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ABSTRACT

We consider an infinite system of coupled stochastic differential equations (SDE) describing dynamics of the following infinite particle system. Each particle is characterized by its position $x \in \mathbb{R}^d$ and internal parameter (spin) $\sigma_x \in \mathbb{R}$. While the positions of particles form a fixed ("quenched") locally-finite set (configuration) $\gamma \subset \mathbb{R}^d$, the spins σ_x and σ_y interact via a pair potential whenever $|x-y| < \rho$, where $\rho > 0$ is a fixed interaction radius. The number n_x of particles interacting with a particle in position x is finite but unbounded in x. The growth of n_x as $|x| \to \infty$ creates a major technical problem for solving our SDE system. To overcome this problem, we use a finite volume approximation combined with a version of the Ovsjannikov method, and prove the existence and uniqueness of the solution in a scale of Banach spaces of weighted sequences. As an application example, we construct stochastic dynamics associated with Gibbs states of our particle system.

I. INTRODUCTION

In recent decades, there has been an increasing interest in studying countable systems of particles randomly distributed in the Euclidean space $\mathbb{R}^d$. In such systems, each particle is characterized by its position $x \in X := \mathbb{R}^d$ and an internal parameter (spin) $\sigma_x \in S := \mathbb{R}^n$, see for example Refs. 34, 32 (Sec. 11), 7, 14, and 15 pertaining to modeling of non-crystalline (amorphous) substances, e.g., ferrofluids and amorphous magnets. Throughout the paper we suppose, mostly for simplicity, that $n = 1$.

Let us denote by $\Gamma(X)$ the space of all locally finite subsets (configurations) of X and consider a particle system with positions forming a given fixed ("quenched") configuration $\gamma \in \Gamma(X)$. Two spins σ_x and σ_y, $x, y \in \gamma$, are allowed to interact via a pair potential if the distance between x and y is no more than a fixed interaction radius $\rho > 0$, that is, they are neighbors in the geometric graph defined by γ and ρ. The evolution of spins is described then by a system of coupled stochastic differential equations (SDE).

Namely, we consider, for a fixed $\gamma \in \Gamma(X)$, a system of stochastic differential equations in $S = \mathbb{R}$ of the following form:

$$d\xi_{x,t} = \Phi_x(\Xi_t)dt + \Psi_x(\Xi_t)dW_{x,t}, \qquad x \in \gamma, \ t \geq 0, \tag{1.1}$$

where $\Xi_t = (\xi_{x,t})_{x \in \gamma}$ and $(W_{x,t})_{x \in \gamma}$ are, respectively, families of real-valued stochastic processes and independent Wiener processes on a suitable probability space. Here the drift and diffusion coefficients Φ_x and Ψ_x are real-valued functions, defined on the Cartesian power $S^\gamma := \{\bar{\sigma} = (\sigma_x)_{x \in \gamma} | \sigma_x \in S, x \in \gamma\}$. Both Φ_x and Ψ_x are constructed using pair interaction between the particles and their self-interaction potentials, see Sec. II, and are independent of σ_y if $|y - x| > \rho$.

The aim of including the diffusion term in (1.1) is two-fold. On the one hand, it allows to consider the influence of random forces on our particle system and, on the other hand, to construct and study stochastic dynamics associated with the equilibrium (Gibbs) states of the system. The Gibbs states of spin systems on unbounded degree graphs have been studied in Refs. 14, 15, and 27, see also references given there.

The case where vertex degrees of the graph are globally bounded (in particular, if γ has a regular structure, e.g., $\gamma = \mathbb{Z}^d$) has been well-studied (in both deterministic and stochastic cases), see e.g., Refs. 2–5, 17, 19, 20, 23, 24, 28, 29, 37, and 39, and references therein. However, the aforementioned applications to non-crystalline substances require dealing with unbounded vertex degree graphs. An important example of such graphs is served by configurations γ distributed according to a Poisson or, more generally, Gibbs measure on $\Gamma(X)$ with a superstable low regular interaction energy, in which case the typical number of "neighbors" of a particle located at $x \in X$ is proportional to $\sqrt{1+\log|x|}$, see e.g., Refs. 38 and 26 (p. 1047).

There are two main technical difficulties in the study of system (1.1). The first one is related to the fact that the number of particles interacting with a tagged particle x is finite but unbounded in $x \in \gamma$. Consequently, the system cannot be considered an equation in a fixed Banach space and studied by standard methods of e.g., Refs. 12 and 16.

The way around it has been proposed in Ref. 11, where a deterministic version of system (1.1) (with $\Psi \equiv 0$) was considered in an expanding scale of embedded Banach spaces of weighted sequences and solved using a version of the Ovsjannikov method.

Originally, the Ovsjannikov method was developed for a linear equation

$$\dot{X}_t = AX_t \tag{1.2}$$

in a scale of densely embedded Banach spaces B_α, $\alpha \in \mathscr{A}$, where $\mathscr{A}$ is a real interval, such that $B_\alpha \subset B_\beta$ if $\alpha < \beta$, and $A\colon B_\alpha \to B_\beta$ is bounded with norm satisfying the estimate

$$\|A\|_{\alpha,\beta} \le L(\beta-\alpha)^{-q}, \;\; \alpha < \beta, \;\; q = 1, \tag{1.3}$$

for any $\alpha, \beta \in \mathscr{A}$. Then, for $X_0 \in B_\alpha$, Eq. (1.2) has a solution $X_t \in B_\beta$, $t < T$, for finite T depending on α and β.

It was noticed in Ref. 11 that, under a stronger norm bound with $q < 1$ in (1.3), the lifetime of the solution $X_t \in B_\beta$ is infinite. That fact allows to find a global uniform bound for a sequence of finite volume approximations of the system of differential equations in question and prove its convergence, thus proving the existence and uniqueness of the global solution of the deterministic version of (1.1).

The first advances in the study of stochastic equations in the scale $\{B_\alpha, \alpha \in \mathscr{A}\}$, were made in Refs. 10 and 8, where, respectively, local and global strong solutions of a general stochastic equation had been constructed. In those works, the coefficients are assumed to be Lipschitz mappings $B_\alpha \to B_\beta$ for any $\alpha < \beta$, with Lipschitz constants $L(\beta-\alpha)^{-q}$, $q = \frac{1}{2}$ and $q < \frac{1}{2}$, respectively. Observe that the threshold value of q here is $\frac{1}{2}$ instead of 1 as in (1.3) because of the presence of the Itô integral, which makes it necessary to work in L^2 spaces instead of L^1.

The results of Refs. 10 and 8 are applicable to system (1.1) only in the case where the drift coefficients $\Phi_x, x \in \gamma$, are globally Lipschitz. However, to construct the dynamics associated with Gibbs states of interacting particle systems, one has to consider the drift coefficients that are only locally Lipschitz. The existence of such dynamics, under certain dissipativity conditions on the drift, is known in the situation of a regular lattice, see Refs. 4 and 5 (observe that those works deal with the more complicated quantum systems but are applicable to classical systems, too, albeit only for the additive noise).

For deterministic systems on unbounded degree graphs, the dissipative case was considered in the aforementioned paper.[11] In the present work, we revisit the volume approximation approach of that paper. However, the presence of stochastic terms requires the application of very different techniques. To prove the convergence of finite volume approximations, we have developed a version of the Gronwall inequality suitable for a scale of Banach spaces. In this way, we have been able to prove the existence and uniqueness of global strong solutions of (1.1) and their component-wise time continuity, in the case of dissipative single-particle potentials.

The structure of the paper is as follows. In Sec. II we introduce the framework and formulate our main results. Section III is devoted to the proof of the existence and uniqueness result for (1.1). In a short Sec. III D, we discuss Markov semigroup generated by the solution of (1.1). In Sec. IV, we study stochastic dynamics associated with Gibbs states of our system.

Finally, the Appendix contains auxiliary results on linear operators in the scales of Banach spaces, estimates of the solutions of system (1.1) and, notably, a crucial for our techniques generalization of the classical comparison theorem and a Gronwall-type inequality, suitable for our framework.

II. THE SETUP AND MAIN RESULTS

Let us fix a configuration $\gamma \in \Gamma(X)$ and a family $(W_{x,t})_{x\in\gamma}$ of independent Wiener processes on a suitable filtered and complete probability space $\mathbf{P} := (\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$. Our aim is to find a strong solution of SDE system (1.1), that is, a family $\Xi_t = (\xi_{x,t})_{x\in\gamma}$ of continuous adapted stochastic processes on $\mathbf{P}$ such that the equality

$$\xi_{x,t} = \zeta_x + \int_0^t \Phi_x(\Xi_s)ds + \int_0^t \Psi_x(\Xi_s)dW_{x,s}, \;\; x \in \gamma, \;\; \zeta_x \in S, \tag{2.1}$$

holds for all $t \in \mathscr{T} := [0,T]$, $T > 0$, almost surely, that is, on a common for all t set of probability 1. The coefficients Φ_x and Ψ_x are defined explicitly in Assumption II.1 below, and $\int_0^t \Psi_x(\Xi_s)dW_{x,s}$ is the continuous version of the Ito integral, cf. Remark II.4.

First, we need to introduce some notations. We fix $\rho > 0$ and denote by n_x, $x \in \gamma$, the number of elements in the set

$$\bar{\gamma}_x := \{y \in \gamma : |x-y| \le \rho\}.$$

Observe that $n_x \ge 1$ for all $x \in \gamma$, because $x \in \bar{\gamma}_x$. We will also use the notation $\gamma_x := \bar{\gamma}_x \backslash \{x\} \equiv \{y \in \gamma : |x-y| \le \rho, y \ne x\}$.

For a fixed $\gamma \in \Gamma(X)$, we will consider the Cartesian product S^{γ} of identical copies S_x, $x \in \gamma$, of S, and denote its elements by $\bar{z} := (z_x)_{x\in\gamma}$, etc. When dealing with multiple configurations $\eta \in \Gamma(X)$, we will sometimes write $\bar{z}_\eta := (z_x)_{x\in\eta}$, to emphasize the dependence on η.

We will work under the following assumption.

Assumption II.1

(A) There exists a constant $C > 0$ such that

$$n_x \le C(1 + \log(1 + |x|)) \quad \text{for all } x \in \gamma. \tag{2.2}$$

(B) The drift coefficients $\Phi_x, x \in \gamma$, have the form

$$\Phi_x(\bar{z}) := \phi(z_x) + \sum_{y\in\bar{\gamma}_x} \varphi_{x,y}(z_x, z_y), \quad \text{for all } x \in \gamma, \tag{2.3}$$

where $\phi : S \to S$ is a measurable function and $\varphi_{xy} : S^2 \to S$ are also measurable functions satisfying uniform Lipschitz condition

$$|\varphi_{xy}(\sigma_1, s_1) - \varphi_{xy}(\sigma_2, s_2)| \le \bar{a}(|\sigma_1 - \sigma_2| + |s_1 - s_2|),$$
$$|\varphi_{xy}(\sigma_1, s_1))| \le \bar{a}(1 + |\sigma_1| + |s_1|),$$

for some constant $\bar{a} > 0$ and all $x, y \in \gamma, \sigma_1, \sigma_2, s_1, s_2 \in S$.

(C) There exist constants $c > 0$ and $R \ge 2$ such that

$$|\phi(\sigma)| \le c(1 + |\sigma|^R), \ \sigma \in S. \tag{2.4}$$

(D) There exists $b > 0$ such that

$$(\sigma_1 - \sigma_2)(\phi(\sigma_1) - \phi(\sigma_2)) \le b(\sigma_1 - \sigma_2)^2, \ \sigma_1, \sigma_2 \in S. \tag{2.5}$$

(E) The diffusion coefficients $\Psi_x, x \in \gamma$, have the form

$$\Psi_x(\bar{z}) := \sum_{y\in\bar{\gamma}_x} \psi_{xy}(z_x, z_y) \quad \text{for all } x \in \gamma, \tag{2.6}$$

where $\psi_{xy} : S^2 \to S$ are measurable functions satisfying uniform Lipschitz condition

$$|\psi_{xy}(\sigma_1, s_1) - \psi_{xy}(\sigma_2, s_2)| \le M(|\sigma_1 - \sigma_2| + |s_1 - s_2|),$$
$$|\psi_{xy}(\sigma_1, s_1))| \le M(1 + |\sigma_1| + |s_1|), \tag{2.7}$$

for some constant $M > 0$ and all $x, y \in \gamma, \sigma_1, \sigma_2, s_1, s_2 \in S$.

The specific form of the coefficients requires the development of a special framework. Indeed, we will be looking for a solution of (2.1) in a scale of expanding Banach spaces of weighted sequences, which we introduce below.

We start with a general definition and consider a family $\mathfrak{B}$ of Banach spaces B_α indexed by $\alpha \in \bar{\mathscr{A}} := [\alpha_*, \alpha^*]$ with fixed $0 \le \alpha_*, \alpha^* < \infty$, and denote by $\|\cdot\|_{B_\alpha}$ the corresponding norms. When speaking of these spaces and related objects, we will always assume that the range of indices is $[\alpha_*, \alpha^*]$, unless stated otherwise. The interval $\bar{\mathscr{A}}$ remains fixed for the rest of this work. We will also use the corresponding semi-open interval $\mathscr{A} := [\alpha_*, \alpha^*)$.

Definition II.2 The family $\mathfrak{B}$ is called a scale if

$$B_\alpha \subset B_\beta \ \textit{and}\ \|u\|_{B_\beta} \le \|u\|_{B_\alpha} \ \textit{for any}\ \alpha < \beta, \ u \in B_\alpha, \ \alpha, \beta \in \bar{\mathscr{A}},$$

where the embedding means that B_α is a dense vector subspace of B_β.

For any $\alpha, \beta \in \mathscr{A}$, we will use the notation

$$B_{\alpha+} := \bigcap_{\beta>\alpha} B_\beta.$$

The two main scales we will be working with are given by the spaces l^p_α of weighted sequences and l^p_α-valued random processes, respectively, defined as follows.

(1) For all $p \ge 1$ and $\alpha \in \bar{\mathscr{A}}$ let

$$l^p_\alpha := \left\{ \bar{z} \in S^\gamma \ \middle|\ \|\bar{z}\|_{l^p_\alpha} := \left(\sum_{x\in\gamma} w(x)^{-1} |z_x|^p \right)^{\frac{1}{p}} < \infty \right\}, \tag{2.8}$$

$$w(x) = e^{a|x|},$$

and $\mathscr{L}^p := \{l^p_\alpha\}_{\alpha\in\mathscr{A}}$ be, respectively, a Banach space of weighted real sequences and the scale of such spaces.

(2) For all $p \ge 1$ and $\alpha \in \bar{\mathscr{A}}$ let $\mathscr{R}^p_\alpha$ denote the Banach space of l^p_α-valued random processes $\bar{\xi}_t$, $t \in \mathscr{T}$, on probability space **P**, with progressively measurable components and finite norm

$$\|\bar{\xi}\|_{\mathscr{R}^p_\alpha} := \left(\sup\left\{\mathbb{E}\|\bar{\xi}_t\|^p_{l^p_\alpha} \mid t \in \mathscr{T}\right\}\right)^{\frac{1}{p}} < \infty,$$

and let $\mathscr{R}^p := \{\mathscr{R}^p_\alpha\}_{\alpha\in\mathscr{A}}$ be the scale of such spaces.

Remark II.3 The choice of exponential weights in the definition of space l^p_α is dictated by the logarithmic growth condition on numbers n_x, cf. (2.2), which in turn is motivated by the fact that it holds for a typical configuration γ distributed according to a Poisson or, more generally, Gibbs measure on $\Gamma(X)$ with a superstable low regular interaction energy, in which case n_x is proportional to $\sqrt{1+\log|x|}$, see e.g., Refs. 38 and 26 (p. 1047). In general, an informal balance condition between n_x and $w(|x|)$ is given by $w(|x|) \approx \exp(\exp(n_x))$, see Sec. 2.2 of Ref. 11 for details.

Remark II.4. Note that for $p \ge 2$, the definition of norms in $\mathscr{R}^p_\alpha$ and l^p_α implies that for any $\bar{\xi} \in \mathscr{R}^p_\alpha$ and any $x \in \gamma$ we have $\mathbb{E}[\int_0^T \xi^2_{x,t} dt] < \infty$. Moreover, since each component of $\bar{\xi}$ is progressively measurable, from the classical theory of integration with respect to the standard Wiener process we see that for all $x \in \gamma$ the integral $\int_0^t \xi_{x,s} dW_s$ is well defined and so is the integral $\int_0^t \Psi_x(\bar{\xi}_s) dW_{x,s}$, because Ψ_x is a finite sum of measurable uniformly Lipschitz functions. Moreover, the process $\int_0^t \Psi_x(\bar{\xi}_s) dW_{x,s}$, $t \in \mathscr{T}$, has a (unique) continuous version.

For all $p \ge 1$ and $\alpha \in \bar{\mathscr{A}}$ we let

$$L^p_\alpha \equiv L^p(\Omega, l^p_\alpha) := \left\{X : \Omega \to l^p_\alpha \mid (\mathbb{E}[\|X\|^p_{l^p_\alpha}])^{\frac{1}{p}} < \infty\right\}$$

be the space of l^p_α-valued p-integrable random variables.

Our main result is the following theorem.

Theorem II.5. *Suppose that Assumption II.1 holds. Then, for all $p \ge R$ and any $\mathscr{F}_0$-measurable $\bar{\zeta} := (\zeta_x)_{x\in\gamma} \in L^p_\alpha$, $\alpha \in \mathscr{A}$, stochastic system (2.1) admits a unique (up to indistinguishability) strong solution $\Xi \in \mathscr{R}^p_{\alpha+}$. Moreover, the map*

$$L^p_\alpha \ni \bar{\zeta} \mapsto \Xi \in \mathscr{R}^p_\beta$$

is continuous for any $\beta > \alpha$.

Remark II.6. Assumption $p \ge R$ ensures that given $\bar{\xi} \in \mathscr{R}^p_\beta$ the random variable $\phi(\bar{\xi}_t)$ is integrable for any $t \ge 0$.

The proof of Theorem II.5 will be given in Sec. III.

Our second main result is about the construction of non-equilibrium stochastic dynamics associated with Gibbs states of our system. We consider a Gibbs measure ν on S^γ defined by the pair interaction $W_{x,y}(\sigma_x, \sigma_y) = a(x-y)\sigma_x\sigma_y$, $\sigma_x, \sigma_y \in S$, $x, y \in \gamma$, where $a : X \to \mathbb{R}$ is a measurable function with compact support and a single particle potential $V : \mathbb{R} \to \mathbb{R}$ satisfying the lower bound

$$V(\sigma) \ge a_V|\sigma|^{R+\varepsilon} - b_V,\ \sigma \in S, \qquad \text{for some } a_V, b_V > 0 \text{ and } \varepsilon > 0,$$

which is supported on l^p_α for some $\alpha \in \mathscr{A}$ and $p \in [R, R+\varepsilon]$, see Sec. IV A for details. Suppose now that ϕ in (2.3) has a gradient form, that is, $\phi = -\nabla V$, and $\begin{cases} \psi_{xy} = 0,\ x \ne y \\ \psi_{xx} = 1 \end{cases}$ for all $x, y \in \gamma$, so that our noise is additive, cf. (2.6). Let T_t be the Markov semigroup defined by the process Ξ_t in a standard way. This semigroup acts in the space $C_b(l^p_{\alpha+})$ of bounded continuous functions on space $l^p_{\alpha+} = \cap_{\beta>\alpha} l^p_\beta$ equipped with the projective limit topology, see Sec. III D below for details.

Theorem II.7. *Gibbs measure ν is a symmetrizing (reversible) distribution for the solution of (2.1), that is,*

$$\int \mathrm{T}_t f(\bar{\zeta}) g(\bar{\zeta}) \nu(d\bar{\zeta}) = \int f(\bar{\zeta}) \mathrm{T}_t g(\bar{\zeta}) \nu(d\bar{\zeta}),\ t \ge 0,$$

for any $\alpha \in \mathscr{A}$ *and* $f, g \in C_b(l^p_{\alpha+})$.

The proof of this result will be given in Sec. IV B.

From now on, the constant $p \geq R$ will be fixed.

III. EXISTENCE, UNIQUENESS AND PROPERTIES OF THE SOLUTION

In this section, we give the proof of Theorem II.5. It will go along the following lines.

(1) Consider a sequence of processes $\{\Xi^n_t\}_{n\in\mathbb{N}}$, $t \in \mathscr{T}$, that solve finite volume cutoffs of system (2.1), and prove their uniform bound in $\mathscr{R}^p_\beta$ for any $\beta > \alpha$. For this, we use our version of the comparison theorem and Gronwall-type inequality in the scale of spaces, which is in turn based on the Ovsjannikov method, see Subsection 3 of the Appendix.

(2) The uniform bound above implies the convergence of sequence Ξ^n, $n \to \infty$, to a process $\Xi = (\xi_x)_{x\in\gamma} \in \mathscr{R}^p_\beta$, $\beta > \alpha$. Our next goal is to prove that the process Ξ solves system (2.1). The multiplicative noise term does not allow to achieve this by a direct limit transition. Therefore, we construct an $\mathbb{R}$-valued process η_t that solves an equation describing the dynamics of a tagged particle x, while processes $\xi_{y,t}$, $y \in \gamma, y \neq x$, are fixed, and prove that $\eta_t = \xi_{x,t}$.

(3) The uniqueness and continuous dependence on the initial data is proved by using our version of a Gronwall-type inequality, as above in part (1). The continuity of components of Ξ will follow from our work on the dynamics of a tagged particle x in Sec. III B.

Finally, in Subsection III D, we introduce Markov semigroup defined by the solution of (2.1).

A. Truncated system

Let us fix an expanding sequence $\{\Lambda_n\}_{n\in\mathbb{N}}$ of finite subsets of γ such that $\Lambda_n \uparrow \gamma$ as $n \to \infty$ and consider the following system of equations:

$$\begin{aligned} \xi^n_{x,t} &= \zeta_x + \int_0^t \Phi_x(\Xi^n_s)ds + \int_0^t \Psi_x(\Xi^n_s)dW_{x,s}, \quad x \in \Lambda_n, \\ \xi^n_{x,t} &= \zeta_x, \quad x \notin \Lambda_n, \quad t \in \mathscr{T}, \end{aligned} \tag{3.1}$$

where $\bar{\zeta} = \{\zeta_x\}_{x\in\gamma} \in L^p_\alpha$, $\alpha \in \mathscr{A}$, is $\mathscr{F}_0$-measurable random initial condition and equality (3.1) holds for all $t \in \mathscr{T}$, $\mathbb{P}$-a.s. Observe that for each $n \in \mathbb{N}$ system (3.1) is a truncated version of our original stochastic system (2.1).

Theorem III.1. *For any* $n \in \mathbb{N}$ *system (3.1) admits a unique (up to indistinguishability) solution* $\Xi^n \in \mathscr{R}^p_\alpha$ *with continuous sample paths.*

Proof. The existence and uniqueness of continuous strong solutions of the non-trivial finite dimensional part of system (3.1) is well-known, see Ref. 30, Chap. 3. The inclusion $\Xi^n \in \mathscr{R}^p_\alpha$ follows then from the fact that $\xi^n_{x,t} = \zeta_x$, $t \in \mathscr{T}$, for $x \notin \Lambda_n$. ■

Our next goal is to show that the sequence $\{\Xi^n\}_{n\in\mathbb{N}}$ converges in $\mathscr{R}^p_\beta$ for any $\beta > \alpha$. We start with the following uniform estimate, which is rather similar to the one from Ref. 27, adapted to the framework of the scale of Banach spaces using our version of the Gronwall inequality.

Theorem III.2. *Let* $\Xi^n = (\xi^n_x)_{x\in\gamma}$, $n \in \mathbb{N}$, *be the sequence of process defined by Theorem III.1. Then for all* $\beta > \alpha$ *we have*

$$\sum_{x\in\gamma} e^{-\beta|x|} \sup_{n\in\mathbb{N}} \sup_{t\in\mathscr{T}} \mathbb{E}[|\xi^n_{x,t}|^p] < \infty. \tag{3.2}$$

Proof It follows from the first part of Lemma A.10 in the Appendix (with $\xi^1 \equiv \xi^n$) that for all $x \in \Lambda_n$ and $t \in \mathscr{T}$ we have

$$\mathbb{E}[|\xi^n_{x,t}|^p] \leq \mathbb{E}|\zeta_x|^p + C_1 n_x^2 \sum_{y\in\bar{\gamma}_x} \int_0^t \mathbb{E}[|\xi^n_{y,s}|^p]ds + C_2^x. \tag{3.3}$$

We remark that inequality above trivially holds for $x \notin \Lambda_n$, because in this case $\xi^n_{x,t} = \zeta_x$ and all terms in the right-hand side of the inequality are non-negative.

We now define a measurable map $\eta^n : \mathscr{T} \to l^1_\alpha$ via the following formula

$$\eta^n_x(t) := \max_{m\leq n} \mathbb{E}[|\xi^m_{x,t}|^p], \quad \forall (t \in \mathscr{T}).$$

It is immediate that its components satisfy inequality similar to (3.3), that is,

$$\eta^n_x(t) \leq \mathbb{E}|\zeta_x|^p + C_1 n_x^2 \sum_{y\in\bar{\gamma}_x} \int_0^t \eta^n_y(s)ds + C_2^x.$$

Set $\theta_x = \mathbb{E}|\zeta_x|^p + C_2^x$ and observe that $(\theta_x)_{x\in\gamma} \in l_\alpha^1$. Then the map η^n fulfills the conditions of Lemma A.8 in the Appendix, which implies that for all $n \in \mathbb{N}$ and $\beta > \alpha$ we have

$$\sum_{x\in\gamma} e^{-\beta|x|} \sup_{t\in\mathcal{T}} \eta_x^n(t) \le K_T(\alpha,\beta) \sum_{x\in\gamma} e^{-\alpha|x|}\theta_x < \infty.$$

Observe that the left-hand side forms an increasing sequence, which implies that it converges and

$$\lim_{n\to\infty} \sum_{x\in\gamma} e^{-\beta|x|} \sup_{t\in\mathcal{T}} \eta_x^n(t) \le K_T(\alpha,\beta) \sum_{x\in\gamma} e^{-\alpha|x|}\theta_x < \infty.$$

Then, for any finite set $\eta \subset \gamma$, we have

$$\sum_{x\in\eta} e^{-\beta|x|} \lim_{n\to\infty} \sup_{t\in\mathcal{T}} \eta_x^n(t) = \lim_{n\to\infty} \sum_{x\in\eta} e^{-\beta|x|} \sup_{t\in\mathcal{T}} \eta_x^n(t) \le K_T(\alpha,\beta) \sum_{x\in\gamma} e^{-\alpha|x|}\theta_x.$$

On the other hand, it is clear that

$$\lim_{n\to\infty} \eta_x^n(t) = \sup_{n\in\mathbb{N}} \max_{m\le n} \sup_{t\in\mathcal{T}} \mathbb{E}[|\xi_{x,t}^m|^p] = \sup_{n\in\mathbb{N}} \sup_{t\in\mathcal{T}} \mathbb{E}[|\xi_{x,t}^n|^p]$$

for any $x \in \gamma$. Thus

$$\sum_{x\in\eta} e^{-\beta|x|} \sup_{t\in\mathcal{T}} \sup_{n\in\mathbb{N}} \mathbb{E}[|\xi_{x,t}^n|^p] \le K_T(\alpha,\beta) \sum_{x\in\gamma} e^{-\alpha|x|}\theta_x.$$

The latter inequality holds for all finite $\eta \subset \gamma$, which implies that

$$\sum_{x\in\gamma} e^{-\beta|x|} \sup_{t\in\mathcal{T}} \sup_{n\in\mathbb{N}} \mathbb{E}[|\xi_{x,t}^n|^p] \le K_T(\alpha,\beta) \sum_{x\in\gamma} e^{-\alpha|x|}\theta_x,$$

and the proof is complete. ■

Theorem III.3. *The sequence $\{\Xi^n\}_{n\in\mathbb{N}}$ is Cauchy in $\mathcal{R}_\beta^p$ for any $\beta > \alpha$.*

Proof. Let us fix $n, m \in \mathbb{N}$ and assume, without loss of generality, that $\Lambda_n \subset \Lambda_m$. We first consider the situation where $x \in \Lambda_n$. It follows from the second part of Lemma A.10 in the Appendix (with $\xi^{(1)} \equiv \xi^n$ and $\xi^{(2)} \equiv \xi^m$) that for all $x \in \Lambda_n$ and $t \in \mathcal{T}$ we have

$$\mathbb{E}|\bar{\xi}_{x,t}^{n,m}|^p \le B n_x^2 \sum_{y\in\bar{\gamma}_x} \int_0^t \mathbb{E}|\bar{\xi}_{y,s}^{n,m}|^p ds, \tag{3.4}$$

$$\bar{\xi}_{x,t}^{n,m} = \xi_{x,t}^n - \xi_{x,t}^m. \tag{3.5}$$

In the case where $x \in \Lambda_m \backslash \Lambda_n$ we see that for all $t \in \mathcal{T}$

$$|\bar{\xi}_{x,t}^{n,m}|^p \le (|\xi_{x,t}^n| + |\xi_{x,t}^m|)^p \le 2^{p-1}|\xi_{x,t}^n|^p + 2^{p-1}|\xi_{x,t}^m|^p,$$

so that

$$\mathbb{E}[|\bar{\xi}_{x,t}^{n,m}|^p] \le 2^p \sup_{n\in\mathbb{N}} \mathbb{E}[|\xi_{x,t}^n|^p] \le 2^p 1_{\Lambda_m\backslash\Lambda_n}(x) \sup_{n\in\mathbb{N}} \sup_{t\in\mathcal{T}} \mathbb{E}[|\xi_{x,t}^n|^p] < \infty \tag{3.6}$$

(cf. Theorem III.2). Combining Eqs. (3.4) and (3.6) and taking into account that $\bar{\xi}_{x,t}^{n,m} = 0$ for $x \notin \Lambda_m$, we obtain the inequality

$$\mathbb{E}[|\bar{\xi}_{x,t}^{n,m}|^p] \le B_1 n_x^2 \sum_{y\in\bar{\gamma}_x} \int_0^t \mathbb{E}[|\bar{\xi}_{y,s}^{n,m}|^p] ds + 2^p 1_{\Lambda_m\backslash\Lambda_n}(x) \sup_{n\in\mathbb{N}} \sup_{t\in\mathcal{T}} \mathbb{E}[|\xi_{x,t}^n|^p]$$

for all $x \in \gamma$ and $t \in \mathcal{T}$. We can now proceed as in the Proof of Theorem III.2. Define a measurable map $\varrho^{n,m} : \mathcal{T} \to l_\alpha^1$ via the formula

$$\varrho_x^{n,m}(t) := \mathbb{E}[|\bar{\xi}_{x,t}^{n,m}|^p], \quad t \in \mathcal{T},$$

and set

$$b_x = 2^p 1_{\Lambda_m\backslash\Lambda_n}(x) \sup_{n\in\mathbb{N}} \sup_{t\in\mathcal{T}} \mathbb{E}[|\xi_{x,t}^n|^p].$$

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Obviously, $(b_x)_{x\in\gamma} \in l^1_{\alpha'}$ for any fixed $\alpha' \in (\alpha,\beta)$. It therefore follows then from Lemma A.8 in the Appendix that

$$\sum_{x\in\gamma} e^{-\beta|x|}\sup_{t\in\mathcal{T}}\rho_x^{n,m}(t) \le K_T(\alpha',\beta)\sum_{x\in\gamma} e^{-\alpha'|x|}b_x.$$

So we have shown that the following inequality holds:

$$\begin{aligned}\|\Xi^n - \Xi^m\|^p_{\mathcal{R}^p_\beta} &\le 2^p K_T(\alpha',\beta)\sum_{x\in\Lambda_m\setminus\Lambda_n} e^{-\alpha'|x|}\sup_{n\in\mathbb{N}}\sup_{t\in\mathcal{T}}\mathbb{E}[|\xi^n_{x,t}|^p] \\ &\le 2^p K_T(\alpha',\beta)\sum_{x\in\gamma\setminus\Lambda_n} e^{-\alpha'|x|}\sup_{n\in\mathbb{N}}\sup_{t\in\mathcal{T}}\mathbb{E}[|\xi^n_{x,t}|^p]. \end{aligned}\tag{3.7}$$

It follows from Theorem III.2 that the right hand side of (3.7) is the remainder of the convergent series (3.2) (with α' in place of β), which completes the proof. ∎

B. One dimensional special case

We have shown in Sec. III A that, for any $\beta > \alpha$, the sequence $\{\Xi^n\}_{n\in\mathbb{N}}$ is Cauchy in the Banach space $\mathcal{R}^p_\beta$ and thus converges in this space. So we are now in a position to define the process

$$\Xi := \overbrace{\lim_{n\to\infty}\Xi^n}^{\text{in } \mathcal{R}^p_\beta}.\tag{3.8}$$

This process is a candidate for a solution of the system (2.1). A standard way to show this would be to pass to the limit on both sides of (3.1). This approach requires however somewhat stronger convergence than that in $\mathcal{R}^p_\beta$. We are going to overcome this difficulty by considering special one-dimensional equations.

Consider an arbitrary $x \in \gamma$. It is convenient to consider elements of S^γ as pairs $(\sigma_x, Z^{(x)})$, where $\sigma_x \in S$ and $Z^{(x)} = (z_y)_{y\in\gamma\setminus x} \in S^{\gamma\setminus x}$. In these notations, we can write $\Phi_x(\Xi_s) = \Phi_x(\xi_{x,s}, \Xi_s^{(x)})$ and $\Psi_x(\Xi_s) = \Psi_x(\xi_{x,s}, \Xi_s^{(x)})$, where

$$\Xi^{(x)} := (\xi_y)_{y\in\gamma\setminus x}.\tag{3.9}$$

Let us now fix process Ξ defined by (3.8) and consider the following one-dimensional equation:

$$\eta_{x,t} = \zeta_x + \int_0^t \Phi_x(\eta_{x,s}, \Xi_s^{(x)})ds + \int_0^t \Psi_x(\eta_{x,s}, \Xi_s^{(x)})dW_x(s),\tag{3.10}$$

for all $t \in \mathcal{T}$, $\mathbb{P}$-a.s. The main goal of this section is to prove that the Eq. (3.10) has a unique solution $\eta_{x,t}$.

Remark III.4. Note that, for a fixed $x \in \gamma$, the principal difference between Eqs. (3.10) and (2.1) is that the process Ξ is fixed in (3.10) and defined by the limit (3.8), which makes (3.10) a one-dimensional equation w.r.t. η_x.

In order to establish the existence of a solution of Eq. (3.10) we need the following auxiliary result.

Theorem III.5. *Let $x \in \gamma$ and ξ_x be an x-component of the process Ξ defined by (3.8). Then sample paths of ξ_x are a.s. continuous and*

$$\mathbb{E}[\sup_{t\in\mathcal{T}}|\xi_{x,t}|^p] < \infty.\tag{3.11}$$

Proof. It is sufficient to show that, for a fixed $x \in \gamma$, the sequence $\{\xi^n_x\}_{n\in\mathbb{N}}$, is Cauchy in the norm $\left(\mathbb{E}[\sup_{t\in\mathcal{T}}|\cdot|^p]\right)^{1/p}$ because then there exists a subsequence $\{\xi^{n_k}_x\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty}\sup_{t\in\mathcal{T}}|\xi^{n_k}_{x,t} - \xi_{x,t}| = 0,\ \ \mathbb{P} - a.s.,$$

which, together with the path-continuity of processes $\xi^{n_k}_{x,t}$, implies the statement of the theorem.

Fix $\bar N \in \mathbb{N}$ such that $x \in \Lambda_{\bar N}$ and $n, m \ge \bar N$ and assume, without loss of generality, that $n < m$ so that $x \in \Lambda_n \subset \Lambda_m$ Consider the process $\bar\xi^{n,m}_{x,t}$ defined in (3.5) and proceed as in the Appendix, Lemma A.10, with $\xi^{(1)} \equiv \xi^n$ and $\xi^{(2)} \equiv \xi^m$. Taking $\sup_{t\in\mathcal{T}}$ of both sides of the equality (A17) we obtain the bound

$$\mathbb{E}[\sup_{t\in\mathcal{T}}|\bar\xi^{n,m}_{x,t}|^p] \le K + \mathbb{E}\left[\sup_{t\in\mathcal{T}}\int_0^t p(\bar\xi^{n,m}_{x,s})^{p-1}\Psi^{n,m}_x(s)dW_x(s)\right],\tag{3.12}$$

where

$$K := Bn_x^2 \sum_{y\in\bar{\gamma}_x} \int_0^T \mathbb{E}[|\bar{\xi}^{n,m}_{y,s}|^p]ds \le Bn_x^2 T \sum_{y\in\bar{\gamma}_x} \sup_{t\in\mathcal{T}} \mathbb{E}[|\bar{\xi}^{n,m}_{y,t}|^p] \tag{3.13}$$

and

$$\Psi^{n,m}_x(s) := \Psi_x(\Xi^n_s) - \Psi_x(\Xi^m_s).$$

Now using first the Burkholder–Davis–Gundy inequality (see Ref. 31) and then the Jensen inequality we see that the following estimate on the stochastic term from (3.12) holds.

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in\mathcal{T}}\int_0^t p(\bar{\xi}^{n,m}_{x,s})^{p-1}\Psi^{n,m}_x(s)dW_x(s)\right] &\le \mathbb{E}\left[\left(\int_0^t (p(\bar{\xi}^{n,m}_{x,s})^{p-1}\Psi^{n,m}_x(s))^2 ds\right)^{\frac{1}{2}}\right]\\ &\le \left(\mathbb{E}\left[\int_0^t (p(\bar{\xi}^{n,m}_{x,s})^{p-1}\Psi^{n,m}_x(s))^2 ds\right]\right)^{\frac{1}{2}}.\end{aligned} \tag{3.14}$$

The integrand in the right-hand side of the above inequality can be estimated in a similar way as (A16), so that we obtain

$$((\bar{\xi}^{n,m}_{x,t})^{p-1}\Psi^{n,m}_x(t))^2 \le 2M^2(n_x+1)^2|\bar{\xi}^{n,m}_{x,t}|^{2p} + 2M^2n_x^2\sum_{y\in\bar{\gamma}_x}|\bar{\xi}^{n,m}_{y,t}|^{2p}.$$

It follows now that inequality (3.14) can be written in the following way:

$$\mathbb{E}\left[\sup_{t\in\mathcal{T}}\int_0^t p(\bar{\xi}^{n,m}_{x,s})^{p-1}\Psi^{n,m}_x(s)dW_x(s)\right] \le C_1\sqrt{\sup_{t\in\mathcal{T}}\mathbb{E}[|\bar{\xi}^{n,m}_{x,t}|^{2p}]} + C_2\sqrt{\sum_{y\in\bar{\gamma}_x}\sup_{t\in\mathcal{T}}\mathbb{E}[|\bar{\xi}^{n,m}_{y,t}|^{2p}]},$$

where

$$C_1 := \sqrt{2p^2M^2(1+n_x)^2T} \quad\text{and}\quad C_2 := \sqrt{2p^2M^2n_x^2T}.$$

Therefore returning to inequalities (3.12) and (3.13) we see that

$$\mathbb{E}[\sup_{t\in\mathcal{T}}|\bar{\xi}^{n,m}_{x,t}|^p] \le Bn_x^2T\sum_{y\in\bar{\gamma}_x}\sup_{t\in\mathcal{T}}\mathbb{E}[|\bar{\xi}^{n,m}_{y,t}|^p] + C_1\sqrt{\sup_{t\in\mathcal{T}}\mathbb{E}[|\bar{\xi}^{n,m}_{x,t}|^{2p}]} + C_2\sqrt{\sum_{y\in\bar{\gamma}_x}\sup_{t\in\mathcal{T}}\mathbb{E}[|\bar{\xi}^{n,m}_{y,t}|^{2p}]}. \tag{3.15}$$

Since $\bar{\gamma}_x$ is finite we can now use Theorem III.3 to conclude that, with a suitable choice of $n, m \in \mathbb{N}$, the right hand side of the inequality (3.15) above can be made arbitrary small hence the proof is complete. ■

Theorem III.6. *Equation (3.10) admits a unique solution.*

Proof. By standard arguments, see e.g., Ref. 1, Proposition 2.9, we conclude that Eq. (3.10) admits a unique local maximal solution η_x such that

$$\eta_{x,t\wedge\tau_n} = \zeta_x + \int_0^{t\wedge\tau_n}\Phi_x(\eta_{x,s\wedge\tau_n},\Xi^{(x)}_{s\wedge\tau_n})ds + \int_0^{t\wedge\tau_n}\Psi_x(\eta_{x,s\wedge\tau_n},\Xi^{(x)}_{s\wedge\tau_n})dW_x(s),$$

for all $t\in\mathcal{T}$, $\mathbb{P}$-a.s. Here $\Xi^{(x)}_{s\wedge\tau_n}$ is as in (3.9) and by construction, for all $n\in\mathbb{N}$, stopping time τ_n is the first exit time of η_x from the interval $(-n,n)$, defined as

$$\tau_n = \begin{cases} T, \text{ if } |\eta_{x,t}| < n,\ t\in[0,T] \\ \inf\{t\in[0,T] : |\eta_{x,t}|\ge n\},\ \text{otherwise}\end{cases}$$

Hence to complete the proof it is sufficient to establish that almost surely $\lim_{n\to\infty}\tau_n = T$. We will prove this fact along the lines of Ref. 1, Theorem 3.1, using the bound (3.11). We begin by using the Itô Lemma to establish the equality

$$\begin{aligned}|\eta_{x,t\wedge\tau_n}|^p &= \int_0^{t\wedge\tau_n} p(\eta_{x,s\wedge\tau_n})^{p-1}\Phi_x(\eta_{x,s\wedge\tau_n},\Xi^{(x)}_{s\wedge\tau_n})ds\\ &\quad+\int_0^{t\wedge\tau_n}\frac{p(p-1)}{2}(\eta_{x,s\wedge\tau_n})^{p-2}(\Psi_x(\eta_{x,s\wedge\tau_n},\Xi^{(x)}_{s\wedge\tau_n}))^2ds\\ &\quad+\int_0^{t\wedge\tau_n}p(\eta_{x,s\wedge\tau_n})^{p-1}\Psi_x(\eta_{x,s\wedge\tau_n},\Xi^{(x)}_{s\wedge\tau_n})dW_x(s),\end{aligned}$$

for all $t \in \mathscr{T}$. Before proceeding we define for convenience the following shorthand notations:

$$\begin{aligned}\bar{\Phi}_x^p(\eta,t) &:= (\eta_{x,t\wedge\tau_n})^{p-1}\Phi_x(\eta_{x,t\wedge\tau_n},\Xi_{t\wedge\tau_n}^{(x)}),\\ \bar{\Psi}_x^p(\eta,t) &:= (\eta_{x,t\wedge\tau_n})^{p-2}(\Psi_x(\eta_{x,t\wedge\tau_n},\Xi_{t\wedge\tau_n}^{(x)}))^2,\\ w_x &:= b+\frac{1}{2}+4\bar{a}^2n_x^2,\\ u_x &:= c+\bar{a}n_x.\end{aligned}$$

An application of Lemma A.9 in the Appendix shows that for all $t \in \mathscr{T}$ we have

$$\begin{aligned}\bar{\Phi}_x^p(\eta,t) &\le |\eta_{x,t\wedge\tau_n}|^{p-2}\left(w_x|\eta_{x,t\wedge\tau_n}|^2+\frac{1}{2}\bar{a}^2n_x\sum_{y\in\gamma_x}|\xi_{y,t}|^2+|\eta_{x,t\wedge\tau_n}|u_x\right)\\ &\le w_x|\eta_{x,t\wedge\tau_n}|^p+\frac{1}{2}\bar{a}^2n_x(\eta_{x,t\wedge\tau_n})^{p-2}\sum_{y\in\gamma_x}|\xi_{y,t\wedge\tau_n}|^2+|\eta_{x,t\wedge\tau_n}|^{p-1}u_x\\ &\le (w_x+2^{p-1}u_x)|\eta_{x,t\wedge\tau_n}|^p+\frac{1}{2}\bar{a}^2n_x|\eta_{x,t\wedge\tau_n}|^{p-2}\sum_{y\in\gamma_x}|\xi_{y,t\wedge\tau_n}|^2+2^{p-1}u_x,\end{aligned} \tag{3.16}$$

where constants b and c are defined in Assumption II.1. In the last inequality we used the simple estimate $C^{p-1}\le(1+C)^{p-1}\le(1+C)^p\le 2^{p-1}(1+C^p)$ for any $C>0$, which holds because $p>1$. We can now use the Hölder inequality and classical estimate $(\sum_{k=1}^m a_k)^N \le m^{N-1}\sum_{k=1}^m a_k^N$ (see e.g., Ref. 25) in conjunction with inequality (3.16) above to see that for all $t\in\mathscr{T}$ we have

$$\begin{aligned}\mathbb{E}[\bar{\Phi}_x^p(\eta,t)] &\le (w_x+2^{p-1}u_x)\mathbb{E}[|\eta_{x,t\wedge\tau_n}|^p]\\ &\quad+\frac{1}{2}\bar{a}^2n_x(\mathbb{E}[|\eta_{x,t\wedge\tau_n}|^p])^{\frac{p-2}{p}}\left(\mathbb{E}\left[\left(\sum_{y\in\gamma_x}|\xi_{y,t\wedge\tau_n}|^2\right)^{\frac{p}{2}}\right]\right)^{\frac{2}{p}}+2^{p-1}u_x\\ &\le (w_x+2^{p-1}u_x)\mathbb{E}[|\eta_{x,t\wedge\tau_n}|^p]\\ &\quad+\frac{1}{2}\bar{a}^2n_x(1+\mathbb{E}[|\eta_{x,t\wedge\tau_n}|^p])n_x^{\frac{p-2}{p}}\left(\mathbb{E}\left[\sum_{y\in\gamma_x}|\xi_{y,t\wedge\tau_n}|^p\right]\right)^{\frac{2}{p}}+2^{p-1}u_x.\end{aligned}$$

In a similar way, we obtain the inequality

$$\bar{\Psi}_x^p(\eta,t)\le 3(M^2(n_x+1)^2+M^2n_x^22^{p-1})|\eta_{x,t\wedge\tau_n}|^p+3M^2n_x^2(\eta_{x,t\wedge\tau_n})^{p-2}\sum_{y\in\gamma_x}|\xi_{y,t\wedge\tau_n}|^2+3M^2n_x^22^{p-1}.$$

Setting

$$A_x := \max\left\{\frac{1}{2}\bar{a}^2n_x^{1+\frac{p-2}{p}},\ 3M^2n_x^2\right\}\left(\sum_{y\in\gamma_x}\mathbb{E}[\sup_{t\in\mathscr{T}}|\xi_{y,t}|^p]\right)^{\frac{2}{p}},$$

we get the bounds

$$\mathbb{E}[\bar{\Phi}_x^p(\eta,t)]\le(w_x+2^{p-1}u_x+A_x)\mathbb{E}[|\eta_{x,t\wedge\tau_n}|^p]+A_x+2^{p-1}u_x$$

and

$$\mathbb{E}[\bar{\Psi}_x^p(\eta,t)]\le(3M^2(n_x+1)^2+3M^2n_x^22^{p-1})+A_x)\mathbb{E}[|\eta_{x,t\wedge\tau_n}|^p]+A_x+3M^2n_x^22^{p-1}.$$

Observe that $A_x<\infty$ by Theorem III.5. Finally letting

$$\begin{aligned}D &:= p(w_x+2^{p-1}u_x+A_x)+\frac{p(p-2)}{2}(3M^2(n_x+1)^2+3M^2n_x^22^{p-1}+A_x),\\ K &:= pT(A_x+2^{p-1}u_x)+\frac{p(p-2)}{2}T(A_x+3M^2n_x^22^{p-1}),\end{aligned}$$

we see that for all $t \in [0, \infty)$ we have

$$\mathbb{E}[|\eta_{x,t\wedge\tau_n}|^p] \le D \int_0^t \mathbb{E}[|\eta_{x,s\wedge\tau_n}|^p] ds + K. \tag{3.17}$$

Observe that constants K and D are independent of the stopping time τ_n.

The rest of the proof is standard and can be completed along the lines of Ref. 1, Theorem 3.1. We give its sketch for the convenience of the reader. Using Gronwall's inequality together with the inequality (3.17) above we see that for all $t \in [0, T]$ we have

$$\mathbb{E}[|\eta_{x,t\wedge\tau_n}|^p] \le K e^{Dt}.$$

It follows from the definition of stopping time τ_n that

$$\mathbb{E}[|\eta_{x,t\wedge\tau_n}|^p] \ge n^p \mathbb{P}(\tau_n < t),$$

so that, for all $t \in [0, T]$,

$$\mathbb{P}(\tau_n < t) \le \frac{1}{n^p} K e^{Dt} \to 0, \;\; n \to \infty.$$

Now convergence in probability and the fact that $\{\tau_n\}_{n\in\mathbb{N}}$ is an increasing sequence imply that almost surely $\lim_{n\to\infty} \tau_n = T$, hence the proof is complete. ■

C. Proof of existence and uniqueness

In this section, we are going to prove Theorem II.5. We will show that, for any $\beta > \alpha$, the process

$$\overbrace{\Xi := \lim_{n\to\infty} \Xi^n}^{\text{in } \mathscr{R}^p_\beta} \tag{3.18}$$

solves system (2.1). For this, we will use auxiliary processes η_x constructed in Theorem III.6.

Proof of the existence. According to Theorem III.6, for each $x \in \gamma$ equation

$$\eta_{x,t} = \zeta_x + \int_0^t \Phi_x(\eta_{x,s}, \Xi_s^{(x)}) ds + \int_0^t \Psi_x(\eta_{x,s}, \Xi_s^{(x)}) dW_{x,s}, \quad \text{for all } t \in \mathscr{T}, \; \mathbb{P} - a.s.$$

where $\Xi_s^{(x)}$ is as in (3.9), admits a unique solution $\eta_{x,t}$. Thus it is sufficient to prove that this solution is indistinguishable from the process ξ_x. The convergence (3.18) implies that, for any fixed $x \in \gamma$,

$$\lim_{n\to\infty} \mathbb{E}|\xi^n_{x,t} - \xi_{x,t}|^p = 0, \;\; t \in \mathscr{T}. \tag{3.19}$$

Therefore, taking into account that both processes ξ_x and η_x are continuous, to conclude this proof it remains to show that, for any $t \in \mathscr{T}$,

$$\lim_{n\to\infty} \mathbb{E}|\xi^n_{x,t} - \eta_{x,t}|^p = 0. \tag{3.20}$$

Let us fix $x \in \gamma$ and $t \in \mathscr{T}$ and assume without loss of generality that $x \in \Lambda_n \subset \gamma$. Define the following processes:

$$\begin{aligned} \Phi^n_x(t) &:= \Phi_x(\xi^n_{x,t}, \Xi^n_t) - \Phi_x(\eta_{x,t}, \Xi^{(x)}_t), \\ \Psi^n_x(t) &:= \Psi_x(\xi^n_{x,t}, \Xi^n_t) - \Psi_x(\eta_{x,t}, \Xi^{(x)}_t), \\ \mathscr{X}^n_{x,t} &:= \xi^n_{x,t} - \eta_{x,t}. \end{aligned}$$

The rest of the proof is rather similar to the Proof of Theorem III.6. The Itô Lemma shows that for all $t \in \mathscr{T}$ we have $\mathbb{P}$-a.s.

$$\begin{aligned} |\mathscr{X}^n_{x,t}|^p = &\int_0^t p(\mathscr{X}^n_{x,s})^{p-1} \Phi^n_x(s) ds + \int_0^t \frac{p(p-1)}{2} (\mathscr{X}^n_{x,s})^{p-2} (\Psi^n_x(s))^2 ds + \\ &+ \int_0^t p(\mathscr{X}^n_{x,s})^{p-1} \Psi^n_x(s) dW_x(s). \end{aligned} \tag{3.21}$$

Using Lemma A.10 in the Appendix, we can see that for all $t \in \mathscr{T}$

$$(\mathscr{X}^n_{x,t})^{p-1}\Phi^n_x(t) \le \left(b + \frac{1}{2} + 4\bar{a}^2 n_x^2\right)|\mathscr{X}^n_{x,t}|^p + \bar{a}^2 n_x |\mathscr{X}^n_{x,t}|^{p-2} \sum_{y \in \gamma_x} (\xi^n_{y,t} - \xi_{y,t})^2,$$

and

$$(\mathscr{X}^n_{x,t})^{p-2}\Psi^n_x(t)^2 \le 2M^2(n_x+1)^2|\mathscr{X}^n_{x,t}|^p + 2M^2 n_x |\mathscr{X}^n_{x,t}|^{p-2} \sum_{y \in \gamma_x} (\xi^n_{y,t} - \xi_{y,t})^2.$$

As in the Proof of Theorem III.6, we see that for all $t \in \mathscr{T}$

$$\mathbb{E}[(\mathscr{X}^n_{x,t})^{p-1}\Phi^n_x(t)] \le \left(b + \frac{1}{2} + 4\bar{a}^2 n_x^2 + A^n_x\right)\mathbb{E}[|\mathscr{X}^n_{x,t}|^p] + A^n_x \tag{3.22}$$

and

$$\mathbb{E}[(\mathscr{X}^n_{x,t})^{p-2}(\Psi^n_x(t))^2] \le (2M^2(n_x+1)^2 + A^n_x)\mathbb{E}[|\mathscr{X}^n_{x,t}|^p] + A^n_x, \tag{3.23}$$

where

$$A^n_x := \max\{\bar{a}^2 n_x,\ 2M^2 n_x\}\mathbb{E}\left[\sum_{y \in \gamma_x} (\xi^n_{y,t} - \xi_{y,t})^2\right].$$

Now, because γ_x is finite and $p \ge 2$ it is clear from Eq. (3.19) that

$$\mathbb{E}\sum_{y \in \gamma_x} (\xi^n_{y,t} - \xi_{y,t})^2 \to 0,\ \ n \to \infty,$$

so we see that $A^n_x \to 0$ as $n \to \infty$. Therefore using inequality (3.22) and (3.23) above we can conclude from Eq. (3.21) that for all $x \in \gamma$ and all $t \in \mathscr{T}$ we have

$$\mathbb{E}[|\mathscr{X}^n_{x,t}|^p] \le C^n_x \int_0^t \mathbb{E}[|\mathscr{X}^n_{x,s}|^p]ds + \bar{A}^n_x,$$

where

$$C^n_x := p\left(b + \frac{1}{2} + 4\bar{a}^2 n_x^2 + A^n_x\right) + \frac{p(p-1)}{2}(2M^2(n_x+1)^2 + A^n_x)$$
$$\bar{A}^n_x := pTA^n_x + \frac{p(p-1)}{2}TA^n_x,$$

and consequently $C^n_x, \bar{A}^n_x \to 0$ on $\mathscr{T}$ as $n \to \infty$. Finally using Gronwall inequality we see that for all $t \in \mathscr{T}$ we have

$$\mathbb{E}[|\mathscr{X}^n_{x,t}|^p] \le \bar{A}^n_x e^{(C^n_x)T},$$

which shows that for all $x \in \gamma$ and uniformly on $\mathscr{T}$

$$\lim_{n\to\infty}\mathbb{E}[|\mathscr{X}^n_{x,t}|^p] = 0.$$

Equation (3.20) now follows immediately hence the proof is complete. ■

Proof of the uniqueness and continuous dependence. Suppose that $\Xi^1_t = (\xi^1_{x,t})_{x\in\gamma}$ and $\Xi^2_t = (\xi^2_{x,t})_{x\in\gamma} \in \mathscr{R}^p_{\alpha+}$, are two solutions of system (2.1), with initial values $\Xi^1_0, \Xi^2_0 \in L^p_\alpha$, respectively. Now, for all $t \in \mathscr{T}$ and all $x \in \gamma$ letting $\bar{\xi}_{x,t} = \xi^{(1)}_{x,t} - \xi^{(2)}_{x,t}$ we see from Lemma A.10 that

$$\mathbb{E}[|\bar{\xi}_{x,t}|^p] \le \mathbb{E}|\bar{\xi}_{x,0}|^p + Bn_x^2 \sum_{y \in \bar{\gamma}_x} \int_0^t \mathbb{E}[|\bar{\xi}_{y,s}|^p]ds.$$

Fix an arbitrary $\beta > \alpha$ and $\alpha_1 \in (\alpha, \beta)$. An application of Lemma A.8 to a bounded measurable map $\kappa : \mathscr{T} \to l^1_{\alpha_1}$ defined by the formula

$$\kappa_x(t) := \mathbb{E}[|\bar{\xi}_{x,t}|^p].$$

shows that

$$\sum_{x\in\gamma} e^{-\beta|x|}\sup_{t\in\mathscr{T}}\kappa_x(t) \le K_T(\alpha_1,\beta)\sum_{x\in\gamma} e^{-\alpha_1|x|}|b_x|,\ \ \beta > \alpha_1,$$

where $b_x = \mathbb{E}|\bar{\xi}_{x,0}|^p$. Therefore we establish that

$$\|\Xi^1 - \Xi^2\|^p_{\mathscr{R}^p_\beta} \equiv \sup_{t\in\mathscr{T}} \mathbb{E}\left[\sum_{x\in\gamma} e^{-\beta|x|}|\bar{\xi}_{x,t}|^p\right] \le K_T(\alpha_1,\beta)\sum_{x\in\gamma} e^{-\alpha_1|x|}\mathbb{E}|\bar{\xi}_{x,0}|^p,$$

which implies both statements. ■

D. Markov semigroup

In this section we denote by $\Xi_t(\bar{\zeta})$ the solution of Eq. (2.1) with initial condition $\bar{\zeta}$. This process generates an operator family $\mathrm{T}_t : C_b(l^p_\beta) \to C_b(l^p_\alpha)$, $\alpha < \beta$, $t \ge 0$, by standard formula

$$\mathrm{T}_t f(\bar{\zeta}) = \mathbb{E} f(\Xi_t(\bar{\zeta})). \tag{3.24}$$

Consider the space $l^p_{\alpha+} = \cap_{\beta>\alpha} l^p_\beta$ equipped with the projective limit topology, which makes it a Polish space see e.g., Ref. 21.

Theorem III.7. *Operator family T_t, $t \ge 0$, is a strongly continuous Markov semigroup in $C_b(l^p_{\alpha+})$ for any $\alpha \in \mathscr{A}$.*

Proof. Continuity of the map $L^p_\alpha \ni \bar{\zeta} \mapsto \Xi(\bar{\zeta}) \in \mathscr{R}^p_\beta$, $\alpha < \beta$, for an arbitrary $T > 0$ (cf. Theorem II.5), implies that operators $\mathrm{T}_t : C_b(l^p_\beta) \to C_b(l^p_\alpha)$, $t \ge 0$, are bounded, which in turn implies their boundedness as operators in $C_b(l^p_{\alpha+})$, for any $\alpha \in \mathscr{A}$. The uniqueness of the solution (cf. Theorem II.5) implies in the standard way the evolution property

$$\mathrm{T}_t\mathrm{T}_s = \mathrm{T}_{t+s}, \;\; t, s \ge 0.$$

Observe that the truncated process $\Xi^n_t(\bar{\zeta})$ generates the strongly continuous semigroup $\mathrm{T}^n_t : C_b(l^p_\alpha) \to C_b(l^p_\alpha)$, for any $\alpha \in \mathscr{A}$. It follows from the convergence

$$\Xi^n_t(\bar{\zeta}) \to \Xi_t(\bar{\zeta}), \;\; n \to \infty,$$

in $\mathscr{R}^p_\beta$ for any $\beta > \alpha$ that

$$\sup_{t\in\mathscr{T}} \left\|\mathrm{T}^n_t f(\bar{\zeta}) - \mathrm{T}_t f(\bar{\zeta})\right\|_{C_b(l^p_{\alpha+})}, \;\; n \to \infty, \text{ for any } f \in C_b(l^p_{\alpha+}) \text{ and } \bar{\zeta} \in l^p_{\alpha+},$$

which in turn implies that $\mathrm{T}_t : C_b(l^p_{\alpha+}) \to C_b(l^p_{\alpha+})$ is strongly continuous. ■

Remark III.8. The dominated convergence theorem implies that

$$\int \mathrm{T}^n_t f(\bar{\zeta})\nu(d\bar{\zeta}) \to \int \mathrm{T}_t f(\bar{\zeta})\nu(d\bar{\zeta}), \;\; n \to \infty, \tag{3.25}$$

for any probability measure ν on $l^p_{\alpha+}$.

IV. STOCHASTIC DYNAMICS ASSOCIATED WITH GIBBS MEASURES

As an application of our results, we will present a construction of stochastic dynamics associated with Gibbs measures on S^γ. Sufficient conditions of the existence of these measures were derived in Ref. 14. For the convenience of the reader, we start with a reminder of the general definition of Gibbs measures, adopted to our framework.

A. Construction of Gibbs measures

In the standard Dobrushin–Lanford–Ruelle (DLR) approach in statistical mechanics,[22,33] Gibbs measures (states) are constructed by means of their local conditional distributions (constituting the so-called Gibbsian specification). We are interested in Gibbs measures describing equilibrium states of a (quenched) system of particles with positions $\gamma \subset X = \mathbb{R}^d$ and spin space $S = \mathbb{R}$, defined by pair and single-particle potentials W_{xy} and V, respectively. We assume the following:

- $W_{xy} : S \times S \to \mathbb{R}$, $x, y \in X$, are measurable functions satisfying the polynomial growth estimate

$$|W_{xy}(u,v)| \le I_W\left(|u|^r + |v|^r\right) + J_W, \quad u, v \in S, \tag{4.1}$$

  and the finite range condition $W_{xy} \equiv 0$ if $|x - y| \le \rho$ for all $x, y \in X$ and some constants $I_W, J_W, R, r \ge 0$. We assume also that $W_{xy}(u,v)$ is symmetric with respect to the permutation of (x,u) and (y,v).

- the single-particle potential V satisfies the bound

$$V(u) \geq a_V |u|^\tau - b_V, \quad u \in S, \tag{4.2}$$

for some constants $a_V, b_V > 0$, and $\tau > r$.

Example IV.1 A typical example is given by the pair interaction in of the form

$$W_{xy}(u,v) = a(x-y)u\, v, \quad u,v \in S,$$

where $a : X \to \mathbb{R}$ is as in Sec. II. In this case, $r = 2$ and so we need $\tau > 2$ in (4.2). The method of Ref. 14 does not allow us to control the case of $\tau = 2$, even when the underlying particle configuration γ is a typical realization of a homogeneous Poisson random field on $\Gamma(X)$.

Let $\mathscr{F}(\gamma)$ be the collection of all finite subsets of $\gamma \in \Gamma(X)$. For any $\eta \in \mathscr{F}(\gamma)$, $\bar{\sigma}_\eta = (\sigma_x)_{x\in\eta} \in S^\eta$ and $\bar{z}_\gamma = (z_x)_{x\in\gamma} \in S^\gamma$ define the relative local interaction energy

$$E_\eta(\bar{\sigma}_\eta|\bar{z}_\gamma) = \sum_{\{x,y\}\subset\eta} W_{xy}(\sigma_x,\sigma_y) + \sum_{\substack{x\in\eta \\ y\in\gamma\setminus\eta}} W_{xy}(\sigma_x, z_y).$$

The corresponding specification kernel $\Pi_\eta(d\bar{\sigma}_\gamma|\bar{z}_\gamma)$ is a probability measure on S^γ of the form

$$\Pi_\eta(d\bar{\sigma}_\gamma|\bar{z}_\gamma) = \mu_\eta(d\bar{\sigma}_\eta|\bar{z}_\gamma) \otimes \delta_{\bar{z}_{\gamma\setminus\eta}}(d\bar{\sigma}_{\gamma\setminus\eta}), \tag{4.3}$$

where

$$\mu_\eta(d\bar{\sigma}_\eta|\bar{z}_\gamma) := Z(\bar{z}_{\gamma\setminus\eta})^{-1} \exp\left[-E_\eta(\bar{\sigma}_\eta|\bar{z}_\gamma)\right] \bigotimes_{x\in\eta} e^{-V(\sigma_x)} d\sigma_x \tag{4.4}$$

is a probability measure on S^η. Here $Z(\bar{z}_\eta)$ is the normalizing factor and $\delta_{\bar{z}_{\gamma\setminus\eta}}(d\bar{\sigma}_{\gamma\setminus\eta})$ is the Dirac measure on $S^{\gamma\setminus\eta}$ concentrated on $\bar{z}_{\gamma\setminus\eta}$. The family $\{\Pi_\eta(d\bar{\sigma}|\bar{z}),\ \eta \in \mathscr{F}(\gamma), \bar{z} \in S^\gamma\}$ is called the Gibbsian specification (see e.g., Refs. 22 and 33).

A probability measure ν on S^γ is said to be a Gibbs measure associated with the potentials W and V if it satisfies the DLR equation

$$\nu(B) = \int_{S^\gamma} \Pi_\eta(B|\bar{z})\nu(d\bar{z}), \quad B \in \mathscr{B}(S^\gamma), \tag{4.5}$$

for all $\eta \in \mathscr{F}(\gamma)$. For a given $\gamma \in \Gamma(X)$, by $\mathscr{G}(S^\gamma)$ we denote the set of all such measures.

By $\mathscr{G}_{\alpha,p}(S^\gamma) \subset \mathscr{G}(S^\gamma)$ we denote the set of all Gibbs measures on S^γ associated with W and V, which are supported on l^p_α.

Theorem IV.2. *Assume that conditions (4.1) and (4.2) are satisfied and $p \in [r,\tau]$. Then the set $\mathscr{G}_{\alpha,p}(S^\gamma)$ is non-empty for any $\alpha \in \mathscr{A}$.*

Proof. It follows in a straightforward manner from condition (2.2) that

$$a_{\gamma,p}(\gamma) = \sum_{x\in\gamma} e^{-\alpha|x|} n_x^{p_1} \sum_{y\in\gamma_x} n_y^{p_1} < \infty$$

for any $p_1, p_2 \in \mathbb{N}$, which is sufficient for the existence of $\nu \in \mathscr{G}_{\alpha,p}(S^\gamma)$ for any $p \in [r,\tau]$, see Refs. 27 and 14. ∎

Remark IV.3. The result of Refs. 27 and 14 is more refined and states in addition certain bounds on exponential moments of $\upsilon \in \mathscr{G}_{\alpha,p}(S^\gamma)$.

Remark IV.4. Conditions of the uniqueness of $\upsilon \in \mathscr{G}_{\alpha,p}(S^\gamma)$ are known only in the case of configuration γ with bounded sequence $\{n_x,\ x\in\gamma\}$. Sufficient conditions of non-uniqueness (phase transition) for Poisson-distributed γ are given in Ref. 14.

B. Construction of the stochastic dynamics

In this section, we will construct a process Ξ_t with invariant measure $\nu \in \mathscr{G}_{\alpha_*,p}(S^\gamma)$ defined by interaction potentials W and V as in Example IV.1. By Theorem IV.2, the set $\mathscr{G}_{\alpha_*,p}(S^\gamma)$ is not empty if $p \in [2,\tau]$. Then, according to the general paradigm, Ξ_t will be a solution of the system (2.1) with the coefficients satisfying the following:

(1) the drift coefficient has a gradient form, that is, $\phi = -\nabla V$ and $\varphi_{x,y}(\sigma_x, \sigma_y) = \nabla_{\sigma_x} W_{x,y}(\sigma_x, \sigma_y)$; moreover, ϕ satisfies Conditions (2.4) and (2.5), a typical example is given by

$$\phi(\sigma) = -\sigma^{2n+1} \text{ for any } n = 1, 2, ..,$$

in which case $R = 2n + 1$ and $\tau = 2n + 2$, cf. (2.4);

(2) for each $x \in \gamma$, the noise is additive, that is, $\Psi_x = id$.

Thus the system (2.1) obtains the form

$$d\xi_{x,t} = \frac{1}{2}\left[\nabla V(\xi_{x,t}) + \sum_{y \in \bar{\gamma}_x} a(x-y)\xi_{y,t}\right]dt + dW_{x,t}, \; x \in \gamma.$$

According to Theorem II.5, this system admits a unique strong solution $\Xi \in \mathscr{R}_{\alpha+}$ for any initial condition $\bar{\sigma}_\gamma \in l^p_\alpha$ with arbitrary $\alpha \in \mathscr{A}$ and $p \geq R$.

A standard way of rigorously proving the invariance of ν would require dealing with Markov processes and semigroups in nuclear spaces. This difficulty can be avoided by using the limit transition (3.25).

Theorem IV.5 *Assume that $p \in [\max\{2, R\}, \tau]$ and let T_t be the semigroup defined by the process Ξ_t, cf. (3.24). Then any $\nu \in \mathscr{G}_{\alpha_*,p}(S^\gamma)$ is a reversible (symmetrizing) measure for T_t, that is,*

$$\int \mathrm{T}_t f(\bar{\sigma}_\gamma) g(\bar{\zeta}) \nu(d\bar{\sigma}_\gamma) = \int f(\bar{\sigma}_\gamma) \mathrm{T}_t g(\bar{\zeta}) \nu(d\bar{\sigma}_\gamma)$$

for all $f, g \in C_b(l^p_{\alpha_*+})$.

Proof. First observe that condition $p \in [\max\{2, R\}, \tau]$ ensures that $\mathscr{G}_{\alpha_*,p}(S^\gamma) \neq \emptyset$ and semigroup T_t is well-defined.

Consider the solution $\Xi^n = (\xi^n_x)_{x\in\gamma}$ of the truncated system (3.1). Its non-trivial part $(\xi^n_x)_{x\in\Lambda_n}$ is a Markov process in S^{Λ_n}. We denote by $\mathrm{T}^{\Lambda_n}_t$ the corresponding Markov semigroup in $C_b(S^{\Lambda_n})$ and observe that, for $f \in C_b(S^\gamma)$ and $f_{\bar{z}_\gamma}(\bar{\sigma}_{\Lambda_n}) := f(\sigma_{\Lambda_n} \times \bar{z}_{\gamma\setminus\Lambda_n})$, we have

$$\mathrm{T}^n_t f(\bar{\sigma}_{\Lambda_n} \times \bar{z}_{\gamma\setminus\Lambda_n}) = \mathrm{T}^{\Lambda_n}_t f_{\bar{z}_\gamma}(\bar{\sigma}_{\Lambda_n}).$$

By standard theory of finite dimensional SDEs, $\mu_{\Lambda_n}(d\bar{\sigma}_{\Lambda_n}|\bar{z}_{\Lambda_n})$ given by (4.4) is a reversible (symmetrizing) measure for the semigroup $\mathrm{T}^{\Lambda_n}_t$. Thus we have

$$\int \mathrm{T}^n_t f(\bar{\sigma}_{\Lambda_n} \times \bar{z}_{\gamma\setminus\Lambda_n}) g(\bar{\sigma}_{\Lambda_n} \times \bar{z}_{\gamma\setminus\Lambda_n}) \mu_{\Lambda_n}(d\bar{\sigma}_{\Lambda_n}|\bar{z}_\gamma) = \int f(\bar{\sigma}_{\Lambda_n} \times \bar{z}_{\gamma\setminus\Lambda_n}) \mathrm{T}^n_t g(\bar{\sigma}_{\Lambda_n} \times \bar{z}_{\gamma\setminus\Lambda_n}) \mu_{\Lambda_n}(d\bar{\sigma}_{\Lambda_n}|\bar{z}_\gamma)$$

for any $\bar{z}_\gamma$. The latter implies in turn that

$$\begin{aligned}\int \mathrm{T}^n_t f(\bar{\sigma}_\gamma) g(\bar{\sigma}_\gamma) \Pi_{\Lambda_n}(d\bar{\sigma}_\gamma|\bar{z}_\gamma) &= \int \mathrm{T}^n_t f(\bar{\sigma}_\gamma) g(\bar{\sigma}_\gamma) \; \mu_{\Lambda_n}(d\bar{\sigma}_{\Lambda_n}|\bar{z}_\gamma) \otimes \delta_{\bar{z}_{\gamma\setminus\Lambda_n}}(d\bar{\sigma}_{\bar{\eta}}) \\ &= \int \mathrm{T}^n_t f(\sigma_{\Lambda_n} \times \bar{z}_{\gamma\setminus\Lambda_n}) g(\sigma_{\Lambda_n} \times \bar{z}_{\gamma\setminus\Lambda_n}) \mu_{\Lambda_n}(d\bar{\sigma}_{\Lambda_n}|\bar{z}_\gamma) \\ &= \int f(\bar{\sigma}_\gamma) \mathrm{T}^n_t g(\bar{\sigma}_\gamma) \Pi_{\Lambda_n}(d\bar{\sigma}_\gamma|\bar{z}_\gamma),\end{aligned}$$

where $\Pi_{\Lambda_n}(d\bar{\sigma}_\gamma|\bar{z}_\gamma) = \mu_{\Lambda_n}(d\bar{\sigma}_{\Lambda_n}|\bar{z}_\gamma) \otimes \delta_{\bar{z}_{\gamma\setminus\Lambda_n}}(d\bar{\sigma}_{\bar{\eta}})$ is the specification kernel, cf. (4.3). Integrating with respect to $\nu(d\bar{z}_\gamma)$ and applying the DLR equation (4.5) we see that

$$\int \mathrm{T}^n_t f(\bar{\sigma}_\gamma) g(\bar{\sigma}_\gamma) \nu(d\bar{\sigma}_\gamma) = \int f(\bar{\sigma}_\gamma) \mathrm{T}^n_t g(\bar{\sigma}_\gamma) \nu(d\bar{\sigma}_\gamma).$$

Passing to the limit as $n \to \infty$ [cf. (3.25)] we obtain the equality

$$\int \mathrm{T}_t f(\bar{\sigma}_\gamma) g(\bar{\sigma}_\gamma) \nu(d\bar{\sigma}_\gamma) = \int f(\bar{\sigma}_\gamma) \mathrm{T}_t g(\bar{\sigma}_\gamma) \nu(d\bar{\sigma}_\gamma).$$

as required. ■

Remark IV.6. *An alternative way to construct stochastic dynamics associated with $\nu \in \mathscr{G}_{\alpha_*,p}(S^\gamma)$ is via the theory of Dirichlet forms. Indeed, ν satisfies an integration-by-parts formula and thus defines a classical Dirichlet form, which is a closed bilinear form in $L^2(S^\gamma, \nu)$. The generator of this form is a non-negative self-adjoint operator in $L^2(S^\gamma, \nu)$ and thus defines a strongly continuous semigroup in $L^2(S^\gamma, \nu)$, which, in turn, defines a Markov process in S^γ with invariant measure ν (so-called Hunt process), see Ref. 6 for details. The SDE approach that we use in our work is, however, more explicit and gives in general better control on properties of the stochastic dynamics.*

Remark IV.7. *Observe that pairs $(\gamma, (\sigma_x)_{x\in\gamma})$ form the marked configuration space $\Gamma(X, S)$. For the mathematical formalism of these spaces and discussion of the existence and uniqueness of Gibbs measures and phase transitions, see Refs. 9, 13, 35, and 36 and references therein. The paper[35] considers in particular the case of marks with values in a path space, which gives a complementary way of defining and studying infinite dimensional interacting diffusions indexed by elements of $\gamma \in \Gamma(X)$.*

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Georgy Chargaziya: Conceptualization (equal); Formal analysis (equal); Investigation (equal); Methodology (equal); Writing – original draft (equal); Writing – review & editing (equal). **Alexei Daletskii**: Conceptualization (equal); Formal analysis (equal); Investigation (equal); Methodology (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

APPENDIX: TECHNICAL DETAILS

1. Linear operators in the spaces of sequences

We start with the formulation of a general result from Ref. 11 on the existence of (infinite-time) solutions for a special class of linear differential equations, which extends the so-called Ovsjannikov method, see e.g., Ref. 18.

Definition A.1 *Let $\mathfrak{B} = \{B_\alpha\}_{\alpha\in\mathscr{A}}$ be a scale of Banach spaces. A liner operator $A : \bigcup_{\alpha\in\mathscr{A}} B_\alpha \to \bigcup_{\alpha\in\mathscr{A}} B_\alpha$ is called an Ovsjannikov operator of order $q > 0$ if $A(B_\alpha) \subset B_\beta$ and there exists a constant $L > 0$ such that*

$$\|Ax\|_{B_\beta} \le \frac{L}{(\beta-\alpha)^q}\|x\|_{B_\alpha},\ x \in B_\alpha, \tag{A1}$$

for all $\alpha < \beta \in \mathscr{A}$. The space of such operators will be denoted by $\mathscr{O}(\mathfrak{B}, q)$.

Theorem A.2 (Ref. 11, Theorem 3.1 and Remark 3.3). *Let $A \in \mathscr{O}(\mathfrak{B}, q)$ with $q < 1$. Then, for any $\alpha, \beta \in \mathscr{A}$ such that $\alpha < \beta$ and $f_0 \in B_\alpha$, there exists a unique continuous function $f : [0,\infty) \to B_\beta$ with $f(0) = f_0$ such that:*

(1) *f is continuously differentiable on $(0,\infty)$;*
(2) *$Af(t) \in B_\beta$ for all $t \in (0,\infty)$;*
(3) *f solves the differential equation*

$$\frac{d}{dt}f(t) = Af(t), \quad t > 0.$$

Moreover,

$$\|f(t)\|_{B_\beta} \le K_t(\alpha,\beta)\|f_0\|_{B_\alpha}, \quad t > 0, \tag{A2}$$

where $K_t(\alpha,\beta) := \sum_{n=0}^{\infty} \frac{L^n t^n}{(\beta-\alpha)^{qn}} \frac{n^{qn}}{n!} < \infty$.

Remark A.3 Let us remark that estimate (A2) generalizes the classical estimate $\|e^{tA}\| \le e^{t\|A\|}$ *for the exponent of a bounded operator* A *in a Banach space and does not take into account possible dissipativity properties of* A.

Remark A.4 Function $K_t(\alpha,\beta)$ *can be estimated in the following way, see Ref. 11:*

$$K_t(\alpha,\beta) \le \sum_{n=0}^{\infty} \frac{L^n e^n t^n}{(\beta-\alpha)^{qn}} \frac{1}{n^{(1-q)n}}. \tag{A3}$$

The r.h.s. of (A3) is an entire function of order $\delta = (1-q)^{-1}$ *and type* $\sigma = (Le)^{\delta}(e\delta)^{-1}(\beta-\alpha)^{-q\delta}$. *Thus, for any* $\varepsilon > 0$, *there exists* $t_\varepsilon > 0$ *such that*

$$K_t(\alpha,\beta) \le e^{(\sigma+\varepsilon)t^{\delta+\varepsilon}} \quad \text{for all } t > t_\varepsilon.$$

The aim of this section is to give a sufficient condition for the linear operator Q, given by an infinite real matrix $\{Q_{x,y}\}_{x,y\in\gamma}$, to generate an Ovsjannikov operator in the scale $\mathscr{L}^1$ of spaces of sequences defined by (2.8).

Theorem A.5 *Assume that* $\{Q_{x,y}\}_{x,y\in\gamma}$ *is such that for all* $x, y \in \gamma$ *we have*

- $Q_{x,y} = 0$ *if* $|x-y| > \rho$;
- *there exist* $C > 0$ *and* $k \ge 1$ *such that*

$$|Q_{x,y}| \le C n_x^k. \tag{A4}$$

Then $Q \in \mathscr{O}(\mathscr{L}^1, q)$ *for any* $q < 1$.

Proof. Since Q is linear, it is sufficient to show that

$$\|Qz\|_\beta \le \frac{L}{(\beta-\alpha)^q} \|z\|_\alpha \tag{A5}$$

for any $\alpha < \beta \in \mathscr{A}$ and $z \in l^1_\alpha$. By the definition of the norm in l^1_β we have

$$\|Qz\|_\beta = \sum_{x\in\gamma} e^{-\beta|x|} \left| \sum_{y\in\gamma} Q_{x,y} z_y \right|.$$

Now, using estimate (A4) we see that

$$\begin{aligned}\|Qz\|_\beta &\le \sum_{x\in\gamma}\sum_{y\in\gamma} |Q_{x,y}| e^{-\beta|x|} |z_y| \le e^{\beta\rho} \sum_{x\in\gamma}\sum_{y\in\bar{\gamma}_x} |Q_{x,y}| e^{-\beta|y|} |z_y| \\ &\le e^{\beta\rho} \sum_{x\in\gamma}\sum_{y\in\bar{\gamma}_x} |Q_{x,y}| e^{-(\beta-\alpha)|y|} e^{-\alpha|y|} |z_y| \\ &\le e^{\alpha^*\rho} U \|z\|_\alpha,\end{aligned} \tag{A6}$$

because $Q_{x,y} = 0$ for $y \notin \bar{\gamma}_x$ and $-|x| \le -|y| + \rho$ for $y \in \bar{\gamma}_x$. Here

$$U := \sup_{y\in\gamma} \sum_{x\in\gamma} |Q_{x,y}| e^{-(\beta-\alpha)|y|}.$$

Our next goal is to estimate the constant U. Using condition (A4) we see that for all $y \in \gamma$

$$\sum_{x\in\gamma} |Q_{x,y}| e^{-(\beta-\alpha)|y|} \le C \sum_{x\in B_y} n_x^k e^{-(\beta-\alpha)|y|}.$$

Observe that there exist constants $M, N \in \mathbb{N}$ such that

$$M < |x| \;\Rightarrow\; n_x \le N^{1/k} |x|^{q/2k}.$$

Then, taking into account that $|x|^{q/2} \le |y|^{q/2} + \rho^{q/2}$ for $x \in B_y$, we obtain, assuming without loss of generality that $|y| > M$, that

$$\begin{aligned}\sum_{x\in B_y} n_x^k &\le \sum_{\substack{x\in B_y\\ |x|>M}} N|x|^{q/2} + \sum_{\substack{x\in\gamma\\ |x|\le M}} n_x^k \\ &\le N\sum_{\substack{x\in B_y\\ |x|>M}} \left(|y|^{q/2}+\rho^{q/2}\right) + P \le Nn_y\left(|y|^{q/2}+\rho^{q/2}\right)+P \\ &\le N^2|y|^{q/2}\left(|y|^{q/2}+\rho^{q/2}\right)+P \le 2N^2|y|^q + N^2\rho^q + P,\end{aligned} \tag{A7}$$

where $P = P(\gamma, M, q) := \sum_{\substack{x\in\gamma\\ |x|\le M}} n_x^q < \infty$. Hence for all $y \in \gamma$ we have

$$\sum_{x\in\gamma} |Q_{x,y}|e^{-(\beta-\alpha)|y|} \le C\left(N^2|y|^q + N^2\rho^q + P\right)e^{-(\beta-\alpha)|y|} \le a_1 + a_2|y|^q e^{-(\beta-\alpha)|y|}$$

with $a_1 = C\left(N^2\rho^q + P\right)$ and $a_2 = CN^2$. Now we see that

$$\begin{aligned}U &\le a_1 + a_2\sup\left\{|y|^q e^{-(\beta-\alpha)|y|} \mid y\in\gamma\right\} \le a_1 + a_2 \sup\left\{\left(he^{-\frac{\beta-\alpha}{q}h}\right)^q \mid h>0\right\} \\ &\le a_1 + a_2\left(\sup\left\{he^{-\frac{\beta-\alpha}{q}h} \mid h>0\right\}\right)^q.\end{aligned} \tag{A8}$$

Hence, we can deduce that function $he^{-\frac{\beta-\alpha}{q}h}$, $h\in\mathbb{R}$, attains its supremum when $\frac{d}{dh}he^{-\frac{\beta-\alpha}{q}h} = 0$ that is when $h = \frac{q}{(\beta-\alpha)}$. Hence it follows from inequality (A8) that

$$U \le \frac{a_1(\alpha^* - \alpha_*)^q + a_2(e^{-1}q)^q}{(\beta-\alpha)^q}.$$

Now, continuing from Eq. (A6) we finally see that (A5) holds with $L = e^{\alpha^*\rho}(a_1(\alpha^*-\alpha_*)^q + a_2 q^q)$, and the proof is complete. ∎

2. Comparison theorem and Gronwall-type inequality

In this section, we prove generalizations of the classical comparison theorem for differential equations and, as a consequence, a version of the Gronwall inequality, that works in our scale of Banach spaces of sequences.

Let us consider the linear integral equation

$$f(t) = \bar{z} + \int_0^t Qf(s)ds, \quad t\in\mathscr{T}, \tag{A9}$$

in $l^1_{\alpha^*}$ where $Q \in \mathscr{O}(\mathscr{L}^1, q)$, $q < 1$, is a linear operator generated by the infinite matrix $\{Q_{x,y}\}_{x,y\in\gamma}$ and $\bar{z} = (z_x)_{x\in\gamma} \in l^1_\alpha$ for some $\alpha < \alpha^*$. It follows from Theorem A.2 that this equation has a unique solution $f \in l^1_{\alpha+}$.

The next result is an extension of the classical comparison theorem to our framework.

Theorem A.6 (Comparison Theorem). *Suppose that $Q_{x,y} \ge 0$ for all $x, y \in \gamma$ and let $g : \mathscr{T} \to l^1_\alpha$ be a bounded map such that*

$$g_x(t) \le z_x + \left[\int_0^t Qg(s)ds\right]_x, \quad t\in\mathscr{T},\ x\in\gamma.$$

Then for all $t\in\mathscr{T}$ and all $x\in\gamma$ we have the inequality

$$g_x(t) \le f_x(t),$$

where $f = (f_x)_{x\in\gamma}$ is the solution of (A9).

Proof. Let $\mathscr{B}_a := \mathscr{B}([0,T], l^1_\alpha)$, $a\in\mathscr{A}$, be the Banach space of bounded measurable functions $\mathscr{T}\to l^1_\alpha$. For any $g\in\mathscr{B}_\alpha$ define the function

$$\mathscr{I}(g)(t) := \bar{z} + \int_0^t Qg(s)ds.$$

It is clear that $\mathscr{I}(g) \in \mathscr{B}_{\alpha+}$, which implies that the composition power $\mathscr{I}^n : \mathscr{B}_a \to \mathscr{B}_{a+}$ is well-defined. It follows from (the proof of) Ref. 11, Theorem 3.1 that

$$\overbrace{\left[\lim_{n\to\infty} \mathscr{I}^n(g)\right]}^{\text{in } \mathscr{B}([0,T], l^1_\beta)} = f, \ \beta > \alpha. \tag{A10}$$

Indeed,

$$\mathscr{I}^n(g)(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} Q^k \bar{z} + Q^n \int_0^t \dots \int_0^{t_{n-1}} g(t_n) dt_n \dots dt_1$$

It was proved in Ref. 11, Theorem 3.1, cf. formula (3.5), that the series $\sum_{n=0}^{\infty} \frac{t^n}{n!} Q^n \bar{z}$ converges uniformly in any $l^1_\beta, \beta > \alpha$, and $\sum_{n=0}^{\infty} \frac{t^n}{n!} Q^n \bar{z} = f(t)$. On the other hand, dividing the interval $[\alpha, \beta]$ into n intervals of equal length and using estimate (A1) on each of them, as in Ref. 11, Theorem 3.1, we obtain the bound

$$a_n := \left\| Q^n \int_0^t \dots \int_0^{t_{n-1}} g(t_n) dt_n \dots dt_1 \right\|_{l^1_\beta} \le \sup_t \|g(t)\|_{l^1_\alpha} \frac{t^n}{n!} D^n n^{qn}$$

with $D := L(\beta - \alpha)^{-q}$. Taking into account that $n! \ge \left(\frac{n}{e}\right)^n$ we see that $a_n \to 0, n \to \infty$, which implies (A10).

We have therefore $\lim_{n\to\infty} \mathscr{I}^n_x(g)(t) = f_x(t)$ for all $x \in \gamma$ and all $t \in \mathscr{T}$. Hence to conclude the proof it is sufficient to fix $x \in \gamma$ and prove by induction that for all $t \in \mathscr{T}$ we have

$$g_x(t) \le \mathscr{I}^n_x(g)(t), \ \forall n \in \mathbb{N}. \tag{A11}$$

The case $n = 1$ is satisfied by the initial assumption on g. Let us now assume that (A11) is true for some $n \ge 1$ and proceed by considering the following chain of inequalities:

$$\begin{aligned}\mathscr{I}^{n+1}_x(g)(t) = z_x + \left[\int_0^t Q(\mathscr{I}^n(g)(s)) ds\right]_x &= z_x + \sum_{y\in\gamma} Q_{x,y} \int_0^t \mathscr{I}^n_y(g)(s) ds \ge z_x + \sum_{y\in\gamma} Q_{x,y} \int_0^t g_y(s) ds \\ &= z_x + \left[\int_0^t Q(g(s)) ds\right]_x \ge g_x(t),\end{aligned}$$

which (since t above is arbitrary) completes the proof. ■

Corollary A.7 (Generalized Gronwall inequality). Suppose in addition that $z_x \ge 0$ for all $x \in \gamma$. Moreover assume that components of the map g are non-negative functions, that is, $g_x(t) \ge 0$ for all $x \in \gamma$ and all $t \in \mathscr{T}$. Then for all $\beta > \alpha$ we have the inequality

$$\sum_{x\in\gamma} e^{-\beta|x|} \sup_{t\in\mathscr{T}} g_x(t) \le K_T(\alpha, \beta) \sum_{x\in\gamma} e^{-\alpha|x|} z_x,$$

where $K_T(\alpha,\beta) = \sum_{n=0}^{\infty} \frac{L^n T^n}{(\beta-\alpha)^{qn}} \frac{n^{qn}}{n!} < \infty$.

Proof. Using Theorem A.6, we see that for all $x \in \gamma$ and all $t \in \mathscr{T}$ we have

$$g_x(t) \le z_x + \left[\int_0^t Q(g(s)) ds\right]_x \le z_x + \left[\int_0^t Q(f(s)) ds\right]_x.$$

Since functions g and therefore f are non-negative we see that for all $x \in \gamma$

$$\sup_{t\in\mathscr{T}} g_x(t) \le z_x + \left[\int_0^T Q(f(s)) ds\right]_x = f_x(T).$$

Hence it follows that

$$\sum_{x\in\gamma} e^{-\beta|x|} \sup_{t\in\mathscr{T}} g_x(t) \le \sum_{x\in\gamma} e^{-\beta|x|} f_x(T) \le \|f(T)\|_{l^1_\beta}.$$

The right-hand side of the inequality above can be estimated using Ref. 11, Theorem 3.1, cf. Theorem A.2. In particular, we get

$$\|f(T)\|_{l^1_\beta} \le \sum_{n=0}^{\infty} \frac{L^n T^n}{(\beta-\alpha)^{qn}} \frac{n^{qn}}{n!} \|\bar{z}\|_{l^1_\alpha} < \infty.$$

Hence letting $K_T(\alpha,\beta) = \sum_{n=0}^{\infty} \frac{L^n T^n}{(\beta-\alpha)^{qn}} \frac{n^{qn}}{n!}$ we see that the proof is complete. ■

Lemma A.8 Consider a bounded measurable map $\rho : \mathcal{T} \to l^1_\alpha$, $\alpha \in \mathcal{A}$, and assume that its components satisfy the inequality

$$\rho_x(t) \le B n_x^k \sum_{y\in\bar{\gamma}_x} \int_0^t \varrho_y(s) ds + b_x,\ \ t \in \mathcal{T},\ \ x \in \gamma, \tag{A12}$$

for some constants $B > 0$ and $k \ge 1$ and $b := (b_x)_{x\in\gamma} \in\ l^1_\alpha$, $b_x \ge 0$. Then we have the estimate

$$\sum_{x\in\gamma} e^{-\beta|x|} \sup_{t\in\mathcal{T}} \varrho_x(t) \le K_T(\alpha,\beta) \sum_{x\in\gamma} e^{-\alpha|x|} b_x \tag{A13}$$

for any $\beta > \alpha$, with $K_T(\alpha,\beta) = \sum_{n=0}^{\infty} \frac{L^n T^n}{(\beta-\alpha)^{qn}} \frac{n^{qn}}{n!} < \infty$, cf. Theorem A.2.

Proof. Inequality (A12) can be rewritten in the form

$$\rho_x(t) \le \sum_{y\in\bar{\gamma}} Q_{x,y} \int_0^t \varrho_y(s) ds + b_x,\ \ t \in \mathcal{T},$$

where

$$Q_{x,y} = \begin{cases} B n_x^k, & |x-y| \le \rho, \\ 0, & |x-y| > \rho \end{cases}$$

for all $x \in \gamma$. We have $\rho \in \mathcal{B}(\mathcal{T}, l^1_\alpha)$, and $|Q_{x,y}| \le B n_x^k$. Therefore using Theorem A.5 we conclude that for any $q \in (0,1)$ matrix $(Q_{x,y})$ generates an Ovsjannikov operator of order q on $\mathcal{L}^1$. Therefore we can now use Corollary A.7 to conclude that (A13) holds. ■

3. Estimates of the solutions

We start with the following auxiliary result.

Lemma A.9 Suppose that $\sigma_1, \sigma_2 \in \mathbb{R}$ and $Z_1, Z_2 \in S^\gamma$. Then for all $x \in \gamma$ we have the following inequalities:

$$|\Psi_x(Z_1) - \Psi_x(Z_2)| \le M(n_x+1)|z_{1,x} - z_{2,x}| + M \sum_{y\in\gamma_x} |z_{1,y} - z_{2,y}|,$$
$$|\Psi(0)| \le M n_x,$$

and

$$|\Phi_x(Z_1)| \le c(1 + |z_{1,x}|^R) + \bar{a} n_x (1 + 2|z_{1,x}|) + \bar{a} \sum_{y\in\gamma_x} |z_{1,y}|,$$
$$(z_{1,x} - z_{2,x})(\Phi_x(Z_1) - \Phi_x(Z_2)) \le \left(b + \frac{1}{2} + 4\bar{a}^2 n_x^2\right)(z_{1,x} - z_{2,x})^2 + \frac{1}{2}\bar{a}^2 n_x \sum_{y\in\gamma_x} (z_{1,y} - z_{2,y})^2,$$

where constants M, c, b and $\bar{a}$ are defined in Assumption II.1.

Proof. The proof can be obtained by a direct calculation using assumptions on Φ and Ψ stated in Sec. II. ■

Let us fix $\alpha \in \mathcal{A}$ and consider two processes $\Xi_t^{(1)} = \left(\xi_{x,t}^{(1)}\right)_{x\in\gamma}$ and $\Xi_t^{(2)} = \left(\xi_{x,t}^{(2)}\right)_{x\in\gamma}$, $\Xi^{(1)}, \Xi^{(2)} \in \mathcal{R}^p_{\alpha+}$, with initial values $\Xi^1_0, \Xi^2_0 \in L^p_\alpha$.

Lemma A.10 Let $p \ge 2$, $x \in \gamma$ be fixed and assume that $\mathbb{R}$-valued processes $\xi_{x,t}^{(1)}, \xi_{x,t}^{(2)}$ satisfy Eq. (2.1). Then there exist universal constants B, C_1 and C_2^x such that

$$\mathbb{E}|\xi_{x,t}^{(1)}|^p \le \mathbb{E}\left|\xi_{x,0}^{(1)}\right|^p + C_1 n_x^2 \sum_{y\in\bar{\gamma}_x} \int_0^t \mathbb{E}|\xi_{y,s}^{(1)}|^p ds + C_2^x \tag{A14}$$

and

$$\mathbb{E}|\bar{\xi}_{x,t}|^p \le \mathbb{E}\left|\bar{\xi}_{x,0}\right|^p + B n_x^2 \sum_{y\in\bar{\gamma}_x} \int_0^t \mathbb{E}|\bar{\xi}_{y,s}|^p ds, \tag{A15}$$

for all $t \in \mathscr{T}$, *where* $\bar{\xi}_{x,t} := \xi^{(1)}_{x,t} - \xi^{(2)}_{x,t}$. *The constants* B, C_1 *and* C_2 *are independent of the processes* $\Xi^{(1)}, \Xi^{(2)}$ *and* $x \in \gamma$. *Moreover* $\bar{C}_2 := \{C_2^x\}_{x\in\gamma} \in l^p_\alpha$.

Proof. We remark that in this proof all inequalities hold for all $t \in \mathscr{T}$ and $\mathbb{P} - a.s.$, that is on the same same set of measure 1. We now start with the proof of inequality (A14). Using Itô Lemma we see that if $x \in \Lambda_n$ then for all $t \in \mathscr{T}$

$$|\xi^{(1)}_{x,t}|^p = |\xi^{(1)}_{x,0}|^p + p\int_0^t (\xi^{(1)}_{x,s})^{p-1}\Phi_x(\Xi^{(1)}_s)ds + \frac{(p-1)p}{2}\int_0^t (\xi^{(1)}_{x,s})^{p-2}(\Psi_x(\Xi^{(1)}_s))^2 ds \\ + p\int_0^t (\xi^{(1)}_{x,s})^{p-1}\Psi_x(\Xi^{(1)}_s)dW_x(s).$$

Now from assumptions (2.4) and (2.5) and Lemma A.9 we can deduce that for all $t \in \mathscr{T}$

$$(\xi^{(1)}_{x,s})^{p-1}\Phi_x(\Xi^{(1)}_s) = (\xi^{(1)}_{x,s})^{p-2}(\xi^{(1)}_{x,s})\Phi_x(\Xi^{(1)}_s) \le \left|\xi^{(1)}_{x,s}\right|^{p-2}\left[\left(b + \frac{1}{2} + 4\bar{a}^2 n_x^2\right)|\xi^{(1)}_{x,s}|^2 + \frac{1}{2}\tilde{a}_x^2\sum_{y\in\gamma_x}|\xi^{(1)}_{y,s}|^2 + \left|\xi^{(1)}_{x,s}(\phi(0) + \bar{a}n_x)\right|\right] \\ \le \left(b + \frac{1}{2} + 4\bar{a}^2 n_x^2\right)|\xi^{(1)}_{x,s}|^p + \frac{1}{2}\tilde{a}_x^2|\xi^{(1)}_{x,s}|^{p-2}\sum_{y\in\gamma_x}|\xi^{(1)}_{y,s}|^2 + \left|\xi^{(1)}_{x,s}\right|^{p-1}(c + \bar{a}n_x) \\ \le \left(b + \frac{1}{2} + 4\bar{a}^2 n_x^2\right)|\xi^{(1)}_{x,s}|^p + \frac{1}{2}\tilde{a}_x^2 n_x \max_{y\in\bar{\gamma}_x}|\xi^{(1)}_{y,s}|^p + (1 + |\xi^{(1)}_{x,s}|)^p(c + \bar{a}n_x).$$

where $\tilde{a}_x := \bar{a}\sqrt{n_x}$ and constants $\bar{a}$, b and c are defined in Assumption II.1. In the last inequality, we used the simple estimate $C^{p-1} \le (1 + C)^{p-1} \le (1 + C)^p$ for any $C > 0$, which holds because $p > 1$. Taking into account that $\max_{y\in\bar{\gamma}_x}|\xi^{(1)}_{y,s}|^p \le \sum_{y\in\bar{\gamma}_x}|\xi^{(1)}_{y,s}|^p$ and using inequality $(1 + \alpha)^p \le 2^{p-1}(1 + \alpha^p)$ we arrive at the following:

$$(\xi^{(1)}_{x,s})^{p-1}\Phi_x(\Xi^{(1)}_s) \le \left(b + \frac{1}{2} + 4\bar{a}^2 n_x^2\right)|\xi^{(1)}_{x,s}|^p + \frac{1}{2}\tilde{a}_x^2 n_x\sum_{y\in\bar{\gamma}_x}|\xi^{(1)}_{y,s}|^p + 2^{p-1}(c + \bar{a}n_x) + 2^{p-1}(c + \bar{a}n_x)|\xi^{(1)}_{x,s}|^p \\ \le \left(b + \frac{1}{2} + 4\bar{a}^2 n_x^2 + 2^{p-1}(c + \bar{a}n_x)\right)|\xi^{(1)}_{x,s}|^p + \frac{1}{2}\bar{a}^2 n_x^2\sum_{y\in\bar{\gamma}_x}|\xi^{(1)}_{y,s}|^p + 2^{p-1}(c + \bar{a}n_x).$$

In a similar way, using assumption (2.7) we obtain the estimate

$$(\xi^{(1)}_{x,s})^{p-2}(\Psi_x(\Xi^{(1)}_s))^2 \le (\xi^{(1)}_{x,s})^{p-2}\left[3M^2(n_x + 1)^2|\xi^{(1)}_{x,s}|^2 + 3M^2 n_x\sum_{y\in\gamma_x}|\xi^{(1)}_{y,s}|^2 + 3|\Psi(0)|^2\right] \\ \le 3M^2(n_x + 1)^2|\xi^{(1)}_{x,s}|^p + 3M^2 n_x^2\sum_{y\in\bar{\gamma}_x}|\xi^{(1)}_{y,s}|^p + 3M^2 n_x^2|\xi^{(1)}_{x,s}|^{p-2} \\ \le 3\left(M^2(n_x + 1)^2 + M^2 n_x^2 2^{p-1}\right)|\xi^{(1)}_{x,s}|^p + 3M^2 n_x^2\sum_{y\in\bar{\gamma}_x}|\xi^{(1)}_{y,s}|^p + 3M^2 n_x^2 2^{p-1}. \tag{A16}$$

Observe that $n_x \ge 1$. Thus there exist constants $C_1, C_2^x > 0$ such that

$$|\xi^{(1)}_{x,t}|^p \le \left|\xi^{(1)}_{x,0}\right|^p + C_1 n_x^2\sum_{y\in\bar{\gamma}_x}\int_0^t|\xi^{(1)}_{y,s}|^p ds + C_2^x + p\int_0^t(\xi^{(1)}_{x,s})^{p-1}\Psi_x(\Xi^{(1)}_s)dW_x(s),$$

which implies that (A14) holds.

The proof of inequality (A15) can be obtained similarly. Using the relation

$$\bar{\xi}_{x,t} = \bar{\xi}_{x,0} + \int_0^t\left(\Phi_x(\Xi^{(1)}_s) - \Phi_x(\Xi^{(2)}_s)\right)ds + \int_0^t\left(\Psi_x(\Xi^{(1)}_s) - \Psi_x(\Xi^{(2)}_s)\right)dW_x(s),$$

$t \in \mathscr{T}$, and applying the Itô Lemma to $|\bar{\xi}_{x,t}|^p$ we obtain the inequality

$$|\bar{\xi}_{x,t}|^p \leq \left|\bar{\xi}_{x,0}\right|^p + Bn_x^2 \sum_{y \in \tilde{\gamma}_x} \int_0^t |\bar{\xi}_{y,s}|^p ds + \int_0^t p(\bar{\xi}_{x,t})^{p-1} \left(\Psi_x(\Xi_s^{(1)}) - \Psi_x(\Xi_s^{(2)})\right) dW_x(s) \tag{A17}$$

for some constant $B > 0$, which implies the result. Finally, $\bar{C}_2 \in l_\alpha^p$ because (see Assumption II.1) for some constant W we have $C_2^x \leq W(1 + \log(1 + |x|))$ and one can use exponential weight to sum up these terms. ■

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