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LOCAL REFLECTIONS OF CHOICE

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Abstract. Under the assumption of small violations of choice with seed S ($\mathsf{SVC}(S)$), the failure of many choice principles reflect to local properties of S, which can be a helpful characterisation for preservation proofs. We demonstrate the reflections of DC, AC_λ, PP, and other important forms of choice. As a consequence, we show that if S is infinite then S can be partitioned into ω many non-empty subsets.

1. Introduction

It is often the case that violating a consequence of choice is easier than verifying that a consequence of choice has been preserved. For example, to violate AC_ω in a symmetric extension, one need only add a countable family with no choice function. On the other hand, to ensure that AC_ω has been preserved one must check 'every' countable family. Blass's *small violations of choice* affords us an alternative approach. In any symmetric extension of a model of ZFC there is a 'seed' S such that $\mathsf{SVC}(S)$ holds: For all non-empty X there is an ordinal η such that $S \times \eta$ surjects onto X (see [1, Theorem 4.3]).[1] In fact, under this assumption, many violations of choice are reflected back and witnessed locally to S. For example, to verify AC_ω, one need only check $\mathsf{AC}_\omega(S)$. Indeed, already in [1] the idea of a local reflection of choice was present, and the concept has appeared variously throughout the literature, summarised by the following. All notation will be introduced in the text as the results are proved.

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[1] Indeed $(\exists S)\mathsf{SVC}(S)$ has since been shown to be *equivalent* to "the universe is a set-generic symmetric extension of a model of ZFC", see [12].

Theorem. *Assume* $\mathsf{SVC}(S)$ *and* $\mathsf{SVC}^+(T)$.

1. (Blass, [1]) AC *is equivalent to "S can be well-ordered".*
2. (Pincus–Blass, [1]) BPI *is equivalent to "there is a fine ultrafilter on* $[S]^{<\omega}$*".*
3. (Karagila–Schilhan, [7]) KWP^*_α *is equivalent to "there is* $\eta \in \mathrm{Ord}$ *such that* $|S| \leq^* |\mathscr{P}^\alpha(\eta)|$*".*
4. (Karagila–Schilhan, [7]) KWP_α *is equivalent to "there is* $\eta \in \mathrm{Ord}$ *such that* $|T| \leq |\mathscr{P}^\alpha(\eta)|$*".*

Continuing this, we prove local equivalents of several common forms of choice, such as the principle of dependent choices and well-ordered choice.

Theorem. *Assume* $\mathsf{SVC}(S)$ *and* $\mathsf{SVC}^+(T)$.

1. (Proposition 3.1) DC_λ *is equivalent to "every λ-closed subtree of* $S^{<\lambda}$ *has a maximal node or a chain of order type λ".*
2. (Proposition 3.2) AC_λ *is equivalent to* $\mathsf{AC}_\lambda(S)$*, which is in turn equivalent to "every function* $g\colon S \to \lambda$ *splits".*
3. (Corollary 3.3) $\mathsf{AC}_{\mathsf{WO}}$ *is equivalent to* $\mathsf{AC}_{<\aleph^*(S)}(S)$.
4. (Proposition 3.6) AC_X *is equivalent to* $\mathsf{AC}_X(S)$.
5. (Proposition 3.8) *Assume that* $\mathrm{cf}(\omega_1) = \omega_1$*. Then* CUT *is equivalent to* $\mathsf{CUT}(T)$.
6. (Proposition 3.11) W_λ *is equivalent to* $\mathsf{W}_\lambda(T)$.
7. (Proposition 3.12) W^*_λ *is equivalent to* $\mathsf{W}^*_\lambda(T)$.
8. (Proposition 3.17) PP *is equivalent to* $\mathsf{PP}\restriction T \wedge \mathsf{AC}_{\mathsf{WO}}$.
9. (Proposition 3.20) $\mathsf{PP}(S) \wedge \mathsf{AC}_{\mathsf{WO}}$ *implies* $\mathsf{SVC}^+(S)$*. Hence,* PP *is equivalent to* $\mathsf{PP}(S) \wedge \mathsf{PP}\restriction S \wedge \mathsf{AC}_{\mathsf{WO}}$.

1.1. Structure of the paper. In Section 2 we go over some preliminaries for the paper. In Section 3 we describe reflections of various consequences of choice in the context of small violations of choice.

2. Preliminaries

We work in ZF. We denote the class of ordinals by Ord. Given a set of ordinals X, we use $\mathrm{ot}(X)$ to denote the order type of X. Given an ordinal α, $\mathrm{cf}(\alpha)$ is the cofinality of α (the least cardinality of a cofinal subset). For sets X, Y, $|X| \leq |Y|$ means that there is an injection $X \to Y$, $|X| \leq^* |Y|$ means that $X = \emptyset$ or there is a surjection $Y \to X$,[2] and $|X| = |Y|$ means that there is a bijection $X \to Y$. For a well-orderable set X, we use $|X|$ to mean $\min\{\alpha \in \mathrm{Ord} \mid |\alpha| = |X|\}$. By a *cardinal* we mean a well-ordered cardinal, that is $\alpha \in \mathrm{Ord}$ such that $|\alpha| = \alpha$. For a set Y

[2] Equivalently, there is a partial surjection $Y \to X$.

and a cardinal κ, we write $[Y]^\kappa$ to mean $\{A \subseteq Y \mid |A| = |\kappa|\}$ and $[Y]^{<\kappa}$ to mean $\{A \subseteq Y \mid |A| < |\kappa|\}$. Given a set X, the *Hartogs number* of X is $\aleph(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleq |X|\}$. Dually, we define the *Lindenbaum number* of X to be $\aleph^*(X) = \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleq^* |X|\}$. It is a theorem of ZF that $\aleph(X)$ and $\aleph^*(X)$ are well-defined cardinals. We denote concatenation of tuples by $\frown$, so if $f\colon \alpha \to X$ and $g\colon \beta \to X$ then $f^\frown g$ is the function $\alpha + \beta \to X$ given by

$$f^\frown g(\gamma) = \begin{cases} f(\gamma) & \gamma < \alpha \\ g(\delta) & \gamma = \alpha + \delta. \end{cases}$$

2.1. Small violations of choice. Introduced in [1], for a set S (known as the *seed*), $\mathsf{SVC}(S)$ is the statement "for all X there is an ordinal η such that $|X| \leq^* |S \times \eta|$", and SVC is the statement $(\exists S)\mathsf{SVC}(S)$. We shall also make use of the injective form, $\mathsf{SVC}^+(S)$ meaning "for all X, there is an ordinal η such that $|X| \leq |S \times \eta|$". See [10] for a more detailed overview of SVC and SVC^+.

FACT. $\mathsf{SVC}^+(S) \implies \mathsf{SVC}(S) \implies \mathsf{SVC}^+(\mathscr{P}(S))$.

2.2. Forcing and symmetric extensions. While no knowledge of forcing or symmetric extensions is required for the main results, forcing is used in the proof of Propositions 3.19 and 3.22. For a thorough introduction to forcing and symmetric extensions one can go to [5, Chapters 14 and 15], and for a more specific overview of the notation and terminology used in the proof of Propositions 3.19 and 3.22, one should see [6]. For a brief overview of the specific ideas used in this paper, continue reading this section.

A *forcing* is a partial order $\langle \mathbb{P}, \leq \rangle$ with maximal element $\mathbb{1}$. We refer to the elements of $\mathbb{P}$ as *conditions*. We force downwards, so we say that q is *stronger* than p (or *extends* p) if $q \leq p$. We say $\mathbb{P}$ has the *countable chain condition* (c.c.c.) if every antichain $A \subseteq \mathbb{P}$ is countable. Given a set X of $\mathbb{P}$-names, we denote by $X^\bullet$ the $\mathbb{P}$-name $\{\langle \mathbb{1}, \dot{x} \rangle \mid \dot{x} \in X\}$. By $\mathrm{Add}(A, B)$ we mean the forcing with conditions that are partial functions $p\colon B \times A \to 2$ such that $|\mathrm{dom}(p)| < |A|$, and $q \leq p$ when $q \supseteq p$. For all B, $\mathrm{Add}(\omega, B)$ is c.c.c. Given a generic filter $G \subseteq \mathrm{Add}(A, B)$, $c = \bigcup G$ is a function $B \times A \to 2$, which we think of as encoding B-many functions $A \to 2$, so for $b \in B$, the bth Cohen subset of A added is $\{a \in A \mid c(b, a) = 1\}$. This is formalised by the name

$$\dot{c}_b = \left\{ \langle p, \check{a} \rangle \mid a \in A,\ p \in \mathrm{Add}(A, B),\ p(b, a) = 1 \right\}.$$

In Proposition 3.19 we consider the Feferman-style model $\mathfrak{N}_{\aleph_1}$ from [11]. This is given by letting G be an L-generic filter of $\mathrm{Add}(\omega, \omega_1)$, and taking the least model $\mathfrak{N}_{\aleph_1} \vDash \mathsf{ZF}$ such that $L \cup \{\langle c_\beta \mid \beta < \alpha \rangle \mid \alpha < \omega_1\} \subseteq M$. Here

c_β is the βth Cohen real $\{n < \omega \mid \bigcup G(\beta, n) = 1\}$ as above. In [11], Truss shows that $\mathfrak{N}_{\aleph_1}$ is a model of $\mathsf{AC_{WO}}$, $V = L(w(\mathbb{R}))$, where $w(\mathbb{R})$ is well-orders of subsets of $\mathbb{R}$, and that every subset of $\mathbb{R}$ in $\mathfrak{N}_{\aleph_1}$ is either well-orderable or contains a perfect subset.

In Lemma 3.22 we consider Cohen's first model M from [2]. This is given by letting G be L-generic for $\mathrm{Add}(\omega, \omega)$ and taking the least model $M \vDash \mathsf{ZF}$ such that $L \cup \{c_n \mid n < \omega\} \subseteq M$. Here c_n is again the nth Cohen real. In [2], Cohen shows that M is of the form $L(A)$, where $A = \{c_n \mid n < \omega\}$ is a Dedekind-finite set of reals. Additional information on Cohen's first model can be found in [4, Sections 5.3 and 5.5].

2.3. Form numbers. Many of the forms of choice mentioned in this text are described and thoroughly examined for interdependence and equivalent statements in [3] (to the extent that the tools at the time allowed). In particular, the consequences of the axiom of choice found are given numerical form numbers, which many still find helpful as a cataloguing tool. We therefore would like to remark the following form numbers of some of the subjects of this paper: AC is Form 1; BPI is Form 14; CUT is Form 31; $\mathrm{cf}(\omega_1) = \omega_1$ is Form 34; $\mathsf{AC_{WO}}$ is Form 40; W_λ is Form 71(α), where $\lambda = \aleph_\alpha$; KWP_n is Form 81(n), where $n < \omega$; AC_λ is Form 86(α), where $\lambda = \aleph_\alpha$; DC_λ is Form 87(α), where $\lambda = \aleph_\alpha$; PP is Form 101; SVC is Form 191.

3. Reflections

3.1. The principle of dependent choices. A *tree* is a partially ordered set $\langle T, \leq\rangle$ with minimum element 0_T such that, for all $t \in T$, $\{s \in T \mid s \leq t\}$ is well-ordered by $\leq$. This gives rise to a notion of rank $\mathrm{rk}(t) = \sup\{\mathrm{rk}(s) + 1 \mid s <_T t\}$, of height $\mathrm{ht}(T) = \sup\{\mathrm{rk}(t) + 1 \mid t \in T\}$, and of levels $T_\alpha = \{t \in T \mid \mathrm{rk}(t) = \alpha\}$. For $x \in T_\alpha$ and $\beta \leq \alpha$, we denote by $x \upharpoonright \beta$ the unique $y \in T_\beta$ such that $y \leq x$. A *chain* is a set $C \subseteq T$ such that for all $s, t \in C$, $s \leq t$ or $t \leq s$. For an ordinal α, T is *α-closed* if every chain in T of order type less than α has an upper bound. For an infinite cardinal λ, DC_λ is the statement "every λ-closed tree has a maximal node or a chain of order type λ".

We note that DC_λ was originally defined in [8] as follows: Let X be a non-empty set and R a binary relation such that for all $\alpha < \lambda$ and all $\langle x_\beta \mid \beta < \alpha\rangle \in X^\alpha$ there is $y \in X$ such that $\langle x_\beta \mid \beta < \alpha\rangle \, R \, y$. Then there is $f \colon \lambda \to X$ such that, for all $\alpha < \lambda$, $f \upharpoonright \alpha \; R \; f(\alpha)$. This alternative definition is equivalent to the one that we are working with [13, Theorem 1].

Given a set X and a limit ordinal α, we endow the set $X^{<\alpha} = \bigcup\{X^\beta \mid \beta < \alpha\}$ with a tree structure given by end-extension: $f \leq g$ if and only if $f \subseteq g$. A *subtree* of $X^{<\alpha}$ is non-empty $T \subseteq X^{<\alpha}$ such that whenever $f \leq g$ and $g \in T$, $f \in T$. In this case, $f \upharpoonright \beta$ meaning "the function f restricted to domain β" is the same as $f \upharpoonright \beta$ meaning "the unique g of rank β such that $g \leq f$".

Proposition 3.1. *Assume* $\mathsf{SVC}(S)$. *For each infinite cardinal* λ, *the following are equivalent*:

1. DC_λ;
2. *every* λ*-closed subtree of* $S^{<\lambda}$ *has a maximal node or a chain of order type* λ.

Proof. Certainly if DC_λ holds then DC_λ holds for subtrees of $S^{<\lambda}$, so assume instead that every λ-closed subtree of $S^{<\lambda}$ has a maximal node or a chain of order type λ. Let T be a λ-closed tree, and let $\eta \in \mathrm{Ord}$ and $f\colon S \times \eta \to T$ be a surjection. We shall define a function $\iota\colon S^{<\lambda} \to T \cup \{\bot\}$, defining $\iota(x)$ by induction on $\mathrm{dom}(x)$, as follows: For $x \in S^\alpha$,

1. let $\iota(x) = \bot$ if there is $\beta < \alpha$ such that $\iota(x \upharpoonright \beta) = \bot$; otherwise
2. if $x = y^\frown\langle s\rangle$, then let $\iota(x) = f(s,\gamma)$, where γ is least such that $f(s,\gamma)$ is an immediate successor of $\iota(y)$, or $\iota(x) = \bot$ if no such γ exists; and
3. if x has limit rank, then let $\iota(x) = \sup\{\iota(x \upharpoonright \beta) \mid \beta < \alpha\}$.

Note that if $\iota(x) \neq \bot$ then $\iota(x) \in T_\alpha$, and that for all $\beta < \alpha$, $\iota(x \upharpoonright \beta) = \iota(x) \upharpoonright \beta$.

Let $A = \{x \in S^{<\lambda} \mid \iota(x) \neq \bot\}$.

Claim 3.1.1. *A is a* λ*-closed subtree of* $S^{<\lambda}$.

Proof. If $x \in A$ and $y < x$, then since $\iota(x) \neq \bot$, $\iota(y) \neq \bot$. Furthermore, $\iota(\emptyset) = 0_T$, so $A \neq \emptyset$ and is indeed a subtree. If $C \subseteq A$ is a chain of length less than λ, then let $b = \bigcup C \in S^{\leq\lambda}$. If $b \in S^\lambda$ then $\{\iota(b \upharpoonright \alpha) \mid \alpha < \lambda\}$ is a chain of order type λ in T. So assume otherwise, that $b \in S^{<\lambda}$. If b has limit rank α, say, then $\iota(b) = \sup\{\iota(b \upharpoonright \beta) \mid \beta < \alpha\} \neq \bot$, since $\iota(b \upharpoonright \beta) \neq \bot$ for all $\beta < \alpha$. Thus, $b \in A$. If instead b has successor rank, then $b \in C$, so certainly $b \in A$ as required. $\square$

If $x \in A$ is a maximal node, then $\iota(x)$ is maximal in T: Otherwise, $\iota(x)$ has an immediate successor, say t, and $t = f(s,\gamma)$ some γ. Then $\iota(x^\frown\langle s\rangle) \neq \bot$, and so x is not maximal in A, contradicting our assumption. Finally, if A has no maximal nodes then there is a chain $C \subseteq A$ of order type λ. In this case, ι"C is a chain in T of order type λ as required. $\square$

3.2. Well-ordered choice. For a set X, AC_X is the statement "if $\emptyset \neq Y$, $\emptyset \notin Y$, and $|Y| \leq |X|$, then $\prod Y \neq \emptyset$", where $\prod Y = \{f\colon Y \to \bigcup Y \mid (\forall y \in Y) f(y) \in y\}$ is the set of choice functions. $\mathsf{AC}_{\mathsf{WO}}$ means $(\forall \alpha \in \mathrm{Ord})\mathsf{AC}_\alpha$. For $\alpha \in \mathrm{Ord}$, $\mathsf{AC}_{<\alpha}$ means $(\forall \beta < \alpha)\mathsf{AC}_\beta$. $\mathsf{AC}_X(A)$ is AC_X for families of subsets of A: If $Y \subseteq \mathscr{P}(A) \setminus \{\emptyset\}$ is non-empty and $|Y| \leq |X|$ then $\prod Y \neq \emptyset$. $\mathsf{AC}_{<\alpha}(A)$ means $(\forall \beta < \alpha)\mathsf{AC}_\beta(A)$.

Given a function $g\colon X \to Y$, we say that g *splits* if there is a partial function $h\colon Y \to X$ such that $\mathrm{dom}(h) = g$"X and $gh(y) = y$ for all $y \in g$"X.

We say that a set X is *Dedekind-finite* if $|\omega| \not\leq |X|$. Otherwise, it is *Dedekind-infinite*.

PROPOSITION 3.2. *Assume* $\mathsf{SVC}(S)$. *For each infinite cardinal* λ, *the following are equivalent*:

1. AC_λ;
2. $\mathsf{AC}_\lambda(S)$;
3. *every function* $g\colon S \to \lambda$ *splits.*

PROOF. Certainly AC_λ implies $\mathsf{AC}_\lambda(S)$. Assuming $\mathsf{AC}_\lambda(S)$, if $g\colon S \to \lambda$ then we define

$$Y = \left\{ g^{-1}(\{\alpha\}) \mid \alpha \in g``S \right\}.$$

Then if $c \in \prod Y$, $h\colon g``S \to S$ given by $h(\alpha) = c(g^{-1}(\alpha))$ splits g.

Finally, assume that every $g\colon S \to \lambda$ splits and let $X = \{X_\alpha \mid \alpha < \lambda\}$ be a collection of non-empty sets. Let $f\colon S \times \eta \to \bigcup X$ be a surjection for some $\eta \in \mathrm{Ord}$. For $\alpha < \lambda$, let $\beta_\alpha = \min\{\beta < \eta \mid (f``S \times \{\beta\}) \cap X_\alpha \neq \emptyset\}$, and let $S_\alpha = \{s \in S \mid f(s, \beta_\alpha) \in X_\alpha\}$. Let $g(s) = \min\{\alpha < \lambda \mid s \in S_\alpha\}$ for all $s \in S$. If $h\colon \lambda \to S$ is a partial function splitting g, then setting $\gamma_\alpha = \min\{\gamma < \lambda \mid h(\gamma) \in S_\alpha\}$ is well-defined, and $C(\alpha) = f(h(\gamma_\alpha), \beta_\alpha)$ is a choice function for X. □

The following corollary was also proved independently by Elliot Glazer (private communication).

COROLLARY 3.3. *Assume* $\mathsf{SVC}(S)$. *The following are equivalent*:

1. $\mathsf{AC}_{\mathsf{WO}}$;
2. $\mathsf{AC}_{<\aleph^*(S)}(S)$.

PROOF. Certainly $\mathsf{AC}_{\mathsf{WO}}$ implies $\mathsf{AC}_{<\aleph^*(S)}(S)$, so assume $\mathsf{AC}_{<\aleph^*(S)}(S)$. Let $g\colon S \to \lambda$, $A = g``S$, and $\alpha = \mathrm{ot}(A) < \aleph^*(S)$. Taking $\iota\colon A \to \alpha$ to be the unique isomorphism, we have $\iota \circ g\colon S \to \alpha$. By $\mathsf{AC}_{<\aleph^*(S)}(S)$, $\iota \circ g$ is split, say by $f\colon \alpha \to S$. Then $f \circ \iota^{-1}$ splits g. Since g was arbitrary, Proposition 3.2 gives us AC_λ. Since λ was arbitrary, we obtain $\mathsf{AC}_{\mathsf{WO}}$ as required. □

COROLLARY 3.4. *Assume* $\mathsf{SVC}(S)$. *Then* $\aleph^*(S) \neq \aleph_0$.

PROOF. If $\aleph^*(S) = \aleph_0$ then there are no surjections $S \to \omega$, so every function $S \to \omega$ has finite image and hence splits. Therefore, AC_ω holds. However, AC_ω implies that for all X, $\aleph^*(X) \neq \aleph_0$. □

We also have the following more direct (albeit longer) proof of this corollary.

ALTERNATIVE PROOF. Assume $\mathsf{SVC}(S)$, where S is infinite (the case of S finite immediately gives $\aleph^*(S) \neq \aleph_0$), and let $f\colon S \times \eta \to S^{\underline{\leq \omega}}$ be a surjection for minimal η, where $S^{\underline{n}}$ is the set of injections $n \to S$ and $S^{\underline{<\omega}} = \bigcup_{n<\omega} S^{\underline{n}}$. For $s \in S$, let

$$X_s = \left\{ \alpha < \eta \mid \langle s, \alpha \rangle \in \mathrm{dom}(f) \wedge (\forall \beta < \alpha) f(s, \beta) \neq f(s, \alpha) \right\}$$

and $\eta_s = \mathrm{ot}(X_s)$. If $\{\eta_s \mid s \in S\}$ is infinite then $|\omega| \leq^* |S|$ as required. Otherwise, $\{\eta_s \mid s \in S\}$ is finite. We consider two cases:

Case 1: There is $s \in S$ such that $\eta_s \geq \omega$. Then we can obtain an injection $g\colon \eta_s \to S^{\underline{<\omega}}$ by setting $g(\alpha)$ to be the αth element of X_s. Hence S is Dedekind-infinite, and in particular $|\omega| \leq^* |S|$.

Case 2: Otherwise. Then η_s is finite for all s, and hence η is finite. We have $\aleph^*(S \times \eta) \geq \aleph^*(S^{\underline{<\omega}}) \geq \aleph_1$, but by the additivity of Lindenbaum numbers,[3] we have $\aleph^*(S \times \eta) = \aleph^*(S)$, and hence $\aleph^*(S) \geq \aleph_1$. □

The idea behind Corollary 3.4 extends to certain other cardinals κ, though we additionally have to assume $\mathsf{AC}_{<\kappa}$, as (unlike $\mathsf{AC}_{<\omega}$) it is not automatic.

PROPOSITION 3.5. *Let κ be a limit cardinal or singular. Assume* $\mathsf{SVC}(S)$ *and* $\mathsf{AC}_{<\kappa}$. *Then* $\aleph^*(S) \neq \kappa$.

PROOF. If $\aleph^*(S) = \kappa$ then, by Corollary 3.3, $\mathsf{AC}_{\mathsf{WO}}$ holds. However, by [10, Theorem 3.4], $\mathsf{AC}_{\mathsf{WO}}$ is equivalent to "for all X, $\aleph^*(X)$ is a regular successor", contradicting that $\aleph^*(S) = \kappa$ is singular or a limit. □

In fact, the method of Proposition 3.2 applies more generally.

PROPOSITION 3.6. *Assume* $\mathsf{SVC}(S)$. *Then for all X, the following are equivalent*:

1. AC_X;
2. $\mathsf{AC}_X(S)$.

PROOF. Certainly AC_X implies $\mathsf{AC}_X(S)$. So assume $\mathsf{AC}_X(S)$ and let $p\colon Y \to X$ be an injection. Let $f\colon S \times \eta \to \bigcup Y$ be a surjection. For $y \in Y$, let

$$\beta_y = \min\left\{ \beta < \eta \mid (\exists s \in S) f(s, \beta) = y \right\},$$
$$S_y = \left\{ \langle s, \beta_y \rangle \mid s \in S, f(s, \beta_y) = y \right\}.$$

By $\mathsf{AC}_X(S)$, we have $c \in \prod\{S_y \mid y \in Y\}$ (noting that $S_y \mapsto p(y)$ is an injection), giving $f \circ c \in \prod Y$. □

QUESTION 3.7. Does AC_X imply "for all Y, if $\emptyset \notin Y$ and $|Y| \leq^* |X|$ then $\prod Y \neq \emptyset$"?

[3] That is, if A is infinite then for all B, $\aleph^*(A \cup B) = \aleph^*(A) + \aleph^*(B)$. In particular, if $n < \omega$ and A is infinite then $\aleph^*(n \times A) = \aleph^*(A)$.

3.3. The countable union theorem. For a set X, we write $\mathsf{CUT}(X)$ to mean "a countable union of countable subsets of X is countable", and CUT to mean the countable union theorem $(\forall X)\mathsf{CUT}(X)$.

Proposition 3.8. *Assume* $\mathsf{SVC}^+(S)$ *and* $\mathrm{cf}(\omega_1) = \omega_1$. *The following are equivalent*:

1. CUT;
2. $\mathsf{CUT}(S)$.

Proof. Certainly CUT implies $\mathsf{CUT}(S)$, so assume $\mathsf{CUT}(S)$. Let $\{A_n \mid n < \omega\}$ be a countable family of countable sets, and let $A = \bigcup_{n<\omega} A_n$, assuming without loss of generality that $A \subseteq S \times \eta$ for minimal η.

For $n < \omega$, let $z_n = \{\alpha < \eta \mid (\exists s \in S)\langle s, \alpha\rangle \in A_n\}$, so $|z_n| \leq^* |A_n| \leq |\omega|$, and so $|z_n| \leq |\omega|$. By minimality of η and $\mathrm{cf}(\omega_1) = \omega_1$, we have $\eta = \mathrm{ot}(\bigcup_{n<\omega} z_n) < \omega_1$. For $n < \omega$, let $B_n = \left\{ s \in S \mid (\exists \alpha < \eta)\langle s, \alpha\rangle \in A_n \right\}$, so $B_n \subseteq S$ is countable for all n. By $\mathsf{CUT}(S)$, $B = \bigcup_{n<\omega} B_n$ is countable, and hence $|A| \leq |B \times \eta| \leq \aleph_0$ as required. □

Question 3.9. Can Proposition 3.8 be improved to "CUT is equivalent to $\mathsf{CUT}(S)$" without assuming that ω_1 is regular? Since the singularity of ω_1 is already a violation of CUT, this is equivalent to "does $\mathsf{SVC}^+(S)$ and $\mathrm{cf}(\omega_1) = \omega$ imply $\neg\mathsf{CUT}(S)$?".

3.4. The axiom of choice. The following was remarked by Blass in [1].

Proposition 3.10 (Blass). *Assume* $\mathsf{SVC}(S)$. *The following are equivalent*:

1. AC;
2. S *can be well-ordered.*

Proof. Certainly AC implies that S can be well-ordered. On the other hand, if S can be well-ordered and $|X| \leq^* |S \times \eta|$ then $|X| \leq |S \times \eta|$, so X can be well-ordered. □

3.5. Comparability. W_X is the statement "for all Y, $|X| \leq |Y|$ or $|Y| \leq |X|$" and W^*_X is the statement "for all Y, $|X| \leq^* |Y|$ or $|Y| \leq^* |X|$". We write $\mathsf{W}^{(*)}_X(B)$ to mean "for all $A \subseteq B$, $|X| \leq^{(*)} |A|$ or $|A| \leq^{(*)} |X|$". Note that "every infinite set is Dedekind-infinite" is equivalent to $\mathsf{W}_{\aleph_0}$.

Proposition 3.11. *Assume* $\mathsf{SVC}^+(S)$. *For each infinite cardinal* λ, *the following are equivalent*:

1. W_λ;
2. $\mathsf{W}_\lambda(S)$.

Proof. Certainly W_λ implies $\mathsf{W}_\lambda(S)$, so assume $\mathsf{W}_\lambda(S)$. Let $X \subseteq S \times \eta$, and let $A = \{s \in S \mid (\exists \alpha < \eta)\langle s, \alpha\rangle \in X\}$. Note that $|A| \leq |X|$, since

$s \mapsto \langle s, \alpha_s \rangle$ is an injection, where α_s is least such that $\langle s, \alpha_s \rangle \in X$. If $|A| \leq |\lambda|$ then $|X| \leq |A \times \eta|$ is well-orderable, so certainly $|\lambda| \leq |X|$ or $|X| \leq |\lambda|$. On the other hand, if $|A| \not\leq |\lambda|$ then $|\lambda| \leq |A| \leq |X|$ as required. □

Replacing $\leq$ by $\leq^*$ in the proof of Proposition 3.11, we obtain Proposition 3.12.

PROPOSITION 3.12. *Assume* $\mathsf{SVC}^+(S)$. *For each infinite cardinal* λ, *the following are equivalent*:

1. W^*_λ;
2. $\mathsf{W}^*_\lambda(S)$.

QUESTION 3.13. As a consequence of Propositions 3.11, 3.12, assuming $\mathsf{SVC}(S)$ (and hence $\mathsf{SVC}^+(\mathscr{P}(S))$), $\mathsf{W}_\lambda(\mathscr{P}(S))$ implies W_λ, and $\mathsf{W}^*_\lambda(\mathscr{P}(S))$ implies W^*_λ. Under the assumption of $\mathsf{SVC}(S)$, can we obtain a 'better' set X such that $\mathsf{W}_\lambda(X)$ implies W_λ? What about the W^*_λ case?

3.6. Boolean prime ideal theorem. The Boolean prime ideal theorem BPI is the statement "every Boolean algebra has a prime ideal", though it has many equivalent forms (see [3, Form 14]). In [1], Blass presents the following local reflection of BPI under the assumption of SVC, attributing the idea behind the proof to Pincus.

PROPOSITION 3.14 (Pincus–Blass, [1]). *Assume* $\mathsf{SVC}(S)$. *The following are equivalent*:

1. BPI;
2. *There is a fine ultrafilter on* $[S]^{<\omega}$. *That is, an ultrafilter* $\mathcal{U}$ *on* $[S]^{<\omega}$ *such that, for all* $s \in S$, $\{a \in [S]^{<\omega} \mid s \in a\} \in \mathcal{U}$.

3.7. Kinna–Wagner principles. For a set X, we define the iterated power set function $\mathscr{P}^\alpha(X)$ by $\mathscr{P}^\alpha(X) = \bigcup_{\beta<\alpha} \mathscr{P}^\beta(X)$ when α is a limit ordinal, and in the successor case $\mathscr{P}^{\alpha+1}(X) = \mathscr{P}(\mathscr{P}^\alpha(X))$. We also extend this notation to Ord, so $\mathscr{P}(\mathrm{Ord})$ is the class of all *sets* of ordinals, $\mathscr{P}^2(\mathrm{Ord})$ is the class of all sets of sets of ordinals, et cetera. For an ordinal α, KWP_α means "for all X there is an ordinal η such that $|X| \leq |\mathscr{P}^\alpha(\eta)|$", and KWP^*_α means "for all X, there is an ordinal η such that $|X| \leq^* |\mathscr{P}^\alpha(\eta)|$". The following observations, from [7], are consequences of the fact that, for all α, there is a definable surjection from $\mathscr{P}^\alpha(\mathrm{Ord})$ onto $\mathscr{P}^\alpha(\mathrm{Ord}) \times \mathrm{Ord}$, and that $\mathrm{Ord} \subseteq \mathscr{P}^\alpha(\mathrm{Ord})$.

PROPOSITION 3.15 (Karagila–Schilhan, [7]). *Assume* $\mathsf{SVC}(S)$. *The following are equivalent*:

1. KWP^*_α;
2. *There is* $\eta \in \mathrm{Ord}$ *such that* $|S| \leq^* |\mathscr{P}^\alpha(\eta)|$.

PROPOSITION 3.16 (Karagila–Schilhan, [7]). *Assume* $\mathsf{SVC}^+(S)$. *The following are equivalent*:

1. KWP_α;
2. *There is* $\eta \in \mathrm{Ord}$ *such that* $|S| \leq |\mathscr{P}^\alpha(\eta)|$.

REMARK. Given that KWP_0 and KWP_0^* are both equivalent to AC, Propositions 3.15, 3.16 give new context to Proposition 3.10.

3.8. The partition principle. The partition principle PP says "for all X and Y, $|X| \leq |Y|$ if and only if $|X| \leq^* |Y|$". Note that the forward implication always holds. By $\mathsf{PP} \restriction X$ we mean the partition principle for subsets of X: If $A, B \subseteq X$ and $|A| \leq^* |B|$ then $|A| \leq |B|$. We instead write $\mathsf{PP}(X)$ to mean "for all A, if $|A| \leq^* |X|$ then $|A| \leq |X|$".

PROPOSITION 3.17. *Assume* $\mathsf{SVC}^+(S)$. *The following are equivalent*:

1. PP;
2. $\mathsf{PP} \restriction S$ *and* $\mathsf{AC}_{\mathsf{WO}}$.

PROOF. Certainly PP implies $\mathsf{PP} \restriction S$, and PP implies "for all X, $\aleph(X) = \aleph^*(X)$", which is equivalent to $\mathsf{AC}_{\mathsf{WO}}$.[4] So instead assume $\mathsf{PP} \restriction S \wedge \mathsf{AC}_{\mathsf{WO}}$. Let $A, B \subseteq S \times \eta$ be such that $|A| \leq^* |B|$, witnessed by $f \colon B \to A$. We treat f as a partial surjection $f \colon S \times \eta \to A$. For $\langle t, \alpha \rangle \in A$, let

$$\varepsilon_{t,\alpha} = \min\left\{ \varepsilon < \eta \mid (\exists s \in S) f(s, \varepsilon) = \langle t, \alpha \rangle \right\}.$$

Let $B^{\langle t,\alpha \rangle} = \{s \in S \mid f(s, \varepsilon_{t,\alpha}) = \langle t, \alpha \rangle\}$, and $B^\varepsilon = \bigcup \{B^{\langle t,\alpha\rangle} \mid \varepsilon_{t,\alpha} = \varepsilon\}$. Let $E = \left\{ \varepsilon < \eta \mid (\exists \langle t, \alpha \rangle \in A) \varepsilon_{t,\alpha} = \varepsilon \right\} = \{\varepsilon < \eta \mid B^\varepsilon \neq \emptyset\}$. For each $\varepsilon \in E$, let $A^\varepsilon = \{\langle t, \alpha \rangle \mid \varepsilon_{t,\alpha} = \varepsilon\}$. Then $|A^\varepsilon| \leq^* |B^\varepsilon|$, witnessed by $s \mapsto f(s, \varepsilon)$. Hence, by assumption, $|A^\varepsilon| \leq |B^\varepsilon|$ and $I^\varepsilon = \{\text{injections } A^\varepsilon \to B^\varepsilon\} \neq \emptyset$. By $\mathsf{AC}_{\mathsf{WO}}$, let $c \colon E \to \bigcup \{I^\varepsilon \mid \varepsilon \in E\}$ be a choice function. Then

$$g\colon \ A \to S \times \eta, \quad \langle t, \alpha \rangle \mapsto \langle c(\varepsilon_{t,\alpha})(t, \alpha), \varepsilon_{t,\alpha} \rangle$$

is an injection. Furthermore, for each $\langle t, \alpha \rangle \in A$, $f(g(t, \alpha)) \in A^{\varepsilon_{t,\alpha}}$. In particular, $f(g(t, \alpha))$ is defined and so g is in fact an injection $A \to B$ as required. □

QUESTION 3.18. Can Proposition 3.17 be improved to "PP is equivalent to $\mathsf{PP} \restriction S$"? Equivalently, is $\mathsf{AC}_{\mathsf{WO}}$ a consequence of $\mathsf{SVC}^+(S) \wedge \mathsf{PP} \restriction S$?

While we do not know if $\mathsf{AC}_{\mathsf{WO}}$ is unnecessary in Proposition 3.17, we cannot weaken the requirement of $\mathsf{SVC}^+(S)$ to $\mathsf{SVC}(S)$, as Proposition 3.19 demonstrates.

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[4] See the first lemma of [9, Theorem 7], proof of which is attributed to Pincus by the author. In fact, the statement of the lemma in [9] is "PP implies $\mathsf{AC}_{\mathsf{WO}}$", but the proof only uses the assumption $(\forall X)\aleph(X) = \aleph^*(X)$, and the converse direction is straightforward. A proof can also be found in [10, Theorem 3.1].

PROPOSITION 3.19. *Let M be the Feferman-style model $\mathfrak{N}_{\aleph_1}$ from* [11]. *That is, for $G \subseteq \mathrm{Add}(\omega, \omega_1)$ an L-generic filter, we set*

$$M = L\big(\{ \langle c_\beta \mid \beta < \alpha \rangle \mid \alpha < \omega_1 \}\big),$$

*where c_β is the β*th *Cohen real introduced by G. Then*

$$M \vDash \mathsf{AC}_{\mathsf{WO}} \wedge \mathsf{PP} \restriction \mathbb{R} \wedge \mathsf{SVC}(\mathbb{R}) \wedge \neg\mathsf{PP}.$$

PROOF. Firstly, by [11, Lemma 2.4], $M \vDash \mathsf{AC}_{\mathsf{WO}}$.

By [11, Theorem 3.2], every set of reals in M can either be well-ordered or contains a perfect subset. If $A, B \subseteq \mathbb{R}$ and $|A| \leq^* |B|$ then: If B can be well-ordered, $|A| \leq |B|$; and if B contains a perfect subset then $|A| \leq |\mathbb{R}| \leq |B|$. Hence $\mathsf{PP} \restriction \mathbb{R}$ holds.

By [11, Lemma 2.3], $M \vDash V = L(w(\mathbb{R}))$, where $w(\mathbb{R})$ is the set of well-orders of subsets of $\mathbb{R}$. We aim to show that $w(\mathbb{R}) \subseteq L(\mathbb{R})$ and so $M \vDash V = L(\mathbb{R})$. In particular, this will show that $M \vDash \mathsf{SVC}(\mathbb{R})$.[5] As a consequence of [11, Theorem 3.1], a set X of reals in M is well-orderable if and only if $X \subseteq L[\langle c_\beta \mid \beta < \alpha \rangle]$ for some $\alpha < \omega_1$. Hence, any well-ordered sequence $f\colon \gamma \to \mathbb{R}$ in M is in fact an element of $L[\langle c_\beta \mid \beta < \alpha \rangle]$ for some α. Since $\alpha < \omega_1$, we may encode the entire sequence $\langle c_\beta \mid \beta < \alpha \rangle$ as a single real c, and so $f \in L[c]$ for some $c \in \mathbb{R}$. Hence, $w(\mathbb{R}) \subseteq L(\mathbb{R}) \subseteq L(w(\mathbb{R}))$, and so $M \vDash V = L(\mathbb{R})$.

It remains to show that $M \vDash \neg\mathsf{PP}$, which we shall do this by showing that there is no injection $[\mathbb{R}]^\omega \to \mathbb{R}$ (noting that $\mathsf{ZF} \vdash |[\mathbb{R}]^\omega| \leq^* |\mathbb{R}|$).

Suppose that $F\colon [\mathbb{R}]^\omega \to \mathbb{R}$ is such an injection, and assume that it is L-definable. Let $\mathbb{P} = \mathrm{Add}(\omega, \omega_1)$ and, for $p \in \mathbb{P}$, we define the support of p, $\mathrm{supp}(p)$, to be $\{\beta < \omega_1 \mid (\exists n < \omega)\langle \beta, n \rangle \in \mathrm{dom}(q)\} \in [\omega_1]^{<\omega}$. Since F is definable in L, F has a $\mathbb{P}$-name $\dot{F}$ such that for all $\sigma \in \mathrm{Aut}(\mathbb{P})$ (where $\mathrm{Aut}(\mathbb{P})$ is the automorphism group of $\mathbb{P}$), $\sigma\dot{F} = \dot{F}$.[6] For $\beta < \omega_1$, we define

$$\dot{c}_\beta = \{ \langle p, \check{n} \rangle \mid p \in \mathbb{P}, n < \omega, p(\beta, n) = 1 \},$$

so $\dot{c}_\beta$ is a name for c_β. Given a permutation π of ω_1, define $\hat{\pi} \in \mathrm{Aut}(\mathbb{P})$ by $\hat{\pi}p(\pi(\alpha), n) = p(\alpha, n)$ for all $p \in \mathbb{P}$ and $\langle \alpha, n \rangle \in \omega_1 \times \omega$. Note that for such automorphisms, $\hat{\pi}\dot{c}_\beta = \dot{c}_{\pi(\beta)}$. Let $\dot{C} = \{\dot{c}_n \mid n < \omega\}^\bullet$.[7] Then $C = \dot{C}^G \in [\mathbb{R}]^\omega \cap M$. Let $p_0 \in \mathbb{P}$ be such that $p_0 \Vdash$ "$\dot{F}$ is an injection $[\dot{\mathbb{R}}]^\omega \to \dot{\mathbb{R}}$".

Suppose that for some $\mathbb{P}$-name $\dot{x}$ and some $p \leq p_0$, $p \Vdash \dot{F}(\dot{C}) = \dot{x}$. Suppose also that for some $q \leq p$ and some $n < \omega$, $q \Vdash \check{n} \in \dot{x}$.

[5] If A is transitive and $M = V(A)$, where $V \vDash \mathsf{ZFC}$ then $M \vDash \mathsf{SVC}([A]^{<\omega})$ (see [1]). However, $\mathsf{ZF} \vdash |[\mathbb{R}]^{<\omega}| = |\mathbb{R}|$.

[6] In fact $\dot{F} = \sigma\dot{F}$ for all automorphisms σ of the Boolean completion of $\mathbb{P}$.

[7] That is, $\dot{C} = \{\langle \mathbb{1}, \dot{c}_n \rangle \mid n < \omega\}$.

Claim 3.19.1. $q\upharpoonright \operatorname{supp}(p)\times\omega \Vdash \check{n}\in\dot{x}$.

Proof. We shall show that for all $r\le q\upharpoonright \operatorname{supp}(p)\times\omega$, $r\nVdash \check{n}\notin\dot{x}$. Let $r\le q\upharpoonright \operatorname{supp}(p)\times\omega$ be arbitrary. Then there is a permutation π of ω_1 such that $\pi``\omega=\omega$, $\pi$ fixes $\operatorname{supp}(p)$ pointwise, and $\operatorname{supp}(r)\cap\pi``\operatorname{supp}(q)\subseteq\operatorname{supp}(p)$ (noting that $\operatorname{supp}(p)$, $\operatorname{supp}(q)$, and $\operatorname{supp}(r)$ are all finite). Then $\hat{\pi}p=p$, $\hat{\pi}\dot{C}=\dot{C}$, $\hat{\pi}\dot{F}=\dot{F}$, and so $\hat{\pi}q\Vdash\dot{F}(\dot{C})=\hat{\pi}\dot{x}$. However, since $\hat{\pi}q\le\hat{\pi}p=p$, $\hat{\pi}q\Vdash\dot{F}(\dot{C})=\dot{x}$ as well, and thus $\hat{\pi}q\Vdash\hat{\pi}\dot{x}=\dot{x}$. Furthermore, $\hat{\pi}q$ and r have a common extension (namely $\hat{\pi}q\cup r$), and so $\hat{\pi}q\cup r\Vdash\check{n}\in\dot{x}$. Therefore, $r\nVdash\check{n}\notin\dot{x}$. Since no $r\le q\upharpoonright \operatorname{supp}(p)\times\omega$ forces $\check{n}\notin\dot{x}$, we must have that $q\upharpoonright \operatorname{supp}(p)\times\omega\Vdash\check{n}\in\dot{x}$. □

By the claim, we may assume that $\dot{x}$ is an $\mathrm{Add}(\omega,\operatorname{supp}(p))$-name.

Let $\beta,\beta'\in\omega_1\setminus\operatorname{supp}(p)$ be such that $\beta<\omega$ and $\beta'\ge\omega$, and let π be the transposition $(\beta\ \beta')$. Then $\hat{\pi}p=p$ and $\hat{\pi}\dot{x}=\dot{x}$, so $p\Vdash\dot{F}(\hat{\pi}\dot{C})=\dot{x}$. However, $p\Vdash\dot{c}_{\beta'}\in\hat{\pi}\dot{C}\setminus\dot{C}$, and so $p\Vdash\hat{\pi}\dot{C}\neq\dot{C}$, contradicting that $p\Vdash$ "$\dot{F}$ is an injection".

In the case that F requires a real parameter, say c, we note that by the c.c.c. of $\mathrm{Add}(\omega,\omega_1)$, c has an $\mathrm{Add}(\omega,\alpha)$-name for some $\alpha<\omega_1$. By working in $L[G\upharpoonright\alpha]$ (where $G\upharpoonright\alpha=\{p\in G\mid\operatorname{dom}(p)\subseteq\alpha\times\omega\}$) rather than L, and noting that the quotient of $\mathrm{Add}(\omega,\omega_1)$ by $\mathrm{Add}(\omega,\alpha)$ (the $\mathrm{Add}(\omega,\alpha)$-name for the 'rest of the forcing', so $\{\langle p\upharpoonright\alpha\times\omega,p\rangle\mid p\in\mathrm{Add}(\omega,\omega_1)\}$) is isomorphic to $\mathrm{Add}(\omega,\omega_1)$, the same result follows (using $\{\dot{c}_\beta\mid\alpha\le\beta<\alpha+\omega\}^\bullet$ instead of C). □

Even though we cannot improve Proposition 3.17 to $\mathsf{SVC}(S)$ as written, we can if we additionally assume the stronger axiom $\mathsf{PP}(S)$, rather than merely $\mathsf{PP}\upharpoonright S$.

Proposition 3.20. $\mathsf{SVC}(S)\wedge\mathsf{PP}(S)\wedge\mathsf{AC}_{\mathsf{WO}}$ *implies* $\mathsf{SVC}^+(S)$.

Proof. Let A be a set. By $\mathsf{SVC}(S)$ there is a surjection $h\colon S\times\eta\to A$ for some ordinal η. For $a\in A$, let $\alpha_a=\min\{\alpha<\eta\mid(\exists s\in S)h(s,\alpha)=a\}$. For $\alpha<\eta$, let $A_\alpha=\{a\in A\mid\alpha_a=\alpha\}$. Then $s\mapsto h(s,\alpha)$ is a (partial) surjection $S\to A_\alpha$ for all α, and so, by $\mathsf{PP}(S)$, $|A_\alpha|\le|S|$. Using $\mathsf{AC}_{\mathsf{WO}}$, we may simultaneously pick injections $i_\alpha\colon A_\alpha\to S$ for all $\alpha<\eta$. Then $a\mapsto\langle i_{\alpha_a}(a),\alpha_a\rangle$ is an injection $A\to S\times\eta$; indeed, if $\langle i_{\alpha_a}(a),\alpha_a\rangle=\langle i_{\alpha_b}(b),\alpha_b\rangle$ then $\alpha_a=\alpha_b$, so $i_{\alpha_a}(a)=i_{\alpha_a}(b)$, which implies $a=b$ as the i_α are injective. A was arbitrary, so we conclude $\mathsf{SVC}^+(S)$. □

Corollary 3.21. *Assume* $\mathsf{SVC}(S)$. *The following are equivalent*:
1. PP;
2. $\mathsf{PP}\upharpoonright S$, $\mathsf{PP}(S)$, *and* $\mathsf{AC}_{\mathsf{WO}}$.

Proof. Certainly PP implies each of $\mathsf{PP}\upharpoonright S$, $\mathsf{PP}(S)$, and $\mathsf{AC}_{\mathsf{WO}}$, so instead assume $\mathsf{PP}\upharpoonright S$, $\mathsf{PP}(S)$ and $\mathsf{AC}_{\mathsf{WO}}$. By Proposition 3.20, $\mathsf{SVC}^+(S)$ holds, and so by Proposition 3.17, PP holds. □

In Lemma 3.22 below, we prove that "$\mathsf{PP}\restriction S \wedge \mathsf{AC}_{\mathsf{WO}}$" cannot be omitted from Corollary 3.21.

PROPOSITION 3.22. *Let $M = L(A)$ be Cohen's first model, where $L \subseteq M \subseteq L[G]$ for L-generic $G \subseteq \operatorname{Add}(\omega, \omega)$. Then*

$$M \vDash \mathsf{SVC}^+(\mathbb{R}) \wedge \mathsf{PP}(\mathbb{R}) \wedge \neg\mathsf{PP}.$$

PROOF. For an overview of Cohen's first model and the proof of $\mathsf{SVC}^+(\mathbb{R})$, see [4, Sections 5.3 and 5.5]. Within is also a proof that there is an infinite Dedekind-finite set of reals in M, contradicting $\mathsf{AC}_{\mathsf{WO}}$ (and hence, by Proposition 3.17, PP).

The proof that $M \vDash \mathsf{PP}(\mathbb{R})$ is due to Elliot Glazer and Assaf Shani. Suppose that $f : \mathbb{R} \to X$ is a surjection and, using $\mathsf{SVC}^+(\mathbb{R})$, assume that $X \subseteq \mathbb{R} \times \eta$ for some minimal η. In $L[G]$, $2^{\aleph_0} = \aleph_1$, and (since η is minimal) f induces a surjection $\mathbb{R} \to \eta$. Therefore $\eta < \omega_2$. In M, $|\omega_1| \leq |\mathbb{R}|$, and so

$$|X| \leq |\mathbb{R} \times \eta| \leq |\mathbb{R} \times \omega_1| \leq |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|. \quad \square$$

Furthermore, the model $\mathfrak{N}_{\aleph_1}$ from Proposition 3.19 shows that we cannot omit the $\mathsf{PP}\restriction S$ requirement from Corollary 3.21.

PROPOSITION 3.23. *Let M be the Feferman-style model $\mathfrak{N}_{\aleph_1}$ from [11] (and Proposition 3.19). Then*

$$M \vDash \mathsf{AC}_{\mathsf{WO}} \wedge \mathsf{PP}(\mathscr{P}(\mathbb{R})) \wedge \mathsf{SVC}^+(\mathscr{P}(\mathbb{R})) \wedge \neg\mathsf{PP}.$$

PROOF. We already saw in Proposition 3.19 that M is a model of $\mathsf{AC}_{\mathsf{WO}} \wedge \mathsf{SVC}(\mathbb{R}) \wedge \neg\mathsf{PP}$. By $\mathsf{SVC}(\mathbb{R})$, $\mathsf{SVC}^+(\mathscr{P}(\mathbb{R}))$ holds.

The proof of $\mathsf{PP}(\mathscr{P}(\mathbb{R}))$ is similar to Lemma 3.22. Suppose $|X| \leq^* |\mathscr{P}(\mathbb{R})|$, witnessed by f. We may assume that $X \subseteq \mathscr{P}(\mathbb{R}) \times \eta$ for minimal η. Since the outer model $L[G] \vDash |\mathscr{P}(\mathbb{R})| = \aleph_2$, and f induces a surjection $\mathscr{P}(\mathbb{R})^M \to \eta$, we must have that $\eta < \omega_3$. In M, $|\omega_2| \leq |\mathscr{P}(\mathbb{R})|$ and so

$$|X| \leq |\mathscr{P}(\mathbb{R}) \times \eta| \leq |\mathscr{P}(\mathbb{R}) \times \omega_2| \leq |\mathscr{P}(\mathbb{R})^2| = |\mathscr{P}(\mathbb{R}^2)| = |\mathscr{P}(\mathbb{R})|. \quad \square$$

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