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On the equivalence of Lurie's ∞ -operads and dendroidal ∞ -operads

Vladimir Hinich¹ | Ieke Moerdijk²

¹Department of Mathematics, University of Haifa, Mount Carmel, Haifa, Israel

²Department of Mathematics, University of Utrecht, Utrecht, The Netherlands

Correspondence

Ieke Moerdijk, Department of Mathematics, University of Utrecht, Budapestlaan 8, Utrecht, The Netherlands.
Email: i.moerdijk@uu.nl

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Abstract

In this paper, we prove the equivalence of two symmetric monoidal ∞ -categories of ∞ -operads, the one defined in Lurie [Higher algebra, available at the author's homepage, <http://math.ias.edu/~lurie/>, September 2017 version] and the one based on dendroidal spaces.

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1 | INTRODUCTION

1.1

In this paper, we return to the question of the comparison of various notions of ∞ -operad occurring in the literature. One such notion is the one defined by Lurie in terms of simplicial sets over the nerve of the category of finite pointed sets, see [section 2]; another is the one defined in terms of dendroidal sets, or dendroidal spaces [5]. Lurie's ∞ -category and the dendroidal one are in fact symmetric monoidal ∞ -categories where the monoidal structures resemble the Boardman–Vogt tensor product of operads. The ∞ -categories themselves are underlying Quillen model structures. The dendroidal model category has been shown in [5] to be Quillen equivalent to the model category of classical simplicial or topological operads.

A first such comparison was made in [7], where it was shown that if one restricts oneself to operads without constants, the Lurie model and the dendroidal one are (Quillen) equivalent at the level of model categories. Moreover, this equivalence respects the monoidal structure of the associated homotopy categories, which is a shadow of the much richer structure of symmetric monoidal ∞ -category.

In a long paper [1], Barwick constructs another ∞ -category based on his notion of operator category, and proves this ∞ -category to be equivalent to Lurie's version mentioned above. A next comparison was studied in [4], where the dendroidal model was shown to be equivalent to Barwick's version at the level of ∞ -categories. Combined with Barwick's equivalence, this gives a composed equivalence between Lurie's ∞ -category and the dendroidal one, now avoiding the condition on the absence of constants of [7]. However, the comparison of [4] does not address the question of equivalence as symmetric monoidal ∞ -categories.

The goal of this paper is to prove a relatively direct and explicit equivalence between two symmetric monoidal ∞ -categories. One is the ∞ -category LOp (short for "Lurie operads") underlying Lurie's model category, the other is the ∞ -category DOp (for "dendroidal operads") underlying the dendroidal model category. To give the reader a rough idea already at this stage, we remark that our proof is based on a functor from level forests to forests, denoted

$$\omega : \mathbb{F} \rightarrow \Phi,$$

see Sections 2.2.2 and 3.1 for the notation. LOp is an ∞ -category of presheaves on \mathbb{F} and DOp is one on Φ , and the equivalence is simply realized by the functors[†]

$$\lambda : \text{DOp} \rightarrow \text{LOp}, \quad \lambda(D)(A) = \text{Map}_{\text{DOp}}(\omega(A), D),$$

$$\delta : \text{LOp} \rightarrow \text{DOp}, \quad \delta(L)(F) = \text{Map}_{\text{LOp}}(i(F), L).$$

Here A, D, F , and L are objects of $\mathbb{F}, \text{DOp}, \Phi$, and LOp , respectively, and i denotes the embedding of objects of Φ as free operads in LOp ; see Section 3.1 for detailed definitions. Our main theorem can then be stated as follows:

Theorem 1.1.1. *The functors λ and δ define an equivalence of symmetric monoidal ∞ -categories*

$$\lambda : \text{DOp} \xrightarrow{\sim} \text{LOp} : \delta.$$

The other comparison proofs mentioned above are also based on the same functor from level forests to forests, but there are several important differences. First of all, our result is an equivalence of symmetric monoidal ∞ -categories, not just of ∞ -categories. To prove this sharper result, we use a colax symmetric monoidal structure on the category of copresheaves on an operad,[‡] which may be of independent interest. Second, our proof uses the category of algebras of an operad in two essential ways. We use the comparison theorem of Pavlov and Scholbach [16], which states that the ∞ -category underlying the category of simplicial algebras over a Σ -free operad P in sets is equivalent to the ∞ -category of algebras over the associated ∞ -operad $\ell(P)$ in the ∞ -category of spaces, and similarly for algebras in a symmetric monoidal model category \mathcal{C} and its underlying ∞ -category \mathcal{C}_∞ . We state this result somewhat cryptically as

$$\text{Alg}_P(\mathcal{C})_\infty = \text{Alg}_{\ell(P)}(\mathcal{C}_\infty),$$

see Section 4.2.1 for a precise formulation. (This result is analogous to an earlier result for linear operads proved in [11].) Second, we prove and use the following reconstruction theorem for

[†] As a mnemonic aid, δ stands for "dendrification," λ for "Lurification."

[‡] Or, as it appears in our paper, the category of presheaves on an anti-operad.

∞ -operads, stating that a map $\mathcal{P} \rightarrow \mathcal{Q}$ between ∞ -operads, which is essentially surjective on colors is an equivalence whenever it induces an equivalence between the associated ∞ -categories of algebras in the ∞ -category of spaces; see Theorem 4.1.1 below. (By the result of [16] just mentioned, this implies the analogous known result for Σ -cofibrant simplicial operads, see [3].)

1.2

To conclude this introduction, let us briefly sketch the contents of this paper. In Section 2, we fix some conventions about our use of ∞ -categorical language, and introduce the ∞ -categories $\mathrm{D}\mathcal{O}\mathcal{P}$ and $\mathrm{L}\mathcal{O}\mathcal{P}$ featuring in our main theorem above. In Section 3, we state and prove a weaker form of the main theorem, ignoring the symmetric monoidal structure for the moment. The proof uses a lemma that is based on the reconstruction theorem, which we postpone until Section 4 where we discuss algebras over an ∞ -operad. In the final Section 5, we address the different symmetric monoidal structures involved, and prove that they are respected by the functors δ and λ . The structure on $\mathrm{D}\mathcal{O}\mathcal{P}$ is defined in terms of shuffles of trees about which we explain some basic facts in the Appendix.

2 | PRELIMINARY DEFINITIONS

2.1 | ∞ -categorical conventions

We present here some basic notation and discuss a few standard recipes for working with ∞ -categories.

In what follows, the word “category” means ∞ -category, operad means ∞ -operad, functor means ∞ -functor, and so on. If we wish to emphasize that an ∞ -category (or an ∞ -operad) can be modeled by a *strict* inner Kan complex, we sometimes refer to it as a *conventional* category (or operad).

The most basic category is the category of spaces \mathcal{S} underlying the Quillen model category of simplicial sets. Given two categories \mathcal{C}, \mathcal{D} , there is a category of functors $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ satisfying the standard equivalence

$$\mathrm{Map}(\mathcal{X}, \mathrm{Fun}(\mathcal{C}, \mathcal{D})) = \mathrm{Map}(\mathcal{X} \times \mathcal{C}, \mathcal{D}).$$

The category of categories Cat can be realized as a full subcategory of the category of simplicial spaces $P(\Delta) = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{S})$, spanned by the simplicial presheaves $X : \Delta^{\mathrm{op}} \rightarrow \mathcal{S}$ satisfying the ∞ -categorical variant of the Segal and completeness properties, see [17]:

1. For any n the natural map

$$X_n \rightarrow X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

is an equivalence.

2. The map $\mathrm{Map}(J, X) \rightarrow \mathrm{Map}(*, X)$ induced by a map $* \rightarrow J$ is an equivalence. ($J \in P(\Delta)$ is the presheaf corresponding to the category having two objects and a unique isomorphism between them.)

This approach allows one to define the opposite of a category defined by a simplicial space $C : \Delta^{\text{op}} \rightarrow \mathcal{S}$ as the composition of C with the functor $\text{op} : \Delta \rightarrow \Delta$ reversing the ordering of a finite totally ordered set.

2.1.1

Another endofunctor of Δ , carrying $[n]$ to the join $[n]^{\text{op}} \star [n] = [2n + 1]$, gives the construction of $\text{Tw}(C)$, the category of twisted arrows in C . The canonical projection

$$\text{Tw}(C) \rightarrow C^{\text{op}} \times C$$

is a left fibration; that is, it is classified by a functor

$$\tilde{Y} : C^{\text{op}} \times C \rightarrow \mathcal{S}.$$

This functor can be rewritten as the Yoneda embedding $Y : C \rightarrow P(C) = \text{Fun}(C^{\text{op}}, \mathcal{S})$.

Dealing with ∞ -categories requires extra care when writing formulas. It is in general not allowed to define functors by describing them on objects and arrows as there is no way to describe all required compatibilities. However, some standard formulas do define functors. For instance, given a functor $f : C \rightarrow D$, one has a functor $\tilde{f} : D^{\text{op}} \times C \rightarrow \mathcal{S}$ defined by the formula $\tilde{f}(d, c) = \text{Map}_D(d, f(c))$. This formula just means that \tilde{f} is defined as the composition

$$D^{\text{op}} \times C \rightarrow D^{\text{op}} \times D \xrightarrow{\tilde{Y}} \mathcal{S}.$$

2.1.2

A map of spaces $f : X \rightarrow Y$ (e.g., modeled by Kan simplicial sets) exhibits X as a *subspace* of Y if f induces an equivalence of X with a union of a subset of connected components of Y . This notion generalizes to any category C : An arrow $f : c \rightarrow d$ is *mono*, if for any $x \in C$ the induced map of spaces $\text{Map}_C(x, c) \rightarrow \text{Map}_C(x, d)$ is an inclusion of a subspace. In the case when $C = \text{Cat}$, we get the notion of subcategory: It is defined by a subspace of objects, and a subspace of morphisms for each pair of objects.

2.1.1 | Subfunctor

The notion of subspace mentioned above allows one to construct a subfunctor of a given functor. In this context, the following elementary result is useful (see [10], 9.2.3).

Proposition. *Let $F : C \rightarrow D$ be a functor. Let, for each $x \in C$, a subobject G_x of $F(x)$ be given, so that for each $a : x \rightarrow y$ the composition $G_x \rightarrow F(x) \rightarrow F(y)$ factors through G_y . Then the collection of subobjects G_y uniquely glues into a subfunctor $G : C \rightarrow D$.*

2.1.4

A marked category is a pair $C^{\natural} = (C, C^{\circ})$ where C is a category and C° is a subcategory of C containing C^{eq} , the maximal subspace of C . The category Cat^+ of marked categories is defined as the full subcategory of $\text{Fun}([1], \text{Cat})$ spanned by the embeddings $C^{\circ} \rightarrow C$. We denote by $C^{\flat} = (C, C^{\text{eq}})$ the category C endowed with the minimal marking. The embedding $\text{Cat} \rightarrow \text{Cat}^+$ carrying C to C^{\flat} has a left adjoint called localization and denoted

$$L : \text{Cat}^+ \rightarrow \text{Cat}. \quad (1)$$

For marked categories represented by a pair of simplicial categories, the localization is represented by the Dwyer–Kan construction.

2.1.5

Many important ∞ -categories appear as the ones *underlying* model categories. One such is the standard model structure on simplicial sets modeling S . Another one is the complete Segal model for the ∞ -category of ∞ -categories Cat that has already been mentioned above.

The ∞ -category of a model category is obtained by a general localization construction as described in Section 2.1.4, which does not enjoy very nice properties. Fortunately, one can often present the ∞ -category underlying a model category as a Bousfield localization[†] of a certain ∞ -category of presheaves of spaces. For instance, Cat can be presented as a Bousfield localization of the ∞ -category $P(\Delta)$ of simplicial spaces. As explained above, this is an ∞ -categorical reformulation of the fact [17] that the model category of complete Segal spaces is obtained by a Bousfield localization (in the sense of model categories) from the Reedy model structure on bisimplicial sets. Below we will present the ∞ -categories DOp and LOp in a similar way, see Sections 2.2.3 and 2.3.5.

Note that by a result of Dugger [6], any ∞ -category underlying a combinatorial model category is in fact equivalent to such a localization of a category $P(C)$ of simplicial presheaves.

Let $\text{CS} \subset \text{Seg} \subset P(\Delta)$ denote the full subcategories of $P(\Delta)$ spanned by the complete Segal spaces and by all Segal spaces, respectively. The following easy observation will be used below.

Lemma 2.1.6. *Let $f : X \rightarrow B$ be an arrow in Seg with $B \in \text{CS}$. Then $X \in \text{CS}$ if and only if the fibers of f are in CS .*

Proof. Recall from Section 2.1 that a Segal space X is complete if and only if the map $\text{Map}(J, X) \rightarrow \text{Map}(*, X)$ induced by a map $* \rightarrow J$, is an equivalence. We have a commutative diagram of spaces

$$\begin{array}{ccc} \text{Map}(J, X) & \longrightarrow & \text{Map}(*, X) \\ \downarrow & & \downarrow \\ \text{Map}(J, B) & \xrightarrow{\sim} & \text{Map}(*, B) \end{array}$$

[†] A localization that has a fully faithful right adjoint.

and we need to show that the top horizontal arrow is an equivalence. It is obviously so if and only if the map of fibers at any $b \in \text{Map}(*, B)$ is an equivalence. These fibers identify with $\text{Map}(J, X_b)$ and $\text{Map}(*, X_b)$ where X_b denotes the fiber of f at $b \in B$. \square

2.2 | Dendroidal ∞ -operads

2.2.1

We begin by recalling the definition of the category Ω introduced in [14] and discussed in detail in [8], Section 3.2, see also the Appendix. The objects of Ω are finite trees, allowed to have “external” edges attached to just one vertex. One of these external edges is specified as the *root* of the tree, the other external edges are called leaves. The edges of a tree connected to two vertices are called *internal* or *inner* edges. The category Ω includes the object η consisting of just one edge that is at the same time the root and the leaf. The choice of a root defines an orientation of each edge in the tree, toward the root. This specifies for each vertex v an outgoing edge $\text{out}(v)$ and a set $\text{in}(v)$ of incoming edges. The cardinality of $\text{in}(v)$ is called the *valence* of the vertex v , and it is allowed to be zero. Vertices of valence zero are called *stumps*. To define the morphisms of the category Ω , we observe that each such tree T defines a (symmetric) colored operad $o(T)$. The colors of $o(T)$ are the edges of T and its operations are generated by the vertices, each vertex v defining an operation from the set $\text{in}(v)$ to $\text{out}(v)$. The morphisms $S \rightarrow T$ in Ω are now defined to be the operad maps $o(S) \rightarrow o(T)$. In particular, this makes o into a full embedding of Ω into the category $\text{Op}(\text{Set})$ of operads in sets.

Alternatively, one may define the morphisms $S \rightarrow T$ in Ω as generated by “elementary morphisms”: these are

- (a) isomorphisms $S \xrightarrow{\sim} T$;
- (b) degeneracies $S \twoheadrightarrow T$ where S is obtained from T by putting a new vertex in the middle of an edge of T ;
- (c) inner face maps $S \twoheadrightarrow T$ where S is obtained from T by contracting an inner edge of T ; and
- (d) outer face maps where S is obtained by chopping off an external vertex from T (i.e., a vertex attached to just one inner edge). In addition, if T is a corolla, that is, a tree with just one vertex, each edge of T defines an elementary morphism $\eta \rightarrow T$.

We observe now that there is a full embedding $\iota : \Delta \hookrightarrow \Omega$ of the simplex category Δ into Ω , which assigns to each object $[n]$ of Δ the linear tree $\iota[n]$ with n vertices (all of valence 1) and $n + 1$ edges. Under this identification of Δ with a subcategory of Ω , degeneracies, inner and outer faces have their usual meaning.

2.2.2

It is convenient to extend the category Ω to include disjoint unions of trees. To this end, we define the category of forests Φ similar to Ω , as the full subcategory of $\text{Op}(\text{Set})$, spanned by $o(F)$ where F is a disjoint union of trees and $o(F)$ is defined as the coproduct of the operads $o(T_i)$ where T_i are the components of F . We will denote by $o : \Phi \rightarrow \text{Op}(\text{Set})$ the full embedding described above, see diagram (28). The category Φ can be alternatively defined as the category obtained from Ω

by formally adjoining finite coproducts. (Note that our definition of Φ is different from that of [7]. First, our category Φ has an empty forest; and, second, the version of [7] has fewer arrows (independence property) than our version.)

2.2.3

The (∞ -) category of dendroidal operads DOp is defined as the full subcategory of the category $P(\Omega)$ of presheaves (of spaces) spanned by the presheaves satisfying Segal and completeness properties:

- (D1) For an inner edge b in T let T_b and T^b be the upper and the lower part of T obtained by cutting T at b . Then $X(T) \rightarrow X(T_b) \times_{X(b)} X(T^b)$ is an equivalence.
- (D2) Completeness: $i^*(X) \in P(\Delta)$ is a complete Segal space.

It will be convenient for us to realize DOp as a full subcategory in $P(\Phi)$ spanned by the presheaves satisfying the above two properties, as well as the extra (also Segal-type) property.

- (D3) The natural map $X(F) \rightarrow \prod X(T_i)$ for a forest F consisting of the trees T_i , is an equivalence. In particular, $X(\emptyset)$ is contractible.

It is a standard fact that the category DOp is equivalent to the one underlying the model category of dendroidal complete Segal spaces, or the equivalent one of dendroidal sets, see [5] or [8].

2.3 | Lurie ∞ -operads

In what follows Fin_* denotes the conventional category of finite pointed sets. We denote by I_* the finite pointed set $I \sqcup \{*\}$ and put $\langle n \rangle = \{1, \dots, n\}_*$. An arrow $f : I_* \rightarrow J_*$ is called *inert* if for any $j \in J$ the set $f^{-1}(j)$ consists of one element.

A functor $p : \mathcal{P} \rightarrow \text{Fin}_*$ is called *fibrous* if the following conditions are satisfied. In what follows, we denote by \mathcal{P}_n (or \mathcal{P}_I) the fiber of p at $\langle n \rangle$ (or at I_*).

- (Fib1) Any inert arrow $f : I_* \rightarrow J_*$ has a cocartesian lifting. As a result, a functor $f_! : \mathcal{P}_I \rightarrow \mathcal{P}_J$ is defined (uniquely up to equivalence).
- (Fib2) For the collection of standard inerts $\rho^i : \langle n \rangle \rightarrow \langle 1 \rangle$ defined by $(\rho^i)^{-1}(1) = \{i\}$, the maps

$$\mathcal{P}_n \xrightarrow{\rho_!^i} \mathcal{P}_1$$

form a product diagram.

- (Fib3) Let $f : \langle m \rangle \rightarrow \langle n \rangle$ be an arrow in Fin_* , $x \in \mathcal{P}_n$ and $r^i : x \rightarrow x_i$ the cocartesian liftings of $\rho^i : \langle n \rangle \rightarrow \langle 1 \rangle$. Then the natural map

$$\text{Map}^f(y, x) \rightarrow \prod_i \text{Map}^{\rho^i \circ f}(y, x_i)$$

is an equivalence for any $y \in \mathcal{P}_m$.

2.3.1

Cocartesian liftings of inerts in Fin_* are called inerts in \mathcal{P} . The category of Lurie operads LOp is the subcategory of $Cat_{/Fin_*}$ consisting of fibrous objects and morphisms preserving inerts, see [12, 2.3.3.28] or [9, 2.6.3]. For example, any conventional (colored) operad P in Set with the set of colors $[P]$ defines an object $\ell(P)$ of LOp , see [12, 2.1.1.7 and 2.1.1.22]. Its morphisms over a map $\alpha : I_* \rightarrow J_*$ in Fin_* are triples (c, d, p) where $c : I \rightarrow [P]$, $d : J \rightarrow [P]$, and $p = (p_j : j \in J)$ where $p_j \in P(c|_{\alpha^{-1}(j)}, d(j))$. These formulas define a functor $\ell : Op(Set) \rightarrow LOp$ identifying $Op(Set)$ with the full subcategory of LOp spanned by fibrous maps $p : C \rightarrow Fin_*$ with C a conventional category, see diagram (28).

2.3.2

The category LOp has a symmetric monoidal structure that is induced from the smash product operation on Fin_* . Following Lurie [12, 2.2.5.9], we will say that a functor $F : (Fin_*)^n \rightarrow Fin_*$ is a smash product functor if $F(\langle 1 \rangle, \dots, \langle 1 \rangle) \approx \langle 1 \rangle$ and F preserves coproducts in each argument. A smash product functor is unique up to a unique isomorphism. An operad multifunctor $(P_1, \dots, P_n) \rightarrow Q$ is defined as a commutative diagram

$$\begin{array}{ccc} P_1 \times \dots \times P_n & \xrightarrow{f} & Q \\ \downarrow & & \downarrow \\ Fin_* \times \dots \times Fin_* & \xrightarrow{F} & Fin_* \end{array}$$

where f carries n -tuples of inerts in $P_1 \times \dots \times P_n$ to inerts in Q and F is a smash product. The notion of operad multifunctor defines on LOp the structure of an operad that turns out to be a symmetric monoidal category, where the multiple tensor product is defined as the target of the universal operad multifunctor out of P_1, \dots, P_n , see [12, 2.2.5.13].

2.3.3

As explained before, the category Cat identifies with the full subcategory of $P(\Delta)$ spanned by the complete Segal objects. In particular, the (conventional) category Fin_* can be viewed as an object of $P(\Delta)$, which we still denote by Fin_* . This identifies $Cat_{/Fin_*}$ with a full subcategory of $P(\Delta)_{/Fin_*}$.

The following general fact allows one to identify the latter with $P(\Delta)_{/Fin_*}$.

Let $Y : C \rightarrow P(C)$ be the Yoneda embedding and let $F \in P(C)$. We define

$$C_{/F} = C \times_{P(C)} P(C)_{/F} \quad (2)$$

and denote by $p : C_{/F} \rightarrow P(C)_{/F}$ the natural projection. Then p is fully faithful and we denote by

$$p_! : P(C_{/F}) \rightarrow P(C)_{/F} \quad (3)$$

the extension of p to a colimit-preserving functor.

Lemma 2.3.4. $p_!$ is an equivalence.

Proof. We apply Corollary 5.1.6.11 in [13] to $p_!$. Its restriction p is fully faithful, and the image of any $\phi : Y(x) \rightarrow F$ is absolutely compact as colimits in $P(C)_{/F}$ are detected by colimits in $P(C)$. Finally, it is clear that the objects $Y(x) \rightarrow F$ generate $P(C)_{/F}$ under colimits. \square

Remark. Note that, by definition (2), $\Delta_{/Fin_*} = \Delta \times_{P(\Delta)} P(\Delta)_{/Fin_*}$. Since Fin_* and all $[n]$ are conventional categories and since the images of conventional categories form a full subcategory of $P(\Delta)$, $\Delta_{/Fin_*}$ is equivalent to the conventional category of simplices in Fin_* .

In this paper, we denote

$$\mathbb{F} = \Delta_{/Fin_*}. \quad (4)$$

Thus, $L\text{Op}$ can be identified with a (nonfull) subcategory of $P(\mathbb{F})$.

2.3.5

Here is another presentation of $L\text{Op}$, this time as a Bousfield localization. The category Fin_* has a marking defined by the collection of inert arrows. This marked category is denoted by Fin_*^{\natural} . There is a fully faithful functor $L\text{Op} \rightarrow \text{Cat}_{/Fin_*^{\natural}}^+$ carrying a fibrous $p : \mathcal{O} \rightarrow Fin_*$ to the marked category \mathcal{O}^{\natural} over Fin_*^{\natural} , with the marking on \mathcal{O} defined by the inerts. By [9, 2.6.4] (based on [12, B.0.20]), $L\text{Op}$ is the Bousfield localization with respect to the class of *operadic equivalences*. An operadic equivalence is defined as a map $f : X \rightarrow Y$ in $\text{Cat}_{/Fin_*^{\natural}}^+$ inducing an equivalence $\text{Map}(Y, \mathcal{O}) \rightarrow \text{Map}(X, \mathcal{O})$ for any fibrous \mathcal{O} .

2.3.1 | Cocartesian arrows

Let $F \in \text{Cat}$. As before, we will identify F with the corresponding complete Segal space in $P(\Delta)$. Lemma 2.3.4 defines a full embedding of $\text{Cat}_{/F}$ into $P(\Delta_{/F})$. Let $p : X \rightarrow F$ be a category over F and let $\mathcal{X} \in P(\Delta_{/F})$ be the corresponding presheaf. By definition, for $A : [n] \rightarrow F$,

$$\mathcal{X}(A) = \text{Map}([n], X) \times_{\text{Map}([n], F)} \{A\}.$$

Fix $\alpha : [1] \rightarrow F$ and let $a \in \mathcal{X}(\alpha)$. We denote by the same letters $a : x \rightarrow y$ and $\alpha : \bar{x} \rightarrow \bar{y}$ the arrows in X and in F . The following lemma is a direct reformulation of the cocartesian property of the arrow a in our language.

Lemma. *The arrow $a \in \mathcal{X}(\alpha)$ is p -cocartesian if and only if for any $\sigma : [2] \rightarrow F$ with $d_2\sigma = \alpha$, $d_0\sigma = \beta$, $d_1\sigma = \gamma$, the map*

$$\{y\} \times_{\mathcal{X}(\bar{y})} \mathcal{X}(\beta) \rightarrow \{x\} \times_{\mathcal{X}(\bar{x})} \mathcal{X}(\gamma)$$

defined as a composition

$$\{y\} \times_{\mathcal{X}(\bar{y})} \mathcal{X}(\beta) \rightarrow \{x\} \times_{\mathcal{X}(\bar{x})} \mathcal{X}(\alpha) \times_{\mathcal{X}(\bar{y})} \mathcal{X}(\beta) \xleftarrow{\sim} \{x\} \times_{\mathcal{X}(\bar{x})} \mathcal{X}(\sigma) \xrightarrow{d_1} \{x\} \times_{\mathcal{X}(\bar{x})} \mathcal{X}(\gamma),$$

is an equivalence.

3 | EQUIVALENCE OF L0p WITH D0p

3.1

In this section, we will construct an equivalence of $(\infty-)$ categories between L0p and D0p. (It will be upgraded to an equivalence of symmetric monoidal categories after some more work.) The construction is based on a functor

$$\omega : \Delta/Fin_* = \mathbb{F} \rightarrow \Phi,$$

see diagram (28), which we will define first. The definition of ω is a variant of the one in [7], which dealt with open trees and forests only.

3.1.1

Consider an object $A : [n] \rightarrow Fin_*$ of \mathbb{F} , that is, a sequence

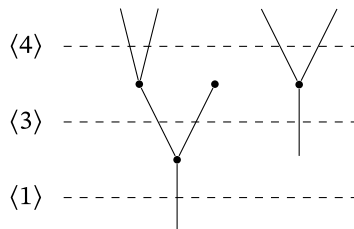
$$A_{0*} \xrightarrow{\alpha_1} A_{1*} \rightarrow \dots \rightarrow A_{n*}$$

of maps between pointed sets. We write $\alpha_{ij} : A_{j*} \rightarrow A_{i*}$ for the composition $\alpha_i \circ \dots \circ \alpha_{j+1}$ (for $i \geq j$). The set of edges of the forest $\omega(A)$ is the disjoint union $\coprod A_i$ of the sets A_i . This set carries a partial order defined for $a \in A_i$ and $b \in A_j$ by

$$a \leq b \text{ iff } j \leq i \text{ and } \alpha_{ij}(b) = a.$$

The roots of $\omega(A)$ are the edges minimal in the above order. For each $a \in A_i$ in this set of edges with $i > 0$, there is a unique vertex v_a in the forest $\omega(A)$ immediately above a . The edge a is the outgoing edge of v_a , while $in(v_a) = \alpha_i^{-1}(a)$. In particular, the set of leaves in the forest can be identified with A_0 . The set of roots of $\omega(A)$ consists of the elements of A_n together with the elements of A_i sent to the basepoint $*$ under $\alpha_{i+1} : A_i \rightarrow A_{i+1}$ for $i = 0, \dots, n - 1$.

Here is an example of the forest corresponding to the map $\langle 4 \rangle \xrightarrow{\alpha_1} \langle 3 \rangle \xrightarrow{\alpha_2} \langle 1 \rangle$ with $\alpha_1(1) = 1 = \alpha_1(2), \alpha_1(3) = 3 = \alpha_1(4), \alpha_2(1) = 1 = \alpha_2(2), \alpha_2(3) = *$.



This defines $\omega : \mathbb{F} \rightarrow \Phi$ on objects. It extends to morphisms in the obvious way: A face map $d_i A \rightarrow A$ induces a morphism $\omega(d_i A) \rightarrow \omega(A)$ which on each component tree is a composition of faces; and a degeneracy map $A \rightarrow s_i A$ induces a morphism $\omega(A) \rightarrow \omega(s_i A)$ which is a composition of degeneracies.

Note the following property of ω .

Lemma 3.1.2. *Any forest $F \in \Phi$ is a retract of some $\omega(A)$ for some $A \in \mathbb{F}$.*

Proof. In order to present a forest F as $\omega(A)$, one has to assign a nonnegative number $h(a)$ to each edge a so that $h(a) = h(b) - 1$ for a immediately under b and so that the a is a leaf precisely when $h(a) = 0$. The first condition is achieved easily; to achieve the second, one may need to enlarge the forest F slightly and construct a forest F' by adjoining a sequence of unary edges on top of leaves of F . Then F' is of the form $\omega(A)$ and F is a retract of F' . \square

3.1.3

The functor ω defines an adjoint pair

$$\omega_! : P(\mathbb{F}) \xrightarrow{\leftarrow} P(\Phi) : \omega^*.$$

The functor $\lambda : \text{DOp} \rightarrow P(\mathbb{F})$ is defined as the restriction of ω^* to DOp . This means that for $D \in \text{DOp}$ and $A \in \mathbb{F}$,

$$\lambda(D)(A) = \text{Map}_{P(\Phi)}(\omega(A), D). \tag{5}$$

Define $i : \Phi \rightarrow \text{LOp}$ as the composition of $o : \Phi \rightarrow \text{Op}(\text{Set})$ with the embedding $\ell : \text{Op}(\text{Set}) \rightarrow \text{LOp}$ discussed in Section 2.3.1. The functor i determines a functor $\text{LOp} \times \Phi^{\text{op}} \rightarrow \mathcal{S}$ that yields $\delta : \text{LOp} \rightarrow P(\Phi)$ by adjunction. This means that, for $P \in \text{LOp}$ and $F \in \Phi$, one has

$$\delta(P)(F) = \text{Map}_{\text{LOp}}(i(F), P). \tag{6}$$

Theorem 3.1.4. *The functors defined above give a pair of quasi-inverse functors*

$$\delta : \text{LOp} \xrightarrow{\leftarrow} \text{DOp} : \lambda. \tag{7}$$

3.1.5

In Section 5, we will extend this equivalence to an equivalence of symmetric monoidal categories. The proof of Theorem 3.1.4 is presented in Sections 3.2–3.5 below. We will first of all verify that $\delta(P) \in \text{DOp}$ for any $P \in \text{LOp}$ and that λ carries DOp to LOp . Then we will construct equivalences $\lambda \circ \delta \rightarrow \text{id}$ and $\text{id} \rightarrow \delta \circ \lambda$.

3.1 | The functor δ has image in DOp

The category DOp is a full subcategory of $P(\Phi)$, so we only have to verify that, for $L \in \text{LOp}$, the presheaf $\delta(L)$ satisfies the conditions (D1), (D2), and (D3). The functor i carries a finite coproduct

of forests to the corresponding coproduct in $\mathbb{L}\mathbb{O}p$ (since both o and ℓ above preserve coproducts). Also, for an inner edge b of a tree T , Proposition 4.3.4 below claims that $i(T)$ is the colimit of the diagram $i(T^b) \leftarrow i(b) \rightarrow i(T_b)$, where T^b and T_b are two halves of the tree T obtained by cutting T along b . These two facts immediately prove the conditions (D1) and (D3). It remains to verify (D2). The simplicial space $\iota^* \circ \delta(L)$ is just the image of L under the functor $P(\Delta_{/Fin_*}) \rightarrow P(\Delta)$ defined by $\langle 1 \rangle \in Fin_*$. It is complete as L represents a category over Fin_* .

3.2 | The functor λ has image in $\mathbb{L}\mathbb{O}p$

3.3.1

Let us, first of all, verify that $\lambda(D) \in \text{Cat}_{/Fin_*}$ for any $D \in \mathbb{D}\mathbb{O}p$. We have to verify that $\lambda(D)$, considered as an object of $P(\Delta_{/Fin_*})$, satisfies the Segal condition and is complete.

For $A : [n] \rightarrow Fin_*$ we denote by A_i the composition $\{i\} \rightarrow [n] \xrightarrow{A} Fin_*$ and by $A_{i-1,i}$ the composition $[1] \xrightarrow{\{i-1,i\}} [n] \xrightarrow{A} Fin_*$. The Segal condition for $\lambda(D)$ means that the natural map

$$\lambda(D)(A) \rightarrow \lambda(D)(A_{01}) \times_{\lambda(D)(A_1)} \dots \times_{\lambda(D)(A_{n-1})} \lambda(D)(A_{n-1,n})$$

is an equivalence. This easily follows from the Segal properties (D1) and (D3) for D formulated in Section 2.2.3.

By Lemma 2.1.6 applied to Fin_* viewed as a complete Segal space, completeness of $\lambda(D)$ means that for any $I_* \in Fin_*$ and the map $\iota_I : \Delta \rightarrow \Delta_{/Fin_*}$ carrying $[n] \in \Delta$ to $[n] \rightarrow [0] \xrightarrow{I_*} Fin_*$, the map $\iota_I^* : P(\Delta_{/Fin_*}) \rightarrow P(\Delta)$ carries $\lambda(D)$ to a complete Segal space. Denote $D_1 = \iota^*(D)$, where $\iota : \Delta \rightarrow \Phi$ is defined in Section 2.2.3. This is the complete Segal space representing the category underlying $D \in \mathbb{D}\mathbb{O}p$. Since $\iota_I^*(\lambda(D)) = D_1^I$, it is a complete Segal space. Thus, $\lambda(D)$ is a category over Fin_* . We will denote it explicitly by $p : \lambda(D) \rightarrow Fin_*$.

Let us now verify that $p : \lambda(D) \rightarrow Fin_*$ is fibrous. The fiber of p at I_* is $\iota_I^*(\lambda(D)) = D_1^I$.

- (Fib1) Given $\alpha : \langle m \rangle \rightarrow \langle n \rangle$ inert, the base change $\lambda(D)_\alpha := [1] \times_{Fin_*} \lambda(D)$ is a category over $[1]$ with fibers D_1^m and D_1^n at 0 and 1, respectively. This is obviously a cocartesian fibration classified by the projection $p_\alpha : D_1^m \rightarrow D_1^n$ determined by the inert α . Therefore, α has a locally cocartesian lifting $a : x \rightarrow p_\alpha(x)$ for each object $x \in D_1^m$. It is now easy to verify the condition of Lemma 2.3.6 that shows that any such a is in fact cocartesian.
- (Fib2) The inert maps $\rho^i : \langle n \rangle \rightarrow \langle 1 \rangle$ give rise to an equivalence $\lambda(D)_n \rightarrow \prod \lambda(D)_1$. This is straightforward.
- (Fib3) It remains to verify the last property of fibrous objects. Fix $A : [1] \rightarrow Fin_*$ defined by an arrow $f : \langle m \rangle \rightarrow \langle n \rangle$. Given $x \in \lambda(D)_m$ and $y \in \lambda(D)_n$, the map space $\text{Map}_{\lambda(D)}^f(x, y)$ can be expressed as the fiber of the natural map

$$\lambda(D)(A) = D(\omega(A)) \rightarrow \lambda(D)(m) \times \lambda(D)(n)$$

at (x, y) . Applying the axiom (D3) to the forest $\omega(A)$, we deduce the required decomposition

$$\text{Map}_{\lambda(D)}^f(x, y) \rightarrow \prod_i \text{Map}_{\lambda(D)}^{\rho^i \circ f}(x, \rho^i(y)).$$

3.3.2

$\mathbb{L}\text{Op}$ is not a full subcategory of $P(\Delta_{/Fin_*})$. This means that we have to verify that, given a map $f : D \rightarrow D'$ with $D, D' \in \mathbb{D}\text{Op}$, the induced map $\omega^*(f) : \omega^*(D) \rightarrow \omega^*(D')$ preserves the inerts. This immediately follows from the description of inerts given above: If $f : \langle m \rangle \rightarrow \langle n \rangle$ is inert and if $f : D \rightarrow D'$ is a map, it induces a commutative square

$$\begin{array}{ccc} D_1^m & \longrightarrow & D_1^n \\ \downarrow & & \downarrow \\ D_1'^m & \longrightarrow & D_1'^n \end{array}$$

with the vertical arrows induced by f and the horizontal arrows being the projections determined by f .

3.3 | An equivalence $\lambda \circ \delta \rightarrow \text{id}$

In this subsection, we construct an equivalence of functors $\beta : \lambda \circ \delta \rightarrow \text{id}$. The construction uses, for any $A \in \mathbb{F}$, the canonical section $s_A : A \rightarrow j(\omega(A))$ in $P(\mathbb{F})$, where $j : \Phi \rightarrow P(\mathbb{F})$ is the composition $\Phi \xrightarrow{i} \mathbb{L}\text{Op} \rightarrow P(\mathbb{F})$, see diagram (28). In more detail, for an operad P in sets the corresponding object $\ell(P)$ in $\mathbb{L}\text{Op}$ can be viewed as a presheaf on \mathbb{F} via the embedding $\mathbb{L}\text{Op} \hookrightarrow P(\mathbb{F})$. By the description given in Section 2.3, the value of this presheaf at $A \in \mathbb{F}$ is precisely the set of operad maps $o(\omega(A)) \rightarrow P$. This yields, for $P = o(\omega(A))$, a canonical section $s_A : A \rightarrow j(\omega(A))$.

We will deduce that β is an equivalence from the following result to be proven in 4.3.

Proposition 3.4.1. *For $A \in \mathbb{F}$ and $L \in \mathbb{L}\text{Op}$, the canonical section*

$$s_A : A \rightarrow j(\omega(A)) \tag{8}$$

in $P(\mathbb{F})$ induces an equivalence

$$\text{Map}_{\mathbb{L}\text{Op}}(i \circ \omega(A), L) \rightarrow \text{Map}_{P(\mathbb{F})}(A, L).$$

3.4.2

Just for now, let us write $g : \mathbb{L}\text{Op} \hookrightarrow P(\mathbb{F})$ for the embedding functor. We will first define a morphism of functors $\beta' : g \circ \lambda \circ \delta \rightarrow g$ from $\mathbb{L}\text{Op}$ to $P(\mathbb{F})$, and then will show that β' factors through a $\beta : \lambda \circ \delta \rightarrow \text{id}$.

Using the standard equivalence

$$\text{Fun}(A, \text{Fun}(B, C)) = \text{Fun}(A \times B, C),$$

we will define instead an equivalence $\tilde{\beta}' : \widetilde{g \circ \lambda \circ \delta} \rightarrow \tilde{g}$ of functors from $\mathbb{L}\text{Op} \times \mathbb{F}^{\text{Op}}$ to \mathcal{S} . The functor $\widetilde{g \circ \lambda \circ \delta}$ carries $(L, A) \in \mathbb{L}\text{Op} \times \mathbb{F}^{\text{Op}}$ to

$$\text{Map}_{\mathbb{D}\text{Op}}(\omega(A), \delta(L)) = \text{Map}_{\mathbb{L}\text{Op}}(i \circ \omega(A), L) \subset \text{Map}_{P(\mathbb{F})}(j \circ \omega(A), g(L)),$$

whereas \tilde{g} carries (L, A) to $\text{Map}_{P(\mathbb{F})}(A, g(L))$.

The functor $\tilde{\beta}'$ is now defined as the precomposition with $s_A : A \rightarrow j \circ \omega(A)$. According to Proposition 3.4.1, $\tilde{\beta}'$, and, therefore, β' , is an equivalence.

Since both $g \circ \lambda \circ \delta(L)$ and $g(L)$ belong to $\text{LOp} \subset P(\mathbb{F})$, the natural equivalence $\beta'_L : g \lambda \delta(L) \rightarrow g(L)$ between them also belongs to LOp , hence is the image under g of a unique equivalence $\beta_L : \lambda \delta(L) \rightarrow L$. (Note that the inclusion $\text{LOp} \rightarrow P(\mathbb{F})$ is fully faithful on equivalences since equivalences automatically preserve cocartesian liftings of inerts.)

3.4.3

Note, for further application, the following consequence of Proposition 3.4.1 which relates two realizations of a forest as an operad, one in DOp and the other in LOp . Define a morphism of functors $\theta : \lambda|_\Phi \rightarrow i$ from Φ to LOp so that its composition with $g : \text{LOp} \rightarrow P(\mathbb{F})$ is given by the natural transformation of functors $\Phi \times \mathbb{F}^{\text{Op}} \rightarrow S$ defined as in Section 3.4.2,

$$\begin{aligned} \text{Map}_{P(\mathbb{F})}(A, \omega^*(F)) &= \text{Map}_\Phi(\omega(A), F) = \text{Map}_{\text{LOp}}(i \circ \omega(A), i(F)) \rightarrow \\ &\text{Map}_{P(\mathbb{F})}(A, j(F)), \end{aligned}$$

where $A \in \mathbb{F}$ and $F \in \Phi$.

Proposition 3.4.4. *The morphism of functors $\theta : \lambda|_\Phi \rightarrow i$ defined above, from the restriction of $\lambda : \text{DOp} \rightarrow \text{LOp}$ to $\Phi \hookrightarrow \text{DOp}$ into $i : \Phi \hookrightarrow \text{LOp}$, is an equivalence.*

3.4 | An equivalence $\text{id} \rightarrow \delta \circ \lambda$

In this subsection, we construct an equivalence of functors $\alpha : \text{id} \rightarrow \delta \circ \lambda$. This will complete the proof of the equivalence of LOp with DOp .

Let us temporarily write $G : \text{DOp} \rightarrow P(\Phi)$ for the embedding. Since this embedding is fully faithful, it is sufficient to construct an equivalence $\alpha' : G \rightarrow G \circ \delta \circ \lambda$ of functors from DOp to $P(\Phi)$. As in Section 3.4, we will construct instead an equivalence of functors

$$\tilde{\alpha}' : \tilde{G} \rightarrow \widetilde{G \circ \delta \circ \lambda}$$

from $\text{DOp} \times \Phi^{\text{Op}}$ to S . The functor \tilde{G} carries $(D, F) \in \text{DOp} \times \Phi^{\text{Op}}$ to $\text{Map}_{\text{DOp}}(F, D)$ whereas $\widetilde{G \circ \delta \circ \lambda}$ carries (D, F) to $\text{Map}_{\text{LOp}}(i(F), \lambda(D)) = \text{Map}_{\text{LOp}}(\lambda(F), \lambda(D))$, the last equivalence following from Proposition 3.4.4.

We define the morphism $\tilde{\alpha}'$ simply as the morphism

$$\text{Map}_{\text{DOp}}(F, D) \rightarrow \text{Map}_{\text{LOp}}(\lambda(F), \lambda(D)) \tag{9}$$

induced by λ . It remains to verify that (9) is an equivalence. By Lemma 3.1.2, we can choose $A \in \mathbb{F}$ so that F is a retract of $\omega(A)$. Then the composition

$$\begin{aligned} \text{Map}_{\text{DOp}}(\omega(A), D) &\rightarrow \text{Map}_{\text{LOp}}(\lambda(\omega(A)), \lambda(D)) = \text{Map}_{\text{LOp}}(i \circ \omega(A), \lambda(D)) \rightarrow \\ &\text{Map}_{P(\mathbb{F})}(A, \omega^*(D)), \end{aligned}$$

is an equivalence. The last map in the composition is also an equivalence by Proposition 3.4.1, so $\tilde{\alpha}'$ is an equivalence for $\omega(A)$, and, therefore, for F .

4 | OPERADIC ALGEBRAS

4.1 | Reconstruction

Recall the category of Lurie operads LOp is a Bousfield localization of $\text{Cat}^+_{/Fin_*^h}$. The latter category is Cat -enriched with the category of functors from X to Y defined by the formula

$$\text{Map}_{\text{Cat}}(K, \text{Fun}^h(X, Y)) = \text{Map}_{\text{Cat}^+_{/Fin_*^h}}(X \times K^b, Y).$$

This Cat -enrichment is used in the definition of the category of operad algebras: Given a pair $\mathcal{P}, \mathcal{Q} \in \text{LOp}$, the category of \mathcal{P} -algebras in \mathcal{Q} , $\text{Alg}_{\mathcal{P}}(\mathcal{Q})$, is defined as $\text{Fun}^h(\mathcal{P}, \mathcal{Q})$. In this subsection, we prove that a Lurie operad $\mathcal{P} \in \text{LOp}$ can be reconstructed from the category of \mathcal{P} -algebras in \mathcal{S} (which is a symmetric monoidal category and therefore can be considered as an object in LOp). More precisely, one has the following.

Theorem 4.1.1. *Let $f : \mathcal{P} \rightarrow \mathcal{Q}$ be a morphism of operads which is essentially surjective on colors. Assume that the functor*

$$f^* : \text{Alg}_{\mathcal{Q}}(\mathcal{S}) \rightarrow \text{Alg}_{\mathcal{P}}(\mathcal{S})$$

is an equivalence. Then f is an equivalence of operads.

Note that the essential surjectivity condition cannot be dropped: The embedding of a category into its Karoubian envelope induces an equivalence of the categories of presheaves! The proof of the theorem is given in 4.1.6.

Note the following easy result.

Lemma 4.1.2. *Let $\alpha : X \rightarrow Y$ be an operadic equivalence in $\text{Cat}^+_{/Fin_*^h}$. Then the map*

$$\text{Fun}^h(Y, \mathcal{S}) \rightarrow \text{Fun}^h(X, \mathcal{S}),$$

where \mathcal{S} is considered as a Lurie operad, is an equivalence.

Proof. Given $K \in \text{Cat}$, the category $\mathcal{S}^K = \text{Fun}(K, \mathcal{S})$ has a cartesian symmetric monoidal structure, so it can be considered as an object of LOp . The operadic equivalence $\alpha : X \rightarrow Y$ induces an equivalence

$$\text{Map}_{\text{Cat}^+_{/Fin_*^h}}(Y, \mathcal{S}^K) \rightarrow \text{Map}_{\text{Cat}^+_{/Fin_*^h}}(X, \mathcal{S}^K).$$

Now the equivalence

$$\text{Map}_{\text{Cat}^+_{/Fin_*^h}}(X, \mathcal{S}^K) = \text{Map}_{\text{Cat}^+_{/Fin_*^h}}(X \times K^b, \mathcal{S})$$

yields an equivalence $\text{Map}(K, \text{Fun}^h(Y, \mathcal{S})) = \text{Map}(K, \text{Fun}^h(X, \mathcal{S}))$. □

Remark 4.1.3. Although Lemma 4.1.2 is sufficient for our purposes, the following more general result can be proven in the same way. Let \mathcal{O} be an arbitrary Lurie operad. Using a full embedding of \mathcal{O} into a symmetric monoidal category $\hat{\mathcal{O}}$, see 5.2 below, an operadic equivalence $\alpha : X \rightarrow Y$ gives rise to an equivalence

$$\text{Fun}^{\text{h}}(Y, \mathcal{O}) \rightarrow \text{Fun}^{\text{h}}(X, \mathcal{O})$$

for any \mathcal{O} . Indeed, the induced symmetric monoidal structure on $\hat{\mathcal{O}}^K$ defines an operad structure on the full subcategory $\mathcal{O}^K := \text{Fun}(K, \mathcal{O}) \times_{\text{Fun}(K, \text{Fin}_*)} \text{Fin}_*$ as in 4.1.2.

4.1.4

Let I be a set and let \mathcal{P} be a Lurie operad. A map $r : I \rightarrow \mathcal{P}_1$ is called a *recoloring* if it induces a surjective map on the equivalence classes of objects of \mathcal{P}_1 . We define a *recolored operad* as an operad \mathcal{P} endowed with a recoloring $r : I \rightarrow \mathcal{P}_1$. Any map $r : I \rightarrow \mathcal{P}_1$ defines a forgetful functor

$$G_r : \text{Alg}_{\mathcal{P}}(S) \rightarrow S^I.$$

A general theorem [12, 3.1.3.5] implies that G_r admits a left adjoint functor of free \mathcal{P} -algebra denoted $F_r : S^I \rightarrow \text{Alg}_{\mathcal{P}}(S)$.

We present below an explicit expression for the free algebra $F_r(X)$ where $p : X \rightarrow I$ is a map of sets, considered as a collection of (discrete) spaces $X_i = p^{-1}(i) \in S$. This is the free \mathcal{P} -algebra generated by the set X of objects such that the color of $x \in X$ is $r(p(x))$. Note that \mathcal{P} -algebras with values in S can be described by functors $A : \mathcal{P} \rightarrow S$ that are *monoid objects* in the sense of [12, 2.4.2.1]. Equivalently, this means that the left fibration $\mathcal{P}_A \rightarrow \mathcal{P}$ classified by A is a left fibration of operads.

Following [12, 2.1.1.20], we denote by $\mathcal{T} \text{riv} \subset \text{Fin}_*$ the subcategory spanned by the inert arrows. This is the trivial operad on one color. For a given set X , we denote by $\mathcal{T} \text{riv}_X$ the coproduct of X copies of the operad $\mathcal{T} \text{riv}$, so that $\mathcal{T} \text{riv}_X$ is the trivial operad on X colors. The objects of $\mathcal{T} \text{riv}_X$ are finite sets over X and the arrows are embeddings of these sets, considered as (inert) arrows in the opposite direction.

The map $c := r \circ p : X \rightarrow \mathcal{P}_1$ extends to $c : \mathcal{T} \text{riv}_X \rightarrow \mathcal{P}$, see [12, 2.1.3.6], and, therefore, gives rise to the functor

$$\bar{c} : \mathcal{T} \text{riv}_X^{\text{op}} \rightarrow \text{Left}(\mathcal{P}) = \text{Fun}(\mathcal{P}, S) \tag{10}$$

with values in the category of left fibrations over \mathcal{P} , carrying $\alpha : U \rightarrow X$ in $\mathcal{T} \text{riv}_X$ to the left fibration $\mathcal{P}_{c\alpha} \rightarrow \mathcal{P}$. We finally denote

$$F_r(X) = \text{colim}(\bar{c}) \in \text{Left}(\mathcal{P}). \tag{11}$$

Rewriting (11) as a functor $F_r(X) : \mathcal{P} \rightarrow S$, we deduce the formula

$$F_r(X)(d) = \text{colim}(\bar{c}_d), \tag{12}$$

where $\bar{c}_d : \mathcal{T} \text{riv}_X^{\text{op}} \rightarrow S$ is the functor carrying $\alpha \in \mathcal{T} \text{riv}_X$ to $\text{Map}_{\mathcal{P}}(c\alpha, d)$.

This can be further rewritten as follows. Let D_* be the image of $d \in \mathcal{P}$ in Fin_* . Define the category $\mathcal{T}riv_{X,D}$ whose objects are the pairs $(\alpha : U \rightarrow X, \beta : U_* \rightarrow D_*)$ and morphisms $(\alpha, \beta) \rightarrow (\alpha' : U' \rightarrow X, \beta' : U'_* \rightarrow D_*)$ defined by an embedding $U' \rightarrow U$ over X so that the corresponding inert arrow $U_* \rightarrow U'_*$ commutes with the β s. One has an obvious forgetful functor $\phi : \mathcal{T}riv_{X,D} \rightarrow \mathcal{T}riv_X$ and a functor

$$\bar{c}_{d,D} : \mathcal{T}riv_{X,D}^{op} \rightarrow \mathcal{S}$$

carrying (α, β) to $Map_{\mathcal{P}}^{\beta}(c \circ \alpha, d)$. The fibers of ϕ are discrete, so obviously \bar{c}_d is a left Kan extension of $\bar{c}_{d,D}$ along ϕ . Therefore, $F_r(X)(d) = \text{colim } \bar{c}_d = \text{colim } \bar{c}_{d,D}$. The category $\mathcal{T}riv_{X,D}$ has a subcategory $\mathcal{T}riv_{X,D}^{act}$ spanned by the pairs (α, β) with β active. This is a groupoid. We denote by $\bar{c}_{d,D}^{act}$ the restriction of $\bar{c}_{d,D}$ to $(\mathcal{T}riv_{X,D}^{act})^{op}$. The embedding $\mathcal{T}riv_{X,D}^{act} \rightarrow \mathcal{T}riv_{X,D}$ is cofinal so that it induces an equivalence of colimits

$$\text{colim } \bar{c}_{d,D}^{act} \rightarrow \text{colim } \bar{c}_{d,D}.$$

We can finally reformulate the description of $F_r(X)(d) = \text{colim } \bar{c}_{d,D}^{act}$ as follows. Let $\mathcal{T}riv_X^{eq}$ be the maximal subgroupoid of $\mathcal{T}riv_X$ (this is just the groupoid of finite sets over X) and let $Map_{\mathcal{P}}^{act}(x, y)$ denote the space of active arrows in \mathcal{P} from x to y . The forgetful functor $\mathcal{T}riv_{X,D}^{act} \rightarrow \mathcal{T}riv_X^{eq}$ having discrete fibers, the left Kan extension of $\bar{c}_{d,D}^{act}$ along it yields the functor

$$\bar{c}_d^{act} : (\mathcal{T}riv_X^{eq})^{op} \rightarrow \mathcal{S}$$

assigning to α the space $Map_{\mathcal{P}}^{act}(c \circ \alpha, d)$. We see that

$$F_r(X)(d) = \text{colim}(\bar{c}_d^{act}). \tag{13}$$

We need yet another version of the above formula.

The functor \bar{c}_d^{act} factors through $\bar{p} : \mathcal{T}riv_X^{eq} \rightarrow \mathcal{T}riv_I^{eq}$ carrying $\alpha : U \rightarrow X$ to $p \circ \alpha$. Therefore, the colimit of \bar{c}_d^{act} can be rewritten as $\text{colim } \bar{X}$ where \bar{X} is the left Kan extension of \bar{c}_d^{act} with respect to \bar{p} . One easily sees that $\bar{X} : (\mathcal{T}riv_I^{eq})^{op} \rightarrow \mathcal{S}$ is defined by the formula

$$\bar{X}(\gamma) = Map_{\mathcal{P}}^{act}(r \circ \gamma, d) \times_{\text{Aut}_I(U)} \text{Hom}_I(U, X) \tag{14}$$

for $\gamma : U \rightarrow I$. In the special case $d \in \mathcal{P}_1$, this can be rewritten as

$$\bar{X}(\gamma) = \mathcal{P}(r \circ \gamma, d) \times_{\text{Aut}_I(U)} \text{Hom}_I(U, X). \tag{15}$$

Proposition 4.1.5. $F_r(X)$ is a free \mathcal{P} -algebra generated by the set X .

Proof. Let $q : \mathcal{Q} \rightarrow \mathcal{P}$ be a left fibration of operads. One has

$$Map_{\text{Cat}_{\mathcal{P}}}(F_r(X), \mathcal{Q}) = \lim_{\alpha \in \mathcal{T}riv_X^{op}} Map_{\text{Cat}_{\mathcal{P}}}(P_{c \circ \alpha}, \mathcal{Q}) = \lim_{\alpha \in \mathcal{T}riv_X^{op}} \mathcal{Q}_{c \circ \alpha} = Map_{\text{Cat}_{\mathcal{P}}}(X, \mathcal{Q}). \quad \square$$

4.1.1 | Proof of 4.1.1

Choose a recoloring $r : I \rightarrow \mathcal{P}_1$. It will automatically give a recoloring $f_{or} : I \rightarrow \mathcal{Q}$. This yields a commutative diagram

$$\begin{array}{ccc}
 \text{Alg}_{\mathcal{P}} & \xleftarrow{f^*} & \text{Alg}_{\mathcal{Q}} \\
 \searrow G_r & & \swarrow G_{f_{or}} \\
 & S^I &
 \end{array} \tag{16}$$

We denote by $f_! : \text{Alg}_{\mathcal{P}} \rightarrow \text{Alg}_{\mathcal{Q}}$ the functor left adjoint (and inverse) to f^* , so we get an equivalence

$$F_{f_{or}} = f_! \circ F_r.$$

This yields an equivalence

$$G_r \circ F_r \rightarrow G_{f_{or}} \circ F_{f_{or}}. \tag{17}$$

The source and the target of the above map are explicitly given as colimits, see formulas (13) and (15). Thus, to yield an equivalence (17), one should have, for any $\gamma : U \rightarrow I$ in $\mathcal{T}riv_I^{eq} = Fin_{/I}^{eq}$ and a map of sets $X \rightarrow I$, an equivalence

$$\mathcal{P}(r \circ \gamma, d) \times_{\text{Aut}_I(U)} \text{Hom}_I(U, X) \rightarrow \mathcal{Q}(r \circ \gamma, d) \times_{\text{Aut}_I(U)} \text{Hom}_I(U, X).$$

Choosing $X \rightarrow I$ large enough for $\text{Aut}_I(U)$ to have an orbit in $\text{Hom}_I(U, X)$ with trivial stabilizer (e.g., choosing $X \rightarrow I$ to be $U \rightarrow I$ itself), we deduce that the map $\mathcal{P}(r \circ \gamma, d) \rightarrow \mathcal{Q}(r \circ \gamma, d)$ has to be an equivalence for all $\gamma : U \rightarrow I$ and $d \in \mathcal{P}_1$. This implies $f : \mathcal{P} \rightarrow \mathcal{Q}$ is an equivalence.

4.2 | Model structures on operad algebras

In this subsection, we present standard results on model structures in categories of algebras and rectification results.

4.2.1

We will use a special case of the rectification theorem of Pavlov–Scholbach. Let \mathcal{C} be a simplicial symmetric monoidal model category. According to [15, A.7], the underlying ∞ -category \mathcal{C}_{∞} inherits a symmetric monoidal structure so that the localization functor $\mathcal{C} \rightarrow \mathcal{C}_{\infty}$ is lax symmetric monoidal.

Let \mathcal{O} be a Σ -free and \mathcal{C} -admissible operad in sets.[†] Then the category of algebras $\text{Alg}_{\mathcal{O}}(\mathcal{C})$ has a projective model structure. One has a functor $L' : \text{Alg}_{\mathcal{O}}(\mathcal{C})^{cf} \rightarrow \text{Alg}_{\ell(\mathcal{O})}(\mathcal{C}_{\infty})$ carrying weak

[†] Pavlov and Scholbach more generally consider simplicial operads.

equivalences of fibrant cofibrant algebras to equivalences, hence inducing a functor

$$L : \text{Alg}_{\mathcal{O}}(C)_{\infty} \rightarrow \text{Alg}_{\mathcal{O}}(C_{\infty}) \tag{18}$$

between the underlying ∞ -categories.

Theorem 4.2.2 (see [16], Theorem 7.11). *Let C be a simplicial symmetric monoidal model category and let \mathcal{O} be a Σ -free C -admissible operad as above. Then the functor L in (18) is an equivalence.*

The following Lemma 4.2.3 is used as an inductive step in the proofs of Proposition 3.4.1 (see Section 4.3) and of Lemma 5.4.2.

Lemma 4.2.3. *Let $A : [1] \rightarrow \text{Fin}_*$ be presented by an arrow $f : I_* \rightarrow J_*$ and let $\mathcal{O} = o(\omega(A))$. Let C be a simplicial model category. We endow $\text{Alg}_{\mathcal{O}}(C)$ with the projective model structure. Then the forgetful functor $\text{Alg}_{\mathcal{O}}(C) \rightarrow C^I$ defined by the source of f induces a fibration of the simplicial categories of fibrant cofibrant objects*

$$p : \text{Alg}_{\mathcal{O}}(C)_*^{cf} \rightarrow (C_*^{cf})^I.$$

Proof. The forest $\omega(A)$ consists of corollas numbered by $j \in J$ and trivial operads η numbered by $f^{-1}(*) \setminus \{*\} \subset I$. The claim immediately reduces to the case when $\omega(A)$ is a single corolla C_n . Thus, from now on we assume $\mathcal{O} = o(C_n)$.

A C_n -algebra in C is given by an arrow $\alpha : D_1 \times \dots \times D_n \rightarrow D_0$ in C . It is a fibrant cofibrant object if D_i are fibrant cofibrant and α is a cofibration. Given two such objects, α as above and $\beta : E_1 \times \dots \times E_n \rightarrow E_0$, the simplicial set $\text{Hom}(\alpha, \beta)$ is defined as the fiber product

$$\prod_{i=1}^n \text{Hom}_C(D_i, E_i) \times_{\text{Hom}(\prod_{i=1}^n D_i, E_0)} \text{Hom}(D_0, E_0).$$

The map $\text{Hom}(\alpha, \beta) \rightarrow \prod_{i=1}^n \text{Hom}_C(D_i, E_i)$ is a fibration because it is obtained by base change from the map $\text{Hom}_C(D_0, E_0) \rightarrow \text{Hom}(\prod_{i=1}^n (D_i, E_0))$, which is itself a fibration because it is defined by the composition with the cofibration α . It remains to verify that the induced map of the homotopy categories

$$\text{Ho}(\text{Alg}_{C_n}(C)_*^{cf}) \rightarrow \text{Ho}(C_*^{cf})^n$$

is an isofibration of conventional categories. This is straightforward. □

4.3 | Proof of Proposition 3.4.1

4.3.1

The map $s_A : A \rightarrow j(\omega(A))$ introduced at the beginning of Section 3.4 induces a map $s'_A : A' \rightarrow i(\omega(A))$ in LOp where $A^b \rightarrow A'$ is an operadic equivalence in $\text{Cat}^+ / \text{Fin}_*^q$; see Section 2.3.5 for the notion of operadic equivalence.

By the reconstruction theorem 4.1.1, it is sufficient to verify that s'_A induces an equivalence of the categories of algebras with values in S ,

$$s'^*_{A} : \text{Alg}_{i(\omega(A))}(S) \rightarrow \text{Alg}_{A'}(S) = \text{Fun}_{\text{Cat}/\text{Fin}_*}(A, S),$$

where A and S on the right-hand side of the formula are considered as objects of Cat/Fin_* (the equality in the last formula follows from the operadic equivalence $A^b \rightarrow A'$ and Lemma 4.1.2).

We will prove that s'^*_{A} is an equivalence by induction, based on Lemma 4.2.3.

4.3.2 | Pruning a simplex

The following procedure of pruning a simplex A will be used. Define $B : [n - 1] \rightarrow \text{Fin}_*$ and $C : [1] \rightarrow \text{Fin}_*$ by the formulas $B = A \circ d_n$, $C = A \circ d_0^{n-1}$. Let $v : [0] \rightarrow \text{Fin}_*$ be defined by $v = C \circ d_1$. The map v is given by an object $V_* \in \text{Fin}_*$. The decomposition $A = B \sqcup^v C$ in Cat/Fin_* gives rise to a commutative diagram

$$\begin{CD} \text{Alg}_{i(\omega(A))}(S) @>>> \text{Alg}_{i(\omega(A))}(S) \times_{S^V} \text{Alg}_{i(\omega(A))}(S) \\ @VVV @VVV \\ \text{Fun}_{\text{Fin}_*}(A, S) @>>> \text{Fun}_{\text{Fin}_*}(B, S) \times_{S^V} \text{Fun}_{\text{Fin}_*}(C, S) \end{CD} \tag{19}$$

so that the lower horizontal arrow is, obviously, an equivalence. In Lemma 4.3.3, we will verify that the upper horizontal arrow is also an equivalence. This will reduce the claim of Proposition 3.4.1 that s'^*_{A} to the case $n = 1$ which is very easy.

Lemma 4.3.3. *The map $\text{Alg}_{i(\omega(A))}(S) \rightarrow \text{Alg}_{i(\omega(A))}(S) \times_{S^V} \text{Alg}_{i(\omega(A))}(S)$ defined by the decomposition $A = B \sqcup^v C$ is an equivalence.*

Proof. Clearly, $o(\omega(A)) = o(\omega(B)) \sqcup^{\mathcal{T} \text{riv}_V} o(\omega(C))$ where $\mathcal{T} \text{riv}_V$ is the trivial operad on V colors. The operad $o(\omega(A))$ is free as an operad in sets. The category $\text{Alg}_{i(\omega(A))}(S)$ is the ∞ -category underlying the simplicial model category $\text{Alg}_{o(\omega(A))}(\text{sSet})$, where the model structure is the projective model structure induced from the standard model structure on the simplicial sets.

The category $\text{Alg}_{o(\omega(A))}(\text{sSet})$ is equivalent to the fiber product

$$\text{Alg}_{o(\omega(B))}(\text{sSet}) \times_{(\text{sSet})^V} \text{Alg}_{o(\omega(C))}(\text{sSet}). \tag{20}$$

Moreover, an arrow $f : X \rightarrow Y$ in $\text{Alg}_{o(\omega(A))}(\text{sSet})$ is a fibration, cofibration, or weak equivalence if and only if its components satisfy the same property in the corresponding model categories $\text{Alg}_{o(\omega(B))}(\text{sSet})$ and $\text{Alg}_{o(\omega(C))}(\text{sSet})$.

Note that one needs to be careful as the fiber product in (20) is taken in the category of conventional categories, and not in Cat . However, the same fiber product formula still holds for the underlying ∞ -categories. Indeed, the ∞ -category underlying $\text{Alg}_{o(\omega(A))}(\text{sSet})$ is the homotopy coherent nerve of the simplicial category of fibrant cofibrant objects which is the fiber product of the simplicial categories of fibrant cofibrant objects of $\text{Alg}_{o(\omega(B))}(\text{sSet})$ and $\text{Alg}_{o(\omega(C))}(\text{sSet})$. By Lemma 4.2.3, this fiber product calculates the fiber product of the corresponding ∞ -categories in Cat .

This completes the proof of Proposition 3.4.1. □

The following result deals with a slightly different type of pruning; it was used in Section 3.2 and its proof is very similar to that of Lemma 4.3.3.

Proposition 4.3.4. *Let b be an inner edge of a tree T . Then $i(T) \in \text{LOp}$ is a pushout*

$$i(T) = i(T^b) \sqcup^{i(b)} i(T_b).$$

Proof. By Theorem 4.1.1, the claim reduces to proving that the natural map

$$\text{Alg}_{i(T)}(\mathcal{S}) \rightarrow \text{Alg}_{i(T^b)}(\mathcal{S}) \times_{\text{Alg}_{i(b)}(\mathcal{S})} \text{Alg}_{i(T_b)}(\mathcal{S})$$

is an equivalence of ∞ -categories. By [16], see Theorem 4.2.2, the ∞ -categories of algebras involved underly the simplicial model categories of algebras with values in sSet . The category $\text{Alg}_{o(T)}(\text{sSet})$ is equivalent to the fiber product

$$\text{Alg}_{o(T^b)}(\text{sSet}) \times_{\text{sSet}} \text{Alg}_{o(T_b)}(\text{sSet})$$

and the reasoning of Lemma 4.3.3 based on Lemma 4.2.3 proves that this equivalence induces an equivalence of the underlying ∞ -categories. \square

5 | MONOIDAL STRUCTURES

5.1

In Section 2.2, we introduced the category DOp underlying a Quillen model category of simplicial presheaves. It is known [8] that the associated homotopy category carries a structure of symmetric monoidal category. Our goal in this section is to explain that this structure can be lifted to the structure of a symmetric monoidal ∞ -category on DOp , and to prove the following sharpening of Theorem 3.1.4.

Theorem 5.1.1. *The functor $\lambda : \text{DOp} \rightarrow \text{LOp}$ is an equivalence of symmetric monoidal categories.*

It follows from this theorem that its inverse $\delta : \text{LOp} \rightarrow \text{DOp}$ is symmetric monoidal as well. Even though we already know that λ and δ form an equivalence of categories, the proof of this stronger theorem is quite involved, due to the fact that (especially in the ∞ -context!) it is difficult to deal with the rich structure of a symmetric monoidal category in a direct way.

5.2 | Preliminaries

5.2.1

We consider symmetric monoidal categories and operads in the sense of Lurie [12], so “operad” means object of LOp here.

It is convenient to define symmetric monoidal categories as commutative algebras in Cat , that is the functors $\text{Fin}_* \rightarrow \text{Cat}$ satisfying the Segal condition. The (covariant) Grothendieck

construction then realizes the category Cat^{SM} of symmetric monoidal categories as the subcategory of LOp whose objects are cocartesian fibrations of operads $p : M \rightarrow \text{Fin}_*$, with the morphisms preserving cocartesian arrows. It is convenient to have another realization, the one connected to the contravariant Grothendieck construction. The following terminology is taken from [2].

Definition 5.2.2. A functor $q : C \rightarrow \text{Fin}_*^{\text{op}}$ is called an *anti-operad* if $q^{\text{op}} : C^{\text{op}} \rightarrow \text{Fin}_*$ is fibrous (i.e., a Lurie operad).

The category of anti-operads will be denoted by Coop . The contravariant realization of Cat^{SM} identifies Cat^{SM} with the subcategory of Coop whose objects are cartesian fibrations and whose arrows preserve the cartesian arrows. The categories of operads and of anti-operads are obviously equivalent. However, if M is a conventional symmetric monoidal category, its operadic realization M^{\otimes} assigns to X_1, \dots, X_n and Y in M the set $\text{Hom}_M(\otimes X_i, Y)$ of operations, whereas its anti-operadic realization ${}^{\otimes}M$ assigns the set $\text{Hom}_M(Y, \otimes X_i)$ of “anti-operations.” The passage to the opposite symmetric monoidal category intertwines between the two realizations: $({}^{\otimes}M)^{\text{op}} = M^{\text{op}\otimes}$. For an operad L , we denote by \hat{L} the symmetric monoidal envelope of L . Passing to opposite categories, we define the enveloping symmetric monoidal category \hat{C} of an anti-operad C . One has canonical embeddings $L \rightarrow \hat{L}^{\otimes}$ and $C \rightarrow {}^{\otimes}\hat{C}$ so that if $C = L^{\text{op}}$, $\hat{C} = \hat{L}^{\text{op}}$.

We will now define two notions intermediate between the world of (anti) operads and the world of symmetric monoidal categories.

Definition 5.2.3. An operad $p : \mathcal{O} \rightarrow \text{Fin}_*$ is called a *lax symmetric monoidal category* if p is a locally cocartesian fibration, see [13, 2.4.2.6].

Definition 5.2.4.

1. An anti-operad $q : C \rightarrow \text{Fin}_*^{\text{op}}$ is called a *colax symmetric monoidal category* if q is a locally cartesian fibration.
2. $q : C \rightarrow \text{Fin}_*^{\text{op}}$ is called a *colax symmetric monoidal category with colimits* if, in addition to the above, the fiber C_1 has colimits and the maps $\otimes_n : C_1^n \rightarrow C_1$ defined by the local cartesian liftings preserve colimits in each argument.

In the conventional setting, a colax symmetric monoidal category C is given by a collection of operations $\otimes_n : C^n \rightarrow C$, with a compatible collection of natural transformations (not necessarily equivalences) of the form

$$\otimes_m \circ (\otimes_{n_1} \times \dots \times \otimes_{n_m}) \rightarrow \otimes_n$$

with $n = \sum n_i$. Note that, since the collections of active and inert arrows in Fin_* form a factorization system, it is sufficient to require that the active arrows have a locally (co)cartesian lifting.

5.2.5 | Day convolution

The result [12, 4.8.1.10] yields the following.

Lemma. *Let C be a symmetric monoidal category. Then the category of presheaves $P(C)$ inherits a symmetric monoidal structure, so that the Yoneda embedding $Y : C \rightarrow P(C)$ is a symmetric monoidal functor, universal among symmetric monoidal functors from C to a symmetric monoidal category with colimits.*

5.2.6 | Presheaves on an anti-operad

We will now define, for any anti-operad C , a full embedding of anti-operads $C \rightarrow P$ where P is a colax symmetric monoidal category with colimits whose underlying category is $P(C_1)$.

Let C be an anti-operad and let \hat{C} be the symmetric monoidal envelope of C . We write C_1 for the category underlying C . Then the full embedding $u : C_1 \rightarrow (\otimes^{\hat{C}})_1 = \hat{C}$ induces an adjoint pair

$$u_! : P(C_1) \rightleftarrows P(\hat{C}) : u^*,$$

where $u_!$ is again a full embedding. In fact, let $f = \text{colim}(Y \circ a)$, $f' = \text{colim}(Y \circ a')$ for $a : K \rightarrow C_1$ and $a' : K' \rightarrow C_1$. Then

$$\text{Map}_{P(C_1)}(f, f') = \lim_{k \in K} \text{colim}_{k' \in K'} \text{Map}(a(k), a'(k')).$$

The same formula describes $\text{Map}_{P(\hat{C})}(u_!(f), u_!(f'))$, so $u_!$ is a full embedding. This implies that $P(C_1)$, as a full subcategory of a symmetric monoidal category, inherits the structure of an anti-operad from $\otimes^{\hat{C}}$. We will denote it by $\otimes P(C_1)$ and we claim that it is a colax symmetric monoidal category. This means that for any f_1, \dots, f_n in $P(C_1)$, the functor

$$f \in P(C_1) \mapsto \text{Map}(u_!(f), \otimes_i u_!(f_i))$$

is representable. This is obviously so as

$$\text{Map}(u_!(f), \otimes_i u_!(f_i)) = \text{Map}(f, u^*(\otimes_i u_!(f_i))).$$

Therefore, the multiple tensor product functor on $P(C_1)$ is defined as the composition

$$P(C_1)^{\otimes n} \xrightarrow{u_!^{\otimes n}} P(\hat{C})^{\otimes n} \xrightarrow{\otimes_n} P(\hat{C}) \xrightarrow{u^*} P(C_1). \quad (21)$$

Note that by construction the map of anti-operads $C \rightarrow \otimes^{\hat{C}}$ factors through the full embedding $\otimes P(C_1) \rightarrow \otimes P(\hat{C})$ and therefore yields a map $C \rightarrow \otimes P(C_1)$.

5.3 | DOp as a symmetric monoidal category

First of all, recall that the conventional category $\text{Op}(\text{Set})$ of operads in sets is symmetric monoidal. Its tensor product is the Boardman–Vogt tensor product of operads, denoted

$$P \otimes_{BV} Q$$

for two operads P and Q .

Next, the full embedding

$$\Phi \rightarrow \text{Op}(\text{Set}), F \mapsto o(F)$$

gives rise to a full sub-anti-operad Φ of $\otimes\text{Op}(\text{Set})$ with $\Phi_1 = \Phi$. Explicitly,

$$\Phi(F; F_1 \dots, F_n) = \text{Hom}_{\text{Op}(\text{Set})}(o(F), o(F_1) \otimes_{BV} \dots \otimes_{BV} o(F_n)).$$

It follows that $P(\Phi)$ has the structure of a colax symmetric monoidal category as explained in Section 5.2.6.

An arrow f in $P(\Phi)$ will be called an *operadic equivalence* if it is carried to equivalence by the localization functor $P(\Phi) \rightarrow \text{DOp}$.

Proposition 5.3.1.

1. Multiple tensor products $\otimes_n : P(\Phi)^n \rightarrow P(\Phi)$ preserve operadic equivalences in each argument.
2. The localization functor $P(\Phi) \rightarrow \text{DOp}$ canonically extends to a map of colax symmetric monoidal categories.
3. The localization functor $P(\Phi) \rightarrow \text{DOp}$ carries associativity constraints to equivalences. Therefore, the colax symmetric monoidal structure on DOp is in fact symmetric monoidal.

Proof. This result easily follows from the properties of shuffles of trees presented in the Appendix. Recall that DOp is a Bousfield localization of $P(\Phi)$ with respect to three types of arrows.

1. $T_d \sqcup^d T^d \rightarrow T$, where d is an inner edge of a tree T .
2. $* \rightarrow J$, embedding of simplicial sets considered as objects of $P(\Phi)$.
3. $\sqcup T_i \rightarrow F$ where $F \in \Phi$ and T_i are the tree components of F .

To prove the first claim, we have to show that for all $f_i \in P(\Phi)$ the functor

$$\otimes_n(f_1, \dots, f_{k-1}, -, f_{k+1}, \dots, f_n) : P(\Phi) \rightarrow P(\Phi) \tag{22}$$

carries the arrows of types 1–3 to operadic equivalences. Since \otimes_n preserves colimits and the localization functor $P(\Phi) \rightarrow \text{DOp}$ preserves colimits, it is enough to verify this claim in the case when f_i are representable, that is, forests. To calculate the tensor product, we can replace each forest with the coproduct of its tree components; in this way, the claim reduces to the case when all f_i are trees. For the arrows of type 1 the result now follows from Proposition A.3.1. The arrow $[0] \rightarrow J$ is carried by (22) to a deformation retract of dendroidal sets, so to an operadic equivalence. The arrows of type 3 are obviously carried to equivalences.

Claim 2. To see that Claim 1 defines a colax symmetric monoidal structure on DOp , we look at the anti-operadic presentation $q : \otimes P(\Phi) \rightarrow \text{Fin}_*^{\text{op}}$ of $P(\Phi)$. Claim 1 implies that the localization of the total category $\otimes P(\Phi)$ with respect to operadic equivalences yields a locally cartesian fibration $\otimes\text{DOp} \rightarrow \text{Fin}_*^{\text{op}}$; moreover, the localization map preserves locally cartesian arrows.

Claim 3. We now look at the morphisms of functors

$$\otimes_q o(\text{id}^k \times \otimes_p \times \text{id}^{q-k-1}) \rightarrow \otimes_{p+q+1} : P(\Phi)^{p+q+1} \rightarrow P(\Phi) \tag{23}$$

describing the associativity constraints. Both source and target preserve colimits on each argument, so the claim is reduced to the case when $f = \{f_i\} \in P(\Phi)^{p+q+1}$ is a collection of trees. Then Proposition A.4.1 implies the result. □

5.4 | Proof of Theorem 5.1.1

Recall [12] that LOp has the structure of a symmetric monoidal category. For two objects L and M , their tensor product is characterized by the property that there is an equivalence

$$\text{Alg}_{L \otimes M}(S) = \text{Alg}_L(\text{Alg}_M(S)).$$

(This characterizes $L \otimes M$ uniquely by the reconstruction theorem 5.1.1.) Since LOp is symmetric monoidal, it has the structure of an anti-operad.

Proposition 5.4.1. *The inclusion $i : \Phi \rightarrow \text{LOp}$ canonically extends to a map of anti-operads.*

Proof. Recall that i is the composition $\Phi \xrightarrow{o} \text{Op}(\text{Set}) \xrightarrow{\ell} \text{LOp}$. Let C be the full subcategory of LOp spanned by the objects $i(F)$, $F \in \Phi$. Since LOp is a symmetric monoidal category, C **acquires receives** the structure of a anti-suboperad. We will verify that C is a conventional anti-operad canonically isomorphic to Φ .

Given a sequence O_1, \dots, O_n of operads in sets, one has a canonical operad multifunctor

$$\ell(O_1) \times \dots \times \ell(O_n) \rightarrow \ell(O_1 \otimes_{BV} \dots \otimes_{BV} O_n),$$

expressing the universal property of Boardmann–Vogt tensor product. In particular, a sequence F_1, \dots, F_n of objects of Φ yields an operad multifunctor

$$iF_1 \times \dots \times iF_n \rightarrow \ell(o(F_1) \otimes_{BV} \dots \otimes_{BV} o(F_n))$$

that induces a map of operads (see Section 2.3.2)

$$\theta : iF_1 \otimes \dots \otimes iF_n \rightarrow \ell(o(F_1) \otimes_{BV} \dots \otimes_{BV} o(F_n)).$$

Thus, it suffices to verify that this map is an equivalence in LOp . Indeed, we would then have an equivalence of anti-operads Φ and C since for any object $F \in \Phi$, the induced map from

$$\text{Hom}_{\text{Op}(\text{Set})}(o(F), o(F_1) \otimes_{BV} \dots \otimes_{BV} o(F_n)) = \text{Map}_{\text{LOp}}(iF, \ell(o(F_1) \otimes_{BV} \dots \otimes_{BV} o(F_n)))$$

to $\text{Map}_{\text{LOp}}(iF, iF_1 \otimes \dots \otimes iF_n)$, will be then an equivalence. The fact that θ is an equivalence now follows by induction from the following lemma. \square

Lemma 5.4.2. *Let $\mathcal{P} = o(F)$ where F is a forest and let \mathcal{Q} be a Σ -free operad in Set . Then the canonical operad bifunctor $\ell(\mathcal{P}) \times \ell(\mathcal{Q}) \rightarrow \ell(\mathcal{P} \otimes_{BV} \mathcal{Q})$ exhibits $\ell(\mathcal{P} \otimes_{BV} \mathcal{Q})$ as a tensor product (in the sense of Lurie) of $\ell(\mathcal{P})$ and $\ell(\mathcal{Q})$.*

Proof. By the reconstruction theorem, it is sufficient to verify that the map $\theta : \ell(\mathcal{P}) \otimes \ell(\mathcal{Q}) \rightarrow \ell(\mathcal{P} \otimes_{BV} \mathcal{Q})$ induces an equivalence of the categories of algebras

$$\theta^* : \text{Alg}_{\ell(\mathcal{P} \otimes_{BV} \mathcal{Q})}(S) \rightarrow \text{Alg}_{\ell(\mathcal{P}) \otimes \ell(\mathcal{Q})}(S).$$

By the rectification theorem, the left-hand side is the ∞ -category underlying the model category

$$\text{Alg}_{\mathcal{P} \otimes_{BV} \mathcal{Q}}(\text{sSet}) = \text{Alg}_{\mathcal{P}}(\text{Alg}_{\mathcal{Q}}(\text{sSet})), \quad (24)$$

whereas, by definition, the right-hand side is

$$\mathbf{Alg}_{\ell(\mathcal{P})}(\mathbf{Alg}_{\ell(\mathcal{Q})}(S)). \tag{25}$$

We denote $C = \mathbf{Alg}_{\mathcal{Q}}(\mathbf{sSet})$. This is a simplicial model category whose underlying ∞ -category is $LC := \mathbf{Alg}_{\ell(\mathcal{Q})}(S)$. We have the localization functor $L : C \rightarrow LC$ and we have to verify that the natural map $\mathbf{Alg}_{\mathcal{P}}(C) \rightarrow \mathbf{Alg}_{\ell(\mathcal{P})}(LC)$ induces an equivalence

$$L(\mathbf{Alg}_{\mathcal{P}}(C)) \rightarrow \mathbf{Alg}_{\ell(\mathcal{P})}(LC). \tag{26}$$

Recall that $\mathcal{P} = o(F)$. We endow $\mathbf{Alg}_{\mathcal{P}}(C)$ with the projective model structure. Note that we cannot use the result of Pavlov–Scholbach [16] as $\mathbf{Alg}_{\mathcal{P}}(C)$ is not a monoidal model category. It is easy to see that a map $f : A \rightarrow A'$ in $\mathbf{Alg}_{\mathcal{P}}(C)$ is a fibration if and only if its restriction to any corolla of F is a fibration. Therefore, the simplicial category $\mathbf{Alg}_{\mathcal{P}}(C)_*^{cf}$ of fibrant cofibrant algebras is the (naive) fiber product of the simplicial categories of algebras over the corollas contained in \mathcal{P} . To simplify the formulas, we will proceed by induction on the number of corollas in F .

We can write $F = F_1 \cup^v F_2$ (pruning/grafting) where F_2 is a corolla containing a root of F , F_1 is a (smaller) forest and $v = \{v_1, \dots, v_k\}$ is a subset of the set of leaves of F_2 whose elements are identified in F with roots of F_1 . This decomposition yields a decomposition of operads $\mathcal{P} = \mathcal{P}_1 \sqcup^v \mathcal{P}_2$ where $\mathcal{P}_j = i(F_j)$ for $j = 1, 2$. We have

$$\mathbf{Alg}_{\mathcal{P}}(C) = \mathbf{Alg}_{\mathcal{P}_1}(C) \times_{C^k} \mathbf{Alg}_{\mathcal{P}_2}(C),$$

where the functors $g_i : \mathbf{Alg}_{\mathcal{P}_i}(C) \rightarrow C^k$ are given by the evaluation at v . According to Lemma 4.2.3, the functor $g_2 : \mathbf{Alg}_{\mathcal{P}_2}(C) \rightarrow C^k$ induces a fibration of the corresponding simplicial categories of fibrant–cofibrant objects. So, applying the functor of homotopy coherent nerve, we get a decomposition

$$L(\mathbf{Alg}_{\mathcal{P}}(C)) = L(\mathbf{Alg}_{\mathcal{P}_1}(C)) \times_{L(C)^k} L(\mathbf{Alg}_{\mathcal{P}_2}(C)),$$

where one of the structure maps is a categorical fibration of quasicategories, so it represents the fiber product in \mathbf{Cat} . Since, by definition, the same decomposition holds for $\mathbf{Alg}_{\mathcal{P}}(LC)$, we deduce that (26) is an equivalence by induction on the number of corollas in \mathcal{P} . □

5.4.3 | Proof of Theorem 5.1.1

The diagram

$$\begin{array}{ccc}
 \Phi & \xrightarrow{i} & \mathbf{LOp} \\
 \downarrow Y & & \uparrow \lambda \\
 P(\Phi) & \xrightarrow{L} & \mathbf{DOp}
 \end{array} \tag{27}$$

with Y the Yoneda embedding, is commutative by Proposition 3.4.4. The composition $i_1 = \lambda \circ L$ preserves colimits. By Proposition 5.4.1, the functor $i : \Phi \rightarrow \mathbf{LOp}$ has a canonical extension to a map of anti-operads. We will show that i_1 canonically extends to a map of colax symmetric monoidal categories with colimits. If $u : \Phi \rightarrow \otimes \Phi$ is the symmetric monoidal envelope of Φ , the

map of anti-operads $i : \Phi \rightarrow {}^\otimes \text{LOp}$ canonically extends to a symmetric monoidal functor $\hat{\Phi} \rightarrow \text{LOp}$ that gives, by Lemma 5.2.5, a colimit preserving symmetric monoidal functor $Y : P(\hat{\Phi}) \rightarrow \text{LOp}$. The composition of Y with $u_1 : P(\Phi) \rightarrow P(\hat{\Phi})$ yields a map of colax SM categories extending i_1 , see Section 5.2.6. Since $Y \circ u_1 = i_1 = \lambda \circ L$ carries operadic equivalences to equivalences, it factors through a symmetric monoidal functor from DOp to LOp extending λ . This proves the theorem.

5.5

We present below, for the convenience of the reader, a diagram presenting some important categories and functors appearing in the paper.

$$\begin{array}{ccccc}
 \mathbb{F} & \xrightarrow{\omega} & \Phi & \xrightarrow{o} & \text{Op}(\text{Set}) \\
 \downarrow \text{Yoneda} & & \swarrow j & \searrow i & \downarrow \ell \\
 P(\mathbb{F}) & \xleftarrow{\quad} & & & \text{LOp}
 \end{array} \tag{28}$$

APPENDIX: SHUFFLES OF TREES

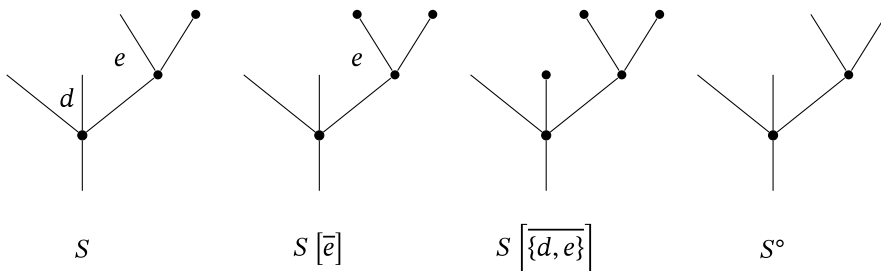
The category dSet of (set-valued) presheaves on Ω carries the “operadic” model structure already mentioned in Section 2.2.3 above and having DOp as the underlying ∞ -category. The tensor product on dSet does not make it a monoidal model category, however, because, for instance, the functors $S \otimes _$ where S is a fixed tree, do not preserve cofibrations (see [8, section 4.3] for a discussion of this point). The smaller category odSet of presheaves on *open* trees does have a homotopically well-behaved tensor product, see [7, section 6.3]. In this appendix, we explain how some of these good homotopical properties of the open trees extend to arbitrary trees. This will imply that DOp is a symmetric monoidal ∞ -category by the argument presented in Section 5.

A.1 | Terminology

Recall from 2.2.3 the category Ω of trees. For a tree S we denote the set of its maximal edges, that is, its leaves and stumps, by

$$\text{max}(S).$$

A tree is open if it has no stumps. For an arbitrary tree S , we write $S^\circ \rightarrow S$ for its “interior”, obtained by chopping off the stumps. So, $S^\circ \rightarrow S$ is bijective on edges. (Warning: the assignment of S° to S is not functorial.) If e is a leaf of S , we denote by $S[\bar{e}]$ the tree obtained by adding a stump on top of e . We will also use the similar notation $S[\overline{\{E\}}]$ for a set E of leaves of S . For example,



The edge e corresponds to a map $e : \eta \rightarrow S$ from the unit tree η , and e extends to a map $\bar{e} : \bar{\eta} \rightarrow S[\bar{e}]$ where $\bar{\eta} = C_0$ is the null-corolla. The tree $S[\bar{e}]$ is the grafting $S \circ_e \bar{e}$ of C_0 onto S at e , and the map

$$S \sqcup^e \bar{e} \rightarrow S[\bar{e}]$$

is a weak equivalence (a Segal map) in the model structure mentioned above. For a set E of leaves, we have a similar weak equivalence

$$S \sqcup^E \bar{E} \rightarrow S[\bar{E}], \tag{A.1}$$

where $S \sqcup^E \bar{E}$ denotes the pushout of $\sqcup_{e \in E} \eta \rightarrow S$ along $\sqcup_{e \in E} \eta \rightarrow \sqcup_{e \in E} \bar{\eta}$.

A.2 | Shuffles

The n -fold tensor product $S_1 \otimes \dots \otimes S_n$ of a sequence of trees S_1, \dots, S_n is a union of “shuffles” (see [8, section 4.4]). A shuffle $A \rightarrow S_1 \otimes \dots \otimes S_n$ is a tree whose edges are (labeled by) n -tuples of edges (e_1, \dots, e_n) of S_1, \dots, S_n , respectively. Not all such tuples will occur in a particular shuffle. But for us, it is important to note that the root of a shuffle A is the n -tuple (r_1, \dots, r_n) of roots, and the set of maximal edges is *exactly* the set of n -tuples of maximal edges in S_i ; that is,

$$\max(A) = \prod_{i=1}^n \max(S_i). \tag{A.2}$$

The fact that $S_1 \otimes \dots \otimes S_n = \bigcup_{j \in J} A_j$ is the union of its shuffles can be expressed as a finite colimit

$$S_1 \otimes \dots \otimes S_n = \operatorname{colim} A_\alpha,$$

where α ranges over nonempty subsets of J , and

$$A_\alpha = \bigcap_{j \in \alpha} A_j$$

is the corresponding intersection of shuffles. Each such finite intersection A_α has property (A.2) and each map $A_\alpha \rightarrow A_\beta$ for $\beta \subseteq \alpha \subseteq J$ is an inner face map, in fact, a map obtained by contracting a set of edges other than the maximal edges or the root.

The structure of the set of shuffles does not depend on the stumps that might occur in the trees S_i . More precisely, for a leaf e in S_i , there is a bijective correspondence between the shuffles of $S_1 \otimes \dots \otimes S_n$ and of $S_1 \otimes \dots \otimes S_i[\bar{e}] \otimes \dots \otimes S_n$, given by $A \mapsto A[\bar{E}_i]$, where $E_i = \{(d_1, \dots, d_n) \mid d_i = e, d_j \text{ are all leaves of } S_j\}$. The same applies to the intersections of shuffles $A_\alpha \mapsto A_\alpha[\bar{E}_i]$.

In particular, for trees S_1, \dots, S_n the tensor product $S_1 \otimes \dots \otimes S_n$ can be reconstructed from the tensor product of their interiors $S_1^\circ, \dots, S_n^\circ$: Precisely, if $S_1^\circ \otimes \dots \otimes S_n^\circ = \bigcup_j A_j = \operatorname{colim} A_\alpha$, then $S_1 \otimes \dots \otimes S_n = \bigcup_j A_j[\bar{E}] = \operatorname{colim} A_\alpha[\bar{E}]$, where

$$E = \{(e_1, \dots, e_n) \mid \text{each } e_j \text{ is a leaf of } S_j^\circ, \text{ at least one } e_i \text{ is a stump of } S_i\}.$$

From this observation, one easily deduces the following lemma.

Lemma A.2.1. For trees S_1, \dots, S_n and the set E as above, the map

$$(S_1^\circ \otimes \dots \otimes S_n^\circ) \sqcup^E \bar{E} \rightarrow S_1 \otimes \dots \otimes S_n$$

is a weak equivalence.

Proof. Write $S_1 \otimes \dots \otimes S_n$ as the colimit $\operatorname{colim} A_\alpha$ of the diagram of finite intersections of shuffles. This is a Reedy cofibrant diagram, so $S_1^\circ \otimes \dots \otimes S_n^\circ$ is also the homotopy colimit. Then $(S_1^\circ \otimes \dots \otimes S_n^\circ) \sqcup^E \bar{E}$ is the colimit of the diagram of the A_α together with the inclusions

$$\bar{\eta} \leftarrow \eta \xrightarrow{e} A_\alpha$$

for each e in E and each α . This colimit is the same as the colimit of the pushouts $A_\alpha \sqcup^E \bar{E}$. But $A_\alpha \sqcup^E \bar{E} \rightarrow A_\alpha[\bar{E}]$ is a weak equivalence for each α , see (A.1), and $S_1 \otimes \dots \otimes S_n = \operatorname{colim}_\alpha A_\alpha[\bar{E}]$ is again the homotopy colimit of the corresponding Reedy cofibrant diagram. Therefore, the weak equivalences $A_\alpha \sqcup^E \bar{E} \rightarrow A_\alpha[\bar{E}]$ for different α yield the one in the lemma. \square

A.3 | Segal condition

Consider trees S_1, \dots, S_n and a further tree T . Let d be an inner edge in T . Cutting the tree T at d results in two trees T^d and T_d of which $T = T^d \circ_d T_d$ is a grafting.

Proposition A.3.1. The map

$$S_1 \otimes \dots \otimes S_n \otimes (T^d \cup T_d) \rightarrow S_1 \otimes \dots \otimes S_n \otimes T$$

is a weak equivalence in the operadic model structure on \mathbf{dSet} .

Proof. The claim is known to hold if all the trees are open, see [7], Lemma 6.3.5. The general case immediately follows from the lemma above, at least if d itself is not a stump in T . If it is, $T_d = \bar{\eta}$ and $T = T^d[\bar{\eta}]$, and the map in the proposition is

$$S_1 \otimes \dots \otimes S_n \otimes (T^d \sqcup^d \bar{d}) \rightarrow S_1 \otimes \dots \otimes S_n \otimes T.$$

But in this case

$$S_1 \otimes \dots \otimes S_n \otimes (T^d \sqcup^d \bar{d}) \cong (S_1 \otimes \dots \otimes S_n \otimes T^d) \sqcup^D \bar{D},$$

where $D = \{(e_1, \dots, e_n, d) \mid e_i \text{ is a leaf in } S_i\}$, and the proposition becomes a special case of the lemma again. \square

A.4 | Associativity

The tensor product of dendroidal sets is not associative. For example, consider trees R, S , and T . If $S \otimes T$ is a union of shuffles, say, $S \otimes T = \bigcup A_j$, then $R \otimes (S \otimes T) = \bigcup R \otimes A_j$ is a union of only a subset of shuffles making up $R \otimes S \otimes T$, see [8, section 4.4]. More generally, if $S_1 \otimes \dots \otimes S_n = \bigcup_{j \in J} A_j$ as above, then the map

$$S_1 \otimes \dots \otimes (S_i \otimes \dots \otimes S_j) \otimes \dots \otimes S_n \longrightarrow S_1 \otimes \dots \otimes S_n \tag{A.3}$$

is an inclusion of the form $\bigcup_{k \in K} A_k \rightarrow \bigcup_{j \in J} A_j$ where $K \subseteq J$.

Proposition A.4.1. *Let S_1, \dots, S_n be trees. Then for any $1 \leq i < j \leq n$ the map (A.3) is a weak equivalence, and similarly for more nested bracketings.*

Proof. This map is a weak equivalence (in fact, inner anodyne) if all the S_i are open, and the same holds for more nested bracketings, see [7, Lemma 6.3.6]. So for general trees S_1, \dots, S_n , the map

$$S_1^\circ \otimes \dots \otimes (S_i^\circ \otimes \dots \otimes S_j^\circ) \otimes \dots \otimes S_n^\circ \longrightarrow S_1^\circ \otimes \dots \otimes S_n^\circ$$

is a weak equivalence. As before, we can write this map as $\operatorname{colim} A_\beta \rightarrow \operatorname{colim} A_\alpha$ where $\beta \subseteq K$ and $\alpha \subseteq J$ are nonempty subsets, and the colimits are the colimits of Reedy cofibrant diagrams. Taking the pushout along $\sqcup_E \eta \rightarrow \sqcup_E \bar{\eta}$ for E as before yields the top map in the diagram

$$\begin{array}{ccc} \operatorname{colim} A_\beta \sqcup^E \bar{E} & \xrightarrow{\sim} & \operatorname{colim} A_\alpha \sqcup^E \bar{E} \\ \downarrow & & \downarrow \\ \operatorname{colim} A_\beta[\bar{E}] & \longrightarrow & \operatorname{colim} A_\alpha[\bar{E}] \end{array} \quad (\text{A.4})$$

The vertical maps are colimits of grafting weak equivalences of the form (A.1), so the bottom map is a weak equivalence as well. But this is precisely the map in the proposition. \square

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