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The influence of cost assumptions on properties of the combined assignment control problem

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Abstract

The uniqueness and existence of a solution to the combined traffic assignment/signal control problem is investigated, particularly with respect to the cost functions used. The two cost functions investigated are the polynomial BPR function and Webster's two term cost function. Properties of three well-known signal control policies are investigated, and a number of new policies are developed, which guarantee a unique solution to the combined problem. The comparative performance of these policies is tested with respect to uniqueness of the resulting green times and total network travel times at the solution. To this end a streamlined version of the iterative assignment control procedure is developed and applied to three networks. It is found that potential theoretical uniqueness and existence problems do not necessarily occur in practical tests, and that enforcement of theoretical properties on signal control policies renders them rather inefficient.

CONTENTS

1 Properties of control policies that ensure an equilibrium

In Smith (1981b) the following expression for Wardrop equilibrium assignment is introduced:

"more costly routes carry no flow" (1)

Just like routes consist of sets of links that can be traversed consecutively, we can envisage signal stages to consist of sets of links that may be given green time simultaneously. We can now define **stage pressures** P_j for all stages, which are made up of the sum of the relevant **link pressures** p_i, just like route costs are made up of the sum of the relevant link costs,

$$P_j = \Sigma_i \, a_{ij} p_i \qquad (2)$$

The link pressures p_i are determined by the control policy employed; they are a function of f_i and λ_i so that

$$p_i = p_i \, (f_i \, , \lambda_i) \qquad (3)$$

and, following the same argument as in (1) we can express signal control policies as follows, subject to minimum green constraints (Smith et al., 1987):

"less pressurised stages receive no green" (4)

Link pressures would be $s_i d_i$ for P_0 and $f_i \partial d_i / \partial \lambda_i$ for delay minimisation. These stage pressures are determined by a summation over all links that have green during that stage, as in (2). The exception is Webster's policy, in which the summation over links is replaced by a determination of the maximum pressurised link i in the stage; the link pressure in that case is $f_i / \lambda_i s_i$.

The condition the flow pattern **f** must satisfy, at equilibrium, may be written as (Smith, 1979b):

$-\mathbf{t}(\mathbf{f}^* \, , \lambda)$ is normal, at $\mathbf{f}^*$, to D (5)

where D is the set of demand-feasible flows.

Using the same arguments for a given control policy, to satisfy (4), green times should follow:

$$\mathbf{p}(\mathbf{f}\ ,\ \lambda^*) \text{ is normal, at } \lambda^*, \text{ to E} \tag{6}$$

where E is the set of allowable green times.

The combined problem, which we investigate here, and in which we look for a set of flows and green times that satisfy (5) and (6) simultaneously, will be solved if

$$(-\mathbf{t}(\mathbf{f}\ ,\ \lambda)\ ,\ \mathbf{p}(\mathbf{f}\ ,\ \lambda)) \text{ is normal, at } (\mathbf{f}\ ,\ \lambda) \text{ to DxE} \tag{7}$$

This condition (7) now enables us to investigate properties of existing control policies, but more importantly, to develop new control policies with advantageous properties, e.g. policies that ensure convergence of the iterative assignment control algorithm to a unique mutual equilibrium.

A straightforward condition on the control policy, that ensures convergence and uniqueness of the resulting equilibrium is:

$$(\mathbf{t}\ ,\ -\mathbf{p}) \text{ is the gradient of a convex function V} \tag{8}$$

so that each $(t_i\ ,\ -p_i)$ must be the gradient of a convex function V_i; and $V = \Sigma_i V_i$. $(t_i\ ,\ -p_i)$ is the gradient of V_i if

$$\partial V_i/\partial f_i = t_i \tag{9}$$

and

$$\partial V_i/\partial \lambda_i = -p_i \tag{10}$$

If V_i is smooth then

$$\partial^2 V_i/\partial f_i d\lambda_i = d^2 V_i/\partial \lambda_i \partial f_i \tag{11}$$

and therefore

$$\partial t_i/\partial \lambda_i = -\partial p_i/\partial f_i \tag{12}$$

Now we can express p_i as follows:

$$p_i(f_i , \lambda_i) = - \int_0^{f_i} (\partial t_i(x , \lambda_i)/\partial \lambda_i)dx + \phi_i(\lambda_i) \tag{13}$$

This opens a world of control policies with different characteristics. The simplest policy is the policy with $\phi_i(\lambda_i) = 0$, so that

$$p_i(f_i , \lambda_i) = - \int_0^{f_i} (\partial t_i(x , \lambda_i)/\partial \lambda_i)dx \tag{14}$$

The policy that gives rise to this pressure definition is equivalent to an approximation to the NDP as suggested by Poorzahedy and Turnquist (1982). We will call this policy an **integrable** policy P_I.

Although the policy P_I gives rise to pressure definitions that are gradients of a function V_i, it is not certain that this function V_i is convex. However, if we allow a $\phi_i(\lambda_i)$ as in (13),

$$V_i(f_i , \lambda_i) = \int_0^{f_i} t_i(x , \lambda_i)dx + \int_0^{\lambda_i} \phi_i(y)dy \tag{15}$$

we can define ϕ_i's that render V_i convex, and thus ensure convergence of the iterative assignment control procedure. To ensure convexity of V_i, the vector pair (t_i , $-p_i$) must be monotone, so that its symmetrized Jacobian is positive semi-definite, and ϕ_i must be chosen to ensure this.

The need for the introduction of a "correction term" ϕ_i and the actual form of it depends on the cost assumptions in the delay curve. This will be discussed in the next Sections; we will call such adapted policies (which contain an appropriate correction term ϕ_i) P_{IM}, as they are both **integrable** and **monotone**.

2 Properties of policies with the BPR cost function

The so-called BPR cost function is extensively used in the USA, and has the following general form for signal-controlled links:

$$t = t_0\,(1 + \alpha(f/\lambda s)^{\beta}) \tag{16}$$

consisting of a free flow travel time t_0 and a delay element $\alpha t_0(f/\lambda s)^{\beta}$.

It is shown in Appendix 1 that for this cost function, Webster's policy and P_0 are policies that are not monotone, so that a unique solution to the combined signal control/assignment problem is not guaranteed.

Furthermore, the integrable policy P_I turns out to be monotone; no correction term ϕ is needed and the policy P_{IM} is therefore of no relevance. For this cost function delay minimisation turns out to be equivalent to P_I (apart from a constant factor) and is therefore monotone too.

Table 1 shows the pressure definitions for each of the policies in conjunction with the BPR cost function.

Table 1 Pressure definitions for various control policies and the BPR cost function

Policy	Pressure
Webster	$f/\lambda s$
delay minimisation	$-\alpha\ t_0\ \beta\ f^{\beta+1}/(\lambda^{\beta+1}s^{\beta})$
P_o	$\alpha\ t_0\ f^{\beta}/(\lambda^{\beta}s^{\beta-1})$
P_I	$\alpha\ t_0\ (\beta/\beta+1)f^{\beta+1}/(\lambda^{\beta+1}s^{\beta})$

3 Properties of policies with Webster's cost function

For signal-controlled networks Webster's delay function is probably most appropriate. This function consists of two parts; the first part is due to the start-stop behaviour of traffic at signals, whilst the second stems from queueing theory:

$$t = C(1-\lambda)^2/[2(1-f/s)] + f/[2\lambda s(\lambda s-f)] \tag{17}$$

To develop policy P_I we can look at each part separately. As Appendix 2 shows the resulting pressure for this policy:

$$-2sC(1-\lambda)\log(1-f/s) + s/(\lambda s-f) - f/(\lambda^2 s) - 1/\lambda \tag{18}$$

is not monotone. To ensure monotonicity a correction term is needed, and policy P_{IM} arises, with pressure definition:

$$-2sC(1-\lambda)\{\log(1-f/s) - 2\} + s/(\lambda s-f) - f/(\lambda^2 s) \tag{19}$$

Neither of these policies is attractive through simplicity; in Smith and Van Vuren (1990) an alternative policy is developed, called P_M, which has the following simple pressure definition, but which still possesses the advantageous monotonicity property:

$$s(1-\lambda) + s/(\lambda s-f) \tag{20}$$

In Heydecker (1983) the fact that neither Webster's policy, delay minimisation, nor P_O are monotone in combination with Webster's delay function, was already established.

Table 2 Pressure definitions for various control policies and Webster's cost definition

Policy	Pressure
Webster	$f/\lambda s$
delay minimisation	$fC(1-\lambda)/(1-f/s) + fs/(\lambda s-f)^2 - 1/\lambda^2 s$
P_O	$sC(1-\lambda)^2/(1-f/s) + s/(\lambda s-f) - 1/\lambda$
P_I	$-2sC(1-\lambda)\log(1-f/s) + s/(\lambda s-f) - f/(\lambda^2 s) - 1/\lambda$
P_{IM}	$-2sC(1-\lambda)\{\log(1-f/s) - 2\} + s/(\lambda s-f) - f/(\lambda^2 s)$
P_M	$sC(1-\lambda) + s/(\lambda s-f)$

In Table 2 all pressure definitions for the various policies in conjunction with Webster's cost definition are summarized. From now on we will call the policies P_0, P_I, P_{IM}, and P_M capacity maximising, following Smith and Van Vuren (1990).

4 Properties of policies with Davidson's cost function

Like Webster's delay function, Davidson's expression for delays tends to infinity when flows reach capacity; though this curve is also based on queueing theory, its form is slightly different from Webster's second term:

$$t = t_0\,[1+K(f/\lambda s-f)] \tag{21}$$

The policy P_I that follows from this cost definition is characterised by the following pressure:

$$s\,\log[\lambda s/(\lambda s-f)] + fs/(\lambda s-f) \tag{22}$$

and this policy turns out to be monotone, so that no correction term ϕ is needed to ensure convergence of the iterative assignment control procedure to a single point. Further calculations in Appendix 3 show again that neither P_0, Webster's policy, nor delay minimisation are monotone with these cost assumptions.
In Table 3 all pressure definitions for the various policies in conjunction with Davidson's cost definition are summarized.

Table 3 Pressure definitions for various control policies and Davidson's cost definition

Policy	Pressure
Webster	$f/\lambda s$
delay minimisation	$-f^2 s/(\lambda s-f)^2$
P_o	$fs/(\lambda s-f)$
P_I	$-s\,\log[\lambda s/(\lambda s-f)] + fs/(\lambda s-f)$

5 A new pragmatic power policy

A simplified expression for delay at a signalised junction is (following closely to Davidson's delay formula):

$$d = 1/(\lambda s-f) \tag{23}$$

which, of course, is too simple to be used in real-life, but which has the property that delays tend to infinity when flows approach capacity.

For this delay expression the three original policies can be expressed as follows:

Webster's $\quad$ Eq $f/\lambda s \equiv$ Eq $f/(\lambda s\text{-}f)$ for small flow f (24)

Delmin $\quad$ Min Σ f.d $\equiv$ Eq $f\ \partial d/\partial\lambda$

$\equiv$ Eq $fs/(\lambda s\text{-}f)^2$ = Eq $f^{1/2}s^{1/2}/(\lambda s\text{-}f)$ (25)

P_o $\quad$ Eq s.d $\equiv$ Eq $s/(\lambda s\text{-}f)$ (26)

and so, for appropriate values of k, all three policies can be expressed as

$$\text{Eq } f^k s^{1\text{-}k}/(\lambda s\text{-}f) \tag{27}$$

Tests described in Van Vuren et al. (1987) and Van Vuren et al. (1990) indicated that

a. Webster's policy performs well under low congestion: either mutual equilibrium has lower average travel time than P_0's stable point.
b. Delay minimisation performs reasonable throughout a range of low to medium congestion.
c. P_0's capacity maximising property is most useful when congestion is considerable.

Thus, if the power k in (27) is related to congestion, this policy can adapt itself to mimic the behaviour of each of the three policies in the most appropriate range of conditions.
The value of k should be close to 1 if junction congestion is low, and close to 0 if congestion is high. An appropriate expression for k is:

$$k = \Sigma(\lambda s\text{-}f)/s \tag{28}$$

with an upper limit of 1. Note that this **power policy** can only be readily applied in combination with cost functions that assume finite capacity for links, as the sign of (λs-f) may change if flows are allowed to exceed capacity. Also, no monotonicity properties can be established for this policy: it is based on pragmatism and rather strong delay assumptions and should be tested thoroughly in a range of circumstances.

6 Tests on a simple network

The characteristics of the existing and newly developed policies will here be tested on a simple two-link network. This test network, as shown in Figure 0, consists of only four links, that make up two routes. The first route is fast, but with a limited capacity, e.g. through a town centre. The second route is longer, but wider, e.g. a bypass. Both routes meet at the end of the town at a signalised junction.

The saturation flow at the junction for the bypass = 4000 pcu/h, whilst the narrow town route has a lower saturation flow of 2000 pcu/h. The bypass is 150m longer, so that at a free flow travel speed of 50 km/h its free flow travel time is 10.8 secs longer than that of the town route.

The following assumptions are further made:

- cycle time = 60 sec;
- no intergreen times;
- two stages, one for each road;
- minimum and maximum green times of 0.5 sec and 59.5 sec respectively.

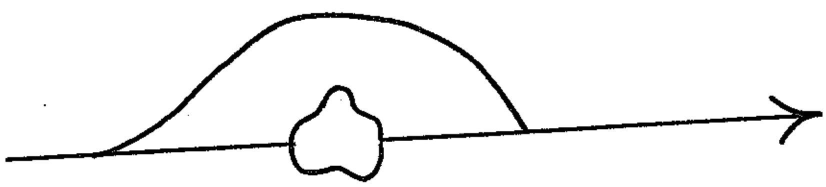

Figure 0: Test network

First I will discuss test results with the BPR cost function, and three control policies P_O, Webster and Delmin (remember that with this cost function the P_I policy is equivalent to delay minimisation, and monotone). Then I will investigate the behaviour of these three original policies plus the capacity maximising P_I, P_{IM} and P_M policies under Webster's cost assumptions for signalised junctions; also the pragmatic power policy will be tested. No comparisons with Davidson's cost function will be made.

In these tests the iterative assignment control procedure is started from both edges of the feasible green time region, to investigate uniqueness of the resulting equilibrium, and the policies' abilities to move away from poor initial settings. In this two-link case the feasible region is straight-forward to determine, and the feasible boundary is determined by the minimum green time constraint. When applying the BPR cost function capacities are unlimited; with Webster's cost function, however, links are capacitated and the feasible green time region is directly dependent on total demand.

6.1 Tests with the BPR cost function

Figures 1, 2 and 3 show information about green time and flow distribution at the mutual equilibrium plus associated excess travel times, related to total network demand and initial green times, using the BPR cost function (16). A comparison is given for the three policies and optimum NDP settings.
First note that, although monotonicity could not be established for either P_0 or Webster's policy, both give rise to single mutually consistent points.

The resemblance for this cost definition between the behaviour of P_0 and Delmin is striking. However, P_0 re-distributes traffic and green time to the wide route earlier, and resulting green times and flows are closer to the optimum. At low and high flow levels both policies give identical (and optimum) results, as expected.

With this polynomial delay function the Webster policy does not achieve any re-distribution to the wide route at all, regardless of the total flow or the initial green time. This can be checked analytically as follows:

Signal control step (Webster's policy)

$f_1/\lambda_1 s_1 = f_2/\lambda_2 s_2$

User equilibrium assignment step

$t_1 \quad = \quad t_2 \quad (t = l + d)$

$l_1 \quad > \quad l_2 \quad$ (free flow travel time)

$\Rightarrow \quad d_1 \quad < \quad d_2 \quad$ by shifting f

$l_1\alpha[(f_1-\Delta f)/\lambda_1 s_1]^\beta < l_2\alpha[(f_2+\Delta f)/\lambda_2 s_2]^\beta$

Signal control step

(reduce λ_1 to compensate for loss of flow; increase λ_2)

$(f_1-\Delta f)/(\lambda_1-\Delta\lambda)s_1 = (f_2+\Delta f)/(\lambda_2+\Delta\lambda)s_2$

Assignment step

(reduce f_1 to compensate for loss of green; increase f_2)

$l_1\alpha[(f_1-\Delta f-\Delta^1 f)/(\lambda_1-\Delta\lambda)s_1]^\beta < l_2\alpha[(f_2+\Delta f+\Delta^1 f)/(\lambda_2+\Delta\lambda)s_2]^\beta$

etc.

In words: flow and green time are persistently re-distributed to the narrow, shorter route until a feasible (minimum green time) boundary is met.

The performance consequence is represented in the average excess travel times in Figure 3. Up to a demand flow of approximately 1500 pcu/h, some 75% of the narrow route's saturation flow, all 3 policies give rise to optimum mutual equilibria, whilst when approaching the wide route's saturation flow first P_0 and then Delmin again perform optimally. The comparative performance of Webster's policy deteriorates when the demand exceeds the narrow route's capacity, because no re-distribution of flow and green time to the wide route is achieved by this policy.

In the intermediate region P_0 performs about 20% better than Delmin, because of the early green time/flow re-distribution. An optimum, however, is not achieved - or even approximated - by application of any of the tested policies in that region.

6.2 Tests with Webster's cost function

Now capacities are finite; also the P_I policy and Delmin are distinct. Both policies are also non-monotone with these cost assumptions. In addition to the three policies tested with the polynomial BPR cost functions four extra policies (P_I, P_{IM}, P_M and the power policy) will now be tested. Therefore, Figures 4, 5 and 6 are more complicated than the corresponding Figures 1 to 3. First note in Figure 4, which depicts the green time distribution at equilibrium as a function of total demand, that because of the capacitated links an infeasible green time region exists.

Webster's policy and delay minimisation show virtually identical behaviour, ending up at one of the feasible boundaries; when demand exceeds the capacity of the narrow route (2000 pcu/h) the lower limiting state will actually be unfeasible and therefore give rise to infinite delays and travel times. This limiting state ceases to exist at a total demand of approximately 2700 pcu/h.

P_0 performs very much like the case described in Van Vuren et al. (1987) in which the sheared dely formula is applied, re-distributing flow and green time so as to always give rise to feasible mutually consistent points.

Of the newly developed policies P_I starts re-distributing flow and green time first. As the policy is not monotone with Webster's cost function two equilibria emerge, a higher one and a lower one. Both follow closely the Webster and Delmin curves, but P_I always gives rise to feasible solutions, because of its capacity-maximising properties. At a flow level of approximately 2400 pcu/h, when excess travel time starts rising rapidly, the lower limiting state merges with the upper solution and ceases to exist.

P_{IM}, the monotone adaptation of P_I, shows a rather rigid behaviour, just like the other monotone policy, P_M. Particularly striking is the rigid green time curve at low flow levels for the P_M policy, caused by the first term of its pressure definition: $sC(1\text{-}\lambda)$. As $s_1 = 2s_2$, λ_1/λ_2 must be close to 2 to satisfy the equal pressure condition when the second term is small. Both policies give rise to unique and feasible solutions.

Of all new policies the power policy shows the most promising behaviour, closely following the optimum settings. A unique flow/green time pattern at mutual equilibrium exists, which at low flow levels supports the narrow route. When the capacity of that route is approached, however, a complete swap-over to the wide route of both green time and flow takes place; note that this swap-over takes place later than for the optimum settings.

The final performance comparison is given by the excess travel times in Figure 6. As observed before with sheared delay assumptions, the two conventional policies may end up in the very adverse situation in which only half the possible amount of traffic can be served. These curves go together with low excess travel times at low demand levels, which steeply increase when the capacity of the narrow route is approached.

On the other hand, because of the two limiting states, if the starting point for the iterative process could be favourably chosen, these policies achieve near-optimum travel times at mutual equilibrium. The integrable policy P_I follows the same pattern, but less extreme.

P_O and the two monotone policies P_{IM} and P_M show a similar behaviour. Of these three P_{IM} performs best at low congestion levels, but rather poorly when demand increases, because of the late re-distribution of flow and the proximity to the infeasible boundary of resulting green times. Of P_O and P_M the first performs better and less rigidly, reacting to flows as well as saturation flows, giving rise to lower excess travel times in low and high congestion. Generally all monotone policies are rather insensitive to existing delays; for a considerable range of demand flows green time is split over both routes, even though all flow is assigned to just one of these. Resulting inefficiencies are the price we pay for theoretical uniqueness and existence of the mutually consistent points.

Overall, the power policy performs best, always ending up at a feasible point but, unlike the three capacity maximising policies, with green times optimally fitted to the flows. This gives rise to optimum behaviour, apart from the demand region between approximately 1860 and 1930 pcu/h; even there average excess travel time is lower than for most other policies.

7 Tests on more realistic networks

7.1 Introduction

Performance of policies on simple networks is not necessary representative of their behaviour in reality. Tests on larger scale networks are needed for a better understanding and they will be presented next. They consist of:

(a) a network as used by Tan, Gershwin and Athans (1979) in their study of optimal signal control, here called the TGA network;

(b) the network of Weetwood, a suburb of Leeds.

With these larger scale networks, simple calculations that sufficed for the two-link case have to make way for more sophisticated algorithms. For the equilibrium assignment the assignment subprogram of SATURN (Van Vliet, 1982) was used. Two adaptations to the program had to be made. Firstly the ability to control signals had to be introduced; secondly cost definitions had to be modified. For the polynomial BPR function this is a straightforward exercise and described in Appendix 4; the infeasibility of link flows above capacity with Webster's cost function, however, is incompatible with the requirements of the Frank-Wolfe algorithm that SATURN employs. An adaptation of Webster's cost function has been devised in order to comply with these requirements. This adaptation will be introduced in Section 7.2. Subsequently in Section 7.3, I will describe the green time control algorithm adopted and in Section 7.4 implementation of the iterative assignment control procedure in the model. After this the results for both networks will be presented. I will discuss convergence of the algorithm, uniqueness of the resulting green times, and the quality of the mutually consistent points in terms of total network travel times at those points.

7.2 An adaptation of Webster's cost function

Webster's cost function has two properties that are incompatible with the Frank-Wolfe algorithm:

(1) links are capacitated

(2) link costs approach infinity when the link flow nears capacity, and they are undefined when the flow exceeds capacity.

The Frank-Wolfe algorithm, as a series of all-or-nothing assignments, needs link costs to be finite and defined throughout the whole flow region, also above capacity. The following adaptation of Webster's cost function is therefore developed.

Given a simulation period T (usually between 30 and 120 mins) the "kink" flow level is determined at which the derivative of Webster's cost function equals the deterministic queueing slope:

$$\partial t/\partial f = T/(2\lambda s) \tag{29}$$

For flow levels above this value the continuation of Webster's curve is replaced by deterministic queueing, thus ensuring existence of a cost definition throughout the whole flow region, though at substantial cost close to or over capacity; and also ensuring a continuous first order derivative. Figure 7 shows Webster's cost function and its approximation; in the applications described next resulting flows that are higher than the kink flow are considered to be infeasible. Appendix 5 presents the relevant mathematical expressions.

7.3 A green time control algorithm

In Smith et al. (1987) the green time optimisation problem was introduced as an assignment problem; see also Section 1. This observation enables us to use a standard assignment algorithm to solve the green time optimisation step in the iterative assignment control loop.

Pressures, as defined by the control policy employed are analogous to **costs**: **link pressures** correspond to **link costs** and **stage pressures**, as a summation over constituent links, correspond to **route costs**. An equilibration of stage pressures can now be sought by swapping green time from less pressurised to more pressurised stages (like an equilibration of route costs is sought by swapping flow from higher cost routes to cheaper ones). As the number of stages at a junction is limited (and known in advance) an algorithm that needs stage enumeration can easily be applied. The algorithm employed here is based on that described by Dafermos and Sparrow (1969) and it works as follows.

For each junction:

(1) determine link pressures (based on flow and green time);

(2) determine stage pressures (by summing over constituent links as determined by the stage matrix);

(3) determine minimum and maximum pressurised stages;

(4) determine an optimum swap of green time from the minimum to maximum pressurised stage, subject to feasibility constraints;

(5) unless convergence is achieved, go to step 1.

This algorithm will determine a set of green splits consistent with a fixed set of flows, as in (6). A number of observations with respect to this algorithm must be made:

- for most control policies a stage pressure is defined in step 2. by a summation over constituting link pressures; for Webster's control policy, however, a stage pressure is determined by the maximum of constituent link pressures;
- determination of an optimum amount of green time to be swapped from the minimum to the maximum pressurised stage is carried out by a golden section search;
- with a polynomial BPR cost function feasibility constraints consist of minimum green times. When employing Webster's cost definition an extra feasibility constraint is introduced, related to link capacities as determined by the λs-value. The feasible boundary for green time reduction of the minimum pressurised stage is set at $\lambda = 0.999$ f/s, so that a link cannot become oversaturated by green time re-distribution; maximum allowed degree of saturation is in effect 99.9%;

- convergence can be monitored via the step size determined for the optimum green swap.

7.4 Implementation aspects of the iterative assignment control procedure

In Smith and Van Vuren (1990) a variant of the iterative assignment control procedure is introduced, which might reduce its computational burden. Instead of carrying out the assignment step till convergence, we might suffice with a single iteration in the assignment, consisting of a direction search via an all-or-nothing load and a subsequent optimum step size search. Even though the assignment objective function would not be minimised in each step, it would definitely be decreased, and the large number of assignment-control iterations should ensure that a mutual equilibrium will be reached in the long run, independent of the actual algorithm employed.

Two implementations have been tested, namely the full implementation that converges each assignment sub-step, and the **streamlined version** that allows only one new route per assignment. The two implementations were tested on the Weetwood network, with a maximum number of assignment-control iterations of 200, the observed OD-matrix and the delay minimising control policy. Resulting computation times with both polynomial delay assumptions and Webster's cost funcion are shown in Table 4

Table 4 Computation times for two implementations of the iterative assignment control procedure and two different cost functions. Weetwood, 1.0 x OD, 200 iterations.

	full implementation	streamlined version
polynomial costs (BPR)	28.21 sec	26.41 sec
Webster's costs	72.92 sec	63.60 sec

First note in Table 4 the difference in computation times between polynomial cost assumptions and Webster's costs; compared with these the computational savings of the streamlined algorithm are limited. This is related to the convergence performance of the iterative assignment control procedure, which is not unlike that of the Frank-Wolfe algorithm. As a rule only in the first few steps of the iterative assignment control procedure a relatively large number of iterations is required to achieve convergence in the assignment, as shown in Figure 8. In later steps signal green time changes and consequent flow changes are so small that single route changes suffice for convergence, governed by the size of the step length λ and the uncertainty in the objective function. This also means that savings in computation time by the streamlined algorithm will be of an absolute, rather than relative nature, as they are achieved in the first few iterations only. The streamlined algorithm has been implemented and used in the test runs described next.

8 Results for the TGA network

The network introduced by Tan et al. (1982) consists of 8 uni-directional links, 6 nodes and 4 OD-pairs. Although still small in size, the network presents a much more realistic situation than the simple network used before; four OD pairs exist and each of the OD pairs has 2 or 3 routes available that do not necessarily pass the signal-controlled junction.

The network is shown in Figure 9; node 3 is signal-controlled. All links have saturation flows of 1500 pcu/h, apart from the link between nodes 4 and 5 which has a capacity of 3000 pcu/h. Link lengths are given in the Figure and the free flow speed is assumed to be 40 km/h.

Some differences with the approach of Tan et al. must be noted:

(1) Tan et al apply the BPR cost function to **all** links in the network; in addition they apply Webster's cost definition to those links that are signal controlled. I have chosen to apply Webster's cost definition **only** to links at signalised junctions, whereas links at non-signalised junctions have their cost calculated according to the sheared delay curve; in effect this will make non-signalised junctions generally less attractive than in the network used by Tan et al.

(2) Cycle times are 60s and 75s respectively for Tan et al and my TGA network. These differences will explain why the results from the two studies differ, even though the conclusions that are drawn from them are very similar.

Tan et al. investigated the behaviour of the iterative assignment control method for the following demand levels:

to / from	5	6
1	800	800
2	800	800

to / from	5	6
1	1200	1200
2	800	800

which I will call demand levels 1 and 2.

I will reproduce these tests for the two cost functions and control policies I described before; and like Tan et al. I will compare the performance of each of these policies with the user optimum. (NB As this network contains only one signal-controlled node with 2 stages the user optimum can be found via a simple one directional search method). In addition I will investigate the following demand patterns:

to / from	5	6
1	400	400
2	400	400

to / from	5	6
1	800	800
2	1200	1200

and I will call these demand levels 3 and 4.

8.1 Results with the BPR cost function

Results for iterative assignment control with the polynomial BPR cost definition and the three relevant control policies are given in Table 5.

Table 5: Resulting green times and total travel times for three policies and varying demand levels, compared with optimum settings; polynomial cost definition

	demand level 1		**demand level 2**		**demand level 3**		**demand level 4**	
	TTT	**G**	**TTT**	**G**	**TTT**	**G**	**TTT**	**G**
Webster	773	0.16	1057	0.26	357	0.35	1075	0.03
Delmin	776	0.20	1059	0.29	357	0.35	1075	0.03
P_o	777	0.21	1060	0.30	357	0.36	1075	0.03
optimum	771	0.11	1057	0.25	357	0.35	1075	0.03

TTT = total network travel time in veh hr/hr

G = green split for link 1-3

Resulting green times turn out not to depend on the initial split and therefore only a single green split and associated total travel time is shown for each policy in Tables 5 to 10. As in the two-link case the behaviour of the three policies is very similar and also very close to the optimum. Of the three policies Webster's gives rise to the most uneven split, generally favouring the in-link from node 2. This can be explained by the observation that the free flow link cost of the alternative route from this node (2-5) is much higher than the route via the signalised junction, certainly compared with the route alternatives that exist for traffic originating at node 1. Whilst for the relations 2-5 and 2-6 this difference is 2.5 miles (360 sec) and 3.5 miles (504 sec), for the relation 1-6 the difference is only 0.5 miles (72 sec). Therefore, with the polynomial cost definition traffic will re-route quicker to the alternative on relations from 1 and Webster's policy (to equalise degrees of saturation) will favour the larger traffic stream from node 2. The extreme behaviour of Webster's policy in the two-link case is not reproduced here, however.

The attempts of P_0 to reroute traffic from 2 away from the signalised junction fail because of the large extra length of the alternative and the comparatively shallow form of the BPR cost function.
This also results in actual oversaturation at the signalised junction for all but the lowest demand levels, irrespective of the control policy used.

This behaviour is further illustrated by Table 6 which shows resulting green splits and total travel times for demand level 2 and a shortened bypass from node 2 to node 5 of 6.25 miles instead of 8 miles. Now P_0 **does** manage to redistribute traffic away from the signalised junction, resulting in an improved behaviour over Webster's policy and delay minimisation. Again, however, the signal-controlled junction is oversaturated, but to a lesser extent, due to the use of a polynomial cost function.

Table 6: Resulting green times and total travel times for three policies and demand level 2; shortened bypass and polynomial cost definition

	TTT	G
Webster	1030	0.30
Delmin	1024	0.35
P_0	1023	0.36

TTT = total network travel time in veh hr/hr
G = green split for link 1-3

8.2 Results with Webster's cost function

With Webster's cost function I investigate the behaviour of 7 policies; the base case is demand level 1 with 800 pcu/h on each OD-relation. Table 7 shows the results; note that these are again unique, independent of the initial green split (even though monotonicity could not be established for five policies).

Table 7: Resulting green times and total travel times for seven policies and Webster's cost function, demand level 1

	TTT	G
Webster	932	0.03
Delmin	932	0.03
P_0	932	0.03
P_I	932	0.03
P_{IM}	932	0.03
P_M	932	0.03
Power	932	0.03
optimum	810	0.39

TTT = total network travel time in veh hr/hr
G = green split for link 1-3

The Table confirms the findings of Tan et al., that iterative assignment control does not find a user optimum for this configuration. Even more, this is irrespective of the policy employed. A full allocation of green time to the in-link from 2 takes place, forcing most traffic from origin 1 to take the alternative route that avoids the signalised junction. Application of Webster's cost function ensures, however, that all link flows are within capacity, mainly by routing nearly all flow from origin 1 to link 1-4.

When the demand level for relations from origin 1 is increased to 1200, however, my findings are rather different from Tan et al. (demand level 2; Table 8). Again the iterative assignment control procedure cannot find the optimum settings according to network design; with these demands, however, the capacity maximising policies perform better. These policies give more green to the in-link from 1, thus forcing traffic from 2 to re-assign to the bypass; the signal-controlled junction is still undersaturated.

<u>Table 8</u>: Resulting green times and total travel times for seven policies and Webster's cost function, demand level 2

	TTT	G
Webster	1436	0.03
Delmin	1436	0.03
P_O	1246	0.30
P_I	1268	0.27
P_{IM}	1244	0.32
P_M	1255	0.30
Power	1268	0.27
optimum	1075	0.76

TTT = total network travel time in veh hr/hr
G = green split for link 1-3

Based on these results Tan et al. reject the iterative assignment control procedure. However, not only do my results show that the use of different control policies can improve its performance, but in addition it is rather limited to base such judgements on just two demand cases.

Therefore I investigate two more demand levels: a low demand level with only 400 pcu/h on each OD relation (demand level 3) and a demand level with increased flows on all relations from origin 2 (demand level 4). Table 9 shows the results for demand level 3.

Table 9: Resulting green times and total travel times for seven policies and Webster's cost function, demand level 3

	TTT	G
Webster	360	0.33
Delmin	360	0.35
P_O	361	0.39
P_I	360	0.34
P_{IM}	363	0.42
P_M	361	0.40
Power	360	0.36
optimum	360	0.31

TTT = total network travel time in veh hr/hr
G = green split for link 1-3

For this demand level all policies perform well, particularly the non-monotone policies that follow and accept flow levels as they are, without attempting flow re-distribution. Again, none of the policies finds exactly the user optimum, but differences are now very small indeed.

The final test is with a similar demand level to the second case, but with a reversed emphasis on origins 1 and 2; shown in Table 10.

Table 10: Resulting green times and total travel times for seven policies and Webster's cost function, demand level 4

	TTT	G
Webster	1148	0.03
Delmin	1148	0.03
P_O	1148	0.03
P_I	1148	0.03
P_{IM}	1148	0.03
P_M	1148	0.03
Power	1148	0.03
optimum	1036	0.14

TTT = total network travel time in veh hr/hr
G = green split for link 1-3

Again all policies end up at a mutually consistent point at the minimum green time boundary for link 1-3, caused by the weight of the OD-flows from origin 2. As before, this is not the user optimum (differences in total travel times exceed 10%), but link flows remain within capacity.

8.3 Conclusions TGA network

Summarising, although the iterative assignment control procedure **for this network** and the demand levels tested never finds optimum signal splits, it does not perform as bad as Tan et al. claim. I do not claim that the procedure is an actual heuristic for the network design problem; it is a practical tool for use in large scale networks, allowing a realistic network description and complex cost functions. The procedure in this case gives rise to sensible signal splits and its extreme behaviour in two of the cases is strongly determined by the network layout. It would be just as easy to construct a network on which the iterative assignment control procedure performs well in conjunction with all or particular policies, and in my view final conclusions should be based on more tests with realistic networks.

Despite the lack of theoretical uniqueness of the resulting mutual equilibrium for a number of policies, the iterative assignment control procedure gives rise to unique settings for all policies on this network. What is shown clearly, and what should matter to the practitioner, is the influence of the control policy employed and the cost assumptions on resulting green splits and accompanying travel times.

Of the control policies investigated the capacity maximising policies probably perform best; the performance, however, is strongly influenced by the quality of the available route alternatives. Of these four policies (P_O, P_I, P_{IM} and P_M), P_O performs best and has the added advantage that it can be applied independently of the cost function employed. The power policy performs promisingly, but needs testing on a larger scale network. Finally, Webster's policy performs most extremely, particularly under Webster's cost definition.

The cost function employed influences the results of the iterative assignment control procedure in two important ways:

(1) It influences the performance of each of the policies, with respect to the quality of the mutual equilibrium reached.

(2) It influences resulting green times, not only for each of the policies, but also the optimum settings. A comparison of green times in Table 5 and those in Tables 6 to 10 will back this up. The question is, of course, which green splits are optimal **in reality**.

9 Results for the Weetwood network

The Weetwood network is of a much larger size than any of the previous networks tested. It consists of 70 zones, 105 nodes and 442 directional links. Of the nodes, 17 are signal controlled with 42 stages in total. The network is depicted in Figure 10; the modelled situation is the AM Peak with strong North-South flows.

As before, this network is tested with:

(a) different cost assumptions
(b) different demand levels
(c) different control policies
(d) different initial green time splits.

Because of the complicated network structure, it is now infeasible to determine optimum settings. It is therefore impossible to state how close to the actual network optimum resulting green time/flow combinations for each of the policies are.

9.1 Results with the BPR cost function

With the polynomial cost function three demand levels have been investigated. The base case is the observed trip matrix, giving rise to an average network speed of 35-40 km/h (dependent on initial green time splits). To allow for a considerable increase in congestion, and because of the shallow form of the cost function, the two other demand levels investigated are for a doubled and trebled OD-matrix, giving rise to speeds of approximately 20 km/h and 10 km/h respectively. Results for these tests are shown in Table 11; as before, three control policies (Webster, Delmin and P_O are tested in interaction with user equilibrium assignment.

The iterative assignment control procedure has been started from two different initial green splits. The Table shows resulting total network travel times at the two mutually consistent points found and the average and maximum differences between resulting green times at those points; cycle time is 100s.

Table 11: Results for the Weetwood network; polynomial cost assumptions

	TTT1	$\delta f1$	$\delta g1$	TTT2	$\delta f2$	$\delta g2$	$\overline{\Delta G}$	ΔG_{max}
ODx1								
Webster	2007	0.001	0.008	2007	0.001	0.003	0.05	0.5
Delmin	2005	0.001	0.003	2005	0.001	0.005	0.06	0.4
P_O	2005	0.001	0.000	2005	0.001	0.000	0.05	0.2
ODx2								
Webster	5280	0.018	0.000	5280	0.020	0.000	0.06	0.2
Delmin	5248	0.015	0.000	5249	0.017	0.010	0.06	0.2
P_O	5258	0.018	0.000	5259	0.018	0.000	0.12	0.5
ODx3								
Webster	14944	0.036	0.003	14945	0.035	0.000	0.03	0.1
Delmin	14276	0.043	0/003	14276	0.041	0.000	0.09	0.4
P_O	14297	0.050	0.008	14292	0.040	0.000	0.10	0.4

TTT1 = total network travel time in veh.hr/hr; start green 1
TTT2 = total network travel time in veh.hr/hr; start green 2
δf = excess travel costs over minimum costs in %
δg = average absolute change in green times in last iteration in sec.
ΔG = average difference in resulting green times in sec.
ΔG_{max} = maximum difference in resulting green times in sec.

The δf and δg columns indicate the level of convergence for the iterative assignment control procedure for the link flows and green times respectively; δf denotes the excess travel costs:

$$\delta = \frac{\text{total network travel costs with current flow pattern}}{\text{total networks costs via minimum routes}}$$

and this is a measure how far we are from an equilibrium, in which case the value of $\delta = 0$. They show how well the procedure has converged, with excess travel costs never more than 0.05% and an absolute average change in green times in the final control iteration of less than 0.01 sec.

As in previous tests with this cost assumption, the results of all three policies in the iterative assignment control procedure are very similar. The maximum difference in travel times between delay minimisation and P_0 is limited to tenths of a percent, and the maximum difference with Webster's policy is less than 5%. Also the resulting green split patterns are virtually independent of the initial splits (even though monotonicity could not be established for either P_0 or Webster's policy). The small differences in green splits resulting from each of the starting points are most likely due to computational inaccuracies.

A closer look at the resulting green splits also reveals that the final splits do not necessarily depend very much on the control policy employed, as Table 12 shows. Although average differences in resulting green splits may run up to some 8 sec between Webster's policy and the two other policies and maximum differences up to 27 sec, particularly striking is the similarity of final green splits for Delmin and P_0. Differences in resulting network travel times are always less than 0.1% and the maximum difference in final green times is 2.0 sec in the 1.0 case and 6.4 sec in the 3.0 case (average differences are 0.7 sec and 1.6 sec respectively), almost the same order of magnitude as the differences resulting from different start greens.

Table 12: Differences in final green times between the three policies; start green 1

	ODx1		ODx3	
	$\overline{\Delta G}$	ΔG_{max}	$\overline{\Delta G}$	ΔG_{max}
Webster-Delmin	3.3	9.4	6.2	21.3
Webster-P_0	3.9	10.9	7.6	27.1
Delmin-P_0	0.7	2.0	1.6	6.4

$\overline{\Delta G}$ = average difference in resulting green times in sec.

ΔG_{max} = maximum difference in resulting green times in sec.

9.2 Results with Webster's cost function

As links are capacitated with Webster's cost function congestion builds up much more rapidly than with polynomial delay assumptions. This is demonstrated by the steeper rising total network travel times in Table 13; in fact no feasible flow/green time pattern (where feasibility is defined as: "with all signal-controlled links below artificial capacity as defined by the "kink" flow in paragraph 7.2") could be found by any of the policies for demand levels higher than 1.2 x observed demand.

N.B. In effect, this is not really a feasibility problem. An appropriately large choice of simulation time T would:

(a) shift the kink flow to the right, as the slope of the over-capacity delays increases;

(b) ensure sufficiently high delays near capacity to re-distribute traffic away from signalized junctions.

Extremely large T's and steep slopes in the cost functions, however, introduce instabilities in both assignment and signal control, and therefore a limited value of 9999 min. was applied to determine the over-capacity slopes of delays at signalized junctions, and 30 min. at all other junctions.

Comparing total network travel times, we can first observe the rather good behaviour of Webster's policy at lower congestion (though never better than Delmin) and the rather poor behaviour when network capacity is approached (OD x 1.2); then total travel times are up to 19% higher than for Delmin. Delay minimisation performs very well and consistently; P_O is as consistent, though resulting travel times are 2-3% higher than those for Delmin. P_I generally performs slightly better than Delmin.
Of the two monotone policies P_{IM} performs very disappointingly, with travel times up to 13% higher than Delmin; P_M performs better, though generally slightly worse than P_O. The power policy again performs encouragingly, with total travel times similar to or lower than Delmin.

Table 13: Results for Weetwood network; Webster's cost assumptions

	TTT1	$\delta f1$	$\delta g1$	TTT2	$\delta f2$	$\delta g2$	$\overline{\Delta G}$	ΔG_{max}
ODx1.0								
Webster	2392	0.056	0.005	2395	0.058	0.008	0.11	1.3
Delmin	2349	0.013	0.000	2350	0.015	0.005	0.05	0.2
P_O	2416	0.057	0.008	2416	0.021	0.008	0.03	0.1
P_I	2343	0.022	0.008	2344	0.023	0.013	0.12	0.9
P_{IM}	2475	0.105	0.005	2475	0.088	0.005	0.09	1.0
P_M	2421	0.037	0.000	2422	0.018	0.005	0.03	0.2
Power	2366	0.019	0.003	2367	0.026	0.003	0.65	10.4
ODx1.1								
Webster	2867	0.027	0.010	2874	0.046	0.005	0.11	0.7
Delmin	2817	0.058	0.000	2819	0.057	0.003	0.12	0.7
P_O	2885	0.026	0.005	2884	0.030	0.003	0.05	0.2
P_I	2787	0.036	0.000	2795	0.070	0.010	2.23	23.4
P_{IM}	3014	0.416	0.005	3017	0.037	0.008	0.05	0.2
P_M	2894	0.056	0.013	2893	0.029	0.023	0.05	0.2
Power	2807	0.049	0.008	2765	0.073	0.000	3.48	27.3
ODx1.2								
Webster	3853	2.346	0.035	4156	2.479	0.010	1.62	9.1
Delmin	3485	0.199	0.013	3443	0.310	0.008	2.15	16.4
P_O	3520	0.372	0.000	3521	0.376	0.023	0.10	0.4
P_I	3370	0.125	0.003	3360	0.106	0.013	0.09	0.5
P_{IM}	3869	0.014	0.000	3901	0.018	0.008	0.09	0.3
P_M	3559	0.072	0.000	3558	0.057	0.000	0.10	0.4
Power	3417	0.110	0.010	3376	0.202	0.005	2.10	16.3

TTT1 = total network travel time in veh.hr/hr; start green 1
TTT2 = total network travel time in veh.hr/hr; start green 2
δf = excess travel costs over minimum costs in %
δg = average absolute change in green times in last iteration in sec.
$\overline{\Delta G}$ = average difference in resulting green times in sec.
ΔG_{max} = maximum difference in resulting green times in sec.

Again the convergence of the iterative assignment control procedure has been monitored via excess travel costs δf and the absolute average change in green times in the final iteration δg. It is important here to note that no stop criterion was applied to the procedure, apart from the maximum number of 500 iterations.
Apart from Webster's policy in the highly congested case the convergence of the assignment process is excellent, indicated by final excess travel costs for all other policies of less than 0.4%. This level of convergence is backed up by the average absolute final change in green times, which in this case is always less than 0.03 sec. Note how the convergence is negatively influenced by an increase in congestion.

An interesting picture is painted by the stability of the seven policies, expressed in $\overline{\Delta G}$ and ΔG_{max}. First note that, not surprisingly, stability tends to decrease with an increase in congestion; this is much more so than with polynomial cost assumptions (Table 11). Least stable in resulting green splits is the power policy, with an average difference of up to 3.5 sec and a maximum difference of up to 27.3 sec in green splits resulting from different starting points, even though this does not express itself in widely differing total travel times. The same argument, but to a lesser extent, is valid for P_I's behaviour and delay minimisation; Webster shows considerable differences in total travel times, as well as in resulting green times when congestion is high. P_O and the two monotone policies P_{IM} and P_M are most stable; particularly striking is the similarity in performance and stability between P_O and P_M.

A vital element of the power policy, the power k, deserves more attention here. I am particularly interested in:

(a) development of k-values during the iterative process and
(b) stability of its final values.

Table 14: Final values for power k dependent on initial green splits; Weetwood network; OD x 1.1

node	value k start green 1	value k start green 2
96	0.61	0.63
100	1.00	1.00
101	0.36	0.39
115	0.43	0.40
122	0.03	0.03
129	0.87	0.75
130	0.78	0.76
131	0.38	0.36
137	1.00	1.00
139	0.84	0.85
140	0.59	0.76
143	1.00	1.00
154	0.93	0.90
155	0.68	0.44
157	0.26	0.21
168	1.00	1.00
170	0.68	0.68

Figures 11 and 12 show the development of k-values per junction through the iterative process in the OD x 1.1 case, which gave rise to most unstable green times for the policy (see Table 13). These graphs show that the k-values change to a certain extent during the process, but settle down to a stable value towards the end of the iterative procedure. Their final value, however, depends on the initial green time settings as Table 14 shows more clearly, as initial timing influences the final flow and green time pattern.

9.3 A more dynamic example

Up to now all numeric examples had a fixed level of demand, for which signals were adjusted according to a chosen control policy. The iterative assignment control procedure can, however, be used to represent:

(a) regular updating of fixed time signal plans after traffic has re-adjusted to changed conditions

(b) performance of vehicle-actuated control over time.

In both cases drivers need time to experience changing conditions and to adjust their route choice accordingly. The assumption of fixed demand is rather restricted, given the current traffic growth of some 2.5% per year. Therefore, in this example, a dynamic adjustment of travel demand is allowed after each signal control step, to represent traffic growth. This traffic growth is set to 0.05% per step; in case (a) this would represent an update of the signal plan every week (maybe rather unrealistically); in case (b) this would represent a learning period for drivers of approximately 1 week.

The iterative assignment control procedure was started for the Weetwood network with a demand level of 1.1 x observed OD flows and again the two different initial green splits. Table 15 shows per policy resulting demand levels at which the flow/green time pattern becomes infeasible; Webster's delay formula was employed.

Table 15: Maximum demand levels that give rise to feasible flow/green time combinations; Weetwood network; Webster's cost function

control policy	maximum demand level; start green 1	maximum demand level; start green 2
Webster	1.203	-
Delmin	1.230	1.228
P_O	1.237	1.240
P_I	1.246	1.243
P_{IM}	1.233	1.241
P_M	1.247	1.249
Power	1.250	1.255

Of all control policies Webster's definitely performs worst: the maximum demand level that still gives rise to feasible flow/green time combinations is only slightly more than 1.2 x observed demand and for one set of start green splits the policy actually never settles down to a feasible solution. The capacity maximising policies (P_O, P_I, P_{IM} and P_M) all perform better than delay minimisation, in that they indeed allow a higher demand level to be processed by the network, but there is a clear influence of initial settings. Also **all** capacity maximising policies should theoretically give rise to equal maximum demand levels, as in the two-link example. The adaptation of Webster's cost function will play a role here. Particularly impressive in their performance are P_I and P_M, but it is the power policy that is really surprising. It outperforms all other policies, even the capacity maximising ones and gives rise to highest feasible demand levels.

Another view on these results is given by Figure 13, which depicts total network travel time per policy against the increasing demand level, for the case with start greens 1.
With Webster's policy the iterative assignment control procedure does not settle down to a feasible solution until a demand level of approximately 1.17 x observed OD flows and at a level just above 1.20 x observed demand at least one of the signal controlled link flows becomes infeasible. During this short feasible region network travel times are higher than for any other policy.
The poor behaviour of the P_{IM} policy, that already emerged from Table 13, is again illustrated. Delay minimisation gives rise to very advantageous settings at lower demand but travel times increase rapidly and infeasibility occurs at a demand of 1.23 x observed OD-flows.
It was observed before that P_O and P_M show a very similar behaviour, which is confirmed by the graph; P_M maintains feasibility longer than P_O. Finally, P_I and the power policy perform very alike, apart from the highest feasible demand levels where the power policy gives rise to lower total travel times; in addition this policy maintains feasibility longest.

9.4 Conclusions for the Weetwood network

The tests on the Weetwood network have shown some interesting, though not surprising characteristics of the iterative assignment control procedure in conjunction with different cost assumptions and different control policies.

In the first place the streamlined version of the iterative assignment control procedure converges extremely well in virtually all cases, as indicated by the values of δf and δg.

Secondly none of the policies shows as extreme a behaviour as in the two-link example, or even the TGA network. A feasible boundary of the flow/green time space is seldom reached, so that delay minimisation in general shows the best behaviour of all policies, despite potential theoretical problems.

The monotone policies are most stable, as expected, but 2-13% less efficient than Delmin. The need for capacity-maximising properties is not apparent in this network; of all four capacity-maximising policies P_0 is preferred. It generally outperforms the other policies and is applicable with **all** cost functions.

Of the remaining policies Webster expresses the most unstable behaviour, particularly at high congestion. The pragmatic power policy's performance is very promising in terms of total network travel times, but rather unstable in resulting green splits; it seldom improves on Delmin.

The cost function employed has at least as important an effect on results as the choice of control policy. With polynomial cost assumptions all three policies tested behave in a very similar way, giving rise to virtually identical network travel times. When employing Webster's cost function the network capacity is limited and the influence of the control policy used on network performance and stability of green splits is much more pronounced.

The influence of the cost assumptions on the results is best illustrated by a comparison of resulting green splits and total network travel times per policy after application of the iterative assignment control procedure with each of the two cost functions, as shown in Table 16.

Table 16: Differences in results with Webster's and polynomial cost assumptions; Weetwood network, OD x 1.0, start green 1

	TTT-BPR	TTT-WEB	$\overline{\Delta G}$	ΔG_{max}
Webster	2007	2392	15.0	63.6
Delmin	2005	2349	12.2	61.9
P_0	2005	2416	10.8	32.7

TTT-BPR = total network travel time in veh.hr/hr; BPR cost function
TTT-WEB = total network travel time in veh.hr/hr; Webster's cost function
$\overline{\Delta G}$ = average difference in resulting green times in sec.
ΔG_{max} = maximum difference in resulting green times in sec.

Because of the congestion characteristics of the two cost functions only the observed case (OD x 1.0) can be compared. Not surprisingly, total network travel times are some 20% higher with Webster's cost definition than under the BPR assumptions, although in fact the shape of the polynomial delay function should be calibrated via the parameters α and β. More importantly, and less dependent on such a calibration, the resulting green times are totally dissimilar under the two cost definitions, as average and maximum green time differences illustrate. This indicates the limited value of modelled green splits for real-life use, **unless a very realistic cost definition is applied**. This will be the subject of the next Chapter.

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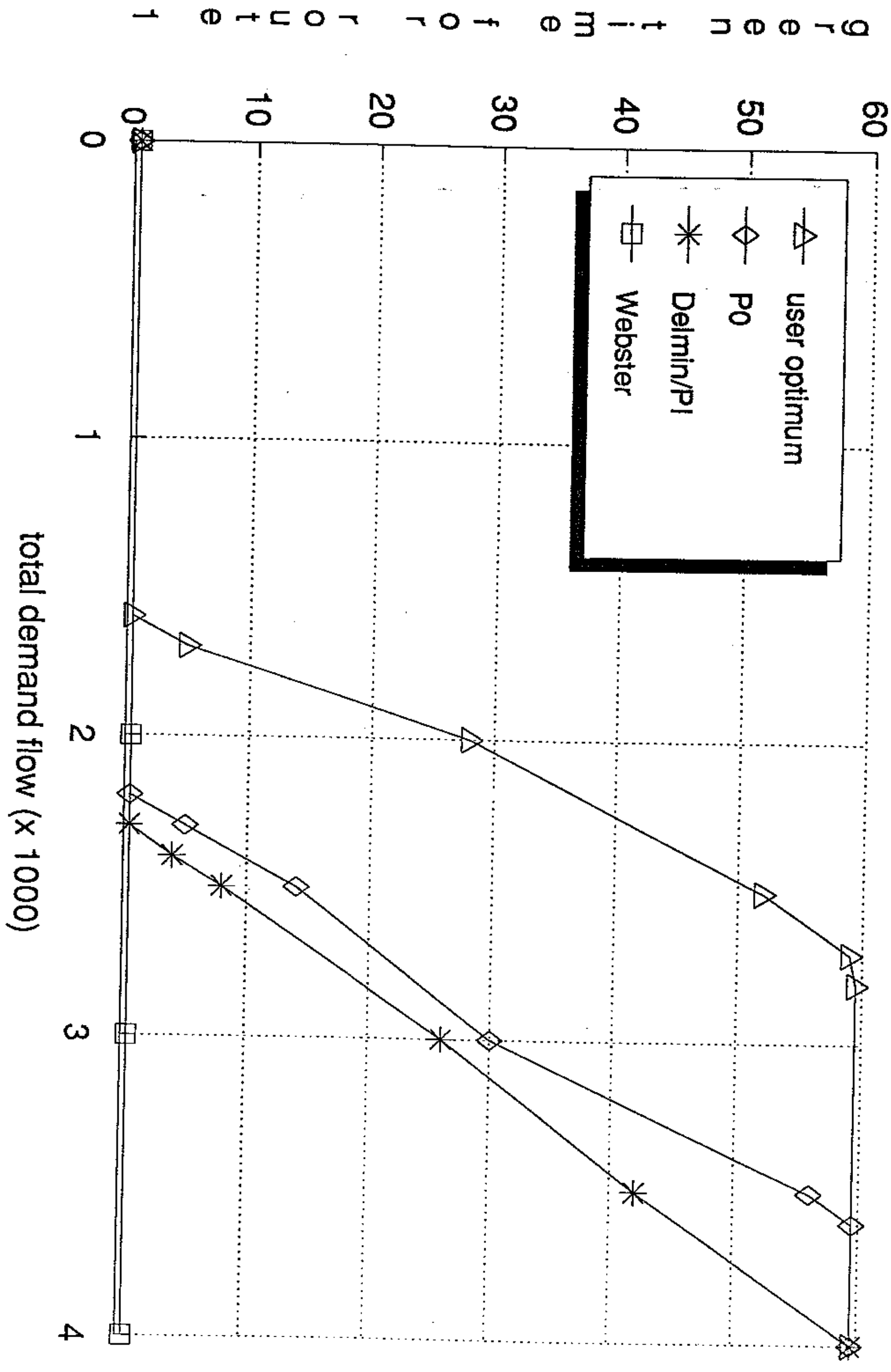
Figure 1 Green time at mutual equilibrium for four policies; 2-link test network; BPR cost function



<u>Figure 2</u> Flows on bypass at mutual equilibrium for four policies; 2-link test network; BPR cost function

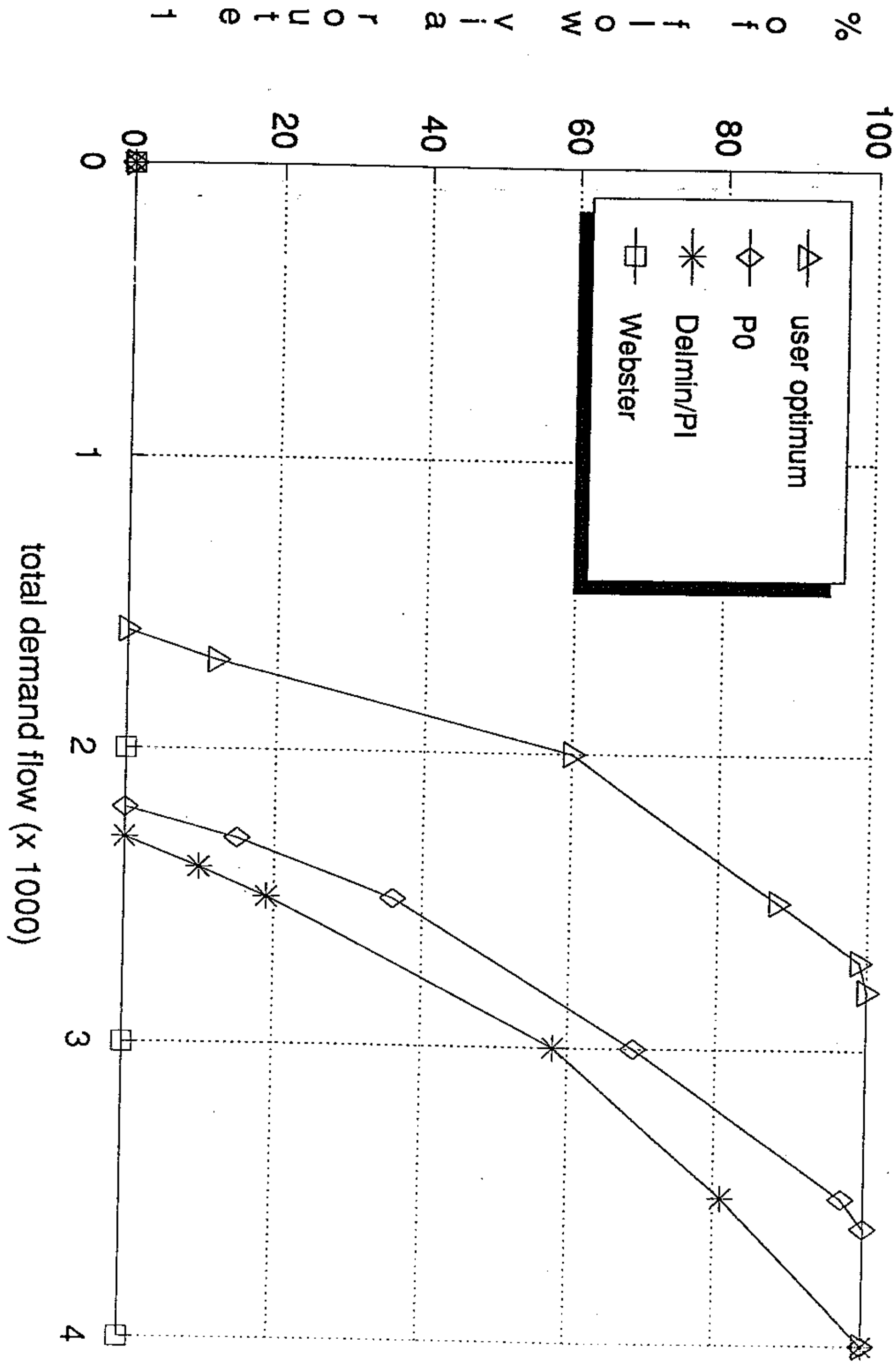
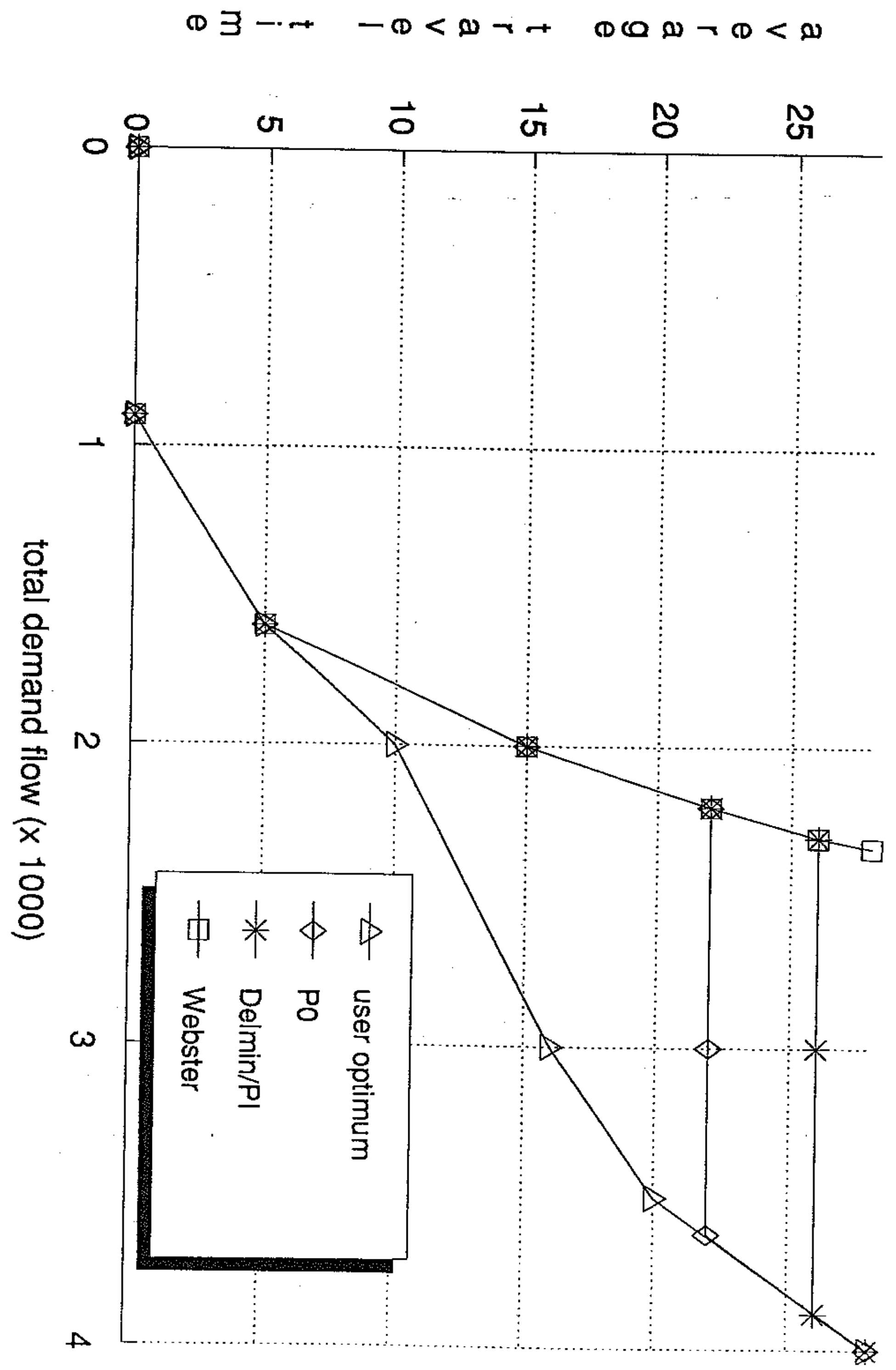


Figure 3 Excess travel time at mutual equilibrium for four policies; 2-link test network; BPR cost function



<u>Figure 4</u> Green time at mutual equilibrium for seven policies; 2-link test network; Webster's cost function

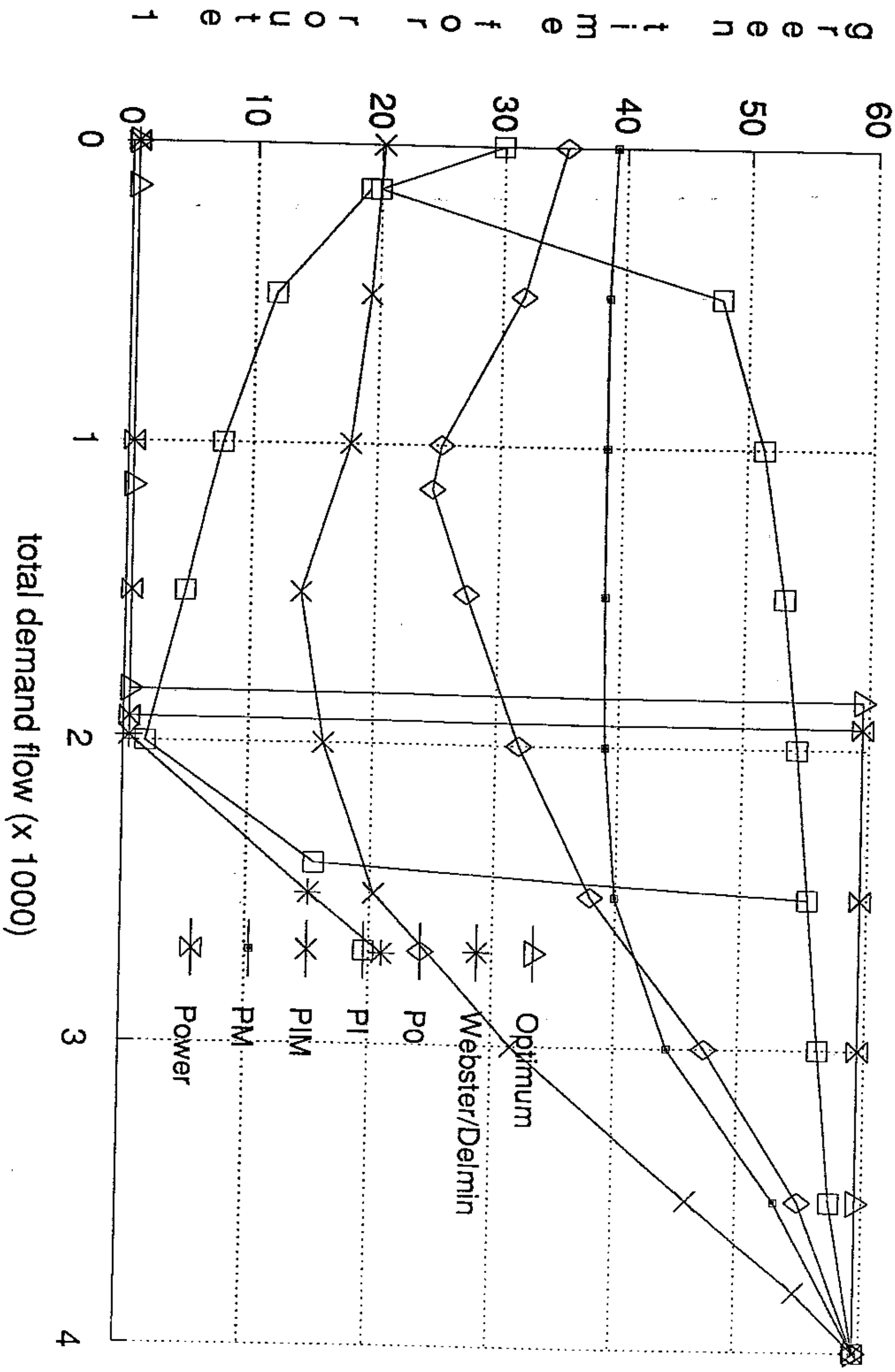


Figure 5 Flows on bypass at mutual equilibrium for seven policies; 2-link test network; Webster's cost function

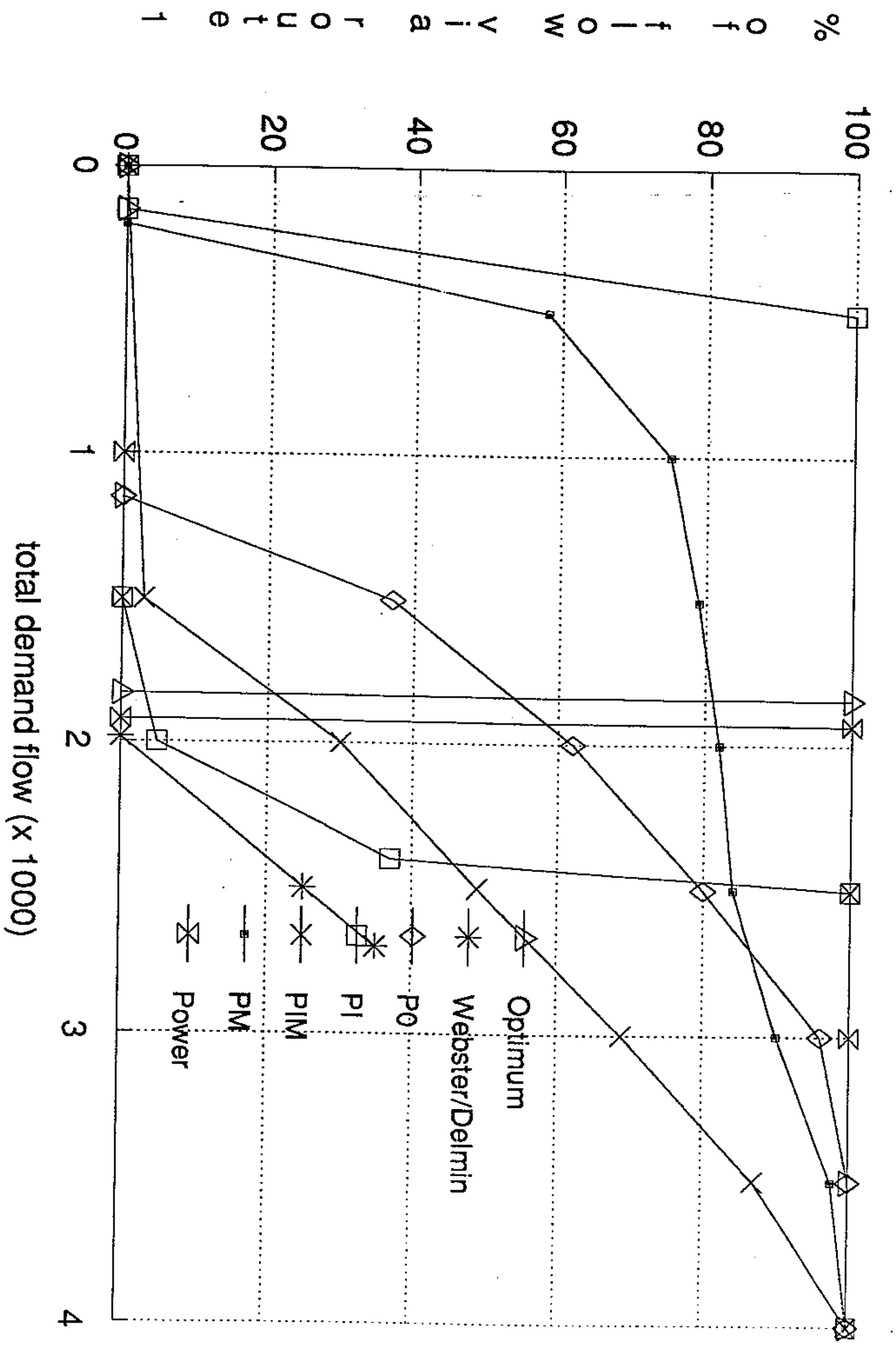


Figure 6 Excess travel time at mutual equilibrium for seven policies; 2-link test network; Webster's cost function

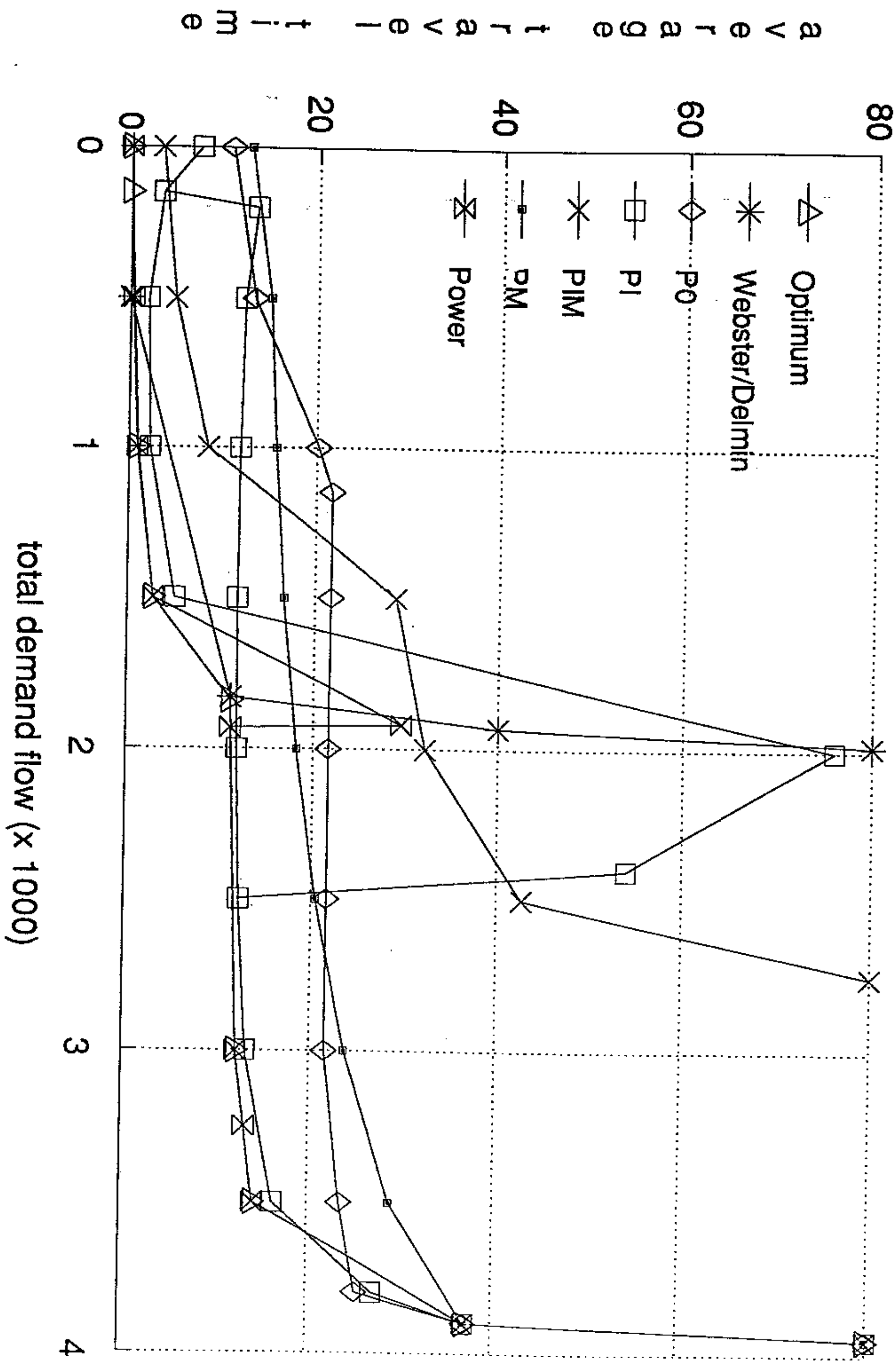
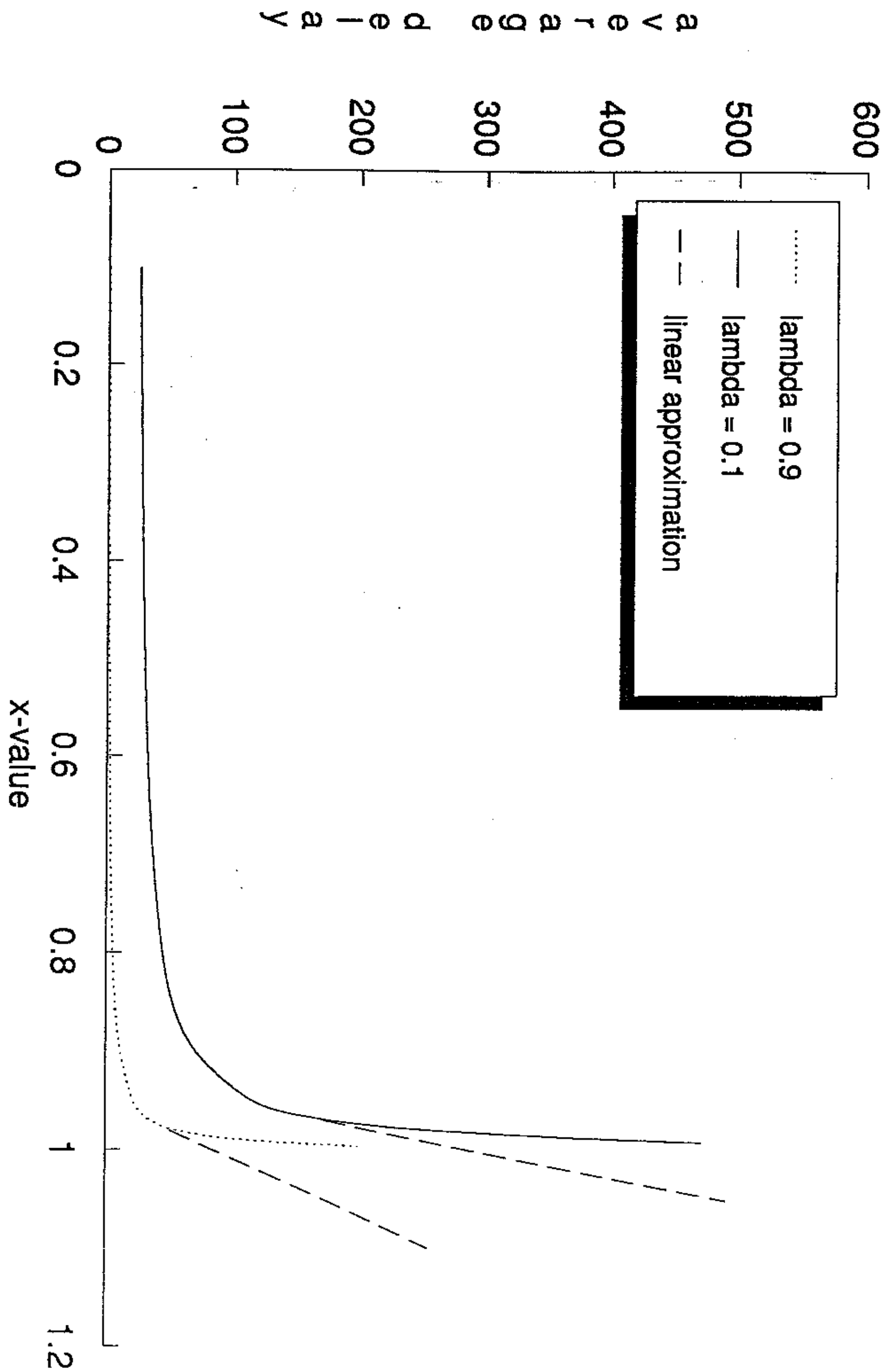


Figure 7 Webster's cost function and its approximation



<u>Figure 8</u> Convergence characteristics of the assignment sub-step in the iterative assignment control procedure; original implementation

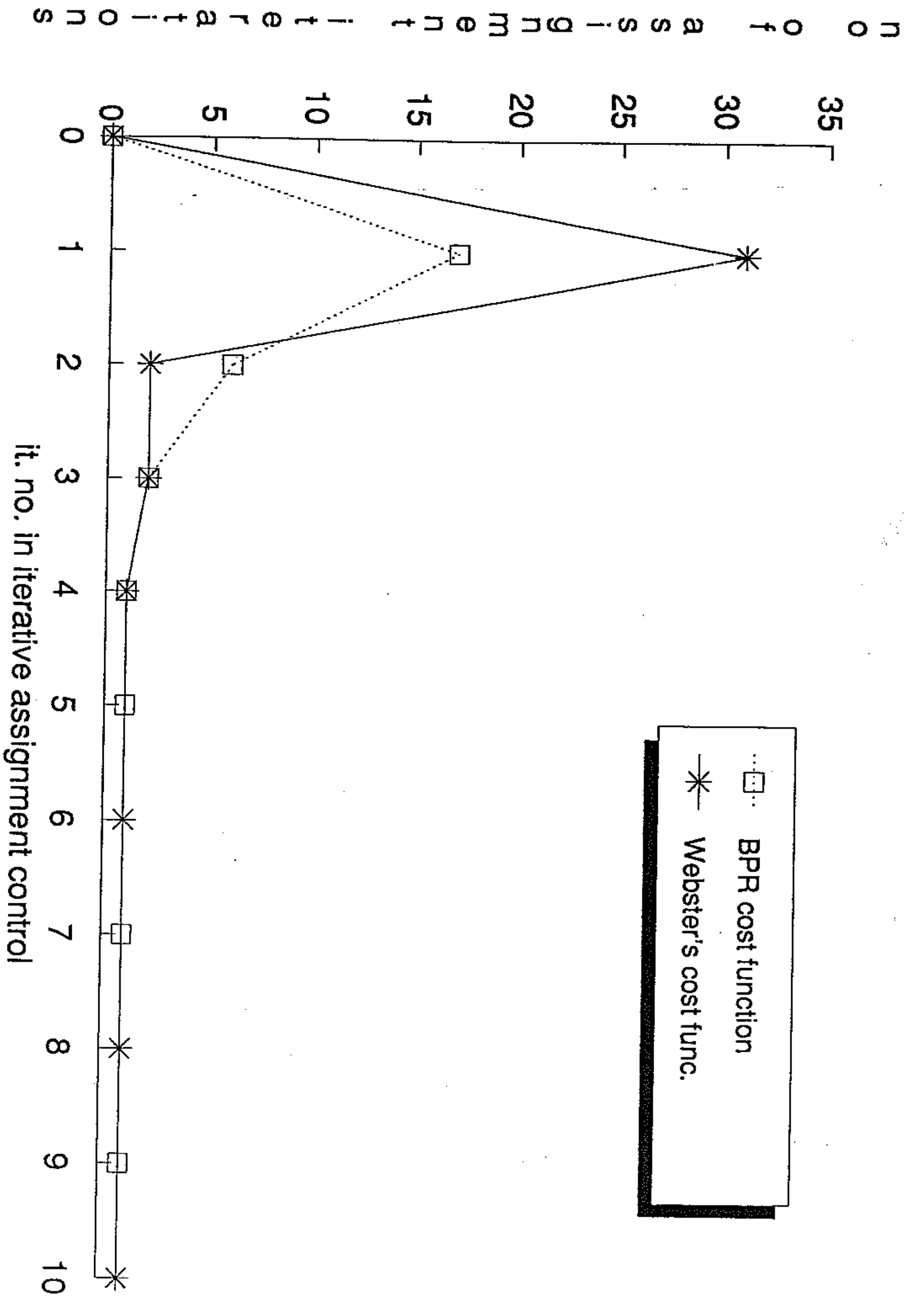


Figure 9 TGA network (Source: Tan et al. 1982)

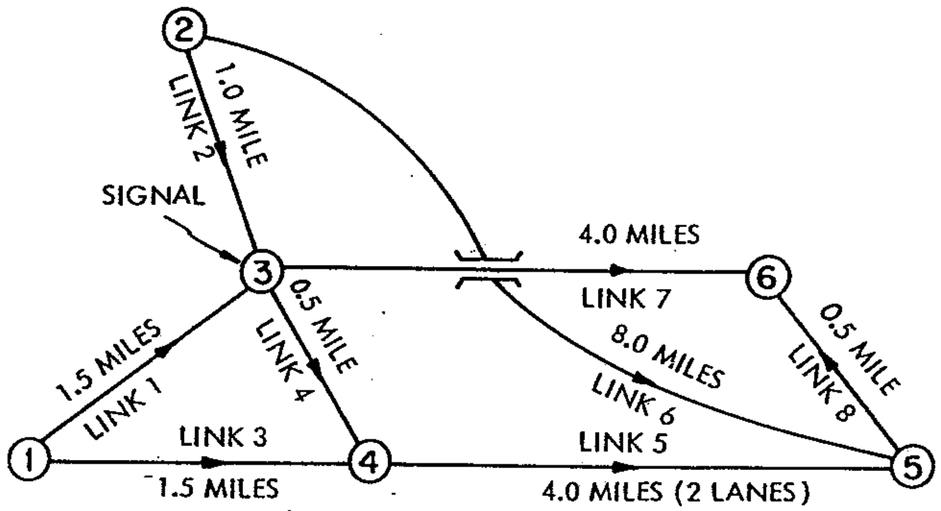


Figure 10 Weetwood network

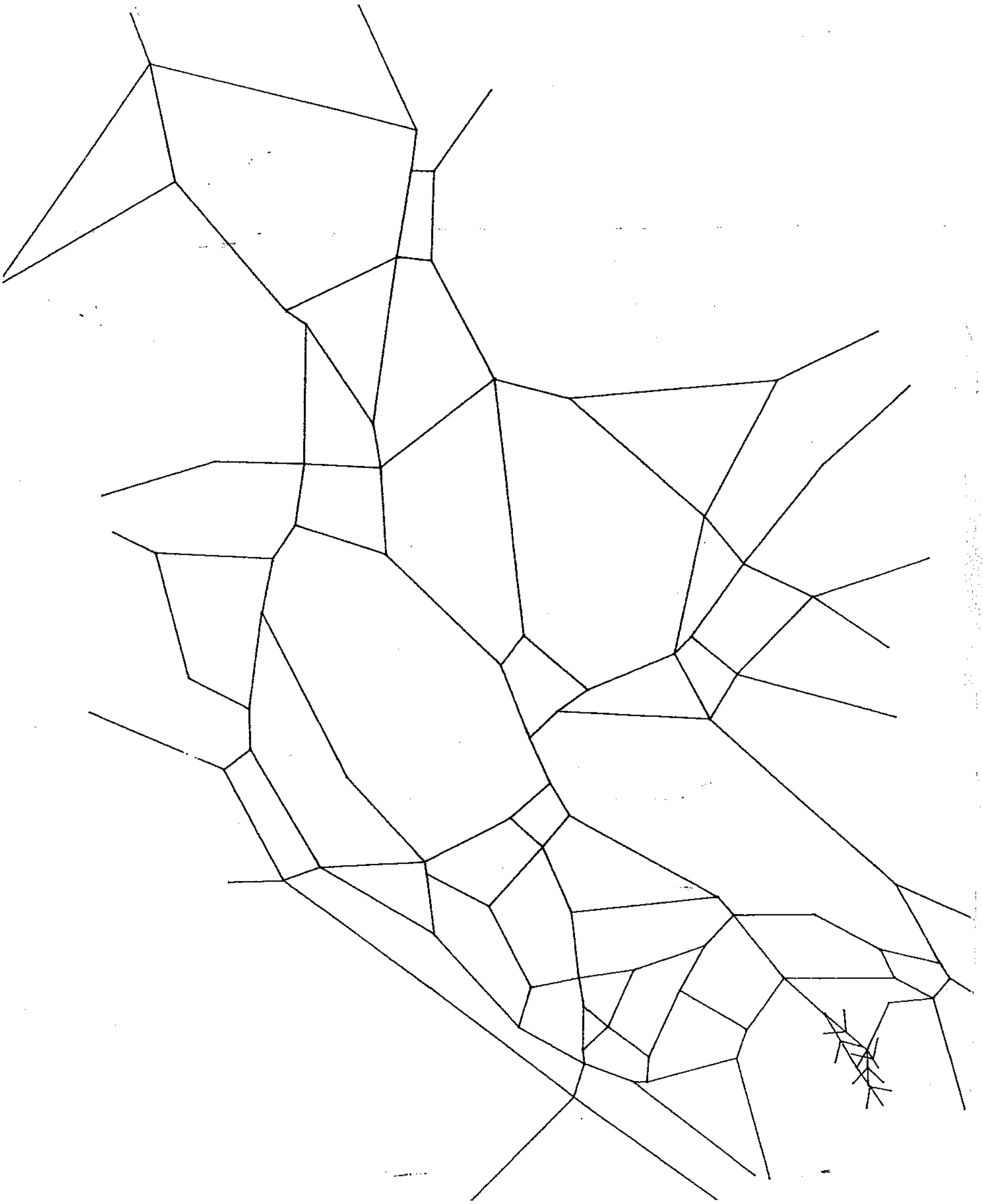


Figure 11 Development of power k in the iterative assignment control procedure; Weetwood network; OD x 1.1; initial settings 1

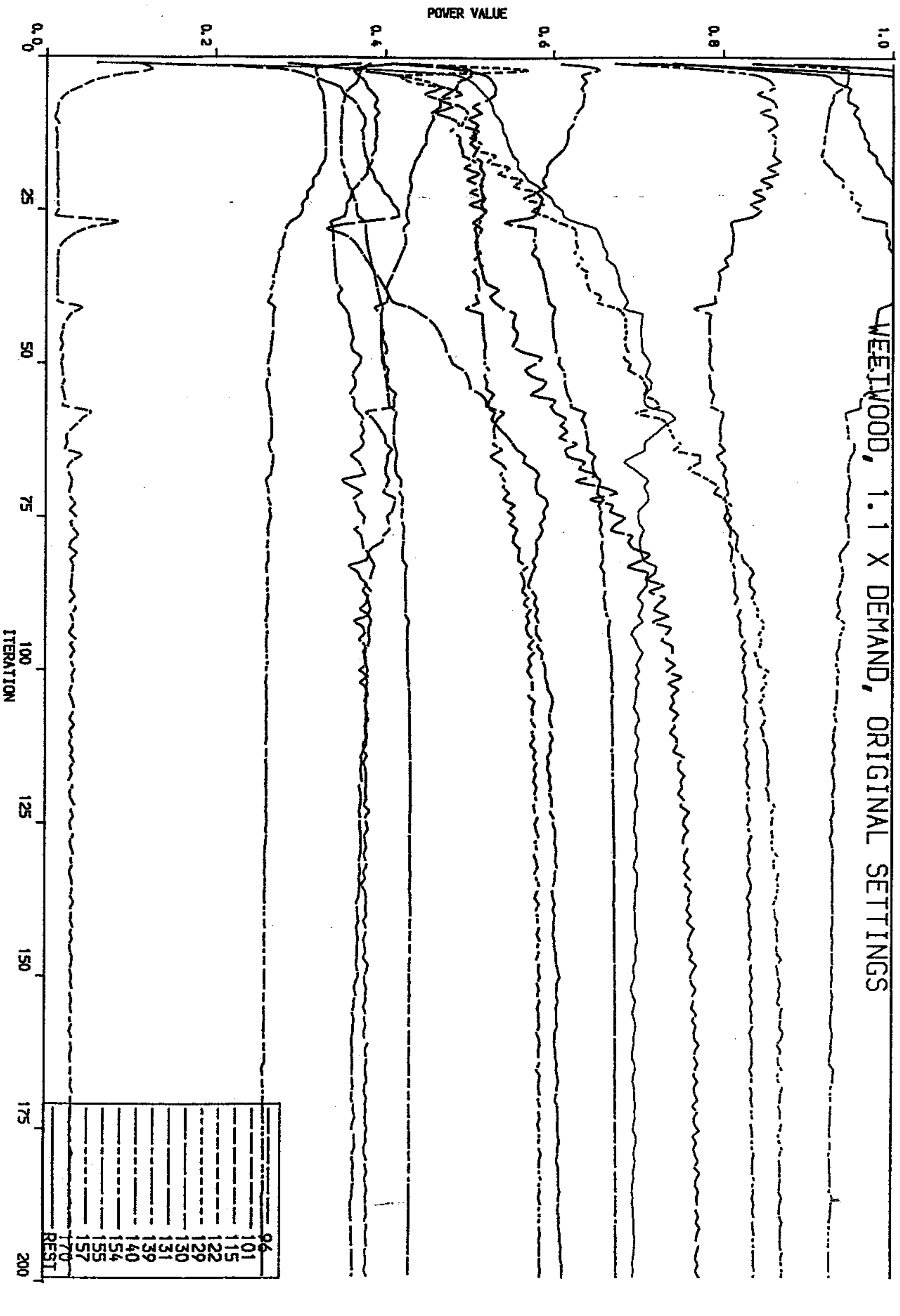


Figure 12 Development of power k in the iterative assignment control procedure; Weetwood network; OD x 1.1; initial settings 2

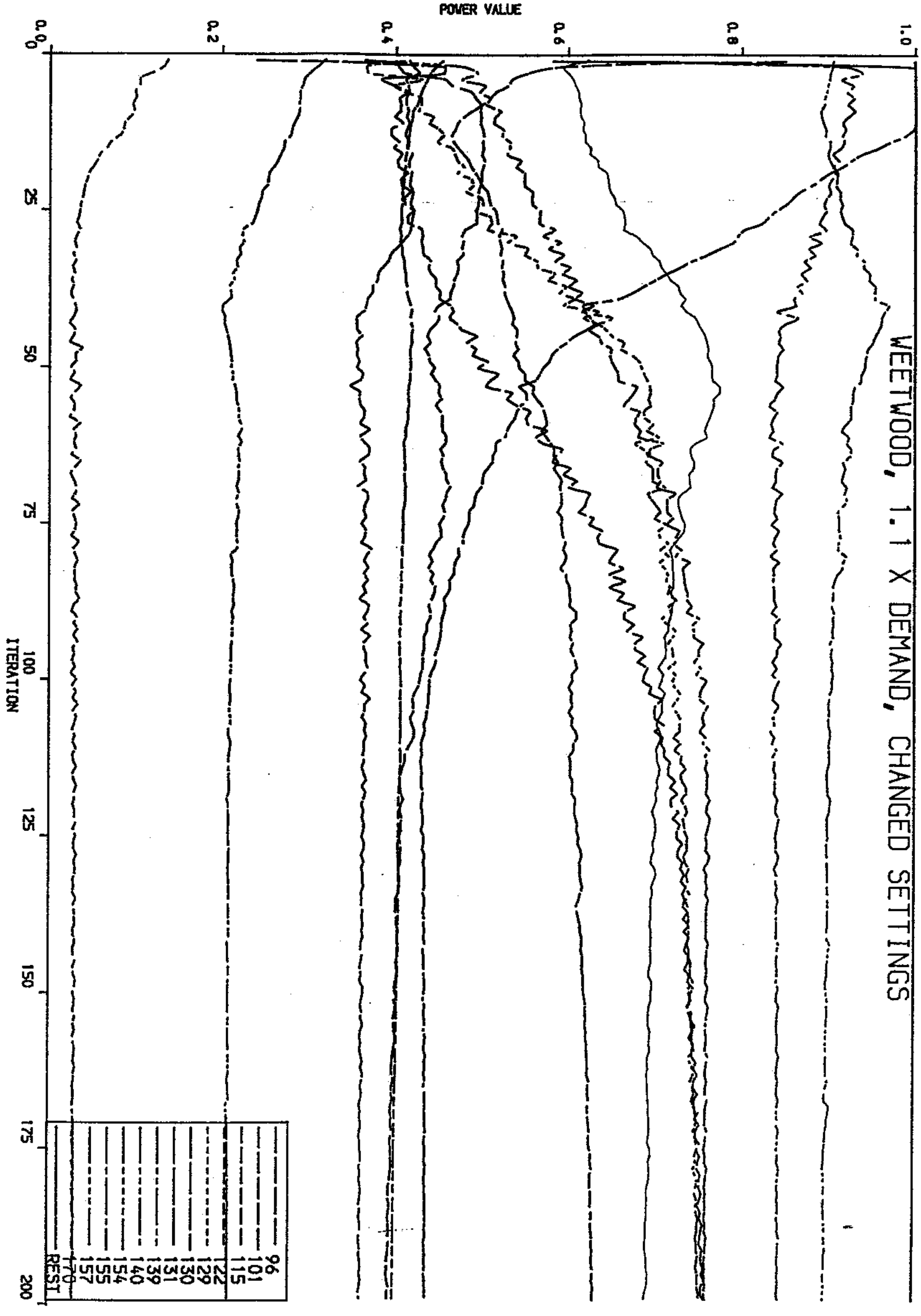
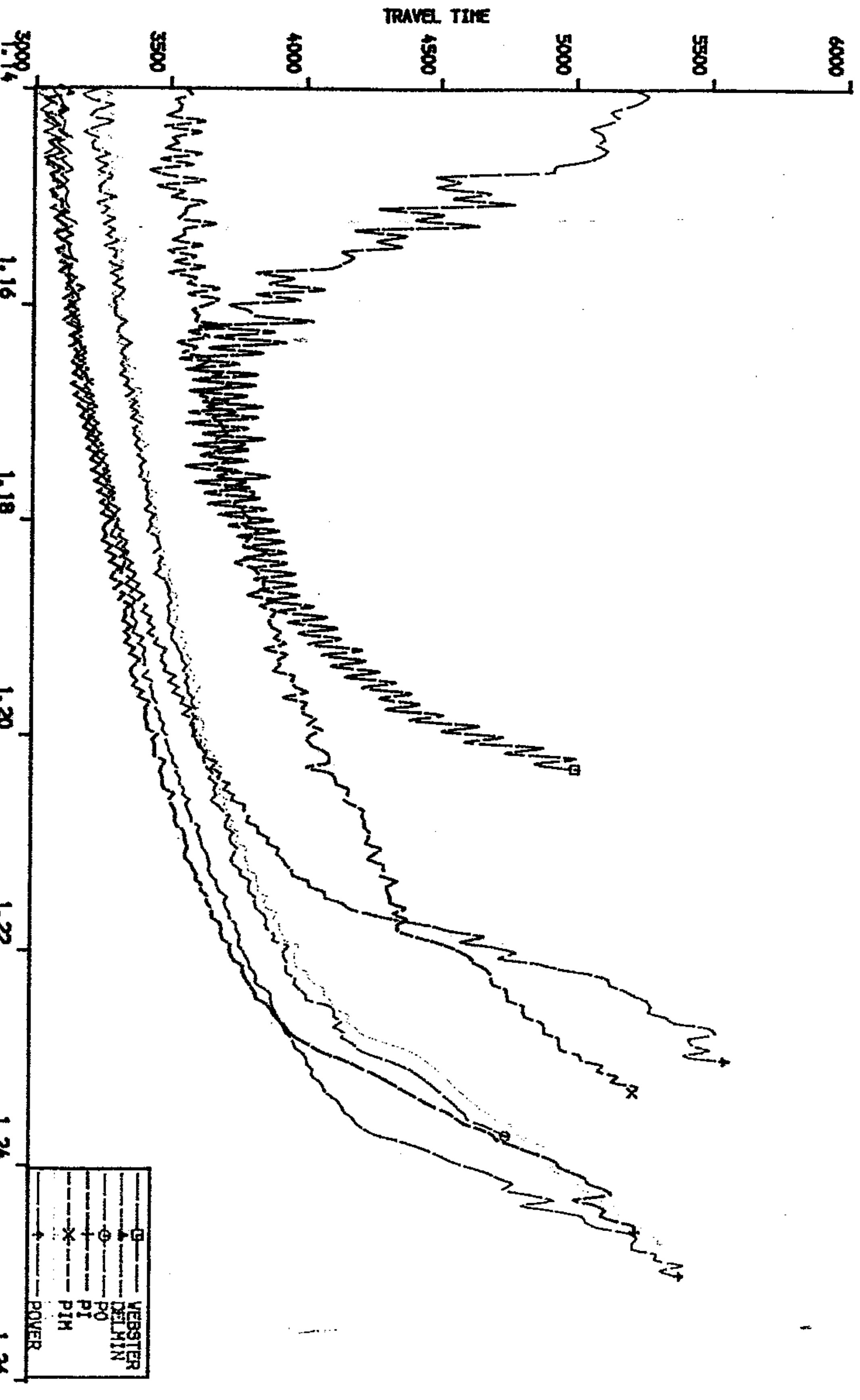


Figure 13 Total network travel time versus travel demand; dynamic loading; Weetwood network; initial settings 1



APPENDIX 1: Monotonicity with BPR delay function

For a solution to the combined assignment signal control problem to exist and to be unique the vector (t,-p) should be the gradient of a convex function V.

(t,-p) is the gradient of function V if

$$\partial V/\partial f = t \text{ and } \partial V/\partial \lambda = -p$$

If V is smooth, and thus

$$\partial^2 V/\partial f \partial \lambda = \partial^2 V/\partial \lambda \partial f$$

we can express p as follows:

$$p = \quad - \int_f \partial t/\partial \lambda + \phi(\lambda)$$

V is convex iff the gradient (t,-p) is monotone so that the Jacobian of this vector is positive semi-definite. The Jacobian is

$$\begin{pmatrix} \partial^2 V/\partial f^2 & \partial^2 V/\partial f \partial \lambda \\ \partial^2 V/\partial \lambda \partial f & \partial^2 V/\partial \lambda^2 \end{pmatrix}$$

Even if (t,-p) is not a gradient monotonicity of (t,-p) is a desirable property, because then a convex set of equilibria is guaranteed to exist; which may be unique in that it may consist of a single point (Smith, 1982).
(t,-p) is monotone iff $\left\| \frac{J+J^T}{2} \right\| \geq 0$; (Smith, 1985).

This is clearly a slightly weaker condition than that mentioned above; here the **symmetrized** Jacobian must be positive semi-definite.

$$t = \alpha t_o \; f^\beta/\lambda^\beta s^\beta$$

$$V = \int_0^f t(x)dx = \alpha t_o \; 1/(\beta+1) \; f^{\beta+1}/\lambda^\beta s^\beta$$

$$p = -\partial V/\partial \lambda = \alpha t_o \; \beta/(\beta+1) \; f^{\beta+1}/\lambda^{\beta+1} s^\beta \qquad \text{for policy } P_I$$

$\partial t/\partial f = \alpha t_o \beta\, f^{\beta-1}/\lambda^\beta s^\beta$
$\partial t/\partial \lambda = -\alpha t_o \beta\, f^\beta/\lambda^{\beta+1} s^\beta$
$\partial p/\partial f = \alpha t_o \beta\, f^\beta/\lambda^{\beta+1} s^\beta$
$\partial p/\partial \lambda = -\alpha t_o \beta\, f^{\beta+1}/\lambda^{\beta+2} s^\beta$

$$J+J^T/2 = \begin{pmatrix} \partial t/\partial f & \partial t/\partial\lambda \\ -\partial p/\partial f & -dp/\partial\lambda \end{pmatrix} \qquad \text{as } \partial t/\partial\lambda = -\partial p/\partial f$$

$$= \alpha t_o \beta\, f^\beta/\lambda^\beta s^\beta \begin{pmatrix} 1/f & -1/\lambda \\ -1/\lambda & f/\lambda^2 \end{pmatrix}$$

$$\left\| \frac{J+J^T}{2} \right\| = 1/\lambda^2 - 1/\lambda^2 = 0$$

so that P_I is monotone

<u>Delmin</u>: $p = f\, \partial t/\partial\lambda$

$$\partial t/\partial\lambda = -\alpha\, t_o\, \beta\, f^\beta/\lambda^{\beta+1} s^\beta$$

$$p = -\alpha\, t_o\, \beta\, f^{\beta+1}/\lambda^{\beta+1} s^\beta = (-\beta+1) \times p_{PI}$$

so (t,-p) is monotone for delay minimisation

<u>P_o</u>

$p = s.d$
$p = \alpha\, t_o\, f^\beta/\lambda^\beta s^{\beta-1}$
$\partial p/\partial f = \alpha\beta\, t_o\, f^{\beta-1}/\lambda^\beta s^{\beta-1}$
$\partial p/\partial\lambda = -\alpha\beta\, t_o\, f^\beta/\lambda^{\beta+1} s^{\beta-1}$

$$J = \alpha\, t_o\, \beta\, f^\beta/\lambda^\beta s^\beta \begin{pmatrix} 1/f & -1/\lambda \\ -s/f & s/\lambda \end{pmatrix}$$

$$J+J^T = \alpha\, t_o\, \beta\, f^\beta/\lambda^\beta s^\beta \begin{pmatrix} 2/f & -1/\lambda - s/f \\ -1/\lambda - s/f & 2s/\lambda \end{pmatrix}$$

$$\left\| \frac{J+J^T}{2} \right\| = 2s/\lambda f - s^2/f^2 - 1/\lambda^2 = -(\lambda s - f)^2 < 0$$

thus P_o not monotone with BPR cost definition.

<u>Webster</u>

$$p = f/\lambda s$$
$$\partial p/\partial f = 1/\lambda s$$
$$\partial p/\partial \lambda = -f/\lambda^2 s$$

$$J = \begin{pmatrix} \alpha\ t_o\ \beta\ f^{\beta-1}/\lambda^{\beta} s^{\beta} & -\alpha\ t_o\ \beta\ f^{\beta}/\lambda^{\beta+1} s^{\beta} \\ -1/\lambda s & f/\lambda^2 s \end{pmatrix}$$

$$\left\| \frac{J+J^T}{2} \right\| = -\ (\alpha\ t_o\ \beta\ f^{\beta}/\lambda^{\beta} s^{\beta}\ -\ 1/s)^2 < 0$$

and so Webster's policy is not monotone with BPR cost definition either.

APPENDIX 2: Monotonicity with Webster's cost definition

$$t = 1/2\ \{C(1-\lambda)^2/(1-f/s) + f/[\lambda s(\lambda s-f)]\}$$

Apart from the factor 1/2, this expression can be divided into 2 terms

$$t_1 = C(1-\lambda)^2/(1-f/s)$$
$$t_2 = f/[\lambda s(\lambda s-f)] = 1/(\lambda s-f) - 1/\lambda s\ [\ = x^2/f(1-x)\ ]$$

<u>First term</u> (t_1)

$$V_1 = \int_f t_1(x)dx$$
$$= -s(1-\lambda)^2\ C\ \log(1-f/s)$$

$$p_1 = -\partial V/\partial\lambda = -2sC(1-\lambda)\log(1-f/s) \quad \text{for policy } P_I$$

$$\partial t_1/\partial f = C(1-\lambda)^2/s(1-f/s)^2$$
$$\partial t_1/\partial\lambda = -2C(1-\lambda)/(1-f/s)$$
$$\partial p_1/\partial f = 2C(1-\lambda)/(1-f/s)$$
$$\partial p_1/\partial\lambda = 2sC\ \log(1-f/s)$$

$$J = \begin{pmatrix} C(1-\lambda)^2/s(1-f/s)^2 & -2C(1-\lambda)/(1-f/s) \\ -2C(1-\lambda)/(1-f/s) & -2sC\ \log(1-f/s)) \end{pmatrix}$$

$$\left\| \frac{J+J^T}{2} \right\| = C(1-\lambda)^2/(1-f/s)^2\ .\ (-2C\ \log(1-f/s) - 4C)$$

Thus, the Jacobian is not positive semi-definite and the first term of policy P_I is not monotone. To ensure montonocity a correction term must be introduced in policy P_{IM}. Properties of this correction term include:

- ϕ is function of λ only
- $\partial\phi/\partial\lambda = 4sC$ (as $-2C\ \log(1 - f/s) \geq 0$)

Thus $p_1 = -2sC(1-\lambda)\log(1-f/s) + 4sC(1-\lambda)$ for policy P_{IM}

$$V_1 = -\int_{\lambda} p(x)dx = sC(1-\lambda)^2\log(1-f/s) - 2sC(1-\lambda)^2$$

$\partial t_1/\partial f$, $\partial t_1/\partial\lambda$, $\partial p_1/\partial f$ as before

$\partial p_1/\partial\lambda = 2sC\ \log(1-f/s) - 4sC$

$$\left\|\frac{J+J^T}{2}\right\| = C(1-\lambda)^2/(1-f/s)^2\ (-2C\ \log(1-f/s) + 4C - 4C)$$

and thus the first term of P_{IM} is monotone

<u>Second term</u> (t_2)

$$V_2 = \int_f t_2(x)dx$$

$$= -\log(\lambda s-f) - f/\lambda s + \log(\lambda s)$$

$$p_2 = -\partial V/\partial\lambda = s/(\lambda s-f) - f/\lambda^2 s - 1/\lambda \quad \text{for policy } P_I$$

$$\begin{aligned} V_2 &= -\log(\lambda s-f) - f/\lambda s + \log(\lambda s) \\ &= -\log[(1 - f/\lambda s).\lambda s] - f/\lambda s + \log(\lambda s) \\ &= -\log(1 - f/\lambda s) - \log(\lambda s) - f/\lambda s + \log(\lambda s) \\ &= -\log(1 - f/\lambda s) - f/\lambda s \end{aligned}$$

$$\begin{aligned} \partial t_2/\partial f &= 1/(\lambda s-f)^2 \\ \partial t_2/\partial\lambda &= -s/(\lambda s-f)^2 + 1/\lambda^2 s \\ \partial p_2/\partial f &= s/(\lambda s-f)^2 - 1/\lambda^2 s \\ \partial p_2/\partial\lambda &= -s^2/(\lambda s-f)^2 + 2f/\lambda^3 s + 1/\lambda^2 \end{aligned}$$

$$J = \begin{pmatrix} 1/(\lambda s-f)^2 & -s/(\lambda s-f)^2 + 1/\lambda^2 s \\ -s/(\lambda s-f)^2 + 1/\lambda^2 s & s^2/(\lambda s-f)^2 - 2f/\lambda^3 s - 1/\lambda^2 \end{pmatrix}$$

$$\begin{aligned} \left\|\frac{J+J^T}{2}\right\| &= -2f/[(\lambda s-f)^2\lambda^3 s] - 1/[\lambda^2(\lambda s-f)^2] + 2/[\lambda^2(\lambda s-f)^2] - 1/\lambda^4 s^2 \\ &= 1/\lambda^2\ \ [(1-2f/\lambda s)/(\lambda s-f)^2 - 1/\lambda^2 s^2] \\ &= 1/\lambda^2\ \ \{(\lambda s-f)^2/[(\lambda s-f)^2\lambda^2 s^2] - f^2/[\lambda^2 s^2(\lambda s-f)^2] - 1/\lambda^2 s^2\} \\ &= -f^2/[\lambda^4 s^2(\lambda s-f)^2] < 0 \ \text{ thus second term of } P_I \text{ not monotone} \end{aligned}$$

Correction term ϕ must be function of λ only, so no ϕ emerges naturally from the above. However, if we try $\phi = 1/\lambda$ (to compensate the integration constant):

$$
\begin{aligned}
p_2 &= s/(\lambda s-f) - f/\lambda^2 s \\
\partial p_2/\partial f &= s/(\lambda s-f)^2 - 1/\lambda^2 s \\
\partial p_2/\partial \lambda &= -s^2/(\lambda s-f)^2 + 2f/\lambda^3 s
\end{aligned}
$$

$$
J = (J+J^T)/2 = \begin{pmatrix} 1/(\lambda s-f)^2 & -s/(\lambda s-f)^2 + 1/\lambda^2 s \\ -s/(\lambda s-f)^2 + 1/\lambda^2 s & s^2/(\lambda s-f)^2 - 2f/\lambda^3 s \end{pmatrix}
$$

$$
\begin{aligned}
\left\| \frac{J+J^T}{2} \right\| &= 1/[\lambda^2(\lambda s-f)^2] \,.\, [-2f/\lambda s + 2 - (\lambda s-f)^2/\lambda^2 s^2] \\
&= 1/[\lambda^2(\lambda s-f)^2] \,.\, [-2f/\lambda s + 2 - 1 + 2f/\lambda s - f^2/\lambda^2 s^2] \\
&= 1/[\lambda^2(\lambda s-f)^2] \,.\, [1 - f^2/\lambda^2 s^2] > 0
\end{aligned}
$$

and so the second term of P_{IM} is monotone too

PM Policy

This policy is characterised by the following pressure definition:

$$p = sC(1-\lambda) + s/(\lambda s-f)$$

This pressure consists of two elements, each one associated with part of Webster's cost function. The term $sC(1-\lambda)$ is associated with Webster's first term; note the similarity with policy P_I's first term. The term $s/(\lambda s-f)$ resembles the second term of P_I and is associated with the second part of Webster's cost function. The resulting vector (t,-p) is not a gradient for this pressure, but monotonicity can be established.

Monotonicity for P_M can again be tested for each part separately.

First term

$$t_1 = C(1-\lambda)^2/(1-f/s)$$
$$p_1 = sC(1-\lambda)$$
$$\partial t_1/\partial f = C(1-\lambda)^2/s(1-f/s)^2$$
$$\partial t_1/\partial \lambda = -2C(1-\lambda)/(1-f/s)$$
$$\partial p_1/\partial f = 0$$
$$\partial p_1/\partial \lambda = -sC$$

$$J = \begin{pmatrix} C(1-\lambda)^2/s(1-f/s)^2 & -2C(1-\lambda)/(1-f/s) \\ 0 & sC \end{pmatrix}$$

$$\left\| \frac{J+J^T}{2} \right\| = C^2(1-\lambda)^2/(1-f/s)^2 > 0$$ so first term of P_M is monotone

Second term

$$t_2 = f/[\lambda s(\lambda s-f)]$$
$$p_2 = s/(\lambda s-f)$$
$$\partial t_2/\partial f = 1/(\lambda s-f)^2$$
$$\partial t_2/\partial \lambda = -s/(\lambda s-f)^2 + 1/\lambda^2 s$$
$$\partial p_2/\partial f = s/(\lambda s-f)^2$$
$$\partial p_2/\partial \lambda = -s^2/(\lambda s-f)^2$$

$$J = \begin{pmatrix} 1/(\lambda s-f)^2 & -s(\lambda s-f)^2 + 1/\lambda^2 s \\ -s/(\lambda s-f)^2 & s^2/(\lambda s-f)^2 \end{pmatrix}$$

$$\begin{aligned} \left\| \frac{J+J^T}{2} \right\| &= s^2/(\lambda s-f)^4 - s^2/(\lambda s-f)^4 + 1/[\lambda^2(\lambda s-f)^2] - 1/(4\lambda^4 s^2) \\ &= 1/(\lambda s-f)^2 - 1/(4\lambda^2 s^2) \\ &\geq 1/\lambda^2 s^2 \quad - 1/(4\lambda^2 s^2) \quad \geq 3/(4\lambda^2 s^2) \quad \geq 0 \end{aligned}$$

and thus second term of P_M also monotone

APPENDIX 3: Monotonicity with Davidson's cost function

$$t = t_o\,(1 + K\,f/\lambda s\text{-}f)$$
$$= t_o + t_o\,K\,(\lambda s/(\lambda s\text{-}f) - 1)$$

Here we concentrate on delay term $(\lambda s/(\lambda s\text{-}f) - 1)$

$$V = \int_{f} t(x)dx$$
$$= -\lambda s \log(\lambda s\text{-}f) - f + \lambda s \log(\lambda s)$$
$$p = -\partial V/\partial \lambda \quad = \lambda s^2/(\lambda s\text{-}f) + s \log(\lambda s\text{-}f) - s \log(\lambda s) - s$$
$$= -s \log[\lambda s/(\lambda s\text{-}f)] + fs/(\lambda s\text{-}f) \quad \text{for policy } P_I$$

$$\partial t/\partial f \quad = \lambda s/(\lambda s\text{-}f)^2$$
$$\partial t/\partial \lambda \quad = -fs/(\lambda s\text{-}f)^2$$
$$\partial p/\partial f \quad = \lambda s^2/(\lambda s\text{-}f)^2 - s/(\lambda s\text{-}f) = fs/(\lambda s\text{-}f)^2$$
$$\partial p/\partial \lambda \quad = s^2/(\lambda s\text{-}f) - \lambda s^3/(\lambda s\text{-}f)^2 + s^2/(\lambda s\text{-}f) - s/\lambda$$
$$= (\lambda s^3\text{-}2fs^{2)}/(\lambda s\text{-}f)^2 - s/\lambda$$

$$J = \begin{pmatrix} \lambda s/(\lambda s\text{-}f)^2 & -fs/(\lambda s\text{-}f)^2 \\ -fs/(\lambda s\text{-}f)^2 & s/\lambda - (\lambda s^3\text{-}2fs^2)/(\lambda s\text{-}f)^2 \end{pmatrix}$$

$$\left\| \frac{J+J^T}{2} \right\| = s^2/(\lambda s\text{-}f)^2\,[1 - (\lambda s\text{-}f)^2/(\lambda s\text{-}f)^2] = 0$$

Thus policy P_I is monotone with Davidson's cost function.

Delmin

$$p = f\,\partial t/\partial \lambda = -f^2 s/(\lambda s\text{-}f)^2$$
$$\partial p/\partial f = -2fs/(\lambda s\text{-}f)^2 - 2f^2 s/(\lambda s\text{-}f)^3 = -2f\lambda s^2/(\lambda s\text{-}f)^3$$
$$\partial p/\partial \lambda = 2f^2 s^2/(\lambda s\text{-}f)^3$$

$$J = \begin{pmatrix} \lambda s/(\lambda s\text{-}f)^2 & -fs/(\lambda s\text{-}f)^2 \\ -2f\lambda s^2/(\lambda s\text{-}f)^3 & 2f^2 s^2/(\lambda s\text{-}f)^3 \end{pmatrix}$$

$$\left\| \frac{J+J^T}{2} \right\| = -[1 - 2\lambda s/(\lambda s\text{-}f)]^2 < 0$$

Therefore, delay minimisation with Davidson's cost assumptions is not monotone.

$\underline{P_o}$

$$\begin{aligned} p &= s.d \\ &= fs/(\lambda s-f) \end{aligned}$$

$$\partial p/\partial f = s/(\lambda s-f) + fs/(\lambda s-f)^2 = \lambda s^2/(\lambda s-f)^2$$

$$\partial p/\partial \lambda = -fs^2/(\lambda s-f)^2$$

$$J = \begin{pmatrix} \lambda s/(\lambda s-f)^2 & -fs/(\lambda s-f)^2 \\ -\lambda s^2/(\lambda s-f)^2 & fs^2/(\lambda s-f)^2 \end{pmatrix}$$

$$\left\| \frac{J+J^T}{2} \right\| = -s^2/(\lambda s-f)^2 < 0$$

And thus the P_o policy is not monotone either.

<u>Webster</u>

$$p = f/\lambda s$$

$$\partial p/\partial f = 1/\lambda s$$

$$\partial p/\partial \lambda = -f/\lambda^2 s$$

$$J = \begin{pmatrix} \lambda s/(\lambda s-f)^2 & -fs/(\lambda s-f)^2 \\ -1/\lambda s & f/\lambda^2 s \end{pmatrix}$$

$$\left\| \frac{J+J^T}{2} \right\| = -[f\lambda s^2/(\lambda s-f)^2 - 1]^2 < 0$$

so that Webster's policy not monotone either with Davidson's delay function.

APPENDIX 4: Polynomial cost implementation

$$t(f) = t_0 [1 + \alpha(f/c)^\beta]$$

$$Z = \int_0^f t(x)dx$$

$$= t_0 f + [t_0\alpha/(\beta + 1)](f/c)^{\beta+1}$$

APPENDIX 5: Webster's adapted cost function implementation

$$t(f) = \frac{C(1-\lambda)^2}{2(1-f/s)} + \frac{1}{2\lambda s-f} - \frac{1}{2\lambda s} \qquad f < kinkf$$

$$= \frac{C(1-\lambda)^2}{2(1-kinkf/s)} + \frac{T(f-kinkf)}{2\lambda s} \qquad f \geq kinkf$$

where kinkf = kink flow as defined in Chapter 7.2.

$$Z = -[Cs(1-\lambda)^2 \log(1-f/s) + \log(1-f/\lambda s) + f/\lambda s] \quad f < kinkf$$

$$= - Cs(1-\lambda)^2 \log(1-kinkf/s) \qquad (a)$$

$$+ \quad C(1-\lambda)(f-kinkf) \qquad (b)$$

$$+ \quad [1/(\lambda s-kinkf) - 1/\lambda s](f-kinkf) \qquad (c)$$

$$- \quad [\log(1-kinkf/\lambda s) + kinkf/\lambda s] \qquad (d)$$

$$+ \quad (f-kinkf)^2 T/\lambda s \qquad (e)$$