vec{f} \forall \vec{x} \theta$, where θ is quantifier free and whose only free variable is the function variable g such that $\mathcal{A} \models_{\mathbb{X}} \phi$ if and only if $(\mathcal{A}, g_{\mathbb{X}}) \models \phi^*(g)$ [12, Lem. 17]. A formula $\psi \in \mathrm{ESO}_{\mathbb{R}}(+, \leq, \mathrm{SUM})$ is almost conjunctive, if for every subformula $(\xi_1 \vee \xi_2)$ of ψ, no numerical term occurs in ξ_i for some $i \in \{1, 2\}$. The proof of Proposition 3 in [12] now yields that model checking for ϕ^* can be done in EXPTIME. The original proof was used to show that data complexity of the model checking is in P (in data complexity the formula is fixed and only the model is considered as an input). In what follows, we sketch the reduction given in [12] pointing out the source of the exponential blow-up. The correctness of the reduction is proven in [12, Prop. 3].

The proof proceeds by describing a process to construct a system of linear inequations S for a given structure $\mathcal{A}$ and formula ϕ^*. We introduce a fresh variable $z_{\vec{a},f}$, for each k-ary function symbol f in $\vec{f}$ and k-tuple $\vec{a} \in A^k$. These variables will range over real numbers. Note that, since k is part of the input, there are exponentially many of such variables.

The system of linear equations S is defined as $S := \bigcup_{s\colon X \to A} S_s$, where X is the set of variables in $\vec{x}$ and S_s is defined as follows. We let θ_s denote the formula obtained from θ by the following simultaneous substitution: If $(\psi_1 \vee \psi_2)$ is a subformula of θ such that no function variable occurs in ψ_i, then $(\psi_1 \vee \psi_2)$ is substituted with $\top$, if

$$\mathcal{A} \models_s \psi_i, \tag{2}$$

and with ψ_{3-i} otherwise. The set S_s is now generated from θ_s together with s. Note that θ_s is a conjunction of first-order or numerical atoms θ_i, $i \in I$, for some index set I. For each conjunct θ_i in which some $f \in \vec{f}$ occurs, add $(\theta_i)_s$ to S_s, where $(\psi)_s$ is defined recursively as follows:

$$(\neg\psi)_s := \neg(\psi)_s, \qquad (iej)_s := (i)_s \, e \, (j)_s, \text{ for each } e \in \{=, <, \leq, +\},$$

$$(f(\vec{z}))_s := z_{s(\vec{z}),f}, \qquad (\mathrm{SUM}_{\vec{z}} i)_s := \sum_{a \in A^{|\vec{z}|}} (i)_{s(\vec{a}/\vec{z})},$$

$$(g(\vec{z}))_s := g^{\mathcal{A}}(s(\vec{z})), \qquad (x)_s := s(x), \text{ for every variable } x.$$

Let θ^* be the conjunction of those conjuncts of θ_s in which no $f \in \vec{f}$ occurs. If $\mathcal{A} \not\models_s \theta^*$, return "No" as an answer to the model checking problem by adding $x \neq x$ to S. The proof of Proposition 3 in [12] now shows that the system of linear inequalities S has a solution if and only if $(\mathcal{A}, g_{\mathbb{X}}) \models \phi^*(g)$.

The desired complexity bound follow from the following observations. Each set of linear inequalities S_s is of polynomial size, but there are exponentially many functions $s\colon X \to A$, and hence the size of S is worst case exponential in the size of the input model and formula. The existence of a solution for S can be checked in polynomial time in the size of S [21]. This yields an EXPTIME decision procedure for the model checking of FO($\approx$). □

We now prove the following lemma, which will be used to prove the upper-bounds in the next three theorems.

LEMMA 3
Let $\mathcal{A}$ be a finite structure and $\phi \in \mathrm{FO}(\perp\!\!\!\perp_{\mathrm{c}}, \sim)$. Then there is a first-order sentence $\psi_{\phi,\mathcal{A}}$ over vocabulary $\{+, \times, \leq, 0, 1\}$ such that ϕ is satisfiable in $\mathcal{A}$ if and only if $(\mathbb{R}, +, \times, \leq, 0, 1) \models \psi_{\phi,\mathcal{A}}$. Moreover, the size of the sentence $\psi_{\phi,\mathcal{A}}$ is exponential in the size of the input.

PROOF. Let ϕ be such that its free variables are from $\vec{v} = (v_1, \ldots, v_k)$. By locality (Prop. 1), we may restrict to the teams over the variables $\{v_1, \ldots, v_k\}$. Define a fresh first-order variable $s_{\vec{v}=\vec{a}}$ for each $\vec{a} \in A^k$. The idea is that the variable $s_{\vec{v}=\vec{a}}$ represents the weight of the assignment s for which $s(\vec{v}) = \vec{a}$. For notational simplicity, assume that $A = \{1, \ldots, n\}$. Thus, we can write $\vec{s} = (s_{\vec{v}=\vec{1}}, \ldots, s_{\vec{v}=\vec{n}})$ for the tuple that contains the variables for all the possible assignments over $\vec{v}$. Define then

$$\psi_{\phi,\mathcal{A}} := \exists s_{\vec{v}=\vec{1}} \ldots s_{\vec{v}=\vec{n}} \left(\wedge_{\vec{a}} 0 \leq s_{\vec{v}=\vec{a}} \wedge 1 = \sum_{\vec{a}} s_{\vec{v}=\vec{a}} \wedge \phi^*(\vec{s}) \right),$$

where $\phi^*(\vec{s})$ is constructed as follows:

- If $\phi(\vec{v}) = R(v_{i_1}, \ldots, v_{i_l})$ or $\phi(\vec{v}) = \neg R(v_{i_1}, \ldots, v_{i_l})$ where $1 \leq i_1, \ldots, i_l \leq k$, then $\phi^*(\vec{s}) := \bigwedge_{s \not\models \phi} s = 0$.
- If $\phi(\vec{v}) = \vec{v}_1 \perp\!\!\!\perp_{\vec{v}_0} \vec{v}_2$ for some $\vec{v}_3$ such that $\vec{v} = \vec{v}_0\vec{v}_1\vec{v}_2\vec{v}_3$, then

$$\phi^*(\vec{s}) := \bigwedge_{\vec{a}_0\vec{a}_1\vec{a}_2} \left(\sum_{\vec{b}_2\vec{b}_3} s_{\vec{v}=\vec{a}_0\vec{a}_1\vec{b}_2\vec{b}_3} \times \sum_{\vec{b}_1\vec{b}_3} s_{\vec{v}=\vec{a}_0\vec{b}_1\vec{a}_2\vec{b}_3} = \sum_{\vec{b}_3} s_{\vec{v}=\vec{a}_0\vec{a}_1\vec{a}_2\vec{b}_3} \times \sum_{\vec{b}_1\vec{b}_2\vec{b}_3} s_{\vec{v}=\vec{a}_0\vec{b}_1\vec{b}_2\vec{b}_3} \right),$$

- If $\phi(\vec{v}) = \sim \theta_0(\vec{v})$ or $\phi(\vec{v}) = \theta_0(\vec{v}) \wedge \theta_1(\vec{v})$, then $\phi^*(\vec{s}) := \neg\theta_0^*(\vec{s})$ or $\phi^*(\vec{s}) := \theta_0^*(\vec{s}) \wedge \theta_1^*(\vec{s})$, respectively.
- If $\phi(\vec{v}) = \theta_0(\vec{v}) \vee \theta_1(\vec{v})$, then

$$\phi^*(\vec{s}) := \exists k \exists t_{\vec{v}=\vec{1}} r_{\vec{v}=\vec{1}} \ldots t_{\vec{v}=\vec{n}} r_{\vec{v}=\vec{n}} \Big(0 \leq k \wedge k \leq 1 \wedge \bigwedge_{\vec{a}} (0 \leq t_{\vec{v}=\vec{a}} \wedge 0 \leq r_{\vec{v}=\vec{a}} \wedge s_{\vec{v}=\vec{a}} + k \times r_{\vec{v}=\vec{a}} = k \times t_{\vec{v}=\vec{a}} + r_{\vec{v}=\vec{a}}) \wedge \theta_0^*(\vec{t}) \wedge \theta_1^*(\vec{r}) \Big).$$

- If $\phi(\vec{v}) = \exists x \theta_0(\vec{v}, x)$, then

$$\phi^*(\vec{s}) := \exists t_{\vec{v}x=\vec{1}1} \ldots t_{\vec{v}x=\vec{n}n} \left(\bigwedge_{\vec{a}b} (0 \leq t_{\vec{v}x=\vec{a}b} \wedge s_{\vec{v}=\vec{a}} = \sum_{c=1}^{n} t_{\vec{v}x=\vec{a}c}) \wedge \theta_0^*(\vec{t}) \right).$$

- If $\phi(\vec{v}) = \forall x\theta_0(\vec{v},x)$, then

$$\phi^*(\vec{s}) := \exists t_{\vec{v}x=\vec{1}1}\dots t_{\vec{v}x=\vec{n}n}\Bigg(\bigwedge_{\vec{a}b}(0 \le t_{\vec{v}x=\vec{a}b} \wedge s_{\vec{v}=\vec{a}} = \sum_{c=1}^{n} t_{\vec{v}x=\vec{a}c} \wedge \bigwedge_{cd} t_{\vec{v}x=\vec{a}c} = t_{\vec{v}x=\vec{a}d}) \wedge \theta_0^*(\vec{t})\Bigg).$$

This completes the proof. □

THEOREM 9
MC (FO($\perp\!\!\!\perp_c$)) is in EXPSPACE and NEXPTIME-hard.

PROOF. For the lower bound, we use the fact that dependence atoms can be expressed by using probabilistic independence atoms. Let $\mathcal{A}$ be a structure and $\mathbb{X}$ be a probabilistic team over $\mathcal{A}$. Then $\mathcal{A} \models_{\mathbb{X}} =\!(\vec{x},\vec{y}) \iff \mathcal{A} \models_{\mathbb{X}} \vec{y} \perp\!\!\!\perp_{\vec{x}} \vec{y}$ [15, Prop. 3]. The NEXPTIME-hardness follows since the model checking problem for FO($=(\dots)$) is NEXPTIME-complete [9, Thm. 5.2].

The upper-bound follows from the fact that when restricted to FO($\perp\!\!\!\perp_c$), the exponential translation in Lemma 3 is an existential sentence, and the existential theory of the reals is in PSPACE. □

THEOREM 10
MC (FO($\sim, \perp\!\!\!\perp_c$)) is in 3-EXPSPACE and AEXPTIME [poly]-hard.

PROOF. We first prove the lower bound through a reduction from the satisfiability problem for propositional team-based logic, i.e. SAT (PL ($\sim$)). The logic PL ($\sim$) is just the usual propositional logic (whose formulas are assumed to be in negation normal form) with propositional team semantics and the Boolean negation. A propositional team is a team whose domain is the two element set $\{0,1\}$. Propositional team semantics for the logic PL ($\sim$) is defined similarly to the usual team semantics, but without reference to any model, because propositional logic does not have relation symbols. If T is a propositional team and p_i is a proposition symbol, we define $T \models p_i$ iff $s(p_i) = 1$ for all $s \in T$, and $T \models \neg p_i$ iff $s(p_i) = 0$ for all $s \in T$. Semantics of the rest of the connectives are defined analogously to the usual team semantics. Given a PL ($\sim$)-formula ϕ, the satisfiability problem asks whether there is a team T such that $T \models \phi$? Let ϕ be a PL ($\sim$)-formula over propositional variables $p_1,\dots,p_n$. For $i \le n$, let x_i be a variable corresponding to the proposition p_i. Let $\mathcal{A}$ be the structure of vocabulary $\tau = \{P\}$ such that $A = \{0,1\}$ and $P^{\mathcal{A}} = \{1\}$. Then, ϕ is satisfiable iff $\exists p_1 \dots \exists p_n \phi$ is satisfiable iff $\mathcal{A} \models \exists x_1 \dots \exists x_n \phi'$, where ϕ' is a FO($\sim$)-formula obtained from ϕ by simply replacing each proposition p_i by the atomic formula $P(x_i)$. This gives AEXPTIME-hardness of MC (FO($\sim$)) (and consequently, of MC (FO($\sim, \perp\!\!\!\perp_c$))) since the satisfiability for PL ($\sim$) is AEXPTIME-complete [14].

The upper-bound follows from the exponential translation from FO($\sim, \perp\!\!\!\perp_c$) to real arithmetic in Lemma 3 and the fact that the full theory of the reals is in 2-EXPSPACE [26]. □

THEOREM 11
SAT (FO($\perp\!\!\!\perp_c, \sim$)) is RE- and VAL (FO($\perp\!\!\!\perp_c, \sim$)) is coRE-complete.

PROOF. It suffices to prove the claim for SAT (FO($\perp\!\!\!\perp_c, \sim$)), since the claim for VAL (FO($\perp\!\!\!\perp_c, \sim$)) follows from the fact that FO($\perp\!\!\!\perp_c, \sim$) has the Boolean negation.

For the lower bound, note that FO($\perp\!\!\!\perp_c, \sim$) is a conservative extension of FO, and hence the claim follows from the r.e.-hardness of SAT(FO) over the finite.

For the upper-bound, we use Lemma 3. Let ϕ be a satisfiable formula of $\mathrm{FO}(\perp\!\!\!\perp_c, \sim)$. We can verify that $\phi \in \mathrm{SAT}(\mathrm{FO}(\perp\!\!\!\perp_c, \sim))$ by going through all finite structures until we come across a structure in which ϕ is satisfiable. Hence, it suffices to show that for any finite structure $\mathcal{A}$, it is decidable to check whether ϕ is satisfiable in $\mathcal{A}$. For this, construct a sentence $\psi_{\mathcal{A},\phi}$ as in Lemma 3. Then $\psi_{\mathcal{A},\phi}$ is such that ϕ is satisfiable in $\mathcal{A}$ iff $(\mathbb{R}, +, \times, \leq, 0, 1) \models \psi_{\mathcal{A},\phi}$. Since real arithmetic is decidable, we now have that $\mathsf{SAT}\,(\mathrm{FO}(\perp\!\!\!\perp_c, \sim))$ is RE-complete. □

COROLLARY 3
$\mathsf{SAT}\,(\mathrm{FO}(\approx))$ and $\mathsf{SAT}\,(\mathrm{FO}(\perp\!\!\!\perp_c))$ are RE- and $\mathsf{VAL}\,(\mathrm{FO}(\approx))$ and $\mathsf{VAL}\,(\mathrm{FO}(\perp\!\!\!\perp_c))$ are coRE-complete.

PROOF. The lower bound follows from the fact that $\mathrm{FO}(\approx)$ and $\mathrm{FO}(\perp\!\!\!\perp_c)$ are both conservative extensions of FO. We obtain the upper bound from the previous theorem, since $\mathrm{FO}(\perp\!\!\!\perp_c, \sim)$ includes both $\mathrm{FO}(\approx)$ and $\mathrm{FO}(\perp\!\!\!\perp_c)$. □

9 Conclusion

We have studied the expressivity and complexity of various logics in probabilistic team semantics with the Boolean negation. Our results give a quite comprehensive picture of the relative expressivity of these logics and their relations to numerical variants of (existential) second-order logic. An interesting question for further study is to determine the exact complexities of the decision problems studied in Section 8. Furthermore, dependence atoms based on various notions of entropy deserve further study, as do the connections of probabilistic team semantics to the field of information theory.

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